Khovanov homology, monopoles, and mutation

Jonathan M. Bloom

Columbia University
jbloom@math.columbia.edu

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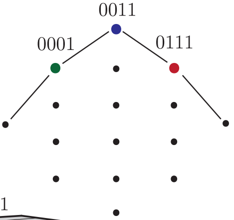
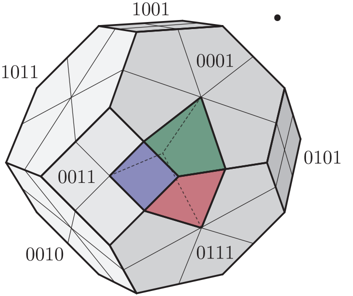
Outline

- Background
  - a) Seiberg-Witten
  - b) Morse theory
  - c) Floer theory
- Theorems
- Connections
  - a) Branched DC
  - b) Mutation
  - c) Exact $\triangle$s
  - d) Metrics
  - e) Polytopes
- Outlook





When worlds collide...

- In 1983, Donaldson exposed a fundamental difference between smooth and topological 4-manifolds by using non-linear equations from gauge theory.
- In 1987, Floer introduced his instanton homology groups as a framework for extending Donaldson theory to integer homology 3-spheres and knots.

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- In 1983, Donaldson exposed a fundamental difference between smooth and topological 4-manifolds by using non-linear equations from gauge theory.
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This collision of topology and physics has given rise to a series of powerful new tools in low-dimensional topology and geometry, known as *Floer homology*. What is it good for?

- 4D Topologist: Smooth structures (Donaldson)
- 3D Topologist: Invariants, algorithms (knot Floer hom.)
- Geometer: Symplectic and contact manifolds (Arnold)
- Analyst: Existence of solutions to PDEs (Weinstein)

Seiberg-Witten invariants

The group $\mathrm{Spin}^c(n)$ is defined by the exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1) \to 1$$

A spin^c-structure on an oriented, Riemannian manifold X is a lift of the principle frame bundle $P_{\mathrm{SO}(4)}$ to a principle spin$^c(4)$ bundle. $spin^c(X)$ admits a free transitive action of $H^2(X;\mathbb{Z})$.

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In 1994, Seiberg and Witten introduced the *S-W monopole equations* on a 4-manifold with spin^c-structure $\mathfrak{s}$.

- Let $S^+ \to X$ be the associated self-dual spinor-bundle.
- Let L be the determinant line bundle.

$\mathcal{C}(X,\mathfrak{s}) = \{(A,\phi)\,|\,A$ is a U(1)-connection on L, $\phi \in \Gamma(S^+)\}$

Definition (SW_4)

$$F_A^+ = q(\phi) + i\omega \qquad\qquad D_A^+\phi = 0$$

Seiberg-Witten

If $b^2_+ \geq 2$, then for generic choice of metric and perturbation:

- The gauge group $\mathcal{C}^\infty(X, S^1)$ acts freely on monopoles.
- The quotient is a finite-dimensional, compact manifold.
- Its cobordism class only depends on the smooth structure.

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In this way, we obtain a number for each spin^c-structure:

$$n(X, \mathfrak{s}) = \#|\mathcal{M}(X, \mathfrak{s})|$$

- Gives strong restrictions on the geometry and topology of symplectic manifolds. Minimal, simply connected symplectic 4-manifolds with $b^2_+ > 1$ must be irreducible. e.g., $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit a symplectic form.
- Used to prove the Thom Conjecture: a complex curve in $\mathbb{CP}^2$ is genus minimizing in its homology class.

Morse homology

Floer homology is modeled on the *Morse homology* of a closed, smooth n-manifold M. Fix a metric g and a Morse-Smale function $f : M \to \mathbb{R}$. Given $a, b \in \mathrm{Crit}(f)$, define the moduli space $\mathcal{M}(a, b)$ of gradient trajectories $\gamma : \mathbb{R} \to M$ from a to b:

$$\frac{d\gamma}{dt}(t) = -\nabla f(\gamma(t)) \qquad \lim_{t\to-\infty} \gamma(t) = a \qquad \lim_{t\to\infty} \gamma(t) = b$$

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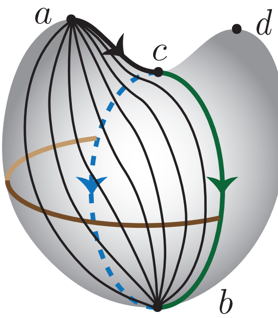
The Morse complex $C(M, f, g)$ is freely generated as an Abelian group by the critical points of f, and the differential ∂ counts gradient trajectories between critical points:

$$C(M, g, f) = \bigoplus_{c \in \mathrm{Crit}(f)} \mathbb{Z}c$$

$$\langle \partial a, b \rangle = \#|\mathcal{M}(a, b)/\mathbb{R}|$$

Morse homology



$$\langle \partial a, c \rangle = 1$$
$$\langle \partial c, b \rangle = 0$$

$$\begin{array}{ccc} \mathbb{Z}a & & \mathbb{Z}d \\ & \searrow \quad \swarrow & \\ & \mathbb{Z}c & \\ & & \\ & \mathbb{Z}b & \end{array}$$

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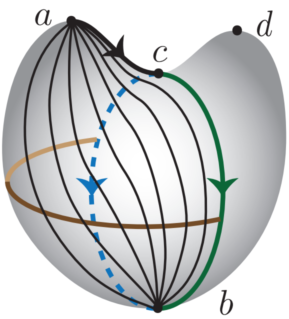

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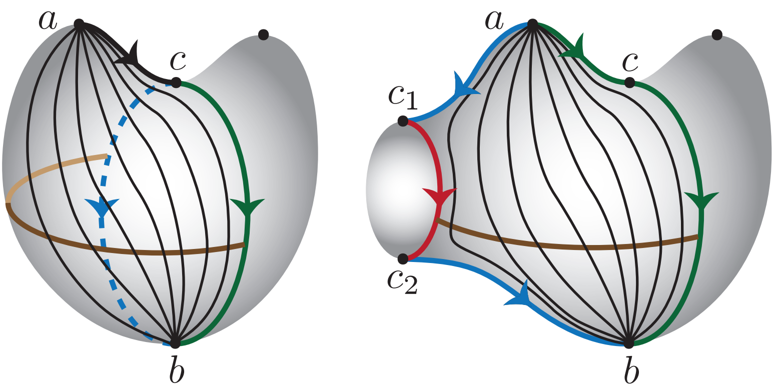
$$\boxed{\begin{tikzcd} \mathbb{Z}a \arrow[dr] & & \mathbb{Z}d \arrow[dl] \\ & \mathbb{Z}c & \\ & \mathbb{Z}b & \end{tikzcd}}$$

$$\begin{aligned}
\langle \partial\partial a, b \rangle &= \sum_{c \in \mathrm{crit}(f)} \#|\mathcal{M}(a,c)/\mathbb{R}| \times \#|\mathcal{M}(c,b)/\mathbb{R}| \\
&= \#\text{ of once-broken gradient trajectories from } a \text{ to } b \\
&= \#\text{ of ends of } \mathcal{M}(a,b)/\mathbb{R} \text{ (compactified)} \\
&= 0, \text{ because this is a compact 1-manifold.}
\end{aligned}$$

Morse homology with boundary

Confession

In monopole Floer homology, the finite-dimensional model is actually Morse homology on a manifold with boundary.



