

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/244989856>

Approximating the mean of a truncated normal distribution

Article · July 2013

Source: arXiv

CITATIONS

7

READS

488

3 authors, including:



[Konstantinos Koutroumbas](#)

National Observatory of Athens

140 PUBLICATIONS 4,078 CITATIONS

[SEE PROFILE](#)



[Athanasios A. Rontogiannis](#)

National Observatory of Athens

146 PUBLICATIONS 2,196 CITATIONS

[SEE PROFILE](#)

Approximating the mean of a truncated normal distribution

Konstantinos D. Koutroumbas Konstantinos E. Themelis Athanasios A. Rontogiannis

Abstract

A non trivial problem that arises in several applications is the estimation of the mean of a truncated normal distribution. In this paper, an iterative deterministic scheme for approximating this mean is proposed. It has been inspired from an iterative Markov chain Monte Carlo (MCMC) scheme that addresses this problem and it can be viewed as a generalization of a recently proposed relevant model. Conditions are provided under which it is proved that the scheme converges to a unique fixed point. Finally, the theoretical results are also supported by computer simulations, which also show the rapid convergence of the method to a solution vector that is very close to the mean of the truncated normal distribution under study.

Index Terms

truncated normal distribution contraction mapping diagonally dominant matrix MCMC method-
truncated normal distribution contraction mapping diagonally dominant matrix MCMC methodsT

I. INTRODUCTION

A non trivial problem that appears in several applications, e.g. in multivariate regression and Bayesian statistics, is the estimation of the mean of a truncated normal distribution. The problem arises in cases where a random vector $x = [x_1, \dots, x_n]^T$ follows a normal distribution $\mathcal{N}(\mu, \Sigma)$, but x is restricted to a closed subset R^n of \mathcal{R}^n . However, the restrictions considered in most of the cases are of the form $a_i \leq x_i \leq b_i$, with $a_i, b_i \in \mathcal{R} \cup \{-\infty, +\infty\}$, $i = 1, \dots, n$. If $a_i \in \mathcal{R}$ and $b_i = +\infty$ ($a_i = -\infty$ and $b_i \in \mathcal{R}$), we

This research has been co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF) - Research Funding Program: ARISTEIA- HSI-MARS-1413.

have a *one-sided truncation* from the left (right), while in the case where both a_i and b_i are reals, we have *two-sided truncation*.

The problem has attracted the attention of several researchers from the sixties. Since then, several deterministic approaches have been proposed with some of them trying to estimate not only the mean but also other moments (such as the variance) of a multivariate truncated normal distribution. These approaches can be categorized according to whether they are suitable for one-sided truncated normal distributions ([18], [21], [2], [6], [8]) or for two-sided truncated normal distributions ([20], [16], [5], [7], [12], [9]), or according to whether they consider the bivariate, $n = 2$, ([18], [20], [16], [12]) or the multivariate case ([21], [2], [5], [6], [7], [8], [9]). In addition, some of these methods put additional restrictions to the distribution parameters (e.g. [21], [2], [6] require that $\mu = 0$). Most of these methods either perform direct integration (e.g. [21]) or they utilize the moment generating function tool (e.g. [9]).

An alternative strategy to deal with this problem has been followed in [17]. More specifically, in [17], a Markov Chain Monte Carlo (MCMC) iterative scheme has been developed. According to this, at each iteration, n successive samplings take place, one from each one of n one-dimensional conditionals of the truncated normal distribution. The mean of the i -th such distribution is the mean of x_i conditioned on the $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$. After performing several iterations, the estimation of the mean of the truncated normal results by performing an averaging over the produced samples. The convergence of this scheme to the mean of the truncated normal distribution is a direct consequence of the Markov chain theory.

The work presented in this paper for approximating the mean of a multivariate truncated normal distribution has been inspired from that of [17]. Specifically, instead of selecting a sample from each one of the above one-dimensional distributions, we select its mean. Thus, the proposed scheme departs from the statistical framework adopted in [17] and moves to the deterministic one.

This work is an extension of a relative scheme used in [22] in the framework of spectral unmixing in hyperspectral images. In addition, a convergence proof of the proposed scheme is given when certain conditions are fulfilled. Experimental results show that the proposed scheme converges significantly faster than the MCMC approach of [17].

The paper is organized as follows. Section 2 contains some necessary definitions and a brief description of the work in [17]. In Section 3, the newly proposed method is described and in Section 4 conditions are given under which it is proved to converge. In Section 5, simulation results are provided and a relevant discussion is presented. Finally, Section 6 concludes the paper.

II. PRELIMINARIES AND PREVIOUS WORK

Let us consider the n -dimensional normal distribution

$$\mathcal{N}(x|\mu, \Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) \quad (1)$$

where the n -dimensional vector μ is its mean and the $n \times n$ matrix Σ is its covariance matrix.

Let R^n be a subset of \mathcal{R}^n with positive Lebesgue measure. We denote by $\mathcal{N}_{R^n}(x|\mu, \Sigma)$ the truncated normal distribution which results from the truncation of $\mathcal{N}(x|\mu, \Sigma)$ in R^n . Speaking in mathematical terms

$$\mathcal{N}_{R^n}(x|\mu, \Sigma) = \begin{cases} \frac{\exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))}{\int_{R^n} \exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)) dW}, & \text{if } x \in R^n \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Note that $\mathcal{N}_{R^n}(x|\mu, \Sigma)$ is *proportional* to $\mathcal{N}(x|\mu, \Sigma)I_{R^n}(x)$, where $I_{R^n}(x) = 1$, if $x \in R^n$ and 0, otherwise.

