16/01/17

The Riemann Zeta Function ①

Ch. 0 Preliminaries

1. Practical Things

- Lecture notes : online notes, in 3 blocks of material

  www.dpmms.cam.ac.uk/~ajh228 , a.j.harper@dpmms.ac...

- Books :

  Titchmarsh, "The theory of the Riemann Zeta Function"

  This covers the first half of the course.

  Ivić, "The Riemann Zeta Function. Theory and Applications"

  This covers the second half of the course.

  Davenport, "Multiplicative Number Theory" GTM

  Iwaniec and Kowalski, "Analytic Number Theory"

  Montgomery and Vaughan, "Multiplicative Number Theory" CUP

- Examples Sheets

  Probably 3 sheets this term, posted online.

  Probably 2 classes this term, 1 in Easter (and revision).

2. Notation and Conventions

We will have some notation to facilitate making estimates.

- We write $f(x) = O(g(x))$, "f is big O of g", if

  $\exists\, C$, constant such that $|f(x)| \leq C g(x)$ $\forall x$

f has order at most g

Here $\forall x$ could mean

i) $\forall x$ for which f, g are defined

OR

ii) $\forall x$ larger than some constant.

We also write $f \ll g$ to mean the same.

- We will also write $f \asymp g$ and say f is <u>of order</u> g if $f \ll g$ and $g \ll f$, or $\frac{1}{c} g(x) \leq |f(x)| \leq c\, g(x) \quad \forall x$
- We write $f(x) = o(g(x))$ as $x \to \infty$, "f is little o of g" if $(g(x) \neq 0)$, $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$
f is of <u>smaller order</u> than g.
- We will write $f(x) \sim g(x)$ as $x \to \infty$, and say f is <u>asymptotic to</u> g if $(g(x) \neq 0)$, $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.
This is the same as saying $f(x) = (1 + o(1))\, g(x)$

<u>Examples</u>

For $x \geq 1$

$100x = O(x^2)$ $\qquad\qquad \log(x) = o(x)$ as $x \to \infty$

$x^3 - 2x + 1 = x^3 + O(x)$ $\qquad\qquad x^3 - 10 \sim x^3$ as $x \to \infty$

We will use c to denote a small positive constant

C " " large " "

ε " " ~~to denote~~ a parameter close to 0.

It is customary, in order to economise on notation, to use these symbols several times in one argument, with different meanings at each use, if this doesn't confuse the reader.

<u>3 Course Content</u>

We focus on parts of the theory with applications to the distribution of primes.

By "distribution of primes", we mean questions like:

- How many primes $\leq x$?

16/01/14 The Riemann Zeta Function ①

- How many primes in the short interval $[x, x + x^{0.99}]$?
- Euclid : the number $\pi(x)$ of primes $\leq x$ satisfies
  $\pi(x) \to \infty$ as $x \to \infty$ (infinitely many primes)
- Chebychev (c. 1850) : $\pi(x) \asymp \frac{x}{\log x}$, $x \geq 2$
- Conjecture of Legendre, Gauss that
  $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$ (*)
- Riemann (1859) proposed that one could prove (*) by complex analysis of the zeta function $\zeta(s)$. He died in 1866 before this could be done.

This course will have 3 main Chapters :

1. Basic Theory of $\zeta(s)$ up to a proof of (*)
   ((*) is the <u>prime number theorem</u>, proved in 1896)
   It turns out that PNT is roughly equivalent to showing that $\zeta(s) \neq 0$ wherever $Re(s) \geq 1$.
2. We will extend the region $\{Re(s) \geq 1\}$, where we know that $\zeta(s) \neq 0$, to the biggest such <u>zero-free region</u> known.
   (Vinogradov, Korobov, independently, 1958) (Very deep)
   As an application, we will improve our estimates for $\pi(x) \sim \frac{x}{\log x}$.
3. In view of (*), it is natural to think that $[x, x + x^{0.99}]$ contains $(1 + o(1)) \frac{x^{0.99}}{\log x}$ primes, as $x \to \infty$.
   Quite shockingly, Hoheisel (1930) managed to prove a result on short intervals.

It turns out that for this application, one doesn't need to know that $\zeta(s)$ has no zeroes with $Re(s)$ close to 1, only that it doesn't have many such zeroes.

In chapter 3, we will prove such zero-density estimates.

Chapter 1 Basic Theory and the PNT

1. First definition of $\zeta(s)$

Definition 1

For $s \in \mathbb{C}$ with $Re(s) > 1$, we define the Riemann-Zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Note that this is absolutely convergent.

18/01/14

Riemann Zeta Function ②

Any series $\sum_{n\geq 1} \frac{a_n}{n^s}$, where $(a_n) \subseteq \mathbb{C}$ and $s \in \mathbb{C}$ is a variable is called a <u>Dirichlet Series</u>.

If only finitely many of the a_n are non-zero, then the resulting finite sum $\sum_{n\leq N} \frac{a_n}{n^s}$ is called a <u>Dirichlet polynomial</u>.

$s = \sigma + it$

$\sum_{n\leq N} \frac{a_n}{n^s} = \sum_{n\leq N} \frac{a_n}{n^\sigma} e^{-it\log n}$ might behave like $\frac{e^{-it\log n}}{e^{-itn}}$?

<u>Lemma 1·2</u>

For any $Re(s) > 1$, any $x \in \mathbb{N}$, we have

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O(|s|x^{-\sigma})$$

Consequently, if $\sigma > 1$, $|t| \geq 2$, then $\zeta(\sigma + it) = O(\log|t|)$

<u>Proof</u>

(We might expect that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \simeq \int_1^\infty \frac{1}{z^s} dz = \left[\frac{z^{1-s}}{1-s}\right]_1^\infty = \frac{1}{s-1}$, but <u>this is not true</u>, because the integral does not approximate the sum well at the start. However, the lemma says that the approximation <u>is</u> good for later terms.)

