Week 8 - Friday

Friday, March 19, 2021 2:27 PM

1) Stability of PDE methods

2) Convolutions $\Leftrightarrow$ finite differencing

Stability: Similar to ODE

Eg. Advection equation : $\partial_t f(x,t) + c\,\partial_x f(x,t) = 0$

suppose $f(x,t) = \underbrace{f^{true}(x,t) + \epsilon(x,t)}$

Von-Neumann Stability Analysis

$$f(x, t+\Delta t) = f(x,t) + \Delta t\, c\, \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad ☆$$

$$\underline{f^{true}(x,t+\Delta t)} + \epsilon(x,t+\Delta t) = \underline{f^{true}(x,t)} + \epsilon(x,t) + c\frac{\Delta t}{\Delta x}\left[\underline{f^{true}(x+\Delta x,t)} + \epsilon(x+\Delta x,t) - \underline{f^{true}(x,t)} - \epsilon(x,t)\right]$$

cancel

assume f^{true} obeys ☆ up to $\Delta x, \Delta t$ error

Blue terms

$$\epsilon(x, t+\Delta t) = \epsilon(x,t) + c\frac{\Delta t}{\Delta x}\left[\epsilon(x+\Delta x,t) - \epsilon(x,t)\right]$$

$\epsilon(x,t) = \int \frac{dk}{2\pi} e^{ikx}\,\tilde{\epsilon}(k,t)$ $\longrightarrow$ assume $\epsilon(x,t) = \underline{E(t)}\, e^{ikx}$ ~ (single) fourier mode

$$E(t+\Delta t)\, \cancel{e^{ikx}} = E(t)\, \cancel{e^{ikx}} + c\frac{\Delta t}{\Delta x}\left[e^{ik\Delta x} - 1\right] E(t)\, \cancel{e^{ikx}}$$

$$\underline{E(t+\Delta t)} = \underline{E(t)}\underbrace{\left[1 + c\frac{\Delta t}{\Delta x}\left(e^{ik\Delta x} - 1\right)\right]}$$

Want $|\;| \le 1$

Is this true for some $\Delta x, \Delta t$?

$$\left| 1 + c\frac{\Delta t}{\Delta x}\left(e^{ik\Delta x} - 1\right) \right| \leq 1$$

$e^{ik\Delta x} = \pm 1$

$+1: \quad \left| 1 + \frac{c\Delta t}{\Delta x} \cdot 0 \right| = 1 \leq 1$

$-1: \quad \left| 1 + -2c\frac{\Delta t}{\Delta x} \right| \leq 1$

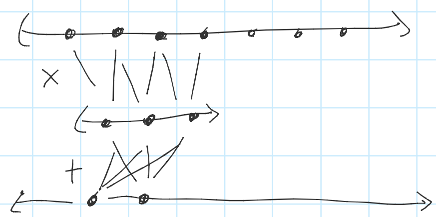
choose Δx, Δt: $\boxed{c\frac{\Delta t}{\Delta x} \leq 1}$

Finite differencing $\Longleftrightarrow$ Convolutions

a_i — Array

w_i — Window function, Kernel

c_i — final, convolved function



$$c_n = \sum_{j=i+n} a_j w_i$$