1 Problem

$$i^i \tag{1}$$

where $i = \sqrt{-1}$.

2 Solution

First we rewrite the term using the exponential (we recall that $a^b = e^{b \ln a}$)

$$i^i = e^{i \ln (i)}. \tag{2}$$

The logarithm of a complex number number can also be written

$$\ln (z) = \ln |z| + i(\arg(z) + 2\pi k) \quad k \in \mathbb{Z} \tag{3}$$

were - if $z = x + iy$ - $|z| = \sqrt{x^2 + y^2}$ and $\arg(z) = \arctan(\frac{y}{x})$. In our case $z = i = iy \Rightarrow y = \mathrm{Im}(z) = 1, x = \mathrm{Re}(z) = 0$.

Thus we have

$$|z| = \sqrt{x^2 + y^2} = \sqrt{0 + 1^2} = 1 \tag{4}$$

$$\arg(z) = \arctan(\frac{y}{x}) = \lim_{u \to \infty} \arctan(u) = \frac{\pi}{2} \tag{5}$$

We substitute equations 4 and 5 into equation 3 which leads to

$$\ln (z) = \ln |z| + i(\arg(z) + 2\pi k) \tag{6}$$

$$= 0 + i(\frac{\pi}{2} + 2\pi k) = i\pi(\frac{1}{2} + 2k) \tag{7}$$

Finally, we substitute equation 7 into equation 2 (we remember that $i \cdot i = -1$) and we have

$$i^i = e^{i \ln (i)} = e^{-\pi(\frac{1}{2} + 2k)} \quad k \in \mathbb{Z} \tag{8}$$

For $k = 0$,

$$i^i = e^{-\frac{\pi}{2}} \approx 0.2079 \tag{9}$$