The monopole category and invariants of bordered 3-manifolds

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Overview

1. Khovanov's arc algebra
2. Monopole category
3. Finite generation
4. Monopole algebra

Khovanov's arc algebra

Khovanov homology

$\textsc{LinkCob}$ is the category of links in S^3 and link cobordisms in $S^3 \times [0,1]$.

Khovanov defines a functor $\mathcal{F} : \textsc{LinkCob} \to \textsc{Vect}$.

$\mathcal{F}(L)$ is the reduced Khovanov homology of L with coefficients in $\mathbb{F}_2$.

Let V be a rank-2 vector space over $\mathbb{F}_2$. For the k-component unlink,

$$\mathcal{F}(\bigcirc \cdots \bigcirc) = \Lambda^{k-1} V$$

The vector spaces $\mathcal{F}(\bigcirc)$, $\mathcal{F}(\bigcirc\bigcirc)$, and $\mathcal{F}(T_{2,3})$ have ranks 1, 2, and 3.

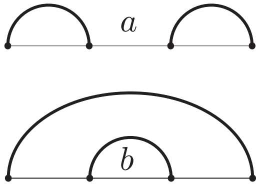
Crossingless matchings

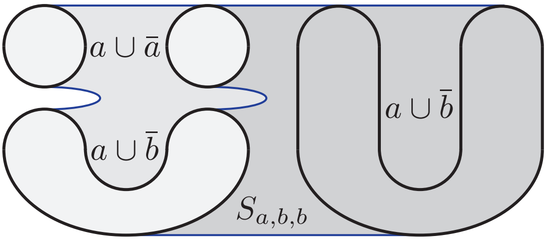
Let B^n be the set of crossingless matchings on $2n$ points.

Let $\bar{b}$ denote the reflection of b across the x-axis.

Then $a \cup \bar{b}$ is a planar unlink and we have well-defined link cobordisms

$$S_{a,b,c} : (a \cup \bar{b}) \sqcup (b \cup \bar{c}) \to a \cup \bar{c}$$





The arc algebra H^n

To 2n **points**, Khovanov associates the ring

$$H^n = \bigoplus_{a,b \in B^n} \mathcal{F}(a \cup \bar{b}).$$

The product of $x \in \mathcal{F}(a \cup \bar{b})$ and $y \in \mathcal{F}(b \cup \bar{c})$ is given by

$$\mathcal{F}(S_{a,b,c}) : \mathcal{F}(a \cup \bar{b}) \otimes \mathcal{F}(b \cup \bar{c}) \to \mathcal{F}(a \cup \bar{c})$$

To a **tangle** T on 2n points, he associates a right H^n-module

$$\mathcal{F}(T) = \bigoplus_{b \in B^n} \mathcal{F}(T \cup \bar{b}).$$

The H^n action is using $\mathcal{F}(S_{T,b,c})$ as above.

To a **tangle cobordism** R, he associated an H^n-module map $\mathcal{F}(R)$.

Pairing theorem

These structures turn $\mathcal{F}$ into a **2-functor** from points, tangles, and tangle cobordisms to algebras, bimodules, and bimodule homomorphisms.

For example, there is a pairing theorem for tangles on $2n$ points:

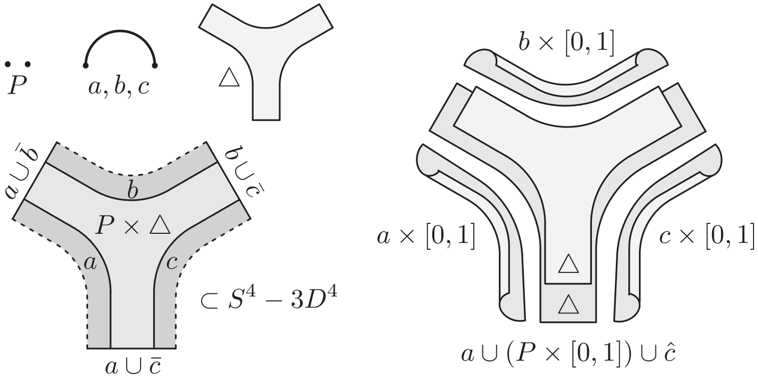
$$\mathcal{F}(T_0) \otimes_{H^n} \mathcal{F}(\bar{T}_1) \cong \mathcal{F}(T_0 \cup \bar{T}_1)$$

The proof uses the **skein relation** to reduce to the case where T_0 and T_1 are crossingless matchings, and from there to $H^n \otimes_{H^n} H^n \cong H^n$.