Note that $\int_n^\infty \frac{dw}{w^{s+1}} = \left[\frac{w^{-s}}{-s}\right]_n^\infty = \frac{1}{s}\frac{1}{n^s}$

therefore $\sum_{n>x} \frac{1}{n^s} = s\sum_{n>x}\int_n^\infty \frac{dw}{w^{s+1}} = s\int_x^\infty \left(\sum_{x<n\leq w} 1\right)\frac{dw}{w^{s+1}}$

(The interchange of summation and integral is justified because everything is absolutely convergent).

So $\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + s\int_x^\infty (\lfloor w \rfloor - x)\frac{dw}{w^{s+1}}$ since $x \in \mathbb{N}$

$$= \sum_{n\leq x} \frac{1}{n^s} + s\int_x^\infty (w-x)\frac{dw}{w^{s+1}} + O\left(|s|\int_x^\infty \frac{dw}{w^{s+1}}\right)$$

since $|w - \lfloor w \rfloor| \leq 1$ $\forall w$

$$= \sum_{n\leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O(|s|x^{-\sigma})$$

$\left|\int^\infty \frac{dw}{w^{s+1}}\right| \leq \int^\infty \frac{dw}{|w^{s+1}|} = \int^\infty \frac{dw}{w^{\sigma+1}} = x^{-\sigma}$

For the second part, just choose $x = \lfloor |t| \rfloor$.

$$\zeta(\sigma+it) = \sum_{n \leq |t|} \frac{1}{n^s} + O\left(\frac{1}{|t|}\right) + O(1)$$
$$= O\left(\sum_{n \leq |t|} \frac{1}{n}\right) + O(1) = O(\log|t|) \qquad \square$$

Remark

This proof has lots of potential for further development as we will see. The zeta function is defined in terms of $(\frac{1}{n^s})_{n \in \mathbb{N}}$. Since each n has a unique expression as a product of primes, we ~~write~~ might hope to express $\zeta(s)$ in terms of $(\frac{1}{p^s})$ only.

Lemma 1·3 (Euler Product)

For any $Re(s) > 1$, we have $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ where $\prod_p (1-p^{-s})^{-1} = \lim_{P \to \infty} \prod_{p \leq P} (1-p^{-s})^{-1}$

Proof

Note that for any prime p, $(1-p^{-s})^{-1} = \sum_{k=0}^{\infty} \frac{1}{p^{ks}}$

Since we can rearrange / multiply out a ~~function~~ finite product of absolutely convergent series, we see that

$$\prod_{p \leq P} (1-p^{-s})^{-1} = \prod_{p \leq P} \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c_P(n)}{n^s}$$

where $c_P(n) := \begin{cases} 1 & \text{if all prime factors of } n \text{ are } \leq P \\ 0 & \text{otherwise} \end{cases}$

(using the Fundamental Theorem of Arithmetic)

$$\therefore \left|\zeta(s) - \prod_{p \leq P} (1-p^{-s})^{-1}\right| \leq \sum_{n \geq 1,\, c_P(n) = 0} \frac{1}{n^{Re(s)}}$$
$$\leq \sum_{n = P}^{\infty} \frac{1}{n^{Re(s)}}$$

Since $Re(s) > 1$, RHS $\to 0$ as $P \to \infty$ $\qquad \square$

17/01/14

Riemann Zeta Function (2)

We have seen that when $Re(s) > 1$, $\zeta(s)$ is simultaneously a (Dirichlet) series which can be approximated and analysed, and a product over primes, which we want to study.

Most research on $\zeta(s)$ is about developing similar properties for other $s \in \mathbb{C}$, and playing them off to get information about $\zeta(s)$ or about primes.

2 Primes and Perron's Formula

$$\pi(x) := \sum_{p \leq x} 1$$

It is technically easier (and completely standard) to study a weighted counting function.

Definition 2.1

We define the von Mangoldt function $\Lambda : \mathbb{N} \to \mathbb{R}$ by
$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{, some } k \geq 1, \text{ } p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

Then we define Chebyshev's Psi Function

$$\Psi(x) = \sum_{n \leq x} \Lambda(n) \quad , x \in \mathbb{R}$$

$$\Psi(x) = \sum_{p \leq x} \log p + \sum_{k=2}^{(\log x)} \sum_{p \leq x^{1/k}} \log p = \sum_{p \leq x} \log p + O(\sqrt{x} \log^2 x) \quad \text{if } x \geq 2$$

since $\sum_{p \leq x^{1/k}} \log p = O(\sqrt{x} \log x)$ if $k \geq 2$, $x \geq 2$

and only the first $O(\log x)$ values k make a contribution.

The reason for introducing $\Lambda(n)$ is because the corresponding Dirichlet series $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$, $Re(s) > 1$, can be related to $\zeta(s)$ much more easily than $\sum_p \frac{1}{p^s}$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{p^k} \left(-\frac{1}{k} \frac{d}{ds} \frac{1}{p^{ks}} \right) = \frac{d}{ds} \left(- \sum_{p^k} \frac{1}{k} \frac{1}{p^{ks}} \right)$$

$$= \frac{d}{ds} \left(- \sum_{p} \log \left(1 - \frac{1}{p^s} \right) \right)$$

Euler Product

$$= \frac{d}{ds} \left(- \log \zeta(s) \right)$$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = - \frac{\zeta'(s)}{\zeta(s)} \quad , \quad Re(s) > 1 .$$

21/01/14

Riemann Zeta Function ③

$$\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$$

i) If we had looked at $\sum_p \frac{1}{p^s}$ we would have ended up with something like $\log \zeta(s)$ which is nasty due to branch cuts.

ii) Since $\zeta(s)$ is an absolutely convergent product (Euler product) when $Re(s) > 1$, and no term in the product vanishes, we know that $\zeta(s) \neq 0$ when $Re(s) > 1$

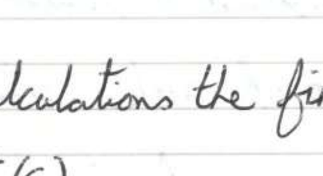
(or the calculations above show that $\frac{\zeta'(s)}{\zeta(s)} \in \mathbb{C}$)

Remark 2·2

We see in these calculations the first connection between primes and vanishings of $\zeta(s)$.

