Problem 1: a) For a symmetric A, the components are

$$\mathsf{A} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} .$$

b) The eigenvalues are roots λ of the polynomial $\det(\lambda\mathsf{A} - I) = 0$. This polynomial is

$$(14 - \lambda)(11 - \lambda) - 4 = 0 \,,$$

and its roots are $\lambda = 10, 15$.

c) The eigenvectors can be found by solving

$$\mathsf{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

for α and β, for both eigenvalues.

For the eigenvalue $\lambda = 10$, I find $\alpha = 1/\sqrt{5}$ and $\beta = 2/\sqrt{5}$.

For the eigenvalue $\lambda = 15$, I find $\alpha = -2/\sqrt{5}$ and $\beta = 1/\sqrt{5}$.

Note that these eigenvectors can be multiplied by -1 and remain eigenvectors.

d) For the matrix

$$\mathsf{B} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

the diagonalized matrix is

$$\begin{aligned} \mathsf{B}^{\mathsf{T}}\mathsf{A}\mathsf{B} &= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix} . \end{aligned}$$

e)

$$\mathsf{B}^{\mathsf{T}}\mathsf{B} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

f) There are a number of ways to find the rotation angle. One method is to notice that the elements of B should correspond to a standard rotation matrix. In this case, the angle is $\cos\theta = 1/\sqrt{5}$, or $\theta = 63.43\,\mathrm{deg} = 1.107\mathrm{rad}$.

g) For this, notice that $\vec{x}^{\mathsf{T}}\mathsf{A}\vec{x} = \vec{x}'^{\mathsf{T}}(\mathsf{B}^{\mathsf{T}}\mathsf{A}\mathsf{B})\vec{x}'$. Expanding this expression and casting into the necessary form, I find $\alpha_1 = \sqrt{5/2}$ and $\alpha_2 = \sqrt{5/3}$.

Problem 2: a) For the 10×10 Hilbert matrix, I find

$$\kappa = \frac{1.75}{1.09 \times 10^{-13}} = 1.6 \times 10^{13}$$

b) Using scipy.linalg.inv I find $\max \left| \mathsf{A}^{-1}\mathsf{A} - I \right| = 8.9 \times 10^{-5}$

c) I find in this case that $\max \left| \mathsf{A}^{-1}\mathsf{A} - I \right| = 4.3 \times 10^{-4}$. This is actually even less accurate!