In the scheme discussed in [17], a Markov Chain Monte Carlo (MCMC) method is proposed, to compute the mean of single or doubly truncated (per coordinate) normal distribution $\mathcal{N}_{R^n}(x|\mu, \Sigma)$, where $R^n = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$. The method relies on the sampling of the n one-dimensional conditionals of the truncated normal distribution. More specifically, letting $\mu_i^* \equiv E[x_i|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ and σ_i^{*2} denoting the expectation and the variance of x_i conditioned on the rest coordinates, respectively¹, and $\mathcal{N}_{[a_i, b_i]}(\mu_i^*, \sigma_i^{*2})$ denoting the (one dimensional) truncated normal distribution which results from the truncation of a normal distribution with mean μ_i^* and variance σ_i^{*2} in $[a_i, b_i]$, the iterative sampling scheme proposed in [17] can be written as

$$\begin{aligned} 1. \quad & x_1^{(t)} \sim \mathcal{N}_{[a_1, b_1]}(E[x_1|x_2^{(t-1)}, \dots, x_n^{(t-1)}], \sigma_1^{*2}) \\ 2. \quad & x_2^{(t)} \sim \mathcal{N}_{[a_2, b_2]}(E[x_2|x_1^{(t)}, x_3^{(t-1)}, \dots, x_n^{(t-1)}], \sigma_2^{*2}) \\ & \vdots \\ n. \quad & x_n^{(t)} \sim \mathcal{N}_{[a_n, b_n]}(E[x_n|x_1^{(t)}, x_2^{(t)}, \dots, x_{n-1}^{(t)}], \sigma_n^{*2}) \end{aligned} \quad (3)$$

where \sim denotes the sampling action and t denotes the current iteration. After performing several, say k , iterations (and after discarding the first few, say k' , ones) the mean of each coordinate is estimated as

$$E[x_i] = \frac{1}{k-k'} \sum_{t=k'}^k x_i^{(t)}, \quad i = 1, \dots, n$$

The quantities μ_i^* and σ_i^{*2} of each one of the above one dimensional conditionals are expressed in terms of the parameters μ and Σ of the non-truncated multidimensional normal distribution as follows

$$\mu_i^* = \mu_i + \sigma_{-i}^T \Sigma_{-i-i}^{-1} (x_{-i} - \mu_{-i}), \quad i = 1, \dots, n \quad (4)$$

¹That is, x_i follows the (non-truncated) i -th conditional of $\mathcal{N}(x|\mu, \Sigma)$.

$$\sigma_i^* = \sigma_{ii} - \sigma_{-i}^T \Sigma_{-i-i}^{-1} \sigma_{-i}, \quad i = 1, \dots, n \quad (5)$$

with Σ_{-i-i} being the $(n-1) \times (n-1)$ matrix that results from Σ after removing its i -th column and its i -th row, σ_{-i} being the i -th column of Σ excluding its i -th element and x_{-i} , μ_{-i} being the $((n-1)$ -dimensional) vectors that result from x and μ , respectively, after removing their i -th coordinates, x_i and μ_i , respectively. Note that μ_i^* depends on all x_j 's *except* x_i .

III. THE PROPOSED MODEL

In the sequel, we focus on the case where R^n is a set of the form $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$, where for each interval $[a_i, b_i]$ it is either, (i) $a_i = -\infty$ and $b_i \in R$ or (ii) $a_i \in R$ and $b_i = \infty$. This means that, along each dimension, the truncation is one-sided and more specifically, case (i) corresponds to right truncation while case (ii) corresponds to left truncation.

The proposed model for estimating the mean of $\mathcal{N}^n(x|\mu, \Sigma)$, is of iterative nature and, at each iteration, it requires the computation of the (one-dimensional) *erfc* function. This model has a close conceptual affinity with the one (briefly) presented in the previous section ([17]). More specifically, instead of utilizing the samples produced by the (one dimensional) distributions $\mathcal{N}_{[a_i, b_i]}(E[x_i|x_1^{(t)}, x_2^{(t)}, \dots, x_{i-1}^{(t)}, x_{i+1}^{(t-1)}, \dots, x_n^{(t-1)}], \sigma_i^{*2})$, we use the corresponding mean values (here denoted by $w = [w_1, w_2, \dots, w_n]^T$).

As it is well known (see e.g. [15]), the mean w of a truncated one dimensional normal distribution $\mathcal{N}_{[a, b]}(\mu^*, \sigma^{*2})$, which has resulted from the (non-truncated) normal distribution with mean μ^* and variance σ^{*2} , is expressed as

$$w = \mu^* + \frac{\phi(\frac{a-\mu^*}{\sigma^*}) - \phi(\frac{b-\mu^*}{\sigma^*})}{\Phi(\frac{b-\mu^*}{\sigma^*}) - \Phi(\frac{a-\mu^*}{\sigma^*})} \sigma^* \quad (6)$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $\Phi(x) = \frac{1}{2} \text{erfc}(-\frac{x}{\sqrt{2}})$ and $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$.

However, since in the present paper we consider the cases where either (i) $a_i \in R$ and $b_i = \infty$ or (ii) $a_i = -\infty$ and $b_i \in R$, let us see now how eq. (6) becomes for each of these cases.

(i) $a \in R$, $b = +\infty$. In this case it is $\frac{b-\mu^*}{\sigma^*} = +\infty$ and as a consequence, $\phi(\frac{b-\mu^*}{\sigma^*}) = 0$, $\Phi(\frac{b-\mu^*}{\sigma^*}) = \frac{1}{2} \text{erfc}(-\frac{b-\mu^*}{\sqrt{2}\sigma^*}) = \frac{1}{2} \text{erfc}(-\infty) = 1$. Thus, taking also into account the definitions of ϕ and Φ and the fact that $\text{erfc}(x) + \text{erfc}(-x) = 2$, eq. (6) gives,

$$w = \mu^* + \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{(\mu^*-a)^2}{2\sigma^{*2}}}}{\text{erfc}(-\frac{\mu^*-a}{\sqrt{2}\sigma^*})} \sigma^* \quad (7)$$

(ii) $a = -\infty$, $b \in R$. In this case it is $\frac{a-\mu^*}{\sigma^*} = -\infty$ and, as a consequence, $\phi(\frac{a-\mu^*}{\sigma^*}) = 0$, $\Phi(\frac{a-\mu^*}{\sigma^*}) = \frac{1}{2}\text{erfc}(-\frac{a-\mu^*}{\sqrt{2}\sigma^*}) = \frac{1}{2}\text{erfc}(\infty) = 0$. Working as in case (i), eq. (6) gives

$$w = \mu^* - \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{(b-\mu^*)^2}{2\sigma^{*2}}}}{\text{erfc}(-\frac{b-\mu^*}{\sqrt{2}\sigma^*})} \sigma^* \quad (8)$$

Eqs. (7) and (8) can be expressed compactly via the following single equation

$$w = \mu^* + \sqrt{\frac{2}{\pi}} \left[f\left(\frac{\mu^* - a}{\sqrt{2}\sigma^*}\right) I_{a \in \mathcal{R}} - f\left(\frac{b - \mu^*}{\sqrt{2}\sigma^*}\right) I_{b \in \mathcal{R}} \right] \sigma^* \quad (9)$$

where $I_{a \in \mathcal{R}}$ ($I_{b \in \mathcal{R}}$) is an indicator function which equals to 1 if $a \in \mathcal{R}$ ($b \in \mathcal{R}$) and 0 otherwise and

$$f(x) = \frac{e^{-x^2}}{\text{erfc}(-x)} \quad (10)$$

Let us now return to the multidimensional case. Since from the (truncated) conditional one-dimensional normals we no longer perform sampling but, instead, we consider their means, eq. (4) is altered to

$$\mu_i^* = \mu_i + \sigma_i^T \Sigma_{-i}^{-1} (w_{-i} - \mu_{-i}), \quad i = 1, \dots, n \quad (11)$$

where w_{-i} results from the current estimate of the (n - dimensional) mean vector w of the truncated normal distribution, after removing its i -th coordinate.

