Exercises 1.1 and 1.2

1. The rule of sum and product

During a local campaign, eight Republican and five Democrat candidates are nominated for president of the school board.

a. If the president is to be one of these candidates, how many possiblities are there for the eventural winner?

b. How many possibleities exist for a pair of candidates (one from each party) to oppose each other for the eventual election?

c. Which counting principle is used in part (a)? in part (b)?

The president is to be one of eight Republicans or five Democrats then any one of the 13 candidates could be president.

There are $5 \times 8 = 40$ different possible races between a Republican and Democrat candidate.

The first question (a) uses the rule of sums. The second question (b) uses the rule of products.

2.

If manufacturing of license plates consists of two letters and four digits, where repetition is allowed. How many plates are there that have only vowels (A, E, I, O, U) and even digits? (0 is an even integer).

There are $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 15,625$ different license plates that only have vowels and even digits.

3.

Buick automobiles come in four models, 12 colors, three engine sizes, and tow transmission types. (a) How many distinct Buicks can be manufactured? (b) If one of the available colors is blue, how many different blue Buicks can be manufactured?

a. There are $4 \times 12 \times 3 \times 2 = 288$ different Buicks.

b. There are $4 \times (1) \times 3 \times 2 = 24$ different blue Buicks.

4.

The board of directors of a pharmaceutical corporation has 10 members. An upcoming stockholders' meeting is scheduled to approve a new slate of company officers (chosen from 10 board members). (a) How many different slates consisting of a president, vice president, secretary, and treasurer can the board present to the stockholders for their approval? (b) Three members of the board of directors [from part (a)] are physicians. How many slates from part (a) have (i) a physician nominated for the presidency? (ii) exactly one physician appearing on the slate? (iii) at least one physician appearing on the slate?

a. Number of slates of company officers

There are 10 members to the board and 4 officer positions. That means there are $\frac{10!}{(10-4)!} = 5,040$ slates of candidates which the stockholders could elect.

b. Three physicians on the board

i. Number of slates with a physician as president

If three members of the board are physicians then each of them could be president. The rest of the slots could be held by any of the other board members.

$$3 \times \frac{9!}{(9-3)!} = 1,512$$

ii. Number of slates with exactly one physician

If one of the slots had to be filled by one of the three physicians, the rule of products states that there are $3 \times 4 = 12$ different ways to fill those offices with a physician. The other three slots are filled with the 7 board members who are not physicians.

$$3 \times 4 \times \frac{7!}{(7-3)!} = 2,520$$

iii. Number of slates with at least one physician

If at least one of the officers in the slate need to be filled with a physician then we can subtract the number of slates without an officer from the total number of slates with all 10 board members.

$$\frac{10!}{(10-4)!} - \frac{7!}{(704)!} = 4,200$$

5.

While on a Saturday shopping spree Jennifer and Tiffany witnessed two men driving away from the front of a jewelry shop, just before a burglar alarm started to sound. Although everything happened rather quickly, when the two young ladies were questioned they were able to give the police the following information about the license plate (which consisted of two letters followed by four digits) on the getaway car. Tiffany was sure that the second letter on the plate was either an O or Q and the last digit was either a 3 or an 8. Jennifer told the investigator that the first letter on the plate was either a C or a G and that the first digit was definitely a 7. How many different license plates will the police have to check?

Given:

- License plate has 2 letters and four digits.
- Tiffany says second letter was an O or Q, last digit was either 3 or 8
- Jennifer says first letter was a C or G, first digit was a 7.

Immediately we see that only two digits remain wild (0 - 9). The number of plates to check is $2 \times 2 \times 2 \times 1 \times 10 \times 10 = 800$.

6.

To raise money for a new municipal pool, the chamber of commerce in a certain city sponsors a race. Each participant pays a $5 entrance fee and has a chance to win one of the different-size trophies that are to be awareded to the first eight runners who finish.

a. If 30 people enter the race, in how many ways will it be possible to award trophies?

b. If Roberta and Candice are two participants in the race, in how many ways can the trophies be awarded with these two runners among the top three?

