Knots for Novices

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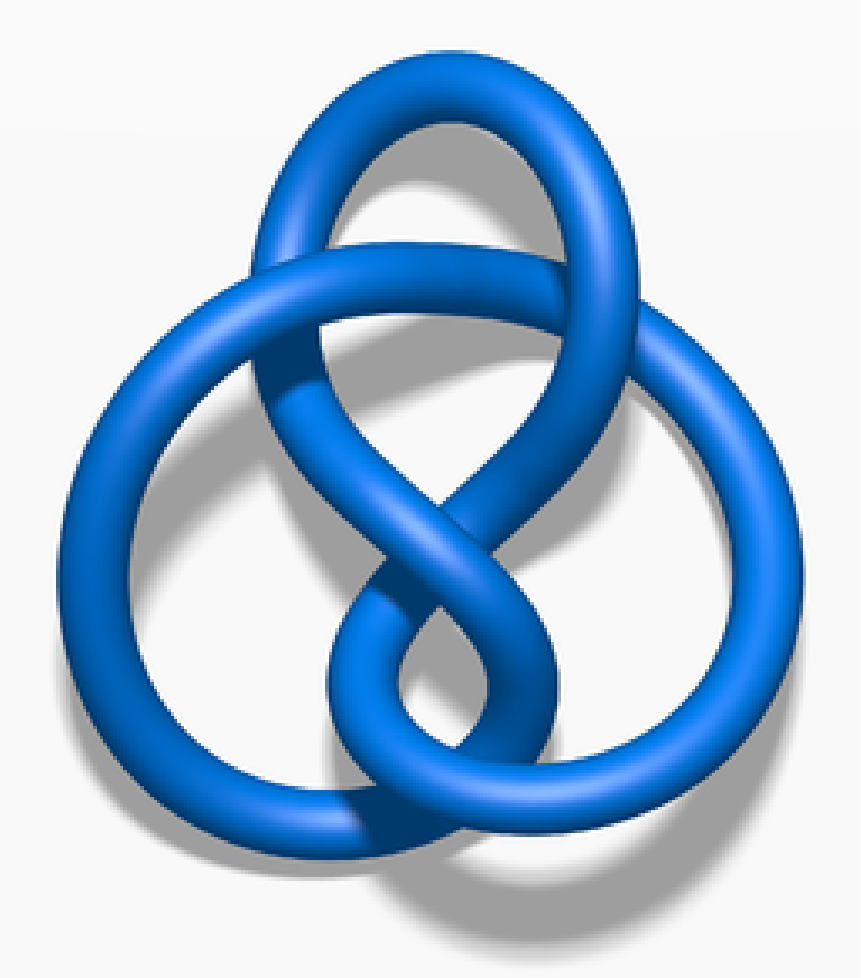
Overview

- What is a knot
  ... and what is not?
- The simplest strands
- Knot theory
- Knot invariants
- Links to the future



Practice

Activity

Make an unknot.

What is a knot...and what is not?

Activity

Tangle up a strand, switch with a partner, and try to untangle.

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Make a simple tangle, connect the ends, switch with a partner, and try to untangle without disconnecting the ends.

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...

Mathematical knots have no loose ends!

Definition (Knots for mathematicians)

A knot is a closed loop in space (with no self-intersections).

Think of a connected the ends of a tangled garden hose!

When are two knots the same?

Activity

Is your knot the same as your neighbor's?

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Definition (Knot type)

Two knots have the same type if one can be continuously transformed into the other without cutting or self-intersection.

Think of moving around an **elastic** rope or hose in space!

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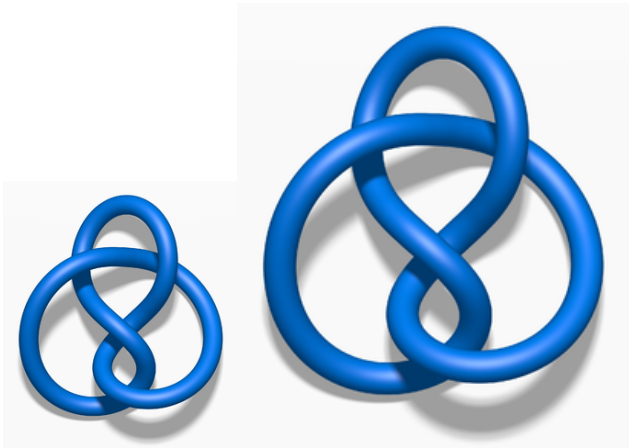

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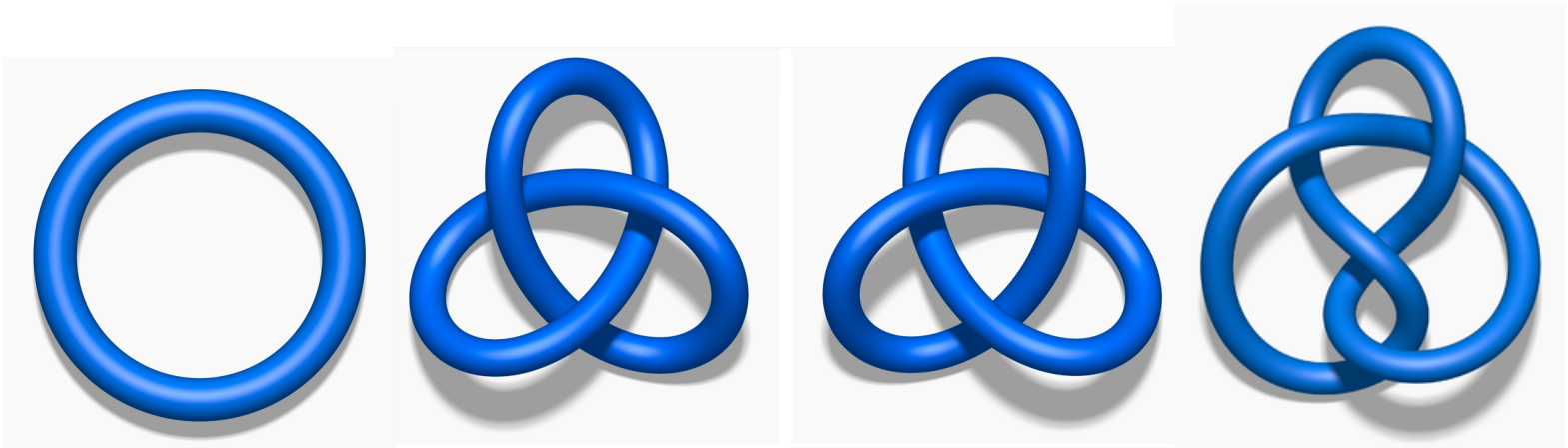

