Week 8 - Monday

Monday, March 15, 2021 1:56 PM

Here be partial differential equations

Start by considering: 2-d, 2nd-order equations:

$$C_{xx}\partial_x^2 f(x,y) + C_{yy}\partial_y^2 f(x,y) + C_{xy}\partial_x\partial_y f(x,y) + C_x\partial_x f(x,y) + C_y\partial_y f(x,y) + A[f(x,y), x, y] = 0$$

Discriminant: $C_{xy}^2 - 4C_{xx}C_{yy} \equiv D$

$D > 0$: Hyperbolic systems	$D < 0$: elliptic	$D = 0$ parabolic
Correspond to 2 "characteristic" directions	No characteristic directions	Single characteristic direction.
• Wave Equation: $\partial_t^2 f - \partial_x^2 f = \text{Source}$ • Maxwell's full equations w/ time-dependence • Gravitational Waves	Poisson Eq: $(\partial_x^2 + \partial_y^2)f = S$ $= \nabla^2 f = S$ • Electro-statics problems • Time-indep. Schrodinger • Gravitating systems	• Advection Eq: $\partial_t f - \partial_x f = S$ • Navier-Stokes (Fluid Eq): $\sim \partial_t f + f\partial_x u = g, v,$ density, ... (source) • Heat Equation: (Diffusion) $\partial_t f + \partial_x^2 f = S$ • Time-dependent Schrodinger
Wave-like behavior Can pose as IVP	~Static systems pose as BVP	diffusive behavior pose as IVP

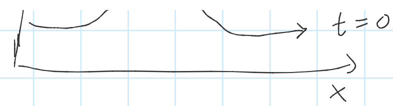
Start off with parabolic systems: advection equation

(initial value problem)

$$\partial_t f(x,t) - c\partial_x f(x,t) = 0 \implies \text{solve as IVP}$$

Know $f(x, t=0)$

What is $f(x, t>0) = ?$



what is $f(x, t>0) = ?$

any function $\underline{g(x+ct)}$ will solve the advection Equation

$$\partial_t g(x+ct) = c g'(x+ct)$$
$$\partial_x g(x+ct) = g'(x+ct)$$
$$\Rightarrow cg' - cg' = 0 \checkmark$$

Corresponds to a left-moving profile.

- c is wave "speed", sign determines direction
- Consider also:

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x) f = 0$$
$$= (\partial_t^2 - c^2 \partial_x^2) f = 0 \Rightarrow \text{wave equation}$$

general sol'ns are: $g(x+ct) + h(x-ct)$

In 3-d, $(\partial_t^2 - c^2 \nabla^2) f = 0$

$\Rightarrow$ how to solve on a computer?

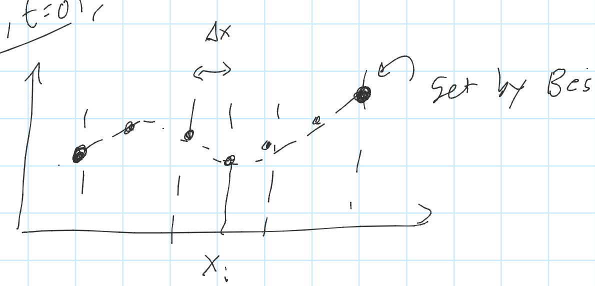
↳ as IVP similar to ODEs

↳ also need to know boundary conditions

(Common)
Boundary Conditions

- "Robin" boundary conditions
  ↳ damping, impeding $\partial_x f = kf$
- Neumann
  ↳ fix derivative $\partial_x f$ @ boundaries
- Dirichlet
  ↳ fix value of f
- Map ∞ in: open boundary
- Mixed versions of ↑

Know $\underline{f(x_i, t=0)}$, want $f(x_i, t=\Delta t)$



$$\partial_t f(x,t) = \frac{f(x, t+\Delta t) - f(x,t)}{\Delta t} = c\partial_x f = c\,\frac{f(x+\Delta x, t) - f(x,t)}{\Delta x}$$

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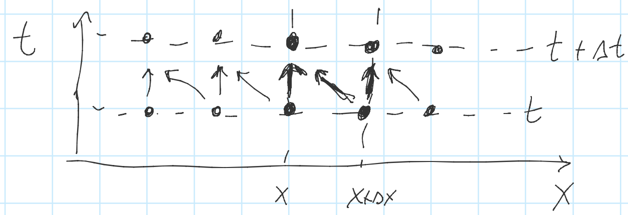
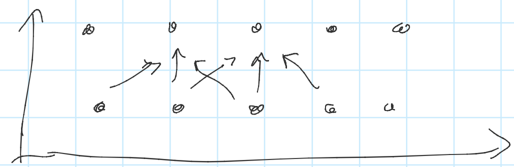
$$f(x,t+\Delta t) = f(x,t) + c\,\frac{\Delta t}{\Delta x}\Big(f(x+\Delta x,t) - f(x,t)\Big)$$

- Euler's method for time-integ.
- "forward" finite difference for $\partial_x f$



Center-finite diff. scheme: $\partial_x f = \frac{f(x+\Delta x,t) - f(x-\Delta x,t)}{2\Delta x} + \mathcal{O}(\Delta x^2)$



Mind your Boundaries: