Exercises 1.3

1.

Calculate $\binom{6}{2}$ and check your answer by listing all the selections of size 2 that can be made from the letters a, b, c, d, e, and f.

$$\binom{6}{2} = 15$$

ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, ef

2.

Facing a four-hour bus trip back to college, Diane decides to take along five magazines from the 12 that her sister Ann Marie has recently acquired. In how many ways can Diane make her selection?

$$\binom{12}{5} = 792$$

3.

Evalueate each of the following:

$$C(10, 4) = 210$$

$$\binom{12}{7} = 792$$

$$C(14, 12) = 91$$

$$\binom{15}{10} = 3003$$

4.

In the Braille system a symbol, such as a lowercase letter, punctuation mark, suffix, and so on, is given by raising a t least one of the dots in the six-dot arrangement.

a. How many different symbols can we represent in the Braille system?

b. How many symbols have exactly three raised dots?

c. How many symbols have an even number of raised dots?

How many different symbols can we represent int he Braille system?

Each dot can be raised or not raised, giving:

$$2^6 = 64 \text{ total patterns}$$

But one of these patterns is the unraised (blank) cell, which is *not* considered a symbol:

$$64 - 1 = 63$$

How many symbols have exactly three raised dots?

Choose 3 of the 6 dots:

$$\binom{6}{3} = 20$$

How many symbols have an even number of raised dots?

Even possibilities: 2, 4, or 6 raised:

$$\binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 31$$

5.

a. How many *permutations* of size 3 can one produce with the letters m, r, a, f, and t?

b. List all the *combinations* of size 3 that result for the letters m, r, a, f, and t.

A. Permutations

$$\frac{5!}{(5-3)!} = 60$$

B. Combinations

$$\binom{5}{3} = 10$$

6.

If n is a positive integer and $n > 1$, prove that $\binom{n}{2} + \binom{n-1}{2}$ is a perfect square.

Compute directly using the binomial coefficient formula:

$$\binom{n}{2} + \binom{n-1}{2} = \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} = \frac{(n-1)(n+n-2)}{2} = \frac{(n-1)(2n-2)}{2} = (n-1)^2$$

Since $(n-1)^2$ is a perfect square for every $n > 1$, the sum is always a perfect square.

Example: for $n = 5$, $\binom{5}{2} + \binom{4}{2} = 10 + 6 = 16 = 4^2$

7.

A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if (a) there are no restrictions? (b) there must be six men and six women? (c) there must be an even number of women? (d) there must be more women than men? (e) there must be at least eight men?

a. No restrictions

If there are no restrictions than we can select any 12 from the 10 men or 10 women.

$$\binom{20}{12} = 125,970$$

b. There must be 6 men and 6 women.

$$\binom{10}{6} \times \binom{10}{6} = 44,100$$

c. There must be an even number of women.

There must be either 2, 4, 6, 8 or 10 women.

$$\begin{aligned}\binom{10}{2}\binom{10}{10} + \binom{10}{4}\binom{10}{8} + \binom{10}{6}\binom{10}{6} + \binom{10}{8}\binom{10}{4} + \binom{10}{10}\binom{10}{2} \\ &= \sum_{i=1}^{5}\binom{10}{12-2i}\binom{10}{2i} \\ &= 63,090\end{aligned}$$

d. There must be more women than men.

If there are more women than men then there must be:

$$\begin{matrix} 2\ m & 10\ w \\ 3\ m & 9\ \ w \\ 4\ m & 8\ \ w \\ 5\ m & 7\ \ w \end{matrix}$$

$$\begin{aligned}\sum_{i=1}^{4}\binom{10}{1+i}\binom{10}{11-i} \\ &= \binom{10}{2}\binom{10}{10} + \binom{10}{3}\binom{10}{9} + \binom{10}{4}\binom{10}{8} + \binom{10}{5}\binom{10}{7} \\ &= 40,935\end{aligned}$$

e. There must be at least eight men.

If there must be at least eight men then there could be 8, 9, or 10 men.

$$\sum_{i=8}^{10} \binom{10}{i}\binom{10}{12-i}$$

$$= \binom{10}{8}\binom{10}{4} + \binom{10}{9}\binom{10}{3} + \binom{10}{10}\binom{10}{2}$$
$$= 10,695$$

8.

