Math 222 Fall 2019 McGill

Section 2 with Dr. Broderick Causley

Typesetting *attempted* by Justin Bondurant-David

"Peut être l'esprit humain va-t'il
être amené à s'occuper sérieusement,
c'est à dire avec précision de l'aléatoire?"

André Malraux
Hôtes de passage

These notes are *slightly* incomplete, and the author shall not be held liable for decreases in GPA, unexpected hairloss, or content that isn't adapted to people who happen to be color-blind.

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Second edition, September 2019

THE ISBN FOR THESE NOTES IS TRIVIAL AND LEFT AS AN EXERCISE TO THE READER

Not intended for consumption

III Calculus 3

11. Infinite Sequences and Series

September 4, 2019

General Information

- broderick.causley@mcgill.ca
- BURN 1017, office hours to be announced
- There will be a slight difference between both classes of 222
- Kahoot might be used in this class.
- Textbook followed is Stewart's Math Var Calculus, 8^{th}

Grading Scheme

- Webworks 15%
- (optional) Midterm 25%(October 24^{th} at 6pm)
- Final Exam 60%

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§11.2 Series

Definition Sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots, a_n, \dots\}$ is a sequence

Example $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\right\}$

(here, a_n is $\frac{n}{n+1}$)
$\rightarrow$ This sequence has a limit $\lim\limits_{n\to\infty} a_n = 1$

$\rightarrow \lim\limits_{n\to\infty} \frac{n}{n+1} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$ ■

note $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$ is different (it's a series)

Definition Series Given $\{a_n\}_{n=1}^{\infty}$, $\quad a_1 + a_2 + a_3 + \dots = \sum\limits_{n=1}^{\infty} a_n$ is an infinite series.

Definition Partial Sums Given $\{a_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} S_k$ is a limit of <u>partial sums</u>

here, $S_k = \sum_{n=1}^{k} a_n$ is a partial sum

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

■ **Example** $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

■

Definition Geometric Series $a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$ is a <u>geometric series</u> $(a \neq 0)$

■ **Example** $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{2}}_{a} \left(\underbrace{\frac{1}{2}}_{r}\right)^{n-1}$ ■

Theorem $\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r}, \text{ if } |r| < 1 \\ \text{diverges, if } |r| \geq 1 \end{cases}$

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + \ldots + ar^{k-1}$$

$$rS_k = ar + ar^2 + \ldots + ar^k$$

$$S_k - rS_k = (a + \cancel{ar} + \ldots + \cancel{ar^{k-1}}) - (\cancel{ar} + \cancel{ar^2} + \ldots + ar^k)$$

$$S_k - rS_k = a - ar^k$$

$$\Rightarrow S_k = \frac{a - ar^k}{1-r}$$

So $\sum_{n=1}^{\infty} ar^{n-1} = \lim_{k\to\infty} \sum_{n=1}^{k} ar^{n-1} = \lim_{k\to\infty} S_k$

$$= \lim_{k\to\infty} \frac{a - ar^k}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \left(\lim_{k\to\infty} r^k\right)$$

The part in brackets only works if $|r| < 1$, thus shrinks to 0.

If $|r| > 1$, this blows up to ∞ (diverges)

if $r = 1$, $a + a + a + \ldots = \infty$ (diverges) ◀

if $r = -1$, $a - a + a - a + a - a \ldots$ (diverges) ◀

<u>**Test for Divergence**</u> $\sum_{n=1}^{\infty} a_n$

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n = \text{DNE}$, then the series does not converge

Theorem If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

R Note If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ does *not* necessarily converges

■ **Example** $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ this diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$ ■

■ **Example** Show that $0.3333\ldots = \frac{1}{3}$

$$\blacktriangleright\ 0.3333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$$

$$= \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \ldots\right)$$

$$= \sum_{n=1}^{\infty} \frac{3}{10}\left(\frac{1}{10}\right)^{n-1}$$

So $\sum_{n=1}^{\infty} \frac{3}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{a}{1-r} = \frac{3/10}{1-1/10} = \frac{3/10}{9/10} = \frac{1}{3}$ ◀ ■

■ **Example** $\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-\ldots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n = \text{DNE}$, so the series diverges ■

September 6, 2019

General Information

- Office hours: Tue/Thu 10:30am-12pm, BURN 1017
- Tutorials! @11:30am, 3:30pm, BURN 1B36
- Webworks → Assignment #1 → due Sept. 18

Mind Warmup

Take a sequence $\{a_n\}_{n=1}^{\infty}$ and add it together for series $\sum_{n=1}^{\infty} a_n$ which as a limit of partial sums is $\lim_{k\to\infty} a_n = \lim_{k\to\infty} s_k$

Geometric series $\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} \text{ (converges)}, |r| < 1 \\ \text{diverges}, |r| \geq 1 \end{cases}$

"finite" ↔ "converging"

"infinite" ↔ "diverging"

If $\lim_{k\to\infty} a_n \neq 0$ or $\lim_{k\to\infty} a_n = \text{DNE}$, then $\sum_{n=1}^{\infty} a_n = \text{diverges}$

§11.3 Integral Test

I'm confused by what I wrote down here. What's the order of operations between the $\geq, \leq, \xrightarrow{implies}$? Also It seems like I possibly didn't transcribe some stuff properly

$$\sum_{n=1}^{\infty} a_n \leftrightarrow \int_1^{\infty} f(x)dx, \qquad f(n) = a_n$$

1. $\sum_{n=1}^{k} a_n \geq \int_1^{k+1} f(x)dx \xrightarrow{implies} \int_1^{\infty} = \infty$
   - $\sum_{n=1}^{\infty} a_n = \infty$
2. $\sum_{n=2}^{k} a_n \leq \int_1^{k+1} f(x)dx \xrightarrow{implies} \int_1^{\infty}$ finite
   - means $\sum_{n=2}^{\infty} a_n$ finite
     - means $\sum_{n=1}^{\infty} a_n$ finite

Definition Integral Test Suppose $f(x)$, where $f(x) = a_n$ is continuous, decreasing, positive on $[1, \infty)$. Then $f'(x) < 0$ or $a_{n+1} \leq a_n$ and $\sum_{n=1}^{\infty} a_n \leftrightarrow \int_1^{\infty} f(x)dx$both converge or both diverge

■ **Example** Does $\sum_{n=1}^{\infty} \frac{5}{n^2+1}$ converge?

► $f(x) = \frac{5}{x^2+1} \leftarrow$ continuous ✓

$\leftarrow$ positive ✓

$\leftarrow$ decreasing ✓

$$f'(x) = 5(x^2+1)^{-2}(2x) = -\frac{10x}{(x^2+1)^2} \leftarrow \text{ always negative on } [1, \infty)$$

$$\int_1^{\infty} \frac{5}{x^2+1} dx = \lim_{t \to \infty} \int_1^t \frac{5}{x^2+1} dx$$

$$= \lim_{t \to \infty} \left(5 \arctan(x) \Big|_1^t \right)$$

$$= \lim_{t \to \infty} (5 \arctan(t) - 5 \arctan(1))$$

$$= 5(\frac{\pi}{2}) - 5(\frac{\pi}{4}) = \frac{5\pi}{4}$$

$\Rightarrow$ Our series $\sum_{n=1}^{\infty} \frac{5}{n^2+1}$ converges ■

R This does not imply that $\sum_{n=1}^{\infty} \frac{5}{n^2+1} = \frac{5\pi}{4}$. It's just saying both converge.

■ **Example** For what values of p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge

▶ $f(x) = \frac{1}{x^p} \leftarrow$ continuous ✓

$\leftarrow$ positive ✓

$\leftarrow$ decreasing ✓

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t x^{-p} dx$$

$$= \lim_{t\to\infty} \left(\frac{x^{1-p}}{1-p} \Big|_1^t \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-p} \frac{t^{1-p}}{1} - \frac{1}{1-p} \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-p} \frac{1}{t^{p-1}} - \frac{1}{1-p} \right)$$

$\frac{1}{t^{p-1}}$ converges if $p-1>0$, so if $p>1$

$\frac{1}{t^{p-1}}$ diverges if $p-1<0$ ■

Exercise Homework: show that when $p=1$, this diverges (use log) ■

$\Rightarrow$ when $p>1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

$\rightarrow$ otherwise $p \leq 1$ diverges

Fact The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p>1$, otherwise diverges

■ **Example** $\sum_{n=1}^{\infty} \left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}} \right)$

▶ $= (e^{\frac{1}{1}} - e^{\frac{1}{2}}) + (e^{\frac{1}{2}} - e^{\frac{1}{3}}) + (e^{\frac{1}{3}} - e^{\frac{1}{4}}) + \dots$

$$\lim_{k\to\infty} \underbrace{\sum_{n=1}^{k} (e^{\frac{1}{n}} - e^{\frac{1}{n+1}})}_{S_k}$$

$$S_k = (e^{\frac{1}{1}} - \cancel{e^{\frac{1}{2}}}) + \dots + (\cancel{e^{\frac{1}{k}}} - e^{\frac{1}{k+1}})$$

$$= e - e^{\frac{1}{k+1}}$$

$\lim_{k\to\infty} S_k = \lim_{k\to\infty} (e - e^{\frac{1}{k+1}}) = e - 1$ ■

September 9, 2019

General Information

- Webwork is due on September 18
- Office hours: T/Th 10:30am-12pm, 1017 BURN

Mind Warmup

$$\sum_{n=1}^{\infty} a_n \text{ converge or diverge?}$$

$\rightarrow$ geometric series ✓

$\rightarrow \lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n = \text{DNE}$ ✓

$\rightarrow$ Integral test $\sum_{n=1}^{\infty} a_n \leftrightarrow \int_1^{\infty} f(x)dx f(n) = a_n$ ✓

$\rightarrow$ p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ✓

§11.4 Comparison Tests

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be two positive series (have positive terms) $a_n > 0$, $b_n > 0$

1. **Comparison Test** If $a_n \leq b_n$ for all n
   (a) $\sum_{n=1}^{\infty} b_n$ converges $\overset{\text{implies}}{\rightarrow}$ $\sum_{n=1}^{\infty} a_n$ converges
   (b) $\sum_{n=1}^{\infty} a_n$ diverges $\overset{\text{implies}}{\rightarrow}$ $\sum_{n=1}^{\infty} b_n$ diverges
2. **Limit Comparison Test** If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$ (not zero and not infinity)

   Then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both diverge

R Most popular comparisons to make
1) p-series
2) geometric series

■ **Example** Determine whether $\sum_{n=1}^{\infty} \frac{3}{5n^2+8n+13}$ converges/diverges

▶ $\frac{3}{5n^2+8n+13} < \frac{3}{5n^2+8n} < \frac{3}{5n^2}$

$$a_n = \frac{3}{5n^2+8n+13}, \quad b_n = \frac{3}{5n^2}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{3}{5n^2} = \frac{3}{5} \qquad \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{convergent } p\text{-series } (p=2)}$$

$\rightarrow$ Comparison test says that $\sum_{n=1}^{\infty} \frac{3}{5n^2}$ converges $\overset{\text{implies}}{\rightarrow}$ $\sum_{n=1}^{\infty} \frac{3}{5n^2+8n+13}$ ◀

■

■ **Example** Test $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ for convergence using the comparison test

I know it would be easier to use the integral test

▶ $\frac{\ln(n)}{n} > \frac{1}{n}$ because $ln(n) > 1$ (at least, after the first couple terms (i.e. $n \geq 3$)

$$\rightarrow a_n = \frac{\ln(n)}{n}, \quad b_n = \frac{1}{n}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{is diverging Harmonic series (or } p\text{-series, } p = 1)$$

$$\rightarrow \text{ Comparison test says } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges } \xrightarrow{\text{implies}} \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \text{ diverges}$$

■

■ **Example** Test $\sum_{n=1}^{\infty} \frac{n^2+2n}{\sqrt{3+4n^5}}$ for convergence/divergence

▶ idea look at dominating terms

$$a_n = \frac{n^2+2n}{\sqrt{3+4n^5}}$$

$$b_n = \frac{n^2}{\sqrt{n^5}} = \frac{n^2}{n^{\frac{5}{2}}} = \frac{1}{n^{\frac{1}{2}}}$$

$$\rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n^2+2n}{\sqrt{3+4n^5}}\right)}{\left(\frac{1}{n^{\frac{1}{2}}}\right)}$$

$$= \lim_{n\to\infty} \frac{(n^2+2n)n^{\frac{1}{2}}}{\sqrt{3+4n^5}} = \lim_{n\to\infty} \frac{n^{\frac{5}{2}}+2n^{\frac{3}{2}}}{\sqrt{3+4n^5}}$$

$$= \lim_{n\to\infty} \frac{\left(n^{\frac{5}{2}}+2n^{\frac{3}{2}}\right)\frac{1}{n^{\frac{5}{2}}}}{\sqrt{3+4n^5}\,\frac{1}{n^{\frac{5}{2}}}} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\sqrt{\frac{3}{n^5}+4}}$$

■

September 11, 2019

Mind Warmup

$$\sum_{n=1}^{\infty} a_n \begin{cases} < \infty \\ = \infty \\ = \text{DNE} \end{cases}$$

geometric ✓

integral test ✓

Comparison Test ✓

§11.5 Alternating Series

Definition Alternating Series A series is <u>alternating</u> if it can be written as

$$\sum_{n=1}^{\infty} (-1)^n b_n \text{ or } \sum_{n=1}^{\infty} (-1)^{n-1} b_n \qquad \text{where } b_n > 0$$

Definition Alternating Series Test If $\sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series and

1. $b_{n+1} < b_n$ (decreasing)
2. $\lim_{n\to\infty} b_n = 0$

$\rightarrow$ then this series converges

■ **Example** Does $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$ converge?

▶ $cos(\pi n) = (-1)^n \rightarrow \frac{\cos(\pi n)}{n} = \frac{(-1)^n}{n} \leftarrow$ alternating ✓

$$b_n = \frac{1}{n} \leftarrow \frac{1}{n+1} < \frac{1}{n}, \quad b_{n+1} < b_n ✓$$

$$\leftarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

$\rightarrow$ the alternating series test says that $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$ converges ■

R $\sum_{n=1}^{\infty}\frac{(-1)^n}{n} \leftarrow$ alternating Harmonic series.

■ **Example** Does $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+e^n}$ converge?

▶ alternating ✓

$$b_n = \frac{1}{n+e^n} \leftarrow \lim_{n\to\infty}\frac{1}{n+e^n} = 0$$

$$\leftarrow b_n \text{ is decreasing}$$

$$\frac{d}{dx}\left(\frac{1}{n+e^n}\right) = -(n+e^n)^{-2}(1+e^n)$$

$$= -\frac{1+e^n}{(n+e^n)^2} < 0 ✓$$

$\rightarrow$ alternating series test says $\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n+e^n}$ converges ■

$\rightarrow$ Let's fine tune some definitions of convergence

§11.6 Absolute Convergence & Ratio/Root test

not expected to know the estimation test? I didn't quite get what he said

Definition Absolutely Convergent $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty}|a_n|$ is convergent

Definition Conditionally Convergent $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it converges, but not absolutely

Theorem $\sum_{n=1}^{\infty}|a_n|$ convergent $\xrightarrow{\text{implies}}$ $\sum_{n=1}^{\infty} a_n$ convergent

$\rightarrow$ An absolutely convergent series is convergent

■ **Example** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \leftarrow$ converges ✓

$\rightarrow$ but $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the diverging Harmonic series

$\rightarrow$ therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent ■

■ **Example** $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ converge?

$$\left| \frac{\cos(n)}{n^2} \right| \leq \frac{1}{n^2}$$

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ is converging p-series ($p = 2$)

$\rightarrow$ So the comparison test says that $\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^2} \right|$ convergent

$\rightarrow$ Hence, $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely convergent ■

September 13, 2019

General Information

- Add drop is coming up on the 17

Mind Warmup

$\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges

$\sum_{n=1}^{\infty} a_n$

-diverges

-converges but not absolutely (conditional)

-converges absolutely

R If a series converges absolutely, it must be convergent

Definition Ratio Test Let $\sum_{n=1}^{\infty} a_n$ be a series

1. If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, $\sum_{n=1}^{\infty} a_n$ <u>absolutely converges</u>
2. If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, $\sum_{n=1}^{\infty} a_n$ <u>diverges</u>
3. If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, $\sum_{n=1}^{\infty} a_n$ <u>Inconclusive</u>

■ **Example** $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{3^n}$ ← determine type of convergence

$$\blacktriangleright \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}((n+1)^2+1)}{3^{n+1}}}{\frac{\cancel{(-1)^n}(n^2+1)}{3^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2+1}{3^{n+1}}\frac{3^n}{n^2+1}\right|$$

$$= \lim_{n\to\infty}\frac{1}{3}\frac{(n+1)^2+1}{n^2+1}$$

$$= \lim_{n\to\infty}\frac{1}{3}\frac{n^2+2n+2}{n^2+1}$$

$$= \frac{1}{3}1 = \frac{1}{3} < 1$$

$\rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{3^n}$ converges absolutely ◀ ■

■ **Example** $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ ← determine the type of convergence

$$\blacktriangleright \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!} \qquad (2n+2)! = (2n+2)(2n+1)(2n)!$$

$$= \lim_{n\to\infty}\frac{(n+1)\cancel{n!}(n+1)\cancel{n!}\cancel{(2n)!}}{\cancel{n!}\cancel{n!}(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$\text{version 1)} = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$$

$$\text{version 2)} = \lim_{n\to\infty}\frac{\cancel{(n+1)}(n+1)}{2\cancel{(n+1)}(2n+1)} = \lim_{n\to\infty}\frac{n+1}{4n+2} = \frac{1}{4}$$

→ absolutely convergent ■

R If you see a factorial, in general you should use the ratio test

$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ → same steps

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 4$ → diverges

$\lim_{n\to\infty}\frac{(2n)!}{(n!)^2} = \ldots \qquad \lim_{n\to\infty} a_n \neq 0$

Definition Root Test Let $\sum_{n=1}^{\infty} a_n$ be a series

1. $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, <u>absolutely convergent</u>

2. $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, <u>divergent</u>
3. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, <u>inconclusive</u>

■ **Example** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{n+5}}{(n+1)^n}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{\cancel{(-1)^{n+1}} 2^{n+5}}{(n+1)^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{2^{n+5}}{(n+1)^n}}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{2^5 2^n}{(n+1)^n}}$$

$$= \lim_{n\to\infty} \sqrt[n]{2^5 \left(\frac{2}{n+1}\right)^n}$$

$$= \lim_{n\to\infty} \underbrace{2^{\frac{5}{n}}}_{=1} \underbrace{\left(\frac{2}{n+1}\right)}_{=0}$$

$$= 1 \cdot 0 = 0 < 1$$

$\rightarrow$ absolutely convergent ■

September 16, 2019

Mind Warmup

$\sum_{n=1}^{\infty} a_n$ converges, good ✓

$\sum_{n=1}^{\infty} |a_n|$ converges, better ✓

;Tools: ratio test, root test, integral test, comparison test, limit comparison test, geometric series, divergence test

§11.8 Power Series

Functions: $\sin(x)$, e^x, $\ln(x)$, x^2+3x+1, $\frac{5x}{x-2}$

Polynomials are the easiest function around

Idea for the week: write complicated functions as polynomials (infinite)

Definition Power Series A <u>power series</u> in x has the form $\sum_{n=0}^{\infty} c_n x^n$, where $c_n \in \mathbb{R}$ are coefficients (the fingerprint)

R $\rightarrow$ note, if we let $x =$ number, we know this already (11.2-11.6)

■ **Example** Let $c_n = 1$ for all n

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$\rightarrow$ try $x = \frac{1}{2}$, $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \dots = \frac{1}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

$\rightarrow$ try $x = 2$, $\sum_{n=0}^{\infty} (2)^n = 1 + 2 + 4 + 8 + \dots$ diverges

$\rightarrow$ therefore $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$ ■

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

Definition A power series in $(x-a)$, or in other words centered at $x=a$, has the form $\sum_{n=0}^{\infty} c_n(x-a)^n$

■ **Example** $\sum_{n=0}^{\infty} (x-13)^n$

$\rightarrow \sum_{n=0}^{\infty} (x-13)^n = \dfrac{1}{1-(1-13)}$, if $|x-13|<1$

$$\begin{aligned} |x-13| &< 1 \\ -1 < x-13 &< 1 \\ -1+13 < x &< 1+13 \\ 12 < x &< 14 \end{aligned}$$

■

■ **Example** For what values of x does $\sum_{n=0}^{\infty} \dfrac{(x-3)^n}{n}$ converge? ▶ $a_n = \dfrac{(x-3)^n}{n}$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\frac{n}{(x-3)^n}\right| = \lim_{n\to\infty}\left|(x-3)\frac{n}{n+1}\right| \\ &= |x-3|\lim_{n\to\infty}\frac{n}{n+1} = |x-3|\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} \\ &= |x-3|\cdot\frac{1}{1-0} = |x-3| \end{aligned}$$

$\rightarrow$ this converges when $|x-3|<1$

$$\begin{aligned} |x-3| &< 1 \\ -1 < x-3 &< 1 \\ -1+3 < x &< 1+3 \\ 2 < x &< 4 \end{aligned}$$

$\rightarrow$ therefore, $\sum_{n=0}^{\infty} \dfrac{(x-3)^n}{n}$ converges (absolutely) when $2<x<4$

$\rightarrow$ not finished!

try $x=2$, $\sum_{n=0}^{\infty} \dfrac{(2-3)^n}{n} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n}$ $\leftarrow$ this is the converging alt. Harmonic series.

try $x=4$, $\sum_{n=0}^{\infty} \dfrac{(4-3)^n}{n} = \sum_{n=0}^{\infty} \dfrac{1^n}{n} = \sum_{n=0}^{\infty} \dfrac{1}{n}$ $\leftarrow$ diverging Harmonic series

So actually, the series converges when $2 \le x < 4$ ■

Theorem Given $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities

1. The series converges only when $x = a$
2. The series converges for all $x \in \mathbb{R}$
3. There is a positive number $R \in \mathbb{R}$ such that this series converges when $|x-a| < R$, and diverges when $|x-a| > R$

September 18, 2019

Mind Warmup

What values of x does $\sum_{n=0}^{\infty} c_n(x-a)^n$ converge?

