Interest Rates

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Contents

4.1 Types of rates

Treasury rates

Treasury rates are the rates an investor earns on Treasury bills and Treasury bonds. These are instruments used by a government to borrow in its own currency.

Overnight rates

Banks are required to maintain a certain amount of cash, known as a reserve, with the central bank. The reserve requirments for a bank at any time depends on its oustanding assets and liabilities. At the end of a day, some financial institutions typically have surplus funds in their accounts with the central bank while others have requirements for funds. This leads to borrowing and lending overnight.

In the U.S. the central bank is the Federal Reserve and the overnight rate is called the *federal funds rate*. The weighted average of the rates in brokered transactions (with weights being determined by the size of the transaction) is termed the *effective federal funds rate*.

4.2 Reference rates

Reference rates are important in financial markets. The parties to transactions frequently enter into contracts where the future interest rate paid or received is uncertain, but will be set equal to the value of an agreed reference interest rate.

The plan is to base reference rates on the following overnight rates:

- **Secured overnight financing rate** (SOFR) – U.S.
- **Sterling overnight index average** (SONIA) – UK
- **Euro short-term rate** (ESTER) – Eurozone
- **Swiss average rate overnight** (SARON) – Switzerland
- **Tokyo overnight average rate** (TONAR) – Japan

Longer rates such as three-month rates, six-month rates, or one-year rates can be determined from overnight rates by compounding them daily.

In the case of SOFR, ther are assumed to be 360 days per year. SOFR overnight rate on the ith business day of a period is r_i ($1 \leq i \leq n$) and the rate applies to d_i days. The (annualized) interest rate for the period is

$$[(1 + r_1\hat{d}_1)(1 + r_2\hat{d}_2)\dots(1 + r_n\hat{d}_n) - 1] \times \frac{360}{D}$$

Chapter examples

Example 4.1 - Calculating interest rates with continuous compounding

Consider an interest rate that is quoted as 10% per annum with semiannual compounding. From equation (4.3) with $m = 2$ and $R_m = 0.1$, the equivalent rate with continuous compounding is

$$R_c = m \ln\left(1 + \frac{R_m}{m}\right) \tag{4.3}$$

$$2 \ln\left(1 + \frac{0.1}{2}\right) = 0.09758$$

or 9.758%

Example 4.2 - Calculating interest rates with quarterly compounding

Suppose that a lender quotes the interest rate on loans as 8% per annum with continouous compounding, and that interest is actually paid quarterly. From equation (4.4) with $m = 4$ and $R_c = 0.08$, the equivalent rate with quarterly compounding is

$$R_m = m(e^{R_c/m} - 1) \tag{4.4}$$

$$4 \times (e^{0.08/4} - 1) = 0.0808$$

or 8.08% per annum. This means that on a $1,000 loan, interest payments of $20.20 would be required each quarter.

Example 4.3 - Calculating the forward rate agreement value

Suppose that the forward SOFR rate for the period between time 1.5 years and time 2 years in the future is 5% (with semiannual compounding) and that some time ago a company entered into an FRA where it will receive 5.8% (with semiannual compounding) and pay SOFR on a principal of $100 million for the period. The 2-year (SOFR) risk-free rate is 4% (with continuous compounding). The value of the FRA is

$$100,000,000 \times (0.058 - 0.050) \times 0.5e^{-0.04 \times 2} = \$369,200$$

Example 4.4 - Calculating the bond price

For the bond in the following table, the bond price, B, is 94,213 and the duration, D, is 2.653, so that equation (4.11) gives

$$\Delta B = -BD\Delta y \tag{4.11}$$

Table 1: Calculation of duration

time	cash.flow	present.value	weight	time.weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333

$$\Delta B = -94.213 \times 2.653 \times \Delta y$$

or

$$\Delta B = -249.95 \times \Delta y$$

When the yield on the bond increases by 10 basis points (=0.1%), $\Delta y = +0.001$. The duration relationship predicts that $\Delta B = -249.95 \times 0.001 = -0.250$, so that the bond price goes down to $94.213 - 0.250 = 93.963$. How accurate is this? Valuing the bond in terms of its yield in the usual way, we find that, when the bond yield increases by 10 basis points to 12.1%, the bond price is

$$5e^{-0.121 \times 0.4} + 5e^{-0.121 \times 1.0} + 5e^{-0.121 \times 1.5} + 5e^{-0.121 \times 2.0}$$
$$+ 5e^{-0.121 \times 2.5} + 105e^{-0.121 \times 3.0} = 93.963$$

which is (to three decimal places) the same as that predicted by the duration relationship.

Example 4.5 - Calculating modified duration

$$\Delta B == BD^* \Delta y \tag{4.13}$$

The bond in Table 1 has a price of 94.213 and a duration of 2.653. The yield, expressed with semiannual compounding is 12.3673%. The modified duration, D^*, is given by

$$D^* = \frac{2.653}{1 + 0.123673/2} = 2.499$$

From equation (4.13),

$$\Delta B = -94.213 \times 2.499 \times \Delta y$$

or

$$\Delta B = -235.39 \times \Delta y$$

When the yield (semiannually compounded) increases by 10 basis points (=0.1%), we have $\Delta y = +0.001$. The duration relationship predicts that we expect ΔB to be $-235.39 \times 0.001 = -0.235$, so that the bond price goes down to $94.213 - 0.235 = 93.978$. How accurate is this? An exact calculation similar to that in the previous example shows that, when the bond yield (semiannually compounded) increases by 10 basis points to 12.4673%, the bond price becomes 93.978. This shows that the modified duration calculation gives good accuracy for small yield changes.

