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Problem 1: **a)** We can minimize the χ^2 value by setting the derivative to zero and proceeding to solve for the coefficients a_i.

$$\begin{aligned}
\frac{d\chi^2}{da_j} &= 2\int_0^1 \left(g(x) - \sum_i a_i x^i\right)(-x^j) = 0 \\
\Rightarrow \qquad & \int_0^1 \left(\sum_i a_i x^{i+j} - x^j g(x)\right) = 0 \\
\Rightarrow \qquad & \sum_i a_i \int_0^1 x^{i+j} = \int_0^1 x^j g(x) \\
\Rightarrow \qquad & \sum_i \frac{1}{i+j+1} a_i = \int_0^1 x^j g(x)
\end{aligned}$$

This last expression has the form $\mathsf{A}\vec{a} = \vec{b}$. The components of the vector b are given by

$$b_j = \int_0^1 x^j g(x)\,.$$

Once given a particular function $g(x)$, these components can be evaluated.

b) We can write out the left-hand side explicitly to gain some insight. The sum runs from $i = 0$ to a given number n. For $n = 2$, we will have three equations,

$$\begin{aligned}
j = 0: &\quad a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 \\
j = 1: &\quad \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 \\
j = 2: &\quad \frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2\,.
\end{aligned}$$

This is of the form $\mathsf{A}\vec{a}$, where

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \qquad \mathsf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

For general n, the matrix will have components exactly as given by the Hilbert matrix,

$$\mathsf{A}_{ij} = \frac{1}{i+j+1}\,.$$