UIMP MASTER IN QUANTUM TECHNOLOGIES
FINAL EXAM COURSE 102776
Quantum Computing: Theory and Practical Applications

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To be returned to the course instructors by February 7th, 2025, 10:00 am (Madrid timezone) via the institutional UIMP email in the virtual campus. Ideally, send it as an attachment of a single email to all course instructors..

Problem 1 (Quantum Gates). Any state of the Block sphere can be constructed from the state $|0\rangle$ as

$$|\psi(\theta,\phi)\rangle = R_z\left(\frac{\pi}{2}+\phi\right) H R_z(\theta)\ H|0\rangle, \quad R_z(\alpha) = e^{-i\frac{\alpha}{2}\sigma^z} \tag{1}$$

where H is the Hadamard gate.

1. Construct the unitary U that implements the transformation

$$|\psi(\theta_2,\phi_2)\rangle = U\,|\psi(\theta_1,\phi_1)\rangle\rangle, \quad \forall\, \theta_{1,2}, \phi_{1,2}$$

Problem 2 (Quantum Noise). Consider a qubit, whose states are described in the computational basis $\{|0\rangle, |1\rangle\}$. Initially, the state of the qubit is given by the operator density $\rho(0) = |0\rangle\langle 0|$. Assume that the qubit interacts with the environment, and that this action can be described by the Kraus operators $K_1 = \sqrt{1-p}\ I_2$, $K_2 = \sqrt{p}\ \sigma_x$, where I_2 is the identity matrix, σ_x the first Pauli matrix, and $0 \leq p \leq 1$. Under the action of these operators, after a time t we obtain

$$\rho(t) = \sum_{i=1}^{2} K_i \rho(0) K_i^\dagger$$

1. Calculate the entropy for the state $\rho(t)$.
2. Is it possible to find a unitary operator $U(t)$ defined on the Hilbert space of the qubit, such that $\rho(t) = U(t)\rho(0)U^\dagger(t)$?
3. Now consider the alternative set of Kraus operators $K_1 = a\,\sigma_z$, $K_2 = b\ \sigma_x$, with a, b some real numbers. What kind of restrictions can be placed on a and b?

Problem 3 (Quantum repetition code). The 3-qubit quantum bit-flip code with stabilizer generators $\{Z_1Z_2, Z_2Z_3\}$ can correct single-qubit Pauli X errors. The 3-qubit quantum phase-flip code with stabilizer generators $\{X_1X_2, X_2X_3\}$ can correct single-qubit Pauli Z errors. Let Y be the Pauli Y gate, which fulfills the following conditions:

$$Y = \mathrm{i}XZ, \qquad Y = SXS^\dagger, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}. \tag{2}$$

In short, Y is Hermitian and identical to X via the unitary change of basis S. Below, we are asked to design quantum error correcting codes for single-qubit Pauli noise channels of the form $\mathcal{E}_U(\rho) = p\rho + (1-p)U\rho U$, where a Pauli error $U \in \{I, X, Y, Z\}$ occurs with some probability $p > 0$.

1. (A different phase-flip code). Show that the repetition code with stabilizer operators $\{Y_1Y_2, Y_2Y_3\}$ can detect and correct single-qubit Pauli Z errors of the form $\mathcal{E}_Z(\rho) = p\rho + (1-p)Z\rho Z$. Describe a quantum circuit for measuring the parity operator Y_1Y_2 of this code.

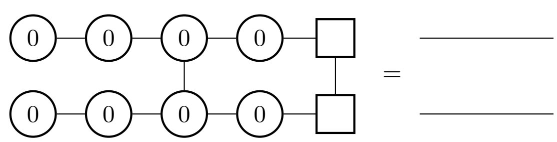
2. (A quantum "Y-flip" code). An experimentalist team has built a faulty quantum computer where qubits are affected by a noise process $\mathcal{E}_Y(\rho) = p\rho + (1-p)Y\rho Y$. Argue that the quantum "Y-flip" code with stabilizer generators $\{Z_1Z_2, Z_2Z_3\}$ can detect and correct single-qubit Pauli Y errors.

3. (A different Bacon-Shor code). Concatenate the repetition code $\{Y_1Y_2, Y_2Y_3\}$ with the bit-flip code $\{Z_1Z_2, Z_2Z_3\}$ to define a modified 9-qubit Bacon-Shor code with stabilizer generators Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, $Y_1Y_2Y_3Y_4Y_5Y_6$, and $Y_4Y_5Y_6Y_7Y_8Y_9$. Argue that this code can correct all single-qubit errors of arbitrary form. (Recall that $\{I, X, Y, Z\}$, the four Pauli matrices, define an operator basis for a single qubit, i.e., any Kraus operator K associated to a single-qubit error can be written as a linear combination $K = a_I I + a_X X + a_Y Y + a_Z Z$ with complex coefficients.)

4. The quantum experimentalists team reports that some mysterious damage has occurred to their quantum computer. We observe that now there is a faulty qubit that is affected by both X and Y errors with even probability: $\mathcal{E}_{X,Y}(\rho) = p\rho + (1-p)/2(X\rho X + Y\rho Y)$, with $p > 0$. On the other hand, Z-errors never occur. Answer: can we use the "Y-flip code" to correct these errors? Is it possible to distinguish X and Y errors by measuring the parity operators Z_1Z_2, Z_2Z_3?

Hints: The stabilizer formalism for quantum codes that we studied in the lectures is quite handy in situations where calculations with wavefunctions become complex. Calculations with stabilizer codes can also be simplified by performing a Clifford change of basis. For instance, in the course, we derived the properties of the phase-flip code by mapping it to a bit-flip code in the Hadamard basis.

Problem 4. Gate teleportation with graph states. The brickwork state is a universal resource for measurement-based quantum computation with applications for blind quantum computing. It is a 2D graph state, with the geometry given in the figure below. Therein, blue balls represent qubits prepared in the $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ state. Edges in the graph represent $CZ = \mathrm{diag}(1, 1, 1, -1)$ gates that are used to create entanglement accross qubits. The qubits with the label "0" are measured with a zero-angle in the X-Y plane: i.e., single-qubit X measurements are performed in all of them.



1. Prove that this measurement pattern implements a two-qubit identity gate modulo some Pauli operation: i.e., it teleports a two-qubit state from left to right and adding some byproduct Pauli operation to it.

2. Imagine the teleported state is measured in the standard basis. Show that the action of a byproduct prior to the measurement Pauli operation is to permute the outcomes in a predictable deterministic way. Discuss: does this jeopardize the outcome of a quantum computation?