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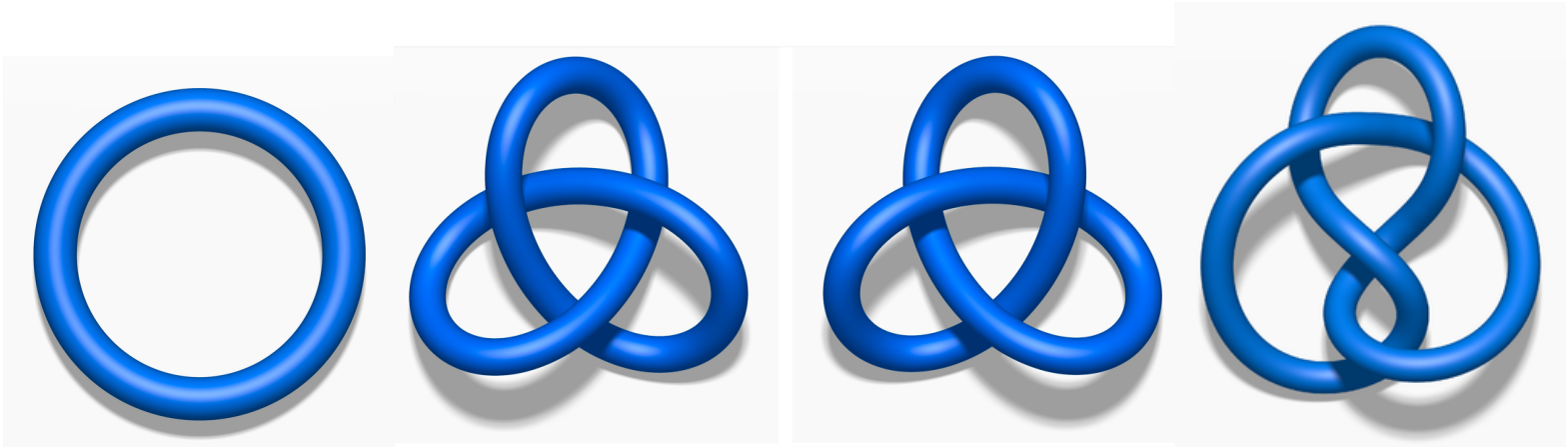


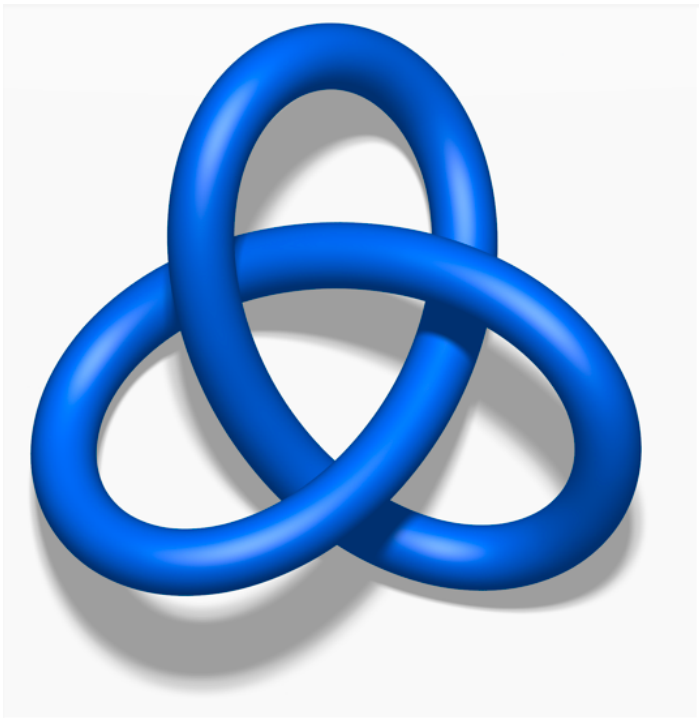
Activity

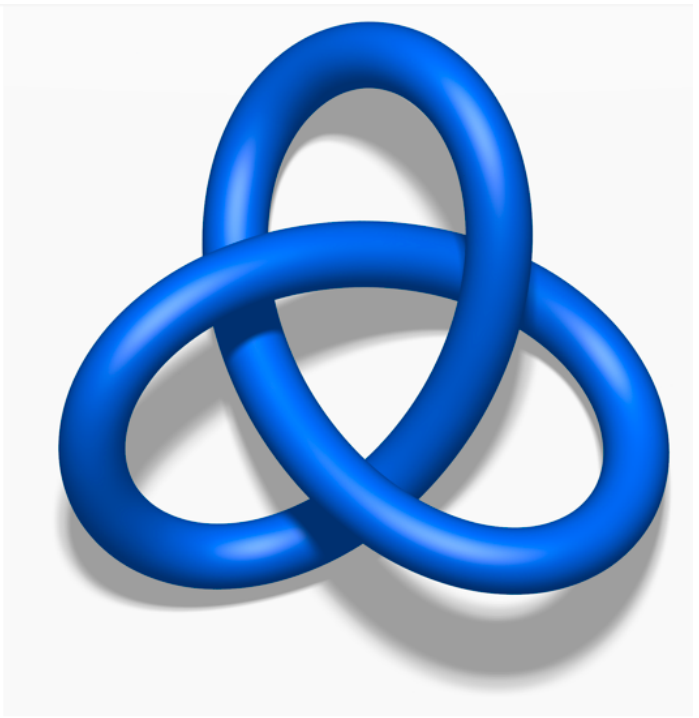
Does your knot have one of these types?

Trefoils

Activity

With a partner, **carefully** *make the left-handed and right-handed trefoil knots below. Are they the same knot type?*





Representing knots with diagrams

Mathematician's represent knots using knot diagrams.

Definition

A knot diagram is a closed loop in the plane with a choice of overstrand and understrand at each crossing (self-intersection).

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Make a different diagram for a trefoil by laying it flat and drawing what you see.

