Ramsey Theory ①

"Can we find some order inside sufficient disorder?"

Chapters

1. Monochromatic Systems
2. Partition Regular Equations
3. Infinite Ramsey Theory

Books

1. Bollobás, "Combinatorics", CUP 1986, for Chapter 3
2. Graham, Rothschild, Spencer, "Ramsey Theory", Wiley 1990

For chapters 1 and 2.

Chapter 1 : Monochromatic Systems

Write $\mathbb{N} = \{1, 2, 3, \dots\}$.

For a set X, we write $X^{(r)}$ for $\{A \subset X : |A| = r\}$

Given a 2-colouring of $\mathbb{N}^{(2)}$ (i.e. a function $c : \mathbb{N}^{(2)} \to \{1, 2\}$), can we always find an infinite monochromatic set M?

(i.e. infinite $M \subset \mathbb{N}$ such that c is constant on $M^{(2)}$)

We write ij for $\{i, j\}$, the edge from i to j, where $i < j$.

Examples

1. Colour ij is red / blue if $i + j$ is even / odd.

   Then $M = \{2, 4, 6, \dots\}$ is monochromatic (red).

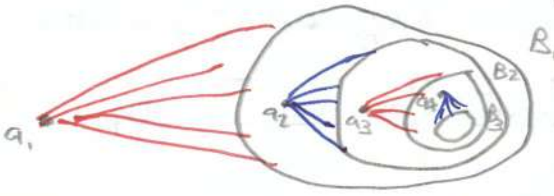
2. Colour ij is red / blue if $\max\{n : 2^n \mid (i+j)\}$ is even / ~~blue~~ odd.

   Then $M = \{4^0, 4^1, 4^2, \dots\}$ is monochromatic.

3. Colour ij red/blue if $i+j$ has an even/odd number of distinct prime factors. No examples of M are known. However...

<u>Theorem 1</u> (Ramsey's Theorem)

Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exists an infinite monochromatic set.



<u>Proof</u>

Choose $a_1 \in \mathbb{N}$ (any will do). There are infinitely many edges from a_1, so infinitely many are the same colour (pigeonhole principle), so there is a set B_1 such that all edges from a_1 to B_1 are colour c_1, and B_1 is infinite.

Choose any $a_2 \in B_1$.

There are infinitely many edges from a_2 to $B_1 \setminus \{a_2\}$, so again, there exists infinite $B_2 \subset B_1 \setminus \{a_2\}$ and c_2 such that all edges from a_2 to B_2 have colour c_2.

Continue inductively.

We obtain $a_1, a_2, a_3, \ldots$ and colours $c_1, c_2, c_3, \ldots$ such that $\forall i$, all edges $a_i a_j$ $(i < j)$ have colour c_i.

But then, infinitely many of the c_i are the same, say $c_{i_1} = c_{i_2} = c_{i_3} = \ldots$. Then $\{a_{i_1}, a_{i_2}, a_{i_3}, \ldots\}$ is monochromatic. $\square$

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Remarks

1. Called a '2-pass' proof.
2. The same proof shows that whenever $\mathbb{N}^{(2)}$ is k-coloured (i.e. we have $c : \mathbb{N}^{(2)} \to [k]$ so there exists an infinite monochromatic se

Alternatively, view the colours as '1' and '2 or 3 or ... or k' and apply theorem 1, getting an infinite set of colour 1 (done) or with colours $2, 3, \dots, k$, so we are done by induction.

3. Having an infinite monochromatic set is stronger than asking for arbitrarily large finite monochromatic set

← arbitrarily large, but not infinite, blue sets

Example

Any sequence $x_1, x_2, \dots$ in $\mathbb{R}$ (or any totally ordered set) has a monotone subsequence. Indeed, 2-colour $\mathbb{N}^{(2)}$ by giving ij colour $\begin{array}{l}\text{up}\\ \text{down}\end{array}$ if $\begin{array}{l}x_i < x_j\\ x_i \geqslant x_j\end{array}$ and apply theorem 1.

What about $\mathbb{N}^{(r)}$, $r = 3, 4, \dots$? If we 2-colour $\mathbb{N}^{(r)}$, do we get an infinite monochromatic set?

e.g. $r = 3$: colour $\mathbb{N}^{(3)}$ by giving ijk ($i<j<k$) colour $\begin{array}{l}\text{red}\\ \text{blue}\end{array}$ if $\begin{array}{l}i \mid (j+k)\\ i \nmid (j+k)\end{array}$.

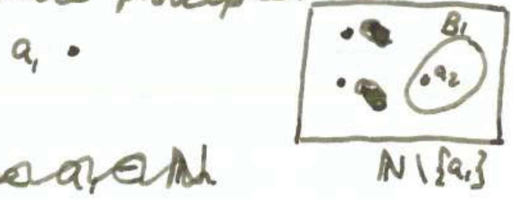
We could take $M = \{2, 4, 8, 16, \dots\}$

Theorem 2 (Ramsey for r-sets)

Whenever $\mathbb{N}^{(r)}$ is 2-coloured, there exists an infinite monochromatic set.

Proof

By induction on r: $r=1$ is the Pigeonhole principle. $r=2$ is Theorem 1.



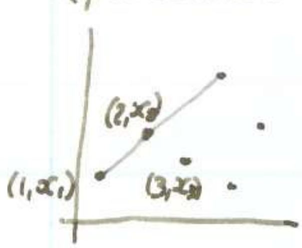
Given a 2-colouring c of $\mathbb{N}^{(r)}$, ~~choose $a_1 \in \mathbb{N}$~~

Choose $a_1 \in \mathbb{N}$. We induce a 2-colouring of $(\mathbb{N}\setminus\{a\})^{(r-1)}$ by $c'(F) = c(F \cup \{a_1\})$. By induction, we have an infinite monochromatic B_1 for this colouring. So all r-sets $F \cup \{a_1\}$, $F \subset B_1$ have the same colouring, c_1 say. Choose $a_2 \in B_1$. By the same argument, there exists infinite monochromatic $B_2 \subset B_1 \setminus \{a_2\}$ such that all r-sets $F \cup \{a_2\}$, $F \subset B_2$ have the same colour, c_2 say. Continue inductively. We obtain points $a_1, a_2, \dots$ such that each r-set $\{a_{i_1}, \dots, a_{i_r}\}$ $(i_1 < \dots < i_r)$ has colour c_{i_r}.

But we have $c_{i_1} = c_{i_2} = \dots$ for some subsequence (by the Pigeonhole principle), whence $M = \{a_{i_1}, a_{i_2}, \dots\}$ $\square$

Example

We saw that given given points $(1, x_1)$, $(2, x_2)$, $(3, x_3)$ we can find a subsequence such that the induced (piecewise-linear) function is monotone.



In fact, we can insist that the induced function is convex or concave. Indeed, 2-colour $\mathbb{N}^{(3)}$ by giving ijk colour concave if  or convex if  and apply theorem 2.

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Surprisingly, Infinite Ramsey (Theorem 2) implies the finite version:

Theorem 3

$\forall m, r$, $\exists N$ such that whenever $[n]^{(r)}$ is 2-coloured, there exists a monochromatic set of size m.

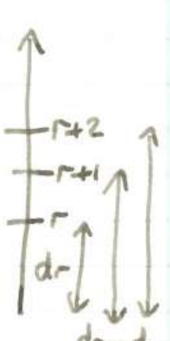
Proof:

Suppose not, so that $\forall n \geq r$, there exists a 2-colouring c_n of $[n]^{(r)}$ without a monochromatic set of size m. We will construct a 2-colouring of $\mathbb{N}^{(r)}$ without a monochromatic set of size m, contradicting Theorem 2 (very strongly).

[If the c_n are nested i.e. $c_n|_{[n-1]^{(r)}} = c_{n-1}$ we can take the union, but they may not be ...]

There are only finitely many ways to 2-colour $[r]^{(r)}$ (2 ways).
So infinitely many of the c_n agree on $[r]^{(r)}$, say $c_n|_{[r]^{(r)}} = d_r$, $\forall n \in B_1$.

There are only finitely many ways to 2-colour $[r+1]^{(r)}$, so infinitely many of the c_n, $n \in B_1$, agree on $[r+1]^{(r)}$; say $c_n|_{[r+1]^{(r)}} = d_{r+1}$ $\forall n \in B_2$.



Continue inductively. We obtain $d_r, d_{r+1}, \ldots$ where $d_r : [n]^{(r)} \to \{1, 2\}$ such that:

i) The d_i are nested

ii) No d_n has a monochromatic m-set (as $d_n = c_{n'}|_{[n]^{(r)}}$, some n')

Now define $c : \mathbb{N}^{(r)} \to (1,2)$ by setting $c(F) = d_n(F)$ for any $n \geq \max F$. Then c has no mono set of size m ※ □

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<u>Remarks</u>

1. The proof gives no bounds on what $n = n(m,r)$ we could take. There <u>are</u> direct proofs, that do give upper bounds.
2. Called a 'compactness argument'. Essentially, we are proving that the space $\{0,1\}^{\mathbb{N}}$ (all 0-1 sequences) with the product topology (i.e. metric $d(f,g) = \frac{1}{\min\{n : f(n) \neq g(n)\}}$) is (sequentially) compact.

What if we coloured $\mathbb{N}^{(2)}$ with infinitely many colours – i.e. we have $c : \mathbb{N}^{(2)} \to X$ for some set X.

Obviously we cannot find an infinite M on which c is constant, e.g. let c be injective (give every edge a different colour).

Can we always find infinite M such that c is either constant on $M^{(2)}$ or injective?

No, colour every edge ij $(i<j)$ colour c_i.

<u>Theorem 4</u> (Canonical Ramsey Theorem)

Let $c : \mathbb{N}^{(2)} \to X$ for some set X. Then $\exists$ infinite $M \subset \mathbb{N}$ such that one of the following holds:

i) c constant on $M^{(2)}$

ii) c injective on $M^{(2)}$

iii) $c(ij) = c(kl) \Leftrightarrow i = k$ $(i,j,k,l \in M,\ i<j,\ k<l)$

iv) $c(ij) = c(kl) \Leftrightarrow j = l$ $(i,j,k,l \in M,\ i<j,\ k<l)$

<u>Note</u>

This generalises Theorem 1. For finite X then ii), iii), iv) cannot arise.

Proof $\qquad i<j \quad k<l$

2-colour $\mathbb{N}^{(4)}$ by giving $ijkl$ colour <u>same</u> if $c(ij)=c(kl)$ and <u>diff</u> if not. By Ramsey for 4-sets (Theorem 2), $\exists$ infinite monochromatic M_1 for this colouring.

<u>If M_1 is colour same:</u>

For any ij and kl in $M^{(2)}$, choose $mn \in M_1^{(2)}$ with $m > j, l$. Then $c(ij) = c(mn)$ and $c(kl) = c(mn)$. Therefore $c(ij) = c(kl)$ so c is constant on $M_1^{(2)}$, case i).

So we may assume that M_1 has colour <u>diff</u>.

Now 2-colour $M_1^{(4)}$ by giving $ijkl$ colour <u>same</u> if $c(il) = c(jk)$ or <u>diff</u> if not.

By Theorem 2, we have infinite monochromatic $M_2 \subset M_1$ (monochromatic for this new colouring).

<u>If M_2 has colour same:</u>

Choose $i<j<k<l<m<n$ in M_2. Then $c(jk) = c(in)$ and $c(lm) = c(in)$, whence $c(jk) = c(lm)$ $\times\!\!\!\times$ since $M_2 \subset M_1$. Thus M_2 is colour <u>diff</u>.

2-colour $M_2^{(4)}$ by giving $ijkl$ colour <u>same</u> if $c(ik) = c(jl)$ and <u>diff</u> otherwise.

We have infinite monochromatic $M_3 \subset M_2$ for this colouring.

If M_3 has colour <u>same</u>, choose $i<j<k<l<m<n$ in M_3. Then $c(il) = c(jn)$ and $c(il) = c(km)$, so $c(jn) = c(km)$ $\times\!\!\!\times$ So M_3 is colour <u>diff</u>.

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2-colour $M_3^{(3)}$ by giving ijk colour <u>same</u> if $c(ij) = c(jk)$ and <u>diff</u> if not.

We have infinite monochromatic $M_4 \subset M_3$ for this colouring.