Practice questions

1. Interest rates and compounding interest

A bank quotes an interest rate of 7% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

Effect of compounding frequency on the value of $100 at the end of 1 year with the interest rate is 7% per annum

1. **Convert to effective annual rate (EAR):**

$$\text{EAR} = \left(1 + \frac{r_q}{m}\right)^m - 1$$

$$= \left(1 + \frac{0.07}{4}\right)^4 - 1$$

$$= .0719$$

2. **Convert to continuous compounding**

$$\text{EAR} = e^{r_c} - 1 \Rightarrow \ln\left(1 + \text{EAR}\right)$$

$$r_c = \ln\left(1 + 0.071361\right)$$

$$\approx 0.0694$$

```
interest.rate <- .07
compound.frequency <- 4
time <- 1
face.value <- 1

# convert to ear
ear <- (1 + interest.rate/compound.frequency)^(compound.frequency) - 1
cat("Annual EAR:", ear, "\n")
```

```
## Annual EAR: 0.07185903
```

```
# convert to continuous compounding
continuous.compounding <- function(r) {exp(r) - 1 - ear}
cc <- uniroot(continuous.compounding, lower=1e-4, upper=1, tol=1e-8)$root
cat("Continuous compounding:", cc, "\n")
```

```
## Continuous compounding: 0.06939455
```

---

2. Bond pricing and zero rates

The 6-month and 1-year zero rates are both 5% per annum. For a bond that has a life of 18 months and pays a coupon of 4% per annum (with semiannual payments and one having just been made), the yield is 5.2% per annum. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.

Given:

- Zero rates (with semiannual compounding): $r_{0.5} = r_{1.0} = 5\%$ per annum
- Bond details:
  - Maturity: 18 months (1.5 years)
  - Coupon: 4% per annum $\rightarrow$ 2% semiannually
  - Yield: 5.2%
  - Face value: Assume $100 (typical unless otherwise specified)
  - Last coupon just paid, so remain payments:
    * 2% at 6 months
    * 2% at 12 months
    * 102% at 18 months (coupon + principal)

Bond price

The semiannual yield rate:

$$Y = \frac{.052}{2} = .026$$

Bond price:

$$B = \frac{2}{(1+0.026)} + \frac{2}{(1+.026)^2} + \frac{102}{(1+.026)^3}$$
$$\approx 98.29$$

18-month zero rate

We now use the zero-coupon bond pricing approach, but we already know the bond price ($98.29) and the zero rates for 6 and 12 months. Let r be the 18-month zero rate (semiannual compounding).

We discount using known zero rates:

$$r_{0.5} = r_{1.0} = 5\% \Rightarrow 2.5\% \text{ per 6 months}$$

$$\text{Price} = \frac{2}{(1.025)^1} + \frac{2}{(1.025)^2} + \frac{102}{(1+r)^3}$$

$$r \approx 2.60\% \text{ semiannual}$$

The annual zero rate is $2 \times 2.60\% = 5.20\%$

```
f <- function(r) 2/(1.025)^1 + 2/(1.025)^2 + 102/(1 + r)^3 - 98.29
uniroot(f, lower=1e-4, upper=1, tol=1e-8)$root * 2
```

```
## [1] 0.05203844
```

3. Compounding rates

An investor receives $1,100 in one year in return for an investment of $1,000 now. Calculate the percentage return per annum with:

a. Annual compounding
b. Semiannual compounding
c. Monthly compounding
d. Continuous compounding

We are only interested in one year return (i.e. n=1)

The general equations for calculating compounding interest:

$$B = A(1 + R/m)^m$$

A. Annual compounding

$$1,100 = 1,000 \cdot (1 + R)$$
$$R = \frac{1,100}{1,000} - 1$$
$$= .10$$

B. Semiannual compounding

$$1,100 = 1,000 \cdot \left(1 + \frac{R}{2}\right)^2$$
$$R = 2 \cdot \left(\sqrt{\left(\frac{1,100}{1,000}\right)} - 1\right)$$
$$= .0976$$

C. Monthly compounding

$$1,100 = 1,000 \cdot \left(1 + \frac{R}{12}\right)^{12}$$
$$R = 12 \cdot \left(\left(\frac{1,100}{1,000}\right)^{1/12} - 1\right)$$
$$= .0957$$

D. Continuous compounding

$$B = Ae^R$$
$$\frac{B}{A} = e^R$$
$$R = \ln\left(\frac{B}{A}\right)$$
$$= \ln\left(\frac{1,100}{1,000}\right)$$
$$= .0953$$

Calculations

```
B <- 1100
A <- 1000
M <- c(1,2,12)

f.asm <- function(r) A * (1 + r/m)^m - B
f.c <- function(r) A * exp(r) - B

for(m in M){
  print(round(uniroot(f.asm, lower=1e-4, upper=1, tol=1e-8)$root, 4))
}
```

```
## [1] 0.1
## [1] 0.0976
## [1] 0.0957
```

```r
round(uniroot(f.c, lower=1e-4, upper=1, tol=1e-8)$root, 4)
```

```
## [1] 0.0953
```

---

4. Forward interest rates

Suppose that risk-free zero interest rates with continuous compounding are as follows:

Table 2: Risk free zero interest rates, per annum

Maturity	Rate
3	0.030
6	0.032
9	0.034
12	0.035
15	0.036
18	0.037

Calculate forward interest rates for the second, third, fourth, fifth, and sixth quarters.