In how many ways can a gambler draw five cards from a standard deck and get (a) a flush (five cards of the same suit)? (b) four aces? (c) four of a kind? (d) three aces and two jacks? (e) three aces and a pair? (f) a full house (three of a kind and a pair)? (g) three of a kind? (h) two pairs?

a. A flush.

Choose the suit 4 ways and choose 5 cards from 13.

$$4 \times \binom{13}{5} = 5,148$$

b. Four Aces.

You must have four aces and any one card.

$$\binom{4}{4}\binom{48}{1} = 48$$

c. Four of a kind.

Choose the rank for the four (13 ways), take all 4 suits, and choose from the remaining 48.

$$13 \times 48 = 624$$

d. Three aces and two jacks.

Choose 3 of the 4 aces and 2 of the 4 jacks:

$$\binom{4}{3}\binom{4}{2} = 24$$

e. Three aces and a pair.

Choose 3 of the 4 aces and choose a pair from the remaining 12 ranks:

$$\binom{4}{3} \times 12 \times \binom{4}{2} = 288$$

f. A full house (three of a kind and a pair).

Choose a triple from 13, and choose a pair from 12:

$$13 \times \binom{4}{3} \times 12 \times \binom{4}{2} = 3,744$$

g. Three of a kind.

Choose a triple from 13, and choose two any singletons from 12 from 1 of 4 suits.

$$13 \times \binom{4}{3} \times 4^2 \times \binom{12}{2} = 54,912$$

h. Two pairs.

Choose 2 ranks for the pairs. For each choose rank choose 2 of 4 suits. Choose the fifth card's rank from the remaining 11 ranks and one of its 4 suits.

$$\binom{13}{2} \times \binom{4}{2}^2 \times 11 \times 4 = 123,552$$

9.

How many bytes contain (a) exactly two 1's; (b) exactly four 1's; (c) exactly six 1's; (d) at least six 1's?

a. exactly two 1's

Choose 2 from 8

$$\binom{8}{2} = 28$$

b. exactly four 1's

Choose 4 from 8

$$\binom{8}{4} = 70$$

c. Exactly six 1's

Choose 6 from 8

$$\binom{8}{6} = 28$$

d. At least six 1's

Choose 6, 7, and 8 from 8

$$\binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 37$$

10.

How many ways are there to pick a five-person basketball team from 12 possible players? How many selections include the weakest and the strongest players?

There are choose 5 from 12 ways.

$$\binom{12}{5} = 792$$

We need the weakest and the strongest already chosen. There are choose 3 from 10 ways to choose the rest.

$$\binom{10}{3} = 120$$

11.

A student is to answer seven out of 10 questions on an examination. In how many ways can he make his selection if (a) there are no restrictions? (b) he must answer the first two questions? (c) he must answer at least four of the first six questions?

Must answer seven out of 10 with...

a. No restrictions.

With no restrictions, the student needs to choose 7 from 10.

$$\binom{10}{7} = 120$$

b. Must answer the first two questions.

The first two must be chosen. Then the student can choose 5 from 8.

$$\binom{8}{5} = 56$$

c. Must answer at least four of the first six questions.

The student must choose at least 4 from the first 6. This breaks down to three cases:

Case 1: 4 from first 6, 3 from last 4

$$\binom{6}{4}\binom{4}{3} = 60$$

Case 2: 5 from first 6, 2 from last 4

$$\binom{6}{5}\binom{4}{2} = 36$$

Case 3: 6 from first 6, 1 from last 4

$$\binom{6}{6}\binom{4}{1} = 4$$

Combine the results

$$\binom{6}{4}\binom{4}{3}+\binom{6}{5}\binom{4}{2}+\binom{6}{6}\binom{4}{1}=100$$

12.

In how many ways can 12 different books be distributed among four children so that (a) each child gets three books? (b) the two oldest children get four books each and the two youngest get two books each?

a. Each child gets three books

choose 3 from 12

$$\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}=\frac{12!}{3!3!3!3!}=369,600$$

b. The two oldest children get four, two youngest get two.

choose 4 from 8 for 2 children; choose 2 from 4 for 2 children

$$\binom{12}{4}\binom{8}{4}\binom{4}{2}\binom{2}{2}=\frac{12!}{4!4!2!2!}=207,900$$