From Morse to Floer

Floer homology is an ∞-dim. version of Morse homology. To a 3-manifold Y and extra data, one associates a configuration space $\mathcal{C}(Y)$ with a functional $F : \mathcal{C}(Y) \to \mathbb{R}$.

1. Stable and unstable manifolds are ∞-dim.
2. Their intersections are finite-dim. and compact.
3. No absolute Morse index, only a relative one.

But if you're lucky, you can still construct a complex generated by the critical points of F and prove that the homology is independent of the choice of extra data.

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A cobordism $W : Y_0 \to Y_1$ also has a configuration space $\mathcal{C}(W)$, with restriction maps r_i to each $\mathcal{C}(Y_i)$. Used to define a chain map on complexes via a kind of "pull-up, push-down". The chain map induces an invariant map on homology.

SW_3

Let Y be a closed, oriented 3-manifold and let $\mathfrak{s} \in \mathrm{spin}^c(Y)$.
Let $S \to Y$ be the associated spinor-bundle.
Let L be the determinant line bundle.

$$\mathcal{C}(Y, \mathfrak{s}) = \{(B, \psi) \mid B \text{ is a } U(1)\text{-connection on } L,\ \psi \in \Gamma(S)\}$$

Definition (SW_3)

$$\rho(F_B) = q(\psi) \qquad D_B\psi = 0$$

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Definition

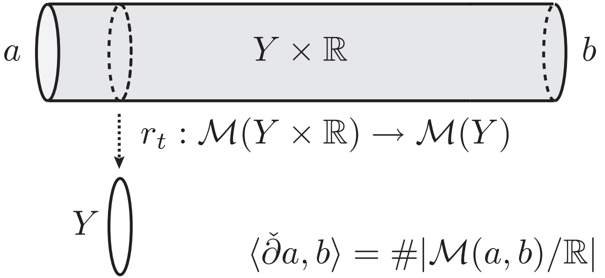
Pairs (B, ψ) with $\psi = 0$ are called *reducible*.

Reducible monopoles correspond to flat connections.
The gauge group $\mathcal{G} = \mathcal{C}^\infty(Y, S^1)$ acts freely on $\mathcal{C}(Y, \mathfrak{s})$ away from reducibles and maps monopoles to monopoles.

Monopole Floer homology

For a closed, oriented, connected 3-manifold Y, the monopole Floer homology group $\widetilde{HM}_\bullet(Y)$ is defined via Morse theory on $CSD : \mathcal{C}(Y, \mathfrak{s}) \to \mathbb{R}$, with $\nabla CSD = SW_3$ and flow equal SW_4:

- $\check{C}(Y)$ is generated by monopoles over Y
- $\check{\partial}$ counts monopoles over $Y \times \mathbb{R}$ modulo translation.



$$\langle \check{\partial} a, b \rangle = \#|\mathcal{M}(a,b)/\mathbb{R}|$$

Example: $\#^k(S^1 \times S^2)$ and positive scalar curvature

Example

Let $Y = \#^k S^1 \times S^2$. Then $\widetilde{HM}_\bullet(Y) \cong \Lambda^* H_1(Y) \cong H_*(T^k)$.

Sketch: Since Y admits a metric of positive scalar curvature, the solutions to the unperturbed equations over the torsion spin^c-structure are reducible (Lichnerowicz) and parameterized by the torus $\mathbb{T}^{b_1} = H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$ via:

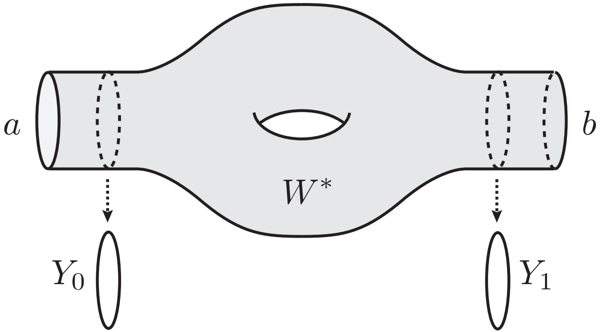
$$\begin{aligned}
\text{monopoles / gauge} &= \text{unitary flat connections / gauge} \\
&= \mathrm{Hom}(\pi_1(Y), S^1) \text{ / conjugation} \\
&= \mathrm{Hom}(H_1(Y), S^1) \\
&= H^1(Y; S^1).
\end{aligned}$$

Perturb slightly to identify $\tilde{C}(Y)$ with Morse complex $C_*(\mathbb{T}^{b_1})$.

Monopole Floer homology

A Reimannian 4-manifold W with boundary $-Y_0 \coprod Y_1$ determines a chain map $\check{m}(W) : \check{C}(Y_0) \to \check{C}(Y_1)$ which counts monopoles on W with infinite cylindrical ends attached.

$$\langle \check{m}(W)a, b \rangle = \#|\mathcal{M}(a, W^*, b)|$$



Monopole Floer homology

These constructions depend on auxiliary choices of metric and perturbation data. However:

- The group $\widecheck{HM}_\bullet(Y)$ is a topological invariant of Y.
- The map $\widecheck{HM}_\bullet(W) : \widecheck{HM}_\bullet(Y_0) \to \widecheck{HM}_\bullet(Y_1)$ induced by $\check{m}(W)$ is a smooth invariant of W.
- If $W = W_2 \circ W_1$ then $\widecheck{HM}_\bullet(W) = \widecheck{HM}_\bullet(W_2) \circ \widecheck{HM}_\bullet(W_1)$.

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Theorem (Kronheimer-Mrowka)

Monopole Floer homology *is a well-defined functor from the smooth, oriented, connected 3-dimensional cobordism category to the category of Abelian groups:*

$$\widecheck{HM}_\bullet : \mathcal{C}^3 \to Ab$$

Heegaard Floer homology

In 2000, Ozsváth and Szabó introduced *Heegaard Floer homology*. The construction starts with a decomposition of Y as a union of handlebodies along a surface Σ_g.

- The complex is generated by the intersection points of two associated Lagrangian tori in $\mathrm{Sym}^g(\Sigma_g)$.
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Despite vastly different origins, Heegaard Floer theory was invented with the following in mind:

Conjecture

Monopole Floer homology $\cong$ Heegaard Floer homology.