Putting now all the previous ingredients together (that is, utilizing eqs. (9), (11) and (5)) we obtain the following iterative scheme

$$\begin{aligned} 1. \quad w_1^{(t)} &= \mu_1^{*(t)} + \sqrt{\frac{2}{\pi}} \left[f\left(\frac{\mu_1^{*(t)} - a_1}{\sqrt{2}\sigma_1^*}\right) I_{a_1 \in \mathcal{R}} - f\left(\frac{b_1 - \mu_1^{*(t)}}{\sqrt{2}\sigma_1^*}\right) I_{b_1 \in \mathcal{R}} \right] \sigma_1^* \\ 2. \quad w_2^{(t)} &= \mu_2^{*(t)} + \sqrt{\frac{2}{\pi}} \left[f\left(\frac{\mu_2^{*(t)} - a_2}{\sqrt{2}\sigma_2^*}\right) I_{a_2 \in \mathcal{R}} - f\left(\frac{b_2 - \mu_2^{*(t)}}{\sqrt{2}\sigma_2^*}\right) I_{b_2 \in \mathcal{R}} \right] \sigma_2^* \\ &\vdots \\ n. \quad w_n^{(t)} &= \mu_n^{*(t)} + \sqrt{\frac{2}{\pi}} \left[f\left(\frac{\mu_n^{*(t)} - a_n}{\sqrt{2}\sigma_n^*}\right) I_{a_n \in \mathcal{R}} - f\left(\frac{b_n - \mu_n^{*(t)}}{\sqrt{2}\sigma_n^*}\right) I_{b_n \in \mathcal{R}} \right] \sigma_n^* \end{aligned} \quad (12)$$

with $\mu_i^{*(t)}$ being computed via eq. (11), where $w_{-i}^{(t)}$ (the only parameter in (11) that varies through iterations) is defined as $w_{-i}^{(t)} = [w_1^{(t)}, \dots, w_{i-1}^{(t)}, w_{i+1}^{(t-1)}, \dots, w_n^{(t-1)}]^T$, that is, the most recent information about w_i 's is utilized. More formally, we can say that the above scheme performs sequential updating and (following the terminology used in [3]) it is a *Gauss Seidel* updating scheme.

It is reminded that, due to the type of truncation considered here (only left truncation or only right truncation per coordinate), the bracketed expression in each equation of (12) contains *only one* non identically equal to zero term. In the sequel, we consider separately the cases corresponding to $a_i \in \mathcal{R}$ and $b_i \in \mathcal{R}$, i.e.,

$$w_i^{(t)} = \mu_i^{*(t)} + \sqrt{\frac{2}{\pi}} f(A_i^{(t)}) \sigma_i^* \quad (13)$$

and

$$w_i^{(t)} = \mu_i^{*(t)} - \sqrt{\frac{2}{\pi}} f(B_i^{(t)}) \sigma_i^* \quad (14)$$

where

$$A_i^{(t)} = \frac{\mu_i^{*(t)} - a_i}{\sqrt{2}\sigma_i^*} \quad (15)$$

and

$$B_i^{(t)} = \frac{b_i - \mu_i^{*(t)}}{\sqrt{2}\sigma_i^*} \quad (16)$$

$i = 1, \dots, n$, respectively, and $f(\cdot)$ is defined as in eq. (10).

IV. CONVERGENCE ISSUES

In this section we provide sufficient conditions under which the proposed scheme is proved to converge². Before we proceed, we give some propositions and remind some concepts that will be proved useful in the sequel.

Proposition 1: Assume that Σ is a symmetric positive definite $n \times n$ matrix, $\sigma_{\neg i}$ is the i -th column of Σ , after removing its i -th element and $\Sigma_{\neg i \neg i}$ results from Σ after removing its i -th row and its i -th column. Also, let s_{ii} be the (i, i) element of Σ^{-1} and $s_{\neg i}^T$ be the $(n-1)$ -dimensional vector resulting from the i -th row of Σ^{-1} after (i) removing its i -th element, s_{ii} , and (ii) multiplying the remaining elements by $-1/s_{ii}$. Then, it holds

- (i) $s_{ii} = \frac{1}{\sigma_{ii} - \sigma_{\neg i}^T \Sigma_{\neg i \neg i}^{-1} \sigma_{\neg i}}$ and
- (ii) $s_{\neg i}^T = \sigma_{\neg i}^T \Sigma_{\neg i \neg i}^{-1}$.

The proof of this proposition is straightforward from the inversion lemma for block partitioned matrices ([19], p. 53) and the use of permutation matrices, in order to define the Schur complement $\sigma_{ii} - \sigma_{\neg i}^T \Sigma_{\neg i \neg i}^{-1} \sigma_{\neg i}$ for each row i of Σ .

Proposition 2: It is

$$-\sqrt{\pi} \leq f'(x) \leq 0, \forall x \in \mathcal{R} \quad (17)$$

where $f'(x)$ denotes the derivative of $f(x)$, which is defined in eq. (10).

²However, it is noted that even if these conditions are slightly violated, the algorithm still works, as is verified by the experiments presented in Section 5.

The proof of proposition 2 is given in the appendix.

Definition 1: A mapping $T : X \rightarrow X$, where $X \subset \mathcal{R}^n$, is called *contraction* if for some norm $\|\cdot\|$ there exists some constant $\alpha \in [0, 1)$ (called *modulus*) such that

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in X \quad (18)$$

The corresponding iteration $x(t+1) = T(x(t))$ is called *contracting iteration*.