1. Only when $x = a$ $\quad (R = 0)$
2. For all values of $x \in \mathbb{R}$ $\quad (R = \infty)$
3. There is an $R \in \mathbb{R}$, where $0 < R < \infty$, such that this series converges when $|x-a| < R$, diverges $|x-a| > R$ $\quad (R = R)$

Definition Radius of Convergence The value of R above is called the radius of convergence for $\sum_{n=0}^{\infty} c_n(x-a)^n$

Definition Interval of Convergence The interval of convergence for $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the set of all x such that the series converges

R $\rightarrow$ note If $0 < R < \infty$, check endpoints

■ **Example** Find the radius/interval of convergence for $\sum_{n=0}^{\infty} \frac{(-5)^n}{\sqrt{n+1}} x^n$

Idea: -find the radius, usually ratio/root test

-If $0 < R < \infty$, check endpoints

$$\blacktriangleright \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-5)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\frac{(-5)^n x^n}{\sqrt{n+1}}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-5)^{n+1}x^{n+1}}{(-5)^n x^n} \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$= \lim_{n\to\infty} \left| 5x \frac{\sqrt{n+1}}{\sqrt{n+2}} \right| = 5|x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}}$$

$$= 5|x| \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{2}{n}}} = 5|x| \frac{\sqrt{1+0}}{\sqrt{1+0}} = 5|x|$$

$\rightarrow$ so this converges if

$$5|x| < 1$$

$$|x| < \frac{1}{5}$$

$$-\frac{1}{5} < x < \frac{1}{5}$$

$\to R = \frac{1}{5}$
$\to$ check endpoints

$\to$ try $x = -\frac{1}{5}$ $\to \sum_{n=0}^{\infty} \frac{(-5)^n}{\sqrt{n+1}} \left(-\frac{1}{5}\right)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ $\to$ this diverges (compare with p-series $p = \frac{1}{2}$)

$\to$ try $x = \frac{1}{5}$ $\sum_{n=0}^{\infty} \frac{(-5)^n}{\sqrt{n+1}} \left(\frac{1}{5}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ $\to$ this converges (use alternating series test)

$\to$ therefore, interval of convergence is

$$\left(-\frac{1}{5}, \frac{1}{5}\right]$$

$$-\frac{1}{5} < x \leq \frac{1}{5} \blacktriangleleft$$

■

§11.10 Taylor Series

(we'll return to 11.9)

main goal $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

drawing on board of tangent line at $x = a$

Missing one board of notes as the prof erased stuff in different order than usual.

future fact: the tangent line is actually just the first two terms of the Taylor Series

Let's Start $\sum_{n=0}^{\infty} c_n (x-a)^n = f(x)$

1a) $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$

1b) If we plug in $x = a$, $f(a) = c_0$

2a) (take a derivative)

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$

2b) If we plug in $x = a$, $f'(a) = c_1$

3a) (Take another derivative)

$f''(x) = 2c_2 + 3cdot2 \cdot c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + \dots$

3b) $f''(a) = 2c_2$

4a+4b) $f'''(a) = 3 \cdot 2 \cdot c_3 = 3!c_3$

5+) $$\underbrace{f^{(n)}(a)}_{n \text{ derivatives of } f(x)} = n!c_n \to c_n = \frac{f^{(n)}(a)}{n!}$$

note $0! = 1$, so $c_0 = f(a)$ is okay

Definition Taylor Polynomial A Taylor polynomial of degree k for $f(x)$ at $x = a$ is $\sum_{n=0}^{k} c_n(x-a)^n = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Definition Taylor Series A Taylor Series for $f(x)$ at $x = a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Definition Maclaurin Series A Maclaurin series for $f(x)$ is just the Taylor series when $a = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

September 20, 2019

Mind Warmup

$\sum_{n=0}^{\infty} C_n(x-n)^n$

<u>Radius of convergence</u>

1. $x=a, R=0$
2. $x\in\mathbb{R}, R=\infty$
3. $|x-a|<R, R=R$

drawing on board "These are the values where the series converges."

$f(x)=3\sin(x)\ f(\frac{1}{3})$ hard

$g(x)=9x^2+3x\ g(\frac{1}{3})$ easy

$T_k(x)=\sum_{n=0}^{k}\frac{f^{(n)}(a)}{n!}(x-a)^n$ K$^{\text{th}}$ degree Taylor polynomial

$\lim_{k\to\infty}T_k(x)=\sum_{n=0}^{k}\frac{f^{(n)}(a)}{n!}(x-a)^n$ Taylor Series

■ **Example** $f(x)=e^x$

1. Find $T_4(x)$ at $x=0$
2. Find Maclaurin series

▶ $$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}(0)}{n!}x^n$$
$$=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$f(x)=e^x\to f'(x)=e^x\to f^{(n)}(x)=e^x\to f^{(n)}(0)=e^0=1$

$\Rightarrow T_4(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4$ ■

▶ Maclaurin series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Question: Does $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$? $\to$ for all values of x?

Question: if yes, what is the error of $e^x-\sum_{n=0}^{k}\frac{x^n}{n!}$?

In general:

- Is $f(x)\approx T_k(x)$? For any x?
- Is $f(x)\approx T_\infty(x)$? For any x?

Definition Let the k^{th}-order remainder for Taylor polynomial $T_k(x)$ of $f(x)$ be

$$R_k(x)=f(x)-T_k(x)$$

Theorem — Taylor's Inequality . (Taylor's inequality) $\to$ how to bound $R_k(x)$
Let $I=(a-R,a+R)$ or any other interval that contains a, and suppose that $\left|f^{(k+1)}(x)\right|\le M$ for all $x\in I$, then
$|R_k(x)|\le\frac{M}{(k+1)}|x-a|^{k+1}$ for all $x\in I$

$\to$ We find M, it's easy.

$\to$ just see how big $f^{(k+1)}(x)$, and take $M=$ maximum

$\to$ We decide I too

Very Important Limit

$\lim_{n\to\infty}\frac{x^n}{n!}=0$ for any choice of x

► Consider $x > 0$, let $M = \lceil x \rceil \leftarrow$ (the ceiling function)

$$\frac{x^n}{n!} = \frac{x}{1}\frac{x}{2}\cdots\frac{x}{n}$$
$$= \underbrace{\frac{x}{1}\frac{x}{2}\cdots\frac{x}{M}}_{B}\underbrace{\frac{x}{M+1}\frac{x}{M+2}\cdots\frac{x}{n-1}}_{C_n}\frac{x}{n} = B\cdot C_n\frac{x}{n}$$

$$x \leq M,\ \text{so } x < M+1 \rightarrow \frac{x}{M+1} < 1$$
$$x < M+2 \rightarrow \frac{x}{M+2} < 1$$

etc.

$\rightarrow$ So now $\frac{x^n}{n!} = B\cdot C_n\frac{x}{n} \leq B\frac{x}{n}$

$0 < \frac{x^n}{n!} \leq B\frac{x}{n} \qquad B\frac{x}{n} \xrightarrow{k\to\infty} 0$

$\rightarrow$ since $\lim_{n\to\infty} B\frac{x}{n} = 0$, squeeze theorem tells us that $\lim_{n\to\infty}\frac{x^n}{n!} = 0$

$\rightarrow$ consider $x < 0$

$$\lim_{n\to\infty}\left|\frac{x^n}{n!}\right| = \lim_{n\to\infty}\frac{|x|^n}{n!} = 0 \text{ from first part } ◄$$
$$e^x \stackrel{?}{=} \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$\rightarrow$ we can show this is true by $\lim_{n\to\infty}|R_k(x)| = 0$ (squeeze theorem)

September 23, 2019

Mind Warmup

$\sum_{n=0}^{\infty} a_n$ -series $\qquad \sum_{n=0}^{\infty} C_n(x-a)^n$ -power series

$T_k(x) = \sum_{n=0}^{k}\frac{f^{(n)}(0)}{n!}(x-a)^n$ - k^{th} order Taylor polynomial at $x = a$

$R_k(x) = f(x) - T_k(x)$ -k^{th} order error

Question: does $f(x) = \lim_{k\to\infty} T_k(x)$

Proposition: If $R_k(x) = f(x) - T_k(x)$, and if $\lim_{k\to\infty} R_k(x) = 0$, then $f(x) =$ Taylor Series

► $R_k(x) = f(x) - T_k(x)$

$\lim_{k\to\infty} R_k(x) = \lim_{k\to\infty}(f(x) - T_k(x))$

$0 = f(x) - \lim_{k\to\infty} T_k(x)$

$\Rightarrow f(x) = \lim_{k\to\infty} T_k(x) = \sum_{n=0}^{\infty}\frac{n^{(n)}(a)}{n!}(x-a)^n$ ◄

Fact: we need $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ for any $x \in \mathbb{R}$

Theorem we need (Taylor's inequality)

Let I be an interval containing $x = a$ and suppose that $\left|f^{(k+1)}(x)\right| \leq M$ for any $x \in I$ Then

$$|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1} \text{ for any } x \in I$$

■ **Example** Show $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $[0,1]$ using $R_k(x)$.

(Last class, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, was Maclaurin series

▶ $I = [0,1]$

$M = ? \longrightarrow f^{(k+!)}(x) = e^x$

$$0 \leq x \leq 1$$

$$e^0 \leq e^x \leq e^1$$

$$1 \leq e^x \leq e \qquad \text{where } e \text{ is our } M$$

So now

$0 \leq |R_k(x)| \leq \frac{M}{(k+1)!}|x-0|^{k+1} = \frac{e}{(k+1)!}|x|^{k+1}$

$$\lim_{k\to\infty} \frac{e}{(k+1)!}|x|^{k+1} = 0$$

$$\left(\lim_{k\to\infty} \frac{x^k}{k!} = 0\right) \text{vspace*0.1cm}$$

$\rightarrow$ So by squeeze theorem,

$\lim_{k\to\infty} R_k(x) = 0$

$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ■

■ **Example** Take $f(x) = \sin(x)$

1. Find Maclaurin series
2. Show $\sin(x) =$ Maclaurin series (i.e. show $R_k(x) \rightarrow 0$)

▶ $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

$$f(x) = \sin(x) = f^{(4)}(x)$$

$$f'(x) = \cos(x) = f^{(5)}(x)$$

$$f''(x) = -\sin(x) = f^{(6)}(x)$$

$$f'''(x) = -\cos(x) = f^{(7)}(x)$$

$\sin(0) = 0$

$\cos(0) = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 - \frac{1}{7!}x^7 + \ldots$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \ldots$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \qquad \text{step 1 } \checkmark$$

▶ we know $|\sin(x)| \leq 1$, $|\cos(x)| \leq 1$

so we also know $\left|f^{(k+1)}(x)\right| \leq 1$ where 1 is our M

so $0 \leq |R_k(x)| \leq \dfrac{M}{(k+1)!}|x-0|^{k+1} = \dfrac{|x|^{k+1}}{(k+1)!}$

$$\lim_{k\to\infty} \frac{|x|^{k+1}}{(k+1)!} = 0$$

$\to$ so by squeeze theorem,

$\lim_{k\to\infty} R_k(x) = 0$

$\Rightarrow$ Hence, $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$\sin(x) = x - \dfrac{1}{6}(x)^3 + \dots$

$\displaystyle\lim_{n\to 0} \frac{\sin(n)}{n} = 1$

$\sin(x) \approx x,$ x is small. ■

September 25, 2019

Mind Warmup

$f(x) \longrightarrow T_k(x) = \displaystyle\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$\longrightarrow f(x) = \displaystyle\lim_{k\to\infty} T_k(x)$ if $\displaystyle\lim_{k\to\infty} R_k(x) = 0$

Q: Now that we can find power series, how can we manipulate them?

§11.9 Representing Functions as Power Series

Given $\displaystyle\sum_{n=0}^{\infty} C_n(x-a)^n$, how can we manipulate it?

1. differentiate/integrate it
2. multiply by copies of $(x-a)^k$
3. replace x by something else (substitution)
4. (multiply/divide two series)

Theorem If $f(x) = \displaystyle\sum_{n=0}^{\infty} C_n(x-a)^n$ with radius of convergence R, then $f(x)$ is differentiable and

1. $f'(x) = \displaystyle\sum_{n=1}^{\infty} C_n(x-a)^{n-1}$
2. $\displaystyle\int f(x)dx = \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1} + C$

R $\to$ note: The radius of convergence is the same, but the endpoints of the interval of convergence could change.

■ **Example** Find Maclaurin series of $\cos(x)$

▶ $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$$\begin{aligned}\cos(x) = \frac{d}{dx}(\sin(x)) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dx}\left(x^{2n+1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \blacktriangleleft\end{aligned}$$

■

■ **Example** Find the Maclaurin series of $x^2\cos(x^2)$

▶ $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$$\begin{aligned}\cos(x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^2)^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}\end{aligned}$$

$$\begin{aligned}x^2\cos(x^2) &= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+2}\end{aligned}$$

■

■ **Example** Find Maclaurin series of $\frac{1}{1+x^3}$

▶ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\begin{aligned}\frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=0}^{\infty}(-x^3)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{3n} \quad \blacktriangleleft\end{aligned}$$

$\rightarrow$ is true if $|x| < 1$, but for us it was $|-x^3| < 1 \Leftrightarrow |x| < 1$ ■

■ **Example** Find Maclaurin series of $\frac{x^2}{4-9x^2}$

▶ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\begin{aligned}
\frac{x^2}{4-9x^2} &= x^2 \cdot \frac{1}{4-9x^2} \\
&= \frac{x^2}{4} \cdot \frac{1}{1-\frac{9}{4}x^2} \\
&= \frac{x^2}{4} \cdot \sum_{n=0}^{\infty} \left(\frac{9}{4}x^2\right)^n \\
&= \frac{x^2}{4} \cdot \sum_{n=0}^{\infty} \frac{9^n x^{2n}}{4^n} \\
&= \sum_{n=0}^{\infty} \frac{9^n x^{2n+2}}{4^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{3^{2n} x^{2n+2}}{2^{2n+2}}
\end{aligned}$$

→ to use $\frac{1}{1-x}$, we needed $|x| < 1$ which for us was $\left|\frac{9}{4}x^2\right| < 1$

$$\begin{aligned}
\left|\frac{9}{4}x^2\right| &< 1 \\
\frac{9}{4}|x^2| &< 1 \\
|x|^2 &< \frac{4}{9} \\
|x| &< \frac{2}{3}
\end{aligned}$$

$R = \frac{2}{3}$ ■

■ **Example** Find Maclaurin series of $\ln(1+x)$

▶ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\begin{aligned}
\ln(x+1) &= \int \frac{dx}{x+1} \\
&= \int \frac{1}{1-(-x)} dx \\
&= \int \sum_{n=0}^{\infty} (-x)^n dx \\
&= \sum_{n=0}^{\infty} (-1)^n \int x^n dx \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right) + C
\end{aligned}$$

→ to remove constant C, we test values of x we know $(x = 0)$

$\ln(0+1) = \sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1} + C$

$\Rightarrow C = 0$

$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ ■

September 27, 2019

Mind Warmup

$\sum_{n=0}^{\infty} a_n$ -series

$C_n(x-a)^n$ -power series

given $f(x)$ at $x=a$, $t_K(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ -Taylor series

§11.10 How to manipulate series (11.9/11.10)

→ Last class, we took derivatives, integrals, we substituted, and we multiplied by $(x-a)^k$

→ What about multiplying/dividing two series together?

■ **Example** Find Maclaurin series of $\frac{3\cos(x)}{1-x}$

▶ $$\begin{aligned}\frac{3\cos(x)}{1-x} &= 3\cos(x)\cdot\frac{1}{1-x}\\ &= 3\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} x^n\right)\\ &= 3\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\dots\right)\left(1+x+x^2+\dots\right)\\ &= 3\left(\underbrace{1-\frac{1}{2}x^2+\frac{1}{24}x^4+\dots}+\underbrace{x-\frac{1}{2}x^3+\frac{1}{24}x^5+\dots}+\underbrace{x^2-\frac{1}{2}x^4+\frac{1}{24}x^6}\right)\\ &= 3+3x+\frac{3}{2}x^2+\dots\end{aligned}$$
we didn't try to find a general formula for this in class

■

■ **Example** Find Maclaurin series of $\tan(x)$

▶ $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x-\frac{1}{6}x^3+\frac{1}{120}x^5+\dots}{1-\frac{1}{2}x^2+\frac{1}{24}x^4+\dots}$

"I expect you to memorize $\sin(x), \cos(x), e^x, \dots$" I didn't really hear everything function that was mentioned here

$$\begin{array}{r|l} & x+\frac{1}{3}x^3+\dots \\ \hline 1-\frac{1}{2}x^2+\frac{1}{24}x^4+\dots & x-\frac{1}{6}x^3+\frac{1}{120}x^5+\dots \\ & -\left(x-\frac{1}{2}x^3+\frac{1}{24}x^5+\dots\right) \\ \cline{2-2} & \frac{1}{3}x^3-\frac{1}{30}x^5+\dots \\ & -\ \frac{1}{3}x^3-\frac{1}{6}x^5+\dots \\ \cline{2-2} & R \end{array}$$

$\tan(x) = x+\frac{1}{3}x^3+\dots$ ■

Some applications

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ = a precise value?