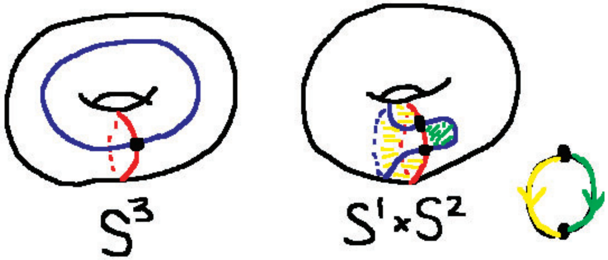
Recently, Katluhan, Lee, and Taubes have announced a proof routed through embedded contact homology.

S^3 and $S^1 \times S^2$

Example

$$\widehat{HF}(S^3) \cong \mathbb{Z} \qquad \widehat{HF}(S^1 \times S^2) \cong H_*(S^1)$$

We start with an admissible Heegaard diagrams:



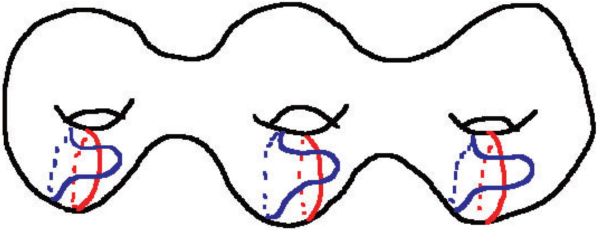
For $S^1 \times S^2$, the two pseudo-holomorphic disks cancel.

$\#^k S^1 \times S^2$

Example

$$\widehat{HF}(\#^k(S^1 \times S^2)) \cong H_*(\mathbb{T}^k) \cong \Lambda^* H_1(\#^k(S^1 \times S^2)).$$

For $k = 3$:



$\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \mathrm{Sym}^3(\Sigma_3) \qquad |\mathbb{T}_\alpha \cap \mathbb{T}_\beta| = 2^3$

Again, pseudo-holomorphic disks cancel in pairs.

More examples

Let P be the Poincaré sphere.

Example

$$\widehat{HF}(S^3) = \widehat{HF}(P) = \mathbb{Z} \qquad \widehat{HF}(L(p,1)) = \mathbb{Z}^p$$

$$\widehat{HF}(S^1 \times S^2) = \mathbb{Z}^2 \qquad \widehat{HF}(T^3) = \mathbb{Z}^6$$

$$\widehat{HF}(\#^k(S^1 \times S^2)) = (\mathbb{Z}^2)^k$$

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Conjecture ("Floer Poincare Conjecture")

If $\widehat{HF}(Y) = \mathbb{Z}$ then Y is a connected sum of copies of $\pm P$.

In 2009, Eftekhary proved that Y is a connected sum of Poincare spheres and hyperbolic homology spheres.

Khovanov homology

In 1999, Khovanov described a new homology theory which associates a bigraded Abelian group $Kh(L)$ to a link $L \subset S^3$.

- The graded Euler characteristic of the reduced version $\widetilde{Kh}(L)$ recovers the Jones polynomial.
- The theory is fully combinatorial, yet powerful enough to compute smooth invariants, like the slice genus of torus knots, that were previously only accessible to Floer theory.

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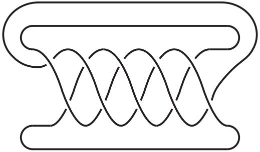
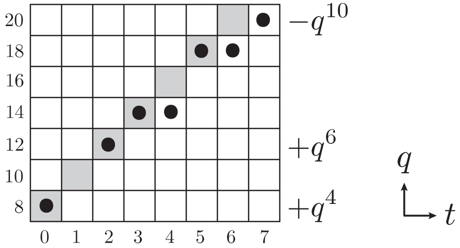
$\widetilde{Kh}(T(3,5);\mathbb{F}_2)$





Functors

These theories have different origins:

- gauge theory (monopole)
- symplectic geometry (Heegaard)
- representation theory (Khovanov)

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But they share an important feature. They are all functors:

$$\widetilde{HM}_\bullet : \mathcal{C}^3 \to Ab$$
$$HF^+ : \mathcal{C}^3 \to Ab$$
$$Kh : \mathcal{L} \to Ab$$

$\mathcal{C}^3$ = Closed, oriented, connected 3-manifolds and cobordisms.
$\mathcal{L}$ = Links in S^3 and link cobordisms in $S^3 \times [0,1]$.
Ab = Abelian groups and group homomorphisms.

Branched double-cover

To relate them, we need to construct a 3-manifold from a link.

Definition

The *branched double-cover* of $L \subset S^3$ is a 3-manifold $\Sigma(L)$ s.t.

- There is a continuous projection $\pi : \Sigma(L) \to S^3$
- $\pi^{-1}(S^3 - L) \xrightarrow{\pi} S^3 - L$ is an honest double cover
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Example

- $\Sigma(U) = S^3$, where U is the unknot.
- $\Sigma(T(p,q)) = \Sigma(2,p,q)$, a Brieskorn $\mathbb{Z}HS^3$ if p and q odd.
- $\Sigma(U_k) = \#^{k-1} S^1 \times S^2$, where U_k is the k-comp. unlink.

Example

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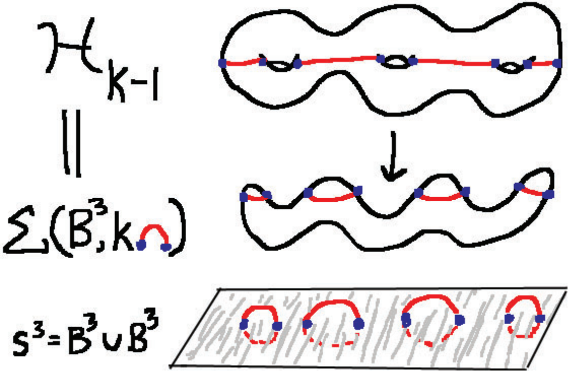

Khovanov to Heegaard Floer

In 2003, Ozsváth and Szabó constructed the first bridge.

- Let $L \subset S^3$ be a link and let $-\Sigma(L)$ be the branched double-cover with reversed orientation.

Theorem (Ozsváth-Szabó)

There is a spectral sequence whose E_2 term is $\widetilde{Kh}(L;\mathbb{F}_2)$ and whose E_∞ term is $\widehat{HF}(-\Sigma(L);\mathbb{F}_2)$.

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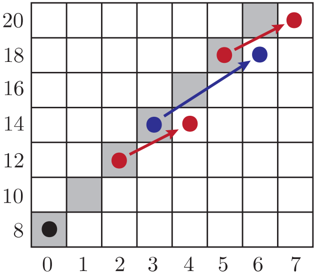
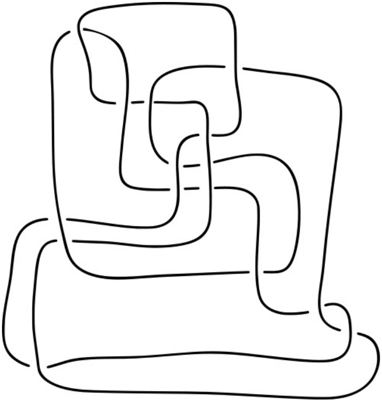
$$\widetilde{Kh}(T(3,5);\mathbb{F}_2)$$
$$\Downarrow$$
$$\widehat{HF}(-\Sigma(2,3,5);\mathbb{F}_2)$$



The Jones polynomial of this knot is 1.