L $\quad$ L_0 $\quad$ L_1

$$\mathcal{F}(L) = \text{cone}\,(\mathcal{F}(S) : \mathcal{F}(L_0) \to \mathcal{F}(L_1))$$

Another view of $S_{a,b,c}$



H^n-action on $\mathcal{F}(T)$: replace crossingless matching a with tangle T.

Module map $\mathcal{F}(R)$ relation: replace $a \times [0,1]$ with tangle cobordism R.

THE MONOPOLE CATEGORY

The monopole category $\mathcal{C}(\Sigma)$

Let Σ be a smooth, connected, oriented surface of genus g.

A bordered 3-manifold over Σ is a smooth, connected, oriented 3-manifold Y together with orientation-preserving smooth collar $\varphi : \Sigma \times [-1, 0] \to Y$.

The **monopole category** $\mathcal{C}(\Sigma)$ of Σ is an A_∞ category.

The **objects** of $\mathcal{C}(\Sigma)$ are all bordered 3-manifolds over Σ.

The **morphisms** are elements of a monopole Floer chain complex:

$$\mathrm{Mor}(Y_0, Y_1) = \hat{C}(Y_0 \cup_\Sigma \bar{Y}_1).$$

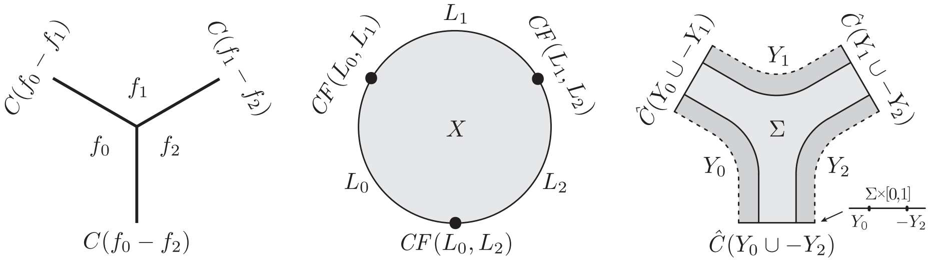
For each $k > 0$ and sequence $Y_0, \ldots, Y_k$, there is a multiplication map

$$\mu_k : \hat{C}(Y_0 \cup_\Sigma \bar{Y}_1) \otimes \cdots \otimes \hat{C}(Y_{k-1} \cup_\Sigma \bar{Y}_k) \to \hat{C}(Y_0 \cup_\Sigma \bar{Y}_k)$$

and these maps satisfy the A_∞ relations.

Context

Our construction is modeled on the **Morse category** of a manifold, as is the **Fukaya category** of a symplectic manifold.



Conjecture: Fuk(Sym$^g(\Sigma)$) and $\mathcal{C}(\Sigma)$ are A_∞ equivalent via a map sending the Lagrangian $\mathbb{T}^\alpha \subset$ Sym$^g(\Sigma)$ to the bordered handlebody (Σ, α).

This is a strengthening of HF $\cong$ HM, itself an analogue of Atiyah-Floer.

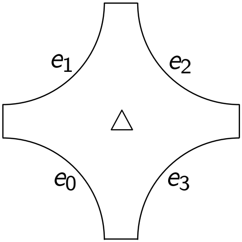
Lipshitz-Ozsváth-Thurston, Lekili-Perutz, Mau-Wehrheim-Woodward

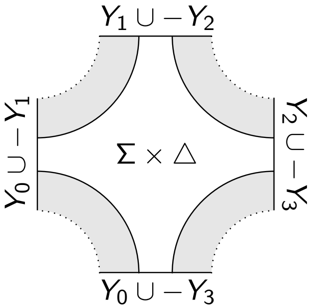
Multiplication maps μ_k

For objects $Y_0, \cdots, Y_k$, the multiplication map

$$\mu_k : \hat{C}(Y_0 \cup_\Sigma \bar{Y}_1) \otimes \cdots \otimes \hat{C}(Y_{k-1} \cup_\Sigma \bar{Y}_k) \to \hat{C}(Y_0 \cup_\Sigma \bar{Y}_k)$$