There are eight trophies and 30 candidates. That means there are:

a.

$$\frac{30!}{(30-8)!} = 235,989,936,000$$

b.

There are $(2 \times 3 = 6)$ ways to place Roberta and Candice among the top three. Then we need to choose one additional from the 28 as a top-3 finisher. Then we order the remaining for the trophy winners.

$$6 \times 28 \times \frac{27!}{(27-5)!} = 1,627,516,800$$

7.

A certain "Burger Joint" advertises that a customer can have his or her hamburger with or without any or all of the following: catsup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, or mushrooms. How many different kinds of hamburger orders are possible?

There are 9 independent condiments. The choice is to include it or don't include it. The number of total choices is $2^9 = 512$.

8.

Matthew works as a computer operator at a small university. One evening he finds that 12 computer programs have been submitted earlier that day for batch processing. In how many ways can Matthew order the processing of these programs if (a) there are no restrictions? (b) he considered four of the programs higher in priority than the other eight and wants to process those four first? (c) he first separates the programs into four of top priority, five of lesser priority, and three of least priority, and he wishes to process the 12 programs in such a way that the top-priority programs are processed first and the three programs of least priority are processed last?

a. no restrictions

$$12! = 479,001,600$$

b. four programs are high priority

$$4! \times 8! = 967,680$$

c. three tiers (4, 5, 3)

$$4! \times 5! \times 3! = 17,280$$

9.

Patter's Pastry Parlor offers eight different kinds of pastry and six different kinds of muffins. In addition to bakery items one can purchase small, medium, or large containers of the following beverages: coffee (black, with cream, with sugar, with cream and sugar), tea (plain, with cream, with sugar, with cream and sugar, with lemon, or with lemon and sugar), hot cocoa, and orange juice. When Carol comes to Patter's, in how many ways can she order:

a. one bakery item and one medium-sized beverage for herself?

b. one bakery item and one conainer of coffee for herself and one muffin and one container of tea for her boss. Ms. Didio?

c. one piece of pastry and one container of tea for herself, one muffin and a container of orange juice for Ms. Didio, and one bakery item and one container of coffee for each of her two assistants, Mr. Talbot and Mrs. Gillis?

Items:

- 8 pastry and six muffins: 14 bakery items
- 3 sizes of: coffee (x4), tea (x6), hot cocoa, and orange juice.

a. one bakery item and one medium-sized beverage

There are 14 different bakery items and 12 different types of medium-sized beverages.

$$14 \times 12 = 168$$

b. one bakery item and one coffee for herself, one muffin and one tea.

There are 14 different bakery items and ($3 \times 3 = 12$) different coffees. There are 6 muffins and ($3 \times 6 = 18$) different teas.

$$(14 \times 12) \times (6 \times 18) = 18,144$$

c. one pastry and one container of tea ($3 \times 6 = 18$) ...

- one pastry (8) and one container of tea ($3 \times 6 = 18$)
- one muffin (6) and one container of orange juice (3)
- one bakery (14) and one container of coffee ($3 \times 4 = 12$)

$$(8 \times 18) \times (6 \times 3) \times (14 \times 12)^2 = 73,156,608$$

10.

Pamela has 15 different books. In how many ways can she place her books on two shelves so that there is at least one book on each shelf? (Consider the books in each arrangement to be stacked one next to the other, with the first book on each shelf at the left of the shelf.)

Key points:

- There are 15 distinct books.
- There are 2 distinct shelves.
- Each shelf must get at least one book.

- On each shelf, the books are arranged where order matters.

Step-by-step approach

For each book, you have to decide two things:

1. Which shelf it goes to.
2. In what position it appears on that shelf.

Here is a way to approach it:

First assign every book to a shelf (allowing empty shelves), then arrange the books on each shelf, and finally subtract the bad cases.

