Math 176 F18: Week 3

Due: Sept. 24, 2018 Nunez, Joseph, `jbnunez@hmc.edu`

Starting with our original ideal I and the polynomial $g \in I(V(I))$, define a new ideal J in the slightly larger polynomial ring $k[x_1, \dots, x_n, x_{n+1}]$, by setting

$$J = \langle f_1, \dots, f_r, x_{n+1}g - 1\rangle$$

1. **4.3.9** (Cont. from last week) There exists no $f \in \mathbb{C}[x, y]$ such that $V(f) = \{(0, 0)\}$. (Class)

   Suppose to the contrary that there does exist some f such that $f(0, 0) = 0$.

   $$f(x, y) = f_0(x) + f_1(x)y + \cdots + f_n(x)y^n$$

   for some n. If for all i, $f_i(x) = 0$, then $f(0, y) = 0$. Then there exists f_j s.t. $f_j \neq 0$, the there exists $M \neq 0$ such that $f_j(M) \neq 0$. Then $f(M, y_0) = 0$ contradicting $V(f) = \{(0, 0)\}$.

2. **4.4.3** Suppose $g \in I(V(I))$, or in other words $g(a_1, \dots, a_n) = 0$ for all $(a_1, \dots, a_n) \in V(\langle f_1, \dots, f_r\rangle)$. For $J = \langle f_1, \dots, f_r, x_{n+1}g - 1\rangle$, show $V(J) = \emptyset$. (Called trick of Rabinowitsch)

   Let $a \in \mathbb{A}^{n+1}(k)$. First, suppose that the first n components of a belong to $V(\langle f_1, \dots, f_r\rangle)$. Since $g \in I(V(I))$, g evaluates to zero over the first n components. Then $x_{n+1}g - 1$ is equal to $x_{n+1}(0) - 1 = -1$, so $V(J)$ cannot contain any elements a for which the first n components of a belong to $V(\langle f_1, \dots, f_r\rangle)$. Next, suppose that the first n components of a do not belong to $V(\langle f_1, \dots, f_r\rangle)$. Then for some $h \in \langle f_1, \dots, f_r\rangle$, h does not evaluate to zero over these elements, so this element cannot be in $V(J)$ since $h \in J$. Therefore there are no a that can belong to $V(J)$, hence $V(J) = \emptyset$.

3. **4.4.4** Assuming the Weak Nullstellensatz, show that J is not a proper ideal and hence that there exists $A_1, \dots, A_r, B$ in $k[x_1, \dots, x_m, x_{n+1}]$ such that

   $$1 = A_1f_1 + \cdots + A_rf_r + B(x_{n+1}g - 1)$$

   By the Weak Nullstellansatz, if J is a proper ideal, then $V(J) \neq \emptyset$. However, as we just showed in the previous problem, $V(J) = \emptyset$, so J is not a proper ideal. Additionally, notice that $f = A_1f_1 + \cdots + A_rf_r \in I$, so for all $a \in V(\langle f_1, \dots, f_r\rangle)$, $f(a) = 0$. Then if we take $x \in \mathbb{A}^{n+1}(k)$ such that a is equal to the first n

elements of x, then $f(x) = 0$ and $x_{n+1}g - 1 = -1$. Then if we set $B = -1$, we get

$$1 = A_1 f_1 + \cdots + A_r f_r + B(x_{n+1}g - 1)$$

4. **4.4.5** Let $x_{n+1} = \frac{1}{y}$. Show there exists $N > 0$ and polynomials $C_1, \ldots, C_r, D$ in $k[x_1, \ldots, x_n, y]$ with

$$y^N = C_1 f_1 + \cdots + C_r f_r + D(g - y)$$

by clearing denominators (Class)

Take $x_{n+1} = \frac{1}{y}$, then

$$1 = A_1 f_1 + \cdots + A_r f_r + B(\frac{g}{y} - 1) = A_1 f_1 + \cdots + A_r f_r + \frac{B(g-y)}{y}$$