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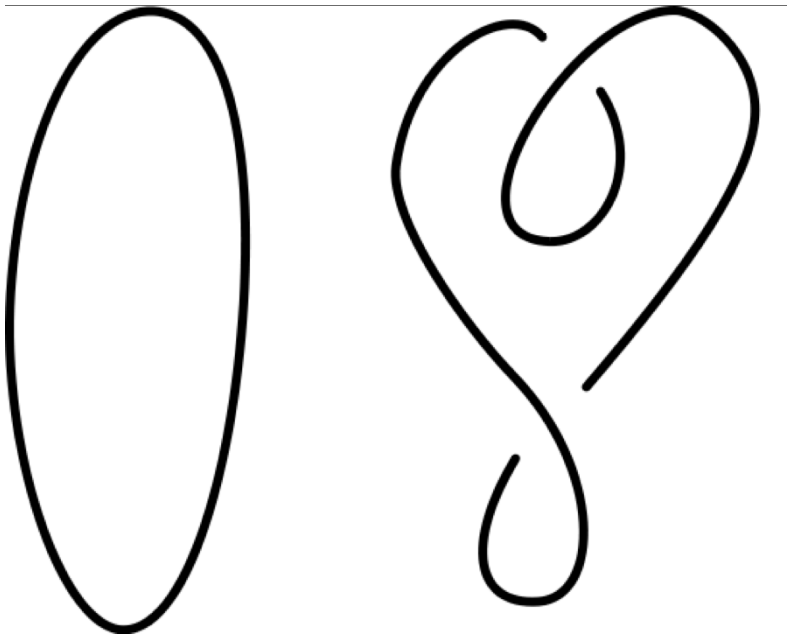
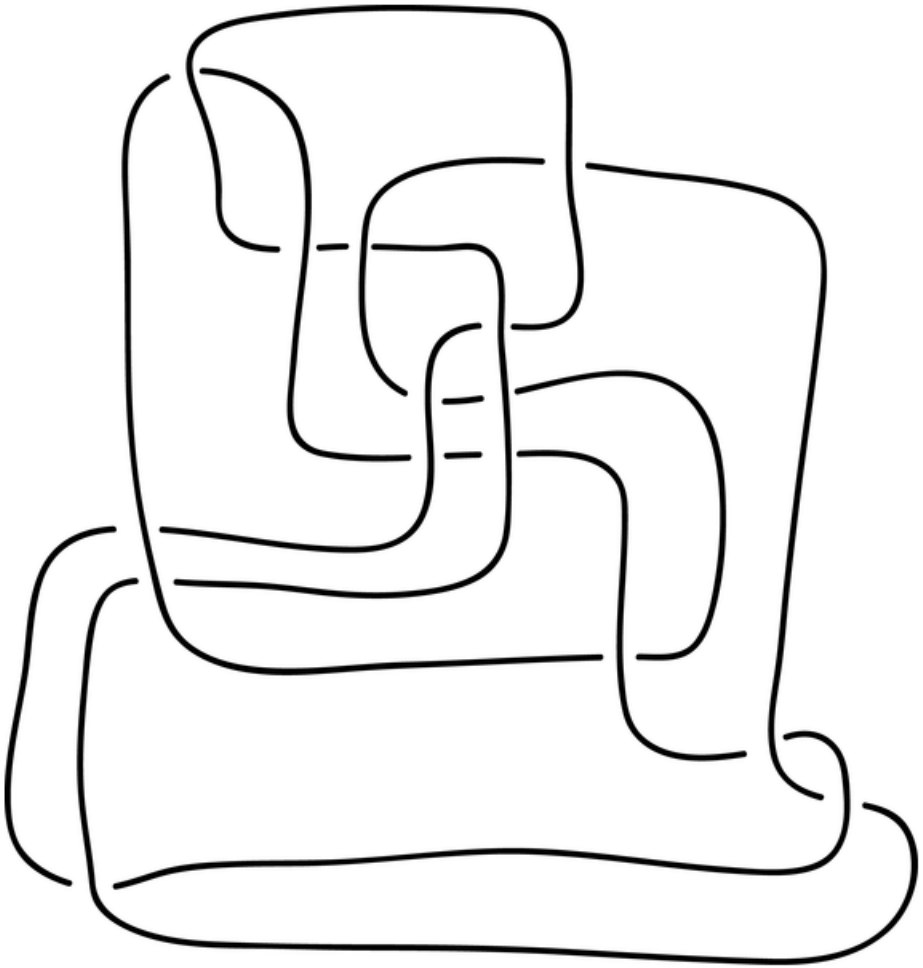
Each knot type may be represented by many different diagrams...in fact, infinitely many.

Two diagrams of the unknot



And another!



Activity

Make a messy diagram of the unknot by laying yours flat!

Fundamental questions of knot theory

Question

How can you tell if a knot is really knotted?

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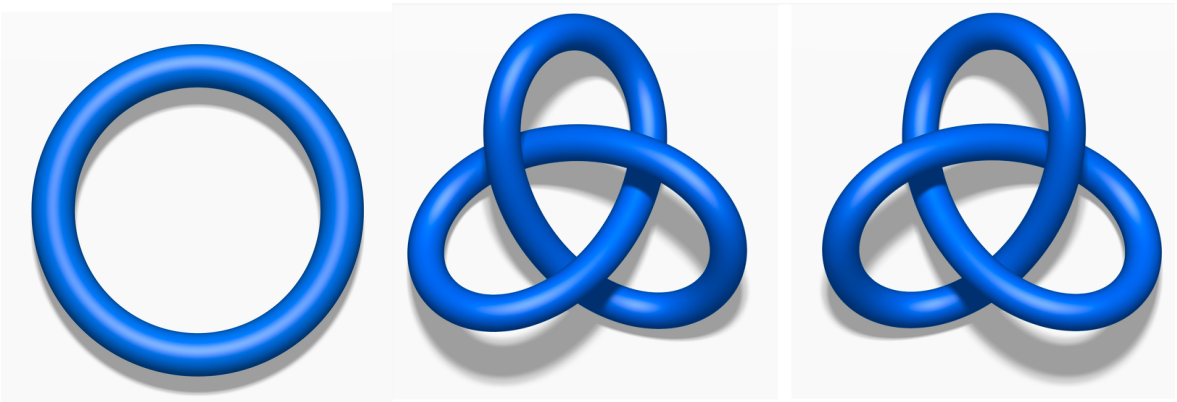
Classification:
Make a table of knot diagrams in which each knot type occurs exactly once.

Identification:
Given a knot diagram, find the unique corresponding knot type in the table.

Crossing number and the unknot

The crossing number of a knot is the minimum number of crossings in any diagram representing it.
The unknot is the unique knot with crossing number 0.
The trefoils have crossing number 3.



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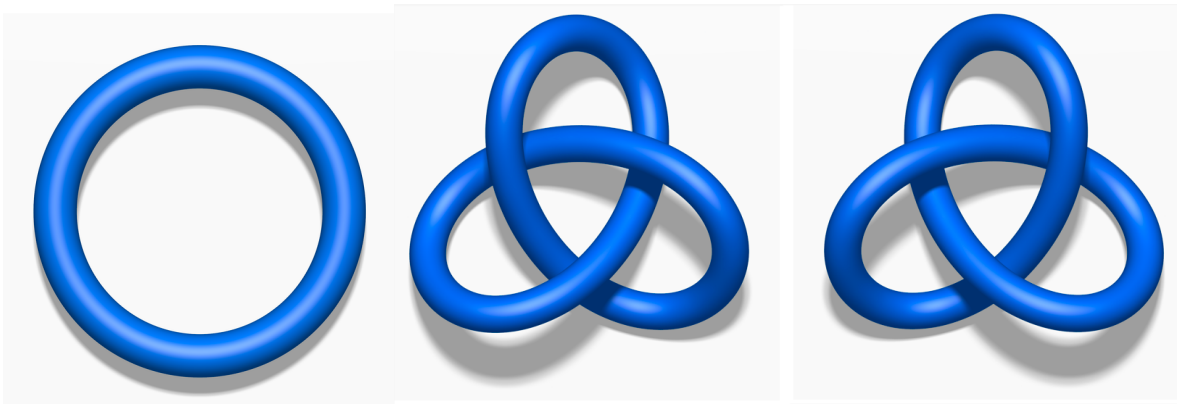
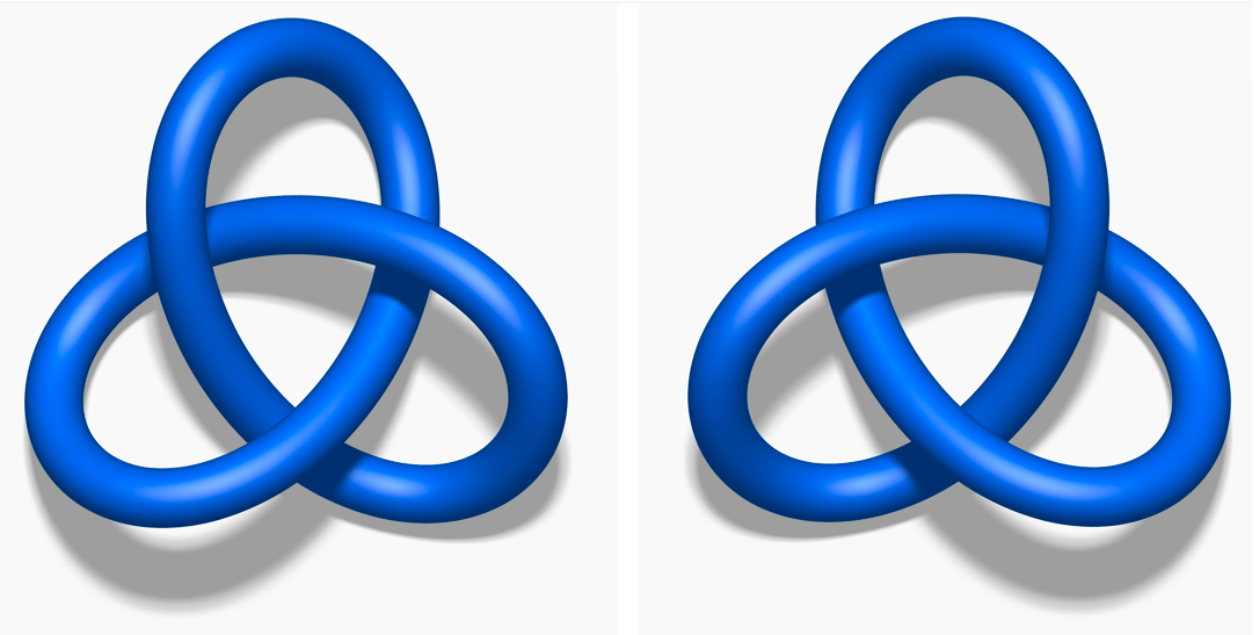