There are 2^15 different ways to assign the books to each shelf. Once the books are assigned, suppose k books go to one shelf, then $15 - k$ books go to the other shelf.

- k books on the first shelf can be arranged in $k!$ ways.
- The $15 - k$ books on the second shelf can be arranged in $(15 - k)!$ ways.

So the total number of arrangements (including cases where one shelf is empty) is:

$$\sum_{k=0}^{15} \binom{15}{k} \cdot k! \cdot (15-k)! = \sum_{k=0}^{15} \frac{15!}{k!(15-k)!} \cdot k! \cdot (15-k)! = \sum_{k=0}^{15} 15! = 16 \times 15!$$

This includes both of the cases where the one of the shelf is empty. Therefore, the number of ways where both shelves have at least one book is:

$$\begin{aligned}\text{Total (allowing empty)} - \text{all on A} - \text{all on B} \\ = 16 \times 15! - 15! - 15! \\ = 16 \times 15! - 2 \times 15! \\ = 14 \times 15!\end{aligned}$$

Alternative method

Imaging the 15 distinct books in a row. Insert one "divider" somehwhere between them (but not at the very beginning or very end) to separate the books that go on the two shelves. The books on the left side go on shelf one, and the books on the right side go on shelf two. There are 14 possible places to put the divider. There are 15! permutations of the books. Thus each permutation can be paird with any of the 14 divider positions:

$$14 \times 15!$$

11.

Three small towns, designated by A, B, and C, are interconnected by a system of two-way roads.

a. In how many ways can Linda travel from town A to town C?

b. How many different round trips can Linda travel from town A to town C and then back to town A?

c. How many of the round trips in part (b) are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C? (For example, if Linda travels from town A to town C along roads R_1 and R_6, then on her return she might take roads R_6 and

R_3, or roads R_7 and R_2, or road R_9, among other possibilities, but she does *not* travel on roads R_6 *and* R_1.)

a.

$$2 + 4 \times 3 = 14$$

b.

$$14 \times 14 = 196$$

c.

$$14 \times 14 - 14 = 182$$

12.

List all the permutations for the letters a, c, t.

$$3! = 6$$

13.

a. How many permutations are there for the eight letters a, c, f, g, i , t, w, x?

b. Consider the permutations in part (a). How many start with the letter t? How many start with the letter t and end with the letter c?

a.

$$8! = 40,320$$

b.

Start with ‘t’: $7! = 5,040$

Start with ‘t’ and end with ‘c’: $6! = 720$

14.

Evaluate each of the following.

a. $P(7, 2) = 42$

b. $P(8, 4) = 1,680$

c. $P(10, 7) = 604,800$

d. $P(12, 3) = 1,320$

15.

In how many ways can the symbols a, b, c, d, e, e, e, e, e be arrranged so that no e is adjacent to another e?

To do this we reorder the other letters between each of the e letters. There are $4! = 24$ ways to do this.

16.

An alphabet of 40 symbols is used for transmitting messages in a communication system. How many distinct messages (lists of symbols) of 25 symbols can the transmitter generate if symbols can be repeated in the message? How many if 10 of the 40 symbols can appear only as the first and/or last symbols of the message, the other 30 symbols can appear anywhere, and repetitions of all symbols are allowed?

The number of distinct messages of 25 symbols with repetitions is 40^{25}. The number of distinct messages of 25 symbols under the given restrictions with repetitions allowed is $40^2 \times 30^{23}$.

17.

In a certain implementation of the programming language Pascal, an identifier consists of a single letter or a letter followed by up to seven symbols, which may be letters or digits. (We assume that the computer does not distinguish between capital and lowercase letters; there are 26 letters and 10 digits). Certain keywords, however, are reserved for commands; consequently, these keywords may not be used as identifiers. If this implementation has 36 reserved words, how many distinct identifiers are possible in this version of Pascal?

