Week 7 - Monday

Monday, March 8, 2021 2:20 PM

1) Error growth in ODEs, "Implicit" methods

2) Numerical integration when "RHS" is known
also known as "quadrature"

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1) Error growth - Stability, quantify how quickly error is growing -

Solve: $f'(x) = A(f(x), x)$

e.g. Euler's Method: $f(x+\Delta x) = f(x) + \Delta x \cdot A(f(x), x) + \mathcal{O}(\Delta x^2)$

Suppose $f(x) = \underbrace{f^{true}(x)}_{\text{Exact}} + \underbrace{\epsilon(x)}_{\text{Error}}$

$$\underline{f^{true}(x+\Delta x)} + \underline{\epsilon(x+\Delta x)} = f^{true}(x) + \epsilon(x) + \Delta x\, A\left(f^{true}(x) + \epsilon(x), x\right) + \mathcal{O}(\Delta x^2)$$

$$= \underline{f^{true}(x)} + \underline{\epsilon(x)} + \underline{\Delta x\, A(f^{true}(x), x)} + \underline{\Delta x\, \epsilon(x) \frac{\partial A}{\partial f}\Big|_x} + \mathcal{O}(\Delta x^2)$$

$$\Rightarrow \quad \underline{\epsilon(x+\Delta x)} = \underline{\epsilon(x)} \underbrace{\left(1 + \Delta x \frac{\partial A}{\partial f}\Big|_x\right)}$$

Want $\left|1 + \Delta x \frac{\partial A}{\partial f}\Big|_x\right| < 1$ for stability (error is shrinking).

"A-stability": how does error behave for $f'(x) = \lambda f(x) = A(f(x), x)$

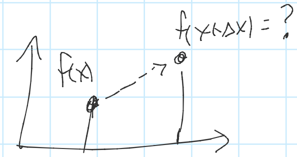
(Sol'n $f(x) = e^{\lambda x}$)

$\Rightarrow \frac{\partial A}{\partial f} = \lambda$

Specific to Euler's Method
$$\begin{cases} \text{When } \lambda > 0, \ |1 + \lambda \Delta x| > 1 \Rightarrow \text{"Unstable"} \\ \lambda < 0, \ |1 - |\lambda| \Delta x| < 1 \Rightarrow \text{Stable depending on } \Delta x \ (\text{need } \Delta x < 2/\lambda) \\ \lambda = \pm i\omega \quad |1 \pm i\omega\Delta x| = \sqrt{1+\omega^2} > 1 \Rightarrow \text{Unstable.} \end{cases}$$

"Implicit methods" | Back to Euler's method: $f(x)$ $f(x+\Delta x) = ?$

"Implicit methods" | Backwards Euler method:



$$f'(x) = A(f(x), x)$$

$$f'_{(x+\Delta x)} = A(f_{(x+\Delta x)}, x+\Delta x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$0 = -f_{(x+\Delta x)} + f(x) + \Delta x\, A(f_{(x+\Delta x)}, x+\Delta x)$$

↳ want to solve for $f_{(x+\Delta x)}$

find a zero of this f^n, eg. using a root finder.

Stability of Backwards Euler

again, $f(x) = f^{true}(x) + \epsilon(x)$

$$f^{true}(x+\Delta x) + \epsilon(x+\Delta x) = f^{true}(x) + \epsilon(x) + \Delta x\, A(f_{(x+\Delta x)} + \epsilon(x+\Delta x), x+\Delta x)$$

$$= f^{true}(x) + \epsilon(x) + \Delta x\, A(f^{true}(x+\Delta x), x+\Delta x) + \Delta x\, \epsilon(x+\Delta x) \left.\frac{\partial A}{\partial f}\right|_{x+\Delta x}$$

$$\epsilon(x+\Delta x) = \epsilon(x) + \epsilon(x+\Delta x) \cdot \Delta x \left.\frac{\partial A}{\partial f}\right|_{x+\Delta x}$$