<u>If M_4 has colour same :</u>

Choose $i<j<k<l$ in M_4. Then $c(ij) = c(jk) = c(kl)$ ※

So M_4 is colour <u>diff</u>.

Now 2-colour $M_4^{(3)}$ by giving ijk colour <u>left same</u> if $c(ij) = c(ik)$, <u>left-diff</u> if not.

We get infinite monochromatic $M_5 \subset M_4$ for this.

Then 2-colour $M_5^{(3)}$ by giving ijk colour <u>right same</u> if $c(jk) = c(ik)$, <u>right diff</u> if not. Infinite mono M_6 for this.

<u>If M_6 left-same right-diff :</u> Case ii)

<u>If M_6 left-same right-diff :</u> Case iii)

<u>If M_6 left-diff right-same :</u> Case iv)

<u>If M_6 left-same right same :</u>

Choose $i<j<k$ in M_6. Then $c(ij) = c(ik) = c(jk)$ ※ □

<u>Remarks</u>

1. We could use just one colouring according to the pattern of colourings of the 2-sets inside the given four-set.
2. For <u>any</u> r, we can show similarly :

For any colouring c of $\mathbb{N}^{(r)}$, $\exists$ infinite monochromatic $M \subset \mathbb{N}$ and $I \subset [r]$ such that $\forall\, i_1 < \dots < i_r$ and $j_1 < \dots < j_r$ in M,

$c(i_1 \dots i_r) = c(j_1 \dots j_r) \iff i_n = j_n \ \forall n \in I$

These 2^r colourings are called the <u>canonical</u> colourings of $\mathbb{N}^{(r)}$.

e.g. $r = 2$, $I = \{1\}$ is case iii)

$I = \{2\}$ is case iv)

$I = \{1, 2\}$ is case ii)

$I = \emptyset$ is case i)

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What if we coloured $\mathbb{N}^{(2)}$ with infinitely many colours?

i.e. we have $c : \mathbb{N}^{(2)} \to X$ for some set X.

Obviously, we cannot find an infinite M on which c is constant.

e.g. let c be injective (give every edge a different colour).

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No, colour every edge ij $(i<j)$ colour c_i.

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Let $c : \mathbb{N}^{(2)} \to X$ for some set X. Then $\exists$ infinite $M \subset \mathbb{N}$ such that one of the following holds:

i) c constant on $M^{(2)}$

ii) c injective on $M^{(2)}$

iii) $c(ij) = c(kl) \Leftrightarrow i = k$ $(i,j,k,l \in M,\ i<j,\ k<l)$

iv) $c(ij) = c(kl) \Leftrightarrow j = l$ $(i,j,k,l \in M,\ i<j,\ k<l)$

Note

This generalises Theorem 1. For finite then ii), iii), iv) cannot arrive.

Proof $\qquad$ $i<j$, $k<l$

2-colour $\mathbb{N}^{(4)}$ by giving $ijkl$ colour same if $c(ij) = c(kl)$ and diff if not. By Ramsey for 4-sets (Theorem 2), $\exists$ infinite monochromatic M_1 for this colouring.

If M_1 is colour same :

For any ij, kl in $M_1^{(2)}$, choose $mn \in M_1^{(2)}$ with $m > j, l$. Then $c(ij) = c(mn)$ and $c(kl) = c(mn)$. Therefore $c(ij) = c(kl)$ so c is constant on $M_1^{(2)}$, case i).

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If M_2 has colour same :

Choose $i<j<k<l<m<n$ in M_2. Then $c(jk) = c(in)$ and $c(lm) = c(in)$, whence $c(jk) = c(lm)$ $\divideontimes$ since $M_2 \subset M_1$. Thus M_2 is colour diff.

2-colour $M_2^{(4)}$ by giving $ijkl$ colour same if $c(ik) = c(jl)$ and diff otherwise.

We have infinite monochromatic $M_3 \subset M_2$ for this colouring.

If M_3 has colour same, choose $i<j<k<l<m<n$ in M_3. Then $c(il) = c(jn)$, and $c(il) = c(km)$, so $c(jn) \underset{\text{since } M_3 \subset M_2}{=} c(km)$ $\divideontimes$ So M_3 is colour diff.

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<u>Van der Waerden's Theorem</u>

We aim to prove that whenever $\mathbb{N}$ is 2-coloured, there exists a monochromatic arithmetic progression of length m, for any m.

e.g. $\{a, a+d, \dots, a+(m-1)d\}$, length m (m members of set)

By Compactness, this is the same as:

$\forall m\ \exists n$ such that when $[n]$ is 2-coloured $\Rightarrow \exists$ monochromatic arithmetic progression of length m.

Indeed, if not, then $\forall n\ \exists$ a 2-colouring c_n of $[n]$ without a monochromatic arithmetic progression of length m. We have infinitely many c_n agreeing on $[1]$, and of these, infinitely many agree on $[2]$. Continuing, we obtain a 2-colouring of $\mathbb{N}$ without a monochromatic arithmetic progression of length m.

One key idea in the proof is to show that $\forall m, k, \exists n$ such that $[n]$ k-coloured contains a monochromatic arithmetic progression of length m.

(A harder result could be easier to prove, if the proof is by induction)

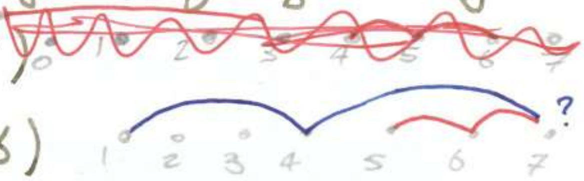
Write $W(m,k)$ for the least such n (if it exists). This is referred to as a "van der Waerden number".

Let $A_1, \dots, A_r$ be arithmetic progressions of length $m-1$, say $A_i = \{a_i, a_i+d_i, \dots, a_i+(m-2)d_i\}$

We say that $A_1, \dots, A_r$ are <u>focussed</u> at f if $a_i + (m-1)d_i = f$ $\forall i$ e.g. $\{1,4\}$ and $\{5,6\}$ are focussed at 7.

If each A_i is monochromatic (for a given colouring) with no two

A_i the same colour, we say that $A_1, \dots, A_r$ are <u>colour-focussed</u>.
(so that if we have any r-colouring and $A_1, \dots, A_r$ are colour-focussed then we get a monochromatic arithmetic progression of length m, by asking "What colour is the focus?")



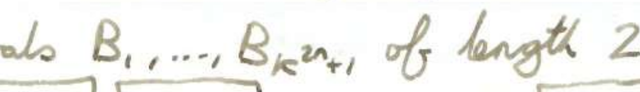
<u>Proposition 5</u> (contained within Theorem 6)
$\forall k$, $\exists n$ such that whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length 3.

<u>Proof</u> $\to 1 \leq r \leq k$

We claim $\forall r$ $\exists n$ such that whenever $[n]$ is k-coloured, $\exists$ either
- A monochromatic arithmetic progression of length 3
  OR
- r colour-focussed arithmetic progressions of length 2.

(Then we are done by setting $r = k$ and looking at the focus)
Proof by induction on r: ($r = 1$ is easy, $n = k+1$)
We will show that if n is suitable for $r-1$, then $(k^{2n}+1)2n$ is suitable for r.
Indeed, given a k-colouring of $[(k^{2n}+1)2n]$ with no monochromatic arithmetic progression of length 3, we break $[(k^{2n}+1)2n]$ into intervals $B_1, \dots, B_{k^{2n}+1}$ of length $2n$: $B_i = [2n(i-1)+1, 2ni]$

Now, there are k^{2n} ways to k-colour a block. There are $k^{2n}+1$ blocks. Hence WLOG B_s and B_{s+t} are coloured identically.



By choice of n, inside B_s, we have $r-1$ colour-focussed arithmetic progressions of length 2, together with their focus.

say $\{a_1, a_1+d_1\}, \ldots, \{a_{r-1}, a_{r-1}+d_{r-1}\}$ focussed at f. But now $\{a_i, a_i+d_i+2nt\}$, $1 \le i \le r-1$ are colour focussed at $f+4nt$ and also $\{f, f+2nt\}$ is monochromatic of a different colour. This gives r colour focussed arithmetic progressions of length 2. $\square$

Remark

1. The idea of looking at "patterns of whole blocks" is called a product argument.
2. The proof gives bounds of the form $W(3,k) \le \underbrace{k^{k^{k^{\cdot^{\cdot^{\cdot^{k^{4k}}}}}}}}_{k \text{ terms}}$
   This is called a "Tower-type" bound.

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Theorem 6 (van der Waerden's Theorem)

$\forall m, k$, $\exists n$ such that whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length m.

Proof

Induction on m (for all k). $m=1$ is trivial (or $m=2$ is true by the Pigeonhole principle, or $m=3$ is Proposition 5).

Given m, we may assume that $W(m-1, k)$ exists for all k.

We claim that $\forall 1 \le r \le k$ $\exists n$ such that whenever $[n]$ is k-coloured we have either:

i) A monochromatic arithmetic progression of length m, or

ii) r colour-focussed arithmetic progressions of length $m-1$.

(Then we are done by setting $r=k$ and looking at the focus)

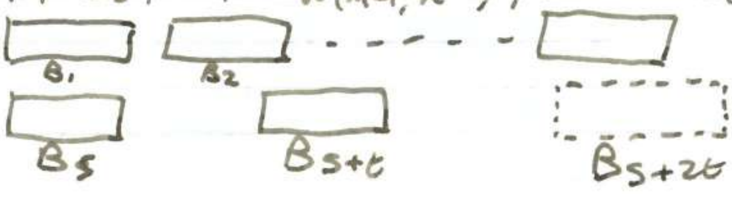
Proof of claim: (induction on r)

$r=1$ is true (taking $n = W(m-1, k)$)

Given n suitable for $r-1$, we will show that $W(m-1, k^{2n})2n$ is suitable for r.

Indeed, given a k-colouring of $[W(m-1, k^{2n})2n]$ without a monochromatic arithmetic progression of length m, we break $[W(m-1, k^{2n})2n]$ up into blocks of length $2n$:

$B_1, B_2, \ldots, B_{W(m-1,k^{2n})}$, where $B_i = [2n(i-1)+1, 2ni]$



Each block may be coloured in k^{2n} ways, so by definition of $W(m-1, k^{2n})$, we have $m-1$ equally spaced blocks with identical colourings; say blocks $B_s, B_{s+t}, \dots, B_{s+(m-2)t}$.
Inside B_s, we have $r-1$ colour-focussed arithmetic progressions of length $m-1$ (by choice of n). Together with their focus (as the blocks are length $2n$), progressions $A_1, \dots, A_{r-1}$ focussed at f, where
$A_i = \{a_i, a_i+d_i, \dots, a_i+(m-2)d_i\}$
But then $\{a_i, a_i+(d_i+2nt), \dots, a_i+(n-2)(d_i+2nt)\}, 1 \leq i \leq r-1$
are colour-focussed at $f+(m-1)2nt$.
Also, $\{f, f+2nt, \dots, f+(n-2)2nt\}$ is monochromatic, of a different colour, so we have r arithmetic progressions of length $m-1$, colour-focussed at $f+(m-1)2nt$. $\square$

The Ackerman or Grzegorczyk hierarchy is the sequence of functions $f_1, f_2, \dots$ (each $\mathbb{N} \to \mathbb{N}$) defined by:
$f_1(x) = 2x$
$f_{n+1}(x) = f_n^{(x)}(1) = \overbrace{f_n(f_n(\dots(f_n(1))\dots)}^{x \text{ times}}$
e.g. $f_2(x) = 2^x$
$f_3(x) = 2^{2^{\cdot^{\cdot^{2}}}}$ height x
$f_4(1) = 2$, $f_4(2) = 2^2 = 4$, $f_4(3) = 65536$
$f_4(4) = 2^{2^{\cdot^{\cdot^{2}}}}$ 65536, $f_4(5) = 2^{2^{\cdot^{\cdot^{2}}}}$ $2^{2^{\cdot^{\cdot^{2}}}}$ 65536
We say that $f: \mathbb{N} \to \mathbb{N}$ is of type n if $\exists\, c, d$ with $f_n(cx) \leq f(x) \leq f_n(dx)\ \forall x$. Our bound on $W(3,k)$ is of type 3.

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For each m, our bound on $W(m,k)$ is of type m.

Then, our bound on $W(m,2) = W(m)$ grows faster than <u>every</u> f_n.

This is often a feature of such "double induction" proofs.