Activity

Try to make a knot with crossing number 1 or 2.

Crossing number

A knot needs at least three crossings to be knotted!
The trefoils are the unique knots with three crossings.

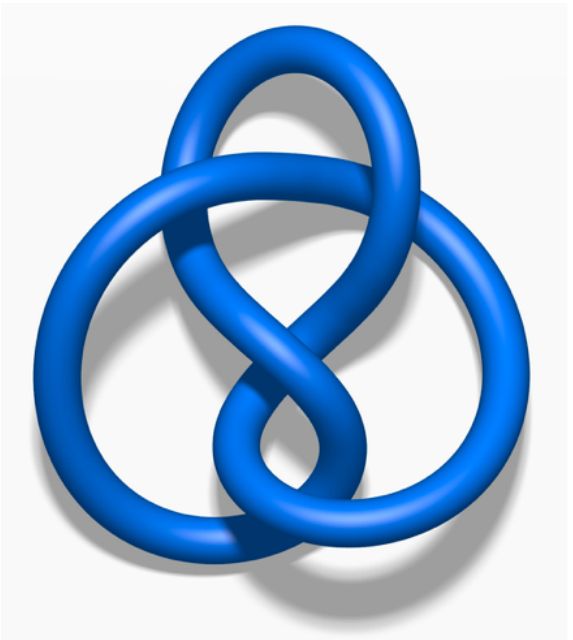


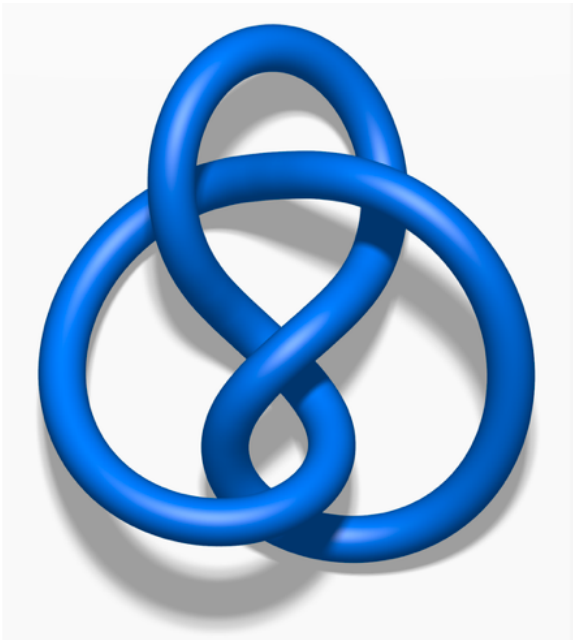
The trefoil knot is **chiral**: not equivalent to its mirror image.

Crossing number 4: The Figure 8

Activity

With a partner, **carefully** *make the left-handed and right-handed Figure 8 knots below. Are they equivalent?*