Khovanov to Heegaard Floer

Let K be a knot with Jones polynomial $V_K(q)$.

Conjecture (The Jones polynomial detects the unknot)

If $V_K(q) = 1$ then K is the unknot.

This conjecture has a categorification.

Conjecture (Khovanov homology detects the unknot)

If $\operatorname{rk} \widetilde{Kh}(K) = 1$ then K is the unknot.

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The spectral sequence implies

$$\operatorname{rk} \widetilde{Kh}(L; \mathbb{F}_2) \geq \widehat{HF}(-\Sigma(L); \mathbb{F}_2).$$

Since $T(3,5)$ is the only knot with $\Sigma(K) = P$, the categorified version would follow from the "Floer Poincaré Conjecture".

Khovanov to Heegaard Floer

Let K^2 denote the 2-cable of K with Seifert framing. Then:

$$\Sigma(K^2) = S^3_0(K \# K)$$

One can relate $S^3_0(K \# K)$ to knot Floer homology of $K \# K$, which is non-trivial in grading $2g$, where g is the genus of K.

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Theorem (Hedden, May 2008)

If K is knotted, then $\operatorname{rk} \widetilde{Kh}(K^2; \mathbb{F}_2) > 2$

Thus Khovanov homology gives an explicit algorithm to determine whether a knot diagram represents the unknot.

Theorem (Grigsby-Wehrli, before May 2008)

Khovanov's categorification of the n-colored Jones polynomial detects the unknot for $n > 1$.

Khovanov to Monopole Floer

Theorem (Bloom, 2008)

There is a spectral sequence whose E_2 term is $\widetilde{Kh}(L;\mathbb{F}_2)$ and whose E_∞ term is $\widetilde{HM}_\bullet(-\Sigma(L);\mathbb{F}_2)$.

Proposition (B.)

There is an absolute $\mathbb{Z}/2\mathbb{Z}$-grading $\check{\delta}$ on the spectral sequence.

- *The differentials each shift $\check{\delta}$ by 1.*
- *On E_2, $\check{\delta}$ coincides with $\delta - (\sigma + \nu)/2$ (mod 2).*
- *On E_∞, $\check{\delta}$ coincides with the usual Floer grading.*

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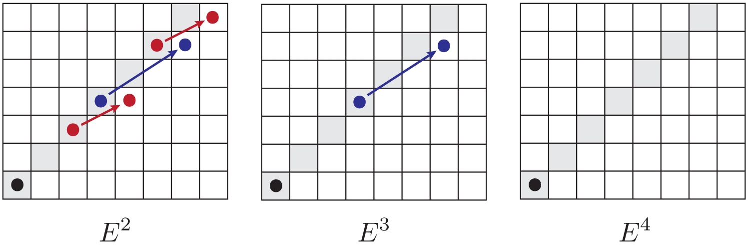
Corollary (B.)

If L is thin, then $\widetilde{Kh}(L) \cong \widetilde{HM}_\bullet(-\Sigma(L))$. Each has rank $\det(L)$.

Khovanov to Monopole Floer

Theorem (B. for analytic invariance, Baldwin for Reidemeister)

The pages E_i for $i \geq 2$ are invariants of the mutation equivalence class of L (as $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-graded vector spaces).



Khovanov to Monopole Floer

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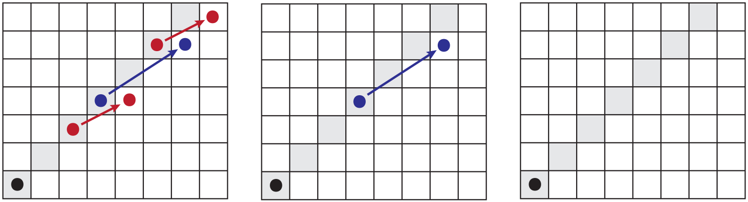

E^2 E^3 E^4

Conjecture

The full Khovanov bigrading extends to the higher pages.

Odd Khovanov homology

Ozsváth, Rasmussen, and Szabó give a new categorification of the Jones polynomial called *odd Khovanov homology*.

- An exterior algebra replaces the original symmetric algebra.
- Coincides with Khovanov homology over $\mathbb{F}_2$.

Conjecture (2007)

There is a spectral sequence from the odd Khovanov homology of L to the Heegaard/monopole Floer homology of $-\Sigma(L)$.

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Conjecture (2007)

There is a spectral sequence from the odd Khovanov homology of L to the Heegaard/monopole Floer homology of $-\Sigma(L)$.

This is work in progress, and motivated the proof of:

Theorem (B., 2009)

Odd Khovanov homology is mutation invariant.

False for Khovanov homology of links; unknown for knots.

Khovanov to Instanton Knot Floer

Let K be a knot in S^3.

Theorem (Kronheimer-Mrowka, 2010)

There is a spectral sequence whose E_2 term is $Kh(K)$ and whose E_∞ term is the instanton knot Floer homology $HKI(K)$.

A deep geometric result, related to the Thom conjecture, implies that this Floer theory detects the genus of the knot:

$$\text{If } K \neq U \text{ then } \operatorname{rk} HKI(K) > 1.$$

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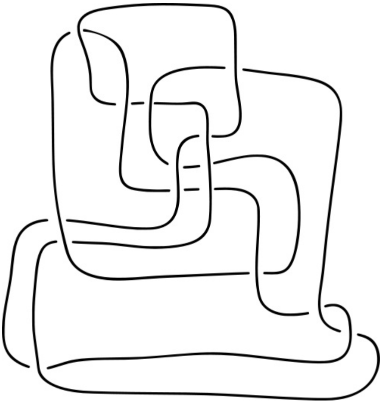
Corollary

Khovanov homology detects the unknot!

In fact, Khovanov homology detects the 2-component unlink, even though the Jones polynomial does not! (Hedden, Ni)

Mathematica tells me that $\operatorname{rk}\widetilde{Kh}(K) = 1$. So K is the unknot!

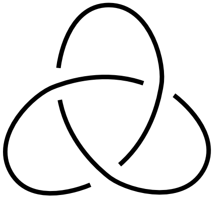


Connections

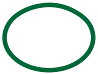
Why should Khovanov and Floer homology be related?
We will link the theories in five steps.

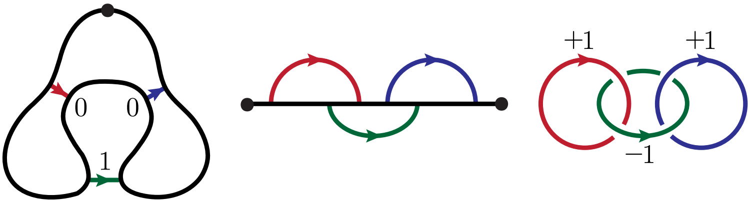
1. Branched double-covers
2. Exact triangles
3. Mutation
4. Families of metrics
5. Polytopes

Surgery diagram for $-\Sigma(L)$.