There can be 26 different one letter identifiers (all identifiers begin with a letter).

$$\begin{aligned} 26 + 26 \times 36^1 + 26 \times 36^2 + \cdots + 26 \times 36^7 \\ &= \sum_{i=0}^{7} 26(36)^i - 36 \\ &= 2.095682e + 12 \end{aligned}$$

18.

Morgan is considering the purchase of a low-end computer system. After some careful investigating, she finds that there are seven basic systems (each consisting of a monitor, CPU, keyboard, and mouse) that meet her requirements. Furthermore, she also plans to buy one of four modems, one of three CD ROM drives, and one of six printers. (Here each peripheral device of a given type, such as the modem, is compatible with all seven basic systems.) In how many ways can Morgan configure her low-end computer system?

Choices

- Seven systems
- Four modems
- Three CD ROMS
- Six printers

$$7 \times 4 \times 3 \times 6 = 504$$

19.

A computer science professor has seven different programming books on a bookshelf. Three of the books deal with C++, the other four with Java. In how many ways can the professor arrange these books on the shelf (a) if there are no restrictions? (b) if the languages should alternate? (c) if all the C++ books must be next to each other? (d) if all C++ books must be next to each other and all the Java books must be next to each other?

a.

$$7! = 5,040$$

b.

$$3! \times 4! = 144$$

c.

d.

$$3! + 4! = 30$$

20.

a. In how many ways can the letters in VISITING be arranged?
b. For the arrangements of part (a), how many have all three I's together?

a.

$$\frac{8!}{3!} = 6,720$$

b.

$$6! = 720$$

21.

a. How many arrangements are there of all the letters in SOCIOLOGICAL?
b. In how many of the arrangements in part (a) are A and G adjacent?
c. In how many of the arrangements in part (a) are all the vowels adjacent?

a.

There is 1 S, 3 O, 2 C, 2 I, 1 L, 1 G, 1 L

$$\frac{12!}{1! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} = 19,958,400$$

b.

$$2 \times \frac{11!}{3! \cdot 2! \cdot 2!} = 3,326,400$$

c.

$$\frac{7!}{2! \cdot 2!} = 1,260$$

22.

How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

The first digit: $\frac{3!}{2!} = 3$ ways.

After placing the first digit, the remaining six digits are chosen from the six leftover digits, which still include one 4 and one 5 left in most cases (or none, if you already used the 6 or 7).

First digit	Remaining	Permutations
5	$\{3, 4, 4, 5, 6, 7\}$	$\frac{6!}{2!} = 360$
6	$\{3, 4, 4, 5, 5, 7\}$	$\frac{6!}{2!\cdot 2!} = 180$
7	$\{3, 4, 4, 5, 5, 6\}$	$\frac{6!}{2!\cdot 2!} = 180$

Therefore the total number is $360 + 2 \times 180 = 720$

23.

Twelve clay targets (identical shape) are arranged in four hanging columns, as shown in Fig. 1.5. There are four red targets in the first column, three white ones int he second column, two green targets in the third column, and three blue ones in the fourth column. To join her college drill team, Deborah must break all 12 of these targets (using her pistol and only 12 bullets) and in so doing must always break the existing target at the bottom of a column. Under these conditions how many different orders can Deborah shoot down (and break) the 12 targets?.

$$\frac{12!}{4! \cdot 3! \cdot 2! \cdot 3!} = 831,600$$

24.

Show that for all integers n, $r \geq 0$, if $n + 1 > r$, then

$$P(n+1, r) = \left(\frac{n+1}{n+1-r} \right) P(n, r).$$

The definition of permutations

$$P(k, r) = k \times (k-1) \times (k-2) \times \cdots \times (k-r+1) = \frac{k!}{(k-r)!} \text{ (for } k \leq r).$$

Left-hand side:

$$P(n+1, r) = (n+1) \times n \times (n-1) \times \cdots \times (n+1-r+1) = (n+1) \times n \times (n-1) \times \cdots \times (n-r+2).$$

Right-hand side:

$$P(n, r) = n \times (n-1) \times \cdots \times (n-r+1).$$

So

$$\frac{n+1}{n+1-r}P(n,r) = \frac{n+1}{n+1-r} \times [n \times (n-1) \times \cdots \times (n-r+1)].$$

Notice that the product $n \times (n-1) \times \cdots \times (n-r+1)$ has exactly r factors, and the smallest (last) factor is $n-r+1$.