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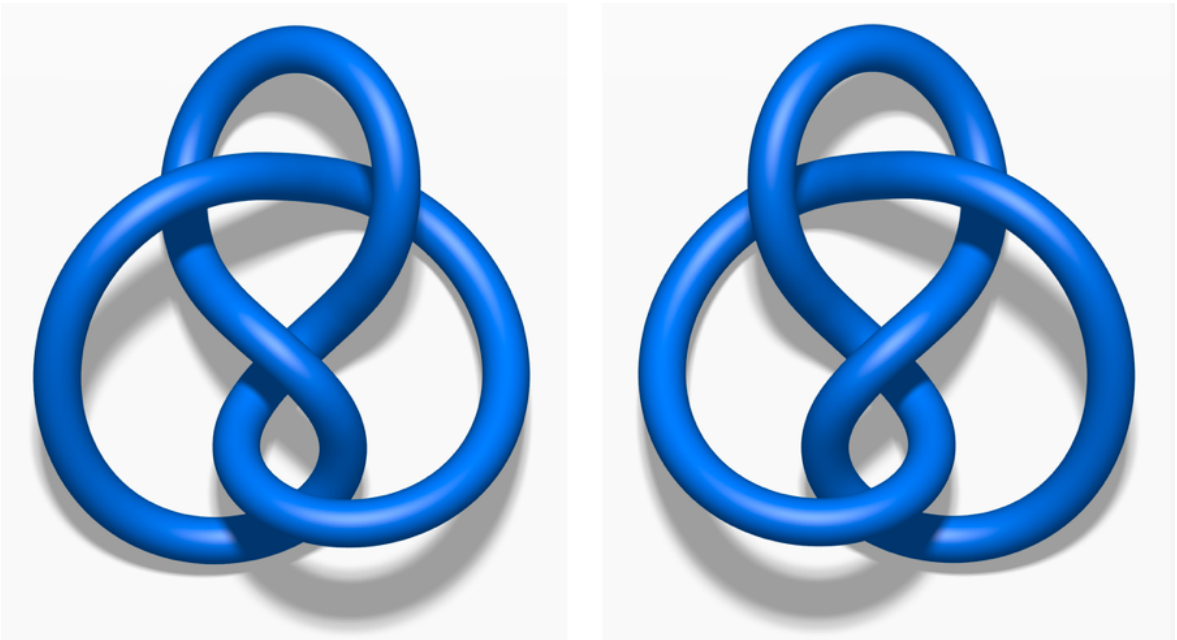
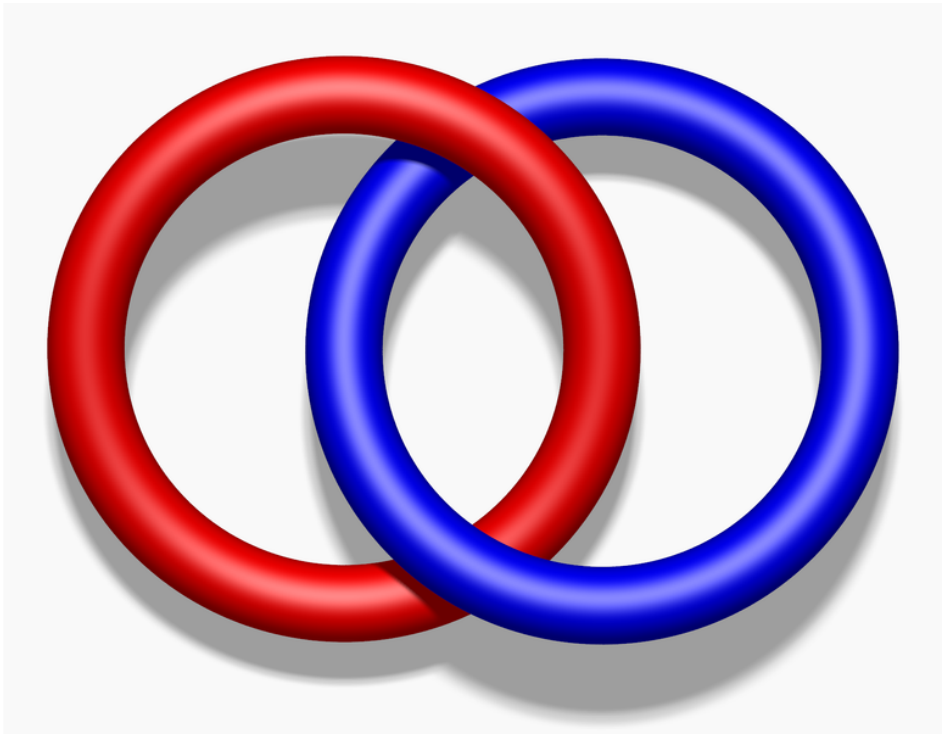


The Figure 8 knot is **achiral**: equivalent to its mirror image.

The Hopf Link

A link has multiple knotted components.
The simplest link (other than the unlink) is the Hopf link:



Activity

Combine unknots with your partner to form the Hopf link.

Torus knots

Definition

A **torus knot (or link)** is a knot that can be drawn on the surface of an inner tube (without self-intersections). The torus knot $T(m, n)$ wraps m times meridianally and n times longitudinally. See T(5,3) below.

Torus knots

Question

Which of the knots we have seen so far are torus knots?

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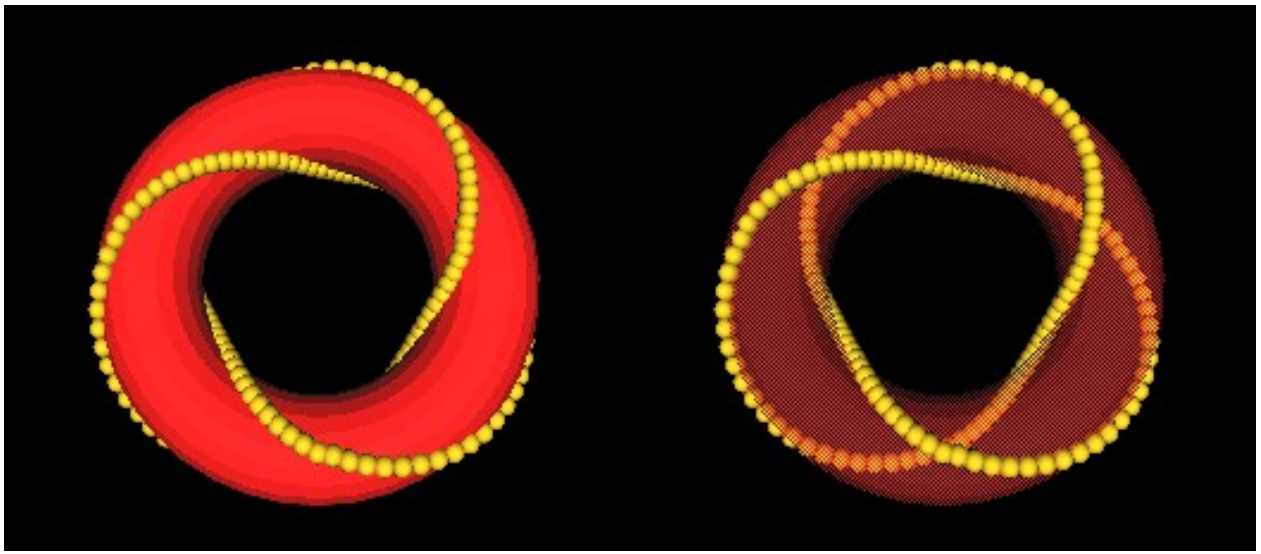
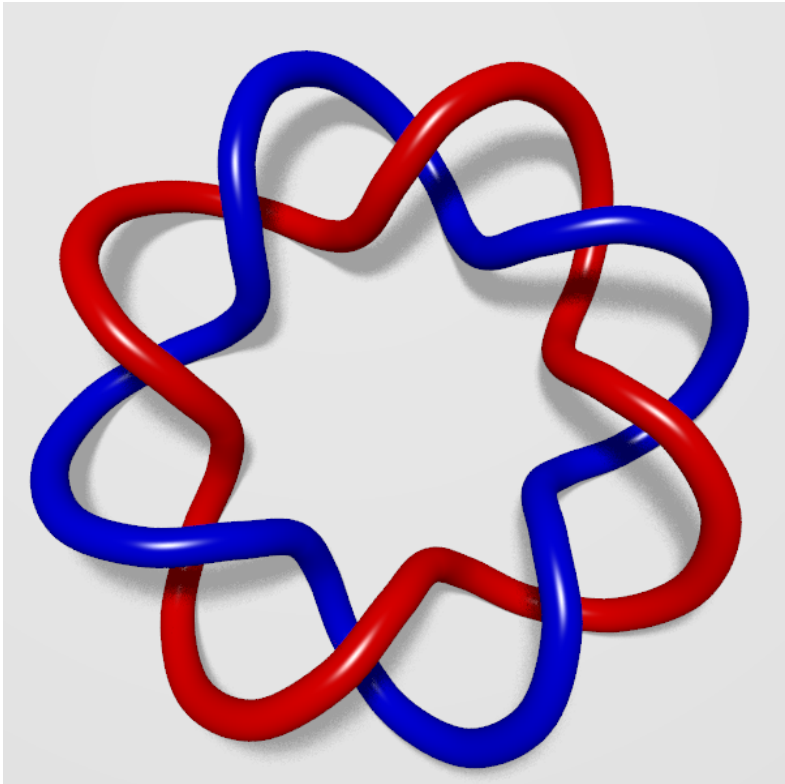


The trefoil is represented by both $T(2,3)$ and $T(3,2)$.