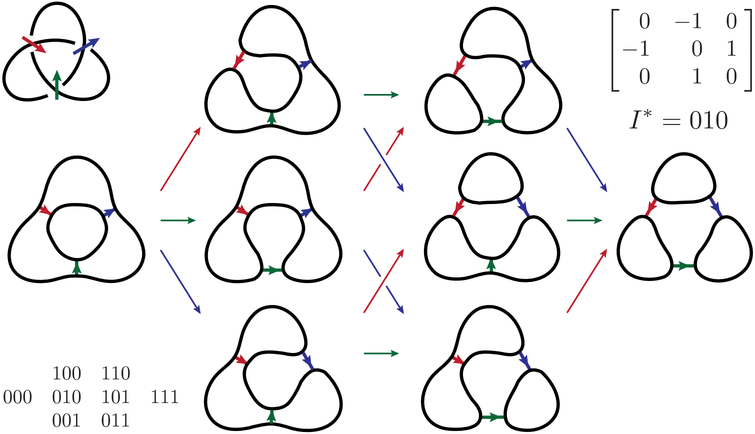
$$-\Sigma(T) = -L(3,1) =$$





Trefoil below

The Khovanov hypercube consists of resolutions $D(I) \subset S^3$ and 1-handle cobordisms $F(IJ) \subset S^3 \times [0,1]$:



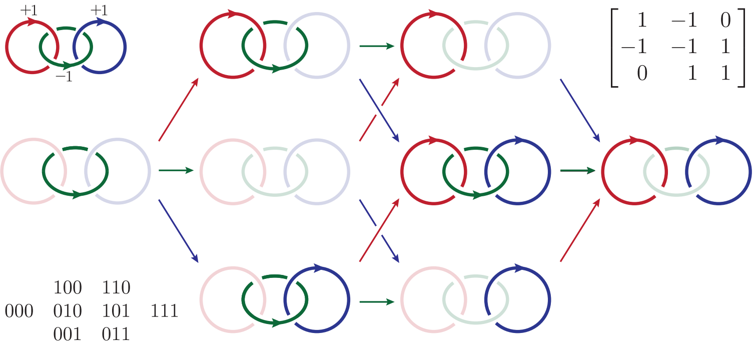
$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$I^* = 010$$

$$\begin{array}{cccc} & 100 & 110 & \\ 000 & 010 & 101 & 111 \\ & 001 & 011 & \end{array}$$

Trefoil above

Applying the branch double-cover functor $-\Sigma$, we obtain a (3+1)-dimensional hypercube of 3-manifolds $Y(I)$ and 2-handle cobordisms $W(IJ)$:



$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

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Branched double-covers

We have two options:

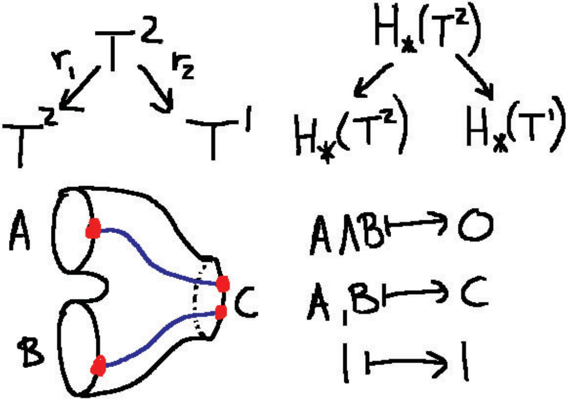
1. Apply $\widetilde{Kh}$ to the (1+1)-dimensional hypercube.
2. Apply $\widetilde{HM}_\bullet$ to the (3+1)-dimensional hypercube.

$$\begin{tikzcd}
\boxed{Y(I) \xrightarrow{W(IJ)} Y(J)} \arrow[r, "\widetilde{HM}_\bullet"] & \boxed{V(I) \xrightarrow{T(IJ)} V(J)} \arrow[d, "\cong"] \\
\boxed{\mathcal{D}(I) \xrightarrow{F(IJ)} \mathcal{D}(J)} \arrow[u, "-\Sigma"] \arrow[r, "\widetilde{Kh}"] \arrow[ur, phantom, "\circlearrowright"] & \boxed{U(I) \xrightarrow{S(IJ)} U(J)}
\end{tikzcd}$$

In fact the two routes yield isomorphic complexes! So $\widetilde{CKh}(D)$ is determined by the choice of resolution and surgery diagram.

Donaldson TQFT

Manifold $M \rightsquigarrow H_*$(moduli space of flat connections / gauge).
Cobordism $X \rightsquigarrow$ Map from restriction of connections to ∂X.



Donaldson TQFT $\Lambda^* H^1$

$$\begin{tikzcd}
\boxed{Y(I) \xrightarrow{W(IJ)} Y(J)} \arrow[rr, "\widetilde{HM}_\bullet"] & & \boxed{V(I) \xrightarrow{T(IJ)} V(J)} \arrow[d] \\
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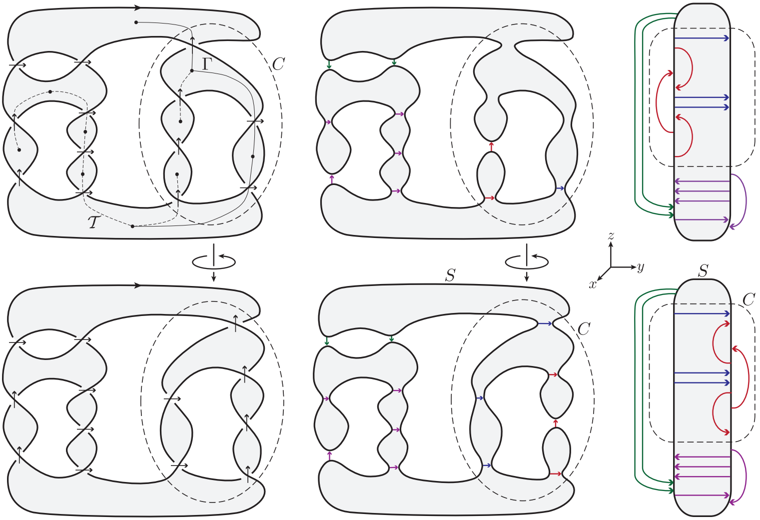
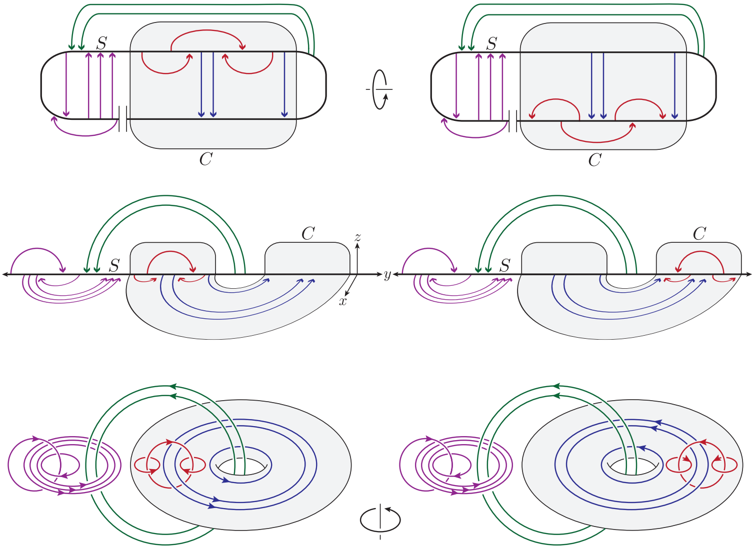
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{Y(I) \xrightarrow{W(IJ)} Y(J)} \arrow[rr, "\Lambda^* H^1"] & & \boxed{V(I) \xrightarrow{T(IJ)} V(J)} \arrow[d, "\Lambda^* \rho"'] \\
\boxed{\mathcal{D}(I) \xrightarrow{F(IJ)} \mathcal{D}(J)} \arrow[u, "-\Sigma"] \arrow[rr, "\Lambda^* H^1"] \arrow[urr, phantom, "\circlearrowright"] & & \boxed{U(I) \xrightarrow{S(IJ)} U(J)}
\end{tikzcd}$$