Shelah (1987) found a proof with induction only on m, giving a bound of $W(m,k) \leq f_4(m+k)$.

Graham offered \$1000 for proof of $W(m) \leq f_3(m) = \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{m}$

Gowers (1998) showed $W(m) \leq 2^{2^{2^{2^{2^{m+9}}}}}$

The best lower bound known is $W(m) \geq \frac{2^m}{8m}$

<u>Corollary 7</u>

Whenever $\mathbb{N}$ is coloured with finitely many colours, some colour class contains arbitrarily long arithmetic progressions. □

What about:

$\mathbb{N}$ finitely coloured $\Rightarrow$ $\exists$ an infinite monochromatic A.P. ?

This is not true, e.g.

Alternatively, list all infinite A.P.s as $A_1, A_2, \ldots$

Choose $x_1, y_1 \in A_1$ ($x_1 \neq y_1$), and name x_1 red, y_1 blue

Choose <u>new</u> points $x_2, y_2 \in A_2$ ($\neq x_1, y_1$), distinct ($x_2 \neq y_2$)

and make x_2 red, y_2 blue. Continue.

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Theorem 8 (Strengthened van der Waerden)

Let $m \in \mathbb{N}$. Then whenever $\mathbb{N}$ is finitely coloured, there exists an arithmetic progression that, together with its common difference, is monochromatic.

Proof (by induction on k, the number of colours)

Given a suitable n for $k-1$ (i.e. n such that whenever $[n]$ is $k-1$ coloured $\exists$ a monochromatic AP + common difference of length m), we will show that $W(n(m-1)+1, k)$ is suitable for k.

Given a k-colouring of $[W(n(m-1)+1, k)]$, we have a monochromatic arithmetic progression of length $n(m-1)+1$, say

$$a,\ a+d,\ a+2d,\ \dots,\ a+n(m-1)d \text{ is red.}$$

$j = \frac{1728\, 4a^3}{4a^3+27b^2}$

If $\underset{\text{common difference}}{d}$ is red, we are done. Similarly, if $\exists\ 1 \le r \le n$

$4a^3 = \frac{1728-j}{j}\, 27b^2$

with $\underset{\text{common difference}}{rd}$ red, we are done. (First term a)

So WLOG, $[d, 2d, \dots, nd]$ is $(k-1)$-coloured, so we are done by induction. $\square$

$\left(\frac{a}{a'}\right)^3 = \left(\frac{b}{b'}\right)^2$

Remarks

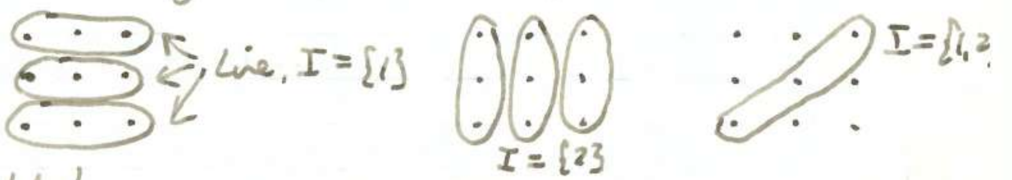
1. Henceforth, we do not care about bounds.
2. Case $m = 2$ is Schur's Theorem:

WNFC (whenever $\mathbb{N}$ is finitely coloured) $\exists\ x, y, z$ monochromatic with $x+y=z$. This can also be deduced from Ramsey's Theorem directly (exercise).

The Hales - Jewett Theorem

Let X be a finite set. A subset L of X^n (the n-dimensional cube on alphabet X) is a line or combinatorial line if
$\exists\ I \subset [n],\ I \neq \emptyset$ and $a_i \in X$, each $i \in [n] \setminus I$ such that
$$L = \{ x = (x_1, \dots, x_n) \in X^n : x_i = a_i\ \forall i \notin I,\ x_i = x_j\ \forall i, j \in I \}$$
We say that I is the set of 'active coordinates'.

e.g. in $[3]^2$



In $[3]^3$, we could have

$\{ (1,1,1), (2,2,1), (3,3,1) \}$, $I = \{1,2\}$

$\{ (1,1,1), (2,2,2), (3,3,3) \}$, $I = \{1,2,3\}$

$\{ (2,3,1), (2,3,2), (2,3,3) \}$, $I = \{3\}$

Note

The definition is unchanged if we permute X.

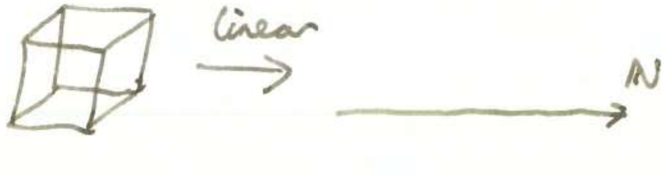
Theorem 9 (Hales - Jewett Theorem)

$\forall k, m$, $\exists n$ such that whenever $[m]^n$ is k-coloured, there is a monochromatic line.

Notes

1. The smallest such n is denoted $HJ(m, k)$.
2. So m-in a row, noughts and crosses, played in enough dimensions, cannot end in a draw. (exercise : player 1 wins)
3. Hales - Jewett $\Rightarrow$ van der Waerden

$[m]^n$



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Ramsay Theory (6)

Indeed, given a k-colouring c of $\mathbb{N}$, induce a k-colouring of $[m]^n$ (n large) by $c'((x_1, \ldots, x_n)) = c(x_1 + \ldots + x_n)$

We have a monochromatic line for c' (n large enough).

Giving a monochromatic A.P in $\mathbb{N}$ of length m

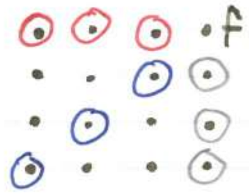
(common difference = # active coordinates)

For a line L in $[m]^n$, write L^- and L^+ for its first and last points (in the ordering $x \leq y$ if $x_i \leq y_i$ $\forall i$)

We say that lines $L_1, \ldots, L_r$ are <u>focused</u> at f if $L_i^+ = f$ $\forall i$

We say that they are <u>colour-focused</u> (for a given colouring) if in addition, each $L_i \setminus \{L_i^+\}$ is monochromatic, and no two are the same colour.

e.g. $[4]^2$



$\exists I \subset [n]$, $I \neq \emptyset$, $a_i \in X$ for $i \in [n] \setminus I$

such that

$$L = \{ x = (x_1, \dots, x_n) \in X^n : x_i = a_i \ \forall i \notin I$$
$$x_i = x_j \ \forall i, j \in I \}$$

I : active coordinates

For $I = \{i_1, \dots, i_n\}$, $k = x_{i_1} = x_{i_2} = \dots = x_{i_n}$, but the value k varies over points x.

For $i \notin I$, $x_i = a_i$, constant over all points x.

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Ramsey Theory (7)

<u>Proof</u> (of Theorem 9)

Induction on m. $m=1$ trivial.

Given $m>1$, we may assume that $HJ(m-1,k)$ exists $\forall k$.

Claim: $\forall\, 1 \le r \le k$ $\exists n$ such that $[m]^n$ k-coloured gives

i) A monochromatic line

OR

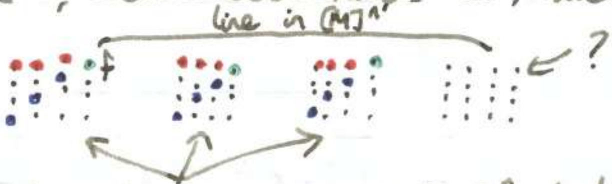
ii) r colour focussed lines

(Then we are done by setting $r=k$ and looking at the focus)

We prove the claim by induction on r. $r=1$ is done, taking $n = HJ(m-1,k)$. Given n suitable for $r-1$, we'll show that $n + HJ(m-1, k^{m^n})$ is suitable for r.

Write n' for $HJ(m-1, k^{m^n})$. Given a k-colouring c of $[m]^{n+n'}$, suppose that we have no monochromatic line.

View $[m]^{n+n'}$ as $[m]^n \times [m]^{n'}$. There are k^{m^n} ways to colour a copy of $[m]^n$, so by definition of n', $\exists$ a line L in $[m]^{n'}$, active coordinates I, such that

$[m]^n$ 

(identically coloured copies of $[m]^n$ (colouring called c'))

$\forall a \in [m]^n$. $\forall b, b' \in L \setminus [L^+]$ we have $c((a,b)) = c((a,b')) = c'(a)$ say.

By definition of n, we have $r-1$ colour focussed lines for c', say $L_1, \dots, L_{r-1}$, active coordinate sets $I_1, \dots, I_{k-1}$ focussed at f.

Let L_i' be the line in $[m]^{n+n'}$ through (L_i^-, L^-), active coordinates $I_i \cup I$. Then $L_1', \dots, L_{r-1}'$ are colour focussed at (f, L^+).

Also, the line through (f, L^-) with active coordinates I_1 is monochromatic of a different colour to the L_i'. $\square$

For $d \geq 1$, a <u>d-dimensional subspace</u> or <u>d-parameter set</u> in X^n is a subset $S \subset X^n$ such that for some disjoint, non-empty $I_1, \dots, I_d \subset [n]$ and some $a_i \in X$, each $i \in [n] \setminus (I_1 \cup \dots \cup I_d)$ we have $S = \{x \in X^n : x_i = a_i \ \forall i \notin I_1 \cup \dots \cup I_d,\ x_i = x_j \ \forall i, j \in I_k, \forall k$

e.g. in $[3]^3$:

$\{(x, y, 1) : x, y \in [3]\}$ is a 2-parameter set.

$\{(x, y, y) : x, y \in [3]\}$

<u>Theorem 10</u> (Extended Hales-Jewett theorem)

$\forall m, k, d$, $\exists n$ such that $[m]^n$ k-coloured $\Rightarrow \exists$ a monochromatic d-parameter set.

(Looks much harder than Hales-Jewett, but..)

<u>Proof</u>

View X^{dn} as $(X^d)^n$, a cube on alphabet X^d.

A line in $(X^d)^n$ (alphabet X^d) corresponds to a d-parameter set in X^{dn} (alphabet X). Then we are done by taking $n = d\,HJ(m^d, k)$. $\square$

Let S be a finite subset of $\mathbb{N}^d$.

A <u>homothetic copy</u> of S is a set of the form $a + \lambda S$ for some $a \in \mathbb{N}^d$, $\lambda \in \mathbb{N}$.

e.g. in $\mathbb{N}$, a homothetic copy of $[1, 2, \dots, m]$ is an arithmetic progression of length m.

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In $\mathbb{N}^2$, a homothetic copy of $\{1,2\} \times \{1,2\}$ is a square.

<u>Theorem 11</u> (Gallai's Theorem)

$\forall d$, $\forall$ finite $S \subset \mathbb{N}^d$, whenever $\mathbb{N}^d$ is finitely coloured, we have a monochromatic, homothetic copy of S.

<u>Proof</u>

Let $S = \{S(1), \dots, S(m)\}$.

Given a k-colouring of $\mathbb{N}^d$, induce a k-colouring of $[m]^n$ (n large) by $c'((x_1, \dots, x_n)) = c(S(x_1) + \dots + S(x_n))$

We have a monochromatic line for c' (n large), which corresponds to a monochromatic, homothetic copy of S (with $\lambda =$ # active coordinates).

<u>Remarks</u>

1. We can also prove this using focussing a a product argument.
2. For say $S = \{1,2\} \times \{1,2\}$, we applied Hales-Jewett (with $m = 4$). What if we had applied extended Hales-Jewett for 2-parameter sets with $m = 2$? We would obtain just a monochromatic rectangle when looking for a square.

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Ramsey Theory ⑧

Chapter 2 : Partition Regular Equations

Let A be an $m \times n$ matrix with rational entries. We say that A is <u>partition-regular</u> if WNFC, $\exists\, x \in \mathbb{N}^n$, monochromatic, with $Ax = 0$. (PR denotes 'Partition Regular')

- e.g. $(1 \; 1 \; -1)$ is PR : WNFC, $\exists$ monochromatic $x, y, z \in \mathbb{N}$ with $(1 \; 1 \; -1)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ i.e. $x+y=z$, Schur's Theorem
- Strengthened van der Waerden says $\begin{array}{ccccc} a & d & a+d & \cdots & a+md \\ \end{array}$ $\begin{pmatrix} 1 & 1 & -1 & & 0 \\ 1 & 2 & & -1 & \\ \vdots & \vdots & & & \ddots \\ 1 & m & 0 & & & -1 \end{pmatrix}$
- $(2 \; 3 \; -5)$ is PR ; take any $x = y = z$

What about $(2 \; 3 \; -6)$?

