Lines

Equation of a line

$(4, -3)$ and $(-1, 1)$

$y = mx + b$
$y - y_0 = m(x - x_0)$

Slope of line

$(-1 - (-3)) = m(1 - 4)$
$m = \frac{(1-(-3))}{-1-4}$
$\boxed{m = -\frac{4}{5}}$

Point Slope

$(y - (-3)) = \frac{-4}{5}(x - 4)$
$y + 3 = -\frac{4}{5}(x - 4)$
$y + 3 = -\frac{4}{5}x + \frac{16}{5}$
$y = -\frac{4}{5}x + \frac{16}{5} - 3$
$y = -\frac{4}{5}x + \frac{16}{5} - \frac{15}{5}$
$\boxed{y = -\frac{4}{5}x + \frac{1}{5}}$

Slope intercept

$y = mx + b$
$-3 = (-\frac{4}{5} \times 4) + b$
$-3 = -\frac{16}{5} + b$
$b = -3 - (-\frac{16}{5})$
$b = -\frac{15}{5} + \frac{16}{5}$
$\boxed{b = \frac{1}{5}}$

Angles

$cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$cos\frac{\pi}{3} = \frac{1}{2}$
$cos\frac{\pi}{2} = 1$

It's not possible to find exact value of $cos\frac{7\pi}{12}$
Then we need to use:

- $cos(\pi - x) = -cosx$
- $cos(\frac{\pi}{2} - x) = sinx$
- $cos(x + y) = cos(x)cos(y) - sin(x)sin(y)$

We can represet $\frac{7\pi}{12}$ as a sum: $\frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$
$cos\frac{7}{12} = cos(\frac{\pi}{4} + \frac{\pi}{3}) = cos(\frac{\pi}{4})cos(\frac{\pi}{3}) - sin(\frac{\pi}{4})sin(\frac{\pi}{3})$
$= \frac{\sqrt{2}}{2}cos(\frac{\pi}{3}) - \frac{\sqrt{2}}{2}sin(\frac{\pi}{3})$
$= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}\sqrt{3}}{4}$
$= \boxed{\frac{\sqrt{2} - \sqrt{6}}{4}}$

Simplify

$(\frac{3y\sqrt{x}}{6x^3y^{-4}})^2$
$(\frac{y\sqrt{x}}{3x^3y^{-4}})^2$
$= ((yx^{\frac{1}{2}})(2x^3y^{-4})^{-1})^2$
$= ((yx^{\frac{1}{2}})(\frac{1}{2}x^{-3}y^4))^2$
$= (yx^{\frac{1}{2}}\frac{1}{2}x^{-3}y^4)^2$
$= (\frac{1}{2}x^{\frac{1}{2}+(-3)}y^{1+4})^2$
$= (\frac{1}{2}x^{\frac{1}{2}-\frac{6}{2}}y^{1+4})^2$
$= (\frac{1}{2}x^{\frac{-5}{2}}y^5)^2$
$= (\frac{1}{4}x^{\frac{-5}{2}+\frac{-5}{2}}y^{10})$
$= (\frac{1}{4}x^{\frac{-10}{2}}y^{10})$
$= (\frac{1}{4}x^{-5}y^{10})$

$$\boxed{= \frac{y^{10}}{4x^5}}$$

Simplify Logarithms

- $log\frac{a}{b} = log(a) - log(b)$
- $log(a \times b) = log(a) + log(b)$
- $log(a^b) = b \times log(a)$
- $x^a x^b = x^{a+b}$

Example:

Expand and simplify $log_4 \frac{a^4 \sqrt[3]{4}}{2e^{-ln(64)}}$
$log_4(a^4 \sqrt[3]{4}) - log_4(2e^{-ln(64)})$
$= ((log_4(a^4)) + (log_4(\sqrt[3]{4}))) - ((log_4(2)) + (log_4(e^{-ln(64)})))$
$= (log_4(a) + (log_4(4^{\frac{1}{3}}))) - (log_4(\sqrt{4}) + ((log_4(2)) + log_4(64))))$
$= (log_4(a^4) + (log_4(4)^{\frac{1}{3}}) - log_4(\sqrt{4}) - (log_4(2)) + ((log(64))))$
$= (log_4(a^4) + log_4(4^{\frac{1}{3}}) - log_4(2) + (log_4(64))$
$= 4log_4(a) + \frac{1}{3} - \frac{1}{2} + 3$
$= \boxed{4log_4(a) + \frac{17}{6}}$

Product Rule Practice

1.