Fact: $T(m,n)$ is always equivalent to $T(n,m)$.

Torus links

T(4,2) is a 2-component link:



Fact: The number of components of the torus knot $T(m, n)$ is the greatest common divisor of m and n.

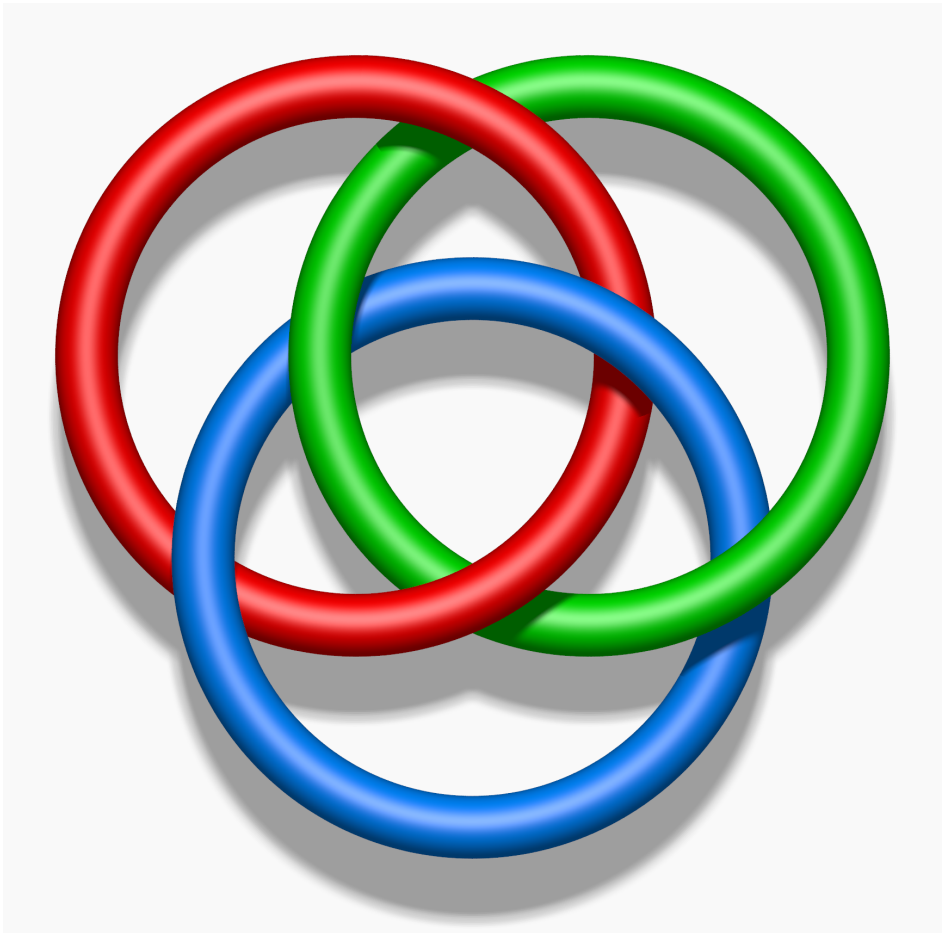
Braids

Torus knot diagrams are examples of **braids**.

Activity

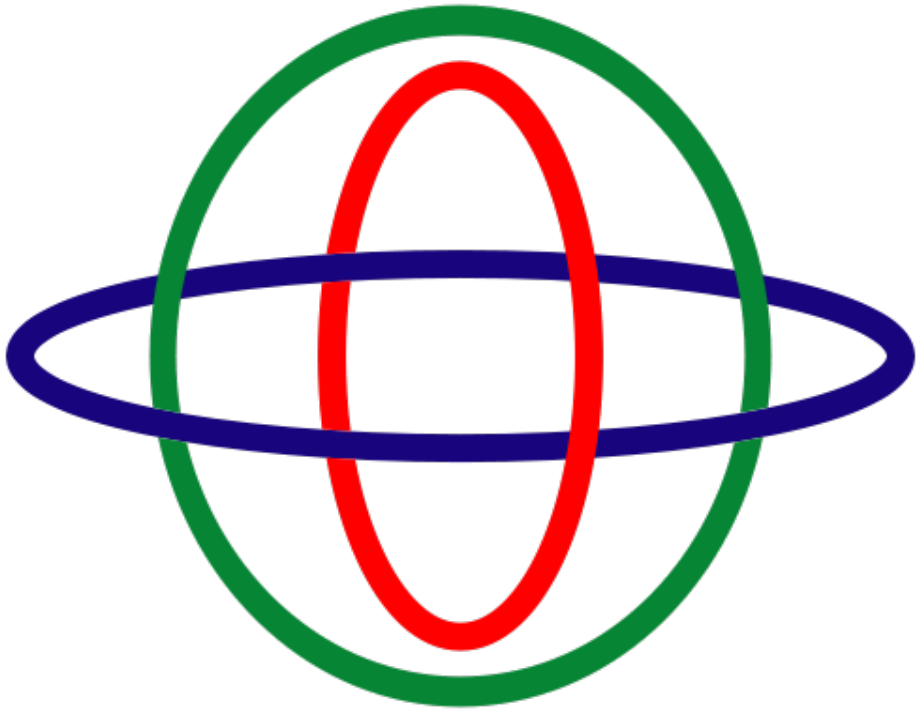
Using three strands, make a standard braid with six crossings. Then connect the top three ends to the corresponding bottom three ends to form a 3-component link.

The Borommean rings



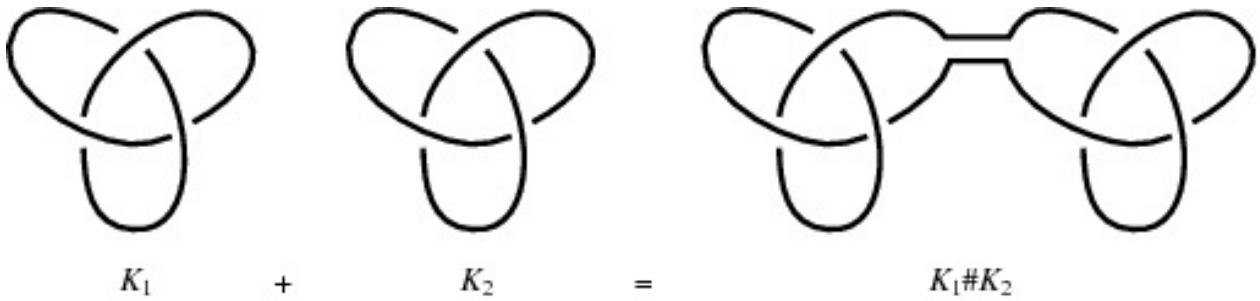
Fact: If you cut any loop, the other two come unlinked!

The Borommean rings



Fact: You cannot make the Borromean rings with rigid circles, but you can make it with ellipses.

