Periodicity in Exponential Function Differences

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Abstract

The taylor series $\sum_{m=0}^{\infty} \frac{x^{a_m}}{a_m!}$ is convergent for set of terms identified by a_m an infinite increasing sequence of positive integers. These functions all behave similar to the exponential function e^x. In particular, if the sequence a_m is an arithmetic progression with common difference d and first term $a_0 < d$, then the function $f(x)$ defined by this taylor series is it's own d^{th} derivative.

Self Derivative Series

Define the function $\mathcal{G}_d^i$ with $d \geq 1$ and $0 \leq i < d$ to be the $i - th$ offset Taylor series as

$$\mathcal{G}_d^i(x) = \sum_{n=0}^{\infty} \frac{x^{dn+i}}{(dn+i)!}.$$

Observe the following familiar functions can be written in terms of these building blocks:

$$e^x = \mathcal{G}_1^0(x)$$

$$\cosh(x) = \mathcal{G}_2^0(x) \text{ and } \sinh(x) = \mathcal{G}_2^1(x),$$

$$\cos(x) = \mathcal{G}_4^0(x) - \mathcal{G}_4^2(x) \text{ and } \sin(x) = \mathcal{G}_4^1(x) - \mathcal{G}_4^3(x).$$

By computing the $d-$ th derivative of $f = \mathcal{G}_d^i$ directly on the taylor series expansion we verify the claim of the abstract that $f^{(d)} = f$. So also we can verify the familiar claims of the above functions being their own first, second and fourth derivatives respectively.

Other functions

If you graph the families $\mathcal{G}_d^i$ with fixed d and $0 \leq i < d$ we see that they cluster together by family of fixed d. In fact, it is startling looking that the graph of the 4 functions $\mathcal{G}_4^i$ overlaid on the same graph appear to be equal for $x >> 0$. The startling fact being that not only are these functions not equal but they criss cross each other as we observe writing $\sin x$ and $\cos x$ as above.

For what follows, it is sufficient to graph the taylor series truncated at 40 terms for $d > 1$ and simply use e^x for $d = 1$. Get a feel for this by observing that

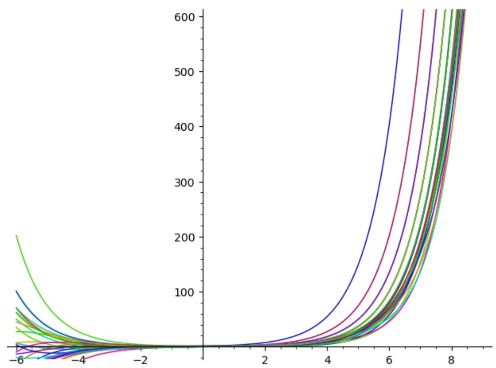

Figure 1: $\mathcal{G}_i$ families for $i = 1, 2, ..., 7$

$\frac{25^{80}}{80!} < 1.0e - 7$. It is sufficient to graph the interval $-3 < x < 25$ to walk through the following.

The expressions for $\sin x$ and $\cos x$ above suggest it is reasonable to look at differences of the various $\mathcal{G}_d^i$. The graph below is of the 3 functions $\mathcal{G}_3^i - \mathcal{G}_3^j$ with $0 \le i < j < 3$. Observe the exponential oscillation to the left and the exponential decay to the right.

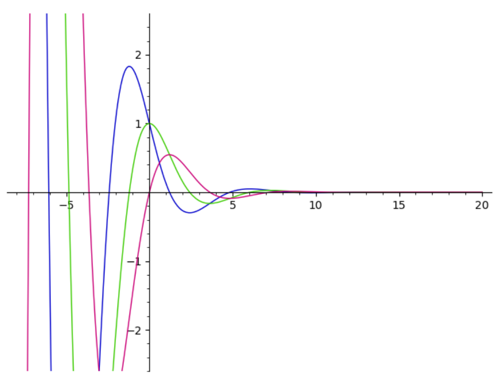

Figure 2: Unscaled graph of $\mathcal{G}_3$ family

Now, here are these same three functions scaled by multiplying by $e^{x/2}$. Specifically

$$e^{x/2}\left(\mathcal{G}_3^i(x) - \mathcal{G}_3^j(x)\right).$$

Observe in this scaled version that we can easily see the periodic zeros very similar to sin and cos. Numerically the zeros are spaced with common difference approximately 3.628.