<u>Remarks</u>

1. A is PR $\Leftrightarrow$ λA is PR (for any $\lambda \in \mathbb{Q} \setminus \{0\}$) so if we wish, we can assume that all entries of A are integers
2. We can also speak of the 'system of equations $Ax = 0$' being partition regular.
3. Not every matrix is PR e.g. $(2 \; -1)$ is not PR

Indeed, if it were PR, we could solve $y = 2x$, x, y monochromatic, in any finite colouring, which is clearly false. For example, colour by whether $\max\{n : 2^n | x\}$ is even or odd

$(\lambda \; -1)$ PR $\Leftrightarrow \lambda = 1$

Which matrices are PR ?

Let A be an $m \times n$ rational matrix with columns $c^{(1)}, \dots, c^{(n)}$.

$$A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ c^{(1)} & c^{(2)} & \dots & c^{(n)} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} . \quad \text{Each } c^{(i)} \in \mathbb{Q}^m$$

We say that A has the <u>column property</u> if $\exists$ a partition $[n] = B_1 \cup \dots \cup B_d$ such that

i) $\sum_{i \in B^{(1)}} c^{(i)} = 0$

ii) $\sum_{i \in B^{(r)}} c^{(i)} \in \langle c^{(i)} : i \in B_1 \cup \dots \cup B_{r-1} \rangle$, $2 \le r \le d$ where $\langle \dots \rangle$ denotes linear span (say over $\mathbb{Q}$).

<u>Examples</u>

1. $\begin{array}{ccc} & B_2 & \\ \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \end{array}$ has CP (the column property). B_1

2. $\begin{pmatrix} 1 & 1 & -1 & & & 0 \\ 1 & 2 & & -1 & & \\ \vdots & \vdots & & & \ddots & \\ 1 & m & 0 & & & -1 \end{pmatrix}$ has CP. Take $B_1 = \{1, 3, 4, 5, \dots, m+2\}$
$B_2 = \{2\}$

3. $(2 \;\; 3 \;\; -5)$ has CP. B_1 contains everything.

4. $(\lambda \;\; -1)$ has CP $\Leftrightarrow \lambda = 1$.

We aim to prove Rado's Theorem:

$$PR \Leftrightarrow CP$$

<u>Notes</u>

i) This can check if A is PR in finite time.

ii) Neither direction is obvious.

(We expect $\Leftarrow$ to be harder)

We start with Rado for a single equation. We want that if $a_1, \dots, a_n$ are non-zero rationals, then $(a_1 \, a_2 \dots a_n)$ is PR $\Leftrightarrow \sum_{i \in I} a_i = 0$ for some $I \neq \emptyset$.

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Ramsey Theory ⑧

Let p be prime. We $(p-1)$ colour $\mathbb{N}$ by giving x the colour $d(x)$, its last non-zero digit in its base p expansion.

For example, if $x = x_n p^n + x_{n-1} p^{n-1} + \dots + x_1 p + x_0$

$(0 \le x_i \le p-1)\ \forall i$. We set $L(x) = \min\{i : x_i = 0\}$

and $d(x) = x_{L(x)}$.

e.g. if x in base p is 30147200070000 then $L(x) = 4$, $d(x) = 7$

<u>Proposition 1</u>

Let $a_1, \dots, a_n \in \mathbb{Q} \setminus \{0\}$. Then $(a_1 \dots a_n)$ PR $\Leftrightarrow \sum_{i \in I} a_i = 0$, some $I \neq \phi$

<u>Proof</u>

We may assume that $a_i \in \mathbb{Z}\ \forall i$ (multiplying up if necessary).

Fix a prime p, $p > \sum_i |a_i|$ and consider the above colouring.

We have monochromatic $x_1, \dots, x_n$ with $\sum_i a_i x_i = 0$, and say $d(x_i) = d\ \forall i$. N.B. x_i can have final digit d in different places.

e.g.

$$\begin{array}{ll} x_1: & \dots\, d\,00000 \\ x_2: & \dots\, d\,00000 \\ x_3: & \dots\dots\, d000 \\ x_4: & \dots d\,0000000 \\ x_5: & \dots\dots\, d000 \end{array} \; I$$

Let $L = \min\{L(x_i) : 1 \le i \le n\}$

and let $I = \{i : L(x_i) = L\}$

Then considering $\sum a_i x_i = 0$, computed in base p, we have

$\sum_{i \in I} d\, a_i \equiv 0\ (p)$ so $\sum_{i \in I} a_i \equiv 0\ (p)$ (p prime)

Therefore $\sum_{i \in I} a_i = 0$ (by choice of p). $\square$

<u>Remarks</u>

1. Or we could have said that for each prime p, we have I with $\sum_{i \in I} a_i \equiv 0\ (p)$ so some set I is used infinitely often, whence $\sum_{i \in I} a_i = 0$

2. We coloured by 'end' mod p. We can also colour by 'start' mod p, but this is harder.

3. No other ways to prove proposition 1 are known.

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Ramsey Theory 4

For the other direction, we start with the first non-trivial case, namely $(1\ \lambda\ -1)$.

<u>Lemma 2</u>

Let $\lambda \in \mathbb{Q}$. Then WNFC $\exists$ monochromatic x, y, z with $x + \lambda y = z$.

<u>Proof</u>

WLOG $\lambda > 0$ (because we can deal with $\lambda = 0$, and we can rewrite for $\lambda < 0$ as $z - \lambda y = x$). Say $\lambda = r/s$ where $r, s \in \mathbb{N}$.

Proceed by induction on k, the number of colours. This is trivial for $k = 1$ (taking $x = 1$, $y = s$, $z = 1 + r$, and $\max(s, 1+r)$ a suitable n). Given n suitable for $k-1$ we show that $sW(nr+1, k)$ is suitable for k. Indeed, given a k-colouring of $[sW(nr+1, k)]$, we have a monochromatic AP of length $nr+1$. Inside $[W(nr+1, k)]$, say $a, a+d, \dots, a+nrd$ are all red.

If any of isd, $1 \leq i \leq n$ are red, we are done: $a + \frac{r}{s}(isd) = a + ird$. So we may assume that $sd, 2sd, 3sd, \dots, nsd$ is $(k-1)$ coloured, and we are done by induction. $\square$

<u>Remark</u>

This is very similar to the proof of Strengthened van der Waerden.

Theorem 3 (Rado for Single Equations)

Let $a_1, \dots, a_n \in \mathbb{Q}\setminus\{0\}$. Then $(a_1\ a_2 \dots a_n)$ is PR $\Leftrightarrow \sum_{i\in I} a_i = 0$ for some $I \neq \emptyset$.

Proof

($\Rightarrow$) This is proposition 1.

($\Leftarrow$) Fix some $i_0 \in I$. For suitable x, y, z, we set $x_{i_0} = x$, $x_i = z\ \forall i \in I\setminus\{i_0\}$, $x_i = y\ \forall i \notin I$.

We want $\sum a_i x_i = 0$, all x_i the same colour (in a given colouring). So we want x, y, z monochromatic such that

$$a_{i_0} x + \Big(\sum_{i\in I\setminus\{i_0\}} a_i\Big) z + \Big(\sum_{i\notin I} a_i\Big) y = 0$$

i.e. $a_{i_0} x - a_{i_0} z + \Big(\sum_{i\notin I} a_i\Big) y = 0$

i.e. $x + \frac{1}{a_{i_0}}\Big(\sum_{i\notin I} a_i\Big) y = z$. Hence we are done by Lemma 2 $\square$

Rado's Boundedness Conjecture

If $m\times n$ matrix A is not PR, then there exists a 'bad' k-colouring for some k. Is k bounded (for fixed m, n) ?

Equivalently, is there a $K = K(m,n)$ such that if an $m\times n$ A is PR for k colours then it is PR.

This is known for 1×3 (Fox, Kleitman, 2006) - 24 colours is enough

The answer is not known for any other case.

Proposition 4

If $m\times n$ A is PR then A has CP

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Ramsey Theory 4

Proof:

WLOG all entries of A are integers. Let $C^{(1)}, \dots, C^{(n)}$ be the columns of A. For a prime p, we have a $(p-1)$-colouring of $\mathbb{N}$ (x has colour $d(x)$) so we have monochromatic $x_1, \dots, x_n$ such that $x_1 C^{(1)} + \dots + x_n C^{(n)} = 0$, say all x_i have colour d

$$\begin{array}{ll} x_1 \quad \dots\dots d000 & \\ x_2 \quad \dots\dots d000 & \} \text{ Rightmost for } B_1 \\ \vdots & \\ x_{n-1} \quad \dots d0000 & \} \text{ Next rightmost for } B_2 \\ x_n \quad \dots d00000 & \quad \text{and so on} \end{array}$$

Partition $[n]$ as $B_1 \cup \dots \cup B_r$ where B_1 consists of the i for which x_i is rightmost ending, and so on, as in the diagram.

For infinitely many p, say all $p \in P$, we get the <u>same</u> (ordered) partitions.

Given $p \in P$, we have $x_1, \dots, x_n$ and d and $B_1, \dots, B_r$ as above, so considering $\sum x_i C^{(i)} = 0$, performed in base p, we have:

where $u \equiv v \ (p)$ if $u_i \equiv v_i \ (p) \ \forall i$

i) $\sum_{i \in B_1} d \, C^{(i)} \equiv 0 \ (p)$

ii) For each $2 \le s \le r$, $p^t \sum_{i \in B_s} d \, C^{(i)} + \sum_{i \in B_1 \cup \dots \cup B_{s-1}} x_i C^{(i)} \equiv 0 \ (p^{t+1})$ some t.

From i) we have $\sum_{i \in B_1} C^{(i)} \equiv 0 \ (p)$ (d invertible mod p).

This holds for all $p \in P$, so $\sum_{i \in B_1} C^{(i)} = 0$.

For $2 \le s \le r$, we have $p^t \sum_{i \in B_s} C^{(i)} + \sum_{i \in B_1 \cup \dots \cup B_{s-1}} (d^{-1} x_i) C^{(i)} \equiv 0 \ (p^{t+1})$ (d^{-1}: inverse of d mod p^{t+1})

Claim:

$$\sum_{i \in B_s} C^{(i)} \in \langle C^{(i)} : i \in B_1 \cup \dots \cup B_{s-1} \rangle$$

Proof of Claim:

Suppose not. Then $\exists u \in \mathbb{Z}^m$ such that $u \cdot C^{(i)} = 0 \ \forall i \in B_1 \cup \dots \cup B_{s-1}$

and $u \cdot \sum_{i \in B_s} c^{(i)} \neq 0$. (Think vector-spaces)

We dot with u: $\quad p^t u \cdot \sum_{i \in B_s} c^{(i)} + 0 \equiv 0 \ (p^{t+1})$

whence $u \cdot \sum_{i \in B_s} c^{(i)} \equiv 0 \ (p)$. This holds for all $p \in P$, so

$u \cdot \sum_{i \in B_s} c^{(i)} = 0$ $\quad$ ※ $\qquad \square$

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Ramsey Theory ⑩

Let $m, p, c \in \mathbb{N}$. A subset $S \subset \mathbb{N}$ is an <u>(m,p,c)-set</u> on <u>generators</u> $x_1, \dots, x_m \in \mathbb{N}$ if

$$S = \Big\{ \sum_{i=1}^{m} \lambda_i x_i \ : \ \exists j \text{ with } \lambda_i = 0 \ \forall i<j,\ \lambda_j = c,\ \lambda_i \in [-p,p] \ \forall i > j \Big\}$$

($[-p,\dots,p]$)

So S is all numbers of the form
$$\text{the rows of } S \left\{ \begin{array}{l} cx_1 + \lambda_2 x_2 + \dots + \lambda_m x_m \ (\lambda_i \in [-p,p] \ \forall i) \\ cx_2 + \lambda_3 x_3 + \dots + \lambda_m x_m \ (\lambda_i \in [-p,p] \ \forall i) \\ \quad \dots \\ c_m x_m \end{array} \right.$$

"Iterated AP + CD with c as well" (like $x + \frac{c}{s} y = z$)

e.g. a $(2,p,1)$ set is $x_1 - px_2,\ x_1 - (p-1)x_2, \dots, x_1 + px_2$, and x_2

This is an AP with CD.

