PDEs Continued:

- Multiple fields,
- Multiple dimensions,
- Elliptic Equations

e.g. Wave equation:

1 space, 1 time: $\underline{\partial_t^2} f - \partial_x^2 f = 0$

write second-order $\partial_t^2 f$: $\begin{cases} \partial_t f = g \\ \partial_t g = \partial_x^2 f \end{cases}$

>1 space, 1 time: $\partial_t^2 f - \nabla^2 f = 0$

write 2nd-order $\partial_t^2 f$: $\begin{cases} \partial_t f = g \\ \partial_t g = \nabla^2 f \end{cases}$

finite-difference version of ∇^2? e.g. 2 spatial dims:

$$\nabla^2 f = \partial_x^2 f + \partial_y^2 f$$

$$\simeq \frac{f(t, x+\Delta x, y) - 2f(t,x,y) + f(t, x-\Delta x, y)}{\Delta x^2} + \frac{f(t, x, y+\Delta y) - 2f(t,x,y) + f(t, x, y-\Delta y)}{\Delta y^2}$$

when $\Delta x = \Delta y$,

$$\nabla^2 f = \frac{f(t, x+\Delta x, y) + f(t, x, y+\Delta y) - 4f(x,t,y) + f(t, x-\Delta x, y) + f(t, x, y-\Delta y)}{\Delta x^2}$$

Relation to convolutions: | 1 spatial dim:

function $f = (f(x_1), f(x_2), f(x_3), f(x_4) \dots)$

window: $w = (1 \quad -2 \quad 1)/\Delta x^2$

convolved $= (\dots \; f(x_1) - 2f(x_2) + f(x_3) \; \dots)$

Window: $w = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} / \Delta x^2$

Convolved function: $c = \left(\cancel{\ } \quad \frac{f(x_1) - 2f(x_2) + f(x_3)}{\Delta x^2}, \quad \ldots \right)$

2 dimensions:

$$f = \begin{pmatrix} f(x_1, y_1) & f(x_1, y_2) & f(x_1, y_3) & \cdots \\ f(x_2, y_1) & f(x_2, y_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$w = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$c = \begin{pmatrix} & \nabla^2 f(x_2, y_2) & \cdots \\ & \vdots & \ddots \end{pmatrix}$$

Wave equation stability: $\begin{cases} \partial_t g = \partial_x^2 f \\ \underline{\partial_t f = g} \end{cases} \leftarrow \begin{cases} f = f_{true} + \epsilon_f \\ g = g_{true} + \epsilon_g \end{cases}$

$$\begin{cases} \partial_t (g_{true} + \epsilon_g) = \partial_x^2 (f_{true} + \epsilon_f) \\ \partial_t (f_{true} + \epsilon_f) = g_{true} + \epsilon_g \end{cases} \Rightarrow \begin{aligned} \partial_t \epsilon_g &= \partial_x^2 \epsilon_f \\ \partial_t \epsilon_f &= \epsilon_g \end{aligned}$$

$$\Rightarrow \begin{aligned} \epsilon_g(t+\Delta t) &= \epsilon_g(t) + \Delta t \, \partial_x^2 \epsilon_f(t) \\ \epsilon_f(t+\Delta t) &= \epsilon_f(t) + \Delta t \, \epsilon_g(t) \end{aligned}$$

Want $|\epsilon_f(t+\Delta t)| \leq |\epsilon_f(t)|$

$$\hookrightarrow \underline{\epsilon_g(t)} = \frac{\epsilon_f(t+\Delta t) - \epsilon_f(t)}{\Delta t}$$

$$\underline{\epsilon_g(t+\Delta t)} = \frac{\epsilon_f(t+2\Delta t) - \epsilon_f(t+\Delta t)}{\Delta t}$$

$$\epsilon_f(t+2\Delta t) - \underline{\epsilon_f(t+\Delta t)} = \epsilon_f(t+\Delta t) - \epsilon_f(t) + \Delta t^2 \partial_x^2 \epsilon_f(t)$$

$$\epsilon_f(t+2\Delta t) - \epsilon_f(t+\Delta t) = \epsilon_f(t+\Delta t) - \epsilon_f(t) + \Delta t^2 \partial_x^2 \epsilon_f(t)$$