We can think of $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ as something like a Fourier transform of $(\Lambda(n))_{n\in\mathbb{N}}$, since if $s = \sigma + it$, then $\frac{1}{n^{\sigma+it}} = \frac{1}{n^\sigma} e^{-it\log n}$ and $e^{-it\log n}$ is like the 'phase' e^{-itn} in a Fourier Series.

So maybe we can formulate some kind of "Fourier inversion" in which we get information about $\psi(x) = \sum_{n\leq x} \Lambda(n)$ by integrating $\sum_n \frac{\Lambda(n)}{n^{\sigma+it}}$



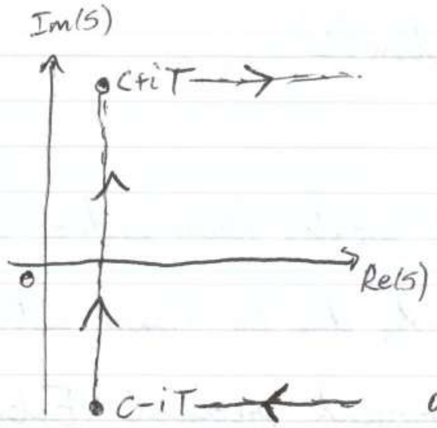
Lemma 2·3

Let $y, c, T > 0$. Define $\delta(y) = \begin{cases} 0 & 0<y<1 \\ \frac{1}{2} & y=1 \\ 1 & y>1 \end{cases}$

Then $\left| \delta(y) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s \frac{ds}{s} \right| \leq \begin{cases} y^c \min\{1, \frac{1}{T|\log y|}\}, & y\neq 1 \\ \min\{1, \frac{c}{T}\} & , y=1 \end{cases}$

Proof

Suppose first that $0 < y < 1$.



$= \frac{y^{Re(s)}}{|s|} \le \frac{y^{Re(s)}}{Re(s)}$

Since $0<y<1$, $\left|\frac{y^s}{s}\right| \to 0$ uniformly as $Re(s) \to \infty$

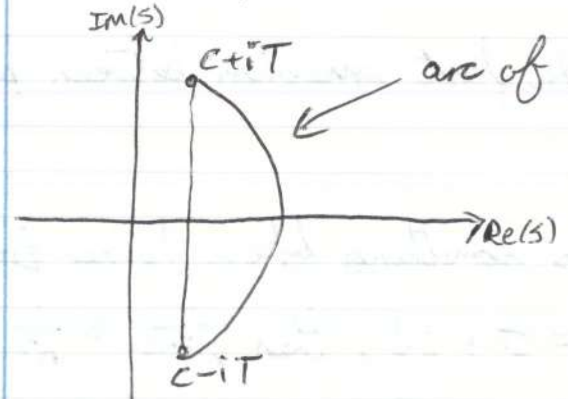
By Cauchy's Residue Theorem, the integral around the contour $= \sum$ residues inside $= 0$.

Therefore $\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = -\frac{1}{2\pi i}\int_{c+iT}^{\infty+iT} \frac{y^s}{s}\,ds + \frac{1}{2\pi i}\int_{c-iT}^{\infty-iT} \frac{y^s}{s}\,ds$

Also, $\left|\int_{c+iT}^{\infty+iT} \frac{y^s}{s}\,ds\right| \le \frac{1}{T}\int_c^{\infty} y^{\sigma}\,d\sigma = \frac{y^c}{T|\log y|}$

We still need to prove the bound y^c (when $0<y<1$)

To do this, consider instead a circular contour.



arc of circle, centre O, radius $|c+iT|$

Again, by the Residue Theorem,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = \frac{1}{2\pi i}\int_{\substack{\text{arc}\\ c-iT \text{ to } c+iT}} \frac{y^s}{s}\,ds$$

$$\left|\frac{1}{2\pi i}\int_{\substack{\text{circular}\\ \text{arc}}} \frac{y^s}{s}\,ds\right| \le \frac{y^c}{2\pi\sqrt{c^2+T^2}}\int_{\substack{\text{circular}\\ \text{arc}}} 1 \le y^c\,\frac{\pi\sqrt{c^2+T^2}}{2\pi\sqrt{c^2+T^2}} \le y^c$$

If $y>1$, do the same but with the contours on the left.

If $y=1$, use real variable methods (Exercise) $\square$

<u>Lemma 2·4</u> (Truncated Perron Formula)

Let $x, c, T > 0$. Suppose $\sum_{n=1}^{\infty} \frac{|a_n|}{n^c}$ is convergent.

Then $$\sideset{}{'}\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right)\frac{x^s}{s}\,ds \quad \circledast$$
$$+\, O\left(x^c \sum_{n=1}^{\infty} \frac{|a_n|}{n^c} \min\left\{1, \frac{1}{T|\log(\frac{x}{n})|}\right\}\right)$$

Here $\sideset{}{'}\sum_{n\le x} a_n$ means that a_x is replaced by $\frac{1}{2}$ if $x \in \mathbb{N}$.

