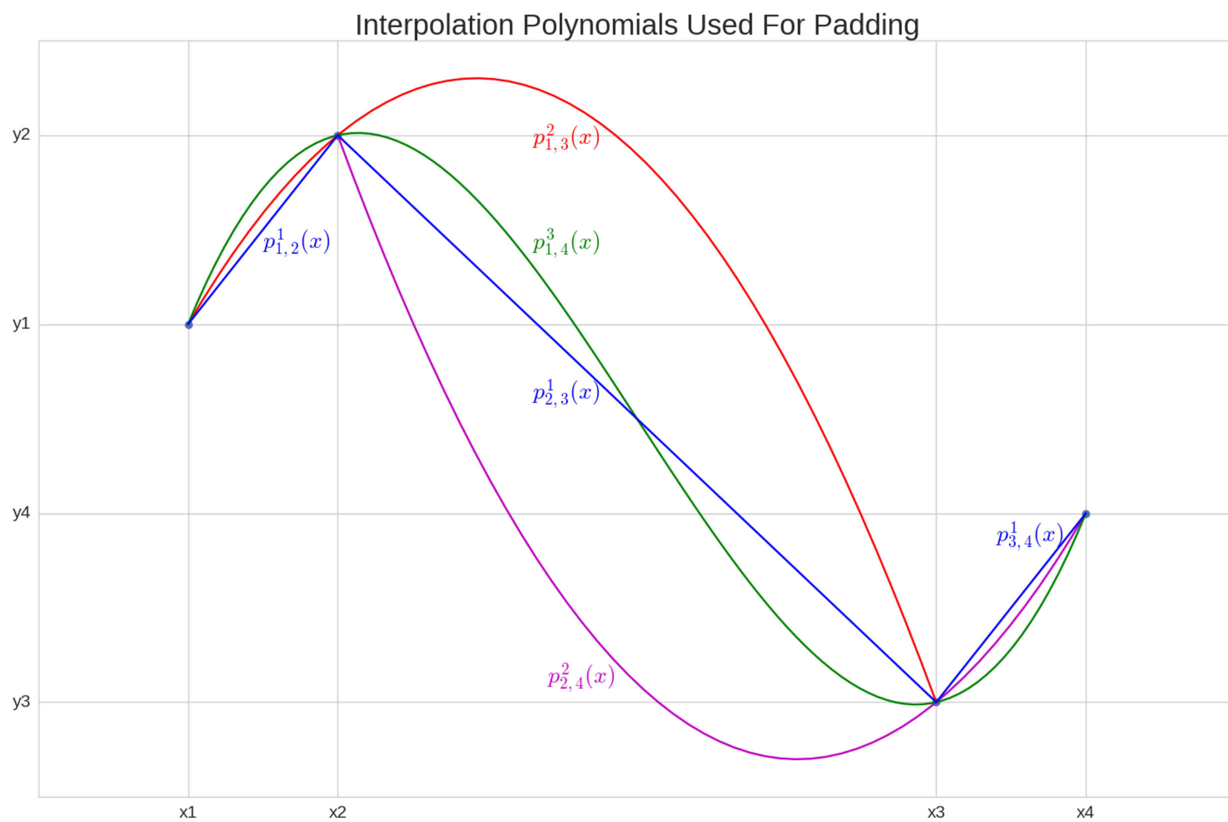
Fast Fourier Series Padding

Theory

Consider four points, $\{(x_n, y_n)\}_{n=1}^{4}$, such that $x_n < x_{n+1}$ as shown in the plot below. A cubic polynomial that starts at x_1 and ends at x_4 and goes through all four points can be obtained by interpolating between lower order polynomials. A kth order polynomial that starts at x_n and ends at x_{n+k} is denoted as $p^k_{n,n+k}(x)$, where $p^0_{n,n}(x) = y_n$. A polynomial of order k+1 is obtained from two order k polynomials using the following formula:

$$p^{k+1}_{n,n+k+1}(x) = \left(\frac{x_{n+k+1}-x}{x_{n+k+1}-x_n}\right) p^k_{n,n+k}(x) + \left(\frac{x-x_n}{x_{n+k+1}-x_n}\right) p^k_{n+1,n+k+1}(x) \qquad (1)$$



The blue lines in the plot, $p^1_{1,2}(x)$, $p^1_{2,3}(x)$, and $p^1_{3,4}(x)$ show the linear interpolations between x_1 and x_2, x_2 and x_3, and x_3 and x_4 respectively. The formulas for these lines are given below.

$$p_{1,2}^1(x) = \left(\frac{x_2 - x}{x_2 - x_1}\right) y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right) y_2 \tag{2a}$$

$$p_{2,3}^1(x) = \left(\frac{x_3 - x}{x_3 - x_2}\right) y_2 + \left(\frac{x - x_2}{x_3 - x_2}\right) y_3 \tag{2b}$$

$$p_{3,4}^1(x) = \left(\frac{x_4 - x}{x_4 - x_3}\right) y_3 + \left(\frac{x - x_3}{x_4 - x_3}\right) y_4 \tag{2c}$$

The quadratic interpolations, $p_{1,3}^2(x)$ and $p_{2,4}^2(x)$, shown in red and magenta interpolate between x_1 and x_3 and x_2 and x_4 respectively. The formulas for these quadratics are shown below.

$$p_{1,3}^2(x) = \left(\frac{x_3 - x}{x_3 - x_1}\right) p_{1,2}^1(x) + \left(\frac{x - x_1}{x_3 - x_1}\right) p_{2,3}^1(x) \tag{3a}$$

$$p_{2,4}^2(x) = \left(\frac{x_4 - x}{x_4 - x_2}\right) p_{2,3}^1(x) + \left(\frac{x - x_2}{x_4 - x_2}\right) p_{3,4}^1(x) \tag{3b}$$

Finally, the cubic polynomial, $p_{1,4}^3(x)$, is shown in green and interpolates between all four points. This cubic function is used to pad the Fourier Series. The formula is given below.

$$p_{1,4}^3(x) = \left(\frac{x_4 - x}{x_4 - x_1}\right) p_{1,3}^2(x) + \left(\frac{x - x_1}{x_4 - x_1}\right) p_{2,4}^2(x) \tag{4}$$

Application to the Fast Fourier Series

Consider a Fast Fourier Series constructed on N points, $\{(x_n, y_n)\}_{n=1}^N$, with uniform x spacing, h, such that $x_{n+1} = x_n + h$. Fourier Series are periodic in nature, meaning that evaluating them at the point, $x_N + h$, will be forced to y_1. In some cases, this transition happens very abruptly when there is a large discrepancy between y_N and y_1. When this occurs, the Fast Fourier Series has a difficult time approximating the original function, which is where padding comes in because it artificially creates a smooth transition between the end of one period and the beginning of the next. If M padding points are to be added, then the four interpolation points are as follows:

$$\{(x_{N-1}, y_{N-1}),\ (x_N, y_N),\ (x_N + Mh, y_1),\ (x_N + (M+1)h, y_2)\} \tag{5}$$

The vector of x values for padding points to be added, x_{pad}, has the following values:

$$x_{pad} = x_N + h[1,2, \dots, M] \tag{6}$$

The vector of y values for padding points, y_{pad}, is obtained by evaluating $p_{1,4}^3(x)$ over x_{pad}:

$$y_{pad} = p_{1,4}^3(x_{pad}) \tag{7}$$

This methodology is implemented in the FFS.fourier_pad() method in the ffs.py module.