The denominator on the right-hand side is

$$n+1-r = (n-r+1)+1.$$

Thus,

$$\frac{n+1}{n+1-r}P(n,r) = (n+1) \times [n \times (n-1) \times \cdots \times (n-r+1)] = (n+1) \times n \times (n-1) \times \cdots \times (n-r+2),$$

which is exactly the product for $P(n+1,r)$.

Therefore,

$$P(n+1,r) = \frac{n+1}{n+1-r}P(n,r).$$

(The condition $n+1 > r$ guarantees $(n+1-r > 0)$, so we are never dividing by zero or a negative integer when interpreting the permutations formulas.)

25.

Find the value(s) of n in each of the following:

a. $P(n,2) = 90$

$$\begin{aligned} P(n,2) &= n \times (n-1) = 90 \\ &= n^2 - n - 90 = 0 \end{aligned}$$

Using the quadratic formula:

$$n = \frac{1 \pm \sqrt{1+360}}{2} = \frac{1 \pm \sqrt{361}}{2} = \frac{1 \pm 19}{2}$$

The two solutions are 10 and -9. n must be positive so it is 10.

b. $P(n,3) = 3P(n,2)$

$$\begin{aligned} P(n,3) &= n \cdot (n-1) \cdot (n-2) \\ P(n,2) &= n \cdot (n-1) \end{aligned}$$

$$\begin{aligned} P(n,3) &= 3 \cdot P(n,2) \\ n \cdot (n-1) \cdot (n-2) &= 3 \cdot n \cdot (n-1) \\ n-2 &= 3 \\ n &= 5 \end{aligned}$$

c. $2 \cdot P(n, 2) + 50 = P(2n, 2)$

$$\begin{aligned}
2 \cdot P(n, 2) + 50 &= P(2n, 2) \\
2 \cdot (n \cdot (n-1)) + 50 &= 2n \cdot (2n-1) \\
2n^2 - 2n + 50 &= 4n^2 - 2n \\
2(n^2 - n + 25) &= 2(2n^2 - n) \\
n^2 - n + 25 &= 2n^2 - n \\
25 &= n^2 \\
n &= \pm 5 \\
n &= 5
\end{aligned}$$

26.

How many different paths in the *xy*-plane are there from (0,0) to (7,7) if a path proceeds one step at a time by going either one space to the right (R) or one spaceupward (U)? How many such paths are there from (2, 7) to (9, 14)? Can any general statement be made that incorporates these two results?

a. (0,0) to (7,7)

There are 7 Rs and 7 Us. Therefore, we can calculate this by $\frac{14!}{7!7!} = 3,432$.

b. (2, 7) to (9, 14)

If we take the distances to the right and up we find that we have the same number of Rs and Us. Therefore the answer is the same.

c.

Generally we can find the paths by counting the number of R moves and number of U moves to get from the first point to the second.

27.

a. How many distinct paths are there from (-1, 2, 0) to (1, 3, 7) in Euclidean three-space if each move is one of the following types?

(H): (x, y, z) $\rightarrow$ (x + 1, y, z); (V): (x,y,z) $\rightarrow$ (x, y + 1, z); (A): (x, y, z) *rightarrow* (x, y, z+1)

b. How many such paths are there from (1, 0, 5) to (8, 1, 7)?

c. Generalize the results in parts (a) and (b).