21/04/14

Riemann Zeta Function ③

<u>Proof</u>

Note that $\sum'_{n \le x} a_n = \sum_n a_n \delta(x/n)$

(Lemma 2·3) $= \sum_n a_n \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^s \frac{1}{s} \, ds$

$+ O\left(x^c \sum_n \frac{|a_n|}{n^c} \min\left\{1, \frac{1}{T|\log x/n|}\right\}\right)$

Since $\sum_n |a_n| \int_{c-iT}^{c+iT} \left|\frac{x^s}{n^s}\right| \frac{1}{|s|} |ds|$ and $\int_{c-iT}^{c+iT} \left|\sum_n \frac{a_n}{n^s}\right| \left|\frac{x^s}{s}\right| |ds|$

are convergent, it is permissible to swap the order of $\sum$ and $\int$

So $\sum'_{n \le x} a_n = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\sum_n \frac{a_n}{n^s}\right) \frac{x^s}{s} \, ds + O(\dots)$ $\square$

<u>Remark 2·5</u>

In fact, Lemma 2·4 is true with $\sum'_{n \le x} a_n$ replaced by $\sum_{n \le x} a_n$; if $x \in \mathbb{N}$ then $|a_x|$ is part of the big O.

$$|y^s| = y^\sigma$$

Moral: Can learn about properties of a series by integrating Dirichlet-series.

Prove $\left|\delta(y) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s \frac{ds}{s}\right| \leq \min\{1, \frac{c}{T}\}$
$y=1$

i.e. $$\left|\frac{1}{2} - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{ds}{s}\right| \leq \min\left\{1, \frac{c}{T}\right\}$$

$$\int_{c-iT}^{c+iT} \frac{ds}{s} = \int_{-T}^{T} \frac{i\,dt}{c+it}$$

$\Rightarrow$ Prove

$$\left|\frac{1}{2} - \frac{1}{2\pi}\int_{-T}^{T} \frac{dt}{c+it}\right|$$

$$\int_{-T}^{T} \frac{c-it}{c^2+t^2}\,dt = \int_{-T}^{T} \frac{c}{c^2+t^2} + i\,\frac{t}{c^2+t^2}\,dt$$

$u = \frac{t}{c}$ $t = uc,\ dt = c\,du$

$$\int_{-T}^{T} \frac{t}{1+(\frac{t}{c})^2}\,dt = \int_{-\frac{T}{c}}^{\frac{T}{c}} \frac{1}{1+u^2}\,du$$

Real Part

$$= \Big[\arctan u\Big]_{-T/c}^{T/c} = 2\arctan \frac{T}{c}$$

$$\int_{-T}^{T} \frac{t}{c^2+t^2}\,dt = \left[\frac{1}{2}\log(c^2+t^2)\right]_{-T}^{T} = 0$$

Im Part

$\Rightarrow$ Prove $$\left|\frac{1}{2} - \frac{\arctan\frac{T}{c}}{\pi}\right| \leq \min\left\{1, \frac{c}{T}\right\}$$

$$\left|\frac{\pi}{2} - \arctan\frac{T}{c}\right| \leq \min\left\{\pi, \frac{\pi c}{T}\right\}$$

$$\left|\frac{\pi}{2} - \arctan u\right| \leq \min\left\{\pi, \frac{\pi}{u}\right\}$$

AIM 2. Extend the fluid-laser analogy through numerical work integrating computational fluid dynamics and nonlinear laser cavity dynamics.

The identification of a laminar/turbulent transition in an optical cavity in recent work by the Novosibirsk group [Nat. Phot. **7**, 783 (2013)] opens the door to useful crossover between the laser and fluid dynamics communities. However, the precise nature of fluid-laser correspondence has not yet been fully explored. I will use computational methods in fluid dynamics to investigate the relationship between optical coherence in lasers and laminar pipe flow, attempting to develop in a fiber laser a more detailed analogy of the hydrodynamic transition to turbulence. By developing an analogue of the dimensionless Taylor-Couette number, I will attempt to extend the theory to cylindrically symmetrical Taylor-Couette flow. I will finally integrate turbulence-suppression techniques from fluid mechanics into a laser context to establish by simulation the utility of this line of inquiry in an experimental context. I will also attempt to extend the work to mode-locked laser cavities.

This aim will require detailed reading of the primary fluid mechanics literature in addition to numerical simulation of lasers. Drawing on two fluid mechanics courses I have already taken, it will help build my knowledge of lasers, fluid dynamics, and numerical simulation, useful for my specific physics research goals as well as the general development of scientific experience. Working directly with members of the group who developed this line of inquiry is likely to make the project much more fruitful than it otherwise would be.

Expected duration: 5 months

AIM 3: Engage with the Novosibirsk community

In line with the Fulbright aim to promote cultural understanding, I will undertake a variety of projects to engage with the community in a culturally-sensitive way taking into account local traditions and regulations. Participation in charitable projects, such as the Chabad soup kitchen and Rotary Club-organized fundraising events for nonprofit organizations, will be excellent opportunities to do good. In the US, I volunteered regularly at blood drives, and I hope to continue doing so in Russia. I am an Eagle Scout and capable of usefully leading and assisting community projects as appropriate.

Doing good in the Novosibirsk area will require flexibility, and for this reason forging links to established community organizations for advice will be essential in advance of my arrival. I have hobbies in climbing and music, and so I may also audition for a choir, such as the Novosibirsk State University Academic Choir, and I will contact my Russian climbing contacts in the US to climb with Russian climbers.

I will harness my energy and dynamism to craft a culturally-sensitive program of community engagement, using contacts I make during my time in Russia, taking into account the need for continuous improvement in Russian, and modifying my plans in line with on-the-ground realities. I will feed back my experience to the US as far as possible.