Proposition 3 ([3], pp. 182-183): Suppose that $T : X \rightarrow X$ is a contraction with modulus $\alpha \in [0, 1)$ and that X is a closed subset of \mathcal{R}^n . Then

- (a) The mapping T has a unique fixed point $x^* \in X$ ³.
- (b) For every initial vector $x(0) \in X$, the sequence $\{x(t)\}$, generated by $x(t+1) = T(x(t))$ converges to x^* geometrically. In particular,

$$\|x(t) - x^*\| \leq \alpha^t \|x(0) - x^*\|, \quad \forall t \geq 0$$

Let us define the mappings $T_i : \mathcal{R}^n \rightarrow \mathcal{R}$, $i = 1, \dots, n$ as

$$T_i(x) = \mu_i^*(x) + \sqrt{\frac{2}{\pi}} \left[f\left(\frac{\mu_i^*(x) - a_i}{\sqrt{2}\sigma_i^*}\right) I_{a_i \in \mathcal{R}} - f\left(\frac{b_i - \mu_i^*(x)}{\sqrt{2}\sigma_i^*}\right) I_{b_i \in \mathcal{R}} \right] \sigma_i^* \quad (19)$$

where

$$\mu_i^*(x) = \mu_i + \sigma_{-i}^T \Sigma_{-i-i}^{-1} (x_{-i} - \mu_{-i}) \quad (20)$$

and

$$A_i(x) = \frac{\mu_i^*(x) - a_i}{\sqrt{2}\sigma_i^*}, \quad B_i(x) = \frac{b_i - \mu_i^*(x)}{\sqrt{2}\sigma_i^*} \quad (21)$$

and the mapping T as

$$T(x) = (T_1(x), \dots, T_n(x))$$

Let us define next the mapping $\hat{T}_i : \mathcal{R}^n \rightarrow \mathcal{R}^n$ as

$$\hat{T}_i(x) = \hat{T}_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, T_i(x), x_{i+1}, \dots, x_n)$$

Performing the sequential updating as described by eq. (12) (one at a time and in increasing order) is equivalent to applying the mapping $S : \mathcal{R}^n \rightarrow \mathcal{R}^n$, defined as

$$S = \hat{T}_n \circ \hat{T}_{n-1} \circ \dots \circ \hat{T}_2 \circ \hat{T}_1 \quad (22)$$

³A point x^* is called *fixed point* of a mapping T if it is $T(x^*) = x^*$.

where \circ denotes composition. Following the terminology given in [3], S is called *the Gauss Seidel mapping based on the mapping T* and the iteration $x(t+1) = S(x(t))$ is called *the Gauss Seidel algorithm based on mapping T* .

A direct consequence of [3, Prop. 1.4, pp.186] is the following Proposition:

Proposition 4: If $T : X \rightarrow X$ is a contraction with respect to the l_∞ norm, then the Gauss-Seidel mapping S is also a contraction (with respect to the l_∞ norm), with the same modulus as T . In particular, if X is closed, the sequence of the vectors generated by the Gauss-Seidel algorithm based on the mapping T converges to the unique fixed point of T geometrically.

Having given all the necessary definitions and results, we will proceed by proving that (a) for each mapping T_i it holds $|T_i(x) - T_i(y)| \leq \|\sigma_{-i}^T \Sigma_{-i-i}^{-1}\|_1 \|x - y\|_\infty$, $\forall x, y \in \mathcal{R}^n$, where $\|\cdot\|_1$, $\|\cdot\|_\infty$ are the l_1 and l_∞ norms, respectively, (b) if Σ^{-1} is diagonally dominant then T is a contraction and (c) provided that T is a contraction, the algorithm $x(t+1) = S(x(t))$ converges geometrically to the unique fixed point of T . We remind here that the $(n-1)$ -dimensional vector $\sigma_{-i}^T \Sigma_{-i-i}^{-1}$ results from the i -th row of Σ^{-1} , excluding its i -th element s_{ii} and dividing each element by the negative of s_{ii} .

Proposition 5: For the mappings T_i , $i = 1, \dots, n$, it holds

$$|T_i(x) - T_i(y)| \leq \|\sigma_{-i}^T \Sigma_{-i-i}^{-1}\|_1 \|x - y\|_\infty, \quad \forall x, y \in \mathcal{R}^n \quad (23)$$

Proof: (a) We consider first the case where $a_i \in \mathcal{R}$. Let us consider the vectors $x, y \in \mathcal{R}^n$. Since μ is constant, utilizing eq. (20) it follows that

$$\mu_i^*(x) - \mu_i^*(y) = \sigma_{-i}^T \Sigma_{-i-i}^{-1} (x_{-i} - y_{-i}) \quad (24)$$

Also, it is

$$A_i(x) - A_i(y) = \frac{1}{\sqrt{2}\sigma_i^*} \sigma_{-i}^T \Sigma_{-i-i}^{-1} (x_{-i} - y_{-i}) \quad (25)$$

Taking the difference $T_i(x) - T_i(y)$ we have

$$\begin{aligned} T_i(x) - T_i(y) &= (\mu_i^*(x) - \mu_i^*(y)) + \\ &+ \sqrt{\frac{2}{\pi}} (f(A_i(x)) - f(A_i(y))) \sigma_i^* \end{aligned} \quad (26)$$

Since f is continuous in \mathcal{R} , the mean value theorem guarantees that there exists $\xi_i \in [\min(A_i(x), A_i(y)), \max(A_i(x), A_i(y))]$ such that

$$f(A_i(x)) - f(A_i(y)) = f'(\xi_i)(A_i(x) - A_i(y)) \quad (27)$$

Substituting eq. (27) to (26) we get

$$\begin{aligned} T_i(x) - T_i(y) &= (\mu_i^*(x) - \mu_i^*(y)) + \\ &+ \sqrt{\frac{2}{\pi}} f'(\xi_i) (A_i(x) - A_i(y)) \sigma_i^* \end{aligned} \quad (28)$$

Substituting (24) and (25) into (28) and after some manipulations it follows that

$$T_i(x) - T_i(y) = (1 + \frac{f'(\xi_i)}{\sqrt{\pi}}) \sigma_{\neg i}^T \Sigma_{\neg i \neg i}^{-1} (x_{\neg i} - y_{\neg i}) \quad (29)$$

Taking absolute values in eq. (29) and applying Hölder's inequality $|a^T b| \leq \|a\|_p \|b\|_q$, for $p = 1$ and $q = \infty$, it follows that

$$|T_i(x) - T_i(y)| \leq \|(1 + \frac{f'(\xi_i)}{\sqrt{\pi}}) \sigma_{\neg i}^T \Sigma_{\neg i \neg i}^{-1}\|_1 \|x_{\neg i} - y_{\neg i}\|_\infty \quad (30)$$

Taking into account that $-\sqrt{\pi} \leq f'(\xi_i) \leq 0$ (from Proposition 2), and the (trivial) fact that $\|x_{\neg i} - y_{\neg i}\|_\infty \leq \|x - y\|_\infty$, it follows that

$$|T_i(x) - T_i(y)| \leq \|\sigma_{\neg i}^T \Sigma_{\neg i \neg i}^{-1}\|_1 \|x - y\|_\infty \quad (31)$$

Thus, the claim has been proved.