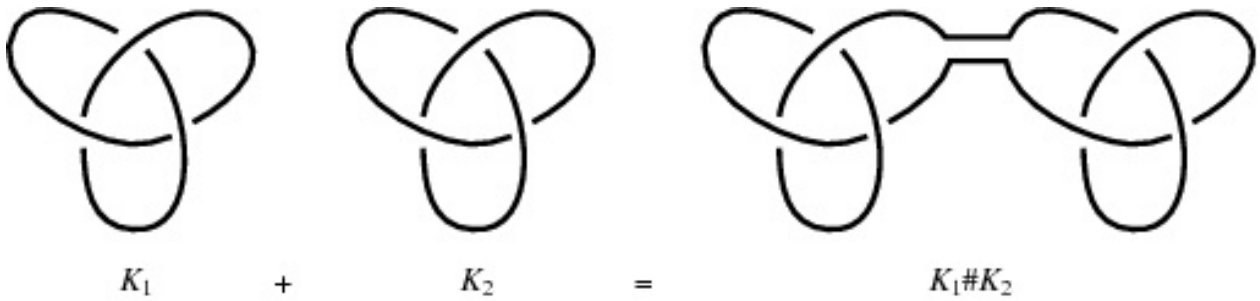
Adding knots



Activity

Add one of your knots to one of your partner's knots.

Adding knots



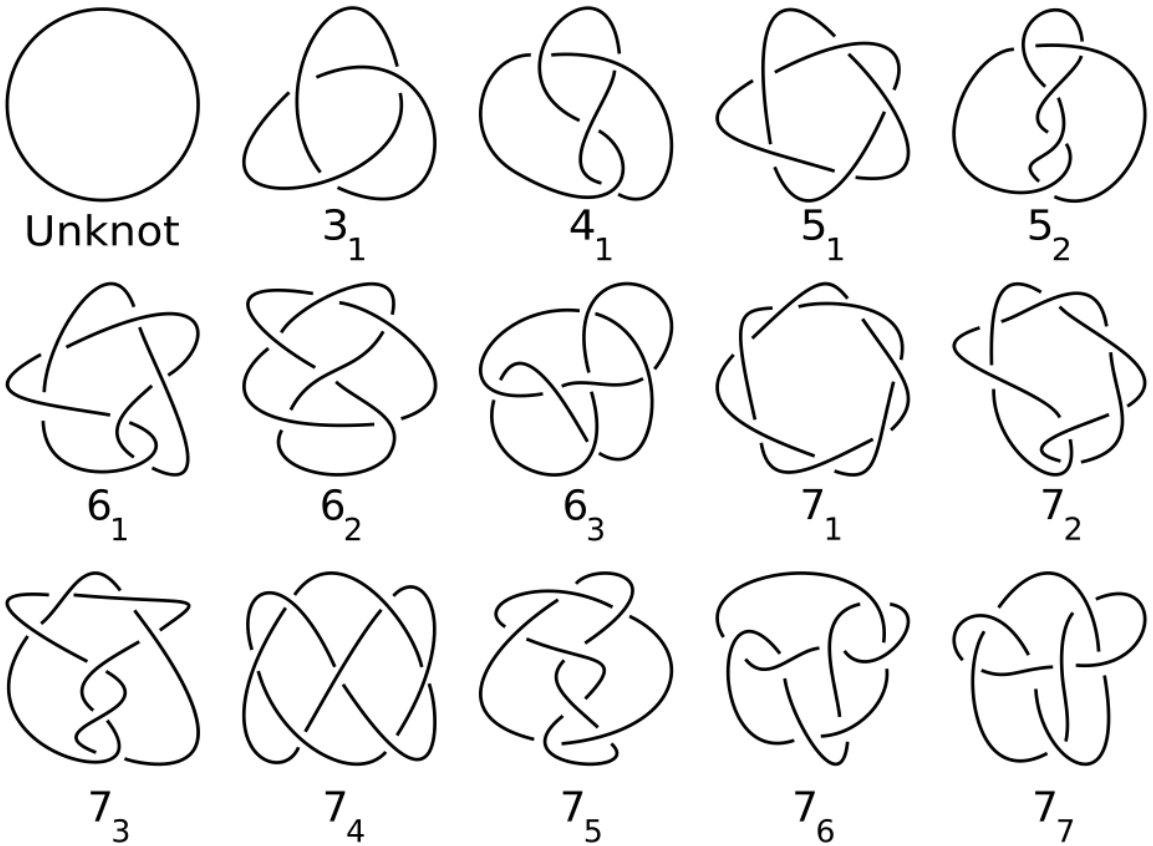
Activity

Add one of your knots to one of your partner's knots.

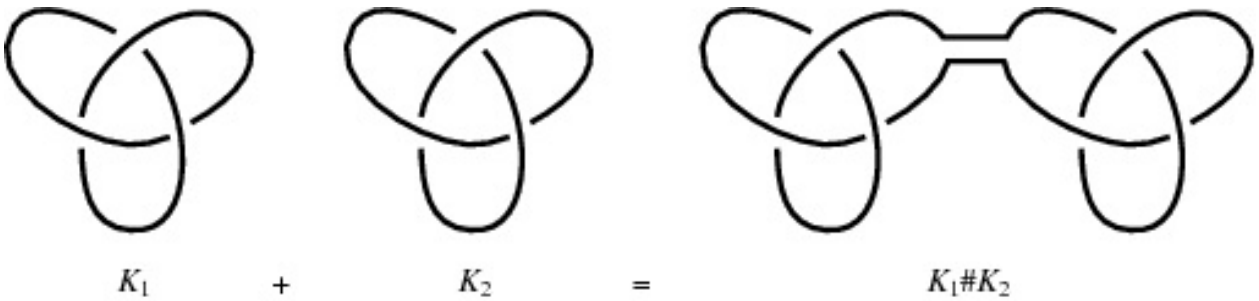
Fact: Every knot has a unique decomposition as a sum of prime knots. The unknot acts like zero for addition.

The knot table

The knot table is ordered by crossing number and shows a representative of each prime knot type, omitting mirror images.



A knot cannot unknot a knot



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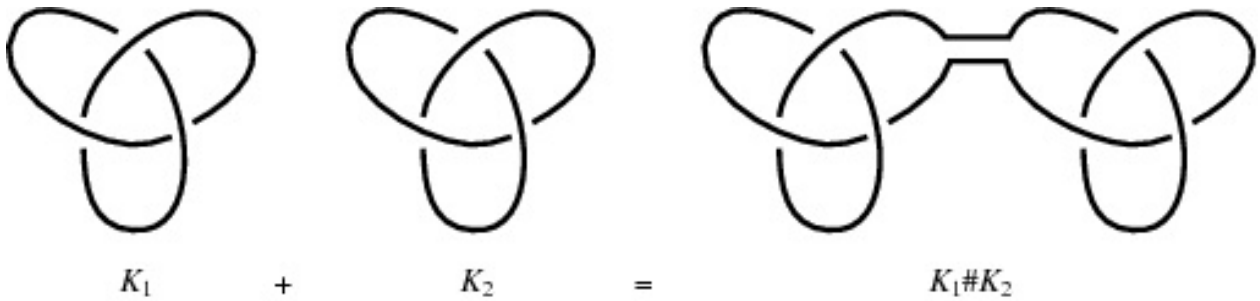


Theorem

The sum of two knots is always knotted (unless both summands are the unknot)!