To calculate the forward rates from the table of zero rates with continuous compounding we use the following formula:

$$f(t_1, t_2) = \frac{R(t_2) \cdot t_2 - R(t_1) \cdot t_1}{t_2 - t_1}$$

using R

```r
f <- function(z, t) (z[-1] * t[-1] - z[-length(z)] * t[-length(t)]) /
    (t[-1] - t[-length(t)])
f(rate, maturity)
```

```
## [1] 0.034 0.038 0.038 0.040 0.042
```

---

5. Forward rate agreements (FRA)

Assuming that SOFR rates are as in the previous problem, what is the value of an FRA where the holder will pay SOFR and receive 4.5% (quarterly compounded) for a three-month period starting in one year on a principal of $1,000,000?

To value a forward rate agreement (FRA) where the holder pays SOFR and receives 4.5% (quarterly compounded).

- The notional is $1,000,000
- The fixed ratre received is 4.5% quarterly compounded
- The floating rate (SOFR) is derived from the zero curve in the previous problem
- The FRA covers a 3-month period starting in 1 year, i.e., from $T_1 = 1.0$ year to $T_2 = 1.25$ years.
- Interest rates are quoted with continuous compounding

Step 1: Extract SOFR forward rate from 1.0 to 1.25 years.

- R(1.0) = 3.5%
- R(1.25) = 3.6%

Now calculate the forward rate using the formula:

$$f(1.0, 1.25) = \frac{R(1.25) \cdot 1.25 - R(1.0) \cdot 1.0}{0.25} = 0.04$$

Step 2: Convert 4.5% quarterly-compounded to continuously compounded

We use:

$$r_{\text{cont}} = m \cdot \ln\left(1 + \frac{R}{m}\right) \quad \text{where } m = 4, \ R = 0.045$$

$$r_{\text{cont}} = 4 \cdot \ln\left(1 + \frac{0.045}{4}\right)$$

$$= 0.4475$$

So the fixed rate (continuously compounded) is approximately 4.475%.

Step 3: Value the FRA

The FRA is valued as the present value of the difference between the fixed and forward floating rate:

$$\text{Value} = N \cdot (r_{\text{fixed}} - f) \cdot (T_2 - T_1) \cdot e^{-R(T_2)) \cdot T_2}$$

Using:

- $N = 1,000,000$
- $r_{\text{fixed}} = 0.04475$
- $f = 0.04$
- $T_2 = 1.25$
- $R(1.25) = 0.036$

$$V = 1,000,000 \cdot (0.04475 - 0.04) \cdot 0.25 \cdot e^{-0.036 \cdot 1.25} = \$1,135$$

---

6. Term structure of interest rates

The term structure of interest rates is upward sloping. Put the following in order of magnitude:

a. The 5-year zero rate
b. The yield on a 5-year coupon-bearing bond
c. The forward rate corresponding to the period between 4.75 and 5 years in the future.

What is the answer when the term structure of interest rates is downward sloping?

When the term structure of interest rates is upward sloping, it means that longer-term interest rates are higher than shorter-term ones.

Upward sloping yield curve

A. The 5-year zero rate   This is the yield on a zero-coupon bond maturing in 5 years. It reflects the average expected short rates over 5 years.

B. The yield on a 5-year coupon-bearing bond   This is a weighted average of yields over the life of the bond. Since some cash flows come earlier, it tends to be lower than the zero rate.

C. The forward rate for 4.75 to 5 years   This reflects the market's expectation of the short-term rate between 4.75 and 5 years in the future. Since the yield curve is rising, this short-term forward rate is higher than both the zero and coupon yields.

Order (from lowest to highest)   b < a < c

Downward sloping yield curve

Now, the term structure implies that future rates are expected to be lower.

The order will be reversed: c < a < b

---

7. Duration

What does duration tell you about the sensitivity of a bond portfolio to interest rates. What are the limitations of the duration measure?

Duration measures a bond's sensitivity to interest rate changes. Specifically, it estimates the percentage change in a bond's price for a 1% (or 100 basis points) change in interest rates, assuming all other factors stay constant.

What duration tells you:

Macaulay duration   The weighted average time to receive the bond's cash flows.

Modified duration   The percentage change in the price for a small change in yield.

$$\text{Modified duration} = \frac{\text{Macaulay duration}}{1 + \text{yield per period}}$$

For example, if a bond has a modified duration of 5, then its price will decrease by approximately 5% if interest rates rise by 1%.

Limits of duration

1. Assumes parallel shifts   Duration assumes interest rates move uniformly across all maturities, which rarely happens in reality (non-parallel shifts in the yield curve are common).

2. Small changes only   It's a linear approximation, so it becomes less accurate for large interest rate changes due to convexity (the curve of the price-yield relationship).

3. Ignores optionality   It doesn't account for bonds with embedded options (like callable or puttable bonds), where cash flows may change as rates change.

4. Static cash flows   Assumes fixed cash flows, which is not true for floating rate bonds or mortgage-backed securities where prepayment risk alters cash flows.

5. Yield vs. curve shifts   Duration typically relates to yield to maturity, but actual risk may come from specific spot rate or forward rate curve changes.

Example

We have a bond with a 5-year maturity, 5% annual coupon (paid once a year), and a face value of $100

This plot is for the bond's price against interest rates from 1% to 10%. The tangent line at the current yield shows how duration approximates price changes.

```
# Bond parameters
face_value <- 100
coupon_rate <- 0.05
maturity <- 5
coupon <- face_value * coupon_rate

# Interest rate range
rates <- seq(0.01, 0.10, by = 0.001)

# Bond price function
bond_price <- function(r, n, c, FV) {
  sum <- 0
  for (i in 1:n) {
    sum <- sum + c / (1 + r)^i
  }
  sum + FV / (1 + r)^n
}

# Calculate prices for each rate
prices <- sapply(rates, bond_price, n = maturity, c = coupon, FV = face_value)

# Chosen yield for duration approximation
r0 <- 0.05
p0 <- bond_price(r0, maturity, coupon, face_value)

# Modified duration (approximate)
mod_duration <- (bond_price(r0 - 0.001, maturity, coupon, face_value) -
                 bond_price(r0 + 0.001, maturity, coupon, face_value)) /
                (2 * 0.001 * p0)

# Tangent line using duration
approx_prices <- p0 - mod_duration * p0 * (rates - r0)

# Plot
plot(rates, prices, type = "l", col = "blue", lwd = 2,
     ylab = "Bond Price", xlab = "Interest Rate",
     main = "Bond Price vs Interest Rate with Duration Approximation")
lines(rates, approx_prices, col = "red", lwd = 2, lty = 2)
legend("topright", legend = c("Actual Price", "Duration Approximation"),
       col = c("blue", "red"), lty = c(1, 2), lwd = 2)
```
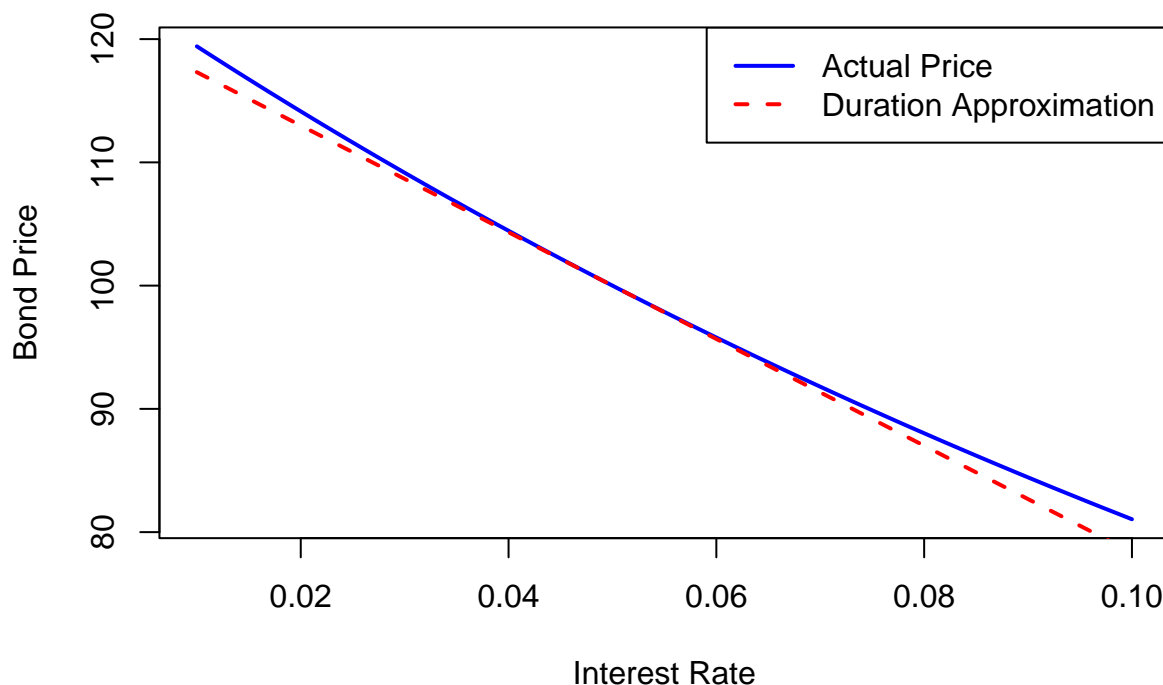
Bond Price vs Interest Rate with Duration Approximation

8. Interest rate comparisons

What rate of interest with continuous compounding is equivalent to 8% per annum with monthly compounding?

$$A \left(1 + \frac{R_1}{m_1}\right)^{m_1} = A \cdot e^{R_2}$$
$$R_2 = 12 \cdot \ln\left(1 + .08/12\right)$$
$$= 0.0797$$

9. Interest paid each quarter with a 4% continuous compounded annual rate

A deposit account pays 4% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a $10,000 deposit?

To find the interest paid each quarter on a $10,000 deposit with a 4% continuously compounded annual rate, but with quarterly payments:

Step 1: Convert the continuous rate to a quarterly effective rate:

$$\text{Effective quarterly rate} = e^{r \times t} - 1$$

where:

- $r = 0.04$ (continuous annual rate)
- $t = \frac{1}{4}$ (a quarter of a year)

The effective quarterly rate:

$$e^{0.04 \times 0.25} - 1 \approx 0.01005$$

Step 2: Calculate the interest paid per quarter

$$\text{interest} = 10,000 \times 0.01005 \approx \$100.50$$

10. Bond price, continuous compounding

Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are, respectively: 4%, 4.2%, 4.4%, 4.6%, and 4.8% per annum, with continuous compounding. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months and pay a coupon of 4% per annum semiannually.