We can find the exact sum for series if we are lucky and have a Taylor expansion for them. (besides geometric, telescoping)

■ **Example** Find the infinite sum

$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (\ln 2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!} \qquad \text{fact } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= e^{-\ln 2}$$

$$= \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \frac{1}{2} \qquad \blacktriangleleft$$

■

■ **Example** Find Maclaurin series of $(1+x)^k$ we did not do this example in class ■

■ **Example** $\int e^{-x^2} dx$ -no good solution

$$\int_0^{\frac{1}{2}} e^{-x^2} dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} dx = \int_0^{\frac{1}{2}} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots\right) dx$$

$$= \left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \dots\right)\Big|_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right) + \frac{1}{10}\left(\frac{1}{32}\right) - \frac{1}{42}\left(\frac{1}{128}\right) + \dots$$

$$= \frac{4133}{8960} + \dots \approx 0.46127$$

■

September 30, 2019

mcgill.ca\tutoring

jacob.beaudry@mail.mcgill.ca (math 222)

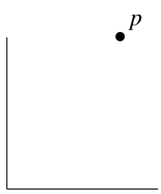
Mind Warmup

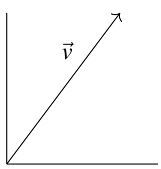
12. Vectors

§12.1 Vectors (12.1-12.3)

$p = (\underbrace{3}_{x}, \underbrace{4}_{y})$

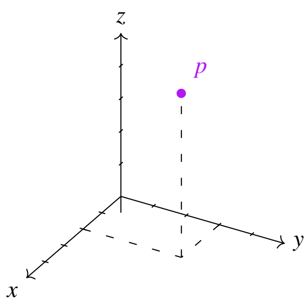
$\vec{v} = < 3,4 >$



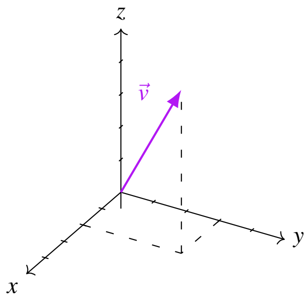


Definition magnitude The length of vector $\vec{v} = < v_1, v_2 >$ is $\|\vec{v}\| = \sqrt{{v_1}^2 + {v_2}^2}$

$p = (\underbrace{2}_{x}, \underbrace{3}_{y}, \underbrace{5}_{z})$



$\vec{v} = < 2,3,5 >$



Definition The length of vector $\vec{v} =< v_1, v_2, v_3 >$ is $||\vec{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

Fact: If $c \in \mathbb{R}$, then $||c\vec{v}|| = |c| \cdot ||\vec{v}||$

■ **Example** Find the length of $\vec{v} =< -7, -14, -14 >$

▶ 1.$||\vec{v}|| = \sqrt{(-7)^2 + (-14)^2 + (-14)^2}$

$= \sqrt{49 + 196 + 196}$

$= \sqrt{441} = 21$

2.$\vec{v} = -7 < 1, 2, 2 >$

$||\vec{v}|| = |-7|||< 1, 2, 2 >||$

$= 7\sqrt{1^2 + 2^2 + 2^2}$

$= 7 \cdot \sqrt{9} = 21$

■

Definition Unit Vector Normalizing vector $\vec{v}$ means rescaling it by $\frac{1}{||\vec{v}||}$.

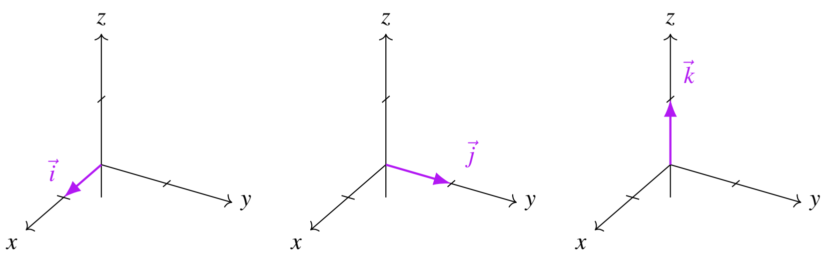
Note the new vector $\frac{\vec{v}}{||\vec{v}||}$ is a unit vector (of length 1)

Definition Basis Vectors (special unit vectors in $\mathbb{R}^3$)

$\vec{i} =< 1, 0, 0 >$

$\vec{j} =< 0, 1, 0 >$

$\vec{k} =< 0, 0, 1 >$



R If $\vec{v} =< a, b, c >$, then $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$

■ **Example** Find the length of $\vec{w} = 2\vec{i} - \vec{j} - 2\vec{k}$

▶ $\vec{w} = < 2, -1, -2 >$

$\|\vec{w}\| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$ ◀ ■

§12.3 Dot Product

Definition Dot Product Let $\vec{u} = < a_1, a_2, a_3 >$, $\vec{v} = < b_1, b_2, b_3 >$

The dot product of $\vec{u}$ and $\vec{v}$ is the number

$$\vec{u} \cdot \vec{v} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$\rightarrow$ Note: also called scalar product, inner product

Facts

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ order doesn't matter
2. $\vec{u} \cdot \vec{u} = (a_1)^2 + (a_2)^2 + (a_3)^2 = \|\vec{u}\|^2$ dot product with itself is related to length
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

Theorem — Angle Formula. Let $\vec{u}$, $\vec{v}$ be nonzero vectors, then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

where θ is the angle between them.

$\rightarrow$ this also gives $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

Note: Two vectors are perpendicular/orthogonal if the angle between them is $\frac{\pi}{2}$ radians, 90°

$\cos\left(\frac{\pi}{2}\right) = 0 \longrightarrow$ what does that tell us??

R **Important**: Two vectors are orthogonal if their dot product is 0

Remember basis vectors $\vec{i}$, $\vec{j}$, $\vec{k}$?

They are orthogonal to each other, i.e

$$\vec{i} \cdot \vec{j} = 0, \quad \vec{j} \cdot \vec{k} = 0, \quad \vec{k} \cdot \vec{i} = 0$$

$\rightarrow$ In other words, the basis vectors are 90° or $\frac{\pi}{2}$ radians away from each other.

October 2, 2019

Mind Warmup

$\vec{v} = < v_1, v_2, v_3 >$ is a vector with length $\|\vec{v}\| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ (basis vectors)

$\vec{u} = < u_1, u_2, u_3 >$

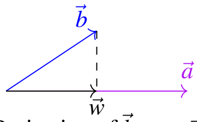
$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$, where θ is angle between $\vec{u}$ and $\vec{v}$

MISSING DIAGRAM

$\rightarrow$ because $\cos\left(\frac{\pi}{2}\right) = 0$, we have that $\vec{u} \cdot \vec{v} = 0$ when $\vec{u} \perp \vec{v}$, $\perp$=orthogonal to

(Scalar) Projections



Projection of $\vec{b}$ onto $\vec{a}$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\|\vec{w}\|}{\|\vec{b}\|} \longrightarrow \|\vec{w}\| = \|\vec{b}\|\cos\theta$$

$\rightarrow$ also know, $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$

$$= \|\vec{a}\|\|\vec{w}\|$$

$\rightarrow \|\vec{w}\| = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} \quad \leftarrow$ the length of $\vec{w}$

$\rightarrow \vec{w}$ same direction as $\vec{a}$

also same direction as $\dfrac{\vec{a}}{\|\vec{a}\|}$

$\longrightarrow$ so $\vec{w} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}\dfrac{\vec{a}}{\|\vec{a}\|}$

R $\quad \text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$ the scalar projection of $\vec{b}$ onto $\vec{a}$ (number)

$\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$ the vector projection of $\vec{b}$ onto $\vec{a}$ (vector)

■ **Example** Find the vector projection of $\vec{b} =< -3,5,8 >$ onto $\vec{a} =< 1,-2,2 >$

▶ $\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a} =$

$$\vec{a}\cdot\vec{b} =< 1,-2,2 > \cdot < -3,5,8 >= -3-10+16 = 3$$

$$\|\vec{a}\| = \sqrt{1^2+(-2)^2+2^2} = \sqrt{9} = 3$$

$$= \frac{3}{(3)^2} < 1,-2,2 >=< \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} > ◀$$

■

Definition Cross Product Let $\vec{u} =< a_1,a_2,a_3 >$, $\vec{v} =< b_1,b_2,b_3 >$ Then the cross product of $\vec{a}$ and $\vec{b}$

The vector $\vec{u}\times\vec{v} =< a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 >$

This comes from

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Facts

1. $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
   - $(\vec{u}\times\vec{v})\cdot\vec{u} = 0$
   - $(\vec{u}\times\vec{v})\cdot\vec{v} = 0$
2. The direction of $\vec{u}\times\vec{v}$ is given by the right hand rule.
   Fingers go from vector $\vec{u}$ to $\vec{v}$, and thumb is pointing in the direction $\rightarrow \vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

3. $||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{v}|| \sin\theta$

   $\rightarrow$ if $\theta = 0$, then $\vec{u} \times \vec{v} =< 0,0,0 >$

4. MISSING DRAWING OF A PARALLELOGRAM

   area of parallelogram is base $\times$ height

   $$||\vec{u}||h = ||\vec{v}||\,||\vec{u}|| \sin\theta = ||\vec{u} \times \vec{v}|| \qquad \sin\theta = \frac{h}{||\vec{v}||}$$ not sure if I transcribed this line properly

■ **Example** Find the cross product of $\vec{j}$ and $\vec{k}$

► $\vec{j} =< 0,1,0 >$

$\vec{k} =< 0,0,1 >$

$$\vec{j} \times \vec{k} =< 1 \cdot 1 - 0 \cdot 0,\ \ 0 \cdot 0 - 0 \cdot 1,\ \ 0 \cdot 0 - 0 \cdot 1 >$$

$$=< 1,0,0 >$$

$$= \vec{i}$$

$$\vec{j} \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= \vec{i}(1) - \vec{j}(0) + \vec{k}(0) = \vec{i}$$

■

October 4, 2019

Mind Warmup (§12.1-12.4)

Vector $\vec{u}$ has length $||\vec{v}||$

$\vec{u} \cdot \vec{v} = ||\vec{u}||\,||\vec{v}|| \cos\theta$

$||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{v}|| \sin\theta$

$\rightarrow \vec{u} \times \vec{v}$ is a third vector which is orthogonal to both $\vec{u}$ and $\vec{v}$

$(\vec{u} \times \vec{v}) \cdot \vec{u} = 0 \qquad (\vec{u} \times \vec{v}) \cdot \vec{v} = 0$

§12.5 Lines and Planes

$y = mx + b$ - equation of a line

$y = 2x - 5$ -slope is 2, y intercept is -5

$r(t) =< t,\ -5 >$ -same line in parametric form

$$= \underbrace{< 0,-5 >}_{y\text{-int}} + t \underbrace{< 1,2 >}_{\text{slope}}$$

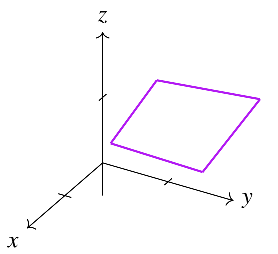
In 3D, we can write

$$r(t) =< x_0 + at, y_0 + bt, z_0 + ct >$$

Definition Parametric Equations of a line The parametric equations of a line through point (x_0, y_0, z_0) and parallel to vector $\vec{v} =< a, b, c >$ is $\begin{cases} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{cases}$

■ **Example** $\vec{r}(t) =< 3-2t, 1+t, 5-8t >$ is an equation for a line which passes through the point $(3,1,5)$ when $t = 0$ and $(1,2,-3)$ when $t = 1$, etc. ■

Planes



(x_0, y_0, z_0) is an actual fixed point on the plane
(x, y, z) is any other general point on the plane
$\vec{q} =< x-x_0, y-y_0, z-z_0 >$ - a vector between two points
$\vec{n}$ the normal vector is orthogonal to the plane
$\rightarrow$ Fact $\vec{n}$ is also orthogonal to $\vec{q}$
$\Rightarrow < a, b, c > \cdot << x-x_0, y-y_0, z-z_0 >= 0$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Definition Scalar Equation of a Plane The scalar equation of a plane through point (x_0, y_0, z_0) with normal vector $\vec{n} =< a, b, c >$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$\rightarrow ax + by + cz + d = 0$ $\quad d = ax_0 - by_0 - cz_0$
$\rightarrow$ now we know for any $ax + by + cz + d = 0$ the normal vector is $< a, b, c >$
$\rightarrow$ aka this plane is perpendicular to $< a, b, c >$

■ **Example** Find the equation of a plane which contains the points $(0,0,0)$, $(3,-2,1)$, $(1,1,1)$
► Two vectors on this plane are

$$\vec{u} =< 3-0, -2-0, 1-0 >=< 3, -2, 1 >$$

$$\vec{v} =< 1-0, 1-0, 1-0 >=< 1, 1, 1 >$$

$\vec{u} \times \vec{v} \leftarrow$ this will give us our normal vector $\vec{u} \times \vec{v} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} =< -3-2, 5 >$ $\quad < -3-2, 5 >$ is $\vec{n}$

$\rightarrow$ so now we have

$$-3x - 2y + 5z + d = 0 \qquad \text{we plug in any point to find } d$$

$\rightarrow -3(0) - 2(0) + 5(0) + d = 0 \rightarrow d = 0$

$$-3x - 2y + 5z = 0$$

■

§12.6 Surfaces

MISSING DRAWING this is not $y = x^2$ in 3 dimensions

■ **Example** Sketch $z = x^2$ in $\mathbb{R}^3$
MISSING DIAGRAM ■

October 7, 2019

1. Sketch $x^2 + y^2 = z^2$ This is a cone. We're going to sketch this using traces

   Use traces

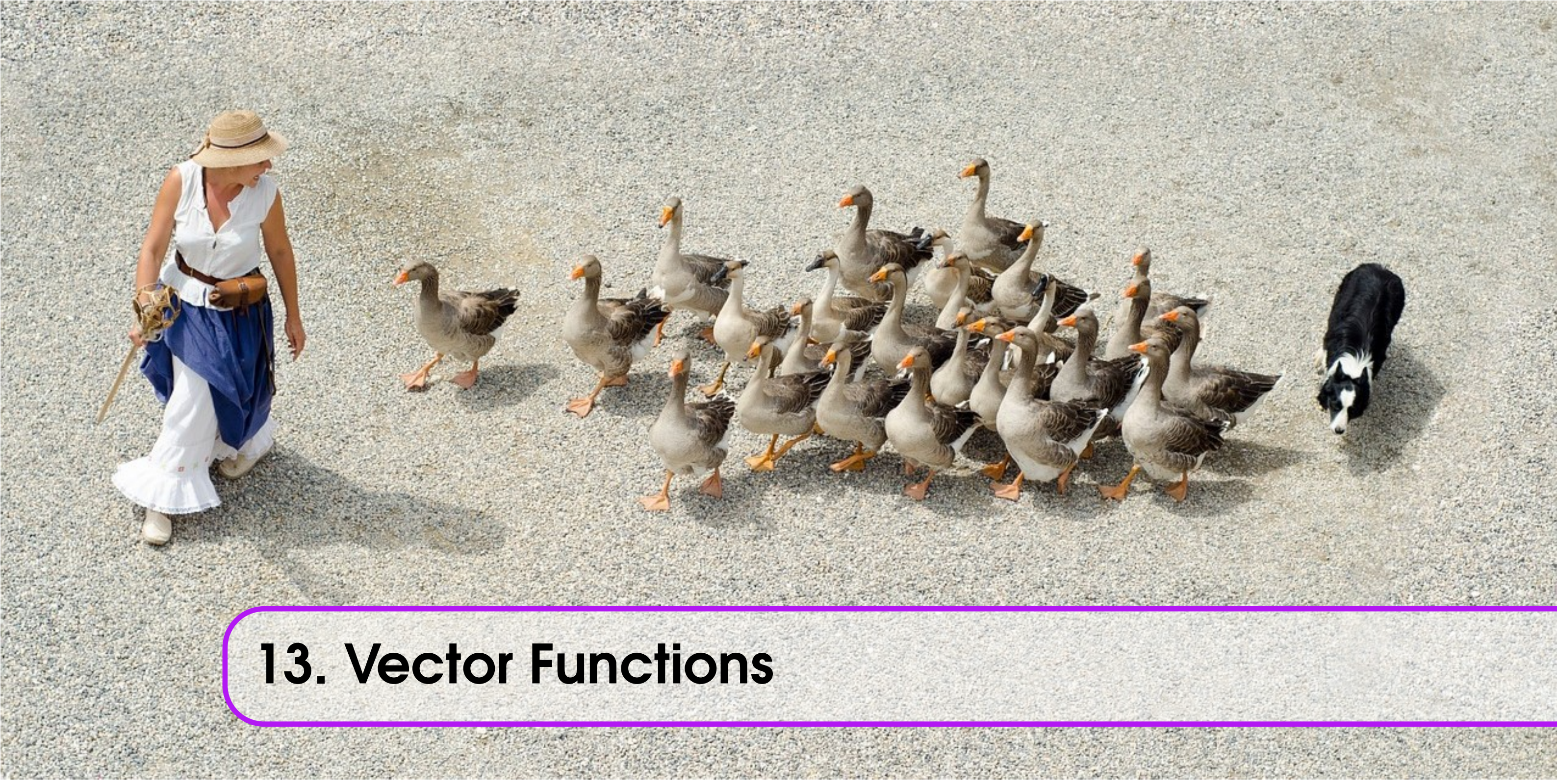
   (a) z-traces

$$\begin{aligned} z = 0 \quad &\Rightarrow x^2 + y^2 = 0 \\ &\Rightarrow x = 0, y = 0 \\ &\Rightarrow \text{ the origin } (0,0) \\ z = 1 \quad &\Rightarrow x^2 + y^2 = 1 \\ &\Rightarrow \text{ circle of radius } 1 \\ z = 2 \quad &\Rightarrow x^2 + y^2 = 4 \\ z = -1 \quad &\Rightarrow x^2 + y^2 = 1 \\ &\Rightarrow \text{ circle of radius } 1 \end{aligned}$$

   MISSING DIAGRAM

   (b) x-traces

$$\begin{aligned} x = 0 \quad &\Rightarrow y^2 = z^2 \\ x = 1 \quad &\Rightarrow 1 + y^2 = z^2 \\ &\Rightarrow 1 = z^2 - y^2 \end{aligned}$$



13. Vector Functions

§13.1 Parametric curves

Definition Vector Valued Function $r : \mathbb{R} \to \mathbb{R}^3$ (or $\mathbb{R}^2$) is called a vector-valued function

$$r(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$$

This gives a set of points $\{r(t) \mid t \in \mathbb{R}\}$ which describes a curve

■ **Example** $x = \cos(t)$, $y = \sin(t)$, $t \in \mathbb{R}$ Sketch this curve.

(a) Plot a few points

MISSING DIAGRAM

$$t = 0 \to (1, 0)$$

$$t = \frac{\pi}{4} \to) \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$t = \frac{\pi}{2} \to (0, 1)$$

■

(b) Find a relationship between x and y

$$\cos^2(t) + \sin^2(t) = 1$$

$$x^2 + y^2 = 1$$

R **Warning** This means $(\cos(t), \sin(t))$ lies on the circle, not that it is the circle

For example, if $0 \leq t \leq \frac{\pi}{2}$, we only have $\frac{1}{4}$ circle.

