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# Probability of Collision Error Analysis

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## Abstract

The decision for the International Space Station (ISS) to maneuver to avoid a potential collision with another space object will be based on the probability of collision,  $P_C$ . The calculation of  $P_C$  requires the covariance of both objects at conjunction. It is well known that the covariance computed by US Space Command is optimistic (too small), especially at altitudes where atmospheric drag is the dominant perturbation, because its computation assumes there are no dynamic model errors. In this paper the effect of errors in the covariance on  $P_C$  and the sensitivity of  $P_C$  to the encounter geometry are investigated.

## Nomenclature

### Subscripts

- s - refers to the ISS
- d - refers to another space object, debris
- 0 - refers to values at conjunction

### Variables

- $P_C$  - probability of collision
- $P$  - covariance
- $A$  - matrix of partial derivatives of the measurements with respect to the initial state
- $\vec{r}, \vec{v}$  - position and velocity
- $\vec{v}_r$  - relative velocity,  $\vec{v}_d - \vec{v}_s$
- $\vec{\rho}$  - relative position vector,  $\vec{r}_d - \vec{r}_s$
- $\vec{e}$  - uncertainty vector
- $\beta$  - angle between ISS and debris orbit planes
- $d$  - horizontal plane separation distance at conjunction
- $h$  - radial separation distance at conjunction

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- $L$  - ISS distance from orbit plane intersection at conjunction
- $R$  - radius of the sphere around the ISS which defines collision

## Introduction

The Space Shuttle (SS) currently performs maneuvers to avoid potential collisions with cataloged space objects whenever the estimated conjunction with an object falls within a box, centered on the estimated SS position, of dimensions  $\pm 5$  km in the in-track direction and  $\pm 2$  km in the radial and out of plane directions. The disadvantage of this criterion is that it does not take into consideration the uncertainty, or accuracy, of the ephemerides of the two objects or the geometry of the conjunction. If the ephemerides are well known then there is no need to perform a collision avoidance maneuver if the estimated miss distance is 4 km. Since a maneuver will disrupt microgravity experiments using this criterion for the International Space Station (ISS) will result in too many maneuvers. In addition, unnecessary maneuvers will waste fuel, a precious commodity for the ISS. Therefore, the ISS has switched from the deterministic SS criterion to a probability based criterion for collision avoidance. In this approach the basis for the collision avoidance maneuver is the probability of collision,  $P_c$ , of the two objects. The calculation of  $P_c$  requires the uncertainty (covariance) of the ephemerides of the two objects at conjunction. Currently, US Space Command calculates a covariance at epoch,  $t_0$ ,

$$P(t_0) = (A^T W A)^{-1} \quad (1)$$

where  $A$  is the matrix of the partial derivatives of the measurements with respect to the state at epoch and  $W$  is a weighting matrix, which typically is a diagonal matrix with the elements being the inverse of the measurement variances. The covariance is propagated by

$$P(t) = \Phi(t, t_0)P(t_0)\Phi^T(t, t_0) \quad (2)$$

where  $\Phi$  is the state transition matrix. The position covariance at epoch is reasonably accurate, but the velocity covariance is very optimistic because only measurement errors are considered in the computation of the covariance, the dynamic model is assumed to be perfect. Since the decision to maneuver must be made several hours before conjunction the covariance will have to be propagated from 4-24 hours. As a result of the assumption of no dynamic model errors and the error in the velocity portion of the covariance the estimated position error at conjunction can be quite optimistic, possibly by an order of magnitude. In order to have an accurate estimate of  $P_c$  a method for accurately including the atmospheric density uncertainty in the computation of the covariance is needed. The atmospheric density uncertainty is generally at least 15-20%, and it has both a temporal and spatial variation. The modeling of the uncertainty must capture both of these variations. An initial approach using a first order stationary Gauss-Markov process to represent the uncertainty is presented in Ref. 2. More accurate values of the sensor measurement errors have been obtained<sup>3</sup> and these have resulted in some improvement of the covariances.

The first launch of the ISS is expected early next year. Until a better method of determining the covariance is obtained an approach for bounding  $P_c$  is needed. In this paper we present a) an approach for determining an upper limit of  $P_c$ , and b) a parametric analysis to determine the sensitivity of  $P_c$  to the encounter geometry and covariance size. Also, presented is an analysis of the Mir-US satellite close approach in September 1997.

### Probability of Collision

The probability of collision is defined as the probability that the debris object will enter the sphere of radius  $R$  during the encounter. Let  $t = 0$  at the estimated point of closest approach, i.e., at conjunction. Referring to Fig. 1, consider a set of perturbed trajectories for the ISS and debris given by  $\bar{r}_{so}$  and  $\bar{r}_{do}$  respectively. Mathematically, we can state this as

$$\tilde{\vec{r}}_{so} = \bar{\vec{r}}_{so} + \bar{\vec{e}}_s, \tilde{\vec{r}}_{do} = \bar{\vec{r}}_{do} + \bar{\vec{e}}_d \quad (3)$$

where  $\bar{\vec{e}}_s$  and  $\bar{\vec{e}}_d$  are the uncertainty vectors for the ISS and debris. For these trajectories conjunction is not at  $t = 0$ .

The following assumptions are made:

- The ISS and debris object motion can be represented by rectilinear motion (straight lines) with constant velocities during the encounter. This is justified because the time duration under consideration is no more than a couple of seconds.
- There is no uncertainty in the velocity during the encounter. This is justified because the velocity uncertainty is usually no more than several meters/second, and the time duration of the encounter is small.
- The position uncertainty during the encounter is constant and equal to the value at the estimated conjunction.
- The position uncertainties can be represented by a Gaussian distribution.

The nominal trajectories near the estimated point of closest approach for both objects are given by

$$\bar{\vec{r}}_{so} = \bar{\vec{r}}_{so} + \bar{\vec{v}}_s t, \quad \bar{\vec{r}}_{do} = \bar{\vec{r}}_{do} + \bar{\vec{v}}_d t \quad (4)$$

Including the position uncertainties of the debris and ISS, the actual (perturbed) positions are

$$\tilde{\vec{r}}_s(t) = \bar{\vec{r}}_{so} + \bar{\vec{v}}_s t, \tilde{\vec{r}}_d(t) = \bar{\vec{r}}_{do} + \bar{\vec{v}}_d t \quad (5)$$

The miss vector between the ISS and debris is

$$\begin{aligned} \tilde{\vec{\rho}}(t) &= \tilde{\vec{r}}_d(t) - \tilde{\vec{r}}_s(t) \\ &= \bar{\vec{r}}_{do} - \bar{\vec{r}}_{so} + (\bar{\vec{v}}_d - \bar{\vec{v}}_s)t + \bar{\vec{e}}_d - \bar{\vec{e}}_s \\ &= \bar{\vec{\rho}}_o + \bar{\vec{e}}_d - \bar{\vec{e}}_s + \bar{\vec{v}}_r t \\ &= \tilde{\vec{\rho}}_o + \bar{\vec{v}}_r t \end{aligned} \quad (6)$$

Assuming the distribution of the errors is Gaussian the probability distribution for  $\bar{\bar{\rho}}$  is given by

$$p(\bar{\bar{\rho}}) = \frac{1}{2\pi(\det P)^{1/2}} \exp[-(\bar{\bar{\rho}} - \bar{\bar{\rho}})^T P^{-1}(\bar{\bar{\rho}} - \bar{\bar{\rho}}) / 2] \quad (7)$$

where

$$P = P_s + P_d \quad (8)$$

The probability of collision at this instant of time is

$$P_c = \frac{1}{2\pi(\det P)^{1/2}} \int_V \exp[-(\bar{\bar{\rho}} - \bar{\bar{\rho}})^T P^{-1}(\bar{\bar{\rho}} - \bar{\bar{\rho}}) / 2] dV \quad (9)$$

where the integral is over the sphere of radius  $R$ .