Mutation invariance I



Mutation invariance II



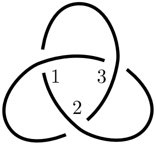
Mutation invariance III

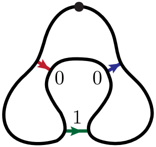
Conclusion

Mutant link diagrams give rise to isotopic surgery diagrams, and therefore isomorphic odd Khovanov complexes.

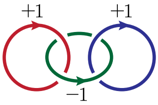
A more concrete construction is given in the paper:

1. To a link diagram D, associate a “linking matrix” A, secretly the linking matrix of a surgery diagram for $-\Sigma(L)$.
2. See that A is invariant under diagrammatic mutation.
3. Reconstruct the odd Khovanov complex from A alone.



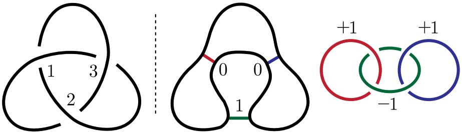


$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



Graph-links

The matrix A may also be viewed as the adjacency matrix of a (bipartite, planar) graph with signed vertices, representing an equivalence class of *graph-link* (Manturov, Ilyutko).



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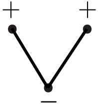

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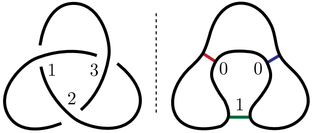

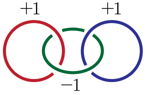

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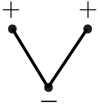


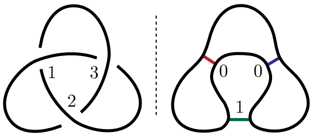
Nikonov proved that the above construction over $\mathbb{F}_2$ is well-defined for arbitrary graph-links, including signed graph-links (which generalize virtual knots). The Khovanov homology of a graph-link categorifies the Kauffman bracket defined by Manturov and Ilyutko. Nikonov lifts the construction to $\mathbb{Z}$ for principally unimodular bipartite graph-links.

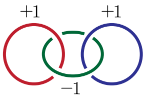
Signature and mutation

Let $k = |I^*| = (n - \text{trace}(A))/2 = \#$ of negative vertices.

Proposition

$$\sigma(L) = \sigma(A) + k - n_- \qquad \det(L) = \det(A) \qquad \nu(L) = \nu(A)$$





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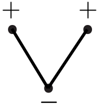

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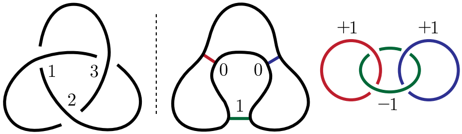

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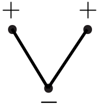


Example

$\sigma(T) = 1 + 1 - 0 = 2$ $\quad$ $\sigma(m(T)) = -1 + 2 - 3 = -2$

1. Mutation invariant definition.
2. Extends to graph-knots.
3. Well-adapted to Khovanov homology.

Exact triangles I

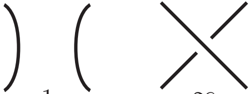
- Khovanov homology has a skein exact triangle.
- Floer homology has a surgery exact triangle.

These are related by branched double-cover:

$$\cdots \to \widetilde{HM}_\bullet(Y(0)) \xrightarrow{\widetilde{HM}_\bullet(W(01))} \widetilde{HM}_\bullet(Y(1)) \to \widetilde{HM}_\bullet(Y) \to \cdots$$

$$\uparrow -\Sigma$$

$$\cdots \to \widetilde{Kh}(D(0)) \xrightarrow{\widetilde{Kh}(F(01))} \widetilde{Kh}(D(1)) \longrightarrow \widetilde{Kh}(D) \to \cdots$$



$0 \qquad 1 \qquad \infty$

Exact triangles II

The surgery exact triangle is a consequence of:

$$\mathrm{MC}\left(\begin{tikzcd} \tilde{C}(Y(0)) \arrow[r, "\tilde{m}(W(01))"] \arrow[loop below, "\tilde{d}(Y(0))"] & \tilde{C}(Y(1)) \arrow[loop below, "\tilde{d}(Y(1))"] \end{tikzcd}\right) \xrightarrow{q.i.} \tilde{C}(Y)$$

$$E_0 = \tilde{C}(Y(0)) \oplus \tilde{C}(Y(1)) \qquad d_0 = \tilde{d}(Y(0)) \oplus \tilde{d}(Y(1))$$

$$E_1 = \widetilde{HM}(Y(0)) \oplus \widetilde{HM}(Y(1)) \qquad d_1 = \widetilde{HM}(W(01))$$

$$E_2 = E_\infty = \widetilde{HM}(Y)$$

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$$\tilde{d}(Y(0)) \qquad \qquad \tilde{d}(Y(1))$$

$$E_0 = \tilde{C}(Y(0)) \oplus \tilde{C}(Y(1)) \qquad d_0 = \tilde{d}(Y(0)) \oplus \tilde{d}(Y(1))$$

$$E_1 = \widetilde{HM}(Y(0)) \oplus \widetilde{HM}(Y(1)) \qquad d_1 = \widetilde{HM}(W(01))$$

$$E_2 = E_\infty = \widetilde{HM}(Y)$$

If we had started from a diagram D with one crossing, then:

1. The complex (E_1, d_1) is isomorphic to $\widetilde{CKh}(D)$.
2. Therefore $E_2 \cong \widetilde{Kh}(L)$.

The spectral sequence

The spectral sequence applies to a diagram D with n crossings.
Let $Y(I) = -\Sigma(D(I))$ for each $I \in \{0,1\}^n$.
The total complex underlying the spectral sequence is:

$$X = \bigoplus_{I \in \{0,1\}^n} \tilde{C}(Y(I))$$

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The total complex underlying the spectral sequence is:

$$X = \bigoplus_{I \in \{0,1\}^n} \tilde{C}(Y(I))$$

The total differential $\tilde{D}$ has a component $\tilde{D}(IJ)$ for each $I \leq J$, and X is filtered by $|I|$ = sum of digits of I.