■ **Example** $r(t) =< \underbrace{\cos(2t)}_{x}, \underbrace{\sin(2t)}_{y} >$

Again $x^2 + y^2 = 1$ ■

Definition Ellipse $r(t) =< a\cos(t), b\sin(t) >$ is an ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{\cancel{a^2}\cos^2(t)}{\cancel{a^2}} + \frac{\cancel{b^2}\sin^2(t)}{\cancel{b^2}}$$

$$= 1$$

■ **Example** Sketch $r(t) =< \underbrace{t^2}_{x}, \underbrace{t^4}_{y} > t \in \mathbb{R}$

We have $y = x^2$
$\Rightarrow r(t)$ is on a parabola
$x \geq 0$ always because $x = t^2$
MISSING DIAGRAM

$$t = 0 \to (0,0)$$
$$t = 1 \to (1,1)$$
$$t = -1 \to (1,1)$$
$$t = -2 \to (4,16)$$

■

■ **Example** Sketch $r(t) =< \underbrace{\cos(t)}_{x}, \underbrace{\sin(t)}_{y}, \underbrace{t}_{z} >$

Observe $x^2 + y^2 = 1 \Rightarrow$ circle
MISSING DIAGRAM You're spiralling out. It's a helix
$x^2 + y^2 = 1$ is a cylinder in $\mathbb{R}^3$
$\Rightarrow r(t)$ is on the cylinder. As t increases, move up the cylinder
MISSING DIAGRAM ■

October 9, 2019

We want to make it past arc-length and curvature before the cut-off in material for the midterm.
Office Hours Thursday: 10:30a.m. - 1:30p.m.
Mind Warmup
$\vec{r}(t) =< x(t), y(t), z(t) >$ - vector function

$$\vec{r}(t) : \mathbb{R} \to \mathbb{R}^3$$

$$\text{to } \to < x(t_0), y(t_0), z(t_0) >$$

$\vec{v}(t) =< \cos t, \sin t >$
$\vec{r}(t) =< \cos 2t, \sin 2t >$
$\vec{r}(t) =< \cos t, \sin t, t >$ helix
MISSING DIAGRAM

§13.2 Derivatives of Parametric Functions

$\to$ think of $\vec{r}(t)$ as the position of a moving object.
  $\to$ how fast is the object moving?
  $\to$ what is the velocity?
  $\to$ what is the derivative of $\vec{r}(t)$

Question If $\vec{r}(t) = < x(t), y(t), z(t) >$ is a vector function, what is the tangent vector at $t = a$

$$\frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}(\underbrace{\vec{r}(t+h)-\vec{r}(t)}_{\text{vector}})$$

MISSING DIAGRAM

$\vec{r}(t)$ - position $\vec{r}\,'(t)$ - velocity (note:$\|\vec{r}\,'(t)\|$ - speed)

$\vec{r}\,''(t)$ - acceleration

$\vec{r}\,'(a)$ - tangent vector at $t = a$

$\vec{T}(a) = \dfrac{\vec{r}\,'(a)}{\|\vec{r}\,'(a)\|}$ - unit tangent vector at $t = a$

$$\vec{l}(a) = \vec{r}(a) + \vec{a}\,' \cdot t$$

Definition Derivative The <u>derivative</u> of $\vec{r}(t) = < x(t), y(t), z(t) >$ is

$$\vec{r}\,'(t) = < x'(t), y'(t), z'(t) >$$

■ **Example** Find the tangent line of the helix $\vec{r}(t) = < 2\cos t, 3\sin t, 5t >$ at the point when $t = \pi$

► $\vec{r}(\pi) = < 2\cos\pi, 3sin\pi, 5\pi >$

$= < -2, 0, 5\pi >$ our "point" on the line

$\vec{r}\,'(t) = < -2\sin t, 3\cos t, 5 >$

$\vec{r}\,'(\pi) = < -2\sin\pi, 3\cos\pi, 5 >$

$= < 0, -3, 5 >$

$\to$ thus, the tangent line is $\vec{l}(t) = < -2, -3t, 5\pi + 5t >$

$< \underbrace{-2, 0, 5\pi}_{\vec{r}(\pi)} > + t < \underbrace{0, -3, 5\pi}_{\vec{r}\,'(\pi)} >$ ■

Intersection of Parametric Curves

$\vec{r}(t), \vec{q}(s)$

■ **Example** Show that $\vec{r}(t) = < t^2 - 1, t^3 - t >$ self-intersects orthogonally.

1. find intersection point
2. find tangent vectors at this point
3. take dot product

► $\begin{cases} t^2 - 1 = s^2 - 1 \\ t^3 - t = s^3 - s \end{cases}$

$t^2 - 1 = s^2 - 1 \longleftarrow t^2 = s^2,$

$s = \pm t$ (use $s = -t$)

$$t^3 - t = s^3 - s \longleftarrow t^3 - t = (-t)^3 - (-t)$$
$$t^3 - t = t^3 + t$$
$$2t^3 - 2t = 0$$
$$2t(t+1)(t-1) = 0$$
$$t = 0, -1, 1$$
$$\vec{r}(0) = < -1, 0 >$$
$$\vec{r}(-1) = < 0, 0 > \leftarrow \text{use these } (t = -1, 1)$$
$$\vec{r}(1) = < 0, 0 > \leftarrow \text{use these } (t = -1, 1)$$

2. $\vec{r}\,'(t) = < 2t, 3t^2 - 1 >$
   first tangent vector $\vec{r}\,'(-1) = < -2, 2 >$
   2^{nd} tangent vector $\vec{r}\,'(1) = < 2, 2 >$
3. $\vec{r}\,'(-1) \cdot \vec{r}\,'(1) = < -2, 2 > \cdot < 2, 2 >$
   $= -2(2) + 2(2) = -4 + 4 = 0$ ■

October 11, 2019

No class Monday!
Midterm: Thursday October 24^{th}, 6-8pm
$\rightarrow$ up to arc length in 13.3 (so no curvature)

Mind Warmup

$\vec{r}(t) < x(t), y(t), z(t) >$
$\vec{r}\,'(t) = < x'(t), y'(t), z'(t) > \leftarrow$ we might call this velocity
$\|\vec{r}\,'(t)\|$ - speed
$\vec{r}\,'(a)$ - tangent vector at $t = a$

1. $\frac{d}{dt}[c\vec{u}(t) + \vec{v}(t)] = c\vec{u}\,'(t) + \vec{v}\,'(t)$
2. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$
3. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$
4. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$
5. $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}\,'(f(t))$

Proposition If $\|\vec{r}(t)\| = k$ (it is constant), then $\vec{r}(t) \perp \vec{r}\,'(t)$ (they are orthogonal) for any t.

► Since $\|\vec{r}(t)\| = k$, we know $\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = k^2$

$$\vec{r}(t) \cdot \vec{r}(t) = k^2$$
$$\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt}(k^2)$$
$$\vec{r}\,' \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$
$$2(\vec{r} \cdot \vec{r}\,'(t)) = 0$$
$$\vec{r} \cdot \vec{r}\,'(t) = 0 \quad \leftarrow \text{ this means orthogonal} \quad ◄$$

Definition Integral The integral of $\vec{r}(t) =< x(t), y(t), z(t) >$ is

$$\int \vec{r}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$$

Proposition $\int \vec{r}\,'(t)dt = \vec{r}(t) + \vec{c}, \quad \vec{c} \in \mathbb{R}^3$

■ **Example** Given $\vec{a}(t) =< \cos t, e^{-t}, t^2 >$ and initial condition $\vec{v}(0) =< 0, 1, 2 >$ find $\vec{v}(t)$

▶ $$\vec{v}(t) = \int \vec{a}(t)dt = \left\langle \int \cos t dt, \int e^{-t}dt, \int t^2 dt \right\rangle$$

$$= \left\langle \sin t + a, -e^{-t} + b, \frac{1}{3}t^3 + c \right\rangle$$

$\vec{v}(0) = \langle 0 + a, -1 + b, 0 + c \rangle = \langle 0, 1, 2 \rangle \quad \Rightarrow a = 0,\ b = 2,\ c = 2$

$$\vec{v}(t) = \left\langle \sin t, 2 - e^{-t}, \frac{1}{3}t^3 + 2 \right\rangle$$

■

§13.3 Arclength (curvature)

Given $y = f(x)$, you may have seen

$$\text{Length} = \int_a^b \sqrt{1 + (f'(x))^2}dx$$

R look $\vec{r}(x) =< x, f(x) >$

$$\vec{r}\,'(x) =< 1, f'(x) >$$

$$\|\vec{r}\,'(x)\| = \sqrt{1 + (f'(x))^2}$$

Definition Arclength The <u>arclength</u> of $\vec{r}(t)$, on $a \le t \le b$ is

$$L = \int_a^b \|\vec{r}\,'(t)\|dt$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

■ **Example** Find the arclength of $\vec{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle$ where $0 \le t \le 1$

▶ $\vec{r}\,'(t) = \left\langle 2, 2t, t^2 \right\rangle$

$$\to \int_0^1 \sqrt{(2)^2 + (2t)^2 + (t^2)^2}dt$$

$$= \int_0^1 \sqrt{4 + 4t^2 + t^4}dt$$

$$= \int_0^1 \sqrt{(2 + t^2)^2}dt$$

$$= \int_0^1 \left(2 + t^2\right) dt = 2t + \frac{1}{3}t^3\Big|_0^1 = \frac{7}{3}$$

■

Ways this can become difficult

1. Find $t = a$, $t = b$ between two points
2. Challenging integrals I'm hinting at trig substitutions
3. Change of variable

October 16, 2019

Mind Warmup

$\vec{r}(t) =< x(t), y(t), z(t) >$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \text{ - unit tangent vector}$$

$$= \left\langle \frac{\vec{x}\,'(t)}{\|\vec{r}\,'(t)\|}, \frac{\vec{y}\,'(t)}{\|\vec{r}\,'(t)\|}, \frac{\vec{z}\,'(t)}{\|\vec{r}\,'(t)\|} \right\rangle$$

$$S = \int_a^b \|\vec{r}\,'(t)\| dt \text{ - arc length, a } \underline{\text{number}}$$

$$S(t) = \int_0^t \|\vec{r}\,'(u)\| du \text{ - arc length, a } \underline{\text{function}}$$

-This is the length of the curve from 0 to t

$\vec{r}(t) =< \cos(t), \sin(t) >$ MISSING DIAGRAM

$$S'(t) = \|\vec{r}\,'(t)\| \text{ FTC II Fundamental theorem of calculus}$$

Definition Curvature This is the magnitude of the rate of change of the unit tangent vector with respect to arclength.

R

- Magnitude / Length
- Rate of change / Derivative
- Unit tangent vector / Length 1
- ArcLength ??

Definition If $\vec{r}(s)$ is parametrized by arclength, the curvature is

$$k(s) = \|T'(s)\|$$

■ **Example** Reparametrize helix $\vec{r}(t) =< \cos(t), \sin(t) >$ with respect to arclength starting from $(1,0,0)$ as t increases.

► $(1,0,0)$ corresponds to $t = 0$

$$S = S(t) = \int_0^t \|\vec{r}(u)\| du$$

$$= \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2 du}$$

$$= \int_0^t \sqrt{\underbrace{(\sin u)^2 + (\cos u)^2}_{1} + 1^2 du}$$

$$= \int_0^t \sqrt{2} du$$

$$= \sqrt{2}t$$

$$\Rightarrow \vec{r}(s(t)) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$ ■

Definition The curvature for any $\vec{r}(t)$ is

$$k(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$$

$$\|\vec{T}\,'(t)\| = \|\frac{d\vec{T}}{ds}\frac{ds}{dt}\| = \|\frac{d\vec{T}}{ds}\|\|\frac{ds}{dt}\|$$

$$= \|\vec{T}\,'(s)\|\|s'(t)\| = k(s)\|\vec{r}\,'(t)\|$$

■ **Example** Show that the curvature of the circle with radius 13 is $k = \dfrac{1}{13}$

▶ $\vec{r}(t) =< 13\cos t, 13\sin t >$

$\vec{r}\,'(t) =< 13\cos t, 13\sin t >$

$$\|\vec{r}\,'(t)\| = \sqrt{169\sin^2 t + 169\cos^2 t} = \sqrt{169} = 13$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \left\langle \frac{-13\sin t, -13\cos t}{13} \right\rangle = \langle -\sin t, \cos t\rangle$$

$$\vec{T}\,'(t) = \langle -\cos t, -\sin t\rangle$$

$$\|\vec{T}\,'(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\Rightarrow \quad k(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{1}{13}$$ ◀

■

Theorem The curvature for any $\vec{r}(t)$ is

$$k(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

- $\vec{r}(t)$
  - $\vec{r}(s)$
    - $\vec{T}(s)$
      - $\|\vec{T}\,'(s)\|$
  - $\vec{r}\,'(t)$
    - $\vec{T}(t)$
      - $\vec{T}\,'(t)$
        - $\dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$
  - $\vec{r}\,'(t), \vec{r}\,''(t)$
    - $\vec{r}\,'(t) \times \vec{r}\,''(t)$
      - $\dfrac{\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$

October 18, 2019

Mind Warmup

Given $\vec{r}(t)$, we know

$\vec{r}'(t)$	Think of this as velocity
$\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$	-unit tangent vector
$K(s) = \|\vec{T}'(s)\|$	-curvature for $\vec{r}(s)$ with respect to arclength
$\vec{K}(t) = \dfrac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$	-curvature, any $\vec{r}$
$\vec{K}(t) = \dfrac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$	-curvature, any $\vec{r}$
$S(t) = \int_0^t \|\vec{r}'(u)\|du$	-arclength function

Using $k(t) = \dfrac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$, what can we say about the curvature of $y = f(x)$?

► $y = f(x)$ corresponds to $\vec{r}(x) = < x, f(x), 0 >$

$$\vec{r}'(x) = < 1, f'(x), 0 >$$

$$\vec{r}''(x) = < 0, f''(x), 0 >$$

$$\vec{r}'(x)\times\vec{r}''(x) = \begin{vmatrix} i & j & k \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \ldots = < 0, 0, f''(x) >$$

$$\|\vec{r}'(x)\times\vec{r}''(x)\| = \sqrt{(f''(x))^2} = |f''(x)|$$

$$\|\vec{r}'(x)\| = \sqrt{1+(f'(x))^2}$$

$$k(x) = \frac{|f''(x)|}{\left(1+(f'(x))^2\right)^{\frac{3}{2}}}$$

§13.4 Tangential and Normal Components of Acceleration

Definition Tangent, Normal, and Binormal Vectors

We'll have $\underbrace{\vec{T}}_{\text{tangent}}, \underbrace{\vec{N}}_{\text{normal}}, \underbrace{\vec{B}}_{\text{binormal}}$ all (unit) vectors

$\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$	- direction the curve is going
$\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$	-direction the curve is turning
$\vec{B}(t) = \vec{T}\times\vec{N}$	-orthogonal to both

MISSING DIAGRAM

Normal Components of acceleration For this section, I'm going to use $v(t)$ for our velocity

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$$

$$\leftrightarrow \vec{v}(t) = \vec{T}(t)\|\vec{v}(t)\|$$

$$a(t) = \vec{v}\,'(t) = \vec{T}\,'(t)\|\vec{v}(t)\| + \vec{T}(t)\,(\|\vec{v}(t)\|)' = \qquad \text{Fact } N(t) = \frac{T'(t)}{\|\vec{T}'(t)\|}$$

$$= \underbrace{(t)\|\vec{T}\,'(t)\|}_{\vec{T}\,'(t)}\|\vec{v}(t)\| + \vec{T}(t)\,(\|\vec{v}(t)\|)' = \qquad \text{Fact } k(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|\vec{T}\,'(t)\|}{\|\vec{v}(t)\|}$$

$$= N(t)\underbrace{k(t)\|\vec{v}(t)\|^2}_{a_N} + T(t)\underbrace{(\|\vec{v}(t)\|)'}_{a_T}$$

$$a(t) = a_N\vec{N}(t) + a_T\vec{T}(t)$$

R trick $T(t)\cdot a(t) = T(t)\cdot(a_N N(t) + a_T T(t))$

$$= a_N\left(\underbrace{T(t)\cdot N(t)}_{=0}\right) + a_T\left(\underbrace{T(t)\cdot T(t)}_{=1}\right)$$

$$T(t)\cdot a(t) = a_T$$

$$\to a_N = k(t)\|\vec{v}(t)\|^2 = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}\|\vec{r}\,'(t)\|^2$$

$$= \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|}$$

$$a_T = T(t)\cdot a(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\cdot\vec{r}\,''(t)$$

$$= \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\vec{r}\,'(t)\|}$$

October 21, 2019

13.3 - remove curvature

13.4 - no tangential/normal components of acceleration

$\to r \leftrightarrow \underbrace{r'}_{=v} \leftrightarrow \underbrace{r''}_{=a}$

Mind Warmup

$$\vec{r}(t) \xrightarrow{\frac{d}{dt}} \underbrace{\vec{r}\,'}_{=v} \xrightarrow{\frac{d}{dt}} \underbrace{\vec{r}\,''}_{=a}$$

$\vec{T}(t), \vec{N}(t)$ - tangent/normal vectors

$$\vec{a}(t) = \underbrace{a_N}_{\frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|}}\vec{N}(t) + \underbrace{a_T}_{\frac{\|\vec{r}\,'(t)\cdot\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|}}\vec{T}(t)$$

■ **Example** Find tangential/normal components of acceleration of $\vec{r}(t) =< t, t^2 >$

▶ MISSING DIAGRAM

$$\vec{v}(t) =< 1, 2t >$$

$\vec{a}(t) =< 0, 2 >$ (always pointing upwards)

$$\vec{r}\,'(t) =< 1, 2t, 0 >$$

$$\vec{r}\,''(t) =< 0, 2, 0 >$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = < 0, 0, 2 >$$

$$a_n = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|} = \frac{\sqrt{0+0+4}}{\sqrt{1+4t^2+0}} = \frac{2}{\sqrt{1+4t^2}}$$

$$a_T = \frac{\|\vec{r}'(t) \cdot \vec{r}''(t)\|}{\|\vec{r}'(t)\|} = \frac{0+4t+0}{\sqrt{1+4t^2}} = \frac{4t}{\sqrt{1+4t^2}}$$

when $t = 0$ $a_N = 2$ (all here)
$a_T = 0$

as $t \to \infty$, $a_N \to 0$
$a_T \to 2$ (all here)

■

$$k(t) = \begin{cases} (1) \\ (2) \to \dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \end{cases}$$

$$k(s) = (1)$$

$$k(x) = (1) \to \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{\frac{3}{2}}}$$

14. Partial Derivatives

§14.1 Functions of Multiple Variables

Definition Function of two variables A <u>function $f(x,y)$ of two variables</u> relates inputs (x_0, y_0) with outputs $f(x_0, y_0)$
The <u>domain</u> of $f(x,y)$ is the set of all allowable inputs (x_0, y_0) and the <u>range</u> is the set of outputs

■ **Example** Find the domain of $f(x,y) = \sqrt{4-x^2-y^2} + \ln(y+1) + e^{-x}$

▶ $4 - x^2 - y^2 \geq 0$

$\leftrightarrow x^2 + y^2 \leq 4$ (everything inside the circle of radius 2)

$y + 1 > 0, y > -1$

$-x \in \mathbb{R}, x \in \mathbb{R}$

■

MISSING DIAGRAM
Tough question: <u>what is the range?</u>

■ **Example** Find domain and range of $f(x,y) = \sqrt{9-x^2-y^2}$

▶ domain is when $9 - x^2 - y^2 \geq 0$

$\leftrightarrow x^2 + y^2 \leq 9$ (everything inside the circle of radius 3)