Expected duration: Continuous

$$\left|\frac{\pi}{2} - \arctan u\right| \le \min\left\{\pi, \frac{\pi}{u}\right\}$$

Trivial for $u < 1$

$(u > 0)$

$$\frac{\pi}{2} - \arctan u - \frac{\pi}{u} \to (1-\pi) u$$

$u > 0$

$$\Rightarrow \arctan(u) + \arctan\left(\frac{1}{u}\right) = \frac{\pi}{2}$$

$$\left|\arctan \frac{1}{u}\right| \le \min\left\{\pi, \frac{\pi}{u}\right\}$$

$t > 1$

$$|\arctan t| \le \min\{\pi, \pi t\}$$

$$f(x) = \arctan x + \arctan \frac{1}{x}$$

$$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

$$f(1) = \frac{\pi}{2}$$

$$\therefore \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2} \quad \forall x > 0$$

23/01/14

Riemann - Zeta Function ④

For any $1 < c \leq 2$, $x > 1$, $1 < T' \leq x$

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds + O\left(x^c\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^c}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\}\right)$$

$$\text{error term} = O\left(\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^c} + x^c\sum_{\frac{x}{2}<n<2x}\frac{\Lambda(n)}{n^c}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\}\right)$$

$\frac{x}{2} < n < 2x \iff |\log\frac{x}{n}| < \log 2$

$$= O\left(\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{\log n}{n^c} + \log x\sum_{\frac{x}{2}<n<2x}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\}\right) \quad (\Lambda(n) \leq \log n)$$

$$= O\left(\frac{x^c}{T(c-1)^2} + \frac{x}{T}\log^2 2x\right)$$

Since $\sum_{\frac{x}{2}<n<2x}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\} \leq \sum_{k=0}^{\log T+1}\sum_{|n-x|<\frac{2^k x}{T}}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\}$ $\to$ but $> \frac{2^{k-1}x}{T}$

Exercise: Compare with Integral

$\sum_{n\geq 1} f(n) = \int_1^{\infty} f(x)\,dx + O(1)$

$$\log\frac{x}{n} = \log\left(1+\frac{x-n}{n}\right) \ll \frac{|x-n|}{x} \quad \text{since } n \asymp x.$$

$$\text{Then } \sum_{k=0}^{\log T+1}\sum_{\frac{2^{k-1}x}{T}<|n-x|<\frac{2^k x}{T}}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\}$$

$$\ll \sum_k \sum_{\frac{2^{k-1}x}{T}<|n-x|<\frac{2^k x}{T}}\min\left\{1, \frac{x}{T|x-n|}\right\}$$

$$\ll \frac{x}{T} + \sum_{k=1}^{\frac{\log T}{\log 2}+1}\sum_{\frac{2^{k-1}x}{T}<|n-x|<\frac{2^k x}{T}}\frac{x}{T\left(\frac{2^k x}{T}\right)} = \frac{x}{T} + \sum_k \frac{1}{2^k}\frac{2^k x}{T}$$

$$\ll \frac{x}{T} + \frac{x}{T}(\log T + 1)$$

It is usual to choose $c = 1 + \frac{1}{\log x}$ in which case both error terms are $O\left(\frac{x^c}{T}\log^2 2x\right) = O\left(\frac{x}{T}\log^2 2x\right)$

$$x^{1+\frac{1}{\log x}} = x\cdot e = O(x)$$

and in the integral, $x^s = x^{c+it} \asymp x$

At this point, one could try to get very precise estimates for $\zeta(1+\frac{1}{\log x}+it)$ and $\zeta'(1+\frac{1}{\log x}+it)$ and substitute.

Instead, we will follow the classical approach, and try to estimate the integral by Cauchy's Residue Theorem.

To do this, we must extend the definition of $\zeta(s)$ to $\{Re(s) > 0\}$.

3 Second Definition of $\zeta(s)$

We will extend the definition of $\zeta(s)$ to a meromorphic function on $\{Re(s) > 0\}$, with a simple pole (of residue 1) at $s=1$, such that the definition agrees with Definition 1·1 on $\{Re(s) > 1\}$

We will do this in a way inspired by Lemma 1·2.

Definition 3·1

For each $s \in \mathbb{C}$ such that $Re(s) > 0$, except for $s=1$, and for any $x > 0$, we define

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \frac{\{w\}}{w^{s+1}}\,dw \qquad (*)$$

where $\{x\} = x - \lfloor x \rfloor$, the fractional part.

The value of RHS is independent of x.

Proof (of well-definition)

We need to prove:

i) That RHS of $(*)$ really is a meromorphic function when $\{Re(s)>0\}$

ii) that RHS of $(*)$ agrees with definition 1·1 when $\{Re(s) > 1\}$

iii) that RHS of $(*)$ takes the same value $\forall x > 0$.

i) is clear; the pole at $s=1$ comes from the term $\frac{x^{1-s}}{s-1}$.

ii) was already checked in the proof of Lemma 1·2, provided that $x \in \mathbb{N}$.

It remains to check iii), and by analytic continuation, it suffices to do this when $Re(s) > 1$. To do this, we calculate that if $x > 0$ is not an integer, and $N = \lfloor x \rfloor + 1$

is the smallest integer exceeding x, then

$$s\int_x^N \{w\}\frac{dw}{w^{s+1}} = s\int_x^N (w-(N-1))\frac{dw}{w^{s+1}}$$

$$= s\left(\left[\frac{w^{1-s}}{1-s}\right]_x^N - (N-1)\left[\frac{w^{-s}}{-s}\right]_x^N\right)$$

$$= \frac{sN^{1-s}}{1-s} - \frac{sx^{1-s}}{1-s} + (N-1)N^{-s} - \frac{N-1}{x^s}$$

$$= -\frac{N^{1-s}}{1-s} + x^{1-s} + \frac{x^{1-s}}{s-1} - \frac{1}{N^s} - \frac{N-1}{x^s}$$

$$\left(\text{since } \frac{s}{1-s} = \frac{s-1+1}{1-s} = \frac{1}{1-s} - 1\right)$$