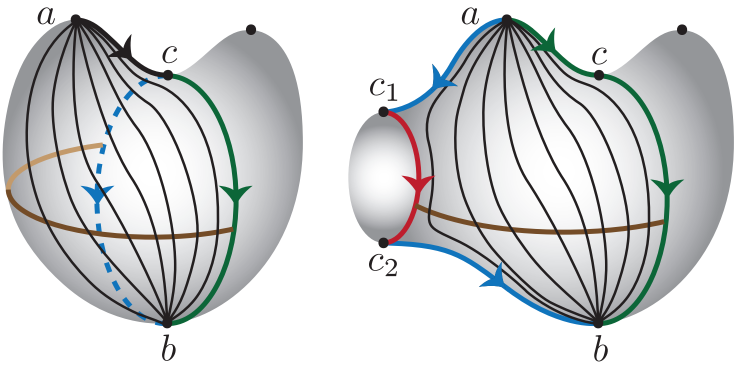
is defined by counting monopoles on a 4-dimensional cobordism $W_{Y_0,\cdots,Y_k}$ over a family of metrics and perturbations parameterized by the $k-2$ dimensional associahedron (point, interval, pentagon,...).





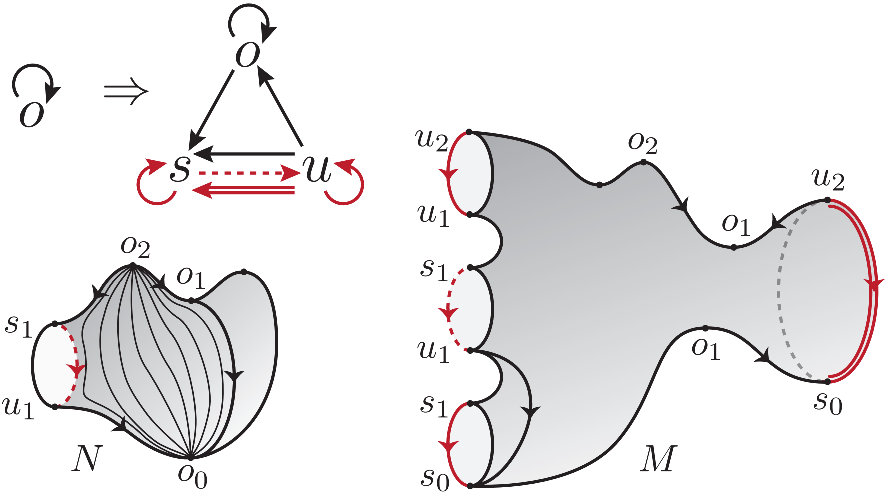
Challenge: reducibles ⇒ boundary

Kronheimer and Mrowka model monopole homology on the Morse homology of a manifold w/ boundary. We model the monopole category on the Morse category of a manifold with boundary.



[B] *The combinatorics of Morse theory with boundary.* http://arxiv.org/abs/1212.6467

Blowing up

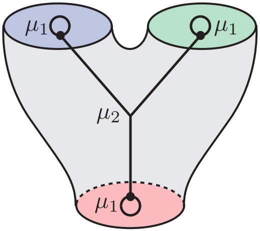


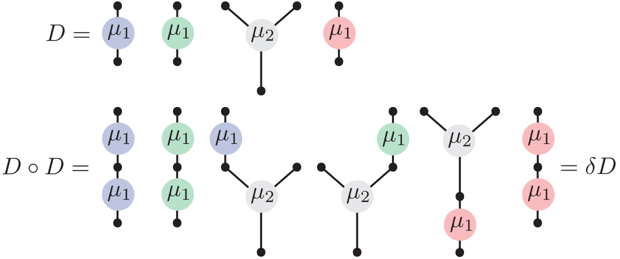
μ_2 as higraph and the structure equation

The path algebra $\mathcal{A}$ of a higraph is a DGA graded by path length.

For an edge ε, the differential $\delta\varepsilon$ is the sum of all length-2 paths from $s(\varepsilon)$ to $t(\varepsilon)$. The differential extends to $\mathcal{A}$ by the Leibniz rule.