Range is $[0, 3]$

$$0 \underbrace{\leq}_{\text{because of square root}} 9 - x^2 - y^2 \leq 9$$

■

Definition The graph of $f(x,y)$ is the surface $z = f(x,y)$

■ **Example** Now sketch surface $z = f(x,y)$, where $f(x,y) = \sqrt{9-x^2-y^2}$

▶ $z = \sqrt{9-x^2-y^2}$

$z^2 = 9-x^2-y^2$

$x^2+y^2+z^2 = 9$ sphere of radius 3 (top half)

MISSING DIAGRAM ■

October 23, 2019

Note, for October 23 especially, my notes might not be complete

Mind Warmup

function $f(x,y)$ with domain (x_0,y_0) and range $f(x_0,y_0)$

-graph of $f(x,y)$ is the surface $z = f(x,y)$

Definition Level Curve A level curve (aka contour line, a z-trace) for $f(x,y)$ are the curves $f(x,y) = k$ where $k \in \mathbb{R}$

Definition Contour map $\to$ A collection of level curves is called a contour map

■ **Example** Sketch a contour map of $f(x,y) = \sqrt{xy}$ MISSING DIAGRAMS ABOVE ■

▶ $k = 1 \quad \sqrt{xy} = 1 \quad y = \dfrac{1}{x}$

$k = 2 \quad \sqrt{xy} = 2 \quad y = \dfrac{4}{x}$

$k = 0 \quad \sqrt{xy} = 0 \quad x = 0$ or $y = 0$

$k = -1 \quad \sqrt{xy} = -1 \quad$ DNE

■ **Example** What are the level surfaces of $f(x,y,z) = x^2+y^2+z^2$

▶ (level surfaces are $x^2+y^2+z^2 = k$) $\to$ the level surfaces $x^2+y^2+z^2 = k$ are just the collection of points that are

$$\begin{array}{lll} k=0 & x^2+y^2+z^2=0 & \text{just } (0,0,0) \\ k=1 & x^2+y^2+z^2=1 & \text{sphere of radius 1} \\ k=4 & x^2+y^2+z^2=4 & \text{sphere of radius 2} \end{array}$$

distance $\sqrt{k}$ from the origin ■

§14.2 Limits/Continuity

Definition The limit of $f(x,y)$ as (x,y) approaches (a,b) is equal to L

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\varepsilon > 0$ (error away from L) there is some $\delta > 0$, such that if a point (x,y) is within distance δ from (a,b), then $f(x,y)$ is within distance ε from L

What does $(x,y) \to (a,b)$ mean?
Before, from Calculus 1
MISSING DIAGRAMS
$\lim_{x\to a} f(x) = L$
$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ Now in Calculus 3
$(x,y) \to (a,b)$ there are an infinite number of paths to take
MISSING DIAGRAMS

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$\to$ this is difficult to show, but the squeeze theorem will help us.

October 25, 2019

Notes for October 25^{th} were made using someone else's notes.
Mind Warmup
$f(x,y) \to$ two variable function

Level curves: $f(x,y) = k$ will help us sketch the graph $z = f(x,y)$

$\lim_{x,y\to a,b} f(x,y) = L$, if for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$0 < \|(x,y) - (a,b)\| < \delta \text{ then } |f(x,y) - L| < \varepsilon$$

■ **Example** Show $\displaystyle\lim_{(x,y)\to(a,b)} \frac{3x^2y}{x^2+y^2} = 0$

▶ Start with $\varepsilon > 0$ (it can be anything)

We need $\delta > 0$ that works for our condition. We'll use $\delta = \dfrac{\varepsilon}{3}$

$$0 < \|(x,y) - (0,0)\| < \frac{\varepsilon}{3} \to \left|\frac{3x^2y}{x^2+y^2}\right| < \varepsilon$$

$$0 < \underbrace{\sqrt{x^2+y^2}}_{\text{length using vector}} < \frac{\varepsilon}{3} \qquad \text{(fact)}$$

$$\left|\frac{3x^2y}{x^2+y^2}\right| = 3|y| \cdot \left|\frac{x^2}{x^2+y^2}\right| \le 3|y|$$

$$= 3\sqrt{y^2} \le 3\sqrt{x^2+y^2} < 3\frac{\varepsilon}{3} = \varepsilon$$

■

Theorem — Squeeze Theorem

. ▶ If $f(x,y) \le g(x,y) \le h(x,y)$ "near" (a,b) and $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L = \lim_{(x,y)\to(a,b)} h(x,y)$,

then also $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = L$ as well

■ **Example** Show $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

$$\begin{array}{ccc} 0 \le \left|\frac{3x^2y}{x^2+y^2}\right| = 3|y|\left|\frac{x^2}{x^2+y^2}\right| \le & 3|y| \\ (x,y)\to(0,0) \downarrow & \downarrow (x,y)\to(0,0) \\ 0 & 0 \end{array}$$

Since $\lim_{(x,y)\to(0,0)} 3|y| = 0$ by squeeze theorem, we know $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$,

and since $= 0$, you can remove the absolute value cause $-0 = +0$ ■

Showing Limit DNE

Pick two “paths" towards the point, if the value of the function is different, then it doesn't exist

use vector functions

> **Theorem** If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and,
> $\vec{r}(t) = \langle x(t), y(t)\rangle$ and is continuous where $\vec{r}(t_0) = \langle a,b\rangle$ for some t_0,
>
> then $\lim_{t\to t_0} f(x(t),y(t)) = L$

We'll use this to find two different paths, to show that a limit DNE.

■ **Example** Find $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$, if it exists

► Along the path $y = 0$ (the x-axis)

$\vec{r}(t) = \langle t, 0\rangle$

so then, $\lim_{t\to 0} \frac{t^4-4(0)^2}{t^2+2(0)^2} = \frac{t^4}{t^2} = t^2 = 0$

► Along the path $x = 0$ (y-axis)

$\vec{r}(t) = \langle 0, t\rangle$

$$\lim_{t\to 0} \frac{(0)^4-4t^2}{(0)^2+2(t^2)} = \frac{-4t^2}{2t^2} = -2$$

Since the two path limits are not the same, the limit DNE.

See tutorial for a path that is a parabola. ■

October 28, 2019

On the topic of webwork 7 exercise 6

$$\int \frac{1}{x}dx = \ln(x) \qquad \text{wolfram will give you this}$$

$$= \ln|x| + c \qquad \text{but this is what it actually is}$$

Mind Warmup

$$\lim_{(x,y)\to(a,b)} f(x,y) = \begin{cases} L, \text{ if all paths led to } L \\ \text{DNE, if two paths lead somewhere different} \end{cases}$$

$\to$ look at figure 6 §14.2

Definition $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

■ **Example** Where is $f(x,y) = (x^2+y^2)\cos(xe^{-y})$ continuous?

► f is continuous on $\mathbb{R}^2$ (everywhere)

→ these are functions we already know are continuous ■

■ **Example** Where is $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

We showed last class that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

→ so then $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = f(0,0)$

→ so then, f is continuous at $(0,0)$

→ it's also continuous everywhere else

→ f is continuous on $\mathbb{R}^2$

■

■ **Example** Where is $f(x,y) = \frac{5}{x^2+y^2-1}$ continuous?

► (let's do this intuitively)

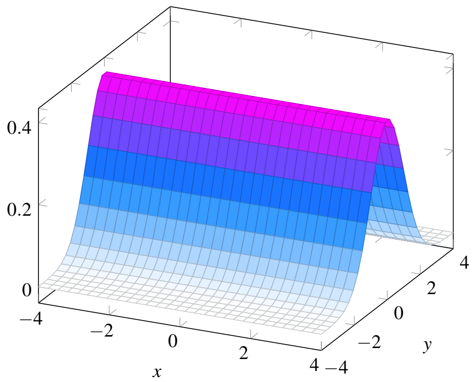
→ we don't want $x^2+y^2-1=0$ $\quad x^2+y^2=1$

→ in otherwords, f is <u>not</u> continuous when $x^2+y^2=1$ (on the unit circle) (and not inside it)

■

MISSING DIAGRAMS

§14.3 Partial Derivatives



→ no change in the x direction

→ there <u>are</u> changes in the y directions

$$f(x,y) = x^2e^y$$

Calculus 1

Def $\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Calculus 3

Definition

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

R $\rightarrow$ Note we use $\frac{\partial f}{\partial x}$ instead of $\frac{df}{dx}$ when we have multiple variables

■ **Example** Let $f(x,y) = x^3 + x^2y^2 - 5y$

Find $f_x(1,0)$, $f_y(1,0)$

► $f_x(x,y) = 3x^2 + 2xy^2$

$f_x(1,0) = 3(1)^2 + 2(1)(0)^2 = 3$

$f_y(x,y) = 2x^2y - 5$

$f_y(1,0) = 2(1)^2(0) - 5 = -5$

■

■ **Example** $f(x,y) = x^3\cos(x^2y^2)$

Find f_x, f_y

► $f_x(x,y) = 3x^2\cos(x^2y^2) - x^3\sin(x^2y^2\left[2xy^2\right]$

$= 3x^2\cos(x^2y^2) - 2x^4y^2\sin(x^2y^2)$

$f_y(x,y) = -x^3\sin(x^2y^2)\left[2x^2y\right]$

$= -2x^5y\sin(x^2y^2)$

■

October 30, 2019

Mind Warmup

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h \to \infty} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial x} = \lim_{h \to \infty} \frac{f(x,y+h)-f(x,y)}{h}$$

■ **Example**

$$f(x,y,z) = x + e^z \ln y$$
$$f_x = 1$$
$$f_y = \frac{e^z}{y}$$
$$f_z = e^z \ln y$$

■

Higher Ordered Derivatives

$$\left.\begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\ f_{xy} &= (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \\ f_{yx} &= (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \\ f_{yy} &= (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \end{aligned}\right\} \text{second-order derivatives}$$

Theorem — Clairaut's Theorem. (order of derivatives usually doesn't matter)
Suppose $f(x,y)$ is defined on a disk D around point (a,b), and that f_{xy} and f_{yx} are continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

■ **Example**

$$f(x,y) = e^{x^2}\sin(x^3 + \tan(3+x^2)) + 2yx^2$$

$\rightarrow$ Find f_{xy}

► $f_{xy} = f_{yx}$

$f_y = 2x^2$

$f_{yx} = (f_y)_x = 4x$ ◄

■

■ **Example**

$$f(x,y,z) = e^{x^2+y^2} + \cos(xz) + (y+z)^3$$

Find f_{xyz}

► $f_{xyz} = 0$, because no term has all three variables ■

$f(x) = |x|$

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$$f(x,y) \begin{cases} \frac{xy(x^2+y^2)}{x^2+y^2} \\ 0 \end{cases}$$

Partial Differential Equations (PDE)

Definition PDE A PDE is an equation involving an unknown function and its partial derivatives.

Laplace Equation $u_{xx} + u_{yy} = 0$

■ **Example** Show that $u(x,y) = e^x \cos y$ is a solution to Laplace's equation.

- $u_x = e^x \cos y$

$u_{xx} = e^x \cos y$

$u_y = -e^x \sin y$

$u_{yy} = -e^x \cos y$

$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$

■

§14.4 Tangent Planes

m - slope in 1D
$\downarrow$
? in 2D
The answer to this in 2D is the gradient

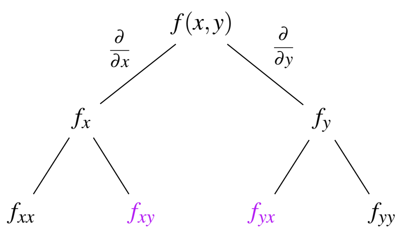
Definition Gradient The <u>gradient</u> of $F(x,y,z)$ is the vector

$$\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$$

$\rightarrow$ to find the equation of a tangent, we need a normal vector and a point
$\rightarrow$ the gradient will be our normal vector
MISSING DIAGRAM
§14.4 Tangent Planes (and Linear Approximations)

November 1, 2019

Mind Warmup



f_{xy} and f_{yx} are the same if both are continuous (Clairaut's theorem)

$F(x,y,z)$, $\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$

R If $\vec{n} = \langle a,b,c \rangle$, and $p = (x_0, y_0, z_0)$ the equation for a plane is

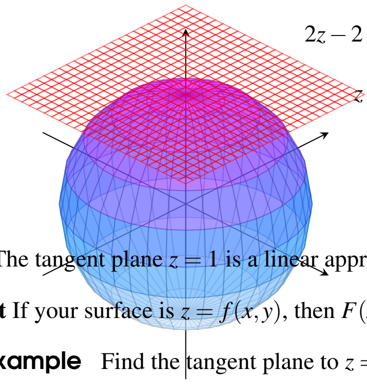
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \qquad ax+by+cz+d = 0$$

$\rightarrow$ For us, if we want to find a tangent plane, we will use $\vec{n} = \underbrace{\nabla F(x_0,y_0,z_0)}_{\substack{\text{1 take gradient} \\ \text{2 Evaluate it at the point } (x_0,y_0,z_0)}}$

■ **Example** Find the tangent plane for the sphere $x^2+y^2+z^2=1$ at the north pole $(0,0,1)$

► $F(x,y,z)=x^2+y^2+z^2-1$
$\nabla F=\langle 2x,2y,2z\rangle$
$\nabla F(0,0,1)=\langle 0,0,2\rangle=\vec{n}$

$$\begin{aligned}\rightarrow\quad 0(x-0)+o(y-0)+2(z-1)&=0\\ 2(z-1)&=0\\ 2z-2&=0\\ z&=1\end{aligned}$$



$\rightarrow$ The tangent plane $z=1$ is a linear approximation to $x^2+y^2+z^2=1$ at the point $(0,0,1)$ ■

Fact If your surface is $z=f(x,y)$, then $F(x,y,z)=f(x,y)-z$, so then $\nabla F=\left\langle f_x,f_y,-1\right\rangle$

■ **Example** Find the tangent plane to $z=x\sin(x+y)$ at $(-1,1,0)$

► $F(x,y,z)=x\sin(x+y)-z$
$\nabla F=\langle\sin(x+y)+s\cos(x+y),x\cos(x+y),-1\rangle$

$$\begin{aligned}F(-1,1,0)&=\langle\sin(-1+1)+(-1)\cos(-1+1),(-1)\cos(-1+1),-1\rangle\\ &=\langle 0-1,-1,-1\rangle=\langle -1,-1,-1\rangle=\vec{n}\end{aligned}$$

$$\begin{aligned}\rightarrow\quad -1(x-(-1))-1(y-1)-1(z-0)&=0\\ -x-1-y+1-z&=0\\ -x-y-z&=0\\ x+y+z&=0\quad ◄\end{aligned}$$

■

Implicit Differentiation

Calc 1: $x^2+y^2-3xy+2=0$, $\dfrac{dy}{dx}=?$

$$\begin{aligned}\rightarrow\quad 2x+2y\frac{dy}{dx}-3y-3x\frac{dy}{dx}+0&=0\\ \frac{dy}{dx}(2y-3x)&=3y-2x\\ \frac{dy}{dx}&=\frac{3y-2x}{2y-3x}\end{aligned}$$

$\rightarrow$ If you have a function (implicit) in three (or more) variables, we treat "other" variables as constants

$$\rightarrow\quad \frac{\partial z}{\partial x},\quad y \text{ is a constant}$$

$$\frac{\partial y}{\partial z},\quad x \text{ is a constant}$$

why? $f_x(x,y)=\lim\limits_{h\to 0}\dfrac{f(x+\overset{\text{only affects } x}{\overbrace{h}},y)-f(x,y)}{h}$

■ **Example** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of $x^2+2y^2+xyz-z^2=1$

$\frac{\partial z}{\partial x}$, treat y as constant

$$2x+0+yz+xy\frac{\partial z}{\partial x}-2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}(xy-2z)=-2x-yz$$

$$\frac{\partial z}{\partial x}=\frac{-2x-yz}{xy-2z}=\frac{2x+yz}{2z-xy}$$

$\frac{\partial z}{\partial y}$, x - constant

$$\to\quad 0+4y+xz+xy\frac{\partial z}{\partial y}-2z\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial y}=\frac{4y+xz}{2z-xy}$$

■

§14.5 Chain Rule

Theorem If partial derivatives f_x, f_y exist near (a,b) and are continuous on (a,b), then f is called differentiable at (a,b)

Theorem — Chain Rule. If $z=f(x_1,x_2,\ldots,x_n)$ is a differentiable function with n variables $x_1,\ldots,x_n$ and if each variable x_i is a differentiable of m variables $t_1,\ldots t_m$, then z is a function of $t_1,\ldots t_m$ where

$$\frac{\partial z}{\partial t_i}=\frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\ldots+\frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

November 4, 2019

Mind Warmup

$f(x)\to\frac{df}{dx}=f'(x)$

$f(x(t))\to\frac{df}{dx}\frac{dx}{dt}=f'(x(t))\cdot x'(t)$

$f(x(t),y(t))\to\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

$f(x_1(t_1,t_2,\ldots,t_m),x_2(t_1,t_2,\ldots,t_m),\ldots,x_n(t_1,t_2,\ldots,t_m))$ $\quad\frac{\partial f}{\partial t_i}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\ldots+\frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$

■ **Example** If $z=x^2\sin(y)$, where $x=s+t$, $y=e^{-t}$

Find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$?

$$\blacktriangleright \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2x\sin(y))(1) + \left(x^2\cos(y)\right)(0)$$

$$= 2x\sin(y)$$

$$= 2(s+t)\sin\left(e^{-t}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2x\sin(y))(1) + \left(x^2\cos(y)\right)\left(-e^{-t}\right)$$

$$= 2(s+t)\sin\left(e^{-t}\right) - e^{-t}(s+t)^2\cos\left(e^{-t}\right)$$

$\rightarrow$ yes, you could have written

$z = (s+t)^2\sin\left(e^{-t}\right)$, and then took derivatives

$\blacksquare$

R Sometimes you cannot substitute

■ Example Let $f(x,y)$ be any differentiable function.

I'm not sure if I mixed up ∂ and d in this example. It seems ok, but I'll need to double check

Prove that $z = f(s^2-t^2, t^2-s^2)$ solves the PDE $t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = 0$

$\rightarrow$ for example, $f(x,y) = xe^y \rightarrow (s^2t^2)e^{t^2-s^2}$

$$\blacktriangleright \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial z}{\partial x}(2s) + \frac{\partial z}{\partial y}(-2s)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{\partial z}{\partial x}(-2t) + \frac{\partial z}{\partial y}(2t)$$

$$\rightarrow t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = \left(\cancel{2st\frac{\partial z}{\partial x}} - \cancel{2st\frac{\partial z}{\partial y}}\right) + \left(\cancel{-2st\frac{\partial z}{\partial x}} + \cancel{2st\frac{\partial z}{\partial y}}\right)$$

$$= 0 \quad \blacktriangleleft$$

$\blacksquare$

The next problem shows that any curve drawn on a surface is orthogonal to the gradient (aka the normal vector of the surface)

Proposition If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a differentiable curve on a surface $F(x,y,z) = 0$, and $\vec{r}(0) = \langle a,b,c\rangle$, then

$$\nabla F(a,b,c)\cdot\vec{r}\,'(0) = 0$$

MISSING DIAGRAM

$\rightarrow$ since $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is on $F(x,y,z) = 0$, then $F(x(t), y(t), z(t)) = 0$

$$\begin{aligned}
\frac{\partial F}{\partial t} &= 0 \\
\frac{\partial F}{\partial t} &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\
&= \frac{\partial F}{\partial x}x'(t) + \frac{\partial F}{\partial y}y'(t) + \frac{\partial F}{\partial z}z'(t) \\
&= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \\
&= \nabla F \cdot \vec{r}\,'(t)
\end{aligned}$$

$\rightarrow$ thus, $0 = \nabla F \cdot \vec{r}\,'(t)$

so when $t = 0$, $\quad 0 = \nabla F(a,b,c) \cdot \vec{r}\,'(0)$

Implicit Differentiation (Part 2)

Let $F(x,y,z) = 0$ be any equation.