Define the  $(x, y, z)$  coordinate system with unit vectors  $(\bar{i}, \bar{j}, \bar{k})$  according to

$$\bar{j} = \frac{\bar{v}_t}{v_r}, \bar{i} = \frac{\bar{\rho}_o}{\bar{\rho}_o}, \bar{k} = \bar{i} \times \bar{j} \quad (10)$$

The geometry of this system is shown in Fig. 2. In this coordinate system the y-component of the nominal miss vector at conjunction is zero. Foster<sup>1</sup> has shown that the probability of collision for the encounter reduces to

$$P_c = \frac{1}{2\pi(\det P^*)^{1/2}} \int_{R-\sqrt{R^2-x^2}}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \exp[-(\bar{s} - \bar{s}_o)^T P^{*-1}(\bar{s} - \bar{s}_o)] dx dz \quad (11)$$

where  $P^*$  is the 2x2 covariance in the  $(\bar{i}, \bar{j}, \bar{k})$  frame and

$$\vec{s} = x\vec{i} + z\vec{k}, \vec{s}_o = x_o\vec{i} + z_o\vec{k} \quad (12)$$

An alternative derivation using a different, but equivalent, definition of  $P_C$ , that leads to the same result is provided in Ref. 4.

### Parametric Analysis

In order to gain insight into the primary factors that affect  $P_C$  a simplified example was developed. Referring to Fig. 3 it is assumed that there is no uncertainty in the ISS position. Eventually when the ISS has a GPS receiver it is expected that its position will be known much more accurately than most of the debris objects. The debris covariance is assumed to be aligned with the in-track, radial and out of plane direction and the radial and out of plane components are assumed to be equal and equal to  $K_d\sigma$ . The in-track component is  $K_d(k_{dv}\sigma)$  as shown below. The factor  $k_{dv}$  incorporates the fact that the in-track component has the most uncertainty.

$$P_d = K_d^2 \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & k_{dv}^2 \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \quad (13)$$

It is also assumed that the debris object is in a near circular orbit so that  $v_s = v_d$ . The miss vector at conjunction is

$$\vec{\bar{\rho}}_o = h\vec{i} - d\vec{k} \quad (14)$$

$L$  is the distance of the ISS at from the intersection of the two orbital planes. The quantity  $d$  is related to  $L$  by

$$d = 2L \sin \frac{\beta}{2} \quad (15)$$

For this case  $P_C$  becomes

$$P_C = \frac{1}{2\pi K_d^2 \sigma \sigma_2} \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \exp \left[ -\frac{(x-h)^2}{2K_d^2 \sigma^2} - \frac{(z-d)^2}{2K_d^2 \sigma_2^2} \right] dx dz \quad (16)$$

$$\sigma_2^2 = \sigma^2 (k_{dv}^2 + 1 - (k_{dv}^2 - 1) \cos \beta)$$

Eq. (16) can be reduced to

$$P_C = \frac{1}{\sqrt{2\pi} K_d \sigma} \int_{-R}^R \exp \left[ -\frac{(x-h)^2}{2K_d^2 \sigma^2} \right] \left\{ \operatorname{erf} \left( \frac{\sqrt{R^2-x^2} + d}{\sqrt{2}\sigma_2} \right) - \operatorname{erf} \left( \frac{-\sqrt{R^2-x^2} + d}{\sqrt{2}\sigma_2} \right) \right\} dx \quad (17)$$

Figures 4-8 show results from this parametric analysis. In all of these figures  $R = 60$  meters and  $\sigma = 20$  meters. Fig. 4 gives  $P_C$  as a function of the size of the covariance. If the conjunction point is outside the sphere and the covariance is very small then the PC is small. As the size of the covariance increases  $P_C$  increases until the uncertainty becomes so large that  $P_C$  begins to decrease. The maximum occurs when the standard deviation in the direction of the miss vector is approximately equal to, but less than, the magnitude of the miss vector. Note the rapid rise in  $P_C$  as the size of the covariance increases until the maximum is approached. This is typical behavior and reinforces the need to accurately model the orbit uncertainty as small changes in uncertainty can create large changes in  $P_C$  in this range. In Fig. 4 with  $L = 100$  meters the conjunction point is inside the sphere. Figure 5 is the same case with  $h=100$  m and the effect of the height differential is evident; it is beginning to dominate the effect of  $L$ . Figure 6 shows the effect of  $\beta$ , the angle between the orbit planes, on  $P_C$ . Since  $\beta=0$  is head on the maximum value of  $P_C$  will occur at  $\beta=0$ . Figures 7 and 8 show the effect of the size of the covariance in the in-track direction for encounter angles of 30 deg. and 90 deg..

An approximate value of  $P_C$  can be made by assuming the probability density is constant over the collision sphere. Any point in the sphere can be used but we will use the center of the sphere. The approximate value is

$$P_C = \frac{R^2}{2(\det P)^{1/2}} \exp \left[ - \left( \frac{\bar{\rho}_o^T P^{-1} \bar{\rho}_o}{2} \right) \right] \quad (18)$$

Figure 9 compares the approximate and exact values for a specific case. As expected the performance is best for the smaller values of  $P_C$  but it is still good over all ranges. In the case  $L=100\text{m}$  when the conjunction is inside the sphere the performance is the worst. However, this is not of concern because in cases like these  $P_C$  is so far above the maneuver threshold an error of 100% has no effect.

### Maximum $P_C$

As stated earlier the covariance provided for satellites at the ISS altitude, where atmospheric drag is the dominant perturbation, can be quite optimistic. The results just presented showed that significant variations in  $P_C$  can occur for small changes in the size of the covariance. For this reason it is worthwhile to determine the maximum possible value of  $P_C$  for an encounter. If  $P_{C\text{max}}$  is below the maneuver threshold then errors in the covariance are of no concern. It was just shown that assuming the probability density function is constant over the sphere, and equal to the value at the sphere center, provides a very good approximation of  $P_C$ . Consider the case where the covariance is multiplied by a constant  $K^2$  and find the value of  $K$  which maximizes  $P_C$ .

$$P_C = \frac{R^2}{2K^2(\det P)^{1/2}} \exp \left[ - \frac{\bar{\rho}_o^T P^{-1} \bar{\rho}_o}{2K^2} \right] \quad (19)$$

$$\frac{\partial P_C}{\partial K} = -2P_C / K + P_C \left[ \frac{\bar{\rho}_o^T P^{-1} \bar{\rho}_o}{K} \right]$$

The value of  $K$  which maximizes  $P_C$  is

$$K^2 = \frac{\bar{\bar{\rho}}_0^T P^{-1} \bar{\bar{\rho}}}{2} \quad (20)$$

Thus,  $P_{C_{\max}}$  occurs when the exponent of the exponential term is  $-1$ . The corresponding value of  $P_{C_{\max}}$  is

$$P_{C_{\max}} = \frac{R^2}{e(\bar{\bar{\rho}}_o^T P^{-1} \bar{\bar{\rho}})(\det P)^{1/2}} \quad (21)$$