Multiply through by y^m, where m was the maximal power of x^{n+1} is the original f_is:

$$y = y^{\alpha_1} A_1 f_1 + \cdots +^{\alpha_r} A_r f_r + y^{m-1} B(g - y)$$

Define each $C_i = y^{\alpha_i} A_i$ and $D = y^{m-1} B$

5. **4.4.6** Letting $y = g$ show that $g^N \in I$ and hence $g \in \mathrm{Rad}(I)$

If we now take $y = g$, then we get

$$y^N = C_1 f_1 + \cdots + C_r f_r + D(0) = C_1 f_1 + \cdots + C_r f_r$$

where $C_i \in k[x_1, \ldots, x_n]$ since $y = g \in k[x_1, \ldots, x_n]$. Thus y^N belongs to $\langle f_1, \ldots, f_r \rangle$. Then $y^N \in I$, thus $y \in \mathrm{Rad}(I)$, so $g \in \mathrm{Rad}(I)$

6. **4.5.1** Prove the Weak Nullstellensatz for $n = 1$. (Attempted)

Let k be an algebraicly closed field and I be a proper ideal of $k[x]$. Suppose that $V(I) = \emptyset$. Then there exist no $x \in k$ such that $f(x) = 0$ for all $f \in I$. However, since $\emptyset$ is trivially a subset of the vanishing set for every polynomial, $V(f) \supset \emptyset \forall f \in k[x]$, so $f \in I$. Thus $I \supseteq k[x]$, hence $I = k[x]$, which contradicts the assumption that I is proper. Thus $V(I) \neq \emptyset$. Thus the Weak Nullstellensatz holds for $n = 1$.

By alg. closure $k[x]$ is a PID, so $I = (p)$ for some $p \in k[x]$. Then $V(I) = V(p)$. Because k is alg. closed, p has $\deg(p) \neq 0$ zeros counting multiplicity. Then $V(p) \neq \emptyset \implies V(I) \neq \emptyset$. Note that if $\deg(p) = 0$ then $V(I) = k$.

7. **4.5.15** (Weak Nullstellensatz—Version 2). Let k be an algebraicly closed field. An ideal I in $k[x_1, \ldots, x_n]$ is maximal if and only if there are elements $a_i \in k$ such that I is the ideal generated by the elements $x_i - a_i$; that is $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. (Attempted)

   Forwards: Let I be a maximal ideal. Then $V(I) \neq \emptyset$. Let $a \in V(I)$. Then $\{a\} \subseteq V(I)$. Then $I(\{a\}) \supseteq I(V(I)) \supseteq I$. Since I is maximal, we must have $I(\{a\}) = I(V(I)) = I$. Since $\{a\}$ is algebraic, $V(\langle x_1 - a_1, \ldots, x_n - a_n \rangle) = \{a\}$, so $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

   Reverse: Let $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. Then $V(I) = (a_1, \ldots, a_n)$. Suppose that $J \supseteq I$. Then $V(J) \neq \emptyset$, but $V(J) \subset V(I)$. $V(J) = \{(a_1, \ldots, a_n)\}$, so $J = I$, hence I is maximal.

8. **4.5.16** Let $I = \langle x^2 + 1 \rangle \subset \mathbb{R}[x]$ and show that $I(V(I)) \neq \text{Rad}(I)$. (Attempted)

   Notice that $V(I) = \emptyset$ since X^2 is nonnegative over the reals, hence $I(V(I))$ contains all polynomials with no real roots. If $p^n = (x^2 + 1)q$, then $x^2 + 1 | p$ because $x^2 + 1$ is prime. $x^2 + 1 = ab$, then $\deg(a) + \deg(b) = 2$. WLOG, if $\deg(a) = 2$, then $\deg(b) = 0$, so b is a unit. Else $\deg(a) = \deg(b) = 1$, so $a = cx + b$, hence $\frac{-d}{c}$ is a root of a, so $\text{Rad}(I)$ is nonempty, hence $\text{Rad}(I) \neq I(V(I))$

9. **4.5.17** Show that $I = \langle x^2 + y^2 \rangle$ and $\langle x, y \rangle$ are radical ideals in $\mathbb{R}[x, y]$ with $V(I) = V(J)$. This demonstrates that the correspondence between algebraic sets and radical ideals is not one-to-one over $\mathbb{R}$.

$$V(I) = \{(x, y) : x^2 + y^2 = 0\} = \{(0, 0)\}$$
$$V(J) = \{(x, y) : x = 0, y = 0\} = \{(0, 0)\}$$

   Take $p \in \text{Rad}(I)$, then for some $n > 0$, $p^n \in I$, hence $p^n = (x^2 + y^2)f(x, y)$. We wish to show that $(x^2 + y^2)$ is prime, hence irreducible. Suppose $(x^2 + y^2) = (ax + by + c)(dx + ey + f)$. Then we would have a nonzer constant. $p \in I$, hence $\text{Rad}(I) \subseteq I$.

