Fast Fourier Series

Fourier Series Fundamentals

Consider a piecewise smooth function, f(x), defined on the range $[x_a, x_b]$. f(x) can be represented by the Fourier Series, $\hat{f}(x)$, as follows:

$$\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n(x - x_a)}{x_b - x_a}\right) + b_n \sin\left(\frac{2\pi n(x - x_a)}{x_b - x_a}\right) \right] = f(x) \tag{1}$$

When f(x) exists in continuous space, the coefficients a_n and b_n can be calculated explicitly, but this project is only concerned with f(x) as a discrete function. As such, the Fast Fourier Transform, a form of the Discrete Fourier Transform, will be used to obtain a finite number of coefficients a_n and b_n.

Fourier Series from the Fast Fourier Transform

Consider two real N-dimensional vectors, $x = \{x_n\}_{n=1}^{N}$ and $y = \{y_n\}_{n=1}^{N}$, where x has uniform spacing, h, such that $x_{n+1} = x_n + h$ and $y = f(x)$. Applying the Fast Fourier Transform (fft) to y will yield a complex N-dimensional vector, $\hat{y} = fft(y) = \{\hat{y}_n\}_{n=1}^{N}$. The first and last N/2 elements of $\hat{y}$ are complex conjugates of one another, so a y vector of length N will yield a Fourier Series of order N/2 because the conjugates essentially provide the same information as one another. For n = 1, 2, …, N/2, the coefficients a_n and b_n are extracted from $\hat{y}$ as follows:

$$a_n = \frac{2 \cdot Real(\hat{y}_n)}{N}, \quad b_n = -\frac{2 \cdot Imag(\hat{y}_n)}{N} \tag{2}$$

These coefficients can then be inserted into the following equation that approximates the discrete vector, $y = f(x)$, as a continuous function:

$$\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{N/2} \left[a_n \cos\left(\frac{2\pi n(x - x_1)}{x_N + h - x_1}\right) + b_n \sin\left(\frac{2\pi n(x - x_1)}{x_N + h - x_1}\right) \right] \approx f(x) \tag{3}$$

The primary differences between (1) and (3) are in the summation limits and the denominators of the cos and sin components of the summation. The reason (3) is only summed from 1 to N/2 was just discussed, and the reason that $x_N + h - x_1$ is in the denominator of cos and sin rather than $x_N - x_1$ has to do with the periodic nature of the Fourier Series, which forces $\hat{f}(x_N + h)$ to equal $\hat{f}(x_1)$.

This methodology is implemented in the FFS.__init__() method in the ffs.py module.