Let the singular values of  $P$  be  $\sigma_1$  and  $\sigma_2$ . Now perform a rotation of coordinates from  $(x, z)$  into  $(u, w)$  so that the  $u$  axis passes through the conjunction point. With  $\rho_o$  as the magnitude of the miss vector and  $\sigma_u$  and  $\sigma_w$  the standard deviations for the  $u$  and  $w$  axes we get

$$P_{C_{\max}} = \frac{1}{e} \left( \frac{R}{\rho_o} \right)^2 \left( \frac{\sigma_1 \sigma_2}{\sigma_w^2} \right) \quad (22)$$

Letting the maneuver threshold be  $\varepsilon$ , the miss distance for which a maneuver is definitely not required is

$$\rho_o > \left( \frac{1}{e\varepsilon} \right)^{1/2} \left( \frac{\sigma_1 \sigma_2}{\sigma_w^2} \right)^{1/2} R \quad (23)$$

Selecting  $\varepsilon = 10^{-4}$  and with  $R = 60\text{m}$  gives

$$\rho_o > 3.64 \left( \frac{\sigma_1 \sigma_2}{\sigma_w^2} \right)^{1/2} \text{ km} \quad (24)$$

Note that

$$\left( \frac{\sigma_1 \sigma_2}{\sigma_w^2} \right) > 1 \quad (25)$$

Using Eq. (24) as the criterion for no maneuver will result in too many maneuvers because of the large separation distance required. However, the value of  $K$  that yields the maximum  $P_C$  is usually much larger than is reasonable. When this occurs the maximum reasonable value of  $K$  should be used.

The previous discussion only considered the size of the covariance. However, our problem is somewhat different since  $P = P_s + P_d$ . Now let

$$P = c^2 P_s + d^2 P_d \quad (26)$$

and find the values of  $c$  and  $d$  which maximize  $P_C$ . Using Eqs. (18) and (26) and setting the partial derivatives of  $P_C$  with respect to  $c$  and  $d$  equal to zero gives

$$\begin{aligned} c^2 &= \frac{Tq^2 - 2np^2}{T^2 - 4mn} \\ d^2 &= \frac{Tp^2 - 2mq^2}{T^2 - 4mn} \end{aligned} \quad (27)$$

where

$$\begin{aligned} p^2 &= |P_s| (\bar{\bar{\rho}}_c - \bar{\bar{\rho}}_o)^T P_s^{-1} (\bar{\bar{\rho}}_c - \bar{\bar{\rho}}_o) \\ q^2 &= |P_d| (\bar{\bar{\rho}}_c - \bar{\bar{\rho}}_o)^T P_d^{-1} (\bar{\bar{\rho}}_c - \bar{\bar{\rho}}_o) \\ m &= |P_s| \\ n &= |P_d| \\ T &= |P_d| \text{Trace}(P_s P_d^{-1}) \end{aligned} \quad (28)$$

$P_{C\max}$  is still given by Eq. (21).

## Mir – US Satellite Encounter

On September 1, 1997 there was a near miss of a US satellite, Object No. 23101, with the Mir. During the encounter the crew went into the escape module. The estimated miss distance was about 800 meters and the angle between the orbital planes was approximately 104 deg. To analyze this encounter orbital data from US Space Command was obtained. These data were the state of the two objects at the estimated conjunction and the covariance. Post processed, as well as 2, 8 and 24 hour predict data were provided. The estimated miss distances at conjunction for these cases were:

Table 1

Estimated Miss Distance

Predict time	Radial	Horizontal	Miss distance (m)
Post Processed	TBD		
2 hour			
8 hour			
24 hour			

For the provided covariance data  $P_c < 10^{-18}$ . Thus, no maneuver would have occurred. Figure 10 shows  $P_c$  as a function of  $K$ , where  $K^2$  multiplies each row of the covariance. The smaller estimated conjunction distance for the post processed data is evident. For this encounter for  $P_c > 10^{-4}$ ,  $K > 12$  for the 8 hour predict and  $K > 28$  for the others. Note again, particularly for the 8 hour predict, the large change in  $P_c$  for small changes in the covariance size.

Figures 11 and 12 are 3-D plots of  $P_c$  vs  $c$  and  $d$  for the 8 hour predict and the post processed data. Within the range shown there is no maximum for the post processed data. This is a result of the smaller post processed data covariance. The maximum values for the 8 hour predict occur at  $c = \text{TBD}$ ,  $d = \text{TBD}$ .

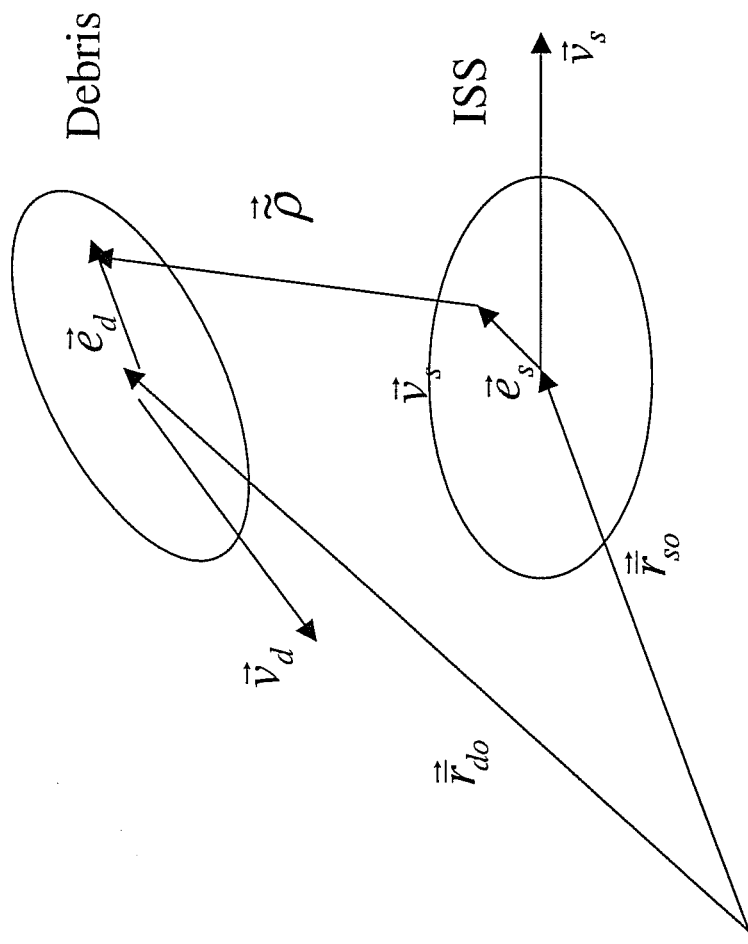
## Conclusions

The conclusions of this analysis of the sensitivity of the probability of collision to the encounter geometry and covariance are:

- Rapid changes in PC can occur with small changes in the covariance size. This reinforces the need to obtain more accurate covariances.
- A good approximation of PC can be obtained by assuming the probability density function is constant over the collision sphere.
- An approach for obtaining the maximum possible PC for any encounter was developed.