Table 3: Zero rates

bond	zero.rate
6-month	0.040
12-month	0.042
18-month	0.044
24-month	0.046
30-month	0.048

$$2e^{-.04 \times 0.5} + 2e^{-.042 \times 1.0} + 2e^{-.044 \times 1.5} + 2e^{-.046 \times 2.0} + 102e^{-.048 \times 2.5} = 98.04$$

11. Bond yield, semiannual compounding

A 3-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond's yield?

The bond yield is the single discount rate that, when applied to all cash flows, gives a bond price equal to its market price.

We can use R to calculate the bond yield

```
cash.flows <- c(4, 4, 4, 4, 4, 104)
iter <- 1:6

f <- function(y) sum(cash.flows * (1 + y)^-iter) - 104

y <- uniroot(f, lower=1e-4, upper=1, tol=1e-8)$root
cat("Semiannual yield:", round(y, 3), "\n")

## Semiannual yield: 0.033
```

```r
cat("Annual yield:", round(y*2, 3))
```

```
## Annual yield: 0.065
```

---

12. Par yield

Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5%, and 7%, respectively. What is the 2-year par yield?

The par yield for a certain bond maturity is the coupon rate that causes the bond price to equal its par value.

```r
iter <- c(.5, 1, 1.5, 2)
zero.rate <- c(.05, .06, .065, .07)

f <- function(C){
  (C/2)*exp(-.05*.5)+(C/2)*exp(-.06*1)+(C/2)*exp(-.065*1.5)+
    (100+(C/2))*exp(-.07*2) - 100
}

uniroot(f, lower=1, upper=10, tol=.001)$root
```

```
## [1] 7.074077
```

---

13. Forward interest rates

Suppose that risk-free zero interest rates with continuous compounding are as follows

maturity	rate
1	0.020
2	0.030
3	0.037
4	0.042
5	0.045

Calculate forward interest rates for the second, third, fourth, and fifth years.

To calculate the forward rates from the table of zero rates with continuous compounding we use the formula:

$$f(t_1, t_2) = \frac{R(t_2) \cdot t_2 - R(t_1) \cdot t_1}{t_2 - t_1}$$

```r
f <- function(z, t) (z[-1] * t[-1] - z[-length(z)] * t[-length(t)]) /
    (t[-1] - t[-length(t)])

forward.rate <- f(rate, maturity)

knitr::kable(data.frame(Maturity=maturity,
                        Zero.rate=rate,
```

```
            Forward.Rate=c(NA, forward.rate)),
     caption="Calculation of forward rates")
```

Table 5: Calculation of forward rates

Maturity	Zero.rate	Forward.Rate
1	0.020	NA
2	0.030	0.040
3	0.037	0.051
4	0.042	0.057
5	0.045	0.057

---

14. Calculating zero rates

A 10-year 8% coupon bond currently sells for $90. A 10-year 4% coupon bond currentl sells for $80. What is the 10-year zero rate? (*Hint*: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.)

We are given:

- Bond 1: 10-year 8% coupon bond, price = $90
- Bond 2: 10-year 4% coupon bond, price = $80

Coupons are likely paid annually unless otherwise stated.

Hint strategy: Take a long position in 2 of the 4% bonds and a short position in 1 of the 8% bonds.

Step 1: Cash flows

- One 8% bond pays $8 per year for 10 years + $100 at maturity.
- One 4% bond pays $4 per year for 10 years + $100 at maturity.

Therefore:

- Long 2 x 4% bonds gives:

  - $8 per year $(4+4)$
  - $200 at maturity $(100 + 100)$

- Short 1 x 8% bond gives:

  - -$8 per year
  - -$100 at maturity

Step 2: Net cash flows (long 2 x 4% bonds, short 1 x 8% bond)

Each year for 10 years:

- Coupons: $+8 - 8 = 0 \Rightarrow$ No net coupon payments.

At maturity:

- Principal repayment: $+200 - 100 = +100 \Rightarrow$ Net $100 received at year 10.

Thus:

- No cash flows until year 10, when you receive $100.

Step 3: Cost to set up the position

- Buy 2 bonds at $80 each $160
- Sell 1 bond at $90 → +$90

Total cost = 160 - 90 = 70

Thus:

- You pay $70 today to receive $100 in 10 years.

Step 4: Implied 10-year zero rate

The equation is:

$$70e^{r \times 10} = 100$$

Solving for r:

$$r = \frac{\ln\left(\frac{10}{7}\right)}{10} \approx 0.0356675$$

3.567% per annum (with continuous compounding)

---

15. Liquidity preference theory and the term structure of interest

Explain carefully why liquidity preference theory is consistent with the observation that the term structure of interest rates tends to be upward-sloping more often than it is downward-sloping.

Liquidity preference theory says that investors prefer short-term bonds over long-term bonds because short-term bonds are more liquid and less risky.

- **Liquidity:** Short-term bonds are easier to sell quickly without a large price discount.
- **Risk:** Short-term bonds are less exposed to interest rate changes (which can cause bond prices to fluctuate).

Because investors generally prefer short-term investments, they must be compensated with a higher yield to be willing to invest in long-term bonds. This extra yield is called a liquidity premium.

So, according to liquidity preference theory:

- The observed long-term interest rates include not only expectations about future short-term rates but also a positive liquidity premium.
- Even if investors expect future short-term rates to stay the same, the liquidity premium causes long-term rates to be higher than short-term rates.