A $(2,p,3)$-set is $3x_1 - px_2,\ 3x_1 - (p-1)x_2, \dots, 3x_1 + px_2$ and $3x_2$

An AP whose middle term is a multiple of 3, and $3 \times$ CD.

<u>Theorem 5</u>

WNFC, there exists a monochromatic (m,p,c)-set (any $m,p,c \in \mathbb{N}$)

<u>Proof</u>

R_i will contain i^{th} <u>row</u> of (M,p,c) set
B_i contains set of good generators so far for rows $1, 2, \dots, i$
A_{i+1} restricts B_i to multiples of c to continue the process

Let $\mathbb{N}$ be k-coloured.

Idea: Go for an (M,p,c)-set, $M = k(m-1)+1$, with each row monochromatic.

Let n be large (large enough for everything to come)

Let $A_1 = \{c, 2c, \dots, \frac{n}{c} c\}$ — either $\lfloor \frac{n}{c} \rfloor$ or choose n with $c \mid n$.

Inside A_1, we have a monochromatic AP

$R_1 = \{cx_1 - n_1 d_1,\ cx_1 - (n_1 - 1)d_1, \dots, cx_1, \dots, cx + n_1 d_1\}$, n_1 large

say of colour k_1.

all later work in here.

Let $B_1 = \{d_1, 2d_1, \ldots, \frac{n_1}{pM} d_1\}$. Note that if $x_2, \ldots, x_M \in B_1$ and $\lambda_2, \ldots, \lambda_M \in [-p, p]$, then $cx_1 + \lambda_2 x_2 + \ldots + \lambda_M x_M \in R_1$ so is colour k_1.

Let $A_2 = \{cd_1, 2cd_1, \ldots, \frac{n_1}{pMc} cd_1\}$. Inside A_2 we have monochromatic AP $R_2 = \{cx_2 - n_2 d_2, cx_2 - (n_2-1)d_2, \ldots, cx_2 + n_2 d_2\}$ with n_2 large. Say R_2 is of colour k_2.

Let $B_2 = \{d_2, 2d_2, \ldots \frac{n_2}{pM} d_2\}$. Note that if $x_1, \ldots, x_M \in B_2$ and $\lambda_3, \ldots, \lambda_M \in [-p, p]$, then $cx_2 + \lambda_3 x_3 + \ldots + \lambda_M x_M \in R_2$, so is colour k_2.

Continuing, we obtain an (M, p, c)-set on generators $x_1, \ldots, x_M$ with each row monochromatic. But then some m rows are the same colour since $M = (m-1)k + 1$, giving a monochromatic (m, p, c)-set. $\square$

<u>Proposition 6</u>

Let $m \times n$ A have CP (columns property). Then $\exists$ (m, p, c) such that every (m, p, c)-set contains a solution to $Ax = 0$.

<u>Proof</u>

Idea: CP enables solution to be easily constructed. Then m, p, c are chosen.

$$A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ c^{(1)} & c^{(2)} & \cdots & c^{(n)} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

We have a partition $[n] = B_1 \cup \ldots \cup B_r$ where $\forall s \geq 2$, $\sum_{i \in B_s} c^{(i)} \in \langle c^{(i)} : i \in B_1 \cup \ldots \cup B_{s-1} \rangle$ (and $\sum_{i \in B_1} c^{(i)} = 0$). We say that

$\sum_{i \in B_s} c^{(i)} = \sum_{i \in B_1 \cup \ldots \cup B_{s-1}} q_{is} c^{(i)}$ for some rationals q_{is} (for each $s = 1, 2, \ldots, r$)

12/11/13 Ramsey Theory ⑩

Define $d_{is} = \begin{cases} 0 & \text{if } i \notin B_1 \cup \dots \cup B_s \\ 1 & \text{if } i \in B_s \\ q_{is} & \text{if } i \in B_1 \cup \dots \cup B_{s-1} \end{cases}$ ('ends with a 1' for each s)

writing the above as a linear dependence

Hence $\sum_{i=1}^{n} d_{is} c^{(i)} = 0$ (for each $1 \leq s \leq r$).

Given $x_1, \dots, x_r \in \mathbb{N}$, put $y_i = \sum_{s=1}^{r} d_{is} x_s$ $(1 \leq i \leq n)$

Then $\sum_i y_i c^{(i)} = \sum_i \sum_s d_{is} x_s c^{(i)}$

a linear combination of known solutions to our equation

want $=0$

$= \sum_s x_s \underbrace{\sum_i d_{is} c^{(i)}}_{\overset{\shortparallel}{0} \text{ all } s} = 0$

So we are done. Set $m = r$, $c =$ lcm of denominators of d_{is}

$p = c \max |d_{is}|$. Then $y_1, \dots, y_n$ are all in the (m,p,c)-set

on generators $x_m, x_{m-1}, \dots, x_1$. □

<u>Theorem ~~7~~</u> (Rado's Theorem)

Let A be an $n \times m$ matrix with rational entries. Then

A PR $\Leftrightarrow$ A has CP.

<u>Proof</u>

($\Rightarrow$) Proposition 4.

($\Leftarrow$) Theorem 5 and Proposition 6. □

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Remark

1. Given Rado, results like Schur or van der Waerden are just trivial CP checks.

columns property

partition regular

2. If a matrix A is PR for all last-digit-base-p colourings, then (by the proof of Rado) we know that A is PR for all colourings.

No direct proof is known.

finite sums   no repetition

For $x_1, \dots, x_m \in \mathbb{N}$, write $FS(x_1, \dots, x_m)$ for $\{\sum_{i \in I} x_i : I \neq \emptyset\}$

The case $(m, 1, 1)$ of Theorem 5 immediately gives:

Theorem 8 (Finite Sums Theorem / Folkman's Theorem / Sanders' Theorem)

$\forall m$, WNFC, $\exists x_1, \dots, x_m$ with $FS(x_1, \dots, x_m)$ monochromatic. □

Remarks

1. Alternatively, check that the matrix has CP.
2. The case $m = 2$ is Schur.
3. What about finding a monochromatic $FP(x_1, \dots, x_m) = \{\prod_{i \in I} x_i : I \neq \emptyset\}$?
   Yes, just look at $\{2^1, 2^2, 2^3, \dots\}$ and apply the finite sums theorem.
4. What about finding monochromatic $FS(x_1, \dots, x_m) \cup FP(x_1, \dots, x_m)$?
   This is unknown.
   The case $m = 2$ would be to find monochromatic $x, y, x+y, xy$.
   This is also unknown.
   except $x = y = 2$
   What about finding x, y with $x + y$, xy the same colour?
   Also unknown.

Corollary 9 (Consistency Theorem)

A, B PR $\Rightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ PR

(i.e. if we can always solve $Ax = 0$ in one colour class, and $By = 0$ in one colour class, then we can solve both in one colour class).

Proof

Trivial by CP. $\square$

Remark

This can also be proved directly (i.e. not via Rado) but this is much harder.

More is true.

Corollary 10

WNFC, some colour class contains solutions to all PR matrices.

Proof

Suppose not. Then we have $\mathbb{N} = D_1 \cup \dots \cup D_r$, where for each i there is a PR matrix A_i such that D_i contains no solution to $A_i x = 0$.

Let $A = \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots & \\ & & & A_r \end{pmatrix}$. Then A is PR by Corollary 9, but no D_i contains a solution to $A_i x = 0$ $\times\!\!\!\times$ $\square$

A set $D \subset \mathbb{N}$ is called partition regular if it contains solutions to all PR matrices. (e.g. $\mathbb{N}$)

So Corollary 10 says that when $\mathbb{N} = D_1 \cup \dots \cup D_r$, then some D_i is PR.

Rado's Conjecture (1933)

If D is PR, $D = D_1 \cup \dots \cup D_k$, then some D_i is PR.

This was proved by Deuber (1973) via (m,p,c)-sets.

Hindman's Theorem

Aim

To show that WMFC, $\exists x_1, x_2, \dots$ with $FS(x_1, x_2, \dots)$ monochromatic

This will be our first infinite PR system.

Filters and Ultrafilters

Roughly "a filter is a notion of which subsets of $\mathbb{N}$ are large" and "an ultrafilter is a more precise one".

A filter is a non-empty $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ such that

i) $\emptyset \notin \mathcal{F}$

ii) $A \in \mathcal{F}$, $B \supset A \Rightarrow B \in \mathcal{F}$ ("$\mathcal{F}$ is an up-set")

iii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$ ("$\mathcal{F}$ closed under finite intersections").

Examples

1. $\{A \subset \mathbb{N} : 4 \in A\}$
2. $\{A \subset \mathbb{N} : 4, 5 \in A\}$
3. Non-example $\{A \subset \mathbb{N} : |A| = \infty\}$, since Odds $\cap$ Evens $= \emptyset$
4. $\{A \subset \mathbb{N} : A^c \text{ finite}\}$, the cofinite filter
5. $\{A \subset \mathbb{N} : \text{Evens} \setminus A \text{ finite}\}$

An ultrafilter is a maximal filter (we can't add any more sets).

Of the above,

1. Is maximal, indeed for any $n \in \mathbb{N}$ we have $\tilde{n} = \{A \subset \mathbb{N} : n \in A\}$ called the "principle ultrafilter at n".

2. No, as 1 ~~explains~~ extends it.

4. No, as 5 extends it.

5. No, as we replace Evens with $\{n : n \text{ a multiple of } 4\}$.

<u>Proposition 11</u>

A filter $\mathcal{F}$ is an ultrafilter $\Leftrightarrow \forall A$, $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

<u>Proof</u>

($\Leftarrow$) We cannot add any new $A \in \mathcal{F}$ as A^c is already in $\mathcal{F}$, whence $A \cap A^c = \emptyset \in \mathcal{F}$ ※

($\Rightarrow$) Given $A \notin \mathcal{F}$, we must have $B \cap A = \emptyset$ for some $B \in \mathcal{F}$ otherwise, we could extend $\mathcal{F}$ to $\{D \subset \mathbb{N} : D \supset A \cap B, \text{ some } B \in \mathcal{F}\}$ ※

So $B \subset A^c$, so $A^c \in \mathcal{F}$. $\square$

19/11/13

Ramsey Theory (12)

Remark

Similarly, if $\mathcal{U}$ is an ultrafilter, $A \in \mathcal{U}$, $A = B \cup C$, then $B \in \mathcal{U}$ or $C \in \mathcal{U}$. Indeed, if not then $B^c, C^c \in \mathcal{U}$ whence $A^c = B^c \cap C^c \in \mathcal{U}$ ※

Theorem 12

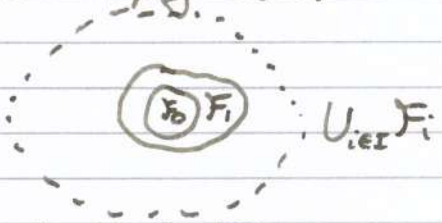
Every filter is contained in an ultrafilter.

Note

Any ultrafilter extending the cofinite filter is non-principal. Conversely, if ultrafilter $\mathcal{U}$ is non-principal, then it extends the cofinite filter, since if finite $A \in \mathcal{U}$ exists, then applying the remark above (repeatedly), we would get $\{n\} \in \mathcal{U}$, some n.

Proof

Given a filter $\mathcal{F}_0$, we seek a maximal filter $\mathcal{F} \supset \mathcal{F}_0$. So, by Zorn's Lemma, it is enough to show that any non-empty chain $\{\mathcal{F}_i : i \in I\}$ has an upper bound.



Put $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$.

Then $\mathcal{F} \supset \mathcal{F}_i$ $\forall i$, so we just need to check that $\mathcal{F}$ is a filter.

i) $\emptyset \notin \mathcal{F}$ since $\forall i \in I$, $\emptyset \notin \mathcal{F}_i$

ii) Given $A \in \mathcal{F}$, $B \supset A$, we have $A \in \mathcal{F}_i$ for some i, so $B \in \mathcal{F}_i$, so $B \in \mathcal{F}$.

iii) Given $A, B \in \mathcal{F}$, we have $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$, for some i, j.

WLOG $F_i \supset F_j$ since we have a chain.

Then $A, B \in F_i$, so $A \cap B \in F_i$, so $A \cap B \in F$. $\square$

Remark

We do need some form of the Axiom of Choice to get non-principa

ultrafilters.

