Blank Words: An alternative method in analyzing paths of space-time curves with use of winding numbers

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Abstract. Within this project, an exploration into mathematical winding numbers is thoroughly researched, beginning with their foundations and fundamentals, and ending with an insight into so-called "Blank Words", which are an application of winding numbers in differential geometry. Blank words are originally a concept introduced by Samuel Blank, who is now a mathematics professor at Northeastern University. Rather than simply observing the winding number at a point (as one would traditionally), we can label arbitrary rays from the point that intersect the curve, and determine a unique transcription of a path (denoted by points) that we call one of the "Blank Word" Representations of the curve. As it turns out, noting the orientation of a particular ray can give insight into the path that is taken, by "cancelling out" positively oriented and negatively oriented rays. Blank words, while being intuitive mathematical constructions, offer deep insights into the paths that curves take through space, and provide a more complex buildup of winding numbers as compared to their more traditional uses in other areas.

1 Video Link:

https://www.youtube.com/watch?v=1UkFNfnwkMM

2 Winding Numbers

Winding numbers arise in a variety of mathematical contexts, which include but are not limited to Differential Geometry and Complex Analysis. The formal definition of a winding number is the amount of times that a curve travels counterclockwise around a point. Important to note is that winding numbers can only be computed with curves that have the quality of being both oriented and closed. That is, traversal of the curve must go in one way only, and the starting point is at the same location as the end point. Also, we can only take winding numbers of a point, and it does not make sense to have a "winding number of a curve". There only exists a notion of a winding number at a point with respect to a curve, and one without the other makes the entire operation of a winding number undefined.

3 Winding Numbers in Differential Geometry

In differential geometry, mathematicians tend to assert the notion that parametric equations of interest are at the very least piecewise differentiable. If a curve does not have this quality, it is very difficult to say anything geometrically about the properties of the curve in question. If we start with the basic notion of winding numbers, we can resort to angles to obtain a more rigorous definition of them.

Recall that by the definition of winding numbers, the winding number at a point x_0 with respect to a curve C is the integer representing the amount of times the the curve travels counterclockwise around the point. In more rigorous terms, we can say that this is equal to the amount of times that $d\theta = 2\pi$ where $d\theta$ is the line segment shown in Fig. 1, and x_0 is the origin.

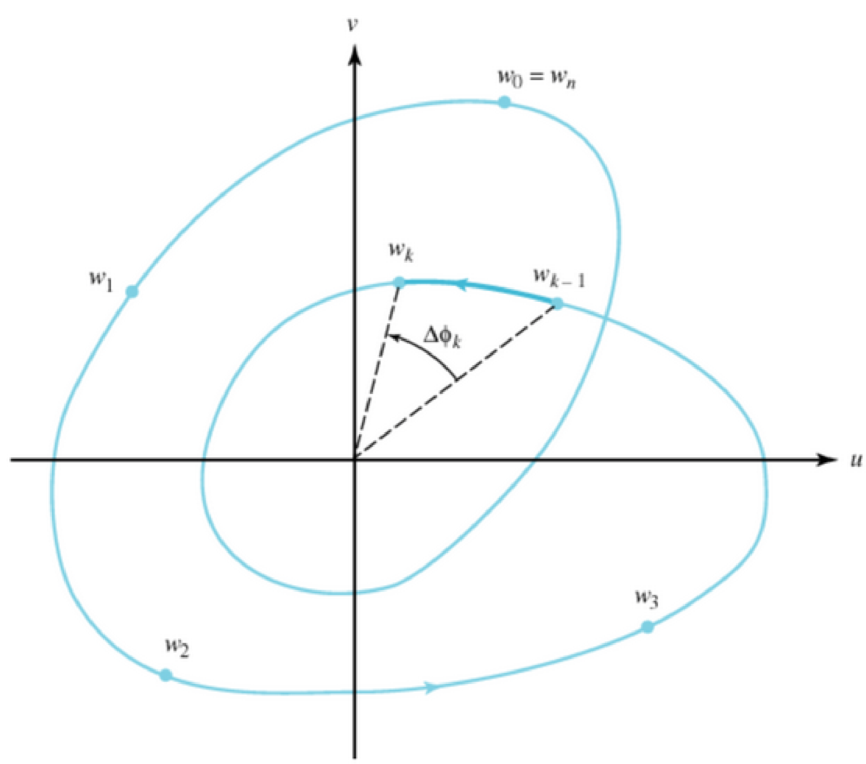

Fig. 1. Representation of change in θ needed for definition of winding number

However, because winding numbers can be less than 0, there must be some form of subtraction occurring in the equation. So, we will say that the previous formula of $d\theta = 2\pi$ is correct, but if $d\theta$ approaches 2π from below the x-axis, we will add 1 to the winding number, and if $d\theta$ approaches 2π from above the x-axis, we will subtract 1 from the winding number. The resultant sum is the total winding number of the point x_0 with respect to the curve C.

4 Winding Numbers in Complex Analysis

As a short aside for mathematical breadth, I will discuss how winding numbers are utilized with the use of complex numbers. In complex analysis, winding numbers are found in the exact same way, however, they have much different uses than in Differential Geometry. The largest application of winding numbers in this field is the very famous Cauchy Integral Theorem, which states (with w being the winding number of x_0 with respect to C) that if f is an analytic function with singular point a on a closed region, then

$$w(x_0, C) * f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a}\, dz.$$

This result shows a connection between closed regions and integration which does not exists for real regions of the x-y plane. Additionally, extensions of this formula exists for when $n \neq 1$ in the expression $(z-a)^n$ listed above. This result tells us that complex differentiation and integration are both well behaved and defined under idealized conditions, which is again not true for the real plane.

