1. Fluid Properties

1.1. Fluid properties.

1.1.1. *Density.*

$$\rho = \frac{m}{V} \quad \begin{cases} \rho_{\text{air}} = 1.23[\text{kg/m}^3] \\ \rho_{\text{water}} = 999[\text{kg/m}^3] \end{cases}$$

1.1.2. *Specific Volume.*

$$\dot{v} = \frac{1}{\rho}$$

1.1.3. *Specific weight.*

$$\gamma = \rho g$$

1.1.4. *Specific gravity.*

$$SG = \frac{\rho}{\rho_{H_20@4^\circ}}$$

1.2. Ideal Gas Law.

$$p = \rho RT$$

p absolute pressure, ρ density, R gas constant (dependant of fluid), T absolute Temperature.

$$\text{Absolute pressure} \\ = \\ \underbrace{\text{Atmospheric pressure}}_{101[\text{kPa}]} + \text{gage pressure}$$

1.3. Fluidity.

1.4. Viscosity. Speed profile :

$$u(y) = U\frac{y}{b}$$

Rate of shearing strain

$$\dot{\gamma} = \frac{U}{b} = \frac{du}{dy}$$

Shearing stress

$$\tau \propto \dot{\gamma}$$
$$\tau = \mu\frac{du}{dy} = \frac{F}{A}$$
$$T = r \times F$$

Newtonian Fluids : μ is constant
Non-Newtonian Fluids : μ is not constant and $\tau \propto \dot{\gamma}$ is no longer applicable.

1.4.1. *Sutherland Equation.*

$$\mu = \frac{CT^{3/2}}{T+S}$$

1.4.2. *Andrade's Equation.*

$$\mu = De^{B/T}$$
$$\ln \mu = \ln D + \frac{B}{T}$$

1.4.3. *Kinematic viscosity.*

$$v = \frac{\mu}{\rho}$$

2. Fluid statics

Pressure is equal at every point in a fluid at rest

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$
$$\frac{\partial p}{\partial x} = \frac{dp}{dz} = -\gamma = -\rho g$$
$$p = \gamma h + p_0$$

2.1. Hydrostatic Force on a Plane Surface.

$$F_R = \gamma h_c A \qquad h_c = \text{centroid}$$
$$y_R = \frac{I_{xc}}{h_c A} + h_c \quad I_x = I_{xc} + Ah_c^2$$
$$I_{xc} = \frac{1}{12}h^3 w$$

2.2. Buoyancy.

$$\mathbf{F_B} = \rho g V_{\text{immerged}}$$
$$F = A \cdot P \quad \text{(exos)}$$
$$P_i V_i = P_f V_f$$

3. Elementary Fluid Dynamics - The Bernoulli Equation

3.1. Bernoulli Equation.

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}$$

$$\frac{P}{\rho} + \int \frac{V^2}{R} dn + gz = \text{constant across a streamline}$$

3.2. Continuity Equation.

$$Q = Q_i = A_i V_i \quad [m^3/s] \quad i = 1, 2, \ldots$$

4. Fluid Kinematics

4.1. Velocity Fields.

$$\frac{dy}{dx} = \frac{v}{u}$$

4.2. Material Derivative.

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + (\mathbf{V} \cdot \nabla)()$$

4.3. **Acceleration.**

$$\begin{aligned}
\mathbf{a} &= \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \\
&= \frac{D\mathbf{V}}{Dt} = a_s\hat{s} + a_n\hat{n} \\
&= \underbrace{\frac{\partial \mathbf{V}}{t}}_{\text{local}} + \underbrace{\mathbf{V}\cdot\nabla(\mathbf{V})}_{\text{convective}} \\
&= \frac{\partial \mathbf{V}}{t} + \begin{pmatrix} u \\ v \\ w \end{pmatrix} \cdot \left(\frac{\partial \mathbf{V}}{\partial x} \quad \frac{\partial \mathbf{V}}{\partial y} \quad \frac{\partial \mathbf{V}}{\partial z} \right) \\
&= \frac{\partial \mathbf{V}}{t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} \\
&= \begin{pmatrix} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix}
\end{aligned}$$

with $\mathbf{V} = (u, v, w)$

4.3.1. *Acceleration in streamline coordinates.*

$$\mathbf{a} = V\frac{\partial V}{\partial s}\hat{\mathbf{s}} + \frac{V^2}{R}\hat{\mathbf{n}}$$
$$a_s = V\frac{\partial V}{\partial s}$$
$$a_n = \frac{V^2}{R}$$