Thus, the term structure tends to slope upward more often because of this built-in liquidity premium.

Summary

- Investors demand a higher return for committing money for longer periods.
- This pushes up long-term rates relative to short-term rates.
- As a result, the yield curve is more often upward-sloping.

---

16. The zero curve

"When the zero curve is upward-sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward-sloping the reverse is true." Explain why this is so.

First, remember:

- The zero rate for a maturity is the yield earned if you invest in a single payment (a zero-coupon bond) at that maturity.

- The par yield is the coupon rate that makes the price of a bond equal to its face value (usually 100)—meaning the bonds sells "at par."

If the zero curve is upward-sloping

- Future zero rates (for longer maturities) are higher than current ones.

- A par bond pays multiple coupons (earlier cash flows) and one final principal.

- These earlier coupons are discounted at lower rates (because they're sooner), while the principal at the end is discounted at a higher zero rate.

- Because the coupons are discounted at lower rates, the present value of the bond is relatively high—but we want the bond price to equal par (100).

- Therefore, the coupon rate (par yield) doesn't need to be as high as the zero rate for the bond to sell for 100.

The result is that when the zero curve is upward-sloping, the par yield is less than the zero rate for that maturity.

If the zero curve is downward-sloping

- Future zero rates are lower than current ones.

- Now, early coupons are discounted at higher rates, and the final payment is discounted at a lower rate.

- This pushes down the present value of early coupons and raises the present value of the final principal payment.

- To balance it out and sell the bond at par (100), the bond must offer a higher coupon rate.

The result when the zero curve is downward-slopoing, the par yield is greater than the zero rate for that maturity.

---

17. Repo markets

Why does a loan in the repo market involve very little credit risk?

A loan in the repo market (repurchase agreement market) involves very little credit risk because it is secured by collateral.

Here's how it works:

- In a repo, the borrower sells securities (like Treasury bonds) to the lender today and promises to buy them back at a slightly higher price later (often overnight or within a few days).

- If the borrower defaults, the lender already holds the securities and can sell them to recover the loaned money.

- The collateral is usually high-quality, liquid assets (like government bonds), so it's easy to sell without much loss.

- Often, there's also a "haircut"—the lender lends less than the full value of the collateral (say, 98% of its value), which provides an extra safety buffer against small price declines.

---

18. Bond analysis

A 5-year bond with a yield of 7% (continuously compounded) pays an 8% coupon at the end of each year.

    a. What is the bond's price?
    b. What is the bond's duration?
    c. Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.
    d. Recalculate the bond's price on the basis of a 6.8% per annum yield and verify that the result is in agreement with your answer to (c).

Given:

- Maturity: 5 years
- Coupon: 8% of face value (usually assume $100 face value, so $8 paid at the end of each year)
- Yield: 7% per annum, continuously compounding

A. Bond Price

$$B = \sum_{i=1}^{5} 8e^{-0.07i} + 100e^{-0.07 \times 5}$$
$$= 103.05$$

B. What is the bond's duration

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i} + n100e^{-yn}}{B}$$
$$= \frac{\sum_{i=1}^{5} i8e^{-0.07i} + 5 \cdot 100e^{-0.07 \times 5}}{103.05}$$
$$= 4.32 \text{ years}$$

C. Use the duration to calculate the effect of a 0.2% decrease in yield

Price change estimation using duration:

$$\Delta P \approx -D \times \Delta y \times P$$
$$= -(4.32)(-0.002)(103.05)$$
$$= 0.890$$

The price is expected to increase by about $0.89

D. Recalculate the bond's price at 6.8% yield

$$B = \sum_{i=1}^{5} 8e^{-0.068i} + 100e^{-0.068 \times 5}$$
$$= 103.95$$

19. Calculating zero rates for bonds using bootstrapping

The cash prices of 6-month and 1-year Treasury bills are 94.0 and 89.0. A 1.5-year Treasury bond that will pay coupons of $4 every 6 months currently sells for $94.84. A 2-year Treasury bond that will pay coupons of $5 every 6 months currently sells for $97.12. Calculate the 6-month, 1-year, 1.5-year, and 2-year Treasury zero rates.

To calculate the zero rates (spot rates) for the given maturities, we'll use the concept of bootstrapping from bond prices. These are the 6-month, 1-year, 1.5-year, and 2-year zero-coupon rates, compounded semiannually.

Step 1: 6-month zero rate

The price of a 6-month Treasury bill is 94.0, and the face value is $100. Let R_0 be the 6-month (0.5-year) zero rate, compounded semiannually.

$$94 = \frac{100}{(1 + R_0/2)^1}$$
$$(1 + R_0/2) = \frac{100}{94} = 1.06383$$
$$R_0 = 2 \times (1.06383 - 1) = 0.12766$$

Step 2: 1-Year zero rate

The price of a 1-year is 89.0. Let R_1 be the 1-year zero rate:

$$89 = \frac{100}{(1 + R_1/2)^2}$$
$$(1 + R_1/2)^2 = \frac{100}{89} = 1.1236$$
$$(1 + R_1/2) = \sqrt{1.1236} = 1.0597$$
$$2 \times (1.0597 - 1) = 0.1194$$

Step 3: 1.5-year zero rate (using bond)

The 1.5-year bond pays $4 every 6 months, so 3 cash flows of $4, $4, and $104.

