Week 11 - Friday

Friday, April 9, 2021 12:32 PM

More on FFTs

- Convolutions: $\int h(t') g(t-t')\, dt' \Longleftrightarrow \tilde{g}(\nu)\, \tilde{h}(\nu)$

  how does some signal h respond to g? g is a window fn or kernel

- Correlations: $\int h^*(t')\, g(t+t')\, dt' \Longleftrightarrow \tilde{g}(\nu)\, \tilde{h}^*(\nu)$

  how much of g is in h?

  if $h = g$, correlation will be $\tilde{g}\tilde{g}^* = |\tilde{g}|^2$

  "autocorrelation"

  Many names, incl. Power Spectrum, Power Spectral Density, Energy Spectrum... Correlation function, Autocorrelation, two-point fn, ...

Computing derivatives in Fourier space

$$\partial_t f(t) = \partial_t \int \tilde{f}(\nu)\, e^{2\pi i \nu t}\, d\nu$$

$$= \int \underbrace{2\pi i \nu\, \tilde{f}(\nu)}\, e^{2\pi i \nu t}\, d\nu$$

just multiply fourier modes to take derivative.

$$\partial_t^2 f(t) = -\int (2\pi\nu)^2\, \tilde{f}(\nu)\, e^{2\pi i \nu t}\, d\nu$$

how does this look for finite differences, + discrete transforms?

DFT version: $\partial_t^2 f(t_n) = \sum_m \tilde{f}(\nu_n)\, e^{2\pi i m n/N} \cdot -(2\pi\nu_n)^2$ ☆

$\nu_n = m/T$, T is a sampling period, $T = N\Delta t$

$$\partial_t^2 f(t) = \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} = \frac{f(t_{n+1}) - 2f(t_n) + f(t_{n-1})}{\Delta t^2}$$

$$= \frac{1}{\Delta t^2} \sum_m \tilde{f}(\nu_m) \left(e^{2\pi i m(n+1)/N} + e^{2\pi i m(n-1)/N} - 2 e^{2\pi i m n/N} \right)$$

$$= \frac{1}{\Delta t^2} \sum_m \tilde{f}(\nu_m) e^{2\pi i m n/N} \Big(\underbrace{e^{2\pi i m/N} + e^{-2\pi i m/N} - 2}_{-4\sin^2\left(\frac{2\pi m}{2N}\right)} \Big)$$

$-4\sin^2\left(\frac{2\pi m}{2N}\right)$ → expand for small $\frac{m}{N}$

$$\simeq -\frac{4\pi^2 m^2}{N^2} \qquad \frac{m}{N} = \Delta t\, \nu_m$$

$$= -4\pi^2 \nu_m^2 \Delta t^2$$

So for small m/N,

$$\simeq \sum_m \tilde{f}(\nu_m) e^{2\pi i m n/N} \cdot -(2\pi \nu_m)^2$$ ⇒ only seems to agree

↳ Can go through same argument for higher-order derivs.

Can use ffts to come up with more accurate methods for computing derivs.

FFts: $N \log N$ vs. $N \cdot$ (# terms in F.D. method)

↑ Worth using fft? ↑