$\frac{d}{dx}xarctanx = \frac{d}{dx}f(x) \times g(x)$

$f(x) = x$
$f'(x) = 1$
$g(x) = arctan(x)$
$g'(x) = \frac{1}{1+x^2}$

$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x)$
$\frac{d}{dx}xarctan(x) = x \times \frac{1}{1+x^2} + arctan(x) * 1$

2.

$\frac{d}{dx}\frac{lnx}{x}$

$f(x) = ln(x)$
$f'(x) = \frac{1}{x}$
$g(x) = x^-1$
$g'(x) = -x^(- 2)$

$\frac{d}{dx} = ln(x) \times (-x^{-2}) + \frac{1}{x} \times x^{-1} = \frac{-ln(x)}{x^2} + \frac{1}{x^2} = \frac{1-ln(x)}{x^2}$

3.

$\frac{d}{dx}4x^5\sqrt{x}$

$f(x) = 4x^5$
$f'(x) = 5 \times 4x^{5-1} = 20x^4$
$g(x) = \sqrt{x}$
$g'(x) = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}4x^4\sqrt{x} = 4x^2 \times \frac{1}{2\sqrt{x}} + 20x^4 \times \sqrt{x} = 2x^{\frac{9}{2}} + 20x^{\frac{9}{2}} = 22x^{\frac{9}{2}}$

Chain Rule

$\frac{d}{dx}f(g(x)) = g'(x) \times f'(g(x))$

1.

$\frac{d}{dx}e^{sin(x)+x^2} : f(x) = e^{g(x)}$

$f(x) = e^x$
$f'(x) = e^x$
$g(x) = sin(x) + x^2$
$g'(x) = cos(x) + 2x$

$\frac{d}{dx}sin(x) + x^2 = (cos(x) + 2x)e^{sin(x)+x^2}$

2.

$\frac{d}{dx}\frac{1}{2+3x^4} : f(x) = \frac{1}{g(x)}$

$f(x) = \frac{1}{x}$
$f'(x) = \frac{-1}{x^2}$
$g(x) = 2 + 3x^2$
$g'(x) = 6x$

$\frac{d}{dx}\frac{1}{2+3x^4} = 6x\frac{-1}{(2+3x^4)^2} = \frac{-6x}{(2+3x^4)^2}$

3.

$\frac{d}{dx}2cos^3(x) : f(x) = 2(g(x))^3$

$f(x) = 2x^3$
$f'(x) = 6x^2$
$g(x) = cos(x)$
$g(x) = -sin(x)$

$\frac{d}{dx}2cos^3(x) = (-sin(x))(6cos(x)^2) = -6(sin(x)(cos(x))^2$

All Local Minima and All Local Maxima

Occurs at critical points so we need f'(x).

1.

$g(x) = xe^{-x^2}$
$g'(x) = -2xe^{-x^2}$
$h(x) = -x^2$
$h'(x) = 2 \times -x^{2-1} = -2x$

$f'(x) = h'(x) \times g(x) + h(x) \times g'(x)$
$f'(x) = (1) \times (e^{-x^2}) + (x) \times (-2xe^{-x^2})$
$f'(x) = ((e^{-x^2}) + (-2x^2e^{-x^2}))$
$\boxed{f'(x) = (1 - 2x^2)e^{-x^2}}$
$P_1 = \frac{1}{\sqrt{2}}$
$P_2 = -\frac{1}{\sqrt{2}}$

2.

since $e^x > 0$ the $(1 - 2x^3)$ is the critical point.
$g(x) = (1 - 2x^2)$
$g'(x) = -4x$
$f''(x) = g'(x)e^{-2x} + g(x)g'()$
$= (-4x)e^{-x^2} + (1 - 2x^2)(-2xe^{-x^2})$
$= (-6x + 4x^3)e^{-x^2}$
$P_1 = \frac{1}{\sqrt{2}}$ (local maxmimum)
$P_2 = -\frac{1}{\sqrt{2}}$ (local mimiumum)