(b) We consider now the case where $b_i \in \mathcal{B}$. Working similarly to the previous case, the difference $B_i(x) - B_i(y)$ is

$$B_i(x) - B_i(y) = -\frac{1}{\sqrt{2}\sigma_i^*} \sigma_{\neg i}^T \Sigma_{\neg i \neg i}^{-1} (x_{\neg i} - y_{\neg i}) \quad (32)$$

while the difference $T_i(x) - T_i(y)$ is expressed as

$$\begin{aligned} T_i(x) - T_i(y) &= (\mu_i^*(x) - \mu_i^*(y)) + \\ &+ \sqrt{\frac{2}{\pi}} (f(B_i(x)) - f(B_i(y))) \sigma_i^* \end{aligned} \quad (33)$$

Utilizing the mean value theorem we have that there exists $\xi_i \in [\min(B_i(x), B_i(y)), \max(B_i(x), B_i(y))]$ such that

$$f(B_i(x)) - f(B_i(y)) = f'(\xi_i) (B_i(x) - B_i(y)) \quad (34)$$

Substituting eq. (34) to (33) we get

$$\begin{aligned} T_i(x) - T_i(y) &= (\mu_i^*(x) - \mu_i^*(y)) + \\ &+ \sqrt{\frac{2}{\pi}} f'(\xi_i) (B_i(x) - B_i(y)) \sigma_i^* \end{aligned} \quad (35)$$

Substituting eqs. (24) and (32) into (35), we obtain

$$T_i(x) - T_i(y) = (1 + \frac{f'(\xi_i)}{\sqrt{\pi}}) \sigma_{-i}^T \Sigma_{-i-i}^{-1} (x_{-i} - y_{-i}) \quad (36)$$

From this point on, the proof is exactly the same with that of (a). Q.E.D.

Proposition 6: The mapping T is a contraction in \mathcal{R}^n , with respect to the l_∞ norm, provided that Σ^{-1} is diagonally dominant.

Proof: Let $x, y \in \mathcal{R}^n$. Taking into account proposition 5, it easily follows that

$$\begin{aligned} \|T(x) - T(y)\|_\infty &\equiv \max_{i=1, \dots, n} \{|T_i(x) - T_i(y)|\} \leq \\ &\leq \max_{i=1, \dots, n} \{ \|\sigma_{-i}^T \Sigma_{-i-i}^{-1}\|_1 \|x - y\|_\infty \} \end{aligned}$$

Now, (a) taking into account that the $(n-1)$ -dimensional vector $\sigma_{-i}^T \Sigma_{-i-i}^{-1}$ results from the i -th row of Σ^{-1} , excluding its i -th element s_{ii} and dividing each element by the negative of s_{ii} and (b) recalling that s_{-i} is the i -th row of Σ^{-1} excluding its i -th element s_{ii} , it is

$$\|\sigma_{-i}^T \Sigma_{-i-i}^{-1}\|_1 = \|\frac{s_{-i}}{s_{ii}}\|_1$$

Provided that Σ^{-1} is diagonally dominant, it is

$$\|s_{-i}\|_1 \leq |s_{ii}|, \quad \forall i$$

or

$$\|\frac{s_{-i}}{s_{ii}}\|_1 < 1, \quad \forall i$$

which proves the claim. Q.E.D.

Theorem 1: The algorithm $x(t+1) = S(x(t))$ converges geometrically to the unique fixed point of T , provided that Σ^{-1} is diagonally dominant.

Proof: The proof is a direct consequence of the propositions 3, 4 and 6 exposed before, applied for $X = \mathcal{R}^n$. Q.E.D.

V. SIMULATION RESULTS

In this section we assess the performance of the proposed iterative scheme using the (widely accepted as reliable) MCMC method described in [17] as benchmark⁴.

⁴In the sequel, all results are rounded to the third decimal.

1st experiment: The purpose of this experiment is to compare the estimates of the mean of a truncated normal distribution obtained by the proposed and the MCMC methods, for dimensions n varying from 2 to 15. To this end, for each dimension n , $k = 30$ different truncated normal distributions (defined by the means and the covariance matrices of the corresponding untruncated normals, as well as their truncation points) have randomly been generated, such that, the corresponding inverse covariance matrix is diagonally dominant. For the i -th such distribution, $i = 1, \dots, k$, both the proposed and the MCMC methods have been applied. Letting w_{ni}^D and w_{ni}^{MCMC} denote the respective resulting estimates, the mean difference per coordinate between the two estimates is computed, i.e., $\delta_{ni} = \frac{\|w_{ni}^D - w_{ni}^{MCMC}\|_1}{n}$ and, averaging over i , the quantity $\Delta_n = \frac{1}{k} \sum_{i=1}^k \delta_{ni}$ is obtained. In figure 1, Δ_n is plotted versus n . From this figure, it can be concluded that the proposed scheme gives estimates that are very close to those given by the MCMC. Thus, the fixed point of the proposed scheme (when the diagonal dominance condition is fulfilled) can be considered as a reliable estimate of the mean of the truncated normal distribution.

Next, in order to show the rapid convergence of the proposed scheme against the MCMC method, we focus on a single example (however, the resulting conclusions are generally applicable). More specifically, Table 1 shows the values of the l_1 norm of the difference $w^{(t)} - w^{(t-1)}$ divided by n , i.e. the quantity $\|w^{(t)} - w^{(t-1)}\|_1/n$, as the number of iterations evolves, for the 5-dimensional left truncated normal distribution defined by $\mu = [2.688, 9.169, -11.294, 4.311, 1.594]^T$,

$$\Sigma = \begin{bmatrix} 0.045 & -0.003 & 0.013 & -0.004 & 0.011 \\ -0.003 & 0.056 & -0.015 & 0.008 & 0.010 \\ 0.013 & -0.015 & 0.074 & -0.001 & 0.004 \\ -0.004 & 0.008 & -0.001 & 0.156 & -0.012 \\ 0.011 & 0.010 & 0.004 & -0.012 & 0.038 \end{bmatrix},$$

while its truncation point is $a = [2.591, 8.891, -11.841, 3.353, 0.629]^T$. It is clear that the proposed method converges rapidly to its estimate, while MCMC exhibits rather slow converge. This is the strong point of the proposed method compared to the MCMC (keeping, however, in mind the requirement for diagonal dominance of the inverse covariance matrix, for the proposed method).