So we see that

$$\left(\sum_{n\le x}\frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \{w\}\frac{dw}{w^{s+1}}\right)$$

$$-\left(\sum_{n\le N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \{w\}\frac{dw}{w^{s+1}}\right)$$

$$= -\frac{1}{N^s} + \frac{x^{1-s}}{s-1} - \frac{N^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^N \{w\}\frac{dw}{w^{s+1}}$$

$$= 0 \quad (\text{since } \{x\} = x-(N-1))$$

So indeed we have the same value $\forall x > 0$ $\square$

25/01/14 Riemann Zeta Function (5)

Definition 3·1

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \{w\} \frac{dw}{w^{s+1}}$$

Using definition 3·1, we can extend the estimates of Lemma 1·2 to hold slightly to the left of the "1-line" ($\{Re(s) = 1\}$).

Lemma 3·2

For any t such that $|t|$ is sufficiently large and any $\sigma > 1 - \frac{100}{\log|t|}$ (say) we have $\zeta(\sigma+it) = O(\log|t|)$. In the same range, we have $\zeta'(\sigma+it) = O(\log^2|t|)$

Proof

If $|t|$ is sufficiently large then $1 - \frac{100}{\log|t|} > 0$ so we can apply definition 3·1. We do this choosing $x = |t|$, and find:

$$\zeta(\sigma+it) = \sum_{n\le|t|} \frac{1}{n^{\sigma+it}} + O\left(\frac{|t|^{\frac{100}{\log t}}}{|t|}\right) + O\left(\frac{1}{|t|^\sigma}\right) + O\left(|\sigma+it|\frac{1}{t^\sigma}\right)$$

$|t|^{\frac{100}{\log t}} = e^{100} = O(1)$

~~large t~~ $|\sigma+it| = O(|t|) + O(\sigma)$ by Δ inequality

$$= \sum_{n\le|t|} \frac{1}{n^{\sigma+it}} + O(1) + O\left(\frac{\sigma}{t^\sigma} + \frac{|t|}{t^\sigma}\right)$$

$$= \sum_{n\le|t|} \frac{1}{n^{\sigma+it}} + O(1) + O(1) + O\left(|t|^{\frac{100}{\log|t|}}\right)$$

$$\zeta(\sigma+it) = \sum_{n\le|t|} \frac{1}{n^{\sigma+it}} + O(1) = O\left(\sum_{n\le|t|} \frac{1}{n^{1-\frac{100}{\log|t|}}}\right) + O(1)$$

$$= O\left(\sum_{n\le|t|} \frac{1}{n}\right) + O(1) = O(\log|t|)$$

To prove the estimate for $\zeta'(\sigma+it)$, just differentiate definition 3·1 with respect to s and estimate as above (choosing $x = |t|$) $\square$

$\log x \log x = \log^2 x$

Next, we obtain a good approximation to the zeta function that is useful further to the left of the 1-line.

Theorem 3·3 (Hardy - Littlewood, 1921)

If $s = \sigma + it$ for any $\sigma > 0$, $t \in \mathbb{R}$, and if $x \geqslant |t|/\pi$,

then $\zeta(\sigma+it) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma})$

Note that the error term $O(x^{-\sigma})$ is much better than in Lemma 1·

(although we must have $x \geqslant |t|/\pi$)

The main ingredient in proving Theorem 3·3 is a Lemma of van der Corput which asserts that certain sums are well approximated by integrals, provided that the summand does not oscillate too fast.

Lemma 3·4 (Special case of van der Corput, 1921)

Let $f(x)$ be a real valued function on an interval $[a,b] \subseteq \mathbb{R}$. Suppose that $f'(x)$ is continuous and monotonic on $[a,b]$, and suppose that $|f'(x)| < \delta$ for some $\delta < 1$.

As usual, write $e(\theta) := \exp(2\pi i \theta)$

Then $\sum_{a < n \leq b} e(f(n)) = \int_a^b e(f(x))\,dx + O\left(\frac{1}{1-\delta}\right)$

Proof (of 3·4)

Idea: Roughly speaking, we shall expand $e(f(n))$ as a Fourier series. It will turn out that the zero Fourier-mode produces the main term of the integral, whilst all of the others contribute to the error term.

More precisely, we have

$$\frac{1}{2}\left(e(f(n)) + e(f(n+1))\right) = \lim_{K \to \infty} \sum_{k=-K}^{K} u_n(k)$$

where $u_n(k) = \int_0^1 e(f(n+x) - kx)\,dx$

25/01/14

Riemann-Zeta Function ⑤

Notice that $u_n(0) = \int_n^{n+1} e(f(x))\,dx$, and if $k \neq 0$, then

$$u_n(k) = \left[\frac{e(f(n+x))\,e(-kx)}{-2\pi i k}\right]_0^1 + \frac{1}{k}\int_0^1 f'(n+x)\,e(f(n+x)-kx)\,dx$$

integration by parts

$$= \underbrace{\frac{1}{2\pi i k}\left(e(f(n)) - e(f(n+1))\right)}_{(\dagger)} + \frac{1}{k}\int_n^{n+1} f'(x)\,e(f(x)-kx)\,dx$$

since $e(-kx) = e(-k(n+x))$.

Here, the first term $(\dagger)$ can be ignored, because the k term cancels the $(-k)$ term.

