Problem 1: [1]

a) The time-domain signal corresponding to $G(\nu_{m'})$ is given by

$$g(t_{n'}) = \sum_{m'=0}^{2N-1} G(\nu_{m'}) e^{2\pi i n' m'/(2N)} .$$

Because $G(\nu_{m'}) = 0$ for odd m', we can change this to a sum over even m' only.

$$g(t_{n'}) = \sum_{m' \in \text{even}}^{2N-1} G(\nu_{m'}) e^{2\pi i n' m'/(2N)} .$$

Letting $m' = 2m$ for a set of integers m, we can sum over m from 0 to $N-1$ in place of m',

$$g(t_{n'}) = \sum_{m=0}^{N-1} G(\nu_{2m}) e^{2\pi i n' 2m/(2N)} .$$

We can then simplify and substitute in our expression for H in place of G,

$$g(t_{n'}) = \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i n' m/N} .$$

But, this is just an expression for the Fourier transform of h! We still need to be careful and consider the cases where $n' \geq N$ separately, but for $n' < N$, we have $g(t_{n'}) = h(t_{n'})$. For $n' \geq N$, we have

$$\begin{aligned}
g(t_{n'}) &= \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i (n'-N+N)m/N} \\
&= \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i (n'-N)m/N} e^{2\pi i m} \\
&= \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i (n'-N)m/N} \\
&= h(t_{n'-N})
\end{aligned}$$

where we have both added and subtracted N in the exponent in the first line, factored out the added N in the second line, then noted that $e^{2\pi i m} = 1$ for an integer m per Euler's identity in the third. The third line is the Fourier transform of h with the time sample taken at $n' - N$.

[1] Note the choice of conventions made in class will result in an unnormalized discrete Fourier transform and inverse, that is, an overall normalization factor of $1/N$ is omitted.

b) For this derivation, it is perhaps easiest to manpulate the final expression in terms of h and show its equivalence to the expression in terms of g. Beginning with the Fourier transform of h, we have that

$$h(t_n) = \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i n m/N} \,.$$

Evaluating this expression at values of n coincident with $n' + N/2$ gives us

$$\begin{aligned} h(t_{n'+N/2}) &= \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i (n'+N/2) m/N} \\ &= \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i n' m/N} e^{\pi i m} \\ &= \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i n' m/N} (-1)^m \,. \end{aligned}$$

Combining these,

$$h(t_{n'}) + h(t_{n'+N/2}) = \sum_{m=0}^{N-1} H(\nu_m) e^{2\pi i n' m/N} (1 + (-1)^m) \,.$$

The combination $(1 + (-1)^m)$ will be zero for odd m, and otherwise 2, so

$$h(t_{n'}) + h(t_{n'+N/2}) = 2 \sum_{m\in\text{evens}}^{N-1} H(\nu_m) e^{2\pi i n' m/N} \,,$$

which is precisely what we want for g, i.e.

$$\begin{aligned} g(t_{n'}) &= \sum_{m'=0}^{N/2-1} G(\nu_{m'}) e^{2\pi i n' m'/(N/2)} \\ &= \sum_{m'=0}^{N/2-1} G(\nu_{m'}) e^{4\pi i n' m'/N} \\ &= \sum_{m\in\text{evens}}^{N-1} G(\nu_{m/2}) e^{2\pi i n' m/N} \\ &= \sum_{m\in\text{evens}}^{N-1} H(\nu_m) e^{2\pi i n' m/N} \end{aligned}$$

for $m = 2m'$. Thus the two sides of the equation in the problem statement are equivalent.