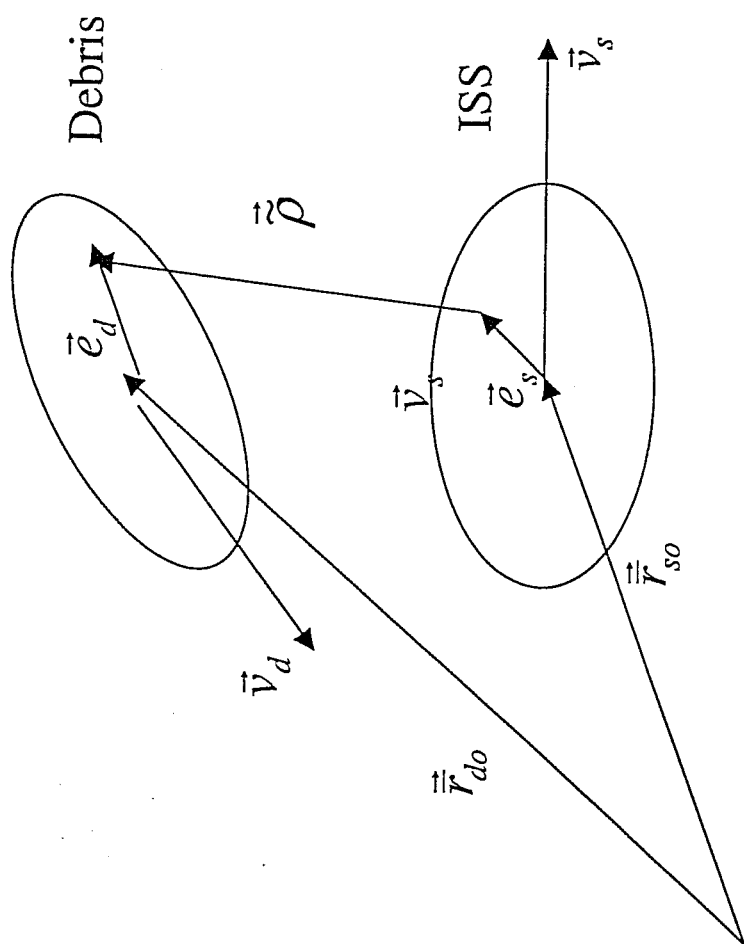
## References

1. Foster, J.L., *A Parametric Analysis of Orbital Debris Collision Probability and Maneuver Rate for Space Vehicles*, NASA JSC-25898, August 1992.
2. Akella, M.R., Junkins, J.L., Alfrend, K.T., *Some Consequences of Orbital Debris Collision Probability and Maneuver Rate for Space Vehicles*, Paper No. 97-606, AAS/AIAA Astrodynamics Specialist Conference, Sun Valley, ID, August 1997.
3. Barker, W.N., *Space Station Debris Avoidance Study, Final Report*, KSPACE 97-47, Kaman Sciences, Colorado Springs, CO, January 31, 1997.
4. Akella, M.R., Alfrend, K.T., *The Probability of Collision Between Space Objects*, to be published.