- $\tilde{D}(II) = \tilde{d}(Y(I))$.
- $\tilde{D}(IJ) = \tilde{m}(W(IJ))$ when $|J| - |I| = 1$.

Therefore $(E_1, d_1) \cong \widetilde{CKh}(\underline{D})$ and $E_2 \cong \widetilde{Kh}(L)$.
The identification $E_\infty \cong \widetilde{HM}_\bullet(-\Sigma(L))$ follows by induction on the surgery exact triangle.

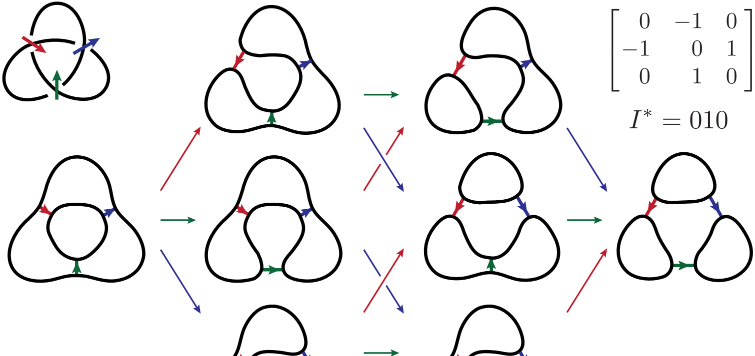
The total differential $\tilde{D}$

In general for $I < J$, $\tilde{D}(IJ)$ is defined to count monopoles on $W(IJ)$ over a family of metrics parameterized by the permutohedron P of dimension $|J| - |I| - 1$, where the metrics degenerate over ∂P.

I will now describe this in more detail.

Review: Trefoil below

The Khovanov hypercube consists of resolutions $D(I) \subset S^3$ and 1-handle cobordisms $F(IJ) \subset S^3 \times [0,1]$:



$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$I^* = 010$$

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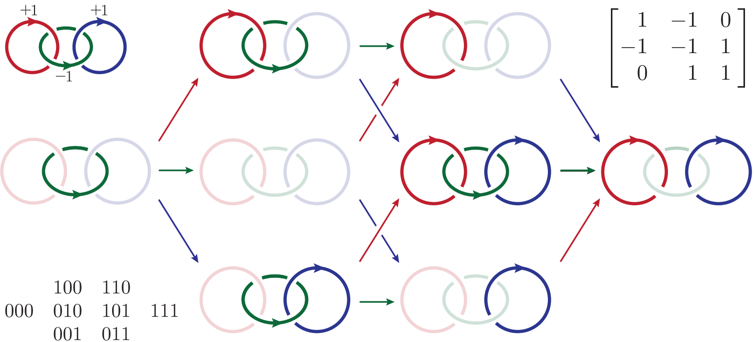
Review Trefoil above

Applying the branch double-cover functor $-\Sigma$, we obtain a $(3+1)$-dimensional hypercube of 3-manifolds $Y(I)$ and 2-handle cobordisms $W(IJ)$:



$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

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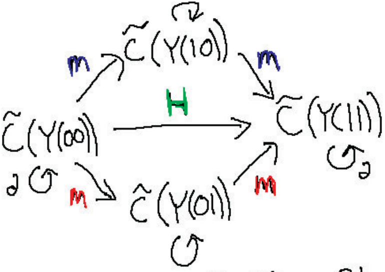
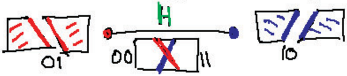
The total differential $\tilde{D}$

Intuitively, for $I \ll J$, the 2-handles forming $W(IJ)$ commute *up to isotopy*, so the corresponding chain maps commute *up to homotopy*. To obtain a complex, we must include homotopies $\tilde{D}(IJ) : \tilde{C}(Y(I)) \to \tilde{C}(Y(J))$ for each square, cube, ...

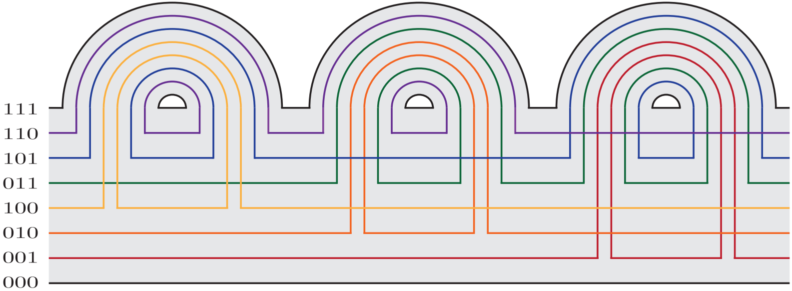


$$\partial H + H\partial + m_{11}^{10} m_{10}^{00} + m_{11}^{01} m_{01}^{00} = 0$$

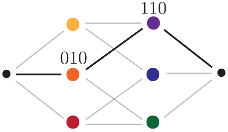


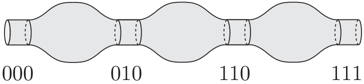
Polytopes

$Y(I)$ and $Y(J)$ intersect iff I and J are not ordered in $\{0,1\}^n$.



$\{0,1\}^3$

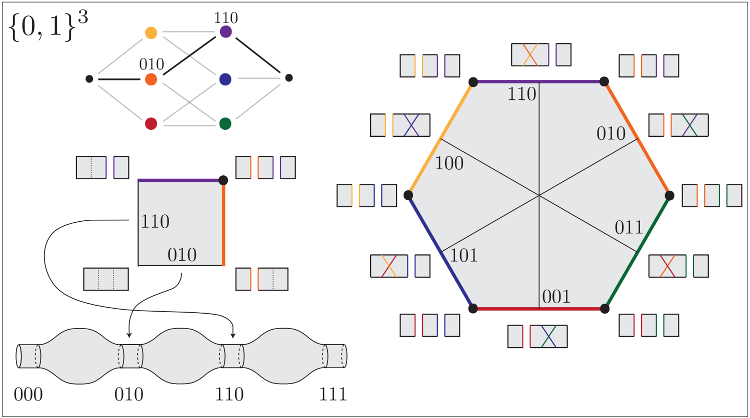




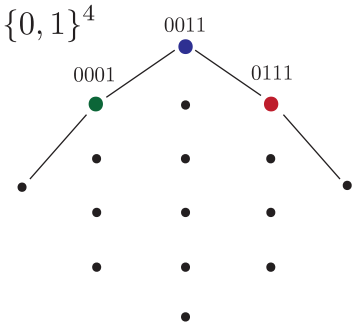
The 2D Permutohedron

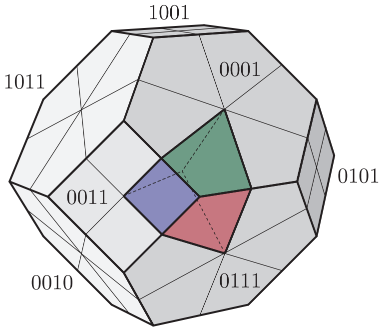
Face poset of $P_2 \xrightarrow{\text{dual}}$ poset of internal chains in $\{0,1\}^3$.