Price = 94.84

We already know:

- $R_0 = 12.766\%$
- $R_1 = 11.94\%$

Let $R_{1.5}$ be the 1.5-year zero rate.

Discounting the first two known cash flows:

$$PV = \frac{4}{(1 + R_0/2)^1} + \frac{4}{(1 + R_1/2)^2}$$
$$= \frac{4}{1.06383} + \frac{4}{(1.0597)^2}$$
$$\approx 3.759 + 3.562$$
$$= 7.321$$

Then,

$$94.84 = 7.321 + \frac{104}{(1 + R_{1.5}/2)^3}$$

$$\frac{104}{(1 + R_{1.5}/2)^3} = 94.84 - 7.321$$

$$= 87.519$$

$$(1 + R_{1.5}/2)^3 = \frac{104}{87.519}$$

$$= 1.1886$$

$$1 + R_{1.5}/2 = (1.1886)^{1/3}$$

$$= 1.0592$$

$$R_{1.5} = 2 \times (1.0592 - 1)$$

$$= 0.1184$$

Step 4: 2-Year zero rate (using bond)

The 2-year bond pays \$5 every 6 months. So cash flows are \$5, \$5, \$5 and \$105.

Price = 97.12

We already know $R_0, R_1, R_{1.5}$. Let R_2 be the 2-year zero rate.

Discount first 3 cash flows:

$$\text{PV} = \frac{5}{(1.06383)} + \frac{5}{(1.0597)^2} + \frac{5}{(1.0592)^3}$$

$$\approx 4.70 + 4.45 + 4.13$$

$$= 13.28$$

Then:

$$97.12 = 13.28 + \frac{105}{(1 + R_2/2)^4}$$

$$\frac{105}{(1 + R_2/2)^4} = 83.84$$

$$(1 + R_2/2)^4 = \frac{105}{83.84}$$

$$= 1.2524$$

$$(1 + R_2/2) = (1.2524)^{1/4}$$

$$= 1.0576$$

$$R_2 = 2 \times (1.0576 - 1)$$

$$= .1152$$

Final answer: zero rates

Maturity	Zero Rate
6 months	12.77%
1 year	11.94%
1.5 years	11.84%

Maturity	Zero Rate
2 years	11.52%

20. Decomposition of an interest rate swap

"An interest rate swap where 6-month LIBOR is exchanged for a fixed rate of 5% on a principal of $100 million for 5 years involves a known cash flow and a portfolio of nine FRAs." Explain this statement.

This statement refers to the decomposition of an interest rate swap into a portfolio of forward rate agreements (FRAs) and explains how the cash flows in the swap can be understood in terms of those simpler instruments.

Interest rate swap basics

An interest rate swap is a contract between two parties to exchange interest payments based on a notional principal. In this case:

- One party pays 6-month LIBOR (a floating rate)
- The other party pays a fixed rate of 5% (compounded semiannually)
- Notional principal is $100 million.
- Duration is 5 years
- Payment frequency is every 6 months (so there are 10 payment periods).

Known cash flow

The fixed rate side (5% annually, or 2.5% every 6 months) is known and predetermined:

$$\text{Fixed payment} = 2.5\% \times \$100\text{million} = \$2.5\text{million every 6 months}$$

So this stream of payments is known in advance and easy to discount or value.

Portfolio of nine FRAs

The floating side (6-month LIBOR) resets every 6 months and is not known ahead of time.

However, it can be decomposed into a series of nine FRAs as follows:

- At each reset date (every 6 months), a new FRA effectively begins, locking in the interest rate for the next 6 months.
- Since there are 10 total payment periods over 5 years, the swap has 9 reset dates, after the first fixing.
- Each FRA is a contract that pays the difference between LIBOR and the forward rate agreed at the start (which would be zero in this synthetic structure), for a 6-month period, on the notional.

So, the floating leg is economically equivalent to a portfolio of nine FRAs, one for each future period where LIBOR will reset (i.e., from 6 months to 5 years in 6-month steps).

Why is this useful?

- This decomposition helps in valuing or hedging the swap.
- The fixed leg is a known annuity.
- The floating leg's value can be built from current forward LIBOR rates using FRAs.

Summary

The floating leg of a swap can be replicated by a portfolio of FRAs, each one starting on a future reset date and lasting for 6 months. Since the swap lasts 5 years and resets semiannually, there are 9 such future floating-rate resets—thus, nine FRAs. The fixed leg is simply a stream of known fixed payments.

---

21. Compounding and interest rates

When compounded annually an interest rate is 11%. What is the rate when expressed with (a) semiannual compounding, (b) quarterly compounding, (c) montly compounding, (d) weekly compoundings, and (e) daily compounding.

To convert an annually compounded rate (here, 11%) to other compounding frequencies:

$$(1 + r_{\text{annual}}) = (1 + r_{\text{periodic}})^m$$

Where:

- $r_{\text{annual}} = 0.11$
- m is the number of compounding periods per year.
- r_{periodic} is the periodic rate.

We solve for $r_{\text{periodic}} = (1 + 0.11)^{1/m} - 1$

Then, multiply by m to get the equivalent nominal rate with that compounding frequency.