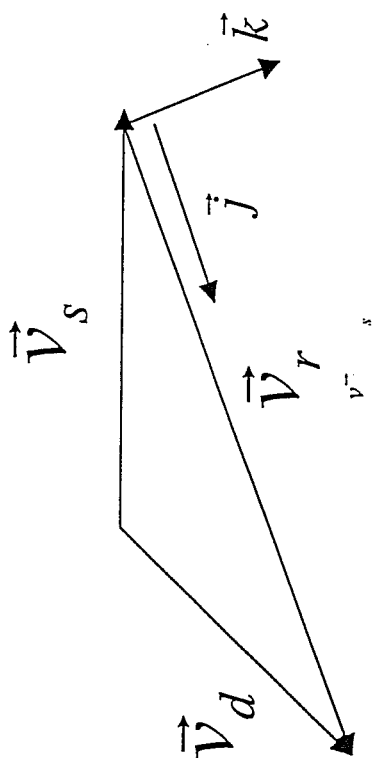




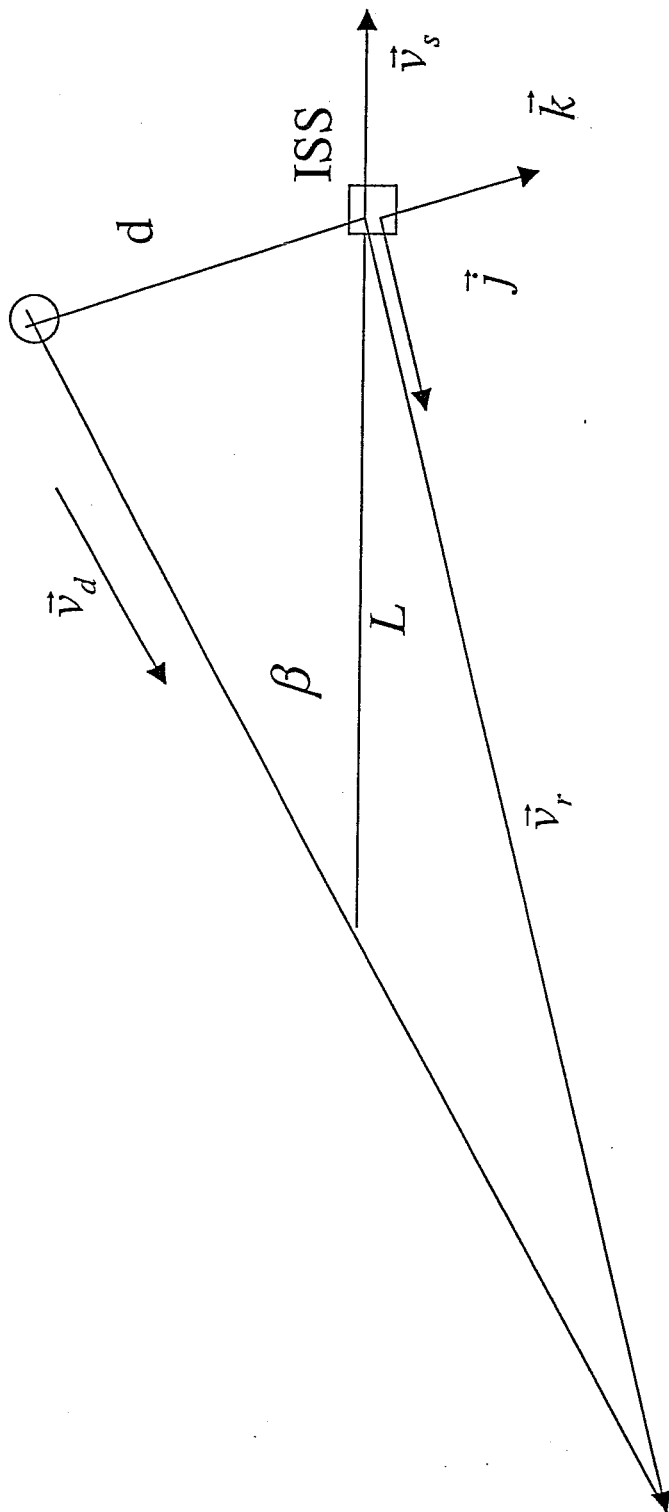
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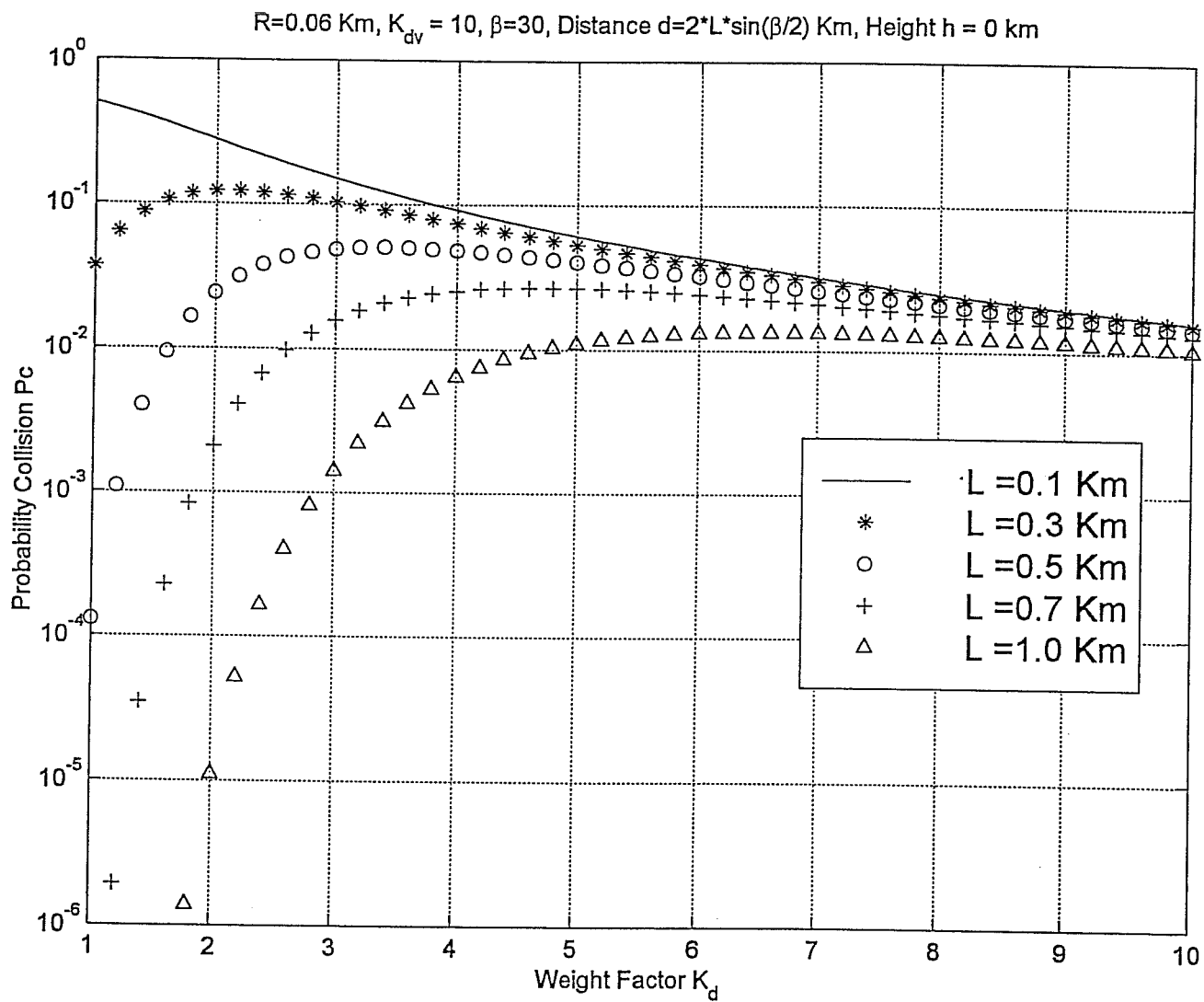
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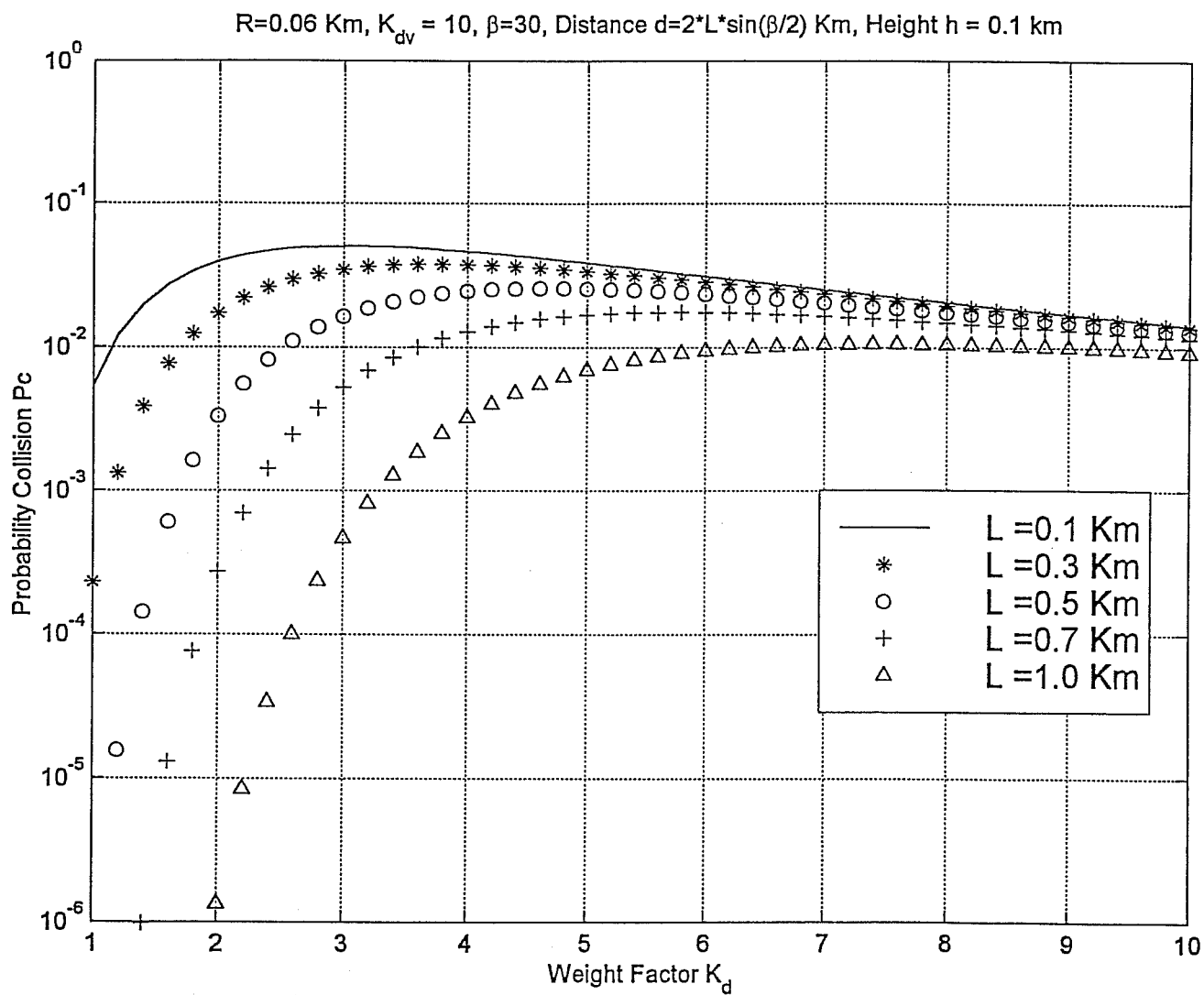
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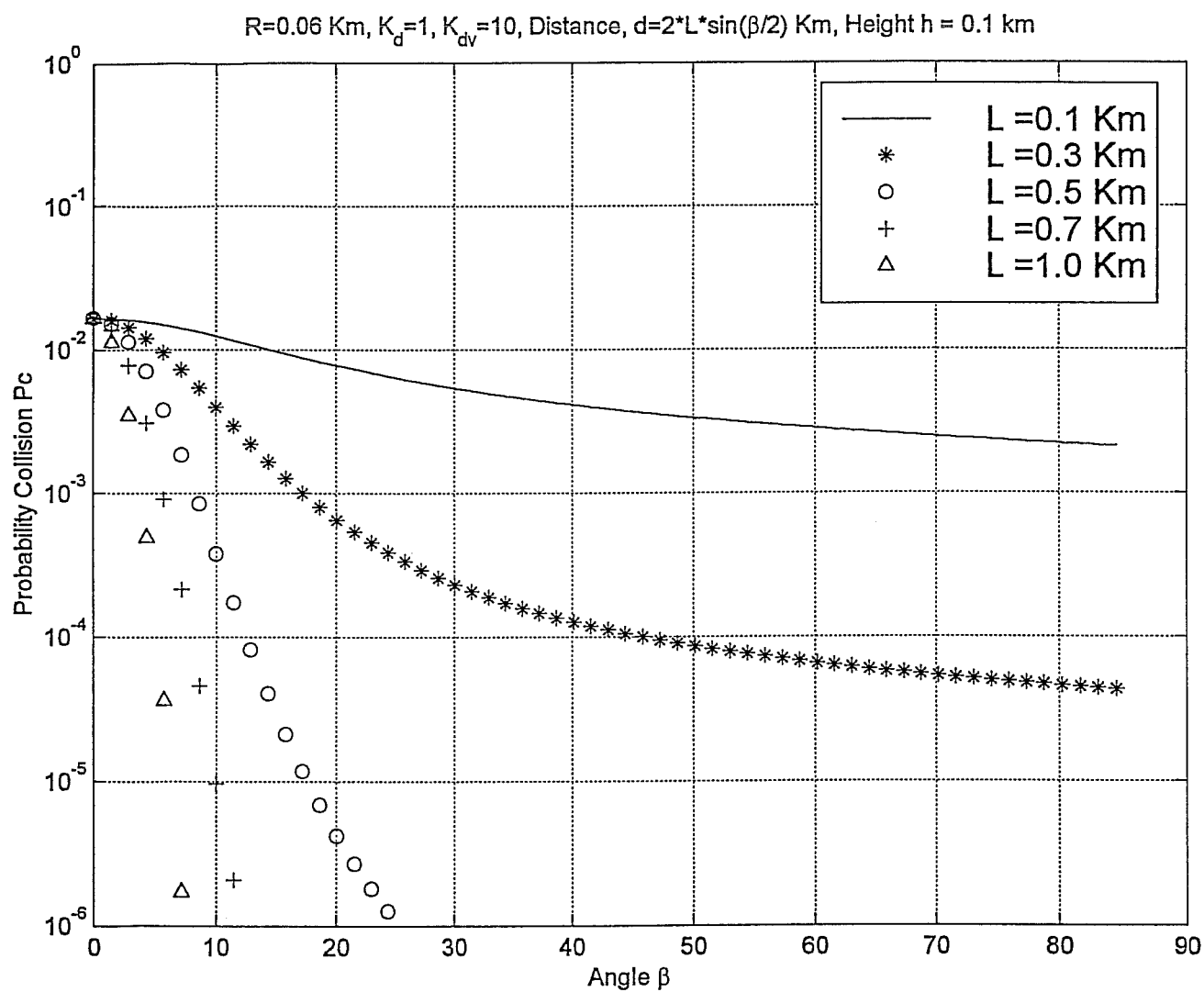