$\sum_{k=-\infty}^{\infty}$ without $k=0$

$$\text{Therefore, } \sum_{a<n\le b} e(f(n)) = \sum_{\lfloor a\rfloor+1\le n\le \lfloor b\rfloor} \left(u_n(0) + \sum_{k\neq 0} u_n(k)\right) + O(1)$$

$$= \int_a^b e(f(x))\,dx + O(1) + \sum_{k\neq 0}\frac{1}{k}\int_{\lfloor a\rfloor+1}^{\lfloor b\rfloor} f'(x)\,(e(f(x)-kx))\,dx$$

$$= \underbrace{\int_a^b e(f(x))\,dx}_{\text{from } u_n(0),\ k=0} + O(1) + \underbrace{\sum_{k\neq 0}\frac{1}{2\pi i k}\int_{\lfloor a\rfloor+1}^{\lfloor b\rfloor}\frac{f'(x)}{f'(x)-k}\frac{d}{dx}e(f(x)-kx)\,dx}_{\text{from } u_n(k),\ k\neq 0}$$

Notice that since f' is monotonic, and $|f'(x)|<1$, then $\frac{f'(x)}{f'(x)-k}$ is also monotonic for each fixed $k\neq 0$.

Check that the following summation Lemma of Abel holds (Exercise

If $c_1 \ge c_2 \ge \dots \ge c_N$, and $d_1,\dots,d_N \in \mathbb{R}$ are arbitrary, then

$$\left|\sum_{n=1}^{N} c_n d_n\right| \le |c_N|\left|\sum_{n=1}^{N} d_n\right| + (c_{N-1}-c_N)\left|\sum_{n=1}^{N-1} d_n\right| + \dots + (c_1-c_2)|d_1|$$

$$\le (c_1 - c_N + |c_N|)\max_{1\le N'\le N}\left|\sum_{n=1}^{N'} d_n\right|$$

Proof (Abel's Lemma)

Write $c_n d_n = c_n\sum_{k=1}^{n} d_k - c_n\sum_{k=1}^{n-1} d_k$

Then use Δ-inequality

28/01/14

Riemann-Zeta Function ⑥

Theorem 3.4

$f:[a,b]\to\mathbb{R}$. $f'(x)$ continuous, monotonic, $|f'(x)|\le\delta$ for some $\delta<1$. Then $\displaystyle\sum_{a<n\le b} e(f(n)) = \int_a^b e(f(x))\,dx + O\left(\frac{1}{1-\delta}\right)$

Proof (continued)

$$\sum_{a<n\le b} e(f(n)) = \int_a^b e(f(x))\,dx + O(1) + \sum_{k\ne 0}\frac{1}{2\pi i k}\int_{\lfloor a\rfloor+1}^{\lfloor b\rfloor}\left(\frac{f'(x)}{f'(x)-k}\right)\frac{d}{dx}\left(e(f(x)-kx)\right)$$

By approximating by Riemann sums and using Abel's Lemma, we get

$$\int_{\lfloor a\rfloor+1}^{\lfloor b\rfloor}\frac{f'(x)}{f'(x)-k}\frac{d}{dx}\left(e(f(x)-kx)\right)dx \le \max_{[\lfloor a\rfloor+1,\lfloor b\rfloor]}\left|\frac{f'(x)}{f'(x)-k}\right| \max_{[\lfloor a\rfloor+1,\lfloor b\rfloor]}\left|\int_{\lfloor a\rfloor+1}^{x}\frac{d}{dx}e(f(x)-kx)\right|$$

$$\le \frac{1}{|k|-\delta}\cdot 1$$

$$\therefore \sum_{a<n\le b} e(f(n)) = \int_a^b e(f(x))\,dx + O(1) + \underbrace{O\left(\sum_{k\ne 0}\frac{1}{|k|(|k|-\delta)}\right)}_{=\,O\left(\frac{1}{1-\delta}\right)} \qquad \square$$

Proof (Theorem 3.3)

Let $N\ge x$ be a large parameter. By Definition 3.1,

$$\zeta(s) = \sum_{n\le N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} + O(N^{-\sigma}) + O\left(|s|\int_N^\infty \frac{dw}{w^{\sigma+1}}\right)$$

In particular, if N is large enough (in terms of x,σ,t) then both big O terms are $O(x^{-\sigma})$.

Now we need to show that $\left(\sum_{n\le N}\frac{1}{n^s}+\frac{N^{1-s}}{s-1}\right)-\left(\sum_{n\le x}\frac{1}{n^s}+\frac{x^{1-s}}{s-1}\right) = O(x^{-\sigma})$

But LHS is $\displaystyle\sum_{x<n\le N}\frac{1}{n^s} - \int_x^N\frac{1}{w^s}\,dw = \sum_{x<n\le N}\frac{1}{n^\sigma}e^{-it\log n} - \int_x^N\frac{1}{w^\sigma}e^{-it\log w}\,dw$

$$= \sum_{x<n\le N}\left(\int_x^n\frac{-\sigma}{v^{\sigma+1}}dv + \frac{1}{x^\sigma}\right)e^{-it\log n} - \int_x^N\left(\int_x^w -\frac{\sigma}{v^{\sigma+1}}\,dv + \frac{1}{x^\sigma}\right)e^{-it\log w}\,dw$$

(Switch orders of integrals / sums)

unchanged - merely rearranged

$$= \frac{1}{x^\sigma}\left(\sum_{x<n\le N} e^{-it\log n} - \int_x^N e^{-it\log w}\,dw\right)$$
$$+ \int_x^N\frac{-\sigma}{v^{\sigma+1}}\left(\sum_{v<n\le N} e^{-it\log n} - \int_v^N e^{-it\log w}\,dw\right)dv \qquad (*)$$

switched integrals / sums

called "Summation by Parts"

Now observe that $e^{-it\log n} = e(f(n))$ where $f(n) = -\frac{1}{2\pi}t\log n$

$f'(x) = -\dfrac{t}{2\pi x}$, monotonic.

