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Explain your solution concisely but clearly. Include all derivation steps. Make sure a fellow student would be able to understand what you mean.

1. Find the largest possible volume V of a cylindrical metal can having a given surface area A.

Solution:

Volume of a cylinder: $V = h\pi r^2$	Surface area of a cylinder: $A = (2\pi r^2) + (2\pi rh)$

1. Isolate h for surface area.

$A = (2\pi r^2) + (2\pi rh)$

$2\pi r^2 = A - 2\pi r^2$

$h = \frac{A-2\pi r^2}{2\pi r}$

2. Insert the volume formula.

$h = \frac{A-2\pi r^2}{2pir}$

$V = h\pi r^2$

$V = (\frac{A-2\pi r^2}{2\pi r})\pi r^2$

$V = \frac{(A\pi r^2)-(2\pi^2 r^4)}{2\pi r}$

$V = \frac{A\pi r^2}{2\pi r} - \frac{2\pi^2 r^4}{2\pi r}$

$V = \frac{Ar}{2} - \pi r^3$

3. Find derivative.

$V(r) = \frac{Ar}{2} - \pi r^3$

$V'(r) = \frac{Ar}{2} - 3\pi r^2$

$V'(r) = \frac{A}{2} - 3\pi r^2$

4. Critical point.

$V'(r) = \frac{A}{2} - 3\pi r^2 = 0$

$3\pi r^2 = \frac{A}{2}$

$r^2 = \frac{A}{(3\times 2)\pi}$

$r = \sqrt{\frac{A}{(3\times 2)\pi}}$

$r = \sqrt{\frac{A}{6\pi}}$

5. Find h for volume

$r = \sqrt{\frac{A}{6\pi}}$

$h = \frac{A-2\pi r^2}{2\pi r}$

$h = \frac{A-2\pi(\sqrt{\frac{A}{6\pi}})^2}{2\pi(\sqrt{\frac{A}{6\pi}})}$

$h = \frac{A-2\pi(\frac{A}{6\pi})}{2\pi(\sqrt{\frac{A}{6\pi}})}$

$h = \frac{A-\frac{2\pi A}{6\pi}}{2\pi\sqrt{\frac{A}{6\pi}}}$

$h = \frac{A-\frac{A}{3}}{2\pi\sqrt{\frac{A}{6\pi}}}$

$h = \frac{\frac{3A}{3}-\frac{A}{3}}{2\pi\sqrt{\frac{A}{6\pi}}}$

$h = \frac{\frac{3A-A}{3}}{2\pi\sqrt{\frac{A}{6\pi}}}$

$h = \frac{\frac{2A}{3}}{2\pi\sqrt{\frac{A}{6\pi}}}$

$h = \frac{2A}{3} \times \frac{1}{\sqrt{\frac{2\pi A}{6\pi}}}$

$h = \frac{2A}{3} \times \frac{\sqrt{6\pi}}{2\pi\sqrt{A}}$

$h = \frac{\sqrt{6\pi A}}{3\pi}$

6. Volume

$V = h \times \pi \times r^2$

$V = \frac{\sqrt{6\pi A}}{3\pi} \times \pi \times (\sqrt{\frac{A}{6\pi}})^2$

$V = \frac{\sqrt{6\pi A}}{3\pi} \times \pi \times \frac{A}{6\pi}$

$V = \frac{\sqrt{6\pi A}\times\pi\times A}{3\pi\times\sqrt{A}\times 6\pi}$

$V = \frac{\sqrt{6\pi A}\times A}{3\times 6\pi}$

$$\boxed{V = \frac{A\sqrt{6\pi A}}{18\pi}}$$

2. Consider the function

$$f(x) = \sqrt{\frac{1}{3+e^x}}$$

1. Find a Taylor approximation of this function around 0 up to order 2.
2. Use the result to approximate $f(0.3)$. How many decimal digits are correct in the approximation?

Solution:

1.

