Week 11 - Monday

Monday, April 5, 2021 12:03 PM

$M^{4/5}$ $F^{4/9}$ → Fourier Transforms

X $F^{4/16}$ → Project (group ok) (~10-15m presentation or paper/Notebook)

1) Random Algs. (Monte-carlo etc) ← $M^{4/19}$ $F^{4/23}$

$M^{4/26}$ $F^{4/30}$ → External speakers

$M^{5/3}$ → Presentation day

2) { python threads, Maybe CLI stuff - tentative, other languages? }

3) Mathematica - Wolfram products day ↳ sympy

→ No graded labs, Short Hw's

Fourier Transforms

- Review of fourier series, transforms, properties, ...
- Discrete decomposition
- Few applications of FTs.

Applications:
- Signal, Image processing + decomposition { • Frequency analysis, filtering • Image compression, ⋮ }
- Simulation techniques
- Differential Eq. Soln's → Useful for "linear" systems

  $\nabla^2 f = \rho \Rightarrow -k^2 \tilde{f} = \tilde{\rho}$

  Solve for f
- Wavelike-diffractive behavior

- Taylor expansions - good for fn's in the vicinity of a point
  ↳ discontinuities, periodic functions ⇒ does poorly
- Fourier Expansions - Work well for periodic, discontinuous functions

Series expansion: $f(t)$, period $= 1$

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(2\pi n t) + \sum_{n=0}^{\infty} b_n \sin(2\pi n t)$$

works for any $f(t)$, $t \in [0,1]$, need finite # mins, maxes, discontinuities

Sins, coss, form an othogonal basis

eg.
$$\begin{cases} \int_0^1 \sin(2\pi m t)\sin(2\pi n t)\,dt = \frac{1}{2}\delta_{mn} \\ \int_0^1 \cos(2\pi m t)\cos(2\pi n t)\,dt = \frac{1}{2}\delta_{mn} \\ \int_0^1 \cos(2\pi m t)\sin(2\pi n t)\,dt = 0 \end{cases}$$

See also:
Hartley Transform
sin-cos / even-odd decomp.

$$a_n = \frac{1}{2}\int_0^1 f(t)\cos(2\pi n t)\,dt$$

$$b_n = \frac{1}{2}\int_0^1 f(t)\sin(2\pi n t)\,dt$$

Usually write as: $f(t) = \sum_{n=\infty}^{\infty} c_n e^{2\pi i n t}$

$$c_n = \begin{cases} (a_n - i b_n)/2 & n > 0 \\ a_0/2 & n = 0 \\ (a_n + i b_n)/2 & n < 0 \end{cases}$$

$$c_n = \int_0^1 f(t)\, e^{-2\pi i n t}\,dt$$

Fourier Transform: Take limit where period $\to \infty$,
$c_n \to$ become a function $\to \tilde{f}(\nu)$

$$\begin{cases} f(t) = \int_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{2\pi i \nu t}\,d\nu \\ \tilde{f}(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \nu t}\,dt \end{cases}$$

$$\widetilde{f(t)+g(t)} \to \tilde{f}(\nu) + \tilde{g}(\nu) \qquad \text{additive}$$

$$\widetilde{f(\alpha t)} \to \frac{1}{|\alpha|}\tilde{f}\left(\frac{\nu}{\alpha}\right) \qquad \text{Scaling}$$

$$\widetilde{f(t-t_0)} \to e^{2\pi i \nu t_0}\tilde{f}(\nu)$$

Conventions:

Unitary	Frequency	
✓	Ordinary	$\tilde{f} = \int f e^{-2\pi i \nu t}\,dt, \quad f = \int \tilde{f} e^{2\pi i \nu t}\,d\nu$
✓	angular	$\tilde{f} = \frac{1}{\sqrt{2\pi}}\int f e^{-i\omega t}\,dt, \quad f = \frac{1}{\sqrt{2\pi}}\int f e^{i\omega t}\,d\omega$
X	angular	$\tilde{f} = \int f e^{-i\omega t}\,dt, \quad f = \frac{1}{2\pi}\int f e^{i\omega t}\,d\omega$

Numerical work.

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m labels freq. $\qquad$ $N =$ # samples

$$\tilde{f}(\nu_m) = \sum_{n=0}^{N-1} f(t_n) e^{-2\pi i n \cdot m/N}$$

$$f(t_n) = \sum_{m=0}^{N-1} \tilde{f}(\nu_m) e^{2\pi i n m/N}$$

n labels time "step" / samples

Proof using completeness relation:

$$\sum_{m=0}^{N-1} e^{2\pi i m/N \cdot (n-n')} = N\delta_{n,n'}$$

- Direct transform requires $\mathcal{O}(N^2)$ operations
- Cooley-Turkey alg: $\mathcal{O}(N\log_2 N)$ operations
  - split sums into $\sum_{\text{even}} + \sum_{\text{odd}}$,

$$\tilde{f}(\nu_n, n < N/2 - 1) = \sum_{\text{even}} + e^{-2\pi i n/N}\sum_{\text{odd}}$$

$$\tilde{f}(\nu_n, n \geq N/2) = \sum_{\text{even}} - e^{-2\pi i n/N}\sum_{\text{odd}}$$

$\mathcal{O}(N)$ sum $\implies \mathcal{O}(N\log N)$.

$O(N)$ sum $\Longrightarrow O(N \log N)$.

repeat even-odd split, $O(\log N)$

Useful for $N = 2^p$

Similar algs useful for $3^p, 5^p, 7^p \ldots$ or combos.

Convolutions: given $h(t)$, $g(t)$

$$(h \otimes g)(t) \equiv \int_{-\infty}^{\infty} h(t')\, g(t-t')\, dt'$$

fourier transform

$$\widetilde{(h \times g)}(\nu) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt\, h(t')\, g(t-t')\, e^{2\pi i \nu t}$$

$$= \int_{-\infty}^{\infty} dt'\, h(t') \underbrace{\int_{-\infty}^{\infty} dt\, g(t-t')\, e^{2\pi i \nu t}}$$

$$= \underbrace{\int_{-\infty}^{\infty} dt'\, h(t')\, e^{2\pi i \nu t'}}\, \tilde{g}(\nu)$$

$$= \tilde{g}(\nu) \cdot \tilde{h}(\nu)$$