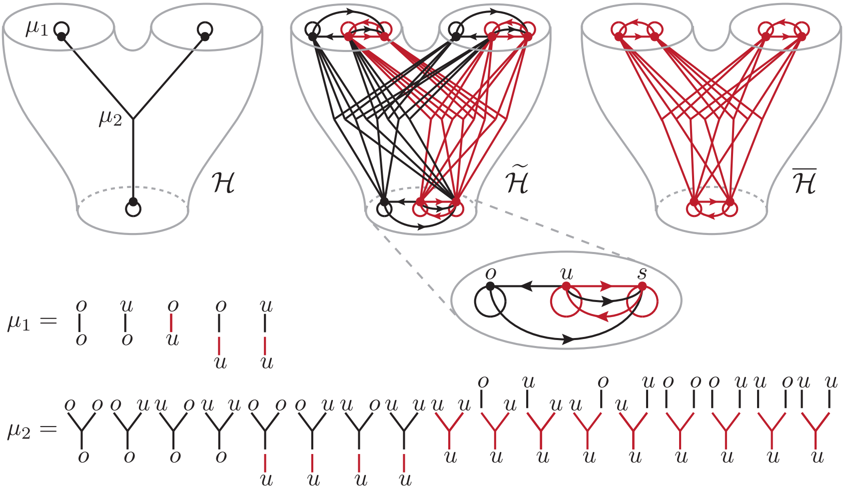
The sum D of all edges satisfies the **structure equation**: $\delta D = D \circ D$



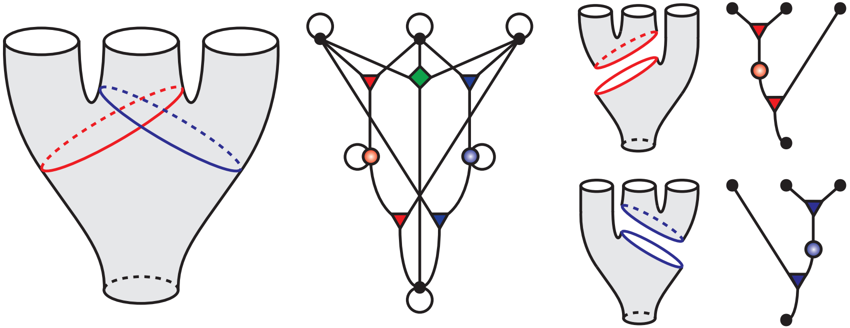


The structure equation is the bridge from geometry to algebra.

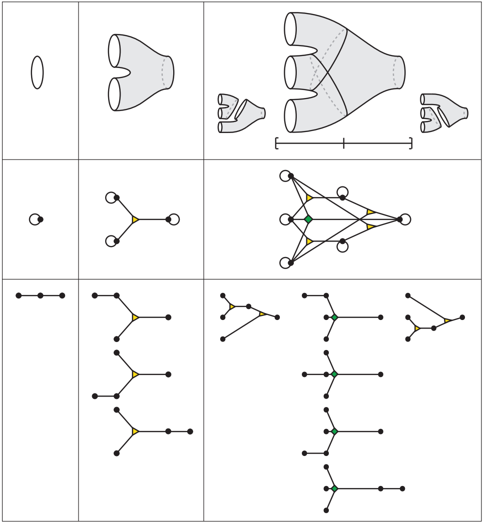
μ_2 as bigraph



μ_3 as higraph



The first three A_∞ relations.



Challenge: coherent data

To define $\mathcal{C}(\Sigma)$, we must make many choices (diffeomorphisms, metrics, perturbations,...) such that the A_∞ relations hold on the nose. Parallel issues arise for the Fukaya category. In our case, $\mathcal{C}(\Sigma)$ is also well-defined up to A_∞ equivalence, and we are working to make this 'natural'.

Our approach to coherence is novel and hinges on Smale's 1964 result that $\mathrm{Diff}(D^2, \partial D^2)$ is contractible. In fact, we need a diffeologically smooth version of this fact that first appeared in 2011.

Jiayong Li and Jordan Alan Watts. *The orientation-preserving diffeomorphism group of S^3 deforms to SO(3) smoothly.* Transformation Groups, Springer, Vol. 16, No. 2, 2011, pp. 537-553.