In the sequel we try to get an indication about the performance of the proposed method when the diagonal dominance requirement does not hold⁵. The following two experiments are in this direction.

⁵It is reminded that no formal proof of the convergence of the proposed scheme has been given for this case.

iterations	MCMC method	Proposed method
1	95.130e-003	6.278e+000
2	72.952e-003	571.150e-003
3	47.163e-003	34.602e-003
4	43.741e-003	3.897e-003
5	35.021e-003	324.702e-006
6	25.441e-003	27.751e-006
7	34.047e-003	1.920e-006
8	14.612e-003	133.025e-009
9	17.539e-003	0
10	21.859e-003	0

TABLE I

THE EVOLUTION OF $\|w^{(t)} - w^{(t-1)}\|/n$ FOR THE MCMC AND THE PROPOSED METHODS, FOR THE FIRST FEW ITERATIONS.

2nd experiment: We consider the three-dimensional case where $\mu = [2.660, 9.307, -3.321]^T$ and

$$\Sigma = \begin{bmatrix} 1.493 & -0.973 & -1.225 \\ -0.973 & 4.463 & 3.429 \\ -1.225 & 3.429 & 8.014 \end{bmatrix},$$

while the truncation point is $a = [2.176, 8.657, -3.990]^T$. Note that, although in this case the inverse of the covariance matrix, which is

$$\Sigma^{-1} = \begin{bmatrix} 0.807 & 0.121 & 0.072 \\ 0.121 & 0.352 & -0.132 \\ 0.072 & -0.132 & 0.192 \end{bmatrix},$$

is not diagonally dominant, the diagonal dominance condition is slightly violated (only in the third row). The estimate of the mean of the truncated normal distribution obtained by the proposed method is $\hat{\mu} = [3.122, 10.509, -1.598]^T$, which is quite close to the estimate obtained by the MCMC (the mean absolute difference per coordinate is 0.048). It is worth noting that in this case, although eq. (31) is not satisfied ($\|\max_i \{\sigma_{-i}^T \Sigma_{-i-i}^{-1}\}\|_1 > 1$, since Σ^{-1} is not diagonally dominant), the quantities $\|(1 + \frac{f'(\xi_i)}{\sqrt{\pi}}) \sigma_{-i}^T \Sigma_{-i-i}^{-1}\|_1, i = 1, \dots, n$, are less than 1 for all pairs of points $(w^{(t)}, w^{(t-1)})$ ⁶. Thus, the (tighter) bound of eq. (30) is satisfied for all the pairs $(w^{(t)}, w^{(t-1)})$, which guarantees the rapid convergence of

⁶The dependence of this quantity from $(w^{(t)}, w^{(t-1)})$ is through ξ .

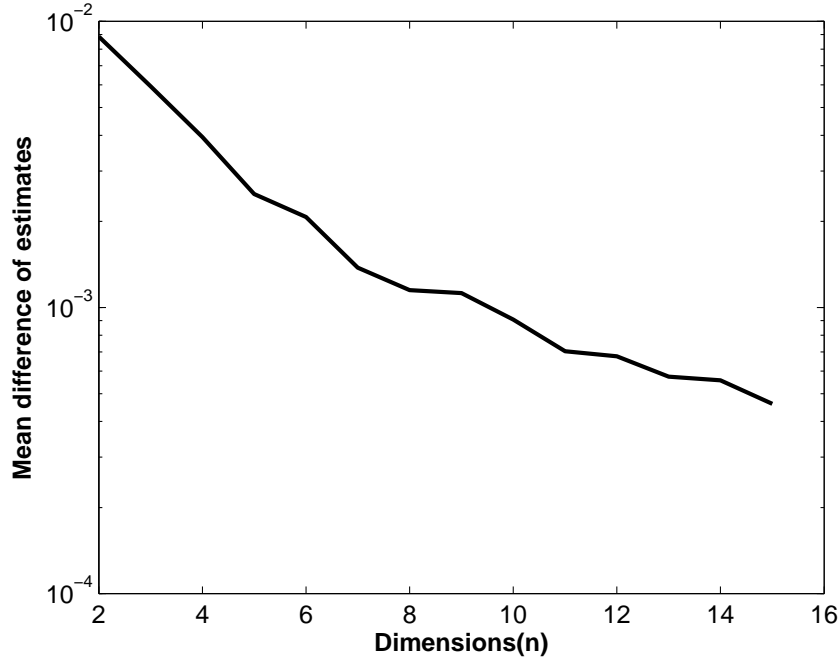


Fig. 1. The mean absolute difference per coordinate between the estimates obtained by the proposed and the MCMC methods, as the dimensionality n varies from 2 to 15. 30 experiments have been conducted for each dimensionality.

the method. However, note that, since $\|(1 + \frac{f'(\xi_i)}{\sqrt{\pi}})\sigma_{-i}^T \Sigma_{-i-i}^{-1}\|_1$ depends on the data (through ξ), it cannot be used as an upper bound in the proof of the contraction.

3rd experiment: We consider now the case where $\mu = [-3.968, -3.141, 8.134]^T$ and

$$\Sigma = \begin{bmatrix} 1.082 & -0.490 & 1.434 \\ -0.490 & 1.088 & -0.052 \\ 1.434 & -0.052 & 2.711 \end{bmatrix},$$

while the truncation point is $a = [-4.541, -3.358, 7.512]^T$. The matrix

$$\Sigma^{-1} = \begin{bmatrix} 7.904 & 3.365 & -4.116 \\ 3.365 & 2.352 & -1.735 \\ -4.116 & -1.735 & 2.513 \end{bmatrix},$$

is non-diagonally dominant and, in addition, the diagonal dominance condition is strongly violated (in the last two rows). We run the proposed method for several different initial conditions. It is noted that, in contrast to the 2nd experiment, the quantity $\|(1 + \frac{f'(\xi_i)}{\sqrt{\pi}})\sigma_{-i}^T \Sigma_{-i-i}^{-1}\|_1$ is now greater than 1 for all

pairs of $(w^{(t)}, w^{(t-1)})$. The algorithm in all cases converges to the vector $\hat{\mu} = [-3.859, -2.610, 8.727]^T$. However, in this case, the mean absolute difference per coordinate between this estimate and the one resulting from MCMC is 0.274, which is significantly greater than that in the previous experiment.