Global Optimization

$f(x) = arctan(x) - \frac{x}{2}$ at interval $[-\sqrt{3}, \frac{1}{\sqrt{3}}]$
$f'(x) = \frac{1}{1+x^2} - \frac{1}{2}$
Zeroes at $x = -1$ and $\cancel{x = 1}$
$f(-\sqrt{3}) = arctan(-\sqrt{3}) - \frac{-\sqrt{3}}{2} = -\frac{\pi}{3} - \frac{-\sqrt{3}}{2} = \frac{3\sqrt{3}-2\pi}{6}$
$f(-1) = arctan(-1) - \frac{-1}{2} = -\frac{\pi}{4} - \frac{-1}{2} = \frac{2-\pi}{4}$
$f(\frac{1}{\sqrt{3}}) = arctan(\frac{1}{\sqrt{3}}) - \frac{\frac{1}{\sqrt{3}}}{2} = \frac{\pi}{6} - \frac{1}{2\sqrt{3}} = \frac{\pi-\sqrt{3}}{6}$
$f(x)$ acheves a maxmimum value of $\frac{\pi-\sqrt{3}}{6} = \frac{1}{\sqrt{3}}$
$f(x)$ acheves a minimum value of $\frac{2-\pi}{4}$ at $x = -1$.

Taylor Approximation

find the second order Taylor approximation of $f(x) = arctan(\sqrt{x})$ around the point $a = 1$.

f'

Let $g_1(x) = \sqrt{x}$
$g_1'(x) = \frac{1}{2\sqrt{x}}$
Then $f_1'(x) = \frac{1}{1+g(x)^2} g_1'(x)$
$= \frac{1}{1+\sqrt{x^2}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}+2x^{\frac{3}{2}}}$

f''

Let $g_2(x) = 2\sqrt{x} + 2x^{\frac{3}{2}}$
$g_2'(x) = \frac{1}{\sqrt{x}} + 3\sqrt{x}.$
Then $f''(x) = -1 \times g_2(x)^{-2} \times g_2'(x)$
$= [-(2\sqrt{x} + 2x^{\frac{3}{2}})^{-2}] \times (\frac{1}{\sqrt{x}} + 3\sqrt{x})$

Solution

$f(1) = arctan(\sqrt{1}) = arctan(1) = \frac{\pi}{4}$
$f'(1) = \frac{1}{2\sqrt{1}+2(1)^{\frac{3}{2}}} = \frac{1}{2+2}\frac{1}{4}$
$f''(1) = -(2\sqrt{1} + 2(1)^{\frac{3}{2}})^{-2}(\frac{1}{\sqrt{1}} + 3\sqrt{1}) = -4^{-2}4 = -\frac{1}{4}$
$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 = \frac{\pi}{4} + \frac{\frac{1}{4}}{1!} + \frac{\frac{-1}{4}}{2!}(x-1)^2$

Optimization word problem

we have $45m^2$ of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.

volume formulas
$V = l \times w \times h$
$V = w^2 \times h$

$V = w^2(\frac{45-w^2}{4w})$
$V = \frac{1}{4}(45w - w3)$

material used:
$M = 1 * w^2 + 4 \times wh$

$45 = w^2 + 4wh$
$h = \frac{45-w^2}{4w}$

our goal is to maximize V, so we differentiate to find critical points
$V' = \frac{1}{4}(45 - 3w^2)$
Zeroes at: $w = \sqrt{15}$ and $\cancel{w = -\sqrt{15}}$
The first derivative test tells us that $\sqrt{15}$ is a maximum.

Solution

we need to solve for h.

$h = \frac{45-w^2}{4w}$
$h = \frac{45-\sqrt{15}^2}{4\sqrt{15}}$
$h = \frac{45-15}{4\sqrt{15}}$
$h = \frac{15}{2\sqrt{15}=\frac{\sqrt{15}}{2}}$
Width of $\sqrt{15}$.
Length of $\sqrt{15}$.
Height of $\sqrt{15}$.