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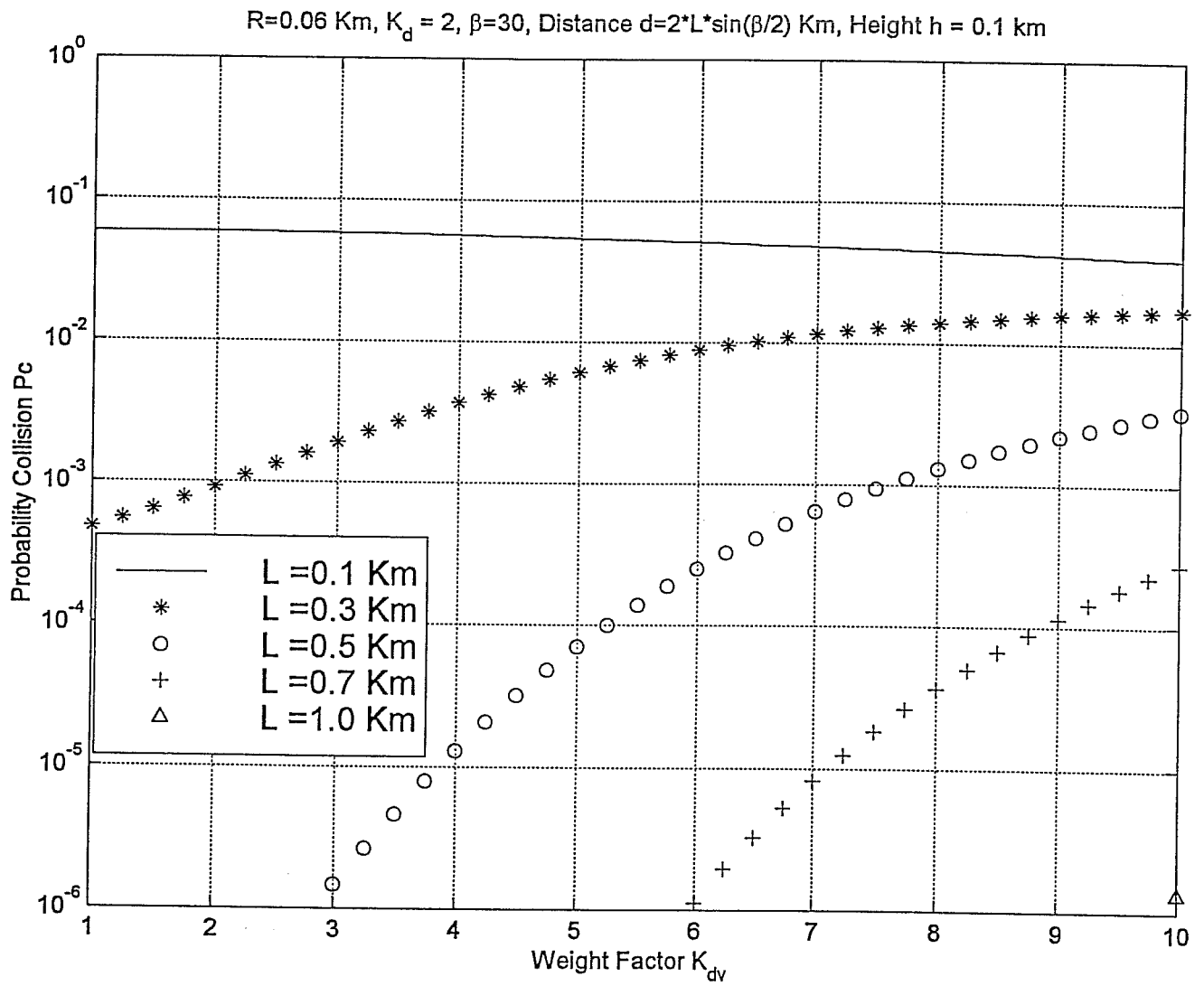