Analysing the previous results we may draw the following conclusions:

- Provided that the inverse of the covariance matrix of the untruncated normal distribution is diagonally dominant, the proposed method gives very accurate estimates of the mean of the truncated normal distribution, using as benchmark the estimates of the MCMC method.
- The proposed method converges much faster (in very few iterations) compared to the MCMC method.
- Even if the diagonal dominance constraint is slightly violated, the algorithm seems to converge to an accurate estimate of the mean of the truncated distribution, as the 2nd experiment indicates.
- When the diagonal dominance condition is strongly violated, the algorithm still (seems to) converge to a vector. However, this vector is a (much) less accurate estimation of the mean of the truncated normal (see the third experiment).
- In the special case, where Σ (and, as a consequence Σ^{-1}) is diagonal, the matrix Σ_{-i-i}^{-1} is (obviously) equal to zero. Thus, both eqs. (4) and (11) imply that $\mu_i^* = \mu_i$ and eq. (5) imply that $\sigma_i^* = \sigma_{ii}$. In other words, in this case, both the proposed and the MCMC methods solve n independent one-dimensional problems, with the deterministic method giving its estimate in a single iteration, since in this case, it reduces to (7) or (8). An additional observation is that the estimates provided by the two methods in this case are almost identical. Letting intuition enter into the scene and generalizing a bit, one could claim that as Σ approaches “diagonality”, the estimates of the mean of both methods are expected to be even closer to each other. This observation, may be an explanation of the decreasing trend observed in figure 1, since, as the dimension increases, *Sigma* moves closer to “diagonality”, due to the diagonal dominance condition.

VI. CONCLUSIONS

In this paper, a new iterative algorithmic scheme is proposed for the approximation of the mean value of a one-sided truncated multivariate normal distribution. The algorithm converges in very few iterations and, as a consequence, it is much faster than the MCMC based algorithm proposed in [17]. In addition, the algorithm is an extension of the algorithm used in [22]. The proof of its convergence is provided for the case where Σ^{-1} is diagonally dominant. However, experimental results indicate that, even if this condition is softly violated, the method still provides accurate estimates of the mean of the truncated normal.

APPENDIX

Proof of Proposition 2:

Utilizing the fact that $\frac{d(\operatorname{erfc}(x))}{dx} = -\frac{2e^{-x^2}}{\sqrt{\pi}}$, simple algebraic manipulations lead to

$$f'(x) = -2xf(x) - \frac{2}{\sqrt{\pi}}f^2(x) \quad (37)$$

Before we proceed, we write down the following inequalities, which hold for $y > 0$ and result from [1, 7.1.13]

$$\frac{\sqrt{\pi}}{2}(y + \sqrt{y^2 + 4/\pi}) \leq \frac{e^{-y^2}}{\operatorname{erfc}(y)} < \frac{\sqrt{\pi}}{2}(y + \sqrt{y^2 + 2}) \quad (38)$$

(A) We will show first that $f'(x) \leq 0$. We separate two cases

(i) $x \geq 0$. Taking into account eq. (37) and the fact that $f(x) \geq 0$, $\forall x \in \mathcal{R}$, the claim follows trivially.

(ii) $x < 0$. Writing $f'(x) = -2f(x)(x + \frac{1}{\sqrt{\pi}}f(x))$, it suffices to show that

$$f(x) \geq -\sqrt{\pi}x \quad (39)$$

Let us set $y = -x > 0$. Then, $f(x) = f(-y) = \frac{e^{-y^2}}{\operatorname{erfc}(y)}$

Taking into account the left inequality in eq. (38), eq. (39) holds if

$$\frac{\sqrt{\pi}}{2}(y + \sqrt{y^2 + 4/\pi}) \geq \sqrt{\pi}y \Leftrightarrow \sqrt{y^2 + 4/\pi} \geq y \Leftrightarrow 4/\pi > 0 \quad (40)$$

Thus, the claim holds also for $x < 0$.

(B) Let us consider now the following, more involved, case $f'(x) \geq -\sqrt{\pi}$. Taking into account eq. (37), it suffices to prove that

$$\frac{2}{\sqrt{\pi}}f^2(x) + 2xf(x) - \sqrt{\pi} \leq 0 \quad (41)$$

Let us consider the second degree polynomial of z

$$\tau(z) = \frac{2}{\sqrt{\pi}}z^2 + 2xz - \sqrt{\pi} \quad (42)$$

Its discriminant is $D = 4(x^2 + 2) > 0$ and its two (real) roots are $z_1 = \frac{\sqrt{\pi}}{2}(-x - \sqrt{x^2 + 2})$ and $z_2 = \frac{\sqrt{\pi}}{2}(-x + \sqrt{x^2 + 2})$, with $z_1 < z_2$.

In order to prove (41) it suffices to show that $z_1 \leq f(x) \leq z_2$, or,

$$\frac{\sqrt{\pi}}{2}(-x - \sqrt{x^2 + 2}) \leq f(x) \leq \frac{\sqrt{\pi}}{2}(-x + \sqrt{x^2 + 2}) \quad (43)$$

since in this case the sign of the second degree polynomial will be the opposite of that of the coefficient of z^2 (i.e., $\frac{2}{\sqrt{\pi}}$). We proceed again by considering separately the cases $x \geq 0$ and $x < 0$.

(i) Let $x < 0$. We set $y = -x > 0$. In this case (43) becomes

$$\frac{\sqrt{\pi}}{2}(y - \sqrt{y^2 + 2}) \leq f(-y) \leq \frac{\sqrt{\pi}}{2}(y + \sqrt{y^2 + 2}) \quad (44)$$

where $f(-y) = \frac{e^{-y^2}}{\operatorname{erfc}(y)}$. Taking into account (38), the right hand side inequality of (44) holds. In order to prove the left hand inequality of (44) (again taking into account (38)), it suffices to prove that

$$\frac{\sqrt{\pi}}{2}(y - \sqrt{y^2 + 2}) < \frac{\sqrt{\pi}}{2}(y + \sqrt{y^2 + 4/\pi})$$

or

$$-\sqrt{y^2 + 2} \leq \sqrt{y^2 + 4/\pi} \quad (45)$$

which trivially holds. Thus, for $x < 0$, it is $z_1 \leq f(x) \leq z_2$ and therefore $\tau(f(x)) \geq 0$. As a consequence (41) also holds.