4.4. **Reynolds Transport Theorem (RTT).**

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \int_{CS} \rho b\mathbf{v}\cdot\mathbf{n}dA$$

5. Finite Control Volume Analysis

5.1. **Continuity Equation.** In RTT, $B = m$ and $b = B/m = 1$

$$\begin{aligned}
\frac{DM_{sys}}{Dt} &= 0 \\
\frac{\partial}{\partial t}\int_{CV}\rho dV + \int_{CS}\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA &= 0
\end{aligned}$$

5.2. **Linear Momentum Equation.**

$$\frac{\partial}{\partial t}\int_{CV}\mathbf{V}\rho dV + \int_{CS}\mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = \sum \mathbf{F}_{\text{ext}}$$

5.3. **Energy Equation.**

$$\frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS} e\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = \left(\dot{Q}_{\text{net}} + \dot{W}_{\text{net}}\right)_{CV}$$
$$\doteq[\text{J/s}]\doteq[\text{F}\cdot\text{m/s}]\doteq[\text{kg}\cdot\text{m/s}^2]$$

$$\frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS}\left(\check{u} + \frac{p}{\rho} + \frac{v^2}{2} + gz\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA$$
$$= \left(\dot{Q}_{\text{net}} + \dot{W}_{\text{net}}\right)_{CV}$$

$$\dot{m}[\check{u}_{out} - \check{u}_{in} + \left(\frac{p}{\rho}\right)_{out} - \left(\frac{p}{\rho}\right)_{in}$$
$$+\frac{1}{2}\left(V_{out}^2 - V_{in}^2\right) + g(z_{out} - z_{in})]$$
$$= \dot{Q}_{net} + \dot{W}_{\text{net}}$$

$$\frac{P_1}{\rho} + \frac{v_1^2}{2} + gz_1 + w_{\text{shaft}} = \frac{P_2}{\gamma} + \frac{v_2^2}{2g} + gz_2 + \text{loss} \quad [\text{m}^2/\text{s}^2]$$

$$\frac{P_1}{\gamma} + \frac{v_1^2}{2g} + z_1 + \underbrace{\frac{\dot{W}_s}{\gamma Q}}_{h_s} = \frac{P_2}{\gamma} + \frac{v_2^2}{2g} + z_2 + \underbrace{\frac{\text{loss}}{g}}_{h_l}$$

turbine head $\quad h_T = -(h_s + h_L)_T$

pump head $\quad h_P = (h_s - h_L)_P$

$$v = \frac{\int v^2\rho\mathbf{v}\cdot\hat{\mathbf{n}}dA}{\overline{v}A}$$

6. Differential Analysis of Fluid Flow

6.1. Volumetric dilatation rate.

$$\frac{1}{V}\frac{d(\delta V)}{dt} = \nabla \cdot \mathbf{V}$$
$$\nabla \cdot \mathbf{V} = 0 \Rightarrow \quad \text{Incompressible fluid}$$

6.2. Vorticity.

$$\mathbf{J} = \nabla \times \mathbf{V}$$
$$\nabla \times \mathbf{V} = 0 \quad \Rightarrow \text{Irrotational fluid}$$

Irrotational, Bernoulli applies otherwise Euler.

6.3. Rate rotation Vector.

$$\omega = \frac{1}{2}\mathbf{J}$$

6.4. Rate of angular deformation.

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

6.5. Stream Function.

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$
$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

6.5.1. *Rate of flow.*

$$Q = \Psi(B) - \Psi(A) \quad [\mathrm{m}^3/\mathrm{s}]$$

6.6. Velocity Potential.

$$\mathbf{v} = \nabla \phi$$

6.6.1. *Conservation of Mass.*

$$\nabla \cdot \mathbf{V} = \nabla^2 \phi = 0$$

6.7. Euler's equations of Motion.

$$\rho \mathbf{g} - \nabla p = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \rho\frac{D\mathbf{v}}{Dt}$$