Finite generation

A **surgery triad** is a triple of 3-manifolds (Y, Y_0, Y_1) such that there is a framed knot K in Y such that Y_i is i-surgery on K.

Significance: Floer homology sends surgery triads to exact triangles.

$$\hat{C}(Y) \cong \text{cone}\left(\hat{m}(W) : \hat{C}(Y_0) \to \hat{C}(Y_1)\right)$$

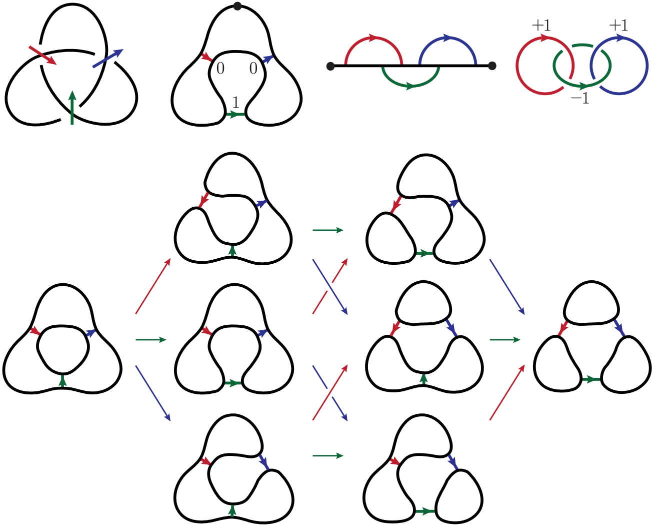
$(S^3, S^1 \times S^2, S^3)$ forms a surgery triad via surgery on the unknot.

i) $\#^k S^1 x S^2$ is **generated** by S^3 through surgery triads.
ii) All links are generated by unlinks through skein triads.
iii) Branched double cover sends unlinks to $\#^k S^1 x S^2$ and
iv) Branched double cover sends skein triads of links to surgery triads.

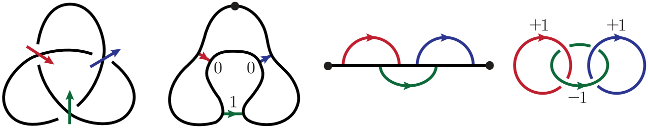
$\Rightarrow$ All branched-double covers are generated by S^3 through surgery triads.

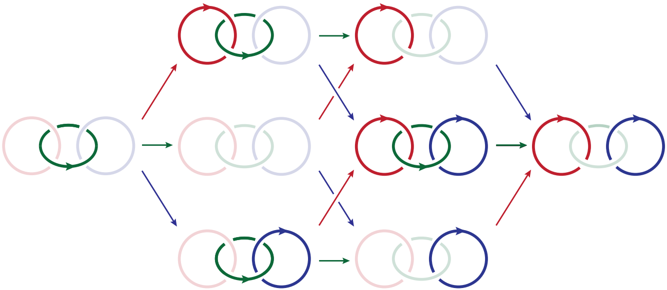
In fact, **all** closed 3-manifolds are generated by S^3 through surgery triads.

Skein triangle below



Surgery triangle above





Bordered manifolds

Bordered mflds that arise as branched covers of $T \subset D^3$ on $2g+2$ points are generated by branched covers of the c_{g+1} crossingless matchings.

Thm: Bordered mflds over Σ_g are generated by n_g bordered handlebodies.

$$n_0 = 1, \quad n_g = 1 + \sum_{i=0}^{g-1} n_i n_{g-1-i}$$

This sequence also equals the binomial transform of the Catalan numbers.

$$n_g = \sum_{k=0}^{g} \binom{g}{k} c_g$$

$c_g =$ 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...
$n_g =$ 1, 2, 5, 15, 51, 188, 731, 2950, 12235, ...

Conjecture: n_g is minimal.

Coincidence?

Khovanov's generators count vertices of the associahedron.
Our bordered generators count vertices of the composihedron.