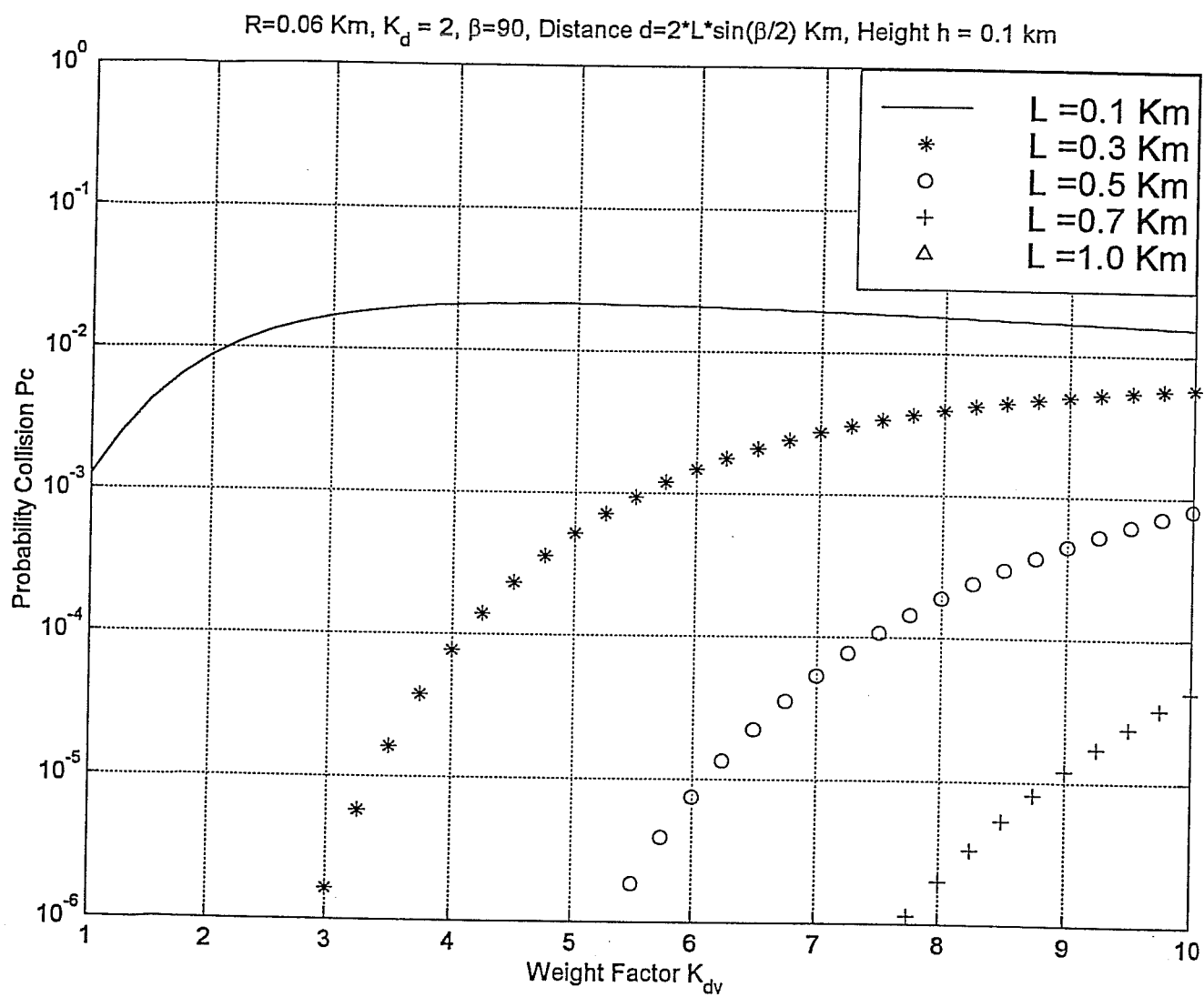
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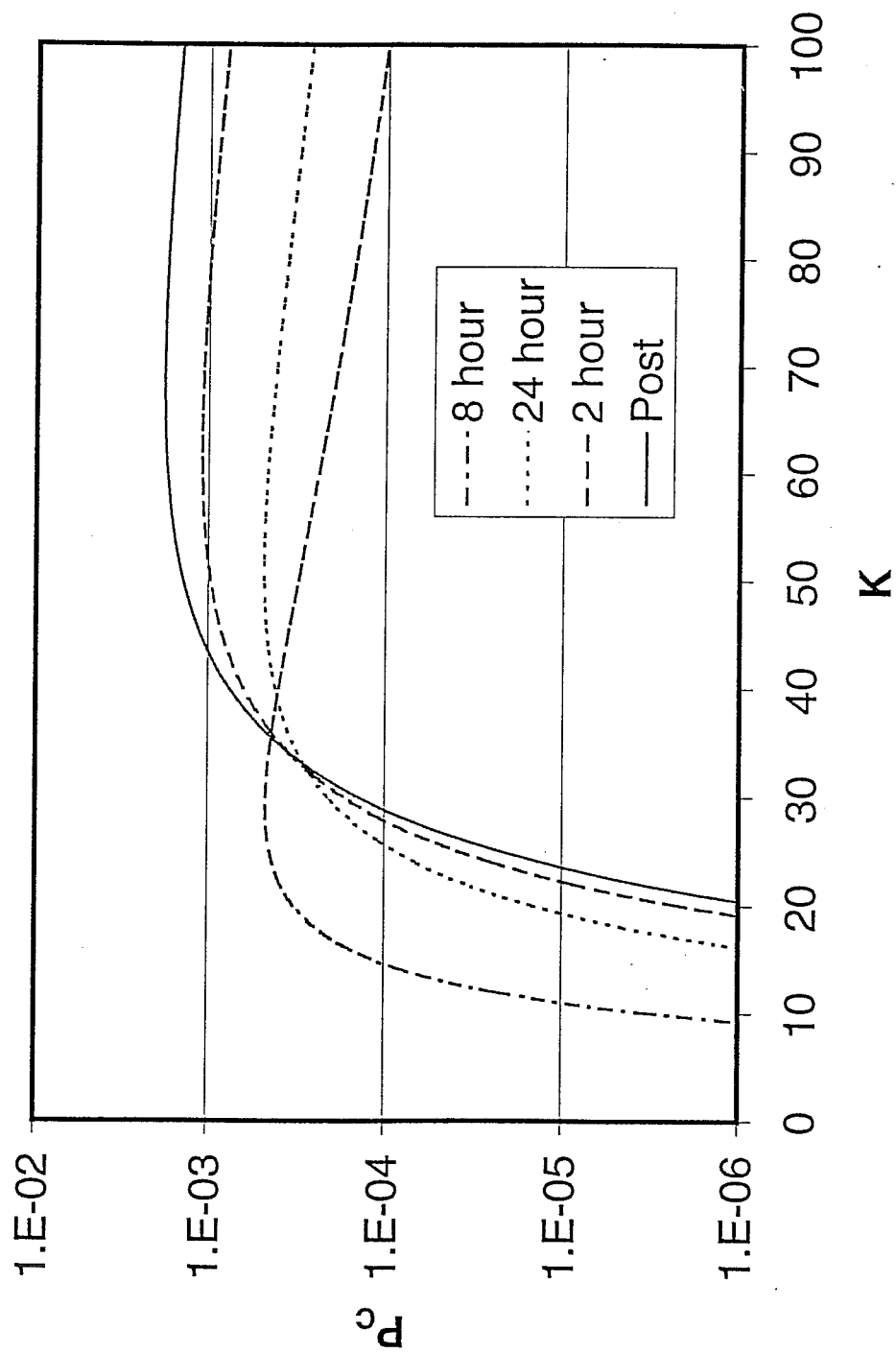


