Physics 427 – Introduction to Computational Physics
James B. Mertens
Homework 4 – due Feb. 18, 2021

Problem 1: In the previous homework we encountered the Hilbert matrix and saw that it is ill-conditioned. This is not just a matrix invented by a mathematician to create problems[1] but instead can appear in a minimization problem. Suppose we are given a known function $g(x)$ and wish to expand it in a finite power series so that

$$g(x) \approx \sum_{i=0}^{n} a_i x^i.$$

To find the coefficients, a_i, we could minimize a "χ^2-like" quantity we define as

$$X^2 \equiv \int_0^1 \left[g(x) - \sum_{i=0}^{n} a_i x^i \right]^2 \mathrm{d}x.$$

Notice that if the integral were replaced by a sum over a finite number of points this would just be the χ^2. When we minimize X^2 with respect to the coefficients a_i we end up with a system of linear equations that can be written in the familiar form

$$\mathsf{A}\vec{a} = \vec{b},$$

where now $\vec{a}$ is a vector with components given by the coefficients a_i. This system of equations can then be solved.

a) Perform the minimization and find the expression for the components of $\vec{b}$. These will depend on $g(x)$, but, given a particular functional form for $g(x)$, the values can be calculated resulting in a known vector $\vec{b}$.

b) Again from the minimization determine the components of the matrix A. You should find that the A_{ij} are precisely the components of the Hilbert matrix. [Note: It can be useful to consider a small n case, such as $n = 2$, to more directly see the structure of the matrix. The results can be generalized to arbitrary n from there.]

Problem 2: Complete the Jupyter notebook assignments.

Problem 3: Complete the "Homework 4 Survey" in the quizzes section of Canvas.

---

[1] Not to say that a mathematician would not create such a matrix for just such a purpose.