(ii) Let $x \geq 0$. The left hand side inequality of (43) holds trivially, since in this case $z_1 < 0 \leq f(x)$. We focus now on the right hand side inequality of (43). Taking into account that (a) $\operatorname{erfc}(x) \leq e^{-x^2}$, for $x > 0$ (see e.g. [4]) and (b) $\operatorname{erfc}(x) + \operatorname{erfc}(-x) = 2$, it is

$$\frac{e^{-x^2}}{\operatorname{erfc}(-x)} = \frac{e^{-x^2}}{2 - \operatorname{erfc}(x)} \leq \frac{1}{2e^{x^2} - 1} \quad (46)$$

Combining the right hand side inequality of (38) with (46), the right hand side inequality of (43) holds if

$$\frac{1}{2e^{x^2} - 1} \leq \frac{\sqrt{\pi}}{2}(-x + \sqrt{x^2 + 2}) \quad (47)$$

or

$$\frac{1}{2e^{x^2} - 1} \leq \frac{\sqrt{\pi}}{2} \frac{2}{x + \sqrt{x^2 + 2}} \quad (48)$$

or

$$2\sqrt{\pi}e^{x^2} - \sqrt{\pi} \geq x + \sqrt{x^2 + 2} \quad (49)$$

Since $e^z > 1 + z$, for $z > 0$ (from the Taylor series expansion), the previous inequality holds if

$$2\sqrt{\pi}(x^2 + 1) - \sqrt{\pi} \geq x + \sqrt{x^2 + 2}$$

or

$$2\sqrt{\pi}x^2 + \sqrt{\pi} \geq x + \sqrt{x^2 + 2} \quad (50)$$

Since $x + \sqrt{2} \geq \sqrt{x^2 + 2}$, the previous inequality holds if

$$2\sqrt{\pi}x^2 - 2x + \sqrt{\pi} - \sqrt{2} > 0 \quad (51)$$

The discriminant of the above second degree polynomial of x is $D_1 = 4 - 8\sqrt{\pi}(\sqrt{\pi} - \sqrt{2}) < 0$. Thus, the above second degree polynomial is always positive since the coefficient of the term x^2 is positive. In other words, (51) holds. Therefore, the right hand side inequality of (43) also holds. Thus, for $x \geq 0$, it is also $z_1 \leq f(x) \leq z_2$ and therefore $\tau(f(x)) \geq 0$. As a consequence (41) also holds. Q.E.D.

Acknowledgment: The authors would like to thank Mr. N. Malamos for his helpful suggestions in the proof of proposition 5.

REFERENCES

- [1] M. Abramowitz, I.A. Stegun, *Handbook of mathematical functions* (Washington: National Bureau of Standards); reprinted 1968 (New York: Dover); online at <http://www.nist.gov/aands> (1968).
- [2] T. Amemiya, "Multivariate regression and simultaneous equations models when the dependent variables are truncated normal", *Econometrica*, 42, 999-1012 (1974).
- [3] D.P. Bertsekas, J.N. Tsitsiklis, *Parallel and distributed computation*, Prentice-Hall (1989).
- [4] M.Chiani, D.Dardari, M.K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels", *IEEE Transactions on Wireless Communications*, 2(4), 840-845, (2003).
- [5] A.K. Gupta, D.S. Tracy, "Recurrence relations for the moments of truncated multinormal distribution", *Communications in Statistics - Theory and Methods*, 5(9), 855-865 (1976).
- [6] L.F. Lee, "On the first and second moments of the truncated multi-normal distribution and a simple estimator", *Economics Letters* 3, 165-169 (1979).
- [7] L.F. Lee, "The determination of moments of the doubly truncated multivariate tobit model", *Economics Letters*, 11, 245-250 (1983).
- [8] P. Leppard, G.M. Tallis, "Algorithm AS 249: Evaluation of the mean and covariance of the truncated multinormal distribution", *Applied Statistics*, 38, 543-553 (1989).
- [9] B.G. Manjunath, S. Wilhelm, "Moments calculation for the doubly truncated multivariate normal density", *arXiv:1206.5387 [stat.CO]* (2012).
- [10] S. Martinez, G. Michon, J. San Martín, "Inverses of ultrametric matrices of Stieltjes type", *SIAM Journal of Matrix Anal. Appl.*, 15, 98-106 (1994).
- [11] J.J. McDonanld, M. Neumann, H. Schneider, M.J. Tsatsomeros, "Inverse M-matrix inequalities and generalized ultrametric matrices", *Linear Algebra Appl.*, 220, 321-341 (1995).
- [12] B. Muthén, "Moments of the censored and truncated bivariate normal distribution", *British Journal of Mathematical and Statistical Psychology*, 43, 131-143 (1990).
- [13] R. Nabben, R.S. Varga, "A linear algebra proof that the inverse of strictly ultrametric matrix is a strictly diagonally dominant Stieltjes matrix", *SIAM Journal of Matrix Anal. Appl.*, 15, 107-113 (1994).
- [14] R. Nabben, R.S. Varga, "Generalized ultrametric matrices - a class of inverse M-matrices", *Linear Algebra Appl.*, 220, 365-390 (1995).
- [15] D.J. Olive, *Applied robust statistics*, <http://books.google.gr/books?id=sswftwAACAAJ>, University of Minnesota (1998).
- [16] M.H. Regier, M.A. Hamdan, "Correlation in a bivariate normal distribution with truncation in both variables", *Australian Journal of Statistics*, 13, 77-82 (1971).

- [17] C.P. Robert, C.P., “Simulation of truncated normal variables”, *Statistics and Computing*, 5, 121-125, (1995).
- [18] S. Rosenbaum, “Moments of a truncated bivariate normal distribution”, *Journal of the Royal Statistical Society Series B (Methodological)*, 23, 405-408 (1961).
- [19] L.L. Scharf, *Statistical Signal Processing*, Englewood Cliffs, NJ:Prentice-Hall (1991).
- [20] S.M. Shah, N.T. Parikh, “Moments of single and doubly truncated standard bivariate normal distribution”, *Vidya (Gujarat University)*, 7, 82-91 (1964).
- [21] G.M. Tallis, “The moment generating function of the truncated multinormal distribution”, *Journal of the Royal Statistical Society, Series B (Methodological)*, Vol. 23(1), 223-229 (1961).
- [22] K. Themelis, A. Rontogiannis, K. Koutroumbas, “A novel hierarchical Bayesian approach for sparse semisupervised hyperspectral unmixing”, *IEEE Transactions on Signal Processing*, 60(2), 585-599 (2012).
- [23] R.S. Varga, R. Nabben, “On symmetric ultrametric matrices”, in *Numerical Linear Algebra*, L. Reichel, A. Ruttan, R.S. Varga (Eds.), De Gryter, Berlin, New York (1993).