We'll use $x = s$, $y = t$, $z = f(s,t)$

$$F(s, t, f(s,t)) = 0$$

$$\begin{aligned}
\frac{\partial F}{\partial s} &= 0 \\
0 &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s} \\
0 &= \frac{\partial F}{\partial x}(1) + \frac{\partial F}{\partial y}(0) + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s} \\
\frac{\partial z}{\partial s} &= \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \rightarrow \text{ because } x = s \\
\text{we have } \frac{\partial z}{\partial x} &= \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}
\end{aligned}$$

November 6, 2019

Mind Warmup

chain rule, $f(x(s,t), y(s,t))$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$F(x,y,z) = 0$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial x}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}$$

■ **Example** $x^2 + 2y^2 + xyz - 2z^2 = 1$

$F(x,y,z) = x^2 + 2y^2 + xyz - 2z^2 - 1$

Last time $2x + 0 + yz + xy\frac{\partial z}{\partial x} - 4z\frac{\partial z}{\partial x} = 0$

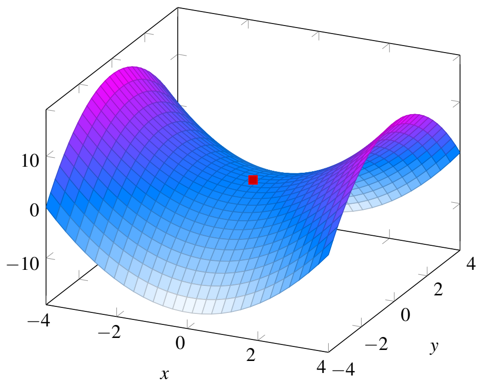
This time $\dfrac{\partial z}{\partial x} = \dfrac{-(2x+0+yz+0)}{0+xy-4z} = \dfrac{yz+2x}{4z-xy}$ ■

§14.6 Directional Derivatives (and Gradient Vector)

R

$$\text{Recall: } \frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$



MISSING DIAGRAM

Definition Directional Derivative Let $\vec{u} = \langle a,b\rangle$ be a unit vector.

The directional derivative in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ah,y+bh)-f(x,y)}{h}$$

Note: we recover $f_x(x,y)$ and $f_y(x,y)$ if $\vec{u} = \langle 1,0\rangle$ and $\vec{u} = \langle 0,1\rangle$ respectively
If $\vec{u}$ is not a unit vector, you can still compute $D_{\vec{u}} f$, but it will be scaled wrong
$\rightarrow$ it will be per $\|\vec{u}\|$, instead of per 1.

Theorem Let $\vec{u} = \langle a,b\rangle$ be a unit vector. Then

$$\begin{aligned} D_{\vec{u}} f(x,y) &= \vec{u}\cdot\nabla f \\ &= a\cdot f_x(x,y) + b\cdot f_y(x,y) \end{aligned}$$

■ **Example** Compute $D_{\vec{u}} f(3,3)$ in the direction $\langle 1,-2\rangle$ if $f(x,y) = \sqrt{1+x^2+y^2}$

► $\vec{u} = \dfrac{\langle 1,-2\rangle}{\|\langle 1,-2\rangle\|} = \dfrac{\langle 1,-2\rangle}{\sqrt{1+4}} = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{-2}{\sqrt{5}} \right\rangle$

$$f_x(x,y) = \frac{1}{2}\left(1+x^2+y^2\right)^{-\frac{1}{2}}[2x] = \frac{x}{\sqrt{1+x^2+y^2}}$$

$$f_x(3,3) = \frac{3}{\sqrt{1+3^2+3^2}} = \frac{3}{\sqrt{19}}$$

$$f_y(x,y) = \frac{y}{\sqrt{1+x^2+y^2}}$$

$$f_y(3,3) = \frac{3}{\sqrt{19}}$$

$$D_{\vec{u}}f(3,3) = \frac{1}{\sqrt{5}}\left(\frac{3}{\sqrt{19}}\right) - \frac{2}{\sqrt{5}}\left(\frac{3}{\sqrt{19}}\right)$$

$$= \frac{-3}{\sqrt{95}} \quad \blacktriangleleft$$

■

■ **Example** Compute $D_{\vec{u}}f(x,y)$ where $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$, and $f(x,y) = x^2+2x+y^2+2y+2$

► $\vec{u}$ is the unit vector "half way" in between the x and y axis

MISSING DIAGRAM

$f(x,y) = (x+1)^2+(y+1)^2$ is the paraboloid centered at $(-1,-1)$

► $f_x(x,y) = 2x+2$

$f_y(x,y) = 2y+2$

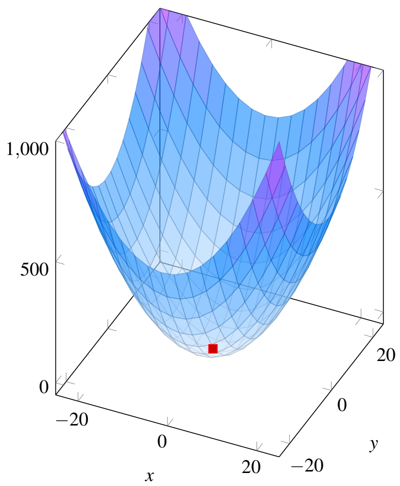
$$D_{\vec{u}}f(x,y) = \frac{\sqrt{2}}{2}(2x+2) + \frac{\sqrt{2}}{2}(2y+2)$$

$$= \sqrt{2}(x+y+2) \quad \blacktriangleleft$$

$f_x(0,0) = 2$

$f_y(0,0) = 2$

$D_{\vec{u}}f(0,0) = 2\sqrt{2} \quad \leftarrow$ this is the steepest



MISSING DIAGRAM

■

November 8, 2019

Mind Warmup

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \leftarrow \text{direction is positive } x\text{-axis or } \vec{u} = \langle 1,0\rangle$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \quad \leftarrow \text{direction is positive } y\text{-axis or } \vec{u} = \langle 0,1\rangle$$

$$D_{\vec{u}} = \lim_{h\to 0} \frac{f(x+ah,y+bh)-f(x,y)}{h} \quad \leftarrow \text{in direction } \vec{u} = \langle a,b\rangle \text{ (unit vector)}$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = a\cdot f_x + b\cdot f_y$$

■ **Example** At which direction $\vec{u}$ is the directional derivative $D_{\vec{u}} f(x,y) = -1$, when $f(x,y) = x\sqrt{y}$, at point $(-1,1)$

► $\vec{u} = \langle a,b\rangle$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \left\langle \sqrt{y}, \frac{x}{2\sqrt{y}} \right\rangle \cdot \langle a,b\rangle$$

$$= a\sqrt{y} + b\frac{x}{2\sqrt{y}} = -1$$

$$= a\sqrt{1} + b\frac{-1}{2\sqrt{1}} = -1 \leftrightarrow a - \frac{b}{2} = -1$$

$$a+1 = \frac{b}{2} \leftrightarrow 2a+2 = b$$

$$\leftrightarrow (2a+2)^2 = b^2$$

$$4a^2+8a+4 = b^2$$

$$a^2+b^2 = 1 \ \text{(because } \vec{u} \text{ is a unit vector)} \sqrt{a^2+b^2} = 1$$

$$a^2+b^2 = 1 \leftrightarrow 1-a^2 = b^2$$

$$\Rightarrow \quad 4a^2+8a+4 = 1-a^2$$

$$5a^2+8a+3 = 0$$

$$(5a+3)(a+1) = 0$$

$$a = \frac{-3}{5} \quad \text{or} \quad a = -1$$

$$b = \frac{4}{5} \qquad\qquad b = 0$$

$$\rightarrow \quad \text{therefore, } \vec{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle \text{ or}$$

$$\vec{u} = \langle -1,0\rangle$$ ◄

Possibly missing things from one board of notes (from here to start of next theorem) as the prof erased stuff in different order than usual.

$$D_{\vec{u}} f(-1,1) = \left\langle 1, \frac{-1}{2} \right\rangle \cdot \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle$$

$$= \frac{-3}{5} - \frac{2}{5}$$

$$D_{\vec{u}} f(-1,1) = \left\langle 1, \frac{-1}{2} \right\rangle \cdot \langle -1,0\rangle$$

$$= -1+0$$

■

Theorem Suppose $f(x,y)$ is differentiable at (x_0,y_0). Then

1. The max value of $D_{\vec{u}}f(x_0,y_0)$ over any choice of $\vec{u}$ is $\|\nabla f(x_0,y_0)\|$, and it occurs in the direction $\nabla f(x_0,y_0)$ $\left(\text{or the unit direction } \dfrac{\nabla f(x_0,y_o)}{\|\nabla f(x_0,y_0)\|}\right)$
2. The min value of $D_{\vec{u}}f(x_0,y_0)$ is $-\|\nabla f(x_0,y_0\|$, and it occurs at direction $-\nabla f(x_0,y_0)$ $\left(\text{or unit } \dfrac{-\nabla f(x_0,y_0)}{\|\nabla f(x_0,y_0)\|}\right)$

■ **Example** If you are on the hill $f(x,y)=\sqrt{4-x^2-y^2}$ at point $(1,-1)$, what direction will you go to descend the fastest? At what rate?

$$\blacktriangleright \quad \nabla f = \left\langle \frac{1}{2}\left(4-x^2-y^2\right)^{-\frac{1}{2}}[-2x], \frac{1}{2}\left(4-x^2-y^2\right)^{-\frac{1}{2}}[-2y]\right\rangle$$

$$= \left\langle \frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}}\right\rangle$$

$$\nabla f(1,-1) = \left\langle \frac{-1}{\sqrt{4-(1)^2-(-1)^2}}, \frac{1}{\sqrt{4-(1)^2-(-1)^2}}\right\rangle$$

$$= \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$$\to \quad \text{direction is } -\nabla f(1,-1) = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

Not sure about what I transcribed. I think it should be

$$\to \quad \text{direction is } -\nabla f(1,-1) = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\rangle$$

(good news is this is already a unit vector)

$$\to \quad \text{the rate is } -\|\nabla f(1,-1)\|$$

$$= -\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{-1}{\sqrt{2}}\right)^2} = -\sqrt{\frac{1}{2}+\frac{1}{2}} = -\sqrt{1} = -1 \quad \blacktriangleleft$$

■

■ **Example**

$$\text{Recall } f(x,y) = x^2+2x+y^2+2y+2$$

$$= (x+1)^2+(y+1)^2$$

at the point $(0,0)$
Find at which direction starting at $(0,0)$, that $D_{\vec{u}}f(0,0)$ will increase the fastest

$$\blacktriangleright \quad f_x(0,0) = 2$$

$$f_y(0,0) = 2$$

$$D_{\vec{u}}f(0,0) = 2\sqrt{2}$$

$$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$\to$ we need to find max using the theorem

$$\to \quad \nabla f = \langle 2x+2, 2y+2\rangle$$

$$\nabla f(0,0) = \langle 2,2\rangle$$

$\to$ theorem says max value is

$$\|\nabla f(0,0)\| = \|\langle 2,2\rangle\| = 2\sqrt{2}$$

and it's in the direction $\nabla f(0,0) = \langle 2,2\rangle$ as a unit vector

$$\begin{aligned}\frac{\nabla f(0,0)}{\|\nabla f(0,0)\|} = \frac{\langle 2,2\rangle}{2\sqrt{2}} &= \left\langle \frac{2}{2\sqrt{2}}, \frac{2}{2\sqrt{2}}\right\rangle \\ &= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \\ &= \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle\end{aligned}$$

■

November 11, 2019

Mind Warmup

- Min and max value of $D_{\vec{u}} f(a,b)$ gives the smallest or largest "slope" at point (a,b)

$\to$ and is solved by $\pm\|\nabla f(a,b)\|$

Today is on min/max of functions

MISSING DIAGRAM

R Recall: $(a, f(a))$ is a local max. $f'(a) = 0$, aka tangent line is flat.
$(b, f(b))$ is local min (same)

§14.7 Min and Max Values

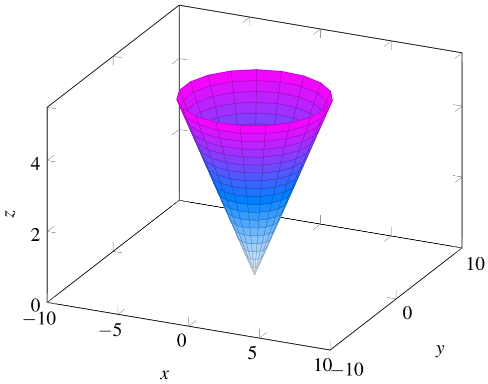
Idea: in multiple variables, we would want all directions to be flat, $f_x(a,b) = 0$ *and* $f_y(a,b) = 0$, aka: flat tangent plane

Definition $f(x,y)$ has a local maximum at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) near (a,b).
Same for local minimum if $f(a,b) \leq f(x,y)$

Theorem If $f(x,y)$ has a local max/min at (a,b). Then
1. $f_x(a,b) = 0$ and $f_y(a,b) = 0$
2. at least one of $f_x(a,b)$, $f_y(a,b)$ does not exist

note: we include "does not exist" for cases like this

MISSING DIAGRAM



Question: How do we find local min/max?

1. find critical points
2. check if min/max

Definition A point (a,b) is a <u>critical point</u> for $f(x,y)$ if

1. $\nabla f(a,b) = \langle 0,0\rangle \qquad (f_x(a,b) = 0 \text{ and } f_y(a,b) = 0)$
2. at least one of $f_x(a,b)$, ${}_y(a,b)$ DNE

■ **Example** $f(x,y) = x^2 - y^2 + 1$ Show the critical point is <u>not</u> a local min/max

▶ $f_x = 2x, \qquad f_x = 0$ when $2x = 0, \qquad x = 0$

$f_y = -2y, \qquad f_y = 0$ when $2y = 0, \qquad y = 0$

$\to$ critical point $(x,y) = (0,0)$

$$f(0,0) = 1$$

$$f(0.1,0) = 0.01 - 0 + 1 = 1.01$$

$$f(0,0.1) = 0 - 0.01 + 1 = 0.99$$

$\to$ near $(0,0)$ has both larger and smaller values.
$\to$ actually, it's a saddle point. ■

Theorem — Second Derivative Test. Let $f(x,y)$ be a function, and (a,b) be a critical point.
Suppose f_{xx}, f_{xy}, f_{yx}, f_{yy} are continuous near (a,b)

Let $D = (f_{xx}(a,b))\,(f_{yy}(a,b)) - (f_{xy}(a,b))^2$

1. $D > 0$ and $f_{xx}(a,b) > 0 \qquad (a,b)$ local min
2. $D > 0$ and $f_{xx}(a,b) < 0 \qquad (a,b)$ local max
3. $D < 0 \qquad (a,b)$ saddle point
4. $D = 0 \qquad$ test inconclusive

Inconclusive? $f(x,y) = x^3 + y^3$

R note $D = \underbrace{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}}_{\text{the Hessian matrix}} = (f_{xx})\,(f_{yy}) - (f_{xy})^2$

Note/Ex Let $z = f(x,y)$, and $f_x(a,b) = 0$, and $f_y(a,b) = 0$ for point (a,b,c)

$\to$ here, $\vec{n} = \nabla F = \langle f_x, f_y, -1 \rangle$

$$\uparrow$$

$$F(x,y,z) = F(x,y) - z$$

$\to \vec{n}(a,b,c) = \langle 0,0,-1 \rangle$

$\to$ plane $0(x-a) + 0(y-b) + 0(z-c) = 0$ $\qquad \underline{z = c}$

In other words, $z = c$ is a flat horizontal tangent plane.

November 13, 2019

Course Evaluations!!

Mind Warmup

(a,b) is a critical point for $f(x,y)$

- $f_x(a,b) = 0$ *and* $f_y(a,b) = 0$
- at least one of $f_x(a,b)$, $f_y(a,b)$ don't exist

Definition 2^{nd} Derivative Test for (a,b)

$$D = (f_{xx}(a,b))\left(f_{yy}(a,b)\right) - \left(f_{xy}(a,b)\right)^2$$

$D > 0$: $f_{xx} > 0$ → local min; $f_{xx} < 0$ → local max

$D < 0$: saddle point

$D = 0$: ?

■ **Example** $f(x,y) = x^4 - 2x^2 + y^2$ Classify all critical points

► $f_x = 4x^3 - 4x$

$$= 4x\left(x^2 - 1\right)$$

$$= 4x(x+1)(x-1)$$

$\to f_x = 0$ when $x = 0, x = -1, x = 1$

$f_y = 2y$

$\to f_y = 0$ when $y = 0$

$\Rightarrow$ critical points $(0,0), (1,0), (-1,0)$

Now $f_{xx} = 12x^2 - 4$

$f_{yy} = 2$

$f_{xy} = f_{yx} = 0$

$$D(0,0) = (-4)(2) - (0)^2 = -8 < 0 \qquad \text{Saddle point } (0,0)$$

$$D(1,0) = (8)(2) - (0)^2 = 16 > 0$$

$$f_{xx}(1,0) = 8 > 0 \qquad \text{Local min } (1,0)$$

$$D(-1,0) = (8)(2) - (0)^2 = 16 > 0$$

$$f_{xx}(-1,0) = 8 > 0 \qquad \text{Local min } (-1,0)$$

◀

■

MISSING DIAGRAM

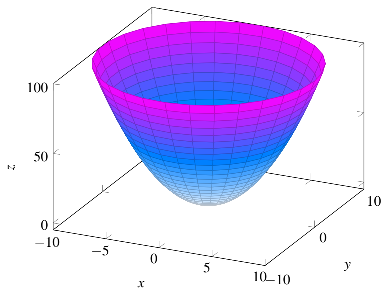
Global Max/Min Values (Absolute)

→ what ablout multiple max/min values?

→ worse, what about functions that increase to infinity

→ Let's consider min/max on a restricted set

$z = x^2 + y^2$



Definition Let $D \subset \mathbb{R}^2$ (D is a subset of $\mathbb{R}^2$, i.e. a square, a triangle, etc.