28.

a. Determine the value of the integer variable *counter* after execution of the following program segment. (Here *i*, *j*, and *k* are integer variables.)

```
counter := 0
for i := 1 to 12 do
  counter := counter + 1
for j := 5 to 10 do
  counter := counter + 2
for k := 15 downto 8 do
```

```
  counter := counter + 3
```

b. Which counting principle is being used?

a. Execute each line of code

1. The first line sets counter = 0.
2. Next we enter the first loop which runs 12 times and increments counter on each iteration: `counter = 0 + 1 x 12 = 12`.
3. The second loop counts from 5 to 10 (6 iterations) and increments counter 2 each iteration: `counter = 12 + 2 x 6 = 24`.
4. Finally, the third loop counts down from 15 to 8 (8 iterations) and increments counter 3 each iteration: `counter = 24 + 3 x 8 = 48`.

The final value of `counter` is 48.

b. The counting principle at work

The Addition principle or the rule of sum. The total is the sum of the contributions from the three disjoint (independent, non-overlapping) processes:

29.

Consider the following program segment where i, j, and k are integer variables.

```
for i := 1 to 12 do
  for j := 5 to 10 do
    for k := 15 downto 8 do
      print(i - j)*k
```

a. How many times is the **print** statement executed?

b. Which counting principle is used?

a. Counting the print statements.

Start with the second inner loop and work your way out.

1. The innermost (k) loop is executed 8 times.
2. The middle (j) loop is executed 6 times.
3. The outer (i) loop is executed 12 times.

The total print statements is $8 \times 6 \times 12 = 576$

b. The counting principle at work.

The **multiplication principle** (also called the rule of product). When a procedure consists of performing k sequential (or nested) independent choices, where the first choice can be made in n_1 ways, the second in n_2 ways, etc..., the total number of possible outcomes is $n_1 \times n_2 \times \ldots \times n_k$.

In this specific problem we have three independent stages so we multiply the number of possibilities at each stage.

30.

A sequence of letters of the form `abcba`, where the expression is unchanged upon reversing order, is an example of a *palindrome* (of five letters). (a) If a letter may appear more than twice, how many palindromes of five letters are there? of six letters? (b) Repeat part (a) under the condition that no letter appears more than twice.

a. 5-letter palindrome where repetitions are allowed.

- Positions 1 and 5 must be the same letter.
- Positions 2 and 4 must be the same letter.
- Position 3 can be any letter.

$$26 \times 26 \times 26 = 26^3 = 17,576$$

b. 5-letter palindrome where repetitions are not allowed.

- Position 1 and 5 must be the same letter.
- Position 2 and 4 must be the same letter and different from the previous letter.
- Position 3 must be different from the previous two letters.

$$26 \times 25 \times 24 = \frac{26!}{(26-3)!} = 15,600$$

31.

Determine the number of six-digit integers (no leading zeros) in which (a) no digit may be repeated; (b) digits may be repeated. Answer parts (a) and (b) with the extra condition that the six-digit integer is (i) even; (ii) divisible by 5; (iii) divisible by 4.

a. 6 digits, no digit repeats and no leading zeros

$$9 \times 9 \times 8 \times 7 \times 6 \times 5 = 9 \times \frac{9!}{(9-5)!} = 136,080$$

b. Digits may be repeated, but no leading zeros

$$9 \times 10^5 = 9e+05$$

i. For both repeating and non-repeating (no leading zeros) even numbers

The choices for the last digit are (0, 2, 4, 6, 8).

a. 6 digits, no digit repeats and no leading zeros

Case 1: last digit = 0

b.

$$9 \times 10^4 \times 5 = 450,000$$

ii. Divisible by 5

For numbers divisible by five the number must end in either 0 or 5 (two options).

Case 1: Last digit = 0

$$9 * 8 * 7 * 6 * 5 = 15,120$$

Case 2: Last digit = 5

$$8 \times 8 \times 7 \times 6 \times 5 = 13,440$$

Combine cases

$$28,560$$

$$9 \times 10^4 \times 2 = 180,000$$

iii. Divisible by 4

$$9 \times \frac{9!}{(9-4)!} \times 2 = 54,432$$

$$9 \times 10^4 \times 2 = 180,000$$