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Explain your solution concisely but clearly. Include all derivation steps. Make sure a fellow student would be able to understand what you mean.

1. Use L'Hôpital's rule as many times as necessary to find the following limit. State *why* you can use L'Hôpital's rule. (What condition needs to be satisfied?)

$$L = \lim_{x \to 0} \frac{(x+1)e^{-x} - 1}{\sin^2 x}$$

Solution:

$$f(0) = \frac{(0+1)e^{-0} - 1}{sin^2(x)} = \frac{(1 \times 1) - 1}{0} = \frac{0}{0}$$

Since this is $\frac{0}{0}$ we can to apply the L'Hopital's rule.

1.

$$f(x) = \frac{g(x)}{h(x)}$$

$$h(x) = sin^2(x)$$

h'(x):

$$h'(x) = [sin^2(x)]\frac{d}{dx} = 2[sin^2(x)]\frac{d}{dx} = 2(sin(x)) \times cos(x) = 2sin(x)cos(x)$$

g'(x):

$$g(x) = (x+1)e^{-x} - 1$$

$$g'(x) = [(x+1)e^{-x}]\frac{d}{dx} - [1]\frac{d}{dx} = ((1)(e^{-x}) + (x+1)(-e^{-x})) - (0)$$

$$= (e^{-x}) + (-xe^{-x} - e^{-x}) = e^{-x} - xe^{-x} - e^{-x} = -xe^{-x}$$

1. Result

$$f'(x) = \frac{g'(x)}{h'(x)} = \frac{-xe^{-x}}{2sin(x)cos(x)}$$

$$f'(0) = \frac{-e^0 \times 0}{2 \times 0 \times 1} = \frac{0}{0}$$

We need to apply the rule again.

Solution:

2.

$$f''(x) = \frac{g''(x)}{h''(x)}$$

g"(x):

$$g''(x) = [-xe^{-x}]\frac{d}{dx} = (e^{-x})(x) + (-e^{x})(1) = (xe^{-x} - e^{-x}) = e^{-x}(x-1)$$

h"(x):

$$h''(x) = [2sin(x)cos(x)]\frac{d}{dx} = 2[sin(x)cos(x)]\frac{d}{dx} = (cos(x))(cos(X)) + (sin(x))(-sin(x)))$$

$$= (cos^2(x) - sin^2(x))$$

$$f''(x) = \frac{e^{-x}(x-1)}{2(cos^2(x) - sin^2(x))}$$

$$f''(0) = \frac{e^0(0-1)}{2(cos^2(0) - sin^2(0))} = \frac{1 \times (-1)}{2(1-0)} = -\frac{1}{2}$$

Now we have an answer we can use!

The value of the limit is:

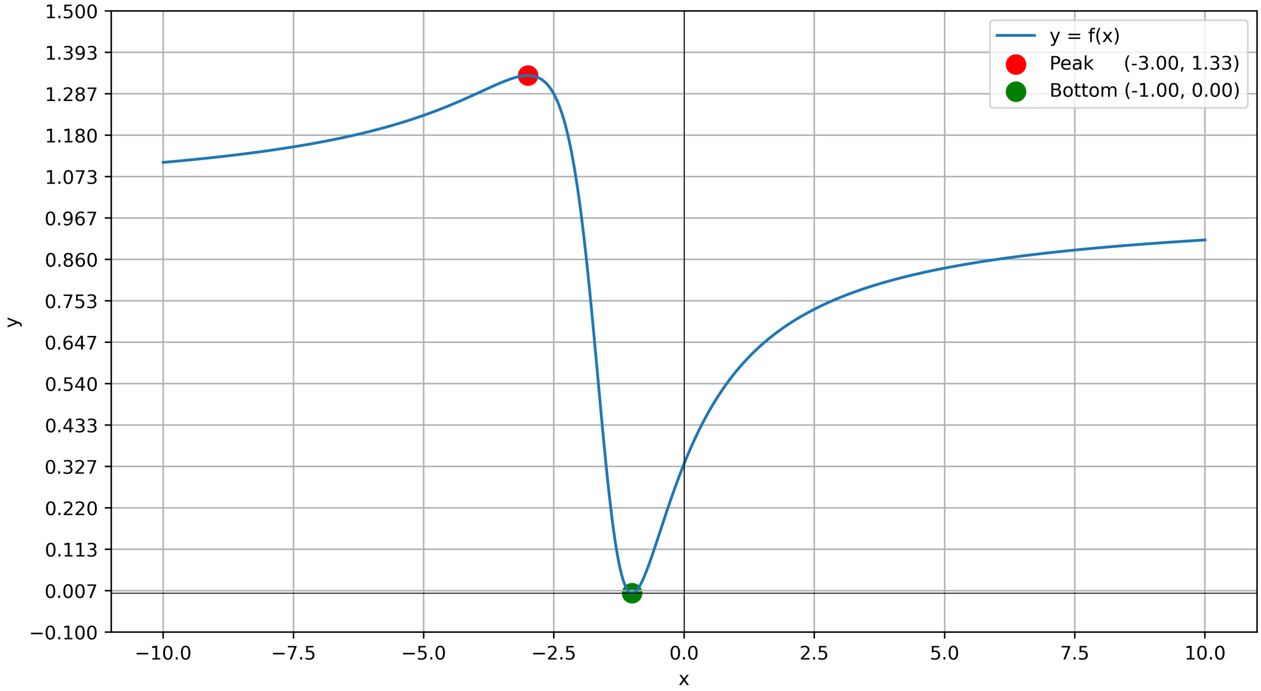
$$L = \lim_{x \to 0} \frac{(x+1)e^{-x} - 1}{2(cos^2(x) - sin^2(x))} = -\frac{1}{2}$$

2. Consider the function
$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 3x + 3}$$

1. Make a sketch of the function and observe where it reaches its smallest and largest value. Use GeoGebra or any other tool for plotting.
2. State what the value of $f'(x)$ is at the points x where $f(x)$ reaches its smallest and largest value ("bottom of the valley", "peak of the mountain").
3. Find a formula for $f'(x)$. Simplify it fully.
4. Solve the equation $f'(x) = 0$. This gives the location of the maximum and minimum.

Solution:

1. Plot of $f(x) = \frac{x^2+2x+1}{x^2+3x+3}$



Solution:

2.

$$f(x) = \frac{g(x)}{h(x)} = \frac{x^2 + 2x + 1}{x^2 + 3x + 3}$$

$$f'(x) = \frac{(f'(x)g(x)) + (f(x)g'(x))}{(g(x))^2}$$

$$= \frac{(2x+2)(x^2+3x+3) - (x^2+2x+1)(2x+3)}{(x^2+3x+3)^2}$$

$$= \frac{(2x^3+6x^2+6x+2x^2+6x+6) - (2x^3+3x^2+4x^2+6x+2x+7)}{(x^4+3x^3+3x^2+3x^3+9x^2+9x+3x^2+9x+9)}$$

$$= \frac{(2x^3+8x^2+12x+6-2x^3-7x^2-8x-3)}{(x^4+6x^3+15x^3+18x+9)}$$

$$= \frac{x^2-4x+3}{x^4+6x^3+15x^2+18x+9}$$

$$= \frac{(x+1)(x+3)}{(x^2+3x+3)^2}$$

Critical Points:

$x = -1$ (Minimum)
$x = -3$ (Maximum)

3.

The formula for $f'(x)$ is:

$$f'(x) = \frac{(x+1)(x+3)}{(x^2+3x+3)^2}$$

See answer 2 for the math.

4.

The critical points are the minimum and maximum. See answer 2 for the math.
$x = -1$ (Minimum)
$x = -3$ (Maximum)