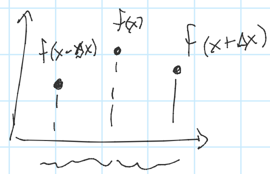
Week 6 - Monday

Monday, March 1, 2021 12:00 PM

Solving ODE's (or systems of)

Exam Recap - how to find $f'(x)$



a) 1) Taylor Series Expansion

expand $f(x+\Delta x) \simeq f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + O(\Delta x^3)$

↳ Right-sided: $\frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x^2)$

$- \quad f(x-\Delta x) \simeq f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + O(\Delta x^3)$

Left-sided

$f'(x) \simeq \frac{f(x) - f(x-\Delta x)}{\Delta x} + O(\Delta x^2)$

Centered finite difference $\quad \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = \frac{\cancel{2\Delta x}}{\cancel{2\Delta x}} f'(x)$

b) Unique parabola going through $f(x-\Delta x)$, $f(x)$, $f(x+\Delta x)$

Should end up w/ same formula!

$\frac{\partial \chi^2}{\partial b} = \frac{\partial}{\partial b} \sum \left(\frac{f(x_i) - b x_i - a}{\sigma_i} \right)^2 = 0$

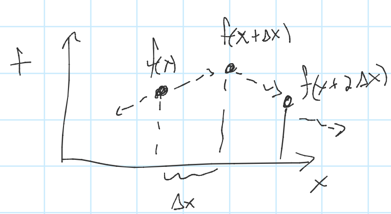
c) fit a line, look @ slope

$$\begin{cases} S_x = \sum_i x_i = (x-\Delta x) + (x) + (x+\Delta x) = 3x \\ S_f \ \sum_i f(x_i) = \end{cases}$$

⋮

Get the same $f'(x)$!

$f'(x) = \text{function of } f, x \equiv A(f(x), x)$



$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} = A(f(x), x)$

☆ $f(x+\Delta x) = f(x) + \Delta x A$ Euler's Method

apply repeatedly to get $f(x+2\Delta x)$

Improvements to Euler's Method?

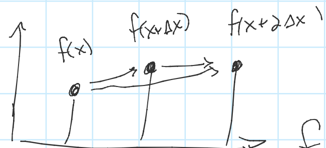
- Choose Δx so error is below some tolerance
  "Adaptive step size"

- If we have f at >1 point, use this info
  "Multi-point" Methods (less common)

- Try to reduce error by taking steps w/ different Δx, cleverly combining.
  "Runge-Kutta" Methods (most common)

"Richardson Extrapolation": take steps Δx, $\Delta x/2$, $2\Delta x$, ...
& extrapolate to $\Delta x \to 0$

trying to solve
$f'(x) = A(f, x)$



$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \mathcal{O}(\Delta x^3)$$

$$f_1 \equiv f((x+\Delta x)+\Delta x) = f(x+\Delta x) + \Delta x f'(x+\Delta x) + \frac{\Delta x^2}{2} \underline{f''(x+\cancel{\Delta x})} + \mathcal{O}(\Delta x^3)$$

$$= f(x) + \Delta x A + \frac{\Delta x^2}{2} f''(x) + \Delta x A(x+\Delta x) + \frac{\Delta x^2}{2} f''(x) + \mathcal{O}(\Delta x^3)$$

$$= f(x) + \Delta x \left(A(x) + A(x+\Delta x) \right) + \underline{\Delta x^2 f''(x)}$$

try to eliminate

$$f_2 \equiv f(x+2\Delta x) = f(x) + 2\Delta x f'(x) + \underline{\frac{4\Delta x^2}{2}} f''(x) + \mathcal{O}(\Delta x^3)$$

take weighted average of f_1, f_2: $c_1 f_1 + c_2 f_2$

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 \cdot \Delta x^2 + c_2 \Delta x^2 \cdot 2 = 0 \end{cases}$$

$\hookrightarrow c_2 = -1, \quad c_1 = 2$

☆ "extrapolated" solution is $2f_1 - f_2 + \mathcal{O}(\Delta x^3)$

☆ error estimate: $|f_1 - f_2| <$ tolerance

☆ Use other step sizes to eliminate Δx^3, Δx^4, ...

☆ Can generate many different integration schemes

- "Midpoint method" – derived above – 2nd-order Runge-Kutta – RK2

$$f_{midpoint}(x+\Delta x) = f(x) + \Delta x\, A\left[f(x) + \frac{\Delta x}{2} A(f(x),x),\ x+\frac{\Delta x}{2}\right] + O(\Delta x^3)$$

- 4th-order RK – RK4

$$f_{RK4}(x+\Delta x) = f(x) + \frac{\Delta x}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) + O(\Delta x^5)$$

$$\begin{cases} k_1 = A(f(x), x) \\ k_2 = A\left(f(x) + \frac{\Delta x}{2} k_1,\ x + \frac{\Delta x}{2}\right) \\ k_3 = A\left(f(x) + \frac{\Delta x}{2} k_2,\ x + \frac{\Delta x}{2}\right) \\ k_4 = A\left(f(x) + \Delta x\, k_3,\ x + \Delta x\right) \end{cases}$$

Given integrate $f'(x)$ in some range, b/w $[a, b]$, $b-a=L$

→ subdivide L into N pieces

$\Delta x \sim \frac{L}{N}$, $N \propto \frac{1}{\Delta x}$

total accumulated error will be

$O(\Delta x^3/\Delta x)$ for RK2

$O(\Delta x^5/\Delta x)$ for RK4

Systems of Multiple Eqn's

$$f'(x) = A(f(x), x)$$

$$\Downarrow$$

$$f'(x) = A_f(f, g, x) \qquad f(x+\Delta x) \simeq f(x) + \Delta x \cdot A_f$$

$$g'(x) = A_g(f, g, x) \quad \text{and} \quad g(x+\Delta x) \simeq g(x) + \Delta x \cdot A_g$$

$$\Downarrow$$

$$\frac{\partial \vec{f}(x)}{\partial x} = \vec{A}(\{\vec{f}\}, x) \quad \text{and} \quad \vec{f}(x+\Delta x) \simeq \vec{f}(x) + \Delta x \cdot \vec{A}$$

Systems w/ higher-order derivatives

$$f^{4}(x) = A(f, x) \;?$$

define: $f'(x) = g$

then: $g'(x) = A(f, x)$

Can work with any order derivative - just successively define new functions to eliminate higher-order derivatives.

Ex: $\frac{\partial^4 f}{\partial x^4} = A(f,x) \Rightarrow \begin{cases} \frac{\partial f(x)}{\partial x} = g(x) \\ \frac{\partial g(x)}{\partial x} = h(x) \\ \frac{\partial h(x)}{\partial x} = j(x) \\ \frac{\partial j(x)}{\partial x} = A(f, x) \end{cases}$