$$\epsilon_f(t+2\Delta t) = 2\epsilon_f(t+\Delta t) - \epsilon_f(t) + \Delta t^2 \partial_x^2 \epsilon_f(t)$$

let $\epsilon_f(t) = E(t)e^{ikx}$, $\quad \partial_x^2 \epsilon_f(t) = \frac{E(t)}{\Delta x^2}\left(e^{ik(x+\Delta x)} - 2e^{ikx} + e^{ik(x-\Delta x)}\right)$

$$= \frac{E(t)e^{ikx}}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$E(t+2\Delta t) = 2E(t+\Delta t) - E(t) + E(t)\frac{\Delta t^2}{\Delta x^2}\underbrace{\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)}_{= -4\sin^2\left(\frac{k\Delta x}{2}\right)}$$

$$|E(t+2\Delta t)| \leq |E(t+\Delta t)|$$

$$\left|2E(t+\Delta t) - E(t)\left[1 + 4\frac{\Delta t^2}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]\right| \leq |E(t+\Delta t)|$$

$$\left|2 - \underbrace{\left(1 + 4\frac{\Delta t^2}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right)}_{\text{always } > 0}\underbrace{\frac{E(t)}{E(t+\Delta t)}}_{\substack{\text{worst-case} \\ = -1}}\right| \leq 1$$

want $|E(t+\Delta t)| \leq |E(t)|$

$\Rightarrow$ No way to guarantee inequality is satisfied.
Not guarantee to be stable.

Friday: Crank-Nicholson
↳ guarantee stability (for most PDEs)

Elliptic Equations: eg. $\underbrace{\nabla^2 f = \rho}$

also eg: (time-indep) Schrodinger Equation,

$$\left(\partial_x^2 - V\right) f = E f$$

one way to solve: cast into matrix form

eg. $\nabla^2 f = \rho$,

$$\underbrace{\frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix}}_{\nabla^2 \text{ (invert)}} \underbrace{\begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \end{pmatrix}}_{(f)} = \underbrace{\begin{pmatrix} \rho(x_1) \\ \rho(x_2) \\ \vdots \end{pmatrix}}_{\rho}$$

invert to solve for f.

also works in >1 dimension! eg. 2-d:

$$\frac{1}{\Delta x^2}\underbrace{\begin{pmatrix} -4 & 1 & & - - & 1 & & \\ 1 & -4 & 1 & & & 1 & \\ & 1 & -4 & 1 & & & 1 \\ & & \ddots & \ddots & \ddots & & \ddots \\ 1 & & & \ddots & \ddots & \ddots & \\ & 1 & & & \ddots & \ddots & \\ & & 1 & & & \ddots & \ddots \end{pmatrix}}_{\nabla^2} \begin{pmatrix} f(x_1, y_1) \\ f(x_2, y_1) \\ f(x_3, y_1) \\ \vdots \\ f(x_1, y_2) \\ f(x_2, y_2) \\ \vdots \end{pmatrix} = \begin{pmatrix} \rho \\ \downarrow \end{pmatrix}$$

Efficient way to solve linear PDEs ↑

for Nonlinear Equations: $\nabla^2 \phi = 2\pi\rho(1+\phi)^5$

$$\nabla^2 \phi_g - 2\pi\rho(1+\phi_g)^5 = R \neq 0$$

Work to minimize residual for a guess ϕ_g

Finite-Element method: directly minimize R, for some interpolating function ϕ_g

method: directly minimize R, for some Interpolating function ϕ_g

More generally: promote to diffusion problem $\Rightarrow$ IVP

$$\hookrightarrow \underbrace{\partial_t \phi = \nabla^2 \phi}_{\text{diffusive behavior}} - \underbrace{2\pi (1+\phi)^5 \rho}_{\text{source term}}$$

Set up with some appx. guess ϕ_g

eventually reaches some steady-state, $\partial_t \phi = 0$

when $\partial_t \phi \to 0$, ϕ now satisfies $\nabla^2 \phi - 2\pi (1+\phi)^5 \rho = 0$.