Order Zero:

$f(0) = \sqrt{\frac{1}{3+e^0}} = \sqrt{\frac{1}{3+1}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Order One:

$f(x) = \sqrt{\frac{1}{3+e^x}} = (\frac{1}{3+e^x})^{\frac{1}{2}} = ((3+e^x)^{-1})^{\frac{1}{2}}$

$f'(x) = -\frac{1}{2}((3+e^x)^{-1})^{\frac{1}{2}} \times e^x = -\frac{1}{2}(3+e^x)^{\frac{3}{2}} \times e^x = -\frac{1}{2(3+e^x)^{\frac{3}{2}}} \times e^x = -\frac{e^x}{2(3+e^x)^{\frac{3}{2}}}$

$f'(0) = -\frac{e^0}{2(3+e^0)^{\frac{3}{2}}} = -\frac{1}{2(3+1)^{\frac{3}{2}}} = -\frac{1}{2(4)^{\frac{3}{2}}} = -\frac{1}{8^{\frac{3}{2}}} = -\frac{1}{16}$

Order Two:

$f'(x) = -\frac{1}{2}e^x(3+e^x)^{-\frac{3}{2}} = -\frac{1}{2}(g(x)h(x))$

$g(x) = (3+e^x)^{-\frac{3}{2}}$

$g'(x) = -\frac{3}{2}(3+e^x)^{-\frac{5}{2}}$

$h(x) = e^x$

$h'(x) = e^x$

$f''(x) = -\frac{1}{2}(g(x)h'(x) + g'(x)h'(x)) = -\frac{1}{2}(((e^x)((3+e^x)^{-\frac{3}{2}}) + (e^x)((-\frac{3}{2}+e^x)^{-\frac{5}{2}})(e^x)))$

$f''(x) = \frac{-3e^x+\frac{1}{2}e^{2x}}{2(3+e^x)^{\frac{5}{2}}}$

$f''(0) = \frac{-3e^0+\frac{1}{2}e^{2\times 0}}{2(3+e^0)^{\frac{5}{2}}} = \frac{-3\times 1+\frac{1}{2}e^0}{2(3+1)^{\frac{5}{2}}} = \frac{-3+\frac{1}{2}}{2(3+1)^{\frac{5}{2}}} = \frac{-\frac{5}{2}}{8^{\frac{5}{2}}} = \frac{-\frac{5}{2}}{64} = -\frac{5}{128}$

$f(0) + f'(0)x + f''(0)x^2 = \frac{1}{2} + (\frac{-\frac{1}{16}}{1})x + (\frac{-\frac{5}{128}}{2})x^2 = \boxed{\frac{1}{2} - \frac{1}{16}x - \frac{5}{256}x^2}$

2.

$$\frac{1}{2} - (\frac{1}{16})(0.3) - (\frac{5}{128})(0.3)^2 = \frac{1}{2} - \frac{1}{16}(0.3) - \frac{5}{256}(0.3)^2 = \boxed{0.4794921875}$$

$$\sqrt{\frac{1}{3+e^{(0.3)}}} = \boxed{0.47947108289}$$

$\boxed{\text{Four decimal digits match}}$.

3. Consider the function

$$f(x) = \arctan x + ax$$

   1. For what values of a does this function have a local minimum?
   2. Find the (x, y) coordinates of the local minimum in terms of a.

Solution:

1.

$f(x) = arctan(x) + ax$
$f'(x) = \frac{1}{x^2+1} + a$
$f''(x) = -\frac{2x}{(x^2+1)^2}$

$\frac{1}{x^2+1} + a = 0$
$\frac{1}{x^2+1} = -a$
$1 = -a(x^2 + 1)$
$1 = -ax^2 - a$
$1 + ax^2 = -a$
$ax^2 = -a - 1$
$x^2 = \frac{-a-1}{a}$
$x = \pm\sqrt{\frac{-a-1}{a}}$
$f''(-1.5) = -\frac{2\times(-1.5)}{((-1.5)^2+1)^2} = \frac{-3}{(2.25+1)^2} = -\frac{3}{10.5625}$
$f''(-1) = -\frac{2\times(-1)}{((-1)^2+1)^2} = \frac{-2}{(1+1)^2} = -\frac{2}{4}$
$f''(-0.5) = -\frac{2\times(-0.5)}{((-0.5)^2+1)^2} = \frac{-1}{(0.25+1)^2} = -\frac{1}{1.5625}$
$f''(0) = -\frac{2\times(0)}{((0)^2+1)^2} = \frac{-0}{(0+1)^2} = 0$

$-\sqrt{\frac{-a-1}{a}}$	Local Minimum	$-1 < a < 0$

2.

$x = -\sqrt{\frac{-a-1}{a}}$
$x = arctan(-\sqrt{\frac{-a-1}{a}}) + a(-\sqrt{\frac{-a-1}{a}}) = -arctan(\sqrt{\frac{-a-1}{a}}) - a(\sqrt{\frac{-a-1}{a}})$

$$\boxed{(-\sqrt{\frac{-a-1}{a}}, (-arctan(\sqrt{\frac{-a-1}{a}}) - a(\sqrt{\frac{-a-1}{a}})))}$$