5 Homotopic invariance of Winding Numbers

In the youtube link above, I made an attempt to show that winding numbers are invariant under any homotopic map by showing that loops can be continuously deformed in a topological sense and yield the same winding number, given that the point which the winding number is being taken of (x_0) is not repositioned.

Homotopic invariance of winding numbers relies crucially on properties of angles being cyclic. We can write the total winding number of a curve at x_0 as

$$w(x_0, C) = \frac{\theta(t_f) - \theta(t_i)}{2\pi}$$

where t_i and t_f are the starting point and endpoint of C respectively. We know that for varying values of t that this is a continuous map, because we are given the fact that a curve is closed and oriented, which only makes sense if there are not discontinuities in the curve itself. Additionally, from the fact that it is continuous, and the fact that we are traversing what is essentially a loop, we know that the difference between $\theta(t_f)$ and $\theta(t_i)$ must be a multiple of 2π, because the loop itself is closed and we are guaranteed to return to the start point after some finite amount of traversal. Building from this, we also know that not only is this difference a multiple of 2π, but it is in fact an integer multiple. We can see through this argument.

Suppose there exists a curve that is closed and oriented. We traverse the curve and compute $\theta(t_f) - \theta(t_i)$ and receive a number that is a multiple of 2π, but not an integer multiple. Based on this information, we know that we have not returned to the start point, as this would yield an integer multiple of 2π. There exists then a contradiction, as a curve where traversal does not yield $\theta(t_f) - \theta(t_i) = 2\alpha\pi$ for some $\alpha \epsilon Z$ cannot be a closed curve. So, when we divide this difference by 2π, we receive α, which is an element of the set of integers Z. So, winding numbers are invariant under homotopy, and they must always be an element of the integers.

6 Blank Words

Blank words present an alternative way of analyzing immersed curves in space without having to visualize them, which can be very prone to error and take a large amount of time. They are a relatively novel idea, and were first articulated by Samuel Blank, who is now a professor at Northeastern University. The concept requires an intuition of winding numbers, so the previous sections were not explained irrelevantly.

For a curve, with interleaved disks, we will label a point within each disk, and draw a ray from each point to its closest boundary. Then, we will label the points with letters A, B, C, ..., Z with more letters allocated as necessary. Then, we will pick an arbitrary starting point, and traverse the curve, noting where we cross the rays. If we cross the ray in a counterclockwise direction, we will simply append the letter corresponding to that ray to the blank words representation, for example, A. Alternatively, if we cross a ray from the clockwise direction, we will append the complement of the letter corresponding to that ray, so for example, $\bar{A}$. We will continue this process until we return to the starting point, at which point we will have a string (in somewhat alphabetic fashion) that we call the "Blank Words" representation. This representation has some useful properties that we can apply to make inferences about the curve.

For sake of example, I will use the Blank Words representation as seen in the YouTube link shown above. The Blank Words representation shown in the video is as follows,

$$BW(C) = [ABCDEF\bar{B}DEFGABD\bar{E}]$$

We can see that this is almost an alphabetical string. Probably the most useful property of blank words is that we have the ability to differentiate between two immersed disks by looking where a substring occurs in the representation where the first character is the complement of the last character or vice versa. More generally, we are looking for a substring that follows the format,

$$BW(C) = [\alpha....\bar{\alpha}]$$

$$or$$

$$BW(C) = [\bar{\alpha}....\alpha]$$

When we find a subtring of this sort in the representation, we can be sure that this in itself defines a disk that is uniquely different from the other immersed disks within the structure.

While not a rigorous proof, we can intuitively think of these specific substrings geometrically as a curve looping in one direction, but then once again going in the other direction opposite of the way it initially came. When this occurs, another unique disk has been made by the curve itself. This is because of the nature of space, and that if we do not go infinitely in one direction, we must return at some point if the curve is closed. Imposing the restriction that the curve is closed allows us to draw the conclusion that if a substring occurs of this particualr type in the blank word representation, then traversing that particular substring will yield a distinct disk in the curve we are traversing. However, something important to note is that the substrings in the Blank Word representation do not define all of the disks in the curve of interest, and there will (usually, depending on complexity) exist disks that are not indicated by the substrings. While this makes the blank word representation not a perfect process persay, the representation still exists as a tool to make analysis of curves much more methodical and simpler.

7 Conclusion

Winding numbers, which arise not only in mathematics but several other fields, show remarkable results and allow for us to extract information about curves that we would otherwise not have been able to. While relatively basic, their existence allows the computation of complex integrals and gives us information about immersed disks in curves. Through building up this concept of winding numbers, we can introduce a new topic of "Blank Words" which has the property of differentiating disks in a complicated curve by simply traversing the curve, lessening the need for visualization and increasing the speed of obtaining information. While their most prevalent property is being able to differentiate disks, Blank Word representations are still a novel topic, and I speculate that they have many useful properties that have not been discovered yet that will increase the amount of information we can receive from the curves they represent.

Bibliography

"Definition of Winding Number, Have Doubt in Definition." Mathematics Stack Exchange, 6 May 2020, math.stackexchange.com/questions/186512/definition-of-winding-number-have-doubt-in-definition. Figure 1 of this paper is used from this Math StackExchange post displaying $d\theta$ when we are discussing winding numbers.

https://mathworld.wolfram.com/ContourWindingNumber.html