- Vertices of $P_2 \leftrightarrow$ maximal chains $\leftrightarrow$ complete paths.
- Facets of $P_2 \leftrightarrow$ internal vertices of $\{0,1\}^3$.

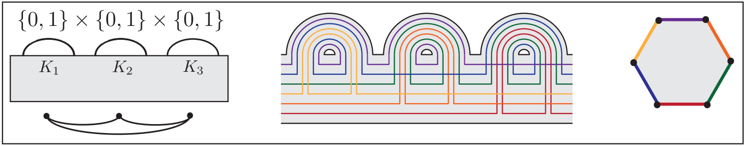


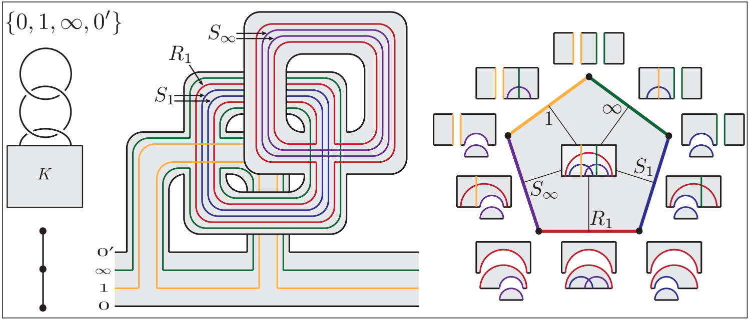
The 3D Permutohedron



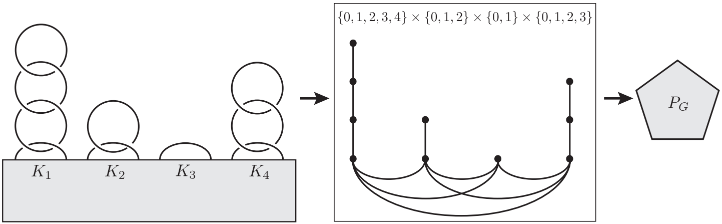


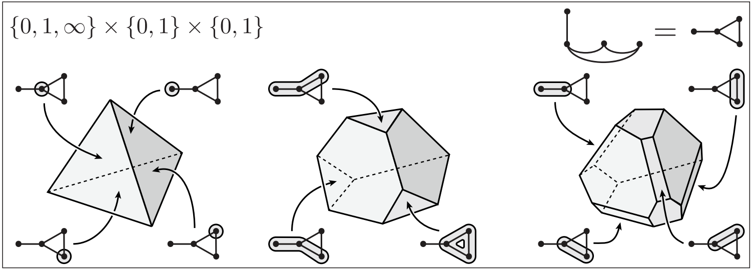
Permutohedra and Associahedra



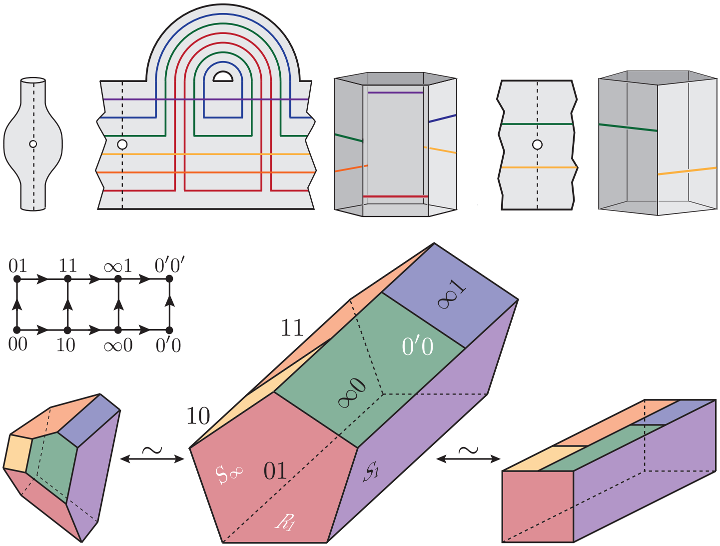


Graph associahedra (Carr, Devadoss)

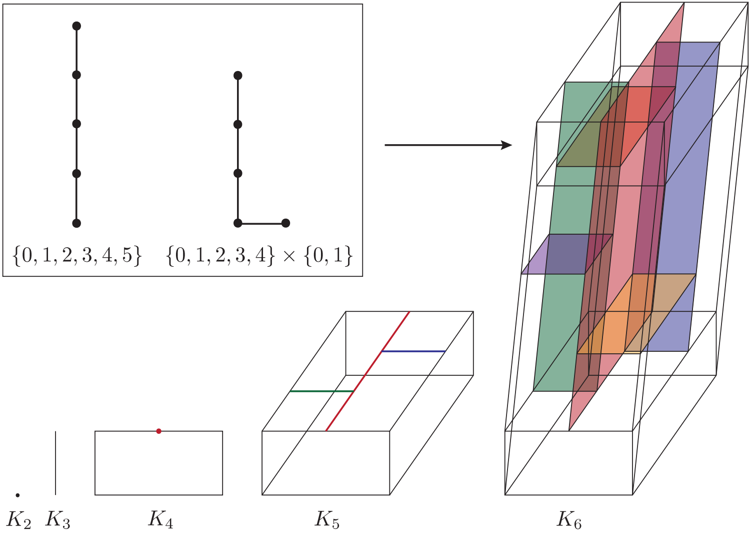




Realization of associahedra



Realization of associahedra



Outlook

Very recently, Szabó defined a combinatorial total differential D on the Khovanov complex of L over $\mathbb{F}_2$. The total homology $H(L)$ is a link invariant, as are the higher pages E_i, as bigraded vector spaces. Calculations by Seed agree with my conjecture for the spectral sequence of torus knots $T(3, 6n \pm 1)$.

Exciting possibility

rk $H(L) =$ *rk* $\widehat{HF}(\Sigma(L))$ *for all links* $L \in S^3$.

1. Rank of $H(L)$ may be an invariant of $\Sigma(L)$ either way.
2. $H(L)$ is a very strong invariant of L, which decategorifies to a sequence of "higher" Jones polynomials.
3. May lead to new ways of computing Floer homology.

Спасибо!