A. Semiannual compounding (m=2)

$$r_{\text{semi}} = (1.11)^{1/2} - 1 = 0.053685$$

The nominal rate:

$$2 \times 5.3685\% = 10.737\%$$

B. Quarterly compounding (m=4)

$$r_{\text{quarterly}} = (1.11)^{(}1/4) - 1 = 0.026407$$

The nominal rate:

$$4 \times 2.6407\% = 10.563\%$$

C. Monthly compounding (m=12)

$$r_{\text{monthly}} = (1.11)^{1/12} - 1 = 0.008748$$

The nominal rate:

$$12 \times 0.8748\% = 10.498\%$$

D. Weekly compounding (m=52)

$$r_{\text{weekly}} = (1.11)^{1/52} - 1 = 0.002025$$

The nominal rate:

$$52 \times 0.2025\% = 10.53\%$$

E. Daily compounding (m=365)

$$r_{\text{daily}} = (1.11)^{1/365} - 1 = 0.000288$$

The nominal rate:

$$365 \times 0.0288\% = 10.512\%$$

---

22. Bond pricing and yield

The table below gives Treasury zero rates and cash flows on a Treasury bond. Zero rates are continuously compounded.

a. What is the bond's theoretical price?
b. What is the bond's yield assuming it sells for its theoretical price?

Maturity (years)	Zero rate	Coupon payment	Principal
0.5	2.0%	$20	
1.0	2.3%	$20	
1.5	2.7%	$20	
2.0	3.2%	$20	$1,000

A. Bond theoretical price

$$20e^{-.02\times.5} + 20e^{-.023\times1} + 20e^{-.027\times1.5} + 1020e{-.032} \times 2 = \$1015.32$$

B. Bond yield

$$20e^{-y\times.5} + 20e^{-y\times1} + 20e^{-y\times1.5} + 1020e{-y} \times 2 = \$1015.32$$

```
f <- function(y) {
  20 * sum(exp(-y * seq(.5, 1.5, by=.5))) + 1020 * exp(-y * 2) - 1015.32
}

uniroot(f, upper=1, lower=1e-4, tol=1e-8)$root
```

```
## [1] 0.03177708
```

---

23. Calculating bond yield

A 5-year bond provides a coupon of 5% per annum payable semiannually. Its price is 104. What is the bond's yield?

The equation for the bond's price using its yield

$$104 = \frac{5}{2} \times \frac{(1 - (1 + y)^{-10})}{y} + \frac{100}{(1 + y)^{10}}$$

using R we can solve for y:

```r
f <- function(y) 5/2 * (1 - (1 + y)^-10)/y + 100 * (1 + y)^-10 - 104
y <- uniroot(f, upper=1, lower=1e-4, tol=1e-8)$root
cat("Semiannual:", y, "\n")
```

```
## Semiannual: 0.02053448
```

```r
cat("Annual:", y * 2, "\n")
```

```
## Annual: 0.04106895
```

---

24. Interest rate compounding

An interest rate is quoted as 5% per annum with semiannual compounding. What is the equivalent rate with (a) annual compounding, (b) monthly compounding, and (c) continuous compounding.

Step 1: Compute the effective annual rate (EAR)

$$\text{EAR} = \left(1 + \frac{0.05}{2}\right)^2 - 1 = .050625$$

A. The equivalent rate with annual compounding

Since EAR is already based on annual growth, the equivalent rate with annual compounding is the EAR: .050625.

B. Equivalent rate with monthly compounding

We solve for the nominal annual rate r with monthly compounding that gives the same EAR:

```r
f <- function(r) (1 + r/12)^12 - 1.050625
uniroot(f, upper=1, lower=1e-4, tol=1e-8)$root
```

```
## [1] 0.04948699
```

C. Equivalent rate with continuous compounding

```r
f <- function(r) exp(r) - 1.050625
uniroot(f, upper=1, lower=1e-4, tol=1e-8)$root
```

```
## [1] 0.04938523
```

---

25. Bond yields and duration

Portfolio A consists of a 1-year zero-coupon bond with a face value of $2,000 and a 10-year zero-coupon bond with a face value of $6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of $5,000. The current yield on all bonds is 10% per annum.

    a. Show that both portfolios have the same duration.
    b. Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.
    c. What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

Given:

- Portfolio A:
    - 1-year zero-coupon bond par value $2,000
    - 10-year zero-coupon bond par value $6,000
- Portfolio B:
    - 5.95-year zero-coupon bond par value $5,000
- Current yield: 10%

Bond duration

The price of a zero-coupon bond:

$$\text{Portfolio A:}$$
$$P_{1 \text{ year}} = 2,000e^{-.1} = 1809.68$$
$$P_{10 \text{ year}} = 6,000e^{-.1 \times 10} = 2207.28$$

Total portfolio value: $1809.68 + 2207.28 = 4016.96$

Duration

$$D_A = \frac{1809.67 \cdot 1 + 2207.28 \cdot 10}{4016.96}$$
$$= 5.95$$

Portfolio B:

$$P_B = 5,000e^{-.1 \times 5.95} = 2757.81$$

Since it is a single zero:

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i}}{B}$$

Since $B = \sum_{i=1}^{n} c_i e^{-yt_i}$

$$D = t = 5.95 \text{ years}$$

---

26. Bond pricing, interest rates and duration

Verify that DerivaGem agrees with the price of the bond in Section 4.6. Test how well DV01 predicts the effect of a 1-basis-point increase in all rates. Estimate the duration of the bond from DV01. Use DV01 and Gamma to predict the effecto of a 200-basis-point increase in all rates. Use Gamma to estimate the bond's convexity. (*Hint:* In DerivaGem, DV01 is dB/dy, where B isthe price of the bond and y is its yield measured in basis points and Gamma is d^2B/dy^2, where y is measured in percent.)