   Take $p \in \text{Rad}(J)$. Then $p^n = xf(x, y) + yg(x, y)$. Then $p^n(0, 0) = 0$, hence $p(0, 0) = 0$, so $p \in J$.

   Note first that all ideals are contained by their radical, since for any I, if $f \in I$, then $f \in \text{Rad}(I)$ since $f^1 \in I$. Thus $I \subseteq \text{Rad}(I)$ for all ideals I. Thus to show an ideal is radical, we need only show that $\text{Rad}(I) \subset I$.

   Let $f \in \text{Rad}(I)$. Then $f^n \in I$.

Hard Fact: let k be an infinite field and

$$R = k[a_1, \ldots, a_n]$$

(polynomials in $a_1, \ldots, a_n$ with coefficients in k, and there may be relations among a_i, e.g. $a_1 a_2 = a_3$)

$$\begin{array}{ccc} \text{ring} : +, \times & -- & \text{algebra} : +, \times, \cdot \\ | & & | \\ \text{group} : + & -- & \text{vector space (module)} : +, \cdot \end{array}$$

BTW: Thm: if R is a finitely generated k-algebra, then $R \cong k[x_1, \ldots, x_n]/I$ for some ideal I.

If R is a field, then R is algebraic over k.

$\forall i = 1, \ldots, n \exists fi \in k[x]$ such that $f_i(a_i) = 0$

BTW: $k \subseteq R$, R is called a field extension of k.

Remark (about David Hilbert): Hilbert is responsible for the proof of Weak Nullstellansatz.

Nullstellansatz: Let k be algebraically closed ($k = \bar{k}$)

1. Every maximal ideal of $A = k[x_1, \ldots, x_n]$ is of the form $m_p = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$ for some point $p = (a_1, \ldots, a_n) \in \mathbb{A}^n_k$, i.e. $m_p = I(p)$
2. Let $J \subseteq A$ be an ideal. If $J \subsetneq A$, then $V(J) \neq \emptyset$, i.e. $J \neq (1) \implies V(J) \neq \emptyset$. Contrapositively, $V(J) = \emptyset \implies V(J) = A$.
3. For any $J \subseteq A$, $I(V(J)) = \text{Rad}(J)$ (Strong NSS)

Proof. 1. Let m be a maximal ideal of $A = k[x_1, \ldots, x_n]$. Then A/m is a field. But A is finietely generated by $x_1, \ldots, x_n$ over k. thus by our hard fact, $k \subseteq A/m$ is an algebraic extension. But k is algebraic closed, so $k = A/m$ (or $\cong$) $k \hookrightarrow_\ell k[x_1, \ldots, x_n] \to_\pi A/m$, $\phi : k \to A/m$. Let $\pi(x_i) = b_i$. Let $a_i = \phi^{-1}(b_i) \in k$. Then $x_i - a_i \in \ker \pi = m$. Thus $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \subseteq m$. But $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$ is a maximal ideal. Thus $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) = m$. Note $k[x_1, \ldots, x_n]/(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \cong k$. $\psi : k[x_1, \ldots, x_n] \to k$, $\psi(x_i) = a_i$, $\psi(f) = f(a_1, \ldots, a_n)$. The difficult part is proving $\ker \psi$.

2. Suppose $J \subsetneq A$. By the ACC, there exists a maximal ideal containing J. By the previous part, there exists $p \in \mathbb{A}_k^n$ such that $p = (a_1, \ldots, a_n)$ and $m = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$. Then $f(p) = 0 \forall f \in m$, so $V(J) \neq \emptyset$.

3. already proved

$\square$