1. We call D bounded if we can draw a circle around it
2. We call D closed if it includes the boundary

   $x^2 + y^2 < 1$ open disk

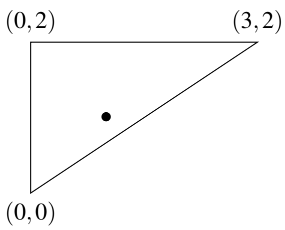
   $x^2 + y^2 \leq 1$ closed disk

Theorem — Extreme Value Theorem. Suppose $f(x,y)$ is continuous on closed, bounded D. Then $f(x,y)$ attains a global max and min

R Important! - you need to consider the boundary. So,

1. find critical points, see how big/small output value is
2. find min/max on the boundary
3. largest/smallest value from 1& 2 are global max/min

■ **Example** Find global min and max of $f(x,y) = x^2 - 2xy + 2y$ on closed triangle with vertices $(0,0)$, $(0,2)$ $(3,2)$



- ▶ $f_x = 2x - 2y = 2(x-y)$

  $\rightarrow f_x = 0$ when $x = y$

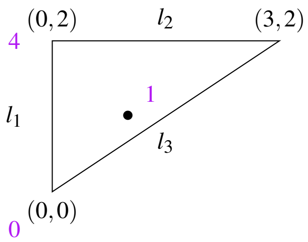
  $f_y = -2x + 2 = 2(1-x)$

  $\rightarrow f_y = 0$ when $x = 1$

  $\rightarrow$ so $y = 1$

$\Rightarrow$ critical point $(1,1)$

$f(1,1) = 1 - 2 + 2 = 1$



$l_1(t) = \langle 0, t\rangle \qquad 0 \le t \le 2$

$l_2(t) = \langle t, 2\rangle \qquad 0 \le t \le 3$

$l_3(t) = \langle t, \frac{2}{3}t\rangle \qquad 0 \le t \le 3$

$\uparrow$

$= \frac{2}{3}x$

$l_1(t) \qquad f(0,t) = 0^2 - 2(0)(t) + 2t$

$= 2t, \qquad 0 \le t \le 2$

$\rightarrow$ min of 0, $t = 0$, $(0,0)$

max of 4, $t = 2$, $(0,2)$

⋮ to be continued ■

November 15, 2019

Mind Warmup

$\rightarrow$ finding global min/max on closed D

1. find critical points, i.e. $0 = f_x = f_y$
2. find extreme values on the boundary of $D*$
3. compare points from 1 & 2 to find largest/smallest

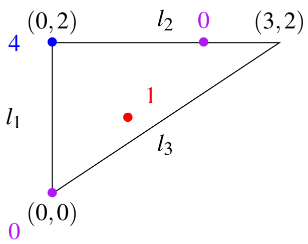
■ **Example — continued.** Find global min/max of $f(x,y) = x^2 - 2xy + 2y$ on closed triangle with vertices $(0,0)$, $(0,2)$, $(3,2)$

▶ critical point $(1,1)$ gave $f(1,1) = 1$

$$l_1(t) = \langle 0, t\rangle \qquad 0 \le t \le 2$$

$$l_2(t) = \langle t, 2\rangle \qquad 0 \le t \le 3$$

$$l_3(t) = \langle t, \frac{2}{3}t\rangle \qquad 0 \le t \le 3$$

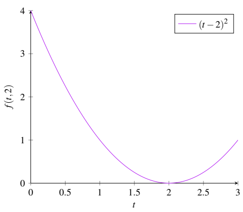


$$l_1(t) \to \text{ min of 0 at } (0,0)$$

$$\to \text{ max of 4 at } (0,2)$$

$$l_2(t): \qquad f(t,2) = t^2 - 4t + 4$$

$$= (t-2)^2, \quad 0 \le t \le 3$$



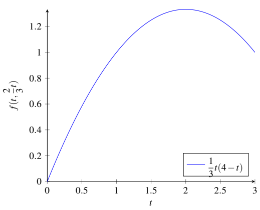
min of 0 when $t = 2$, $(2,2)$

max of 4 when $t = 0$, $(0,2)$

$$l_3(t): \qquad f(t, \frac{2}{3}t) = t^2 - \frac{4}{3}t + \frac{4}{3}t$$

$$= -\frac{1}{3}t^2 + \frac{4}{3}t$$

$$= \frac{1}{3}t(4-t), \quad 0 \le t \le 3$$



min of 0 when $t = 0$, $(0,0)$

max of $\frac{4}{3}$ when $t = 2$, $(2, \frac{4}{3})$

$\Rightarrow$ comparing all points

global max is 4 at $(0,2)$

global min is 0 at $(0,0)$ and $(2,2)$ ◀

■

§14.8 Lagrange Multipliers

Idea: compare gradient vectors $\leftrightarrow$ it's where there is a common tangent.

MISSING DIAGRAM

Contour map for $f(x,y)$

$y = g(x)$

Here is where the output for the input $y = g(x)$ is the largest

$\to$ they are tangent so the gradient vectors agree

$$\nabla f = \lambda \nabla g$$

Method of Lagrange Multipliers

To find min/max of $F(x,y)$ restricted to $g(x,y) = 0$ (provided min/max exist, and $\nabla g \neq 0$ anywhere),

$$1 \begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \quad \text{solve these simultaneously for } x, y, 1$$

2 Compare all solutions for largest smallest

■ **Example** Find min/max of $f(x,y) = xy^2$ on the circle $x^2 + y^2 = 1$

► $g(x,y) = 0 \qquad x^2 + y^2 - 1 = 0$

$$\begin{aligned} \nabla f &= \langle y^2, 2xy \rangle \\ \lambda \nabla g &= \lambda \langle 2x, 2y \rangle \end{aligned} \quad =$$

$$\begin{cases} y^2 = \lambda 2x & (1) \\ 2xy = \lambda 2y & (2) \\ x^2 + y^2 - 1 = 0 & (3) \end{cases}$$

$$= \quad \begin{aligned} &2xy - 2\lambda y = 0 \\ &2y(x - \lambda) = 0 \end{aligned}$$

$$\text{Case } y = 0\text{: } \xrightarrow{(3)} x^2 + 0 - 1 = 0, \qquad x = \pm 1$$

$$\Rightarrow (1,0), (-1,0)$$

$$\text{Case } x = \lambda\text{: } \xrightarrow{(1)} y^2 = 2\lambda^2, \qquad y = \pm\sqrt{2}\lambda$$

$$\xrightarrow{(3)} (\lambda)^2 + 2\lambda - 1 = 0$$

$$3\lambda^2 = 1, \qquad \lambda = \pm \frac{1}{\sqrt{3}}$$

$$\blacktriangleright \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}} \right), \left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}} \right)$$

$$\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}} \right), \left(-\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}} \right)$$

$\Rightarrow f(\pm 1, 0) = 0 \qquad f = xy^2$

$$\Rightarrow \left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}} \right) = +\frac{2}{3\sqrt{3}} \text{ (max)}$$

$$\Rightarrow \left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}} \right) = -\frac{2}{3\sqrt{3}} \text{ (min)}$$

■

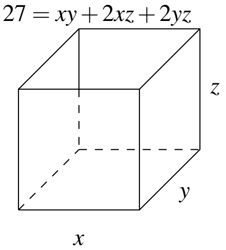
November 18, 2019

Mind Warmup

Find min/max over $D \subset \mathbb{R}^2$

1. interior of D, find critical points
2. boundary of D,
   a) observe each boundary piece of D
   b) Lagrange multipliers $\nabla F = \lambda \nabla g$

■ **Example** A box without a lid is to be made from 27m^2 of cardboard.
What is max volume of box? ► Find max $V(x,y,z) = xyz$
on $27 = xy + 2xz + 2yz$



$g(x,y,z) = xy + 2xz + 2yz - 27$
$\nabla f = \lambda \nabla g$

$$\langle yz, xz, xy \rangle = \lambda \langle y+2z, x+2z, 2x+2z \rangle \begin{cases} yz = \lambda y + 2\lambda z \\ xz = \lambda x + 2\lambda z \\ xy = 2\lambda x + 2\lambda y \\ xy + 2xz + 2yz = 27 \qquad (4) \end{cases}$$

Important trick: use symmetry

$$\begin{cases} xyz = \lambda xy + 2\lambda xz & (1) \\ xyz = \lambda xy + 2\lambda yz & (2) \\ xyz = 2\lambda xz + 2\lambda yz & (3) \end{cases}$$

$(1) - (2) \qquad 2\lambda xz - 2\lambda yz = 0$

$$2\lambda z(x-y) = 0 \quad \rightarrow \quad x = y$$

$(2) - (3) \qquad \lambda xy - 2\lambda xz = 0$

$$\lambda x(y - 2z) = 0 \quad \rightarrow \quad y = 2z$$

$(4) \qquad (2z)(2z) + 2(2z)z + 2(2z)z = 27$

$$4z^2 + 4z^2 + 4z^2 = 27$$

$$z^2 = \frac{27}{12} = \frac{9}{4}$$

$$z = \pm\frac{3}{2}, \qquad y = 3, \qquad x = 3$$

$$\Rightarrow \text{ max volume } = xyz = (3)(3)\left(\frac{3}{2}\right) = \frac{27}{2}\text{m}^2$$

■

■ **Example** Find min/max $f(x,y) = x^2 + y^2 - 2x - 5$ on $x^2 + 2y^2 \leq 16$

1. find critical points

$$\begin{cases} f_x = 2x-2 = 2(x-1) = 0 & \text{when } x = 1 \\ f_y = 2y = 0 & \text{when } y = 0 \end{cases}$$

$\rightarrow$ critical point $(1,0)$

2. boundary of D, $x^2+2y^2 = 16$

$$\nabla f = \lambda \nabla g$$

$$\langle 2x-2, 2y\rangle = \lambda \langle 2x, 4y\rangle$$

$$\begin{cases} 2x-2 = 2\lambda x & (1) \\ 2y = 4\lambda y & (2) \\ x^2+2y^2 = 16 & (3) \end{cases}$$

$$(2) \quad 2y-4\lambda y = 0, \quad 2y(1-2\lambda) = 0$$

$$\rightarrow y = 0 \overset{(3)}{\rightarrow} x^2 = 16, \quad x = \pm 4$$

$$\rightarrow \lambda = \frac{1}{2} \overset{(1)}{\rightarrow} 2x-2 = x \leftrightarrow x = 2$$

$$\overset{(3)}{\rightarrow} 2y^2 = 12, \quad y = \pm\sqrt{6}$$

$$f = x^2+y^2-2x-5$$

$$f(1,0) = 1+0-2-5 = -6$$

$$f\left(2, \pm\sqrt{6}\right) = 4+6-4-5 = 1$$

$$f(4,0) = 16+0-8-5 = 3$$

$$f(-4,0) = 16+0+8-5 = 19$$

$\rightarrow$ max of 19 at $(-4,0)$
$\rightarrow$ min of -6 at $(1,0)$

alternative for boundary,

$$x^2+2y^2 = 16$$

$$y^2 = 8-\frac{1}{2}x^2$$

$$f = x^2+y^2-2x-5$$

$$= x^2+\left(8-\frac{1}{2}x^2\right)-2x-5$$

$$= \frac{1}{2}x^2-2x+3 \text{ on } [-4,4]$$

■

Lagrange for two restrictions

Find min/ max of $f(x,y,z)$ restricted to $g(x,y,z) = 0$ and $h(x,y,z) = 0$

$$\Rightarrow \quad \nabla f = \lambda \nabla g + \mu \nabla h$$

November 20, 2019

Mind Warmup

Post solutions for # 9.2-9.4, 9.12

Lagrange multipliers $\to$ boundary of $D \to x^2 + y^2 = 2$

Critical points $\to$ interior of $D \to x^2 + y^2 \leq 2$

15. Multiple Integrals

Topics:

- Double Integrals
- Iterated integrals
- Fubini's Theorem
- Polar Coordinates
- Surface Area
- How to Visualize/Sketch
- Triple Integrals
- Change of coordinates (cylindrical/spherical)
- Volume of a Solid

§15.1 Double Integrals (over rectangles)

Let $R = [a,b] \times [c,d]$

$= \{(x,y) \text{ s.t. } a \leq x \leq b, c \leq y \leq d\}$

MISSING DIAGRAM

Calculus 1 $\int_a^b f(x)dx = \lim \sum_{i=1}^{n} \underbrace{f(x_i{}^*)}_{\text{height}} \underbrace{\Delta x}_{\text{base}}$

(adding thin rectangles for area)

Calculus 3 $\iint\limits_R f(x,y)dxdy = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{m} \sum j = 1^n \underbrace{f(x_i{}^*, y_j{}^*)}_{\text{height}} \underbrace{\Delta x}_{\text{width}} \underbrace{\Delta y}_{\text{length}}$

(adding thin rectangular prisms for volume)

$\rightarrow$ so $\iint\limits_R f(x,y)dxdy$ can be seen as the volume under $z = f(x,y)$, defined over R

→ also, the places where $f(x,y) > 0$ will give positive contribution, where $f(x,y) < 0$ will give negative contribution

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sin(x)\sin(y)dxdy = 0 \qquad \text{(think about it)}$$

Iterated Integrals

$$\begin{aligned} V &= \int_c^d \int_a^b f(x,y)dxdy \\ &= \int_i^d \left(\int_a^b f(x,y)dx\right)dy \\ &= \int_c^d A(y)dy, \qquad \text{where } A(y) = \int_a^b f(x,y)dx \end{aligned}$$

→ this means evaluate the inner integral first, keeping y constant

→ geometrically it is:

MISSING DIAGRAM

■ **Example**

$$\begin{aligned} \int_0^1 \int_2^3 (2x+y)dxdy &= \int_0^1 \left(\int_2^3 (2x+y)dx\right)dy \\ &= \int_0^1 \left(x^2+xy\right)\Big|_{x=2}^{x=3} dy \\ &= \int_0^1 (9+3y-4-2y)dy \\ &= \int_0^1 (5+y)dy \\ &= \left(5y+\frac{1}{2}y^2\right)\Big|_0^1 \\ &= 5+\frac{1}{2}-0-0 \\ &= \frac{11}{2} \end{aligned}$$

■

Theorem — Fubini. If $f(x,y)$ is continuous over $R = [a,b]\times[c,d]$, then

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

■ **Example**

$$\int_1^2 \int_0^{\frac{\pi}{2}} y\sin(xy)dydx$$

► Solving $\int_0^{\frac{\pi}{2}} y\sin(xy)dy$ is tough (Integral By Parts) so we switch using Fubini

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_1^2 y\sin(xy)dxdy &= \int_0^{\frac{\pi}{2}}\left(\int_1^2 y\sin(xy)dx\right)dy\\ &= \int_0^{\frac{\pi}{2}}\left(-\cos(xy)\Big|_{x=1}^{x=2}\right)dy\\ &= \int_0^{\frac{\pi}{2}}\left(-\cos(2y)+\cos(y)\right)dy\\ &= \left(-\frac{1}{2}\sin(2y)+\sin(y)\right)\Big|_0^{\frac{\pi}{2}}\\ &= \cancel{-\frac{1}{2}\sin(\pi)}+\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cancel{+\frac{1}{2}\sin(0)}\cancel{-\sin(0)}\\ &= 1\end{aligned}$$

■

■ **Example**

$$\begin{aligned}\int_0^1\int_0^1 e^{2x+3y}dxdy &= \int_0^1\left(\int_0^1 e^{2x}e^{3y}dx\right)dy\\ &= \int_0^1 e^{3y}\left(\int_0^1 e^{2x}dx\right)dy\\ &= \left(\int_0^1 e^{2x}dx\right)\left(\int_0^1 e^{3y}dy\right)\end{aligned}$$

$$\int_a^b\int_c^d f(x)g(y)dxdy = \left(\int_a^b g(y)dy\right)\left(\int_c^d f(x)dx\right)$$

■

November 22, 2019

Mind Warmup

Notation today:

$R=[a,b]\times[c,d]$ a rectangle

D general domain/region

$$\iint_R f(x,y)dxdy = \int_c^d\int_a^b f(x,y)dxdy = C$$

R → this number can be seen as the volume under $z=f(x,y)$, defined over R

$\iint_R f(x,y)dxdy = \iint_R f(x,y)dydx$if $f(x,y)$ is continuous on R (Fubini)

→ R will be written differently for $dxdy$ or $dydx$

§15.2 Double Integrals over General Regions $(D \subset \mathbb{R}^2)$

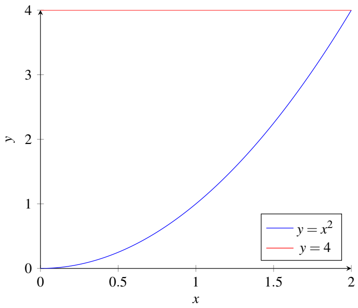
Recall $\displaystyle\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$

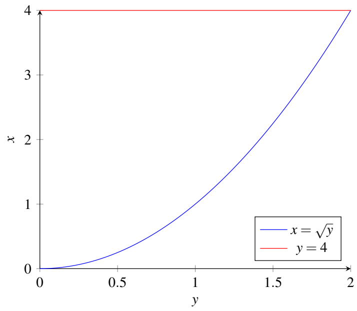
Goal $\displaystyle\int_0^2 \int_{x^2}^4 f(x,y)dydx = \int_0^4 \int_0^{\sqrt{y}} f(x,y)dxdy$

Notation: dA will be either $dxdy$ or $dydx$ (we need to decide what's first)

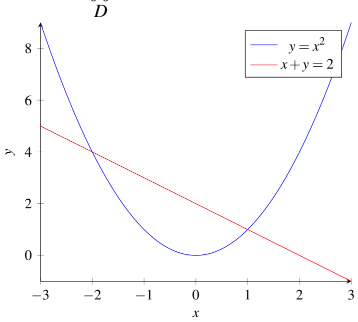
■ **Example** Show $\displaystyle\int_0^2 \int_{x^2}^4 f(x,y)dydx = \int_0^4 \int_0^{\sqrt{y}} f(x,y)dxdy$ using a sketch

▶





Evaluate $\displaystyle\iint_D 2xydA$ where D is the region enclosed by $y = x^2$, $x + y = 2 \leftrightarrow y = -x + 2$



$$y = y$$

$$x^2 = -x + 2$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$x = -2, \quad x = 1$ needed this to see where to begin and end

$$\begin{aligned}
\int_{-2}^{1}\int_{x^2}^{-x+2} 2xydydx &= \int_{-2}^{1}\left(\int_{x^2}^{-x+2} 2xydy\right)dx \\
&= \int_{-2}^{1}\left(xy^2\Big|_{y=x^2}^{y=-x+2}\right)dx \\
&= \int_{-2}^{1}\left(x(-x+2)^2 - x\left(x^2\right)^2\right)dx \\
&= \int_{-2}^{1}\left(x^3-4x^2+4x-x^5\right)dx \\
&= \frac{x^4}{4}-\frac{4x^3}{3}+\frac{4x^2}{2}-\frac{x^6}{6}\Big|_{-2}^{1} \\
&= \frac{-45}{4} \qquad \blacktriangleleft
\end{aligned}$$

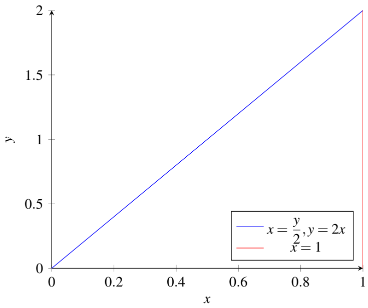
note: if we chose $dA = dxdy$

$$\underbrace{\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} 2xydxdy}_{=0} = \underbrace{\int_1^4\int_{-\sqrt{y}}^{-y+2} 2xydxdy}_{=\frac{-45}{4}}$$

■

■ **Example** Compute $\int_0^2\int_{\frac{y}{2}}^{1} y\cos\left(x^3-1\right)dxdy$ by switching the order of integration.

► (we can do this because of Fubini)



$$\int_0^1 \int_0^{2x} y\cos\left(x^3-1\right)dydx = \int_0^1 \left(\int_0^{2x} y\cos\left(x^3-1\right)dy\right)dx$$

$$= \int_0^1 \left(\frac{1}{2}y^2\cos\left(x^3-1\right)\Big|_{y=0}^{y=2x}\right)dx$$

$$= \int_0^1 2x^2\cos\left(x^3-1\right)dx$$

$$u = x^3-1$$

$$du = 3x^2dx$$

$$\frac{2}{3}du = 2x^2dx$$

$$= \int_{-1}^0 \cos(u)\frac{2}{3}du$$

$$= \frac{2}{3}\sin(u)\Big|_{-1}^0$$

$$= \cancel{\frac{2}{3}\sin(0)} - \frac{2}{3}\sin(-1)$$

$$= \frac{-2}{3}\sin(-1)$$

$$= \frac{2}{3}\sin(1)$$

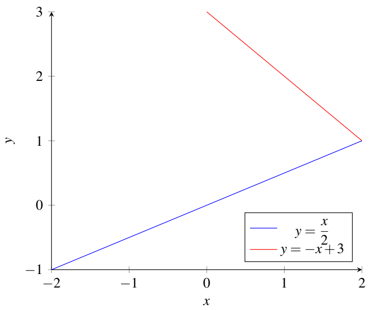
■

■ **Example** (15 2.64) Determine D and switch the order of integration of

$$\underbrace{\int_0^1\int_0^{2y}}_{D_1} f(x,y)dxdy + \underbrace{\int_1^3\int_0^{3-y}}_{D_2} f(x,y)dxdy$$

$$\int_0^2\int_{\frac{1}{2}x}^{3-x} f(x,y)dydx$$

Diagram below incomplete and possibly wrong



■

November 25, 2019

Mind Warmup

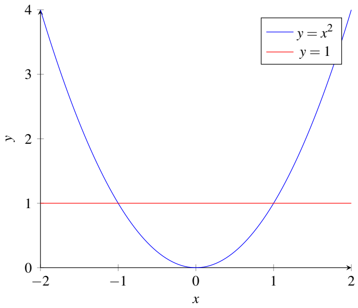
$$\iint_D f(x,y)dA \quad \leftarrow \text{ seen as volume under } z=f(x,y)\text{, over } D.$$

$\leftarrow$ some consider D as the base, and $f(x,y)$ as the height.

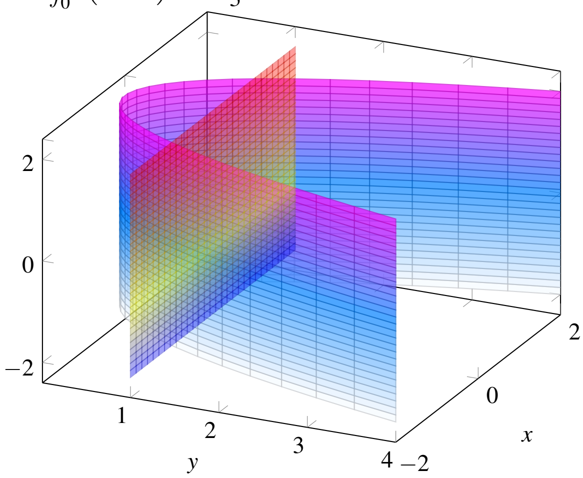
$\rightarrow$ dA is $dxdy$ or $dydx$ (one direction might be impossible to solve)

$$\text{recall} \quad \int_0^2 \int_{\frac{y}{2}}^1 y\cos\left(x^3-1\right)dxdy$$

$$= \int_0^1 \int_0^{2x} y\cos\left(x^3-1\right)dydx$$



$$\text{area}=\int_0^1 \left(1-x^2\right)dx = \frac{2}{3}$$



$$\text{Volume} \quad \int_0^1 \int_{x^2}^1 4dydx = \frac{8}{3}$$

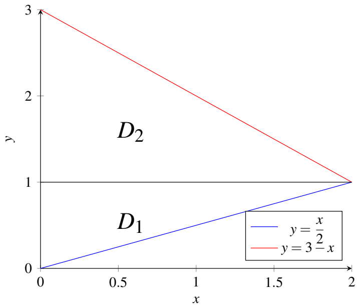
$\rightarrow$ main idea for different volumes

1. different $z=f(x,y)$
2. different D

■ **Example** Determine D and switch the order of integration of

$$\underbrace{\int_0^1 \int_0^{2y}}_{D_1} f(x,y)dxdy = \underbrace{\int_0^3 \int_0^{3-y}}_{D_2} f(x,y)dxdy$$

$\rightarrow$ want to make $D = D_1 + D_2 =$

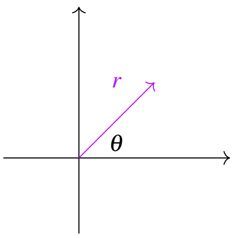


$$= \underbrace{\int_0^2 \int_{\frac{1}{2}x}^{3-x}}_{D} f(x,y)dydx$$

$\rightarrow$ so we turned two integrals into one.

■

§15.3 Polar Coordinates



$$(x,y) = (1,1)$$
$$r = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\theta = \frac{\pi}{4}$$
$$\tan\theta = \frac{1}{1}$$
$$\theta = \arctan(1) = \frac{\pi}{4}$$

$(r,\theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$

$(x,y) = (1,1)$

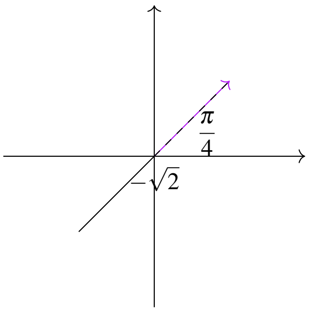
Polar Coordinates

(r,θ), $r_1, \theta \in \mathbb{R}$

r can be negative

$$(r,\theta) = \left(-\sqrt{2}, \frac{\pi}{4}\right)$$

$$(x,y) = (-1,-1)$$



Definition Polar Equations

$$x = r\cos\theta, \quad y = r\sin\theta$$

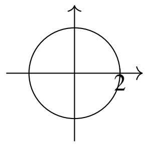
$$x^2 + y^2 = r^2, \quad r = \pm\sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

So we can think of some curves we know

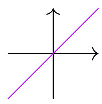
1. $r = 2$, all points have radius

   $r = 2 \leftrightarrow x^2 + y^2 = 4$



2. $\theta = \frac{\pi}{4}$, all points midway between x and y

   $\theta = \frac{\pi}{4} \leftrightarrow y = x$

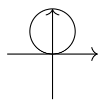


3. $r = \sin\theta \rightarrow r^2 = r\sin\theta \rightarrow x^2 + y^2 = y$

   $\rightarrow x^2 + y^2 - y = 0 \rightarrow x^2 + y^2 - y + \frac{1}{4} - \frac{1}{4} = 0$

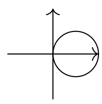
   $x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

   $r = \sin\theta \leftrightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$



4. $r = \cos\theta$

   $r = \cos\theta \leftrightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$



$$\iint_D f(x,y)\,\underbrace{dxdy}$$

$$= \iint_D f(r\cos\theta, r\sin\theta)\,\underbrace{rdrd\theta}$$

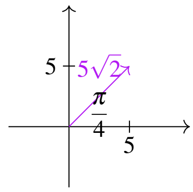
MISSING DIAGRAM

November 27, 2019

We're going to remove surface area

Mind Warmup

Cartesian $(x,y) = (5,5)$



Polar $(r,\theta) = \left(5\sqrt{2}, \frac{\pi}{4}\right)$

$$\iint\limits_{D_{x,y}} f(x,y)dxdy = \iint\limits_{D_{r,\theta}} f(r\cos\theta, r\sin\theta)rdrd\theta$$

$$\left.\begin{aligned} x &= r\cos\theta \\ x &= r\sin\theta \end{aligned}\right\} x^2+y^2=r^2$$

Question: Where did this double integral come from?

Question: How to use it?

Definition Change of Variables (double integrals)

If we let $x = g(u,v), yh(u,v)$, then we change the integral by

$$\iint\limits_{D_{x,y}} f(x,y)dxdy = \iint\limits_{D_{u,v}} f\left(g(u,v),h(u,v)\right) \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\substack{\text{this is called the} \\ \text{Jacobian, and we} \\ \text{take the determinant} \\ \text{(positive)}}} dudv$$

■ **Example** Let $x = r\cos\theta$, $y = r\sin\theta$. Find change of variable

▶ $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial r} = \sin\theta$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r$$

$$\iint f(x,y)dxdy = \iint f(r\cos\theta, r\sin\theta)\,|r|drd\theta$$

■

1. replace $dxdy$ by $rdrd\theta$
2. replace x,y with $r\cos\theta, r\sin\theta$
3. write D in terms of r,θ.

■ **Example** $\iint\limits_D d(x+y)dxdy$, where D is the quarter circle in the first quadrant of radius 3, as follows.

So here, $0 \le r \le 3$ and $0 \le \theta \le \frac{\pi}{2}$

MISSING DIAGRAM

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}}\int_0^3 (r\cos\theta, r\sin\theta)\, r dr d\theta &= \int_0^{\frac{\pi}{2}}\int_0^3 r^2(\cos\theta+\sin\theta)\, dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_{r=0}^{r=3} d\theta \\
&= \int_0^{\frac{\pi}{2}} 9(\cos\theta+\sin\theta)\, d\theta \\
&= 9(\sin\theta-\cos\theta)\Big|_0^{\frac{\pi}{2}} \\
&= 9\left(\sin\frac{\pi}{2} - \cancel{\cos\frac{\pi}{2}} - \cancel{\sin 0} + \cos 0\right) \\
&= 18
\end{aligned}
$$

■

$\int_0^3 \int_0^{\sqrt{9-x^2}} (x+y) dydx$

■ **Example** Evaluate $\underbrace{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}}_{D} \cos\left(x^2+y^2\right) dydx =$

$y = -\sqrt{1-x^2}$ $y^2 = 1-x^2$

MISSING DIAGRAM

So here, $0 \le r \le 1$ and $\pi \le \theta \le 2\pi$

$$
\begin{aligned}
&= \int_\pi^{2\pi}\int_0^1 \cos\left(r^2\right) r dr d\theta \\
&= \underbrace{\left(\int_\pi^{2\pi} d\theta\right)}_{=\pi}\left(\int_0^1 r\cos\left(r^2\right) dr\right) \\
&= \pi\int_0^1 r\cos\left(r^2\right) dr =
\end{aligned}
$$

$$
\begin{aligned}
u &= r^2 \\
du &= 2rdr \\
\frac{du}{2} &= rdr
\end{aligned}
$$

$$
\begin{aligned}
&= \pi\int_0^1 \cos(u)\frac{du}{2} \\
&= \frac{\pi}{2}\sin(u)\Big|_0^1 \\
&= \frac{\pi}{2}\sin(1) - \cancel{\frac{\pi}{2}\sin(0)} \\
&= \frac{\pi}{2}\sin(1)
\end{aligned}
$$

■

R $\rightarrow$ Any time you see x^2+y^2, or parts of circles, it's probably a good idea to switch to polar coordinates.

$$\iint f(x,y)dxdy = \iint f(r\cos\theta, r\sin\theta)\, rdrd\theta$$

$$\begin{tikzcd}[column sep=small] & D_{r\theta} \arrow[dl, <->] \\ D_{x,y} \arrow[dr, <->] & \\ & D_{y,x} \end{tikzcd}$$

November 29, 2019

Class

$$\left.\begin{array}{l}\text{Monday, Dec 2}\\ \text{Tuesday, Dec 3}\end{array}\right\}\text{10:30-11:30 (here)}$$

Office Hours

$$\left.\begin{array}{l}\text{Wednesday, Dec 4}\\ \text{Wednesday, Dec 11}\end{array}\right\}\text{10:30-12:00}$$

Burnside 1017

Mind Warmup

$$\iint_D f(x,y)dA = \iint_D f(r\cos\theta, r\sin\theta)\, rdrd\theta \qquad \leftarrow \text{volume with base area } D \text{ and height } f$$

Area of (Polar) Regions

R **Fact:** area = $\iint_D 1dA$ $\leftarrow$ gives the area of region D

Where does this come from?

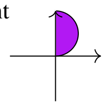
Volume = base area $\times$ height

$\rightarrow$ so if height = 1

Volume = base area

$$\Rightarrow \text{area} = \iint_D 1dxdy = \iint_D 1dydx = \iint_D rdrd\theta$$

■ **Example** Find the area of $r = 2\sin\theta$ in the first quadrant



$$r^2 = 2r\sin\theta$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 - 2y + 1 - 1 = 0$$

$$x^2 + (y-1)^2 = 1$$

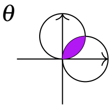
$$\begin{aligned}
\text{area} &= \int_0^{\frac{\pi}{2}} \int_0^{2\sin\theta} r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} r^2 \Big|_{r=0}^{2\sin\theta} \right) d\theta \\
&= \int_0^{\frac{\pi}{2}} 2\sin^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} (1-\cos 2\theta)\, d\theta \qquad\qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \\
&= \left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} - \cancel{\frac{\pi}{2}\sin(\pi)} - \cancel{0} + \cancel{\frac{1}{2}\sin(0)} \\
&= \frac{\pi}{2}
\end{aligned}$$

We already know this is half the area of a circle, with radius 1

$$= \frac{1}{2}\left(\pi r^2\right) = \frac{1}{2}\left(\pi 1^2\right) = \frac{\pi}{2}$$

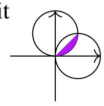
■

■ **Example** Find the area between polar curves $r = \sin\theta$ and $r = \cos\theta$



$$\begin{aligned}
r^2 &= r\cos\theta \\
x^2 + y^2 &= x \\
x^2 - x + y^2 &= 0 \\
x^2 - x + \frac{1}{4} - \frac{1}{4} + y^2 &= 0 \\
\left(x - \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4}
\end{aligned}$$

What I'll solve instead is half the area and double it



We know $\sin\theta = \cos\theta$ when $\theta = \frac{\pi}{4}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = 1 = \frac{1}{1}$

$\arctan(1) = \frac{\pi}{4}$

MISSING DIAGRAM

$$\begin{aligned}
\text{area } &= 2\int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} r\,dr\,d\theta \\
&= \frac{2}{2}\int_0^{\frac{\pi}{4}} r^2\Big|_{r=0}^{r=\sin\theta} d\theta \\
&= \int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\,d\theta \\
&= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\Big|_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} - \cancel{0} + \cancel{\frac{1}{4}\sin(0)} \\
&= \frac{\pi}{8} - \frac{1}{4}
\end{aligned}$$

MISSING DIAGRAM

$$\theta = 0$$

$$0 \le r \le \sin\theta$$

$$\theta = \frac{\pi}{4}$$

$$0 \le r \le \sin\theta$$

$$\theta = \frac{\pi}{8}$$

■

Definition Triple Integrals

If E is a region in $\mathbb{R}^3$, then

$$\iiint_E 1\,dx\,dy\,dz \text{ volume of region } E$$

$$\iiint_E f(x,y,z)\,dx\,dy\,dz$$

gives the mass of E where $f(x,y,z)$ is the density at each point $(x,y,z) \in E$

■ **Example** Compute

$$
\begin{aligned}
&\int_0^2 \int_0^{\frac{\pi}{2}} \int_0^3 xy^2 \cos(z)\,dydzdx \\
&= \int_0^2 \int_0^{\frac{\pi}{2}} \frac{1}{3}xy^3 \cos(z)\Big|_{y=0}^{y=3} dzdx \\
&= \int_0^2 \int_0^{\frac{\pi}{2}} 9x\cos(z)\,dzdx \\
&= \int_0^2 9x\sin(z)\Big|_0^{\frac{\pi}{2}} dx \\
&= \int_0^2 9x\,dx \\
&= \frac{9x^2}{2}\Big|_0^2 \\
&= 18
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\int_0^2 x\,dx\right)\left(\int_0^{\frac{\pi}{2}} \cos(z)\,dz\right)\left(\int_0^3 y^2\,dy\right) \\
&= (2)(1)(9) \\
&= 18
\end{aligned}
$$

■

December 2, 2019

Exam

- formula sheet online
- breakdown (by general topic) will be online
- Kahoot tomorrow

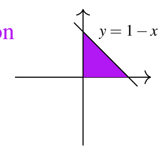
Mind Warmup

$$\iiint_E f(x,y,z)\underbrace{dxdydz}_{dV}, \quad E \subset \mathbb{R}^3$$

■ **Example** Let E be the region below $x+y+z=1$ in the first octant $(x,y,z \geq 0)$

1. Find volume of E
2. Find mass of E with density $\rho = 12 - 6z$

▶ (1) $\rho(x,y,z) = 12 - 6z \iint \left(\int_0^{1-x-y} 1\,dz\right) dxdy$ since $z = 1 - x - y$ is the upper function



When $z = 0$, $x + y = 1$, it is what E is above

$$\begin{aligned}
\text{volume } &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 dzdydx \\
&= \int_0^1 \int_0^{1-x} (1-x-y)dydx \\
&= \int_0^1 \left(y - xy - \frac{1}{2}y^2\right)\Big|_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left(1 - x - x(1-x) - \frac{1}{2}(1-x)^2\right) dx \\
&= \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx \\
&= \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\Big|_0^1 \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \\
&= \frac{1}{6}
\end{aligned}$$

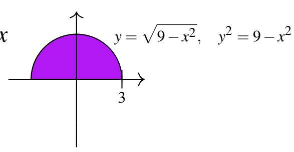
MISSING DIAGRAM

▶ (2)
$$\begin{aligned}
\text{mass } &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (12-6z)dzdydx \\
&= \int_0^1 \int_0^{1-x} \left(12z - 3z^2\right)\Big|_{z=0}^{z=1-x-y} dydx \\
&= \int_0^1 \int_0^{1-x} 12(1-x-y) - 3(1-x-y)^2 dydx \\
&= \int_0^1 \frac{-12}{2}(1-x-y)^2 + \frac{3}{3}(1-x-y)^3\Big|_{y=0}^{y=1-x} dx \\
&= \int_0^1 -6(0)^2 + 1(0)^3 + 6(1-x)^2 - 1(1-x)^3 dx \\
&= \int_0^1 6(1-x)^2 - (1-x)^3 dx \\
&= -2(1-x)^3 + \frac{1}{4}(1-x)^4\Big|_0^1 \\
&= -0 + 0 + 2 - \frac{1}{4} = \frac{7}{4}
\end{aligned}$$

→ note mass > volume, this is because $12 - 6z > 1$ over E ■

■ Example

Compute $\underbrace{\int_{-3}^{3} \int_0^{\sqrt{9-x^2}}}_{R} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2} dzdydx$



$0 \leq r \leq 3 \qquad 0 \leq \theta \leq \pi$

MISSING DIAGRAM

I knew this is where the upper function met the xy-plane when $z = 0$, because $0 = 9 - x^2 - y^2$

$\to$ Switch to polar coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \text{this means } dzdydx \text{ becomes } rdzdrd\theta$$

$$\underbrace{\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}}_{\text{top half circle}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dzdydx$$

$$= \underbrace{\int_{0}^{\pi}\int_{0}^{3}}_{\text{top half circle}}\int_{0}^{9-r^2}\sqrt{r^2}\,rdzdrd\theta$$

$$= \int_{0}^{\pi}\int_{0}^{3}\underbrace{\int_{0}^{9-r^2}}\, r^2dzdrd\theta$$

$$= \int_{0}^{\pi}\int_{0}^{3} zr^2\Big|_{0}^{9-r^2} drd\theta$$

$$= \int_{0}^{\pi}\int_{0}^{3} 9r^2 - r^4 drd\theta$$

$$= \int_{0}^{\pi} 3r^3 - \frac{1}{5}r^5\Big|_{0}^{3} d\theta$$

$$= \int_{0}^{\pi}\left(81 - \frac{243}{5}\right)d\theta$$

$$= \int_{0}^{\pi}\frac{162}{5}d\theta$$

$$= \frac{162\pi}{5}$$

■

Spherical coordinates is not an expectation on the final exam

$$\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sin\theta} rdrd\theta$$

December 3, 2019

We did some questions on kahoot