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Differential Equations ①

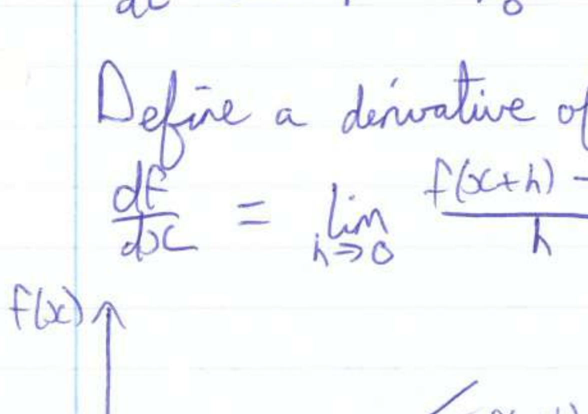
Newton's Law of cooling:
The rate of change of the temperature of a body is proportional to the difference in temperature between the body and its surroundings.

Temp of body $T(t)$ dependent variable
Time t independant variable
Temp of surroundings T_0 constant

$$\frac{dT}{dt} \propto T - T_0 \qquad \frac{dT}{dt} = -k(T - T_0), \quad k > 0$$

Define a derivative of $f(x)$ wrt x

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ ⇒ slope of the line at a single poi



Right and left hand derivatives must be equal for f to be differentiable

$$\lim_{h \to 0-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

E.g. $f(x) = |x|$ not differentiable at $x = 0$

$$\frac{df}{dx} \equiv f'(x) \equiv \frac{d}{dx} f(x)$$

$$\frac{d}{dx}\left(\frac{df}{dx}\right) \equiv \frac{d^2f}{dx^2} \equiv f''(x) \equiv f^{(2)}(x)$$

Note !!! $f'(2x)$ means $\frac{df}{dy}$, $y = 2x$

$$= \frac{df(2x)}{dx} \cdot \frac{dx}{dy} = \frac{1}{2} \frac{df(2x)}{dx}$$

$f(x) = o[g(x)]$ as $x \to x_0$

if $\lim_{x \to x_0} \frac{f}{g} = 0$

e.g. $x = o(\sqrt{x})$ as $x \to 0$
$\sqrt{x} = o(x)$ as $x \to \infty$

$f(x) = \underset{\text{"is of order"}}{O}[g(x)]$ as $x \to x_0$

if $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$

$\sin 2x = O(x)$ as $x \to 0$
$x = O(\sqrt{x})$ as $x \to 0$

Note $f(x) = o[g(x)] \Rightarrow f(x) = O[g(x)]$ but not vice versa

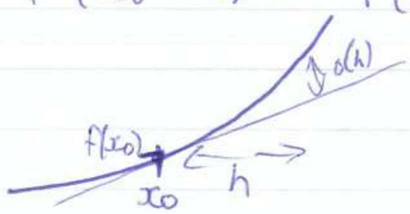
Tangent line at x_0

evaluated at

$$\left.\frac{df}{dx}\right|_{x_0} = \frac{f(x_0+h) - f(x_0)}{h} + \overbrace{\frac{o(h)}{h}}^{\text{error}}$$

no need for signs or constants for little o

$$f(x_0+h) = f(x_0) + h\left.\frac{df}{dx}\right|_{x_0} + o(h)$$



Equation of tangent line at x_0 of $y = f(x)$
Replace x by $x_0 + h$
$y(x) = y_0 + m(x - x_0)$, $m = \left.\frac{df}{dx}\right|_{x_0} = \frac{df}{dx}(x_0)$

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Differential Equations ②

Recap: i) $f(x) = o[g(x)]$ as $x \Rightarrow x_0$
$\Leftrightarrow \lim \frac{f(x)}{g(x)} = 0$

ii) $f(x) = O[g(x)]$ as $x \Rightarrow x_0$
$\Leftrightarrow \frac{f(x)}{g(x)}$ remains bounded

Note, all o are also O as o means tends to zero is a bound therefore also O

iii) $f(x_0 + h) = f(x_0) + h\frac{df}{dx}\Big|_{x_0} + o(h)$

Chain rule

Consider $f(x) = F[g(x)]$ e.g. $f(x) = \sin(x^2 - x + 2)$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{F[g(x+h)] - F[g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{F[g(x) + \underbrace{h\frac{dg}{dx} + o(h)}_{\text{treat as } h}] - F[g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left\{F[g(x)] + \left[h \cdot \frac{dg}{dx} + o(h)\right]F'[g(x)] + o(h) - F[g(x)]\right\}$$

$$= \lim_{h \to 0}\left(\frac{dg}{dx} \times F'[g(x)] + \frac{o(h)}{h}\right) = \frac{dg}{dx} \times F'[g(x)]$$

→ relies on finite $\frac{dg}{dx}$ at the point in question an also $\frac{dF}{dg}$ being finite. In other words, both the inner and outer functions must be differentiable.

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

e.g. $\frac{d}{dx}\left\{\sin(x^2 - x + 2)\right\} = \cos(x^2 - x + 2) \times (2x - 1)$

Product Rule

$f(x) = u(x)\,v(x)$, $\frac{df}{dx} = u'v + uv'$

Leibnitz's rule   Coefficients are from Pascal's triangle → binomial

$$\begin{aligned}
f &= uv \\
f' &= u'v + uv' \\
f'' &= u''v + u'v' + u'v' + uv'' \\
f'' & u''v + 2u'v' + uv'' \\
f''' & u'''v + 3u''v' + 3u'v'' + uv'''
\end{aligned}$$

$$f^{(n)}(x) = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v'' + \dots$$

$$\dots + {}^nC_r\, u^{(n-r)}v^{(r)} + \dots + uv^{(n)}$$

Where $^nC_r = \frac{n!}{(n-r)!\,r!}$

Taylor Series

Recall $f(x+h) = f(x) + hf'(x) + o(h)$

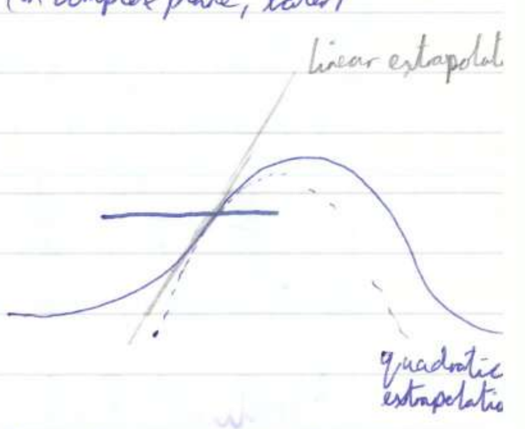
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots$$
$$\dots + \frac{h^n}{n!}f^{(n)}(x) + E_n$$

NB. f must be $n+1$ times differentiable → (in complex plane, later)

then Taylor's Theorem states that

$$E_n = O(h^{n+1}) \text{ as } h \to 0$$

$(\text{so } E_n = o(h^n))$



A Taylor series provides a local approximation to a function.

Contrast with a global approximation e.g. Fourier Series

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Differential Equations (2)

Alternative form

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \dots$$
$$\dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + E_n$$

Taylor series of $f(x)$ about the point $x = x_0$. A local approximat of the function near x_0.

Finding Coefficients

WLOG, consider an expansion of $f(x)$ about $x = 0$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$f'(x) = a_1 + 2a_2 x + \dots$$
$$f''(x) = 2a_2 + 3\times 2 a_3 x + \dots$$

$$f(0) = a_0 \qquad f'''(0) = 3\times 2 a_3$$
$$f'(0) = a_1 \qquad f^{(n)}(0) = n!\, a_n$$
$$f''(0) = 2a_2$$
$$a_n = \frac{f^{(n)}(0)}{n!} \qquad QED$$

L'Hopital's Rule

Suppose $f(x)$ and $g(x)$ are differentiable at $x = x_0$ and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$

The limit $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ provided $g'(x_0) \neq 0$

Proof

From Taylor Series (Linear part)

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + o(x - x_0)$$
$$g(x) = g(x_0) + (x - x_0) g'(x_0) + o(x - x_0)$$

$$\lim_{x\to x_0} \frac{f}{g} = \lim_{x\to x_0} \frac{f' + \frac{o(x-x_0)}{x-x_0}}{g' + \frac{o(x-x_0)}{x-x_0}} = \frac{f'(x_0)}{g'(x_0)}$$

Proof of L' Hopital's Rule

f, and g are continuous, differentiable at x_0, $g'(x) \neq 0$

$\lim_{x \to x_0} f(x_0) = \lim_{x \to x_0} g(x_0) = 0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x \to x_0} \frac{\frac{f(x) - f(x_0)}{x - x_0}}{\frac{g(x) - g(x_0)}{x - x_0}}$$

$$= \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

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Differential Equations ③

RECAP

Chain Rule

$$\frac{d}{dx} f[g(x)] = f'(g)\,g'(x)$$

Product Rule

$$\frac{d}{dx}(uv) = uv' + vu'$$

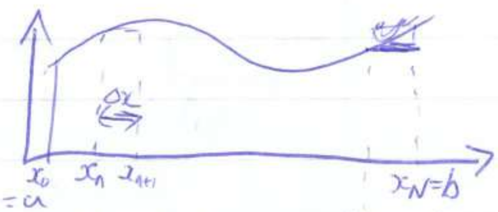
Taylor Series

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \dots + (x-x_0)^n\frac{f^{(n)}(x_0)}{n!} + O\left[(x-x_0)^{n+1}\right]$$

L'Hopital $$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$ If $\lim f = \lim g = 0$ or ∞, $g' \neq 0$ and ratio limits exist

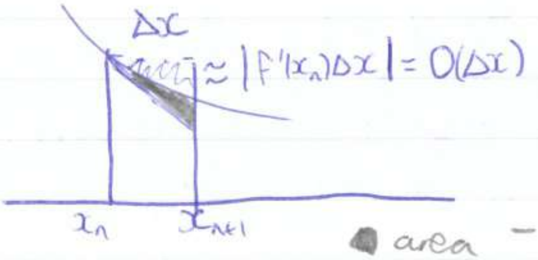
__Integration__ An integral is a __sum__.

$$\int_a^b f(x)\,dx = \lim_{\Delta x\to 0}\sum_{n=0}^{N-1} f(x_n)\Delta x$$



Area under the graph from x_n to x_{n+1} $\quad \Delta A_n = f(x_n)\Delta x + O(\Delta x^2)$

provided f is differentiable at x_n



Error in area $= O(\Delta x^2)$

area - error, $O(x^3)$ using Taylor Series

Area for a to b $\displaystyle\lim_{N\to\infty}\left[\sum_{n=0}^{N-1} f(x_n)\Delta x + O(N\Delta x^2)\right]$

Note $\Delta x = \frac{b-a}{N}$, $N = \frac{b-a}{\Delta x}$ $\qquad O(N\Delta x^2) = O(\Delta x)$

$$\int_a^b f(x)\,dx$$

Fundamental Theorem of Calculus

t is a 'dummy variable'
'internal variable' to the sum

$$F(x) = \int_a^x f(t)\,dt$$

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{1}{h} \left\{ \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right\}$$

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} \frac{1}{h}\left[f(x)h + O(h^2) \right]$$

$$= f(x)$$

Notation $F(x) = \underbrace{\int f(x)\,dx}_{\text{WRONG NOTATION}} \quad \int^x f(t)\,dt$

Similarly $\frac{d}{dx} \int_x^b f(t)\,dt = -f(x)$ SHOW YOURSELF

$$\frac{d}{dx} \int_a^{g(x)} f(t)\,dt = f[g(x)]\,g'(x)$$

Integration by substitution

Integration is an art of recognition. If the integrand contains a function of a function it can sometimes aid recognition to substitute for the inner function. Especially helpful if we can recognise the structure of the chain rule.

$$\int \frac{1-2x}{\sqrt{x-x^2}}\,dx \qquad u = x - x^2 \qquad du = (1-2x)\,dx$$

$$= \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{x-x^2} + C$$

~~Trigonometric Substitutions~~

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Differential Equations ③

Trigonometric Substitutions

Useful Identities	Part of integrand	Substitution
$\cos^2\theta + \sin^2\theta \equiv 1$	$\sqrt{1-x^2}$	$x = \sin\theta$, $dx = \cos\theta \, d\theta$
$1 + \tan^2\theta \equiv \sec^2\theta$	$1 + x^2$	$x = \tan\theta$, $dx = \sec^2\theta \, d\theta$
$\cosh^2 u - \sinh^2 u \equiv 1$	$\sqrt{1+x^2}$	$x = \sinh u$, $dx = \cosh u \, d$
	$\sqrt{x^2-1}$	$x = \cosh u$, $dx = \sinh u \, d$
$1 - \tanh^2 u \equiv \operatorname{sech}^2 u$	$1 - x^2$	$x = \tanh u$, $dx = \operatorname{sech}^2 u \, du$

$$\int \sqrt{2x - x^2} \, dx = \int \sqrt{1 - 1 + 2x - x^2} \, dx = \int \sqrt{1 - (x^2 - 2x + 1)} \, dx$$

$$= \int \sqrt{1 - (x-1)^2} \, dx = \int \sqrt{1 - \sin^2\theta} \cos\theta \, d\theta = \int \cos^2\theta \, d\theta$$

$x - 1 = \sin\theta$, $x = 1 + \sin\theta$
$dx = \cos\theta \, d\theta$

$$= \int \tfrac{1}{2}\cos 2\theta + \tfrac{1}{2} \, d\theta = \tfrac{1}{4}\sin 2\theta + \theta + c$$

$$= \tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta + C$$

$$= \tfrac{1}{2}(x-1)\sqrt{2x - x^2} + \tfrac{1}{2}\arcsin(x-1) + C$$

By Parts

Product rule $(uv)' = u'v + uv'$

$$\int uv' \, dx = \int (uv)' - u'v \, dx = uv - \int u'v \, dx$$

e.g. $\int_0^\infty x e^{-x} \, dx$ $\quad u = x$, $u' = 1$ $\quad v' = e^{-x}$, $v = -e^{-x}$

$$= \left[x e^{-x}\right]_0^\infty - \int_0^\infty -e^{-x} \, dx = \left[-x e^{-x}\right]_0^\infty - \left[e^{-x}\right]_0^\infty$$
$$= 1$$

$\int \ln x \, dx$ $\quad u = \ln x$, $u' = \frac{1}{x}$ $\quad v' = 1$, $v = x$ $\quad\quad \int \arcsin x \, dx$

$$= x\ln x - \int \frac{1}{x} \times x \, dx$$
$$= x\ln x - \int 1 \, dx = x\ln x - x + C$$

inverse
also for inverse trig + hyp

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Differential Equations ④

Functions of several variables

Partial differentiation:

Consider a function $f(x, y)$

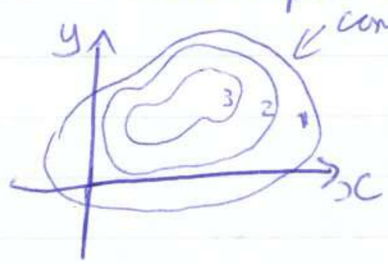
e.g. - height of terrain / a hill
- pressure (temperature) at sea level

} function of east/west / north, south coordinates

- density of a gas $= f(\text{temp}, \text{pressure})$

Represent such functions either on a graph

or as a contour plot



contours, curves along which f = constant

Q: What is the slope of a hill?
A: Depends which direction you are facing.

Begin by finding the slopes in directions parallel to the axes.
The partial derivative of $f(x, y)$ wrt x
= the rate of change of f wrt x, keeping y constant

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Similarly $$\left.\frac{\partial f}{\partial y}\right|_x = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

Calculating partial derivatives:

$$f(x, y) = x^2 + y^3 + e^{xy^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x + y^2 e^{xy^2} \qquad \left.\frac{\partial f}{\partial y}\right|_x = 3y^2 + 2xye^{xy^2}$$

Can also find 2nd partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = 2 + y^4 e^{xy^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 6y + 2x e^{xy^2} + 4x^2 y^2 e^{xy^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= 2y e^{xy^2} + 2xy^3 e^{xy^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2y e^{xy^2} + 2xy^3 e^{xy^2}$$

It is a general rule (in Euclidean space) that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

NB To be careful, we indicate which variable, or variables are being held constant, but if no indication, we assume everything is constant except the variable we are differentiating with respect to.

e.g. $f = f(x, y, z)$

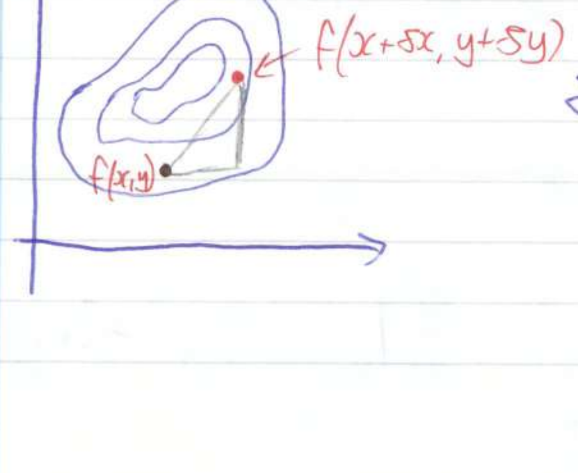
$$\frac{\partial f}{\partial x} \equiv \left.\frac{\partial f}{\partial x}\right|_{y,z} \neq \left.\frac{\partial f}{\partial x}\right|_{y} \text{ in which } z \text{ may vary}$$

Alternative notation $f_x \equiv \frac{\partial f}{\partial x}$ $\quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

<u>Chain rule</u>

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y)$$

$$= f(x+\delta x, y+\delta y) - f(x+\delta x, y) + f(x+\delta x, y) - f(x,y)$$



$$\delta f = \frac{\partial f}{\partial y}(x+\delta x, y) \cdot \delta y + o(\delta y) + \frac{\partial f}{\partial x}(x,y) \cdot \delta x + o(\delta x)$$

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Differential Equations ④

$$= \left[\frac{\partial f}{\partial y}(x,y) + o(\delta x)\right]\delta y + o(\delta y)$$
$$+ \frac{\partial f}{\partial x}(x,y)\,\delta x + o(\delta x)$$

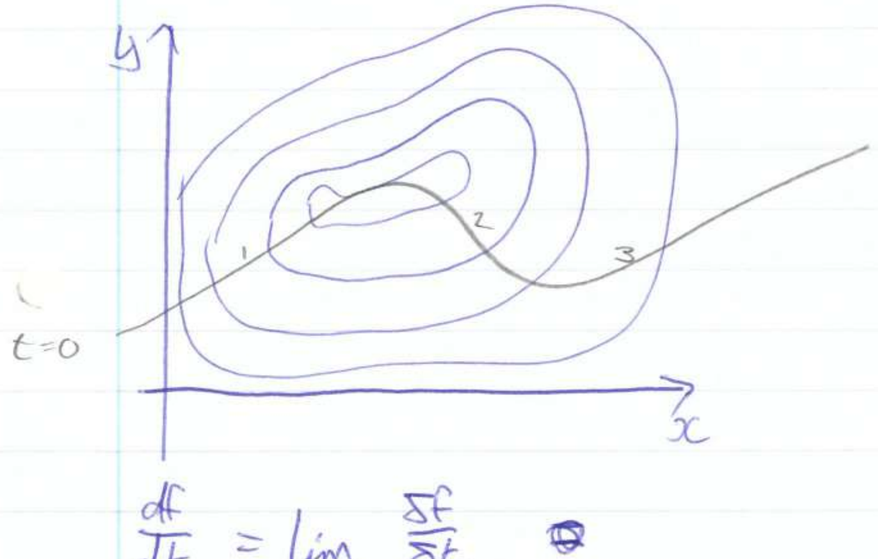
$$(*) \quad \delta f = \frac{\partial f}{\partial x}(x,y)\,\delta x + \frac{\partial f}{\partial y}(x,y)\,\delta y + o(\delta x, \delta y)$$

Take limit as $\delta x \to 0$, $\delta y \to 0$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

This is the chain rule in differential form. We understand it as a shorthand for $(*)$ knowing that we shall either sum terms or divide by another infintesimal quantity before taking the limit.

E.g. $$\int \Box\, df = \int \Box \frac{\partial f}{\partial x}dx + \int \Box \frac{\partial f}{\partial y}dy$$



Along a path, $(x,y) = [x(t), y$ where t is a parameter along the path (e.g. time)

$$f(x,y) = f[x(t), y(t)]$$

$$\frac{df}{dt} = \lim_{\delta t \to 0} \frac{\delta f}{\delta t}$$
$$= \lim_{\delta t \to 0}\left[\frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{o(\delta x, \delta y)}{\delta t}\right]$$
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} \qquad \text{Chain rule}$$

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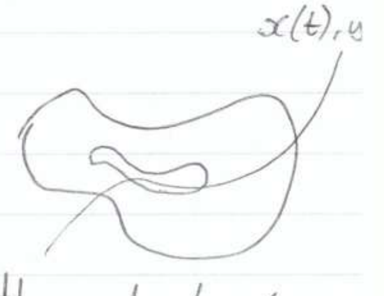
Differential Equations (5)

Recap

$$\frac{\partial f}{\partial y}(x+\delta x, y)\,\delta y = \left[\frac{\partial f}{\partial y}(x,y) + \frac{\partial^2 f}{\partial x \partial y} + o(\delta x)\right]\delta y$$



Chain Rule $\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

Short for

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + o(\delta x, \delta y)$$

Along a path:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

Path can also be given $y = y(x)$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$ – by division + taking limit $\quad f = f[x, y(x)]$

Change of variables: e.g. $x = (r,\theta)$, $y = (r,\theta)$

$$f = f[x(r,\theta), y(r,\theta)]$$

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \cdot \left.\frac{\partial y}{\partial r}\right|_\theta$$

Similarly $\left.\frac{\partial f}{\partial \theta}\right|_r = \left.\frac{\partial f}{\partial x}\right|_y \cdot \left.\frac{\partial x}{\partial \theta}\right|_r + \left.\frac{\partial f}{\partial y}\right|_x \cdot \left.\frac{\partial y}{\partial \theta}\right|_r$

in 3d, will be a contour which is a surface

<u>Implicit Differentiation</u>: $\quad F(x, y, z) = \text{constant}$

It implicitly defines $z = z(x,y)$
or $x = x(y,z)$
or $y = y(x,z)$

e.g. $\quad xy^2 + yz^2 + z^5 x = 5 \quad *$

Solve for x $\quad x = \frac{5 - yz^2}{y^2 + z^5}$ $\quad$ explicitly

Could also find $y = (x,z)$ by solving the quadratic $\Rightarrow$ function with two branches
but we cannot find $z = z(x,y)$, would have to solve a quintic

Find $\left.\frac{\partial z}{\partial x}\right|_y$ by differentiating wrt x holding y constant.

$$y^2 + 2yz\left.\frac{\partial z}{\partial x}\right|_y + 5z^4 x\cdot\left.\frac{\partial z}{\partial x}\right|_y + z^5 = 0$$

$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{y^2 + z^5}{2yz + 5xz^4}$$

In general, think of $F(x, y, z(x,y)) = \text{constant}$

Chain rule in differential form:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

$$\frac{\partial F}{\partial x}\Big|_y = \frac{\partial F}{\partial x}\underbrace{\cancel{\frac{\partial x}{\partial x}}\Big|_y}_{1} + \frac{\partial F}{\partial y}\underbrace{\cancel{\frac{\partial y}{\partial x}}\Big|_y}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\Big|_y = 0$$

$$\frac{\partial F}{\partial x}\Big|_y = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\Big|_y = -\frac{\frac{\partial F}{\partial x}\big|_{y,z}}{\frac{\partial F}{\partial z}\big|_{x,y}}$$

Similarly:

$$\frac{\partial x}{\partial y}\Big|_z = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}$$

$$\frac{\partial y}{\partial z}\Big|_x = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}}$$

$$\frac{\partial x}{\partial y}\Big|_z \frac{\partial y}{\partial z}\Big|_x \frac{\partial z}{\partial x}\Big|_y = -1$$

Note: Normal rules apply provided the same variables are being held constant.

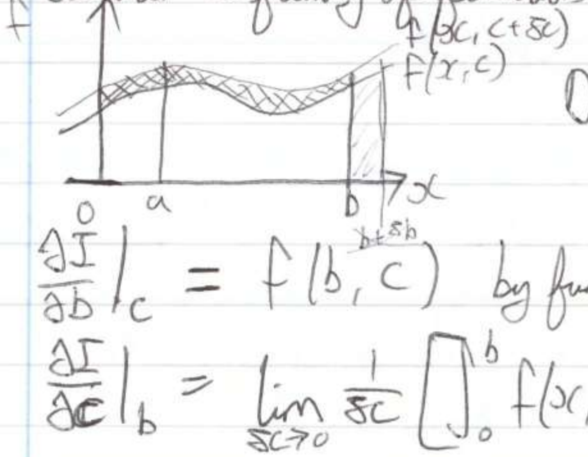
$$(x, y) \to (r, \theta)$$

$$\frac{\partial r}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial r}} \quad \text{because} \quad \frac{\partial r}{\partial x}\Big|_y \neq \frac{1}{\frac{\partial x}{\partial r}\big|_\theta}$$

But $$\frac{\partial r}{\partial x}\Big|_y = \frac{1}{\frac{\partial x}{\partial r}\big|_y} \quad ✓$$

<u>Differentiation of an integral:</u> with respect to a parameter

Consider a family of functions $f(x, c)$



Define a function $I(b,c) = \int_0^b f(x,c)\,dx$

$$\frac{\partial I}{\partial b}\Big|_c = f(b, c) \quad \text{by fundamental theorem of calculus}$$

$$\frac{\partial I}{\partial c}\Big|_b = \lim_{\delta c \to 0} \frac{1}{\delta c}\left[\int_0^b f(x, c+\delta c)\,dx - \int_0^b f(x,c)\,dx\right]$$

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Differential Equations ⑤

$$= \lim_{\delta c \to 0} \int_0^b \frac{f(x, c+\delta c) - f(x,c)}{\delta c} \, dx$$

$$= \int_0^b \frac{\partial f}{\partial c}\bigg|_x \, dx$$

Consider :

$$I[b(x), c(x)] = \int_0^{b(x)} f[y, c(x)] \, dy$$

$$I(x) = \int_0^x e^{-x^2 y} \, dy$$

$$\frac{dI}{dx} = \underbrace{f(b,c)\frac{db}{dx}}_{\text{varying the limit}} + \underbrace{\frac{dc}{dx}\int_0^{b(x)} \frac{\partial f}{\partial c}\bigg|_y \, dy}_{\text{varying function}}$$

$$\frac{dI}{dx} = e^{-x^3} + \int_0^x -2xy\, e^{-x^2 y} \, dy$$

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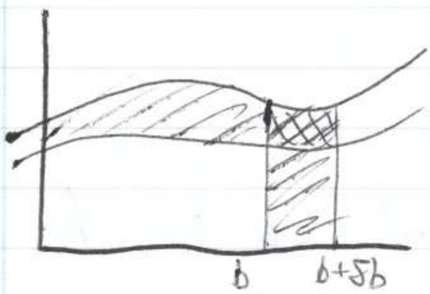
Differential Equations (6)

$$I[b(x), c(x)] = \int_0^{b(x)} f[y, c(x)]\, dy$$

$$\frac{dI}{dx} = \frac{\partial I}{\partial b}\frac{db}{dx} + \frac{\partial I}{\partial c}\frac{dc}{dx}$$

$$= f(b, c)\frac{db}{dx} + \frac{dc}{dx}\int_0^b \frac{\partial f}{\partial c}\Big|_y\, dy$$



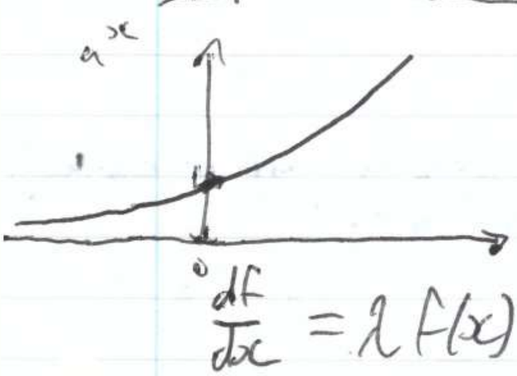
i) $I = \int_0^{\lambda} e^{-x^2} dx$, $\frac{dI}{d\lambda} = e^{-\lambda^2}$

ii) $I = \int_0^1 e^{-\lambda x^2} dx$, $\frac{dI}{d\lambda} = \int_0^1 -x^2 e^{-\lambda x^2} dx$

iii) $I = \int_0^{\lambda} e^{-\lambda x^2} dx$

$$\frac{dI}{d\lambda} = e^{-\lambda^3} + \int_0^{\lambda} -x^2 e^{-\lambda x^2} dx$$

<u>Exponential Function</u> $f(x) = a^x$, $a > 0$, a is constant



$$\frac{df}{dx} = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} a^x\left[\frac{a^h - 1}{h}\right]$$

$$= a^x \lim_{h\to 0}\frac{a^h - 1}{h} = \lambda a^x$$, assuming limit exist

$\frac{df}{dx} = \lambda f(x)$ where $\lambda = \lim_{h\to 0}\frac{a^h - 1}{h}$ = constant = $f'(0)$

__Define__ $f(x) = \exp(x) \equiv e^x$ by $\frac{df}{dx} = f(x)$ with $f(0) = 1$.

Proof that $e = \lim_{k \to \infty} (1 + \frac{1}{k})^k$ is on example sheets.

$y = a^x = e^{x \ln a}$, then $\frac{dy}{dx} = \ln a \times e^{x \ln a} = \ln a \; a^x$

$\lambda = \ln a$

First order, linear differential equations.

$$\frac{d}{dx}(e^{\lambda x}) = \lambda (e^{\lambda x})$$

$e^{\lambda x}$ is an eigenfunction of the differential operator $\frac{d}{dx}$

The functional form is unchanged by the operator, only the magnitude is changed.

Any linear homogeneous ordinary differential equation with constant coefficients has solutions of the form $e^{\lambda x}$.

e.g. $\quad 5y' - 3y = 0 \quad *$

__Linear__ the dependent variable appears only linearly

$x^2 y' + y \sin x = e^x$ is linear,

$y y' + x y = 5$ non linear

__Homogeneous__ $y = 0$ is a solution.

__Constant Coefficients__ independent variable does not appear explicitly

__First order__ No higher derivatives than 1st are involved

$y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$

In example $*$ $5\lambda e^{\lambda x} - 3e^{\lambda x} = 0$, $\lambda = \frac{3}{5}$

as $e^{\lambda x} \neq 0$ so $y = e^{\frac{3}{5}x}$ is a solution

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Differential Equations 6

i) Because the equation is linear and homogeneous, any multiple of a solution is also a solution.

$\Rightarrow$ $y = Ae^{\frac{3}{5}x}$ is also a solution for any constant a.

ii) An n^{th} order linear differential equations has (only) n independent so

Therefore $y = Ae^{\frac{3}{5}x}$ is the most general solution to (*)

Can determine A by applying a boundary condition, i.e. y at x = (

<u>Discrete Equations</u> $5y' - 3y = 0$, $y = y_0$ when $x = ($

Approximate by $\dfrac{5y_{n+1} - y_n}{h} - 3y_n = 0$ with $y(0) = y_0$
with $x_n = nh$

Simple Euler Method

$y_{n+1} = (1 + \frac{3}{5}h)\, y_n$ (compound interest formula)

repeatedly $y_n = (1 + \frac{3}{5}h)^n y_0$

$= (1 + \frac{3}{5}\frac{x}{n})^n y_0$

Take limit as $n \to \infty$ $\quad y(x) = \lim_{n \to \infty} y_0 (1 + \frac{3x}{5n})^n$

$= y_0 e^{\frac{3x}{5}}$

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Differential Equations (7)

Series solution

Try a solution of the form: $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} a_n x^{n-1}$ $\qquad 5y' - 3y = 0$

$5(xy') - 3xy = 0 \Rightarrow \sum a_n [5n - 3x] x^n = 0$

Coefficient of x^n: $5n a_n - 3a_{n-1} = 0$

$n = 0$: $\quad 0 \cdot a_0 = 0 \Rightarrow a_0$ is arbitrary

$n > 0$ $\quad a_n = \frac{3}{5n} a_{n-1} = \cancel{\left(\frac{3}{5n}\right)^2 a_{n-2}} \left(\frac{3}{5n}\right)\left(\frac{3}{5(n-1)}\right) a_{n-2}$

$= \left(\frac{3}{5}\right)^n \frac{1}{n!} a_0$

$$y = a_0 \sum_{n=0}^{\infty} \frac{\left(\frac{3x}{5}\right)^n}{n!} = a_0 e^{\frac{3x}{5}}$$

Forced equations - Inhomogeneous

i) Constant forcing $\quad 5y' - 3y = 10$

Can spot a steady (equilibrium) solution $\quad y = y_p = -\frac{10}{3}, \quad y_p' = 0$

y_p 'particular' 'steady' solution

Write $y = y_p + y_c \quad \Rightarrow 5y_c' - 3y_c = 0$

$y = -\frac{10}{3} + Ae^{\frac{3x}{5}}$

ii) Eigenfunction forcing In a radioactive rock, isotope A decays into isotope B at a rate proportional to the number, a, of remaining nuclei of A, and B decays to C, at a rate proportional to the number b, of remaining nuclei B.

y_c 'deviation from equilibrium' complementary function

$\frac{da}{dt} = -k_a a \qquad \frac{db}{dt} = k_a a - k_b b$

$\Rightarrow \frac{da}{dt} + k_a a = 0 \qquad \frac{db}{dt} = k_a a_0 e^{-k_a t} - k_b b$

$a = a_0 e^{-k_a t} \qquad \frac{db}{dt} + k_b b = k_a a_0 e^{-k_a t}$

$\left(\frac{d}{dt} + k_b\right) b = \ldots$

Note: forcing is an eigenfunction of the differential operator on the LHS so try a particular integral.

$b_p = Ce^{-k_a t} \Rightarrow -k_a C + k_b C = k_a a_0 \Rightarrow C = \frac{k_a a_0}{k_b - k_a}$

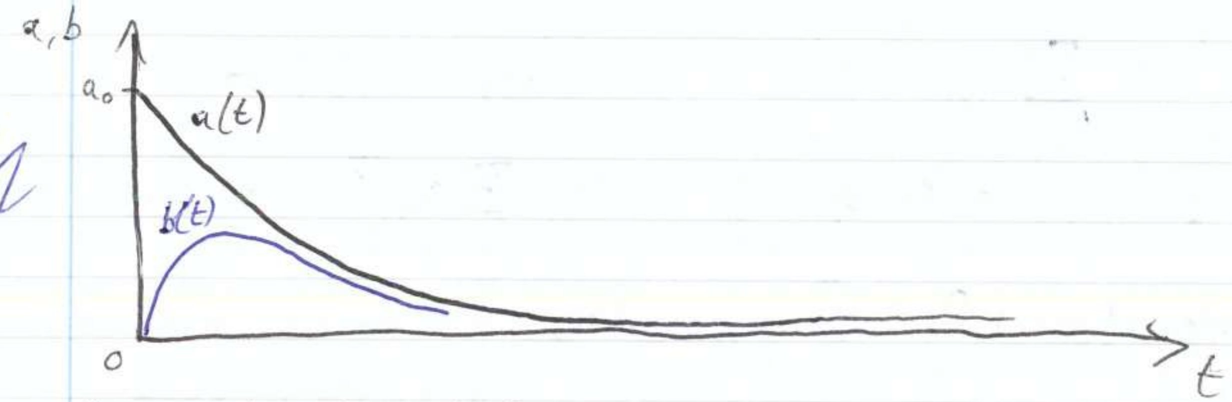
provided $k_a \neq k_b$

Write $b = b_p + b_c$

$$b_c' + k_b b_c = 0$$

$$b_c = De^{-k_b t}$$

$$b = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + De^{-k_b t}$$



Suppose $b = 0$ at $t = 0$

$$b = \frac{k_a}{k_b - k_a} a_0 \left(e^{-k_a t} - e^{-k_b t} \right)$$

$$\frac{b}{a} = \frac{k_a}{k_b - k_a} \left[1 - e^{(k_a - k_b)t} \right].$$

This allows a rock to be dated from the relative proportions of certain isotopes.

<u>Non-constant coefficients</u>

General form: $a(x)\, y' + b(x)\, y = c(x)$

Divide by $a(x)$ to get standard form: $y' + p(x) y = f(x)$

Integrating Factor

Multiply by $\mu(x)$ so $\mu y' + \mu p y = \mu f$

LHS is given to the product rule applied to $(\mu y)'$ if

$\mu p = \mu'$, $p = \frac{\mu'}{\mu}$, $\int p\, dx = \int \frac{1}{\mu} \frac{d\mu}{dx}\, dx$

$\ln \mu = \int p\, dx$ $\cancel{(+c)}$ $\mu = A e^{\int p\, dx}$

$A(\mu y)' = A \mu f$ $\mu y = \int \mu f\, dx$ etc

e.g. $xy' + (1-x)y = 1$, $y' + (\frac{1}{x} - 1)y = \frac{1}{x}$

IF $\mu = \exp\left(\int (\frac{1}{x} - 1)\, dx\right) = e^{\ln x - x} = xe^{-x}$

$(xe^{-x} y)' = e^{-x}$

$xe^{-x} y = -e^{-x} + c$ $y = \dfrac{c - e^{-x}}{xe^{-x}} = -\frac{1}{x} + \frac{c}{x} e^{x}$

y is finite, $x = 0$
$\Rightarrow c = 1$
$y = -\frac{1}{x} + \frac{1}{x} e^{x}$

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Differential Equations (8)

Nonlinear first order

In general, a first order ordinary differential equation has the form
$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0$$

Separable Equations

The equation is separable if it can be manipulated into the form
$$q(y)\,dy = p(x)\,dx$$
in which case, solution can be found by integration
$$\int q(y)\,dy = \int p(x)\,dx$$

E.g. $(x^2y - 3y)\frac{dy}{dx} - 2xy^2 = 4x$
$$\frac{dy}{dx} = \frac{4x + 2xy^2}{x^2y - 3y} = \frac{2x(2+y^2)}{y(x^2-3)}$$
$$\Rightarrow \int \frac{y}{2+y^2}\,dy = \int \frac{2x}{x^2-3}\,dx \qquad \frac{1}{2}\ln(2+y^2) = \ln(x^2-3) + c$$
$$(2+y^2)^{\frac{1}{2}} = A(x^2-3)$$

Exact Equations

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0$$
is an exact equation if an only if
$Q(x,y)\,dy + P(x,y)\,dx$ is an exact differential of a funct
$f(x,y)$. i.e.

Trivial soluti

$$\exists\, f(x,y),\ df \neq P\,dx + Q\,dy$$
in which case, $df = 0$ from the differential equation, so f = constan

Suppose there exists such a function $f(x,y)$

Chain Rule
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \quad \frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q$$
Note
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

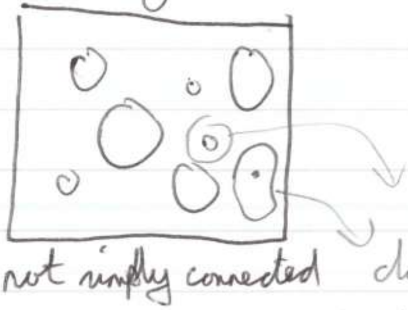
True (proof not given) that if $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ throughout a simply connected domain D the $P\,dx + Q\,dy$ is an exact differential of a single valued function $f(x, y)$ in D.

Note

i) The reverse implication follows locally from the chain rule.

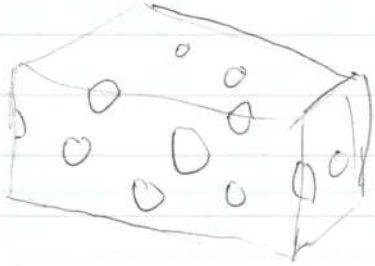
ii) What is a simply connected domain? "A domain with no holes"

a) Slice of swiss cheese   b) A whole swiss cheese



not simply connected   closed path

can be shrunk to a point continuously without leaving the domain



is simply connected

Every closed path in the block can be shrunk to a point (providing no holes all the way through)

Example

$$6y(y-x)\frac{dy}{dx} + (2x - 3y^2) = 0$$

$$\Rightarrow (2x - 3y^2)\,dx + 6y(y-x)\,dy = 0$$

$$P = 2x - 3y^2, \quad Q = 6y(y-x)$$

$$\frac{\partial P}{\partial y} = -6y \quad \frac{\partial Q}{\partial x} = -6y$$

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x - 3y^2 \qquad \frac{\partial f}{\partial y} = 6y(y-x) = -6xy + 6y^2$$

$$\Rightarrow f = x^2 - 3xy^2 + g(y) \qquad \Rightarrow \frac{\partial f}{\partial y} = -6xy + g'(y) \qquad g'(y) = 6y^2$$

$$g = 2y^3 + C$$

$$f = x^2 - 3xy^2 + 2y^3 + C$$

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Differential equations (8)

Solution of the equation is f = constant

$x^2 - 3xy^2 + 2y^3 + C$ = constant

E.g. $\frac{dy}{dt} = t(1-y^2)$

$\int \frac{1}{1-y^2}\,dy = \int t\,dt$

$\operatorname{artanh} y = \frac{1}{2}t^2 + C$

$y = \tanh\left(\frac{1}{2}t^2 + C\right)$

$\frac{1}{2}\ln\left(\frac{1+y}{1-x}\right) = \frac{1}{2}t^2 + C$

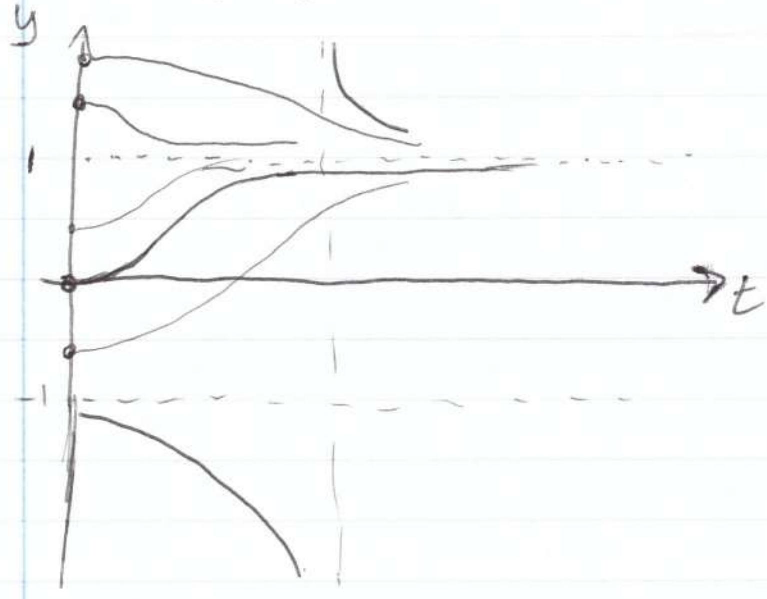
$\frac{1+y}{1-y} = Ae^{t^2}$ $\quad A = e^{2c}$

$y = \frac{A - e^{-t^2}}{A + e^{-t^2}}$

If we have an initial condition, we can determine A.

e.g if $y(0) = 0$, $A = 1$.

A = 1



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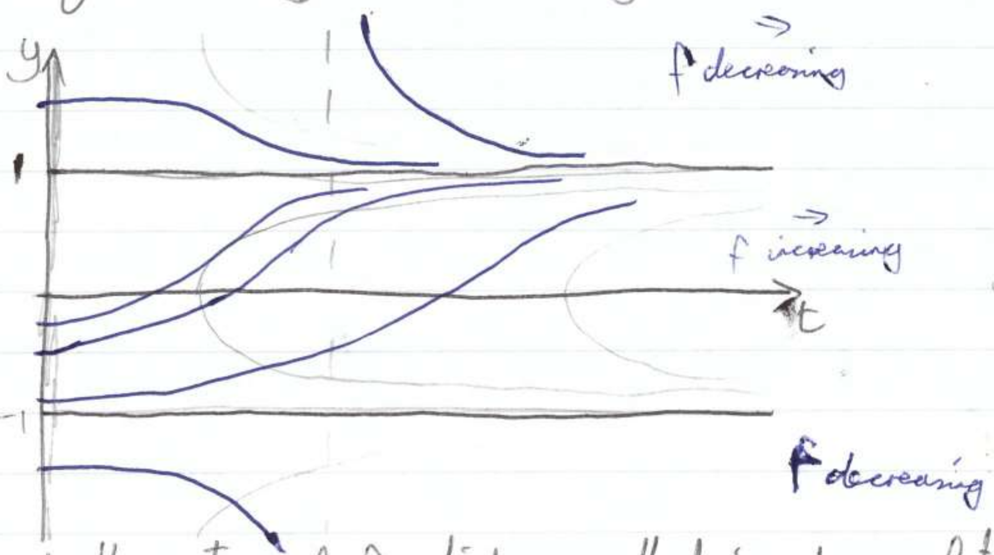
Differential Equations (9)

$y' = t(1-y^2)$, Generally $\frac{dy}{dt} = f(y,t)$

Note first that $y' = 0$ where $y = \pm 1$ or $t = 0$. Note also that $\frac{dy}{dt} > 0$ for $-1 < y < 1$, $t > 0$ and negative for $y < -1, y > 1, t > 0$



Consider the contours of f which are called isoclines, of the differential equation

$$t(1-y^2) = c \quad , \quad t = \frac{c}{1-y^2} = \frac{c}{(1+y)(1-y)}$$

Note that as $|y| \to \infty$, $\frac{dy}{dt} \approx -ty^2$, $-\frac{dy}{y^2} = t\,dt$, $\frac{1}{y} = \frac{1}{2}t^2 - D$

Note if $f(y,t)$ is single valued curves do not cross. $y = \frac{1}{\frac{1}{2}t^2 - D}$

$y = 1$ is a stable attractor, $y = -1$ is an unstable attractor.

Equilibria and stability

Fixed points (equilibrium points) are where $\frac{dy}{dt} = 0$ for all t $\Rightarrow f(y,t) = 0$ for all t. In our example these are $y = \pm 1$.
We can see from the solution curves that as time increases, solutions converge towards $y = +1$, a stable fixed point but diverge from $y = -1$, an unstable fixed point.

W

Perturbation analysis – to determine stability and nature of solutions close to fixed point

$\frac{dy}{dt} = f(y,t)$. y is a fixed point, i.e. $f(a,t) = 0$

Write $y = a + \varepsilon(t)$ ← perturbation

Substitute: $\frac{d\varepsilon}{dt} = f(a+\varepsilon, t)$

$$\frac{d\varepsilon}{dt} = f(a,t) + \varepsilon \frac{\partial f}{\partial y}(a,t) + O(\varepsilon^2)$$

$$\Rightarrow \frac{d\varepsilon}{dt} \approx \left[\frac{\partial f}{\partial y}\right]_\varepsilon \quad \text{linear equation}$$

In example $f = t(1-y^2)$, $\frac{\partial f}{\partial y} = -2yt = \begin{cases} -2t, & y=1 \\ 2t, & y=-1 \end{cases}$

Near $y=+1$, $\dot{\varepsilon} = -2t\varepsilon$, $\varepsilon = \varepsilon_0 e^{-t^2} \Rightarrow 0$ as $t \to \infty$

$\underbrace{\text{Perturbation } \varepsilon \text{ decays as } t \to \infty \Rightarrow y=1 \text{ is stable.}}$

This is true for sufficiently small ε_0.

Near $y=-1$, $\dot{\varepsilon} = 2t\varepsilon \Rightarrow \varepsilon = \varepsilon_0 e^{t^2} \to \infty$ as $t \to \infty$

Perturbation ε grows ("to infinity") as $t \to \infty$ for arbitrarily small $|\varepsilon_0| > 0$

So $y=-1$ is unstable.

Autonomous Systems

$\dot{y} = f(y)$, independent of t. Then, near a fixed point $y=a$, $f(a)=0$, write

$y = a + \varepsilon(t) \Rightarrow \dot{\varepsilon} = \frac{df}{dy}(a) \cdot \varepsilon \equiv k\varepsilon$ say

$\varepsilon = \varepsilon_0 e^{kt}$

Fixed point is stable or unstable according to whether $\frac{df}{dy}(a)$ is +ve or −ve.

(stable if is −ve, unstable is +ve)

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Differential Equations (9)

Example - Chemical reaction kinetics.

e.g. $NaOH + HCl \Rightarrow H_2O + NaCl$

new water

molecules # a b c c

Initially $a = a_0$, $b = b_0$, $c = 0$ $\propto ab$

If the reactants are in dilute solution (e.g. water) then the reaction rate

Reaction rate is linear in both a and b.

$\Rightarrow \frac{dc}{dt} = \lambda (ab)$ for some λ

$\frac{dc}{dt} = \lambda (a_0 - c)(b_0 - c)$

$$\dot{y} = f(y)$$

$y(t) = a + \varepsilon(t)$, $f(a) = 0$, a a fixed point

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{dy}{dt} = f(a+\varepsilon) \approx f(a) + \varepsilon \left.\frac{df}{dy}\right|_a$$

$$\dot{\varepsilon} = \left.\frac{df}{dy}\right|_a \varepsilon$$

$\Rightarrow$ Stability for $\left.\frac{df}{dy}\right|_a$ -ve

Instability for $\left.\frac{df}{dy}\right|_a$ +ve

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Differential Equations (10)

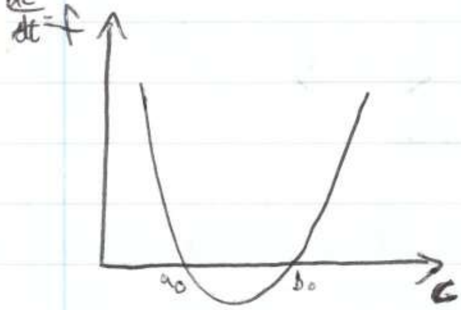
$$NaOH + HCl \to H_2O + NaCl$$

	$NaOH$	HCl	H_2O	$NaCl$
	a	b	c	c
Initially	a_0	b_0	0	0

$$\frac{dc}{dt} = \lambda ab = \lambda(a_0 - c)(b_0 - c) = f(c)$$

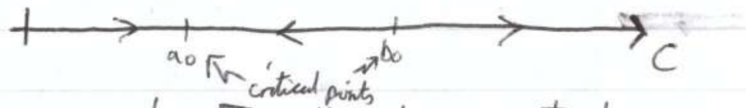
We can plot $\frac{dc}{dt}$ as a function of c

wlog $a_0 < b_0$



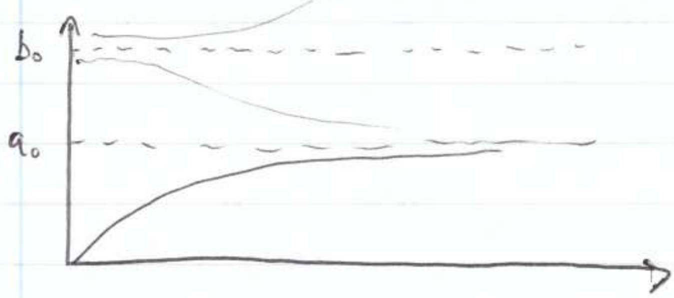
Determine the <u>phase portrait</u>. The dimension of the relevant phase space is equal to the order of the differential system.

Phase portrait



Arrows point in the direction of increasing t. From the phase portrait we can see easily that $c = a_0$ is a stable fixed point, $c = b_0$ an unstable fixed point

Exercise, show $$c = \frac{a_0 b_0\left[1 - e^{-(b_0 - a_0)\lambda t}\right]}{b_0 - a_0 e^{-\lambda(b_0 - a_0)t}}$$



<u>Logistic equation</u> - A simple model of population dynamics

Population y, birth rate αy, death rate βy

$$\Rightarrow \frac{dy}{dt} = (\alpha - \beta) y \quad \Rightarrow y = y_0 e^{(\alpha - \beta)t}$$

population increases or decreases exponentially depending whether birth rates exceed death rates.

Fighting for limited resources

Probability of some food being found $\propto y$

same food being found by two individuals $\propto y^2$

If food is scarce, then fight (to the death).

Death rate due to fighting $\propto y^2$, $= ry^2$

$r = \alpha - \beta$

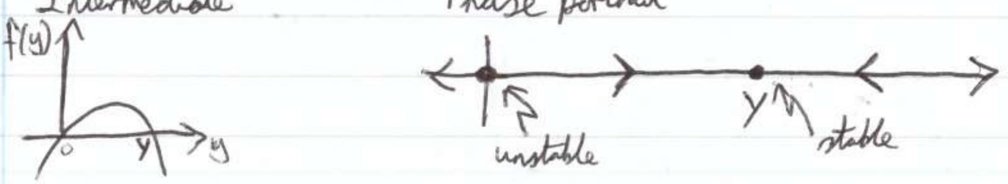
$$\frac{dy}{dt} = (\alpha - \beta)y - ry^2, \quad \dot{y} = ry\left(1 - \frac{y}{Y}\right) \qquad Y = \frac{r}{r}$$

differential logistic equation

Phase portrait

Intermediate

Phase portrait



When population is small, $\dot{y} \approx ry$, no competition, exponential growth. Eventually, a stable equilibrium $y = Y$ is reached.

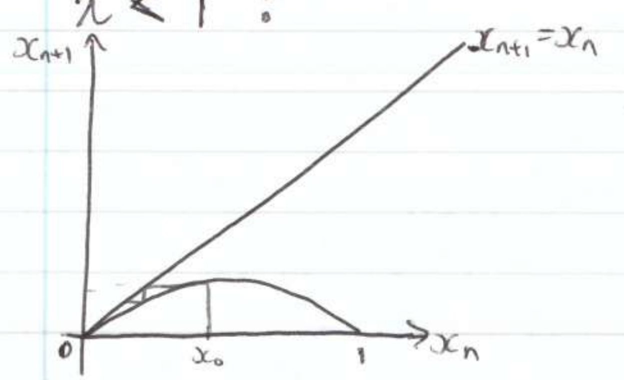
Discrete Equations Evolution of species may occur discretely (e.g. birth in spring, death in winter) rather than continuously. So a better model might be

$$x_{n+1} = \lambda x_n (1 - x_n)$$

Discrete logistic equation, or difference map. $x_{n+1} = f(x_n)$

Behaviour

$\lambda < 1$:



$$x_{n+1} = x_n$$
$$\Rightarrow f(x_n) = x_n$$
$$\lambda x_n (1 - x_n) = x_n$$
$$x_n = 0, \; x_n = 1 - \frac{1}{\lambda}$$

From picture, $x = 0$ is a stable fixed point.

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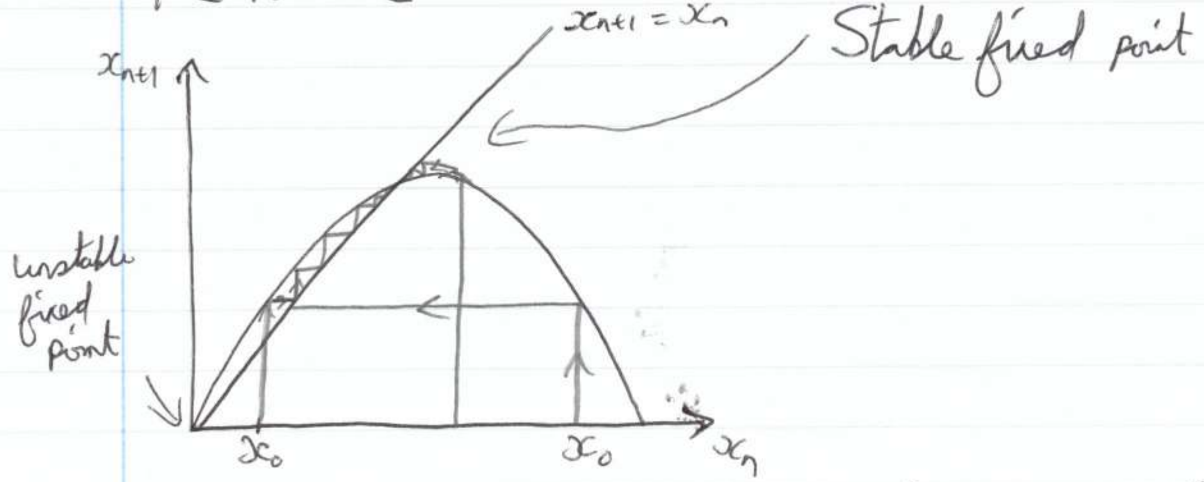
Differential Equations (10)

$1 < \lambda < 2$



~~Stability~~ Stability suppose $x_n = X$ is a fixed point

Write $x_n = X + \varepsilon_n$ perturbation

$$X + \varepsilon_{n+1} = f(X + \varepsilon_n)$$

$$X + \varepsilon_{n+1} = f(x) + \varepsilon_n f'(x) + O(\varepsilon_n^{\,2})$$

$$\varepsilon_{n+1} \approx f'(x)\ \varepsilon_n$$

Fixed point is stable if $\left|\frac{\varepsilon_{n+1}}{\varepsilon_n}\right| < 1$ for all n

$\Rightarrow |f'(x)| < 1$

For logistic equation $f = \lambda x(1-x)$

$f' = \lambda - 2\lambda x$

$x = 0$, $f' = \lambda$, so $x = 0$ is stable $\Leftrightarrow |\lambda| < 1$

$x = 1 - \frac{1}{\lambda}$ is stable if $|\lambda - 2\lambda + 2| < 1 \Leftrightarrow |2 - \lambda| < 1$

$\Leftrightarrow 1 < \lambda < 3$

$\frac{\varepsilon_{n+1}}{\varepsilon_n} = f'(x) = 2 - \lambda$, >0 for $\lambda < 2$

<0 for $\lambda > 2$

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Relationship between logistic equation and logistic map

Logistic equation:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{Y}\right).$$

Approximate the left-hand side to give

$$\begin{aligned}
\frac{y_{n+1} - y_n}{\Delta t} &\approx ry_n\left(1 - \frac{y_n}{Y}\right) \\
\Rightarrow y_{n+1} &\approx y_n + r\Delta t y_n\left(1 - \frac{y_n}{Y}\right) \\
&= (1 + r\Delta t)y_n - \frac{r\Delta t}{Y}y_n^2 \\
&= (1 + r\Delta t)y_n\left[1 - \left(\frac{r\Delta t}{1 + r\Delta t}\right)\frac{y_n}{Y}\right]
\end{aligned}$$

Write

$$\lambda = 1 + r\Delta t, \qquad x_n = \left(\frac{r\Delta t}{1 + r\Delta t}\right)\frac{y_n}{Y}$$

Then

$$x_{n+1} = \lambda x_n(1 - x_n),$$

which is the logistic map.

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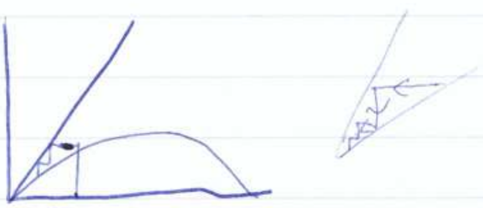
Differential Equations (11)

$x_{n+1} = \lambda x_n(1 - x_n)$



$\lambda < 1$

$1 < \lambda < 2$

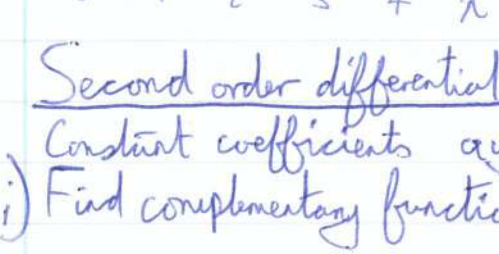
$2 < \lambda < 3$

$3 < \lambda < 1 + \sqrt{6} \approx 3.449$

Oscillatory convergence to a limit cycle

At $\lambda = 1 + \sqrt{6} \approx 3.449$.. the limit cycle gives way to a four cycle and at a little larger value of λ, to an 8 cycle and so on ad infinitum.

Stability Diagram



Second order differential equations

Constant coefficients $ay'' + by' + cy = f(x)$ $\quad$ a, b, c constant

i) Find complementary functions, which satisfy the homogeneous equation

$$ay'' + by' + cy = 0$$

ii) Find a particular integral that satisfies the full equation

Complementary functions

Recall that $e^{\lambda x}$ is an eigenfunction of $\frac{d}{dx}$ and hence also $\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right)$

Therefore the complementary functions have the form:

$y_c = e^{\lambda x}$, $y_c' = \lambda e^{\lambda x}$ $y_c'' = \lambda^2 e^{\lambda x}$

multiplied by $e^{\lambda x}$, $e^{\lambda x} \neq 0$ eigenvalue

$\Rightarrow a\lambda^2 + b\lambda + c = 0$ characteristic equation

There are two (possibly complex) solutions of the characteristic equation.

If they are distinct, λ_1, λ_2 say, $\lambda_1 \neq \lambda_2$, then there are two independent complementary functions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

If λ_1, λ_2 are distinct, y_1 y_2 are linearly independent and complete.

They form a basis of the space of solutions of the homogeneous equations.

The general complementary function is:

$$y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

E.g.

$y'' - 5y + 6y = 0$ Try an eigenfunction solution $y = e^{\lambda x}$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\Rightarrow y_c = Ae^{2x} + Be^{3x}$

$y'' + 4y = 0$ Try $y = e^{\lambda x}$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$ $y_c = Ae^{2ix} + Be^{-2ix}$

$y_c = A(\cos 2x + i \sin 2x) + B(\cos 2x - i \sin 2x)$

$y_c = (A+B)\cos 2x + i(A-B)\sin 2x$

$y_c = \alpha \cos 2x + \beta \sin 2x$

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Differential Equations (11)

Degeneracy:

$y'' - 4y' + 4y = 0$ Try $y = e^{\lambda x}$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda = 2$ or 2, But e^{2x} and e^{2x} are clearly not independent.

So these in particular are not complete.

Detuning: Consider $y'' - 4y' + (4 - \varepsilon^2)y = 0$

Try an eigenfunction solution $y = e^{\lambda x}$

$\lambda^2 - 4\lambda + 4 - \varepsilon^2 = 0 \Rightarrow \lambda = 2 \pm \varepsilon$

$$y_c = Ae^{(2+\varepsilon)x} + Be^{(2-\varepsilon)x} = e^{2x}(Ae^{\varepsilon x} + Be^{-\varepsilon x})$$

$$= e^{2x}\left[(A+B) + \varepsilon x(A-B) + O(A\varepsilon^2, B\varepsilon^2)\right]$$

Choose $A + B = \alpha$, independent of ε

$\varepsilon(A - B) = \beta$, independent of ε

$A = \frac{1}{2}(\alpha + \frac{\beta}{\varepsilon})$, $B = \frac{1}{2}(\alpha - \frac{\beta}{\varepsilon})$

$= O(\frac{1}{\varepsilon})$ $B = O(\frac{1}{\varepsilon})$ as $\varepsilon \to 0$

$$y_c = e^{2x}\left[\alpha + \beta x + O(\varepsilon)\right]$$

linear equations with constant coefficients

$$\to e^{2x}\left[\alpha + \beta x\right] \text{ as } \varepsilon \to 0$$

A demonstration of a general rule that if $y_1(x)$ is a degenerate complementary function, then $y_2(x) = xy_1(x)$ is a complementary function.

02/11/10

Differential Equations (12)

Method of finding second complementary functions (degenerate cases):
If $y_1(x)$ is a complementary function of a homogeneous linear 2nd order ODE, look for another solution of the form $y_2(x) = v(x)\, y_1(x)$

Note that $v'(x)$ will satisfy a first order equation.

E.g. $y'' - 4y' + 4y = 0$, $y_1 = e^{2x}$

Try $y_2 = v(x)e^{2x}$

$y_2' = (v' + 2v)e^{2x}$

$y_2'' = (v'' + 4v' + 4v)e^{2x}$

$$\Rightarrow v'' + 4v' + \underbrace{4v - 4(v' + 2v) + 4v}_{} = 0$$

Cancel because $y_1 = e^{2x}$ is a complementary function

$v'' = 0$, $v' = A$, $v = Ax + B$

So $y_2(x) = (Ax+B)e^{2x}$

Note that y_2 may include arbitrary amounts of y_1.

This method works for any linear homogeneous ODEs, constant coefficients not need

Phase Space A differential equation of n^{th} order determines the n^{th} derivative $y^{(n)}(x)$, and hence, all other higher derivatives in terms of $x, y(x), y'(x) \ldots y^{(n-1)}$

We can think of this in terms of a solution vector:

$$\underline{Y} = \begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix}$$

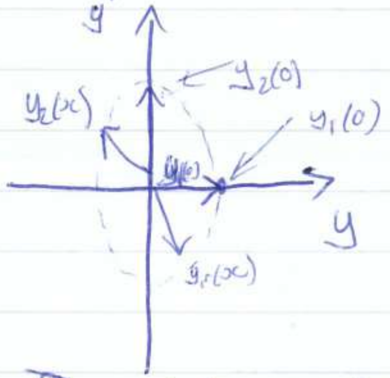
n dimensional vector. Defining a point (for each value of x) in an n dimensional phase space. $Y(x)$ traces out a trajectory in phase space.

$y'' + 4y = 0$

$y_1 = \cos 2x$, $y_2 = \sin 2x$

$y_1' = -2\sin 2x$ $y_2' = 2\cos 2x$

The solution vectors are $\underline{Y_1} = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix}$ $\underline{Y_2} = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}$



The solutions $y_1(x)$ and $y_2(x)$ are independent solutions of the differential equation or linear if the vectors ~~x~~ y_1 and y_2 are linearly independent i.e. If the Wronskian ~~Determ~~ Determinant:

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & \dots & y_n \end{vmatrix} \neq 0$$

For a 2nd order equation $W(x) = \begin{vmatrix} y_1, & y_2, \\ y_1' & y_2' \end{vmatrix}$

$$\text{e.g. } W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$$

Or

$y_1 = e^{2x}$, $y_2 = xe^{2x}$

$$W = \begin{Bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x-2x)e^{4x} \end{Bmatrix} = e^{4x}(1+2x-2 \quad = e^{4x} \neq 0$$

Abel's Theorem

Write equation in standard form $y'' + p(x)y' + q(x)y = 0$
If p, q are continuous then either ~~the Wronski~~ $W \equiv 0$ or $W \neq 0$ for any value of x.

02/11/10

Differential Equations (12)

Suppose y_1 and y_2 are two solutions. Then $y_2(y_1'' + py_1' + qy_1) = 0$

$y_1(y_2'' + py_2' + qy_2) = 0$

subtract $\Rightarrow (y_2 y_1'' - y_1 y'') + p(y_2 y_1' - y_1 y_2') = 0$

$$-W' - pW = 0$$
$$\Rightarrow W' + pW = 0$$
$$\Rightarrow W = W_0 e^{-\int p\,dx}$$

The exponential is never zero so $W_0 = 0$ or $W \neq 0$ for any x.

<u>Note</u> Any <u>linear</u> homogeneous n^{th} order differential equation can be written in the form

$$\underline{Y}' + \underline{A}(x)\underline{Y} = 0$$

It can be shown that $W' + Tr(A)W = 0$, $W = W_0 e^{-\int Tr(A)\,dx}$

and Abel's Theorem holds.

04/11/10

Differential Equations (13)

Particular Integrals

Method 1 - Guesswork

	$f(x)$	$y_p(x)$
eigenfunction →	e^{mx}	Ae^{mx}
forcing	$\sin kx$, $\cos kx$	$A\sin kx + B\cos kx$
	x^n	$q_n(x) = a_n x^n + \dots + a_1 x + a_0$
polynomial	$p_n(x)$	

Remember that equation is linear, so we can superpose solutions corresponding to different forcings.

E.g. $y'' - 5y' + 6y = 2x + e^{4x}$

$$y_{p_1} = ax + b + ce^{4x}$$
$$y_p' = a + 4ce^{4x}$$
$$y_p'' = 16ce^{4x}$$
$$16ce^{4x} - 5(a + 4ce^{4x}) + 6(ax + b + ce^{4x}) = 2x + e^{4x}$$

Coefft e^{4x}: $16c - 20c + 6c = 1 \Rightarrow c = \frac{1}{2}$

x^0: $-5a + 6b = 0 \Rightarrow b = \frac{5}{18}$

x^1: $6a = 2$, $a = \frac{1}{3}$

General solution: $y = Ae^{3x} + Be^{2x} + \frac{1}{2}e^{4x} + \frac{1}{3}x + \frac{5}{18}$

Note!!

Can apply boundary conditions with only the complete solution $y = y_c + y_p$

Resonance Consider $\ddot{y} + \omega_0^2 y = \sin \omega_0 t$, $y_c = A\sin\omega_0 t + B\sin\omega_0 t$

Here the forcing is linearly dependent on the eigenfunctions of the homogeneous ODE (i.e on the complimentary functions).

$y_p = C\sin\omega_0 t + D\cos\omega_0 t$ will give $\ddot{y}_p + \omega_0^2 y_p = 0$ so we can't balance forcing

This example is a simple harmonic oscillator forced at its natural (resonant) frequency.

Detuning

Consider $\ddot{y} + \omega_0^2 y = \sin \omega t$ $\qquad \omega \neq \omega_0$

$y_p = c(\sin \omega t - \sin \omega_0 t)$ $\Leftarrow$ no cosine needed, no need to take account of first derivative

$\ddot{y}_p = c(-\omega^2 \sin \omega t + \omega_0^2 \sin \omega_0 t)$

Substitute

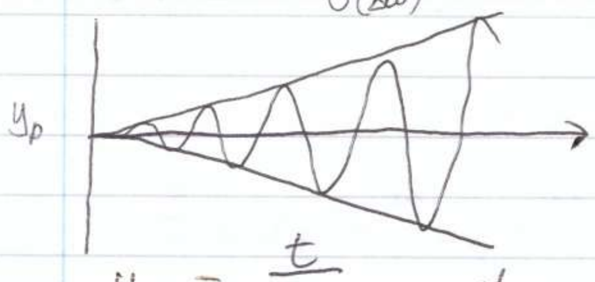
$\Rightarrow c(\omega_0^2 - \omega^2) = 1$

$$\Rightarrow y_p = \frac{\sin \omega t - \sin \omega_0 t}{\omega_0^2 - \omega^2} = \frac{2\cos\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)}{\omega_0^2 - \omega^2}$$

$\omega_0 - \omega = \Delta\omega$

$$y_p = -\frac{2}{(\omega+\omega_0)\Delta\omega} \times \cos\left[\left(\omega_0 - \frac{\Delta\omega}{2}\right)t\right]\sin\left(\frac{\Delta\omega}{2}t\right)$$

If the forcing is at a frequency close to the natural frequency we get beating, and as $\Delta\omega \to 0$, the 'envelope' tends to ∞ and we just see initial linear growth



Mathematically:

$\Delta\omega \to 0$

$$y_p \to -\frac{2}{\omega_0+\omega_0}\cos(\omega_0 t) \times \left(\frac{t}{2}\right) = -\frac{t}{2\omega_0}\cos \omega_0 t$$

$$y_p = \frac{t}{2\omega_0}\cos \omega_0 t$$

General rule: If forcing is a linear combination of complementary functions then the particular integral has an amplitude proportional to t times the non resonant guess (relates to ODEs with constant coefficients).

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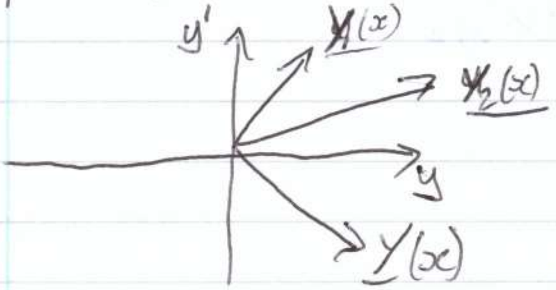
Differential Equations (13)

Method 2

Let $y_1(x)$, $y_2(x)$ be linearly independent functions of the ODE.

$y'' + p(x)\,y' + q(x)\,y = f(x)$

The solution vectors $\underline{Y_1} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$ and $\underline{Y_2} = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$ form a basis of the phase space (solution space).



We can write

$$\underline{Y_p}(x) = u(x)\,\underline{Y_1}(x) + v(x)\,\underline{Y_2}(x)$$

Then

$$y_p = uy_1 \cancel{+ vy_2} \qquad + vy_2 \qquad (1)$$

$$y_p' = uy_1' \qquad + vy_2' \qquad (2)$$

$$y_p'' = uy_1'' + u'y_1' \qquad + vy_2'' + v'y_2'$$

(1) Apply product rule

$$y_p' = uy_1' + u'y_1 + vy_2' + v'y_2$$

Compare with (2) $\Rightarrow y_1 u' + y_2 v' = 0$

$$y'' + p y' + q y = f$$

$$y_p = u y_1 + v y_2$$

$$y_p' = u y_1' + v y_2' \qquad \Rightarrow y_1 u' + y_2 v' = 0$$

$$\cancel{y_1' u' + y_2 v'}$$

$$y_p'' = u y_1'' + u' y_1' + v y_2'' + v' y_2'$$

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Differential Equations (14)

$$\underline{Y_1}(x) = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \qquad \underline{Y_2}(x) = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$$

$$\underline{Y_p} = u(x)\,\underline{Y_1}(x) + v(x)\,\underline{Y_2}(x)$$

$$y_p = u y_1 + v y_2$$

$$y_p' = u y_1' + v y_2' \qquad \Rightarrow y_1 u' + y_2 v' = 0 \quad (1)$$

Differential Equation

$$y'' + p(x) y' + q(x) y = f(x)$$

$$y_p'' = u y_1'' + u' y_1' + v y_2'' + v' y_2'$$

Sub (1) (2) and (3) into differential equation

$$\Rightarrow y_1' u' + y_2' v' = f(x) \qquad (2)$$

(1) and (2) give $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix}$$

So solution exists providing $W \neq 0$

$$\Rightarrow u' = -\frac{y_2}{W} f \qquad v' = \frac{y_1}{W} f$$

Eg. $y'' + 4y = \sin 2x$

$$\left.\begin{array}{l} y_1 = \sin 2x \\ y_2 = \cos 2x \end{array}\right\} W = -2$$

$$y_p = u \sin 2x + v \cos 2x$$
$$y_p' = u\, 2\cos 2x + v(-2\sin 2x)$$
$$y_p'' = \ldots$$

Sub and solve to find $u' = \frac{\cos 2x \sin 2x}{2}$, $v' = -\frac{\sin^2 2x}{2}$

$$u = -\tfrac{1}{16}\cos 4x \quad , \quad v = \tfrac{1}{16}\sin 4x - \frac{x}{4}$$

$$y_p = \tfrac{1}{16}\left(-\cos 4x \sin 2x + \sin 4x \cos 2x\right) - \frac{x}{4}\cos 2x$$

$$= \underline{\tfrac{1}{16}\sin 2x - \tfrac{1}{4}x\cos 2x} \rightarrow \text{found earlier by 'detuning'}$$

piece of complementary function

Homogeneous Equations (linear equidimensional equations)

$$ax^2y'' + bxy' + cy = f(x)$$

with a, b, c constants.

Complementary functions.

Note $y = x^k$ is an eigenfunction of the operator $x\frac{d}{dx}$

1. To solve $ax^2y'' + bxy' + cy = 0$
   Try $y = x^k$, $y' = kx^{k-1}$, $y'' = k(k-1)x^{k-2}$
   $$ak(k-1) + bk + c = 0 \quad \Rightarrow k = k_1, k_2$$
   $$y_c = Ax^{k_1} + Bx^{k_2}$$

2. Write $z = \ln x$, show $a\frac{d^2y}{dz^2} + (b-a)\frac{dy}{dz} + cy = f(e^z)$
   So this transformation converts an equidimensional equation into one with constant coefficients.

06/11/10

Differential Equations (14)

Characteristic equation $a\lambda^2 + (b-a)\lambda + c = 0$

$$\lambda = k_1, k_2$$

$y_c = Ae^{k_1 z} + Be^{k_2 z} \Rightarrow$ same solution

If roots of the characteristic equation are equal then $y_c = e^{kz}, ze^{kz}$

$y_c = x^k, x^k \log x$

And if there is a resonant forcing proportional to x^{k_1} or x^{k_2} then there is a particular again with logarithmic growth; form $x^{k_1}\log x$ or $x^{k_2}\log x$

Difference Equations for discrete variables

$$ay_{n+2} + by_{n+1} + cy_n = f_n$$

Solve in a similar way to differential equations by exploiting linearity and eigen functions.

Difference operator $D[y_n] = y_{n+1}$ has eigenfunctions $y_n = k^n$

because $D[y_n] = D[k^n] = k^{n+1} = k\cdot k^n = ky_n$

To solve the difference equation, first look for complementary functions satisfying $ay_{n+2} + by_{n+1} + cy = 0$

Try $y_n = k^n$ $\quad ak^{n+2} + bk^{n+1} + ck^n = 0$

$$ak^2 + bk + c = 0$$

$$\Rightarrow k = k_1, k_2$$

General complementary function $y_n^{(c)} = Ak_1^n + Bk_2^n$ if $k_1 \neq k_2$

$$= (A+Bn)k_1^n \text{ if } k_1 = k_2$$

~~Particular integrals~~

3

Particular Integrals (difference equations)

f	$y_n^{(p)}$
λ^n	$A\lambda^n$, $\lambda \neq k_1, k_2$
k_1^n, k_2^n	Ank_1^n, Bnk_2^n
n^p	$An^p + Bn^{p-1} + \dots + Cn + D$

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Differential Equations (15)

Difference Equations

Difference operator $D[y_n] = y_{n+1}$ has an eigenfunction $y_n = k^n$

E.g. Fibonacci Sequence $y_n = y_{n-1} + y_{n-2}$, $y_0 = y_1 = 1$

$$y_{n+2} - y_{n+1} - y_n = 0$$

$$D^2[y_n] - D[y_n] - y_n = 0 \qquad \varphi_2 = -\frac{1}{\varphi_1}$$

$$\Rightarrow k^2 - k - 1 = 0 \qquad k = \frac{1 \pm \sqrt{5}}{2}, \varphi_1, \varphi_2$$

General solution $y_n = A\varphi_1^{\,n} + B\varphi_2^{\,n}$

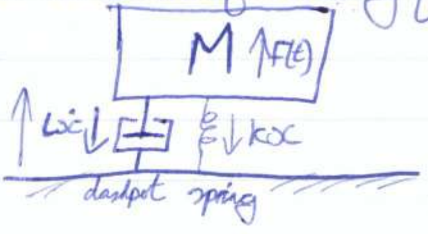
Initial conditions $y_0 = 1 = A + B$

$$y_1 = 1 = A\varphi_1 + B\varphi_2 \qquad \Rightarrow A = \frac{\varphi_1}{\sqrt{5}}, \; B = -\frac{\varphi_2}{\sqrt{5}}$$

$$\Rightarrow y_n = \frac{\varphi_1^{\,n+1} - \varphi_2^{\,n+1}}{\sqrt{5}}$$

Transients and damping

In many physical systems there is some sort of restoring force and some damping. E.g car suspension



Newton's second law

$$M\ddot{x} = F - kx - L\dot{x}$$

$$\ddot{x} + \frac{L}{M}\dot{x} + \frac{k}{M}x = \frac{F(t)}{M}$$

Note: τ is dimensionless

Write $t = \sqrt{\frac{M}{k}}\,\tau$ where $(\dot{\ })$ means $\frac{\partial}{\partial \tau}$, $K = \frac{L}{2\sqrt{kM}}$

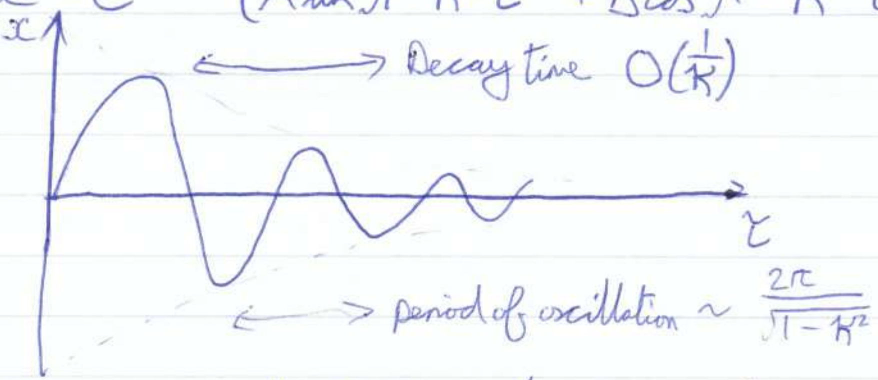
$$\ddot{x} + 2K\dot{x} + x = f(\tau) \qquad f = \frac{F}{k}$$

There is a single parameter K determining the behaviour of the system.

Free (natural response) $f = 0$, $\ddot{x} + 2K\dot{x} + x = 0$

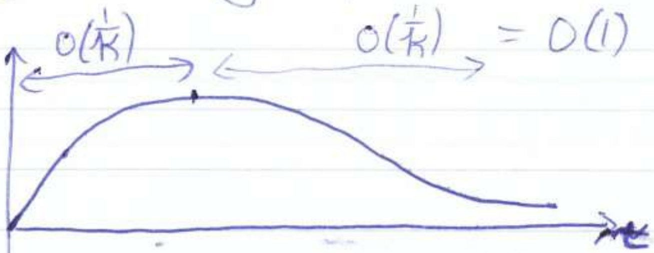
Try $x = e^{\lambda t} \Rightarrow \lambda^2 + 2K\lambda + 1 \Rightarrow \lambda = -K \pm \sqrt{K^2 - 1} = \lambda_1, \lambda_2$

$K<1$ $x = e^{-K\tau}\left(A\sin\sqrt{1-K^2}\,\tau + B\cos\sqrt{1-K^2}\,\tau\right)$

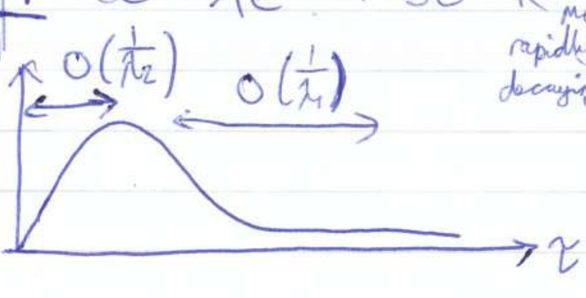


If we increase the damping (or decrease the mass or spring constant) then the period increases and the decay decreases.
As $K \to 1$, period $\to \infty$

$K=1$ (critically damped) $x = (A + B\tau)e^{-K\tau}$



$K>1$ $x = Ae^{\lambda_1\tau} + Be^{\lambda_2\tau}$ more rapidly decaying $\quad \lambda_{1,2} = -K \pm \sqrt{K^2-1} < 0$



Possible to get a large initial increase in amplitude before the eventual slow decay
In a forced system, the complementary functions determine the early time transient response while the particular integral determines the long term "asymptotic" response

09/11/10

Differential Equations (15)

E.g. $\ddot{x} + 2K\dot{x} + x = \sin\tau \qquad K \neq 0$

try $x = C\sin\tau + D\cos\tau$ for particular integral

$C = 0,\ D = -\frac{1}{2K}$

$\Rightarrow x = Ae^{\lambda_1\tau} + Be^{\lambda_2\tau} - \frac{1}{2K}\cos\tau \sim -\frac{1}{2K}\cos\tau$ as $\tau \to \infty$

because $Re(\lambda_{1,2}) = 0$

Note the forced response is out of phase with the forcing.

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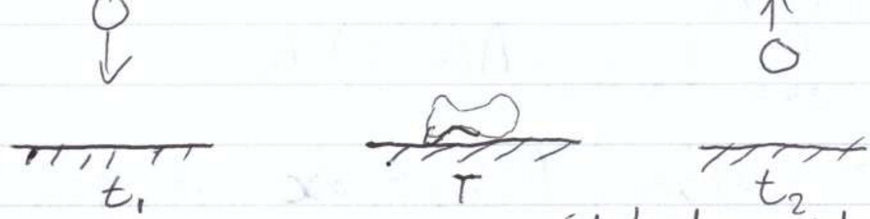
Differential equations (16)

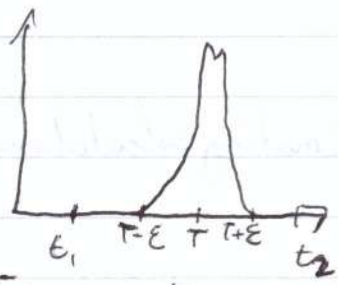
Impulses and point forces

Consider a ball bouncing on the ground



Force exerted on the ball is $F(t)$

Often don't know or wish to know details of $F(t)$ but note that it only acts for a time of $O(\varepsilon)$ much less than the total time scale of the system.



It is mathematically possible and convenient to imagine the force acting instantaneously at $t=T$, i.e. $\varepsilon \to 0$. Using Newton's 2nd Law

$m\ddot{x} = F(t) - mg$. Integrate for $T-\varepsilon$ to $T+\varepsilon$

$$\int_{T-\varepsilon}^{T+\varepsilon} m\frac{d^2x}{dt^2}\,dt = \int_{T-\varepsilon}^{T+\varepsilon} F(t)\,dt - \int_{T-\varepsilon}^{T+\varepsilon} mg\,dt$$

area under force curve

$$\left[m\frac{dx}{dt}\right]_{T-\varepsilon}^{T+\varepsilon} = \overset{\text{impulse}}{I} - 2mg\varepsilon \quad \text{where } I = \int_{T+\varepsilon}^{T+\varepsilon} F(t)\,dt$$

only property of F influencing macroscopic behaviour

If contact time 2ε is very small then mathematically we neglect it and write

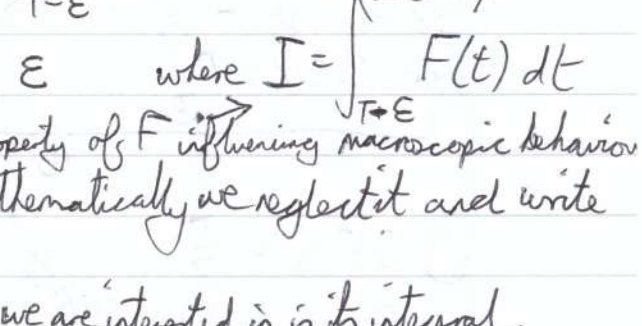
$$\left[m\frac{dx}{dt}\right]_{T-}^{T+} = I$$

Important The only feature of $F(t,\varepsilon)$ we are interested in is its integral.

Mathematically, we consider a family of functions $D(t;\varepsilon)$ such that

$\lim_{\varepsilon\to 0} D(t;\varepsilon) = 0$ for all $t = 0$.

$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} D(t;\varepsilon)\,dt = 1$



E.g. $D(t;\varepsilon) = \frac{1}{\varepsilon\sqrt{\pi}}e^{-\frac{t^2}{\varepsilon^2}}$

as $\varepsilon \to 0$, $D(0;\varepsilon) \to \infty$ so $\lim_{\varepsilon\to 0} D(t;\varepsilon)$ is not a function it is undefined.

Nonetheless we define the Dirac Delta Function by $\delta(x) = \lim_{\varepsilon \to 0} D(x;\varepsilon)$ on the understanding that we can only use its integral properties.

e.g $\int_{-\infty}^{\infty} g(x)\,\delta(x)\,dx$

means

$$\left.\begin{aligned} &\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} g(x)\,D(x;\varepsilon)\,dx \\ &= g(0)\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} D(x;\varepsilon)\,dx = g(0) \end{aligned}\right\} *$$

Note! no formal proof here

Provided g is continuous.

This gives us a convenient way of representing and making calculations involving impulses or point forces.

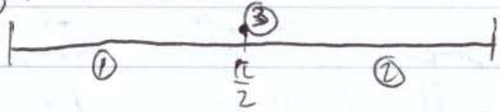
$$m\ddot{x} = -mg + I\,\delta(t-T)$$
$$x = x_0,\ \dot{x} = 0,\ t = 0.$$

$*$ In general $\int_a^b g(x)\,\delta(x-c)\,dx = g(c)$ if $c \in (a,b)$
$= 0$ if $c < a,\ c > b$

<u>Example</u> Point force

Solve $y'' - y = 3\,\delta(x - \frac{\pi}{2})$, $y = 0$ at $x = 0, \pi$

for $0 \le x \le \pi$



① $0 \le x < \frac{\pi}{2}$ $\quad y'' - y = 0$, $y = A\sinh x + B\cosh x$

$y = 0$ @ $x = 0 \Rightarrow B = 0$

② $\frac{\pi}{2} < x \le \pi$ $\quad y'' - y = 0$ $\quad y = C\sinh(\pi - x) + D\cosh(\pi - x)$

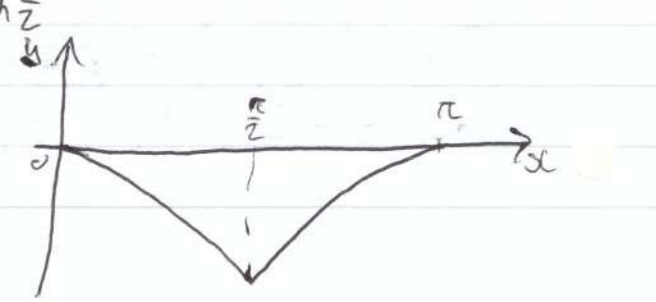
$y = 0$ @ $x = \pi \Rightarrow D = 0$

③ $x = \frac{\pi}{2}$ y is continuous $\Rightarrow A = C$

Integrate from $\frac{\pi}{2}-$ to $\frac{\pi}{2}+$.

$$\left[y'\right]_{\frac{\pi}{2}-}^{\frac{\pi}{2}+} = 3 \Rightarrow A = -\frac{3}{2\cosh\frac{\pi}{2}}$$

$$y = \begin{cases} -\dfrac{3\sinh x}{2\cosh\frac{\pi}{2}} & 0 \le x < \frac{\pi}{2} \\[2ex] -\dfrac{3\sinh(\pi - x)}{2\cosh\frac{\pi}{2}} & \frac{\pi}{2} < x \le \pi \end{cases}$$



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Differential Equations (17)

$\delta(x) = 0, \; x \neq 0$

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$ $\qquad \int_a^b g(x)\,\delta(x-c)\,dx = \begin{cases} g(c) & \text{if } a<c<b \\ 0 & \text{if } c<a \text{ or } c>b \end{cases}$

$\cancel{\left[x^4/3\right]_3^4}$ $ay'' + by' + cy = I\delta(x-d)$

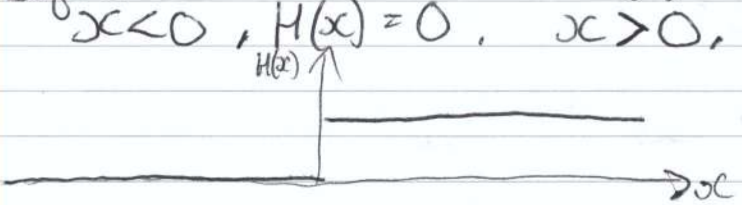
Then $ay'' + by' + cy = 0$ if $x > d$, $x < d$

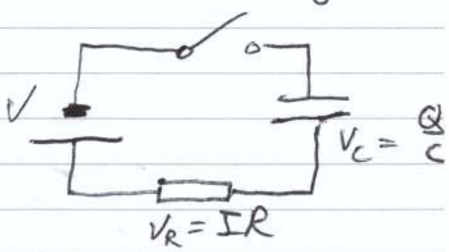
b $[y]_{d-}^{d+} + a[y']_{d-}^{d+} = I$

<u>Heaviside Step Function</u> $H(x)$ $\qquad H(x) = \int_{-\infty}^{x} \delta(x')\,dx'$

Define

$x < 0$, $H(x) = 0$. $x > 0$, $H(x) = 1$. $H(0)$ is undefined.





Can apply the Fundamental theorem of calculus gives

$\frac{dH}{dx} = \delta(x)$, useful for switching problems

$$V H(t) = IR + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C}$$

$$\dot{Q} + \frac{1}{RC}Q = \frac{V}{R}H(t)$$

$$\Rightarrow \ddot{Q} + \frac{1}{RC}\dot{Q} = \frac{V}{R}\delta(t)$$

Note, Q is continuous at $t=0$ but $\dot{Q}$ jumps by $\frac{V}{R}$

<u>Series Solutions</u>

Consider equations of the form $p(x)\,y'' + q(x)\,y' + r(x)\,y = 0$

$x = x_0$ is an <u>ordinary point</u> of the DE if $\frac{q}{p}$ and $\frac{r}{p}$ have Taylor series about x_0 (i.e infinitely differentiable at x_0). Otherwise it is a singular point.

If x_0 is a singular point, but the equation can be written in the form

$$P(x)(x-x_0)^2 y'' + Q(x)(x-x_0)\,y' + R(x)\,y = 0$$

where $\frac{Q}{P}$ and $\frac{R}{P}$ have Taylor series about x_0, then x_0 is a <u>regular</u> singular point

<u>Examples</u>

i) $(1-x^2)y'' - 2xy' + 2y = 0$. $x=0$ is an ordinary point.
$x = \pm 1$ are regular singular points

ii) $\sin x\, y'' + \cos x\, y' + 2y = 0$
$x = n\pi$ is a singular points, all regular. All other points are ordinary.

iii) $(1+\sqrt{x})y'' - 2xy' + 2y = 0$.
$x=0$ is an irregular singular point.

<u>Theorem</u> If x_0 is an ordinary point then the equation has 2 linearly independent solutions of the form
$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$
convergent in some neighbourhood of x_0.

If x_0 is a regular singular point then the equation has at least 1 solution of the form
$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\sigma} \quad \text{Frobenius Series, } a_0 \neq 0$$

Eg. $(1-x^2)y'' - 2xy' + 2y = 0$ $\qquad x=0$, ordinary point
We will find a series solution about $x=0$.
Write
$$(1-x^2)\underline{x^2y''} - 2x^2\underline{xy'} + 2x^2\underline{y} = 0$$

Try $y = \sum_{n=0}^{\infty} a_n x^n$
Sub:
$$\sum a_n\left[(1-x^2)n(n-1) - 2x^2 n + 2x^2\right]x^n = 0$$

Coefficient of x^n gives a general recurrence relation.
$$n(n-1)a_n - \left[(n-2)(n-2-1) + 2(n-2) - 2\right]a_{n-2} = 0$$

$$\boxed{n(n-1)a_n = n(n-3)a_{n-2}} \qquad \begin{array}{l} n=0, \ 0\cdot a_0 = 0 \Rightarrow a_0 \text{ is arbitrary.} \\ n=1, \ 0\cdot a_1 = -2a_1 = 0 \Rightarrow a_1 \text{ is arbitrary} \end{array}$$

If $n>1$: $a_n = \frac{n-3}{n-1}a_{n-2}$
often, this is the end of the story.

$$a_n = \frac{n-3}{n-1}a_{n-2} = \frac{n-3}{n-1}\cdot\frac{n-5}{n-3}a_{n-4}$$

$$\Rightarrow a_{2k} = -\frac{1}{2k+1}a_0, \quad a_{2k+1} = 0, \quad k \geq 1$$

$$y = a_0\left[1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} \dots\right] + a_1 x$$
$$= a_0\left[1 - \frac{x}{2}\ln\frac{1+x}{1-x}\right] + a_1 x$$

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Differential Equations (18)

$(1-x^2)y'' - 2xy' + 2y = 0$ (1)

$p\,y'' - qy' + ry = 0$

If $\frac{q}{p}$, $\frac{r}{p}$ have Taylor series at $x = x_0$ then x_0 is an ordinary point

(1) is singular at $x = 1$ as $\frac{q}{p} = -\frac{2x}{1-x^2}$

but $(x-1)^2\frac{q}{p}$ and $(x-1)^2\frac{r}{p}$ have Taylor Series, so $x=1$ is a regular singular point

E.g. $4xy'' + 2(1-x^2)y' - xy = 0$

$x = 0$ is a regular singular point.

First write

$4(x^2y'') + 2(1-x^2)(xy') - x^2(y) = 0$

Try $y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$, $a_0 \neq 0$

$$\sum a_n\left[4(n+\sigma)(n+\sigma-1) + 2(1-x^2)(n+\sigma) - x^2\right]x^{n+\sigma} = 0$$

Coefficient of $x^{n+\sigma}$ gives

$$\left[4(n+\sigma)(n+\sigma-1) + 2(n+\sigma)\right]a_n + a_{n-2}\left[-2(n+\sigma-2) - 1\right] = 0$$

$$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2}$$

The case $n=0$ gives the indicial equation, which determines the index σ

$2\sigma(2\sigma-1)a_0 = 0$ but we decided $a_0 \neq 0$

$\Rightarrow \sigma = 0,\ \sigma = \frac{1}{2}$

Try $\sigma = 0$

$2n(2n-1) = (2n-3)a_{n-2}$

$n = 0 \Rightarrow 0\cdot a_0 = 0 \Rightarrow a_0$ is arbitrary

$n > 0 \Rightarrow a_n = \frac{2n-3}{2n(2n-1)}a_{n-2}$, note $a_1 = 0 \Rightarrow a_n = 0$ if n is odd.

$a_{2k} = \frac{4k-3}{4k(4k-1)}a_{2k-2}$

$$y = a_0\left[1 + \frac{x^2}{4\cdot3} + \frac{5}{8\cdot7\cdot4\cdot3}x^4 + \dots\right]$$

$\sigma = \frac{1}{2}$ $(2n+1)(2n)a_n = (2n-2)a_{n-2}$

$n=0$, $0\cdot a_0 = 0 \Rightarrow a_0$ is arbitrary (call it b_0)

$n=1$, $6a_1 = 0 \Rightarrow a_1 = 0$

$n > 1$ $a_n = \frac{n-1}{n(2n+1)}a_{n-2}$ $\Rightarrow y = b_0 x^{\frac{1}{2}}\left(1 + \frac{1}{2\cdot5}x^2 + \frac{3}{2\cdot5\cdot4\cdot9}x^4 + \dots\right)$

Behaviour near x_0:
Indicial equation has two roots (for this 2nd order equation) say σ_1, σ_2

i) $\sigma_2 - \sigma_1$ is not an integer then there are two linearly independent Frobenius series solutions

ii) $\sigma_2 - \sigma_1$ is an integer, then there is one solution of the form
$$y_1 = (x-x_0)^{\sigma_2} \sum_{n=0} a_n (x-x_0)^n$$
where $Re(\sigma_2) \geq Re(\sigma_1)$

The other solution is of the form $y_2 = \sum_{n=0} b_n (x-x_0)^{n+\sigma_1} + \ln(x-x_0)\, y_1$
stated without proof.

Example $xy'' - y = 0$
$(x^2 y'') - x(y) = 0 \qquad P = 1,\ R = -x,\ \frac{R}{P} = -x$ a Taylor series

So $x = 0$ is a regular singular point

$$\sum a_n x^{n+\sigma} \left[(n+\sigma)(n+\sigma-1) - x\right] = 0$$

Coefficient of $x^{n+\sigma}$ $\quad (n+\sigma)(n+\sigma-1)\, a_n = a_{n-1}$

$n=0$ gives indicial equation $\Rightarrow \sigma(\sigma-1)\, a_0 = 0$
$a_0 \neq 0 \Rightarrow \sigma = 0, 1$

$\sigma = 1$: $(n+1)n\, a_n = a_{n-1}$
$n = 0$: $0 \cdot a_0 = 0 \Rightarrow a_0$ is arbitrary
$n > 0$: $a_n = \frac{1}{n(n+1)} a_{n-1}$
$$a_n = \frac{1}{(n+1)(n!)^2} a_0$$
$$y_1 = a_0 x \left[1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \dots\right]$$

$\sigma = 0$: $n(n-1)\, a_n = a_{n-1}$
$n = 0$: $0 \cdot a_0 = 0 \Rightarrow a_0$ arbitrary
$n = 1$: $0 \cdot a_1 = a_0 \Rightarrow a_0 = 0$ but we chose $a_0 \neq 0$ $\times$

Suppose we allow $a_0 = 0$. Then $0 \cdot a_1 = 0 \Rightarrow a_1$ is arbitrary.

$n \geq 1$, $a_n = \frac{1}{n(n-1)} a_{n-1} = \frac{1}{n(n-1)!^2} a_1$
$$y_2 = a_1 \left[x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{144} + \dots\right] = y_1$$ which is the solution we found already.

The other independent solution is actually
$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n$$

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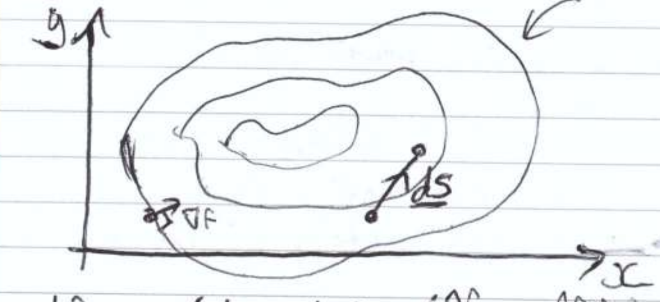
Differential Equations (19)

Directional Derivatives

$z = c$ contours of $f(x,y)$



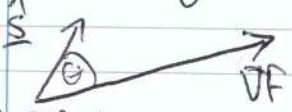
Consider an infinitesimal displacement $\underline{ds} = (dx, dy)$
The change in $f(x,y)$ during that displacement
$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$df = (dx, dy) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \underline{ds} \cdot \underline{\nabla f}$ where $\underline{\nabla f} \equiv \text{grad } f$ has Cartesian components $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. ∇f is the gradient of f.
Write $\underline{ds} = ds\, \hat{\underline{s}}$ where $|\hat{\underline{s}}| = 1$

Then $df = ds\, \hat{\underline{s}} \cdot \underline{\nabla f}$, $\boxed{\frac{df}{ds} = \hat{\underline{s}} \cdot \underline{\nabla f}}$ $*$

$\frac{df}{ds}$ is the directional derivative of f in the direction of $\hat{\underline{s}}$.

<u>The Gradient vector</u> $\text{grad } f \equiv \nabla f$ is defined by $*$. It is a <u>vector</u> with the following properties:



$$\frac{df}{ds} = |\hat{\underline{s}}||\nabla f| \cos\theta = |\nabla f| \cos\theta \le \overset{\max \theta}{\Downarrow}{|\nabla f|}$$

i) ∇f has magnitude equal to the maximum rate of change of $f(x,y)$ with distance in the x-y plane.
ii) It has direction in which f increases most rapidly.
iii) If $\underline{ds}$ is a displacement along a contour of f then $\frac{df}{ds} = 0 \Leftarrow$ No change in f along a contour
$\Rightarrow \hat{\underline{s}} \cdot \nabla f = 0 \Rightarrow \nabla f$ orthogonal to the contour

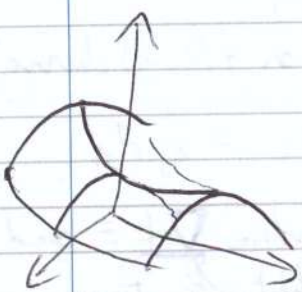
Examples of gradient vectors

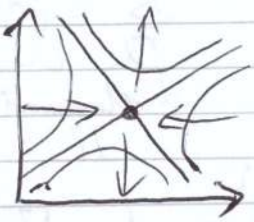
If $\underline{\Phi}$ is the gravitational potential, $\underline{F} = -\nabla \underline{\Phi}$ is the gravitational force.
If $T(x, y, z)$ is temperature then heat flows by conduction in the direction of $-\nabla T$, so heat flux $\underline{q} = -k \nabla T$ — thermal conductivity

Stationary Points

There is always one direction in which $\frac{df}{ds} = 0$ namely parallel to a contour of f
Local maxima and minima have $\frac{df}{ds} = 0$ for all directions.
$\Rightarrow \hat{\underline{s}} \cdot \nabla f = 0$ for all $\hat{\underline{s}} \Rightarrow \nabla f = 0$
In cartesians this translates to $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

but $\nabla f = 0$ also at saddle points (pringle points)





Note: contours are locally elliptical at maxima and minima, whereas they are locally hyperbolae at saddle points.

Note: contours cross only at saddle points.

Taylor Series for multivariable functions

Consider a finite displacement δs along a straight line in the x-y plane. Then $\delta s \frac{d}{ds} \equiv \underline{\delta s} \cdot \nabla$

The Taylor series along the line is

$$f(s) = f(s_0 + \delta s) = f(s_0) + \delta s \frac{df}{ds} + \frac{1}{2}\delta s^2 \frac{d^2f}{ds^2} + \dots$$

$$= f(s_0) + \underline{\delta s}\cdot\nabla f + \frac{1}{2}\delta s^2 (\hat{\underline{s}}\cdot\nabla)(\hat{\underline{s}}\cdot\nabla) f + \dots$$

where $\underline{\delta s}\cdot\nabla f = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}$ $\qquad \hat{\underline{s}} = (\hat{s}_x, \hat{s}_y)$

$$\delta s^2 (\hat{\underline{s}}\cdot\nabla)(\hat{\underline{s}}\cdot\nabla) f$$

$$= \delta s^2 \left(\hat{s}_x \frac{\partial}{\partial x} + \hat{s}_y \frac{\partial}{\partial y}\right)\left(\hat{s}_x \frac{\partial f}{\partial x} + \hat{s}_y \frac{\partial f}{\partial y}\right)$$

$$= \delta x^2 f_{xx} + \delta x\, \delta y\, f_{yx} + \delta y\, \delta x\, f_{xy} + \delta y^2 f_{yy}$$

$$(\underline{\delta s}\cdot\nabla)(\underline{\delta s}\cdot\nabla) f = (\delta x, \delta y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

where $[\nabla\nabla f] \equiv \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ is called the Hessian Matrix.

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Differential Equations ㉑

Systems of Linear Equations

Consider two dependent variables $y_1(t)$ and $y_2(t)$

$$\dot{y}_1 = ay_1 + by_2 + f_1(t)$$
$$\dot{y}_2 = cy_1 + dy_2 + f_2(t) \Rightarrow \dot{\underline{Y}} = \underline{M}\,\underline{Y} + \underline{F}$$
$$\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ \underline{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \underline{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Equivalence to higher order equations

$$\ddot{y}_1 = a\dot{y}_1 + b\dot{y}_2 + \dot{f}_1 = a\dot{y}_1 + b(cy_1 + dy_2 + f_2) + \dot{f}_1$$

$$\ddot{y}_1 = a\dot{y}_1 + bcy_1 + bd\left\{\dot{y}_1 - ay_1 - f_1\right\} + bf_2 + \dot{f}_1$$
$$\ddot{y}_1 - (a+d)\dot{y}_1 + (ad - bc)y_1 = bf_2 - df_1 + \dot{f}_1$$

Conversely $\ddot{y} + A\dot{y} + By = f$ $\qquad \left.\begin{matrix} y_1 = y \\ y_2 = \dot{y} \end{matrix}\right\} = \underline{Y} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$

$y_1 = y,\ y_2 = \dot{y}$ $\qquad \dot{y}_1 = y_2 \quad \dot{y}_2 = f - Ay_2 - By_1$

$$\dot{\underline{Y}} = \begin{pmatrix} 0 & 1 \\ -B & -A \end{pmatrix}\underline{Y} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Any n^{th} order ODE is equivalent to a system of n first order ODEs

Consider $\dot{\underline{Y}} - M\underline{Y} = \underline{F}$. Try complementary function $\underline{Y}_c = \underline{v}e^{\lambda t}$

$\lambda\underline{v} - M\underline{v} = 0 \Rightarrow$ possible values of λ are eigenvalues of M, $\underline{v}$ is an eigenvect

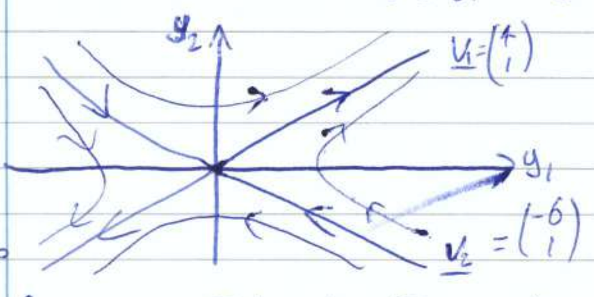
E.g. $\dot{\underline{Y}} - \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}\underline{Y} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}e^t$. Find $\underline{Y}_c$: try $\underline{v}e^{\lambda t}$, $\det(A - \lambda I) = 0$

$\begin{vmatrix} -4-\lambda & 24 \\ 1 & -2-\lambda \end{vmatrix} = 0 \qquad 8 + 6\lambda + \lambda^2 - 24 = 0,\ \lambda = 2, -8$

$\lambda = 2 \qquad \begin{pmatrix} -6 & 24 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \underline{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \underline{Y}_c = A\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{2t} + B\begin{pmatrix} -6 \\ 1 \end{pmatrix}e^{-8t}$

$\lambda = -8 \qquad \begin{pmatrix} 4 & 24 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \underline{v} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

Phase Space Trajectories



unknown constant vector

could then determine A and B from some initial conditions

Particular Integral Try $\underline{Y}_p = \underline{u}e^t$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}e^t - \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}e^t = \begin{pmatrix} 4 \\ 1 \end{pmatrix}e^t$$

use inverse $\Rightarrow \begin{pmatrix} 5 & -24 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} 3 & 24 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

General solution: $\underline{Y} = A\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{2t} + B\begin{pmatrix} -6 \\ 1 \end{pmatrix}e^{-8t} - \begin{pmatrix} 4 \\ 1 \end{pmatrix}e^t$

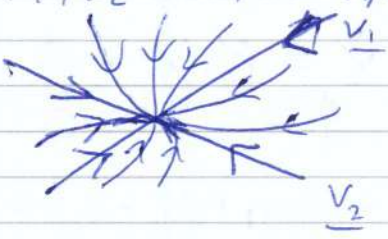
Other linear phase-plane portraits:

General solution to $\dot{\underline{Y}} = M\underline{Y}$ is $\underline{Y} = A\underline{v_1}e^{\lambda_1 t} + B\underline{v_2}e^{\lambda_2 t}$

① λ_1, λ_2 real, $\lambda_1\lambda_2 < 0$ gives a saddle as before

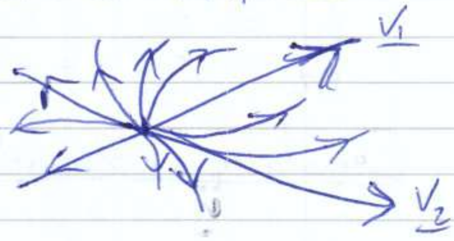
② λ_1, λ_2 real, $\lambda_1 \times \lambda_2 > 0$ eg $|\lambda_1|\,|\lambda_2|$, $\cancel{\lambda_1}$

$\lambda_i < 0$



nodes

$\lambda_i > 0$



③ λ_1, λ_2 complex conjugates

$Re(\lambda_1) < 0$  stable spiral $Re(\lambda_1) > 0$  unstable spiral

$Re(\lambda_1) = 0$  ← centre

Differential Equations 22

General Non-Linear ODEs

In general, a 2nd order ODE can be written

$\dot{x} = f(x, y, t)$ $\qquad$ $\dot{y} = f(x, y, t)$

An autonomous system of equations can be written

$\dot{x} = f(x, y)$ $\qquad$ $\dot{y} = f(x, y)$

so the independent variable does not appear explicitly.

An n^{th} order, non autonomous system can be converted into an $(n+1)^{th}$ order autonomous system by treating the former independent variable as a dependent variable, e.g. write $z = t$

$\dot{x} = f(x, y, z)$ $\qquad$ $\dot{y} = f(x, y, z)$, $\quad \dot{z} = 1 = h(x, y, z)$

Equilibrium (Fixed points) for 2nd order autonomous systems

$\dot{x} = \dot{y} = 0$

$f(x_0, y_0) = g(x_0, y_0) = 0$, solve simultaneously

Stability Write $x = x_0 + \alpha$ $\qquad y = y_0 + \beta$

Substitute to find $\dot{\alpha} = f(x_0 + \alpha, y_0 + \beta)$ if $\alpha, \beta \ll 1$

$$\dot{\alpha} = f(x_0, y_0) + \alpha f_x(x_0, y_0) + \beta f_y(x_0, y_0) + O(\alpha^2, \beta^2)$$

Similarly for $\dot{\beta}$ $\Rightarrow \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ as $f(x_0, y_0) = 0$, $g(x_0, y_0) = 0$

Example Population dynamics: Predators - Prey

Prey: $\dot{x} = Ax - Bx^2 - Cxy$

births - deaths $\qquad$ mutual competition $\qquad$ killed by predators

Predators: $\dot{y} = -\delta y + \epsilon xy$

e.g. $\dot{x} = 8x - 2x^2 - 2xy$

$\dot{y} = -y + xy$

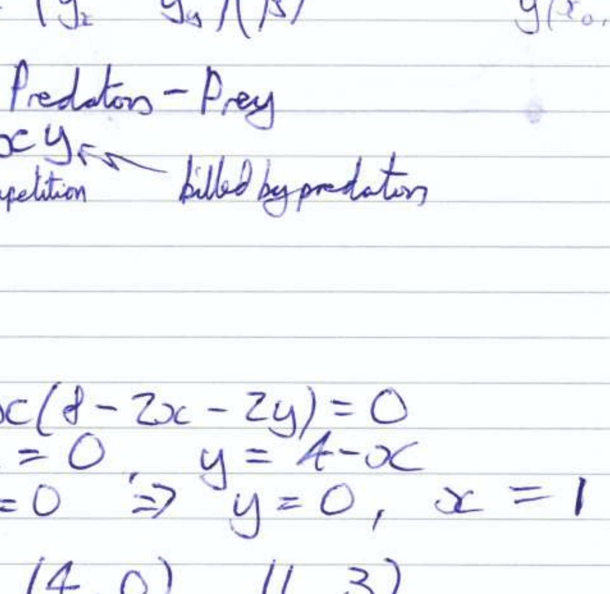
Fixed Points $\quad \dot{x} = 0 \Rightarrow x(8 - 2x - 2y) = 0$

$\Rightarrow x = 0, \quad y = 4 - x$

$\dot{y} = 0 \Rightarrow y(x - 1) = 0 \Rightarrow y = 0, \quad x = 1$

Fixed points at $(0, 0)$, $(4, 0)$, $(1, 3)$

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$



Near $(4,0)$ $x = 4+\alpha$, $y = \beta$ $\qquad \begin{pmatrix}\dot\alpha\\ \dot\beta\end{pmatrix} = \begin{pmatrix}-8 & -8\\ 0 & 3\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$$\dot\alpha = (4+\alpha)(8-8-2\alpha-2\beta) \approx -8\alpha - 8\beta$$
$$\dot\beta = \beta(3+\alpha) \approx 3\beta$$

eigenvalues $-8, 3$

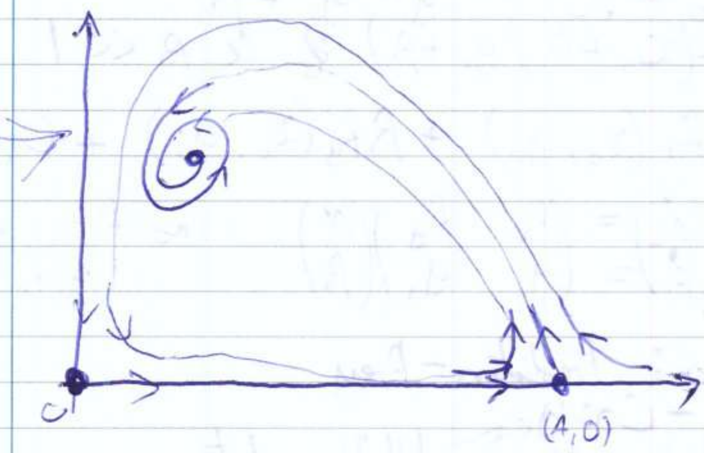
eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}8\\-11\end{pmatrix}$

Phase portrait around $(4,0)$

Near $(1,3)$ write $x = 1+\alpha$, $y = 3+\beta$

$\begin{pmatrix}\dot\alpha\\ \dot\beta\end{pmatrix} = \begin{pmatrix}-2 & -2\\ 3 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ $\qquad$ Characteristic polynomial $\lambda^2 + 2\lambda + 6 = 0$

(1,3)



Lecture 23

Partial Differential Equations – Hyperbolic (wave) equations

order for $y(x,t)$ $\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x}$, $\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0$ unforced

Recall that along a path $x = x(t)$, $\frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{dx}{dt} + \frac{\partial y}{\partial t}\frac{dt}{dt} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\frac{dx}{dt}$

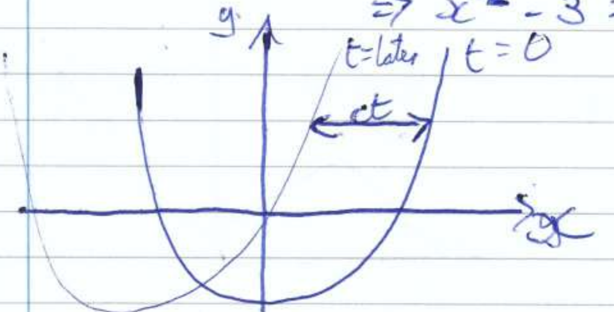
Choose to travel along a particular path defined by $\frac{dx}{dt} = -c$. Then along that path, $\frac{dy}{dt} = 0$. This method converts a PDE into several ODEs. The path is defined by $x = -ct + x_0$, $x + ct = x_0$ (constant)
Along the path, $y = A$, (constant).

There is a function $f(x_0)$ that determines the value of y on each path so $y = f(x_0) = f(x+ct)$. This is the general solution of the partial differential equation.

Usually, initial conditions are given eg. $\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x}$ with $y(x,0) = x^2 - 3$

$\Rightarrow y = f(x+ct)$

$\Rightarrow x^2 - 3 = f(x)$ $\Rightarrow$ $\underline{f(x+ct) = (x+ct)^2 - 3}$



wave velocity $= -c$

E.g. $\frac{\partial y}{\partial t} + 5\frac{\partial y}{\partial x} = e^{-t}$ $\qquad y(x,0) = e^{-x^2}$

The "characteristic equation" defining the paths or "characteristics" of the PDE is $\frac{dx}{dt} = 5 \Rightarrow x = 5t + x_0 \Rightarrow x_0 = x - 5t$
Along these paths, $\frac{dy}{dt} = e^{-t}$, $y = A - e^{-t}$

At $t=0$, $y = A - 1$, $x_0 = x$, $A - 1 = e^{-x_0^2}$, $A = 1 + e^{-x_0^2}$
$\Rightarrow y = (1 + e^{-x_0^2}) - e^{-t}$

$\Rightarrow y = 1 + e^{-(x-5t)^2} - e^{-t}$

Second Order Wave equations

$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ (mass x acceleration $\propto$ curvature)

$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0$, $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)y = 0$
operators commute as the coefficients are constant, so $y = f(x+ct)$ is a soln and $y = g(x-ct)$ is also a solution. The equation is linear so solutions can be superposed.

$y = f(x+ct) + g(x-ct)$

Exercise Show that $\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = -4c^2 \frac{\partial^2 y}{\partial \alpha \partial \beta}$ $\qquad \alpha = x+ct$, $\beta = x-ct$

Hence $\frac{\partial^2 y}{\partial \alpha \partial \beta} = 0$, $\frac{\partial y}{\partial \beta} = h(\alpha)$

$$\Rightarrow y = f(\alpha) + g(\beta) = f(x+ct) + g(x-ct) \qquad (f' = h)$$

Example $y = \frac{1}{1+x^2}$, $\frac{\partial y}{\partial t} = 0$ at $t = 0$

$y \to 0$ as $x \to \pm\infty$

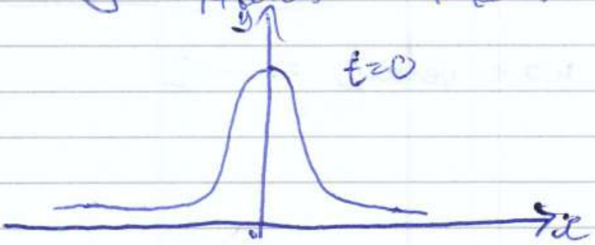
Therefore, at time $t=0$ we have $f(x) + g(x) = \frac{1}{1+x^2}$

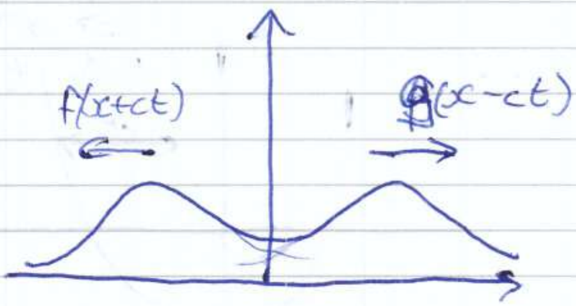
$$cf'(x) - cg'(x) = 0 \Rightarrow f' = g'$$
$$\Rightarrow f = g + \text{constant}$$

$g \neq 0$ by applying $y \to 0$ as $x \to \pm\infty$

$f = g = \frac{\frac{1}{2}}{1+x^2} \Rightarrow y = f(x+ct) + g(x-ct)$

$$y = \frac{\frac{1}{2}}{1+(x+ct)^2} + \frac{\frac{1}{2}}{1+(x-ct)^2}$$





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Differential Equations (24)

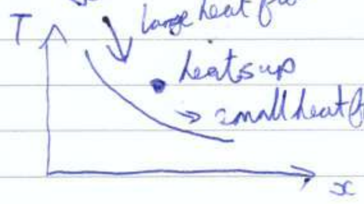
Hyperbolic Equations $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + F$

$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = F$ Mnemonic $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ No connection but similar appearance

Elliptic Equations Laplace's Equation $F=0$ $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 y}{\partial x^2} = F$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Poisson's Equation $F \neq 0$

Parabolic Equation - Diffusion Equation $\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$ $y = ax^2$

Note that heat flux $\propto -\frac{\partial T}{\partial x}$



$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$
where $T(x,t)$ is temperature and K is called diffusivity

Example An infinitely long bar petered at one end

$x=0$ $T(x,t)$ $x \to \infty$

Suppose $T(x,0) = 0$ $T(0,t) = H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$
There is a similarity solution of the differential equation in which $T(x,t) = \theta(\eta)$ where $\eta = \frac{x}{2\sqrt{Kt}}$

Then $\frac{\partial T}{\partial t} = \frac{d\theta}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{x}{4\sqrt{K}\,t^{\frac{3}{2}}}\theta'(\eta) = -\frac{\eta}{2t}\theta'(\eta)$

$\frac{\partial T}{\partial x} = \frac{d\theta}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{1}{2\sqrt{Kt}}\theta'(\eta)$, $\frac{\partial^2 T}{\partial x^2} = \frac{1}{2\sqrt{Kt}}\theta''(\eta)\frac{\partial\eta}{\partial x} = \frac{1}{4Kt}\theta''(\eta)$

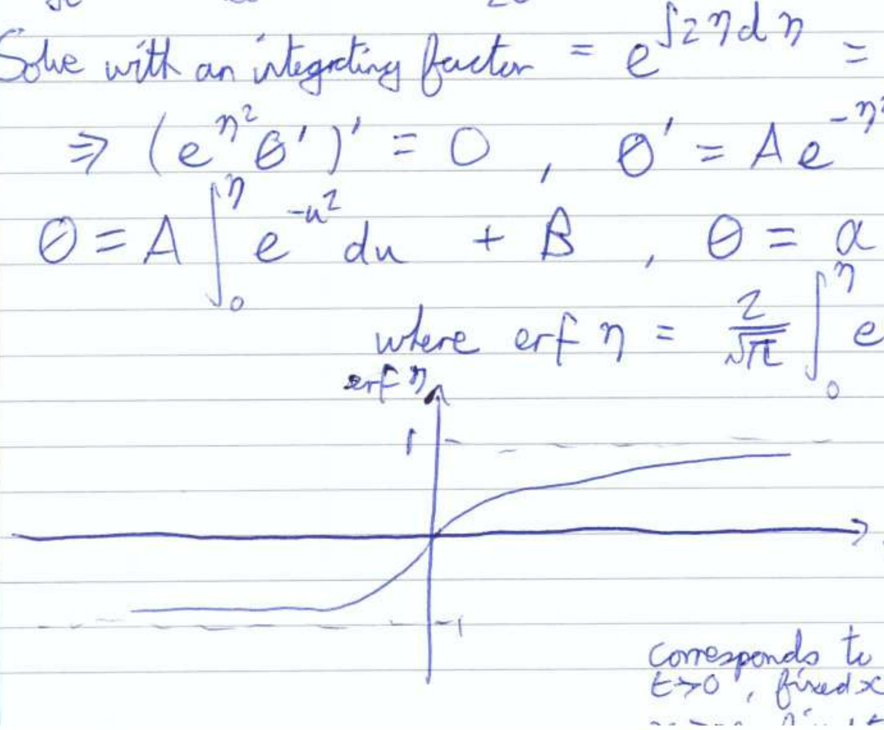
$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} \Rightarrow -\frac{\eta}{2t}\theta' = \frac{K}{4Kt}\theta'' \Rightarrow \theta'' + 2\eta\theta' = 0$

Solve with an integrating factor $= e^{\int 2\eta\, d\eta} = e^{\eta^2}$

$\Rightarrow (e^{\eta^2}\theta')' = 0$, $\theta' = Ae^{-\eta^2}$

$\theta = A\int_0^{\eta} e^{-u^2}\,du + B$, $\theta = \alpha \operatorname{erf}\eta + B$

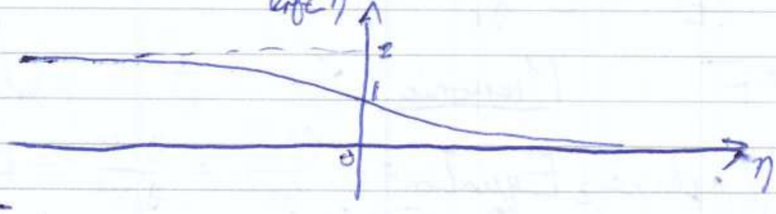
where $\operatorname{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-u^2}\,du \to 1$ as $\eta \to \infty$



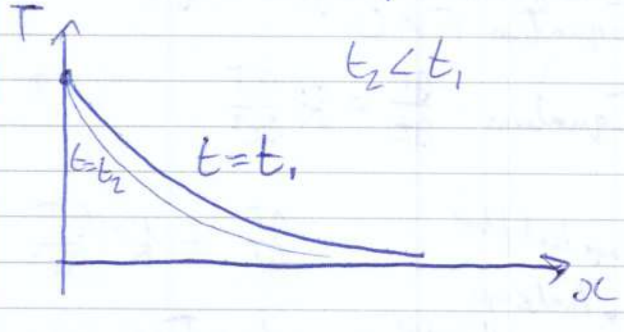
$\theta(0) = 1$
$\Rightarrow B = 1$
$\theta(\infty) = 0 \Rightarrow \alpha = -1$

corresponds to $t \to 0$, fixed x
$x \to \infty$, fixed t

$$\Rightarrow \Theta = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)$$



$$T = \mathrm{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)$$



The solutions at all times are similar, they have the same functional form but have a scale in the x direction that depends on time. The decay length is proportional to $\sqrt{\kappa t}$