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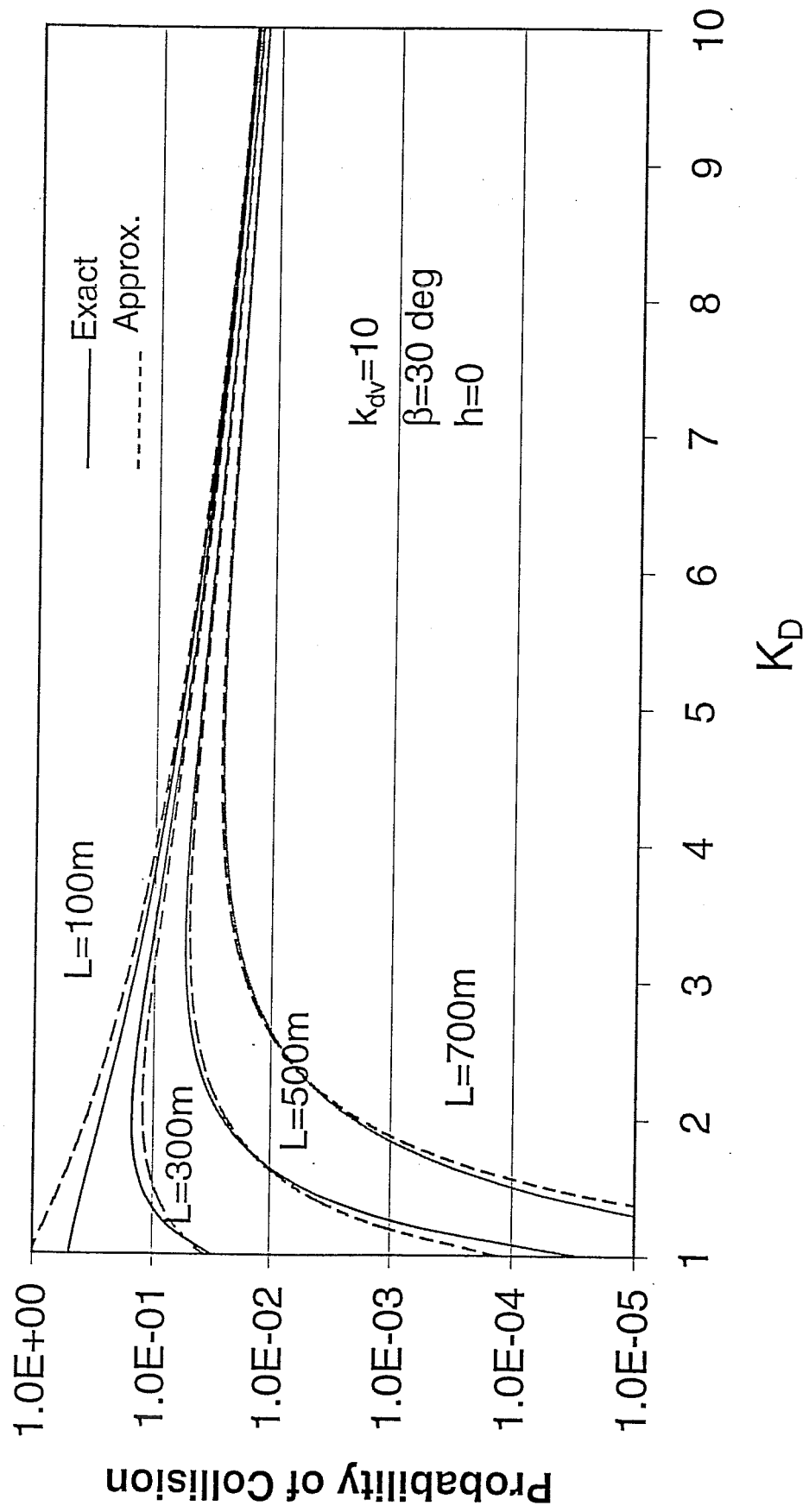




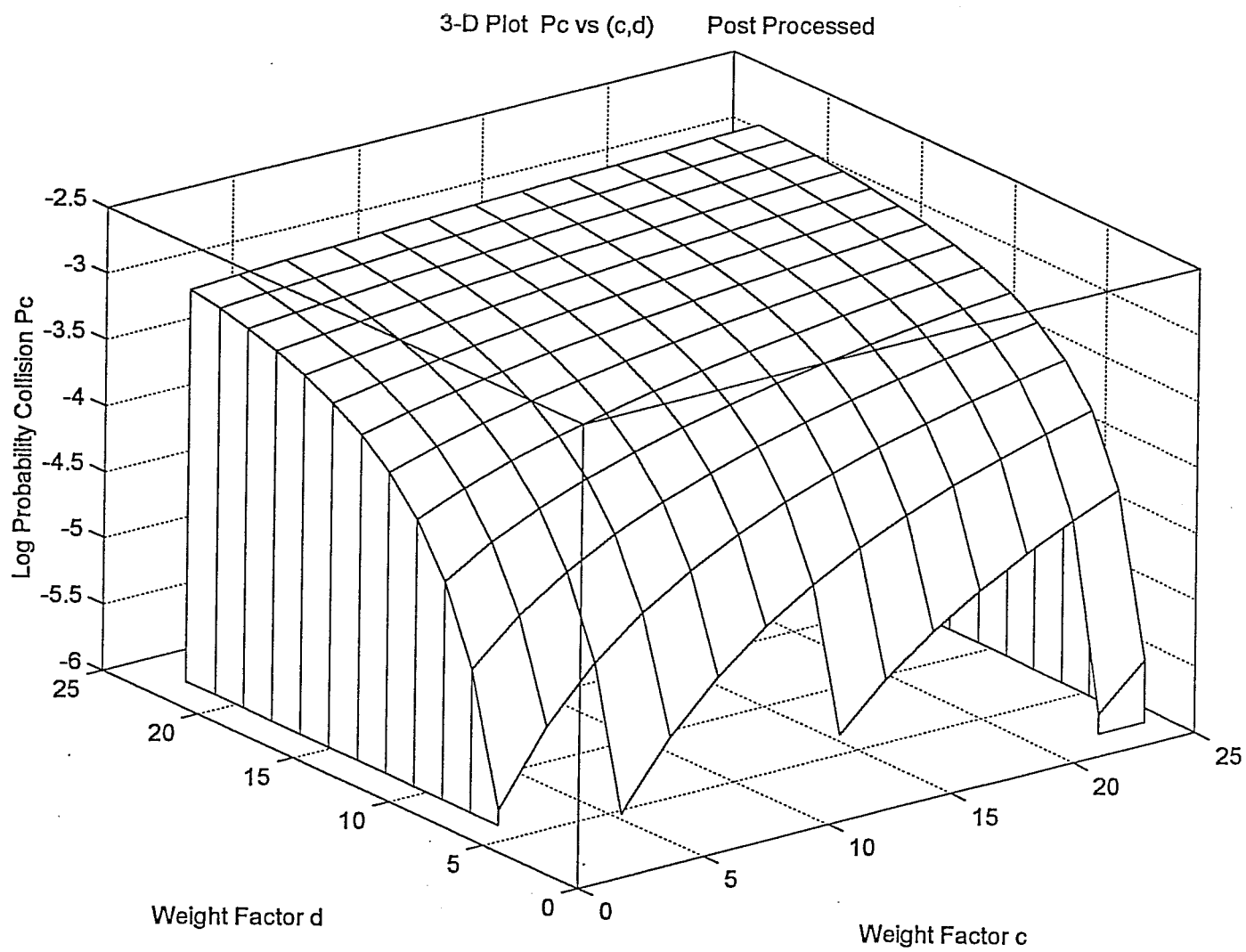
# Mir - US Sat Near Collision $P_C$ vs Covariance Size



# Comparison of Approximate and Exact $P_c$



11



12<sup>th</sup>

3-D Plot  $P_c$  vs (c,d) 8 hour Prediction