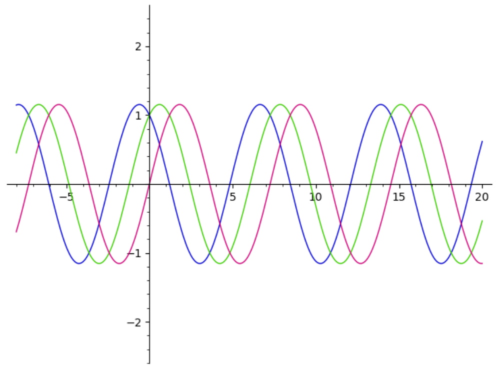

Figure 3: Scaled graph of $\mathcal{G}_3$ family

Conjecture 1: There exists a periodic function $\mathcal{H}_3^{i,j}$ such that

$$\lim_{x \to \infty} e^{x/2} \left(\mathcal{G}_3^i(x) - \mathcal{G}_3^j(x) \right) - \mathcal{H}_3^{i,j}(x) = 0.$$

However they are not mutually derivitives of each other when the scaling factor is included.

Observation 2: The difference $\mathcal{G}_2^0(x) - \mathcal{G}_2^1(x)$ converges to 0 from above as $x \to \infty$. There is no oscillation. This is easily verified by writing sinh and cosh as linear combinations of e^x and e^{-x} in the usual way.

Observation 3: The 6 differences $\mathcal{G}_4^i(x) - \mathcal{G}_4^j(x)$ need no scaling and are exactly periodic and mutually self-differential with a small caveat. Aside from the differences for sin and cos, these functions grow exponentially to the left. The difference functions are "visually periodic" for $x > 5$. As above in Conjecture 1 we can make this precise with a periodic limiting function $\mathcal{H}_4^{i,j}$.

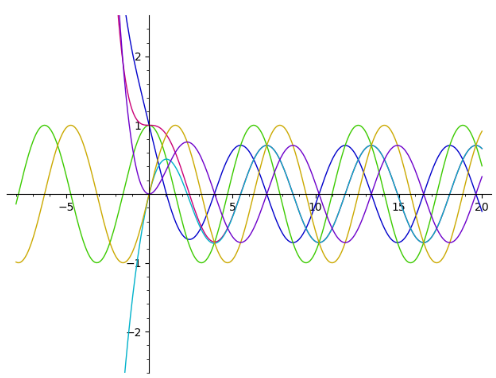

Figure 4: Unscaled graph of $\mathcal{G}_4$ family

With correct scaling for each $d \geq 5$, we observe this same sort of periodic behavior for $x >> 0$. The following table summarizes the numeric observation of required scaling and resulting periods and amplitudes.

d	scale	period	amplitudes
3	0.500	7.256	1.155
4	--	2π	0.707 1.000
5	-0.309	6.606	0.470 0.761
6	-0.500	7.255	0.334 0.578 0.667
7	-0.623	8.036	0.248 0.447 0.557
8	-0.707	8.886	0.191 0.354 0.462 0.501
9	-0.766	9.775	0.152 0.286 0.385 0.438
10	-0.809	10.690	0.123 0.235 0.324 0.381 0.401
11	-0.841	11.622	0.102 0.196 0.274 0.330 0.360
12	-0.866	12.566	0.086 0.165 0.235 0.288 0.322 0.333

Remaining Questions

Question 1: Given d can we compute the scale, period and amplitudes?

I have not been able to see any precise pattern in the scaling factor as d varies beyond the general decreasing and it possibly approaches -1.

Question 2: Do the functions

$$e^{\gamma x}\left(\mathcal{G}_d^i(x) - \mathcal{G}_d^j(x)\right)$$

always converge to a sinusoidal periodic function as $x \to \infty$?

Question 3: Does

$$\left(\sum_{n=0}^{\infty} \frac{x^{nd}}{(nd)!}\right) e^{-x} \to \frac{1}{d}$$

for all d as $x \to \infty$?

Question 4 (bonus): The abstract considered any sequence of exponents in a taylor series. What kind of conditions on a sequence a_m with $a_{m+1} > a_m$ enable us to make any interesting observations about $\sum_{m=0}^{\infty} \frac{x^{a_m}}{a_m!}$?