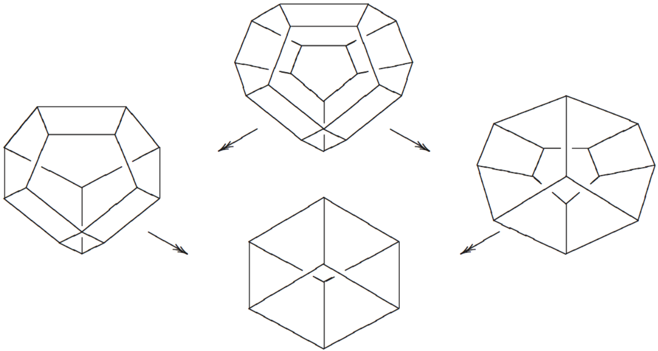


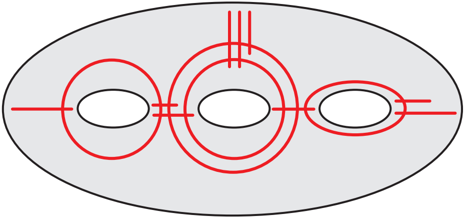
Figure 1: The cast of characters. Left to right: $\mathcal{CK}(4)$, $\mathcal{J}(4)$, 3-d cube, and $\mathcal{K}(5)$.

Stefan Forcey, *Quotients of the multiplihedron as categorified associahedra*, Homotopy, Homology and Applications, vol. 10(2), 227256, 2008.

The generators

Our proof of finite generation proceeds as follows:

1) Bordered manifold = (bordered handlebody) $\cup$ (compression body).
2) Bordered handlebodies are described by mapping classes of Σ.
3) Mapping classes are described by Dehn twists.
4) Dehn twists correspond to boundary parallel ± 1 surgeries.
5) Kirby calculus reduces such configurations of 0-surgeries to a set of n_g.



A simpler description of the generators

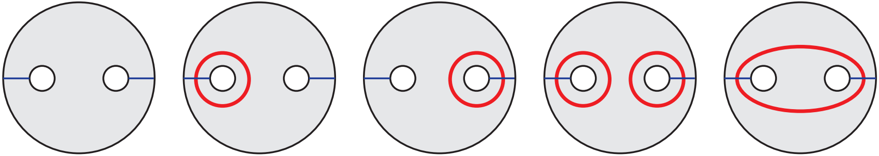
After hearing us talk, Lucas Culler (MIT) came up with an alternative approach that gives a simple description of n_g generators:

Let D_g be a disk D with g punctures and an arc connecting each to ∂D.

Let γ be simple closed multicurve which intersect each arc at most once.

The generators result from 0-surgery on $\gamma \times \{0\} \subset D_g \times [-1,1]$.

For g $= 2$, there are 5 generators:



The monopole algebra

The monopole algebra

Now that we have a set $\mathcal{H}(\Sigma) = \{H_\alpha\}$ of finitely many generators for $\mathcal{C}(\Sigma)$, we can equivalently work with A_∞ algebras and modules:

$$\mathcal{A}(\Sigma) = \bigoplus_{\alpha,\beta} \mathcal{F}(H_\alpha \cup_\Sigma \bar{H}_\beta) \qquad \mathcal{C}(Y) = \bigoplus_{\beta} \mathcal{F}(Y \cup_\Sigma \bar{H}_\beta)$$

Here $\mathcal{C}(Y)$ is an invariant of the bordered manifold Y up to quasi-isomorphism, and the pairing theorem holds by reduction to Khovanov's approach. For $g = 0$, this is the Künneth formula.

We can compute the homology algebra of $\mathcal{A}(\Sigma)$ explicitly. It naturally contains a copy of the arc algebra H^{g+1}.

MCG(Σ) acts on $\mathcal{A}(\Sigma)$ and in fact yields a faithful linear-categorical action in every genus except possibly $g = 2$. Here we rely on recent work of Corrin Clarkson and HM $\cong$ HF.

2+1+1

HM is a "2-functor" from the "2-category" of surfaces, cobordisms of surfaces, and cobordisms of cobordisms of surfaces to the "2-category" of A_∞ algebras, bimodules, and maps of bimodules.

A 3-dimensional cobordism between n surfaces yields an A_∞ n-module.

Self-gluing is related to Hochschild homology.

Master version may have applications to computing 4-manifold invariants.

Topological approach goes through in other Floer theories and may give algebraic insight into equivalences.