Alexander's Theorem

Theorem (J.W. Alexander, 1920s)

Every link can be represented as a closed braid.

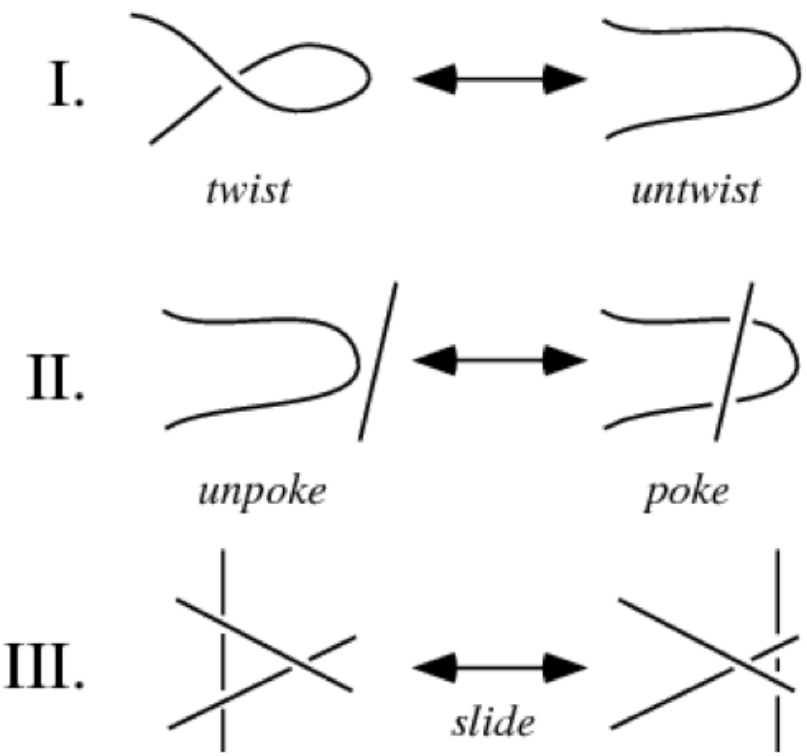
`http://www.youtube.com/watch?v=h5IErq3m1ns`

Reidemeister's Theorem

Theorem (Reidemester, Alexander-Briggs, 1927)

Two diagrams represent the same knot type if and only if the diagrams are related by a sequence of Reidemeister moves.

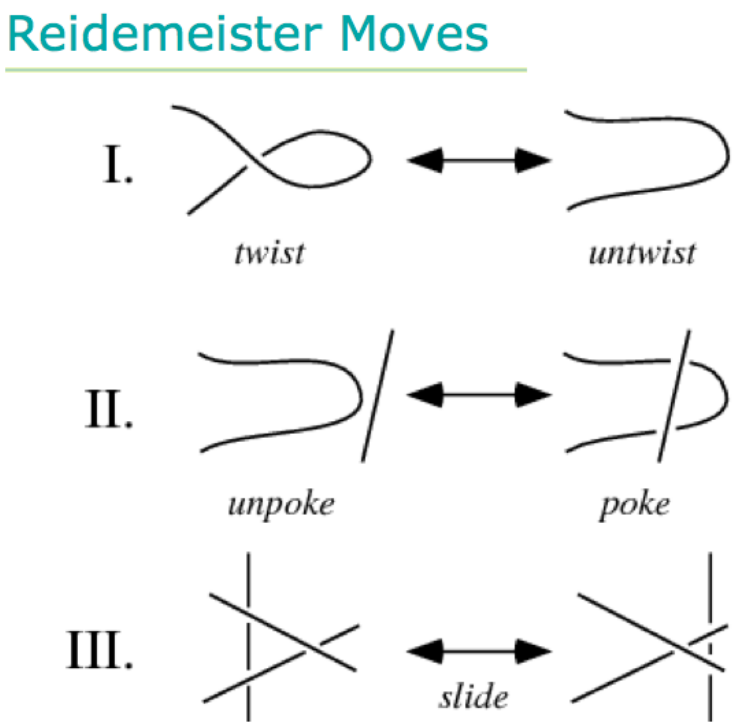
Reidemeister Moves



Reidemeister's Theorem

Activity

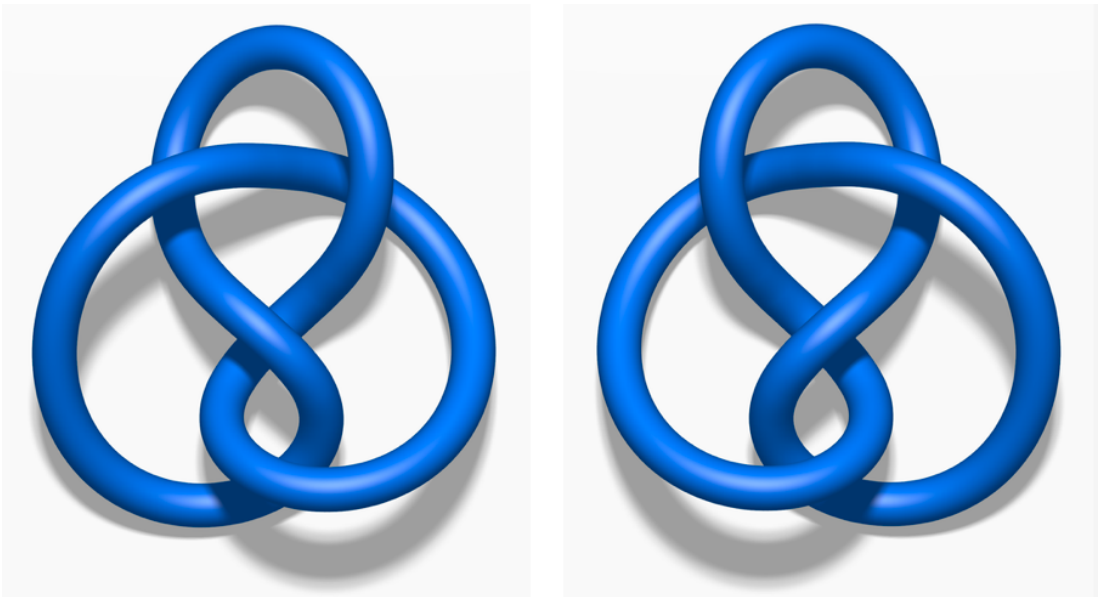
Try each of the three moves!



Those Figure 8's

Activity

Find a sequence of Reidemeister moves taking the Figure 8 to its mirror image.



http://www.youtube.com/watch?v=psxo559wJuU

Tricolorability

Definition (Tricolorable)

A knot diagram is **tricolorable** if you can color its edges so that

- All three colors are used.
- At each crossing, either all three colors are the same or all three colors are different.

A knot is tricolorable if it can be represented by a tricolorable knot diagram.

Fact: For a fixed knot type, either all the knot diagrams are tricolorable or none are tricolorable.

Linking number