$$\epsilon(x+\Delta x) = \epsilon(x) \frac{1}{\left|1 - \Delta x \frac{\partial A}{\partial f}\right|}$$

again, consider $\frac{\partial A}{\partial f} = \lambda$

Want $\left|\frac{1}{1-\lambda\Delta x}\right| < 1$

or $|1 - \lambda\Delta x| > 1$

$\lambda > 0$, unstable or stable depending on Δx

$\lambda < 0$, stable

$\lambda = \pm i\omega$, $|1 \pm i\omega| = \sqrt{1+\omega^2}$, stable

⇒ better than forwards Euler for stability.

Other implicit methods can have even better stability. Python: Radau, BDF

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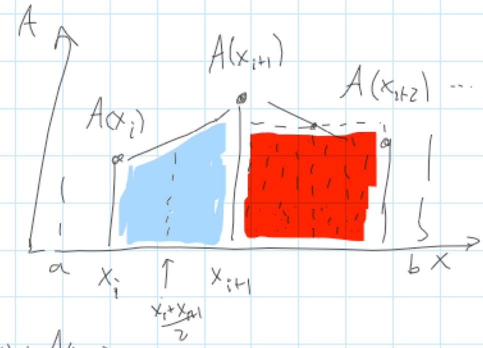
2) Numerical Integration: $A(\cancel{A}(x), x) \Rightarrow A(x)$

Solving $f'(x) = A(x)$

Want to know $I = \int_a^b A(x)\, dx$



Suppose we can tabulate $A(x)$

- Trapezoid Rule:

$$I = \sum_i \Delta x_i \frac{A(x_i) + A(x_{i+1})}{2} = T, \quad \text{error is: } -\frac{1}{12}(\Delta x)^3 A''(x)$$

- Midpoint Rule:

$$\sum_i \Delta x_i\, A\left(\frac{x_i + x_{i+1}}{2}\right) = M, \quad \text{error is: } \frac{1}{24}(\Delta x)^3 A''(x)$$

$$C_T T + C_M M = I$$

$$\left.\begin{aligned} C_T + C_M &= 1 \\ C_T\left(-\tfrac{1}{12}(\Delta x)^3 A''\right) + C_M \tfrac{1}{24}\Delta x^3 A'' &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_T &= 1/3 \\ C_M &= 2/3 \end{aligned}$$

$$I = \sum_i \Delta x_i \left(\frac{A(x_i) + A(x_{i+1})}{3 \cdot 2} + \frac{4}{6} A\left(\frac{x_i + x_{i+1}}{2}\right) \right) = \sum_i \frac{\Delta x_i}{6}\left(A(x_i) + 4A\left(\frac{x_i + x_{i+1}}{2}\right) + A(x_{i+1}) \right)$$

Simpson's rule,

Integration of 2nd-order poly spline.

$\Rightarrow$ "Romberg Integration" is generalization to higher order

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Special case: $I = \int_a^b A(x)\, dx$, what if $a \to -\infty$, $b \to \infty$

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$b \to \infty$
or $A(x) \to \pm\infty$?

eg. $$I = \int_{1/b}^{1/a} \frac{1}{t^2} A\left(\frac{1}{t}\right) dt,$$ now if $a \to \infty$, $b \to \infty$,

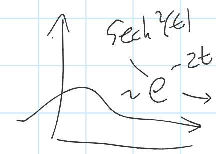
for a, b infinite

$$I = \int_c^d A(x(t)) \frac{dx}{dt}\,dt$$

choose $\frac{dx}{dt} \to 0$ where $A \to \infty$

"Tanh" rule: $x = \frac{1}{2}(b-a) + \frac{1}{2}(b-a)\tanh(t)$

$$\frac{dx}{dt} = \frac{1}{2}(b-a)\,\text{sech}^2(t)$$



→ falls to zero quickly!

⇒ Python: contains implementations of Romberg Integration,
scipy.integrate.quad handles many ∞'s