The set of all ultrafilters is denoted $\beta\mathbb{N}$. We can put a topology on $\beta\mathbb{N}$, given by a base of open sets:

$$C_A = \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}, \ A \subset \mathbb{N}.$$

This is a base:

i) $\bigcup_{A \subset \mathbb{N}} C_A = \beta\mathbb{N}$

ii) $C_A \cap C_B = C_{A \cap B}$ (since $A, B \in \mathcal{U} \Leftrightarrow A \cap B \in \mathcal{U}$)

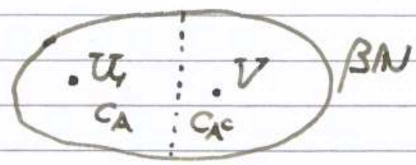
Then the open sets are all sets of the form $\bigcup_{i \in I} C_{A_i} = \{\mathcal{U} : A_i \in \mathcal{U}, \text{some } i$

Basic closed sets are the C_A (because $(C_A)^c = C_{A^c}$ since $A \notin \mathcal{U} \Leftrightarrow A^c \in \mathcal{U}$). So the closed sets are of the form

$$\bigcap_{i \in I} C_{A_i} = \{\mathcal{U} : A_i \in \mathcal{U} \ \forall i\}$$

Each principal $\tilde{n}$ is isolated. Indeed, $C_{\{n\}} = \{\tilde{n}\}$.

Also, the $\tilde{n}$, for $n \in \mathbb{N}$, are dense in $\beta\mathbb{N}$. Indeed, $\tilde{n} \in C_A \Leftrightarrow n \in A$. Then, we can view $\mathbb{N}$ as a subset of $\beta\mathbb{N}$ by identifying $n \in \mathbb{N}$ with $\tilde{n} \in \beta\mathbb{N}$.



Theorem 13 $\beta\mathbb{N}$ is compact Hausdorff

First, we show that $\beta\mathbb{N}$ is Hausdorff. Given distinct $\mathcal{U}, \mathcal{V}$, $\exists A \subset \mathbb{N}$ with $A \in \mathcal{U}$, $A \notin \mathcal{V}$. Then $\mathcal{U} \in C_A$, $\mathcal{V} \in C_{A^c}$

19/11/13 Ramsey Theory 12

For compactness, given closed sets F_i, $i \in I$, with the finite intersection property (any finite intersection $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$), we must show that $\bigcap_{i \in I} F_i \neq \emptyset$.

WLOG, each F_i is basic, say $F_i = C_{A_i}$

Hence the sets A_i, $i \in I$, have the finite intersection property.

Indeed, $C_{A_{i_1} \cap \dots \cap A_{i_n}} = C_{A_{i_1}} \cap \dots \cap C_{A_{i_n}} \neq \emptyset$, whence $A_{i_1} \cap \dots \cap A_{i_n} \neq \emptyset$

Hence the A_i, $i \in I$, generate a filter:

$\mathcal{F} = \{ B \subset \mathbb{N} : B \supset A_{i_1} \cap \dots \cap A_{i_n},\ i_1, \dots, i_n \in I \}$.

Let ultrafilter $\mathcal{U}$ extend $\mathcal{F}$. Then, $\forall i$, $A_i \in \mathcal{F} \subset \mathcal{U}$,

i.e. $\mathcal{U} \in C_{A_i}$ $\square$

<u>Remark</u>

$\beta\mathbb{N}$ is actually the largest compact Hausdorff space in which $\mathbb{N}$ is dense ("the largest compactification of $\mathbb{N}$"). More precisely,

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook] \arrow[d, "f"'] & \beta\mathbb{N} \arrow[dl, dotted, "g"] \\ X & \end{tikzcd}$$

compact Hausdorff

given $f : \mathbb{N} \to X$, any compact Hausdorff space, $\exists!\ g : \beta\mathbb{N} \to X$ that extends f (not hard to prove).

<u>Ultrafilter Quantifiers</u>

For an ultrafilter $\mathcal{U}$, and a property $p(x)$ $(x \in \mathbb{N})$, write

$\forall_{\mathcal{U}} x\ p(x)$ if $(x \in \mathbb{N} : p(x)) \in \mathcal{U}$.

"For $\mathcal{U}$-most x, $p(x)$"

e.g. If $\mathcal{U} = \tilde{n}$ then $\forall_{\mathcal{U}} x\ p(x) \iff p(n)$

For any non-principal $\mathcal{U}$, $\forall_{\mathcal{U}} x : x > 10 \Leftarrow \{x \in \mathbb{N} : x > 10\} \in \mathcal{U}$

__Warning__

$\forall_{\mathcal{U}} x$ and $\forall_{\mathcal{V}} x$ don't commute, even if $\mathcal{U} = \mathcal{V}$.

For example, let $\mathcal{U}$ be non-principal.

Then $\forall_{\mathcal{U}} x \; \forall_{\mathcal{U}} y \;\; x < y$, indeed "$\forall_{\mathcal{U}} y \;\; x < y$" holds for all $x \in \mathbb{N}$.

But $\forall_{\mathcal{U}} y \; \forall_{\mathcal{U}} x \;\; x < y$ is false.

Indeed "$\forall_{\mathcal{U}} x \;\; x < y$" holds for __no__ $y \in \mathbb{N}$.

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Ramsey Theory (13)

$\forall_{\mathcal{U}}$ has nice properties.

Proposition 14

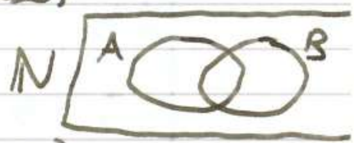
Let $\mathcal{U}$ be an ultrafilter, p, q statements. Then

i) $(\forall_{\mathcal{U}} x)(p(x) \text{ and } q(x)) \iff (\forall_{\mathcal{U}} x)p(x) \text{ and } (\forall_{\mathcal{U}} x)q(x)$

ii) $(\forall_{\mathcal{U}} x)(p(x) \text{ or } q(x)) \iff (\forall_{\mathcal{U}} x)p(x) \text{ or } (\forall_{\mathcal{U}} x)q(x)$

iii) $((\forall_{\mathcal{U}} x)p(x)) \text{ false} \iff (\forall_{\mathcal{U}} x)(p(x) \text{ false})$

Proof



Let $A = \{x \in \mathbb{N} : p(x)\}$, $B = \{x \in \mathbb{N} : q(x)\}$

i) $A \cap B \in \mathcal{U} \iff A \in \mathcal{U} \text{ and } B \in \mathcal{U}$

ii) $A \cup B \in \mathcal{U} \iff A \in \mathcal{U} \text{ or } B \in \mathcal{U}$

iii) $A \notin \mathcal{U} \iff A^c \in \mathcal{U}$ $\square$

Definition

$\mathcal{U} + \mathcal{V} := \{A : (\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(x+y \in A)\}$

e.g. $\tilde{n} + \tilde{m} = \widetilde{(n+m)}$

or without quantifiers

$\mathcal{U} + \mathcal{V} = \{A : \{x : \{y : x+y \in A\} \in \mathcal{V}\} \in \mathcal{U}\}$

Note that $\mathcal{U} + \mathcal{V}$ is an ultrafilter:

- $\emptyset \notin \mathcal{U} + \mathcal{V}$.
- If $A \in \mathcal{U} + \mathcal{V}$ and $B \supset A$ then $B \in \mathcal{U} + \mathcal{V}$.
- If $A, B \in \mathcal{U} + \mathcal{V}$ then $(\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(x+y \in A)$

AND $(\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(x+y \in B)$

So $(\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(x+y \in A \text{ and } x+y \in B)$ (Prop 14, twice)

i.e. $(\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(x+y \in A \cap B)$

- If $A \notin \mathcal{U} + \mathcal{V}$ then NOT $(\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(x+y \in A)$

so $(\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(\text{NOT } x+y \in A)$ (Prop 14 twice)

i.e. $(\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(x+y \in A^c)$ $\square$

We have that $+$ is also associative:

$$(\mathcal{U} + \mathcal{V}) + \mathcal{W} = \{A \subset \mathbb{N} : (\forall_{\mathcal{U}} x)(\forall_{\mathcal{V}} y)(\forall_{\mathcal{W}} z)(x+y+z \in A)\}$$
$$= \mathcal{U} + (\mathcal{V} + \mathcal{W})$$

Also, $+$ is left continuous i.e. for fixed $\mathcal{V}$, the mapping $\mathcal{U} \mapsto \mathcal{U} + \mathcal{V}$, $\beta\mathbb{N}$ to $\beta\mathbb{N}$, is continuous.

$$\beta\mathbb{N} \xrightarrow{\mathcal{U} \mapsto \mathcal{U}+\mathcal{V}} \beta\mathbb{N}$$

Indeed, given a basic open set C_A,

$\mathcal{U} + \mathcal{V} \in C_A \Leftrightarrow A \in \mathcal{U} + \mathcal{V}$

$\Leftrightarrow \{x : (\forall_{\mathcal{V}} y)(x+y \in A)\} \in \mathcal{U}$

$\Leftrightarrow \mathcal{U} \in C_{\{x : (\forall_{\mathcal{V}} y)(x+y \in A)\}}$

preimage of basic open set is open

Remark

In fact, $+$ is not commutative or right continuous.

The key to Hindman will be

Lemma 15 (Idempotent Lemma)

$\exists\, \mathcal{U} \in \beta\mathbb{N}$ with $\mathcal{U} + \mathcal{U} = \mathcal{U}$

Note

Stone-Cech compactification of $\mathbb{N}$

All we will use about $\beta\mathbb{N}$ is compactness, Hausdorff, non-emptiness, and that $+$ is associative and left continuous.

Proof

set of all possible $x+y$, $x, y \in M$

Idea: We go for a minimal $M \subset \beta\mathbb{N}$ with $M + M \subset M$ and hope that $M = \{x\}$ for some x.

21/11/13

Ramsey Theory ⑬

There exists a compact, non-empty $M \subset \beta\mathbb{N}$ with $M+M \subset M$ (e.g. $M = \beta\mathbb{N}$) and we seek a minimal such M.

By Zorn, it is enough to check that if $\{M_i : i \in I\}$ is a chain of such sets then so is $M = \bigcap_{i\in I} M_i$

i) M is non-empty, because the M_i are closed sets with the Finite Intersection Property (remember that compact $\Leftrightarrow$ closed in a compact Hausdorff space) normality of compact-Hausdorff spaces useful

ii) M is an intersection of closed sets, so is closed.

iii) $\forall x, y \in M$: $x, y \in M_i \ \forall i$, so $x+y \in M_i \ \forall i$, so $x+y \in M$.

Let M be a minimal such set. Fix $x \in M$, and we will show that $x+x = x$.

$\{y+x : y \in M\}$

Claim : $M + x = M$

Proof of Claim : We have $M + x \subset M$. (since $M+M \subset M$)

Also, $M+x \neq \emptyset$ (as $M \neq \emptyset$).

$M+x$ is compact (a continuous image of compact set M under $+$)

$(M+x)+(M+x) = (M+\underbrace{x+M}_{M+x \,\supset\, M})+x \subset M+x$

Hence $M+x = M$ by minimality of M.

So $\exists y \in M$ with $y+x = x$.

Now, let $N = \{ y \in M : y+x = x \}$

Claim : $N = M$ (then we are done as $x \in N \Rightarrow x+x = x$)

Proof of Claim : We have $N \subset M$. by definition

Also, $N \neq \emptyset$ (by the above, $y \in N$)

N is closed (since it is the inverse image of $\{x\}$ under continuous map $+$)

Also, $y, z \in N \Rightarrow (y+z)+x = y+(z+x) = y+x = x$

so $y+z \in N$. so $N+N \subset N$

Hence $N = M$, by minimality of M. $\square$

Remarks

1. Hence $M = \{x\}$ by minimality.
2. Does $\beta\mathbb{N}$ have any finite (non-trivial) subgroups?
   e.g. $\mathcal{U}$ with $\mathcal{U}+\mathcal{U} \neq \mathcal{U}$ but $\mathcal{U}+\mathcal{U}+\mathcal{U} = \mathcal{U}$
   This is the Finite Subgroup Problem. The answer is no (Zelenyuk, 1996)
3. Can one ultrafilter absorb another?
   i.e. can we have $\mathcal{U}, \mathcal{V}$ with $\mathcal{U}+\mathcal{U}, \mathcal{U}+\mathcal{V}, \mathcal{V}+\mathcal{U}, \mathcal{V}+\mathcal{V} = \mathcal{U}$
   This is called the Continuous Homomorphism Problem - unknown.