6.8. Navier-Stokes Equations.

6.8.1. *Cartesian Coordinates.*

$$\rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{v} = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right)$$

6.8.2. *Shearing stresses.*

$$\begin{cases} \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{cases}$$

6.8.3. *Cylindrical Coordinates.*

$$r : \rho g_r - \frac{\partial p}{\partial r}$$
$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right]$$
$$= \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r}\right)$$
$$\phi : \rho g_\phi - \frac{1}{r}\frac{\partial p}{\partial \phi}$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2}\right]$$
$$= \rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi}{r}\right)$$
$$z : \rho g_z - \frac{\partial p}{\partial z}$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$
$$= \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z}\right)$$

6.8.4. *Nabla operator in cylindrical coordinates.*

$$\nabla = (\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}) \quad r = \sqrt{\rho^2 + z^2}$$

6.8.5. *Poiseuille's Law.*

$$Q = \frac{\pi R^4 \Delta p}{8\mu l}$$
$$\overline{v} = \frac{R^2 \Delta p}{8\mu l} \qquad Q = \pi R^2$$
$$v_{\max} = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z} = \frac{R^2 \Delta p}{4\mu l} \qquad v_{\max} = 2\overline{v}$$
$$\frac{v_z}{v_{\max}} = 1 - \left(\frac{r}{R}\right)^2$$

7. Dimensional Analysis

7.1. Buckingham Pi Theorem.

$$\begin{aligned} u_1 &= f(u_2, \ldots, u_k) \\ &\rightarrow \\ \Pi_1 &= \phi(\Pi_2, \ldots, \Pi_k - r) \end{aligned}$$

where r is the number of repeating variables.

7.2. Similitude and Models.

$$\Pi_{i_{\text{model}}} = \Pi_i \quad \forall i$$

$\mu \doteq FL^{-2}T, \rho = FL^{-4}T^2$

8. Viscous Flow in Pipes

8.1. Reynold's Number.

$$Re = \frac{\rho v D}{\mu} = \frac{vD}{\nu} < 2100$$

8.1.1. *Fully Developed Profile.*

$$\begin{array}{rl} \text{laminar} & \frac{l_e}{D} = 0.06Re \\ \text{turbulent} & \frac{l_e}{D} = 4.4(Re)^{1/6} \end{array}$$

8.2. **Laminar Pipe Flow (Poiseuille improved).**

$$v = \frac{(\Delta p - \gamma l \sin\theta)D^2}{32\mu l}$$

8.3. **Major Losses (to be added to Bernoulli).**

$$h_{L_{\text{Major}}} = f\frac{l}{D}\frac{v^2}{2g}$$

8.4. **Minor Losses.**

$$h_{L_{\text{Minor}}} = K_L\frac{v^2}{2g} \quad K_L = \frac{\Delta p}{\frac{1}{2}\rho v^2}$$

8.5. **Moody Chart.**

$$f \leftrightarrow \frac{\epsilon}{D} \leftrightarrow Re$$

9. Boundary Layers, Drag & Lift

9.1. **Lift & Drag.**

9.1.1. *Drag.*

$$\begin{aligned} \mathscr{D} &= \int dFx = \int p\cos\theta dA + \int \tau_\omega \sin\theta dA \\ c_{\mathscr{D}} &= \frac{\mathscr{D}}{\frac{1}{2}\rho U^2 A} \end{aligned}$$

9.1.2. *Lift.*

$$\begin{aligned} \mathscr{L} &= \int dFy = -\int p\sin\theta dA + \int \tau_\omega \cos\theta dA \\ c_{\mathscr{L}} &= \frac{\mathscr{L}}{\frac{1}{2}\rho U^2 A} \end{aligned}$$

9.2. **Boundary Layer for Laminar Flow.**

$$\delta = c\sqrt{x} \propto \sqrt{\frac{\nu}{U}}\sqrt{x}$$

$$\tau_\omega = 0.332U^{3/2}\sqrt{\frac{\rho\mu}{x}}$$

$$\mathscr{D} = \int \tau_\omega dA$$

$$\text{Power} = \frac{\text{Energy}}{t} = \mathscr{D}v$$