26/11/13 Ramsey Theory (14)

Theorem 16 (Hindman's Theorem)

WNFC, $\exists x_1, x_2, \ldots$ with $FS(x_1, x_2, \ldots)$ monochromatic.

Remark

$\mathcal{U}$ is doing "lots of passes and choosing" for us.

Proof

Let $\mathcal{U}$ be an idempotent ultrafilter. We have $A \in \mathcal{U}$ for some colour class A. We'll find $FS(x_1, x_2, \ldots)$ in A.

$(\forall_{\mathcal{U}}\, y)(y \in A)$.

So $(\forall_{\mathcal{U}}\, x)(\forall_{\mathcal{U}}\, y)(x + y \in A)$ since $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

So $(\forall_{\mathcal{U}}\, x)(\forall_{\mathcal{U}}\, y)(FS(x, y) \subset A)$ by Proposition 14.

Choose x_1 such that $(\forall_{\mathcal{U}}\, y)(FS(x_1, y) \subset A)$

(possible since we have a $\mathcal{U}$-big set of such x_1)

Inductively, suppose we have chosen $x_1, \ldots, x_n$ such that

$(\forall_{\mathcal{U}}\, y)(FS(x_1, \ldots, x_n, y) \subset A)$

For each $z \in FS(x_1, \ldots, x_n)$ we have $(\forall_{\mathcal{U}}\, y)(z + y \in A)$

so $(\forall_{\mathcal{U}}\, x)(\forall_{\mathcal{U}}\, y)(x + y + z \in A)$ since $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

THINK! Definition of addition, set is "$A - z$"

Thus $(\forall_{\mathcal{U}}\, x)(\forall_{\mathcal{U}}\, y)(FS(x_1, \ldots, x_n, x, y) \subset A)$ by Proposition 14.

Choose x_{n+1} such that $(\forall_{\mathcal{U}}\, y)(FS(x_1, \ldots, x_{n+1}, y) \subset A)$ $\square$

(as before, possible since we have a $\mathcal{U}$-big set)

Remarks

1. Very few infinite PR systems are known. No "$\Leftrightarrow$" characterisation is known.

2. An example is the Milliken-Taylor Theorem: WNFC

$\exists x_1, x_2, \ldots$ such that $FS_{1,2}(x_1, x_2, \ldots)$ is monochromatic.

Here $FS_{1,2}(x_1, x_2, \ldots) = \left\{ \sum_{i \in I} x_i + \sum_{i \in J} 2x_i : \begin{array}{l} I, J \text{ finite, non empty} \\ \max I < \min J \end{array} \right\}$

Similarly for $FS_{1,3,7,1}(x_1, x_2, \dots)$ etc.

3. Sadly, the Consistency Theorem fails for infinite PR systems.
It was proved in 1995 that Hindman and Milliken-Taylor$_{1,2}$ are inconsistent. Hence, there is no "universal" ∞ PR system.

Chapter 3: Infinite Ramsey Theory

We know that for any $r = 1, 2, 3, \dots$, whenever $\mathbb{N}^{(r)}$ is 2-coloured, there exists an infinite monochromatic set. What if we coloured the *infinite* subsets of $\mathbb{N}$?

For any infinite set $M \subset \mathbb{N}$, write $M^{(\omega)} = \{L \subset M : L \text{ infinite}\}$

So, if we 2-colour $\mathbb{N}^{(\omega)}$, must there exist a monochromatic $M \in \mathbb{N}^{(\omega)}$ (i.e. $M^{(\omega)}$ is all one colour).

1 e.g. 2-colour $\mathbb{N}^{(\omega)}$ by giving M colour red if $\sum_{x \in M} \frac{1}{x}$ is convergent and blue if $\sum_{x \in M} \frac{1}{x}$ is divergent.

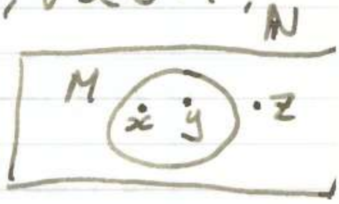
We could take $M = \{2^n : n = 0, 1, 2, \dots\}$

Proposition 1

There is a 2-colouring of $\mathbb{N}^{(\omega)}$ with no infinite monochromatic set.

Proof

We seek a 2-colouring c such that $\forall M \in \mathbb{N}^{(\omega)}$, $\forall x \in M$,
$c(M \setminus \{x\}) \neq c(M)$.



Notice that $c(M \setminus \{x, y\}) = c(M)$ and
$c(M \cup \{z\}) \neq c(M)$.

Define a relation $\sim$ on $\mathbb{N}^{(\omega)}$ by:
$L \sim M$ if $|L \triangle M| < \infty$

This is clearly an equivalence relation. Let the equivalence classes be the $E_i : i \in I$.

In each class E_i, fix an element M_i. Colour $\mathbb{N}^{(\omega)}$ by:

For each $M \in \mathbb{N}^{(\omega)}$ we have a unique M_i with $M \sim M_i$.

Colour M red if $|M \triangle M_i|$ even and
blue if $|M \triangle M_i|$ odd.

chosen representatives

Remark

We do need some form of the Axiom of Choice.

A 2-colouring of $\mathbb{N}^{(\omega)}$ corresponds to a partition $Y \cup Y^c$ of $\mathbb{N}^{(\omega)}$.

We say that Y is Ramsey if $\exists M \in \mathbb{N}^{(\omega)}$ with $M^{(\omega)} \subset Y$ or $M^{(\omega)} \subset Y^c$. i.e. M is a monochromatic subset

So Proposition 1 says that not all sets are Ramsey.

But are "nice" sets Ramsey?

because basic open sets for $\{0,1\}^{\mathbb{N}}$ are

We have a metric on $\mathbb{N}^{(\omega)}$:

$$d(L,M) = \begin{cases} 0 & \text{if } L = M \\ \frac{1}{\min(L \triangle M)} & \text{if } L \neq M \end{cases}$$

Equivalently, we have $\mathbb{N}^{(\omega)} \subset \mathcal{P}(\mathbb{N}) \longleftrightarrow \{0,1\}^{\mathbb{N}}$ which has product topology. So a basic neighbourhood of a point $M \in \mathbb{N}^{(\omega)}$ is $\{L \in \mathbb{N}^{(\omega)} : L \cap [n] = M \cap [n]\}$, $n = 1, 2, \ldots$

Equivalently, the basic open sets are, for each finite $A \subset \mathbb{N}$, the set $\{M \in \mathbb{N}^{(\omega)} : A \text{ is an initial segment of } M\}$.

This is called the product or usual or τ topology.

Our first aim is to show that open sets are Ramsey.

Write $\mathbb{N}^{(<\omega)} = \{A \subset \mathbb{N} : A \text{ finite}\}$

For $M \in \mathbb{N}^{(\omega)}$, $A \in \mathbb{N}^{(<\omega)}$, write

$(A, M)^{(\omega)} = \{L \in \mathbb{N}^{(\omega)} : A \text{ is an initial segment of } L \text{ and } L \setminus A \subset M\}$

"Start as A ; carry on in M".

Fix $Y \subset \mathbb{N}^{(\omega)}$. We say that M <u>accepts</u> A (into Y) if $(A, M)^{(\omega)} \subset Y$.

We say that M <u>rejects</u> A if no $L \in M^{(\omega)}$ accepts A (into Y).

<u>Notes</u>

1. M need not accept or reject A.

   Example: Take $M \in \mathbb{N}^{(\omega)}$, $A = \{1\}$, $L \subsetneq M$, $L \in M^{(\omega)}$, $Y = (A, L)^{(\omega)}$

2. If M accepts A then every $L \in M^{(\omega)}$ also accepts A.
3. If M rejects A then every $L \in M^{(\omega)}$ also rejects A.
4. If M accepts A, then M also accepts $A \cup B$, for any $B \in M^{(<\omega)}$ with $\min B > \max A$.

Look at definitions

$$(\emptyset, M)^{(\omega)} = M^{(\omega)}$$

* Gowin-Priory Explanation *

(starred part)

Check indices

Want to reject a finite subset of $\{a_1, a_2, \dots\}$ e.g. $A = \{a_2, a_7, a_{\dots}$

$M_{n+1}^{(\omega)} \ni \{a_{n+1}, a_{n+2}, \dots\}$

M_{n+1} rejects $A \Rightarrow \{a_{n+1}, a_{n+2}, \dots\}$ rejects A

$\Rightarrow \{a_1, a_2, \dots\}$ rejects A

(since $a_1, \dots, a_n$ are not considered in $(A, M)^{(\omega)}$ type sets

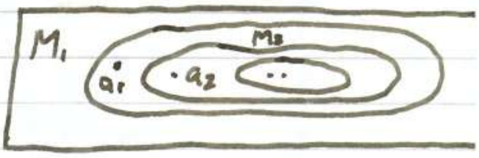
28/11/13

3PM Wed Jan 22nd

Ramsey Theory (13)

<u>Lemma 2</u> (Galvin Prikry Lemma)

Fix $Y \subset \mathbb{N}^{(\omega)}$. Then $\exists M \in \mathbb{N}^{(\omega)}$ such that either M accepts $\emptyset$ or M rejects all of its finite subsets.



<u>Proof</u>

* Suppose that no $M \in \mathbb{N}^{(\omega)}$ accepts $\emptyset$, i.e. $\mathbb{N}$ rejects $\emptyset$. We will
* find $a_1 < a_2 < \dots$ in $\mathbb{N}$ and $M_1 \supset M_2 \supset \dots$ with $a_n \in M_n$ $\forall n$
* and M_n rejects all subsets of $\{a_1, \dots, a_{n-1}\}$ $\forall n$. Then we are
* done since $\{a_1, a_2, \dots\}$ rejects all of its finite subsets. finite subset ends at say a_n. Consider M_{n+1}. Look at 3 on previous page

Put $M_1 = \mathbb{N}$, so M_1 rejects $\emptyset$. Base case

Having chosen $M_1, \dots, M_k$ and $a_1, \dots, a_{k-1}$ suitably, we seek $a_k \in M_k$, $a_k > a_{k-1}$ and $M_{k+1} \subset M_k$ such that M_{k+1} rejects all subsets of $\{a_1, \dots, a_k\}$.

(This is automatic for all subsets of $\{a_1, \dots, a_{k-1}\}$ since $M_{k+1} \subset M_k$)

Let $b_1 \in M_k$, $b_1 > a_{k-1}$. We cannot put $a_k = b_1$, $M_{k+1} = M_k \cup \{b_1\}$ or done so M_k fails to reject some subset of $\{a_1, \dots, a_{k-1}, b_1\}$, say $t_1 \cup \{b_1\}$, where $t_1 \subset \{a_1, \dots, a_{k-1}\}$

Thus some $N_1 \in M_k^{(\omega)}$ accepts $t_1 \cup \{b_1\}$.

Choose $b_2 \in N_1$, $b_2 > b_1$. We cannot put $a_k = b_2$, $M_{k+1} = N_1$ or done so N_1 fails to reject some subset of $\{a_1, \dots, a_{k-1}, b_2\}$, say $N_2 \in N_1^{(\omega)}$ accepts $t_2 \cup \{b_2\}$, for some $t_2 \subset \{a_1, \dots, a_{k-1}\}$. Continue.

We obtain $M_k \supset N_1 \supset N_2 \supset \dots$ and $b_1 < b_2 < \dots$
($b_1 \in M_k$, $b_n \in N_{n-1}$ $\forall n \geq 2$) and

1

$t_i \cup \{b_i\}$ etc

//

$E_1, E_2, \dots \subset \{a_1, \dots, a_{k-1}\}$ such that N_n accepts $E_n \cup \{b_n\}$.

WLOG $E_n = E$ $\forall n$ (Passing to a subsequence), for some $E \subset \{a_1, \dots, a_{k-1}\}$. So $\{b_1, b_2, \dots\}$ accepts E, contradicting M_k rejecting E ※ □

<u>Theorem 3</u>

Let $Y \subset \mathbb{N}^{(\omega)}$ be open. Then Y is Ramsey.

<u>Proof</u>

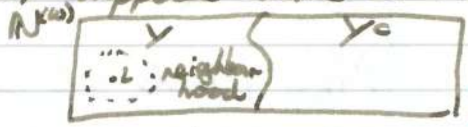
Choose M as given by Galvin-Prikry. If M accepts $\emptyset$: we have $M^{(\omega)} \subset Y$. so done.

If M rejects all of its finite subsets:

We must have $M^{(\omega)} \subset Y^c$. Indeed, suppose some $L \in M^{(\omega)}$ has $L \in Y$.

or just since τ defined by a metric

~~compact Hausdorff space?~~



Since Y is open, ~~some~~ neighbourhood of L is contained in Y. this is a neighbourhood of ?

So $\exists$ an initial segment A of L with $(A, \mathbb{N})^{(\omega)} \subset Y$, so certainly $(A, L)^{(\omega)} \subset Y$, contradicting M rejecting A ※ □

overkill. We define a new topology

<u>Remark</u>

Since Y is Ramsey $\Leftrightarrow$ Y^c Ramsey, we now have all closed sets Ramsey.

<u>Definition</u>

The * or <u>Ellentuck</u> or <u>Mathias</u> topology on $\mathbb{N}^{(\omega)}$ has basic open sets $(A, M)^{(\omega)}$ with $A \in \mathbb{N}^{(<\omega)}$, $M \in \mathbb{N}^{(\omega)}$.

This is a base: $(A, M)^{(\omega)} \cap (A', M')^{(\omega)}$

$$= \emptyset \text{ or } (A \cup A', M \cap M')^{(\omega)}$$

This is *stronger* than τ i.e. we have more open sets.

Is it true that τ has basic open sets $(A, \mathbb{N})^{(\omega)}$ $\quad A \in \mathbb{N}^{(<\omega)}$

$*$ " " " " $(A, L)^{(\omega)}$ $\quad L \in \mathbb{N}^{(\omega)}$

Theorem 3'

Let $Y \subset \mathbb{N}^{(\omega)}$ be $*$-open. Then Y is Ramsey.

Proof

The same as Theorem 3, removing the 'overkill'.

Definition

Ramsey: completely Ramsey? with $L = \phi$.

We say that $Y \subset \mathbb{N}^{(\omega)}$ is completely Ramsey if $\forall A \in \mathbb{N}^{(<\omega)}$ and $M \in \mathbb{N}^{(\omega)}$, $\exists L \in M^{(\omega)}$ with $(A, L)^{(\omega)} \subset Y$ or Y^c.

Not all Ramsey sets are completely Ramsey. For example, take the non-Ramsey Y from Proposition 1, and let Y' be:

$$Y' = Y \cup \{M \in \mathbb{N}^{(\omega)} : 1 \notin M\}$$

Then Y' is Ramsey: $\{2, 3, 4, \dots\}^{(\omega)} \subset Y'$, but Y' is not completely Ramsey since there is no M with $(\{1\}, M)^{(\omega)} \subset Y'$ or Y'^c.

Theorem 4

If Y is $*$-open, then Y is completely Ramsey.

Proof

Given $A \in \mathbb{N}^{(<\omega)}$, $M \in \mathbb{N}^{(\omega)}$, we seek $L \subset M$ with $(A, L)^{(\omega)} \subset Y$ or Y^c.

We "view $(A, M)^{(\omega)}$ as a copy of $\mathbb{N}^{(\omega)}$".

Let $M = \{m_1, m_2, \dots\}$ where $m_1 < m_2 < \dots$ and WLOG $m_1 > \max A$.

We define $f : \mathbb{N}^{(\omega)} \to (A, M)^{(\omega)}$, $N \mapsto A \cup \{m_i : i \in N\}$

This is clearly a homeomorphism in the $*$- topology.

Let $Y' = \{N \in \mathbb{N}^{(\omega)} : f(N) \in Y\}$ $= f^{-1}(Y)$

Then Y' is $*$-open (as Y is $*$-open).

So $\exists L \in \mathbb{N}^{(\omega)}$ with $L^{(\omega)} \subset Y'$ or Y'^c (Y' Ramsey)

i.e. $(A, f(L))^{(\omega)} \subset Y$ or Y^c $\square$

03/11/12

Ramsey Theory (16)

A subset Y of a topological space X is <u>nowhere dense</u> if Y is not dense on any open set - i.e. the closure $\overline{Y}$ has empty interior.

i.e. $\forall$ open $O \neq \emptyset$, $\exists O' \subset O$, $O' \neq O$, with $O' \cap Y = \emptyset$.

e.g. in $\mathbb{R}$: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ or $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{0\}$, but not $\mathbb{Q} \cap (0,1)$

every open set has a neighbourhood not meeting Y

<u>Proposition 5</u>

Let $Y \subset \mathbb{N}^{(\omega)}$. Then

Y $*$-nowhere dense $\Leftrightarrow$ $\forall (A,M)^{(\omega)}$, $\exists L \subset M$ with $(A,L)^{(\omega)} \subset Y^c$

So that Y is completely Ramsey

<u>Note</u>

The "$\Leftrightarrow$" is a good sign that the topology and combinatorics are meshing nicely.

<u>Proof</u>

The RHS says: every $(A,M)^{(\omega)}$ contains an $(A,L)^{(\omega)} \subset Y^c$.

LHS says: every $(A,M)^{(\omega)}$ contains a $(B,L)^{(\omega)} \subset Y^c$.
(underlined statement above)

($\Leftarrow$) is now trivial.

($\Rightarrow$) We know that $\overline{Y}$ is completely Ramsey (since $\overline{Y}$ is closed).
so $\overline{Y}^c$ is open $\Rightarrow$ completely Ramsey

So $(A,M)^{(\omega)}$ contains $(A,L)^{(\omega)} \subset \overline{Y}$ or $\overline{Y}^c$.

Hence $(A,L)^{(\omega)} \subset \overline{Y}^c$ (because $\overline{Y}$ has no interior).
definition of nowhere dense

So $(A,L)^{(\omega)} \subset Y^c$. because $Y^c \supset (\overline{Y})^c$ $\square$

We say that $Y \subset X$ is <u>meagre</u> or of <u>first category</u> if it is a countable union of <u>nowhere dense</u> sets.
very small

e.g. in $\mathbb{R}$, $\mathbb{Q}$ is meagre.

Think of 'meagre' as <u>quite small</u>.

e.g. Baire Category: X a non-empty complete metric space means that X itself is not meagre in X.

Theorem 6

Let $Y \subset \mathbb{N}^{(\omega)}$ be $*$-meagre. Then $\forall\, (A, M)^{(\omega)}$ (basic open set) $\exists\, L \subset M$ with $(A, L)^{(\omega)} \subset Y^c$ (so Y is completely Ramsey). In particular, Y is $*$-nowhere dense.

Y meagre. For every open set O, we have open $O' \subset O$, $O' \subset Y^c$. i.e. we can "remove Y from O".

Proof

We have $Y = \bigcup_{n=1}^{\infty} Y_n$ with each Y_n $*$-nowhere dense.

Given $(A, M)^{(\omega)}$, we have $M_1 \subset M$ with $(A, M_1) \subset Y_1^c$ (Proposition 5). exactly the statement

Choose $x_1 \in M_1$, $x_1 > \max A$. By Proposition 5 twice, we get $M_2' \subset M_1$ with $(A, M_2')^{(\omega)} \subset Y_2^c$ and then $M_2 \subset M_2'$ with $(A \cup \{x_1\}, M_2)^{(\omega)} \subset Y_2^c$. By Proposition 5 four times, we get $M_3 \subset M_2$ with $(A, M_3)^{(\omega)}$, $(A \cup x_1, M_3)^{(\omega)}$, $(A \cup x_2, M_3)^{(\omega)}$, $(A \cup x_1 \cup x_2, M_3)^{(\omega)}$.

Continuing, we obtain $M_1 \supset M_2 \supset \dots$ and $x_1 < x_2 < \dots$ with $x_n \in M_n$ $\forall n$, and $(A \cup F, M_n)^{(\omega)} \subset Y_n^c$, $\forall F \subset \{x_1, \dots, x_{n-1}\}$

So $(A, \{x_1, x_2, \dots\})^{(\omega)} \subset Y_n^c$ $\forall n$, so $\subset Y^c$. $\square$

We say that $Y \subset X$ is a Baire set or has the property of Baire if $Y = O \,\Delta\, M$ for some open O, meagre M.

"Y is nearly an open set."

Examples

1. $(0,1) \setminus \mathbb{Q}$ in $\mathbb{R}$. $= (0,1) \,\Delta\, (\mathbb{Q} \cap (0,1))$

$\mathbb{Q}$ is meagre: write as union of individual points. so $\mathbb{Q} \cap [0,1]$ is also meagre

2. Any open Y. $M = \phi$

3. Any closed Y. We have $Y = \text{Interior}(Y) \,\Delta\, (Y \setminus \text{Interior } Y)$
Interior $(Y) = (\overline{Y^c})^c$ "biggest open set inside Y"
(Interior (Y) contains no non-empty open set)

4. The Baire sets form a σ-Algebra (closed under complements and countable unions. Indeed

- Y Baire $\Rightarrow Y = O \,\Delta\, M$ (O open, M meagre)
$\Rightarrow Y^c = O^c \Delta M = (O' \Delta M') \Delta M = O' \Delta (M' \Delta M)$

- $Y_1, Y_2, \ldots$ Baire $\Rightarrow Y_n = O_n \Delta M_n$ (O_n open, M_n meagre)
$\Rightarrow \bigcup_{n=1}^{\infty} Y_n = \bigcup_{n=1}^{\infty} O_n \,\Delta\, M$, for some $M \subset \bigcup_{n=1}^{\infty} M_n$ so that M is meagre.

So Baire is a bit like measurable.

Theorem 7

Let $Y \subset \mathbb{N}^{(\omega)}$. Then Y is completely Ramsey
$\Leftrightarrow Y$ is $*$-Baire.

Notes

i) Hence any τ-Borel set (Borel meaning in the σ-algebra generated by the open sets) is Ramsey:
Y τ-Borel $\Rightarrow Y$ $*$-Borel $\Rightarrow Y$ $*$-Baire $\Rightarrow Y$ completely Ramsey $\Rightarrow Y$ Ramsey.

ii) Any set that we 'write down' will invariably (nearly always) be Borel.

iii) e.g. $\exists$ infinite M such that all ∞ $L \subset M$, we have $\sum_{n \in L} \frac{1}{\pi^n}$ having infinitely many 7s in its decimal expansion.

OR finitely many 7s. (easy to check that the colouring is Borel).

Proof

($\Leftarrow$) We have $Y = W \Delta Z$, W $*$-open, Z $*$-meagre.

$*$open $\Rightarrow$ completely Ramsey

Given $(A, M)^{(\omega)}$, $\exists L \subset M$ with $(A, L)^{(\omega)} \subset W$ or W^c, and

$\exists N \subset L$ with $(A, N)^{(\omega)} \subset Z^c$. Hence either $(A, N)^{(\omega)} \subset Z^c \cap W \subset Y$

Z is $*$-meagre

or $(A, N)^{(\omega)} \subset Z^c \cap W^c \subset Y^c$. because $Y = W \Delta Z$

($\Rightarrow$) We have $Y = \mathrm{Int}(Y) \,\Delta\, (Y - \mathrm{Int}(Y))$.

It is enough to show that $Y - \mathrm{Int}(Y)$ is nowhere dense.

Given a basic open $(A, M)^{(\omega)}$, we have $L \subset M$ with

$(A, L)^{(\omega)} \subset Y$ or Y^c (as Y is completely Ramsey).

- If $(A, L)^{(\omega)} \subset Y$ we have $\underbrace{(A, L)^{(\omega)} \subset \mathrm{Int}(Y)}_{\text{definition of } \mathrm{Int}(Y) \text{ since } (A,L)^{(\omega)} \text{ is open.}}$, so $(A, L)^{(\omega)}$ misses $Y \setminus \mathrm{Int}(Y)$
- If $(A, L)^{(\omega)} \subset Y^c$, certainly $(A, L)^{(\omega)}$ misses $Y \setminus \mathrm{Int}(Y)$ $\square$

Remark

Without Theorem 6, this proof would say

Y completely Ramsey $\Leftrightarrow$ Y = Open Δ Nowhere dense.

But then we would not know that the Completely Ramsey sets form a σ-algebra.