$|f'(z)| \leqslant \frac{|t|}{2\pi x} \leqslant \frac{1}{2}$ since $x \geqslant |t|/\pi$

So by Lemma 3·4 ($\delta = \frac{1}{2}$), we get

$$(*) = O(x^{-\sigma}) + O\left(\int_x^N \frac{\sigma}{v^{\sigma+1}}\,dv\right) = O(x^{-\sigma}) \qquad \square$$

Theorem 3·3 says that we can approximate $\zeta(\sigma+it)$ by a Dirichlet polynomial with $\asymp |t|$ terms (we usually say "of length $|t|$"). It turns out that one can understand Dirichlet polynomial of length $|t|$, evaluated where $\mathrm{Im}(s) \asymp |t|$, quite well.

See Chapter 3 of the course.

4 Prime Number Theorem

Recall. Chebyshev's Psi function $\psi(x) = \sum_{n \leq x} \Lambda(n)$

where $\Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{otherwise} \end{cases}$

We are going to prove the following fundamental result.

Theorem 4·1 (PNT, Hadamard, de la Vallée Poussin, 1896)

As $x \to \infty$, we have $\psi(x) \sim x$.

More precisely, $\psi(x) = x + O(x\exp(-c\log^{1/10} x))$

(Actually, Hadamard and de la Vallée Poussin did not originally obtain this quantitative result with error term, but later, de la Vallée Poussin obtained a better error term $O(x\exp(-c\sqrt{\log x}))$. This is usually called the Prime Number Theorem with Classical error term.

We will prove this later.)

From Theorem 4·1 and summation by parts (exercise), one obtains Corollary 4·2.

28/01/14

Riemann-Zeta Function (6)

Corollary 4.2

As $x \to \infty$, $\pi(x) \sim \int_2^x \frac{dt}{\log t}$

The function $\int_2^x \frac{dt}{\log t}$ is called the Logarithmic Integral $Li(x)$ and one can use integration by parts to expand

$$Li(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\right)$$

general term $\frac{q!\,x}{\log^{q+1} x}$

To prove Theorem 4.1 we need a new ingredient:

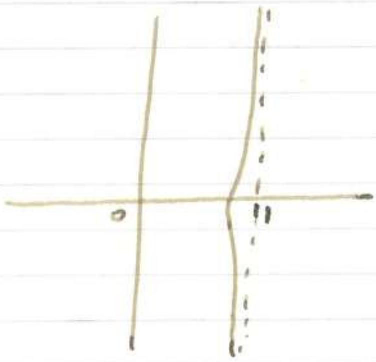
Theorem 4.3 (Weak Zero-free region) Hadamard, de la Vallée Poussin 1896

There exists a small absolute constant $c > 0$ such that the following is true:

For any $t \in \mathbb{R}$, $\sigma \geq 1 - \frac{c}{\log^9(|t|+2)}$, we have

$$\left|\frac{1}{\zeta(\sigma+it)}\right| = O(\log^7(|t|+2))$$

In particular, $\zeta(\sigma + it) \neq 0$.



30/01/14

Riemann-Zeta Function (7)

Most of our recent work has used the series expansion of $\zeta(s)$. In contrast, the main ingredient in the proof of Theorem 4·3 is the Euler product. We will show that this continues to have an influence a little to the left of $\{Re(s) > 1\}$.

Proof of Theorem 4·3

We may assume throughout that $\sigma - 1$ is $\leq$ a small +ve constant, since otherwise the result is trivial. We may also assume that $|t| \geq$ a small positive constant, since otherwise we are close to the pole at $s = 1$, in which case $\frac{1}{\zeta(s)}$ is certainly small.

Let $\sigma' > 1$ be a number, to be chosen later (in terms of σ, t).

For any $t' \in \mathbb{R}$, we have

$|z| = |e^{\log z}| = |\exp(\log|z| + i\arg(z))| = e^{\log|z|} = e^{Re(\log z)}$

$Re(\log z) = \log|z|$

$Re\left(\log\left(1 - \frac{1}{p^{\sigma'+it'}}\right)\right) = \log\left|1 - \frac{1}{p^{\sigma'+it'}}\right|$.

$$|\zeta(\sigma'+it')| = \exp(Re \log \zeta(\sigma'+it'))$$

$$= \exp\left(-Re \sum_p \log\left(1 - \frac{1}{p^{\sigma'+it'}}\right)\right) \quad \text{(Euler Product } \sigma' > 1\text{)}$$

p prime

$$= \exp\left(\sum_{p^k} \frac{\cos(t'k\log p)}{k p^{k\sigma'}}\right) \quad \text{use } \log(1-z) = -\left(z + \frac{z^2}{2} + \dots\right)$$

Key Idea

Consider the following product:

$$\zeta(\sigma')^3 |\zeta(\sigma'+it)|^4 |\zeta(\sigma'+2it)| = \exp\left(\sum_{p^k} \frac{3 + 4\cos(kt\log p) + \cos(2kt\log p)}{k p^{k\sigma'}}\right)$$

no imaginary part so expands as above.

$$= \exp\left(\sum_{p^k} \frac{2(1+\cos(kt\log p))^2}{k p^{k\sigma'}}\right) \quad \text{as } 1 + \cos 2\theta = 2\cos^2\theta$$

$$\geq 1$$

Therefore the only way that $\zeta(\sigma'+it)$ can be very small is if $\zeta(\sigma')$, $\zeta(\sigma'+2it)$ are very large.

More precisely, $|\zeta(\sigma'+it)| \geq \dfrac{1}{\zeta(\sigma')^{3/4}|\zeta(\sigma'+2it)|^{1/4}}$

$$>> \frac{(\sigma'-1)^{3/4}}{\log^{1/4}(|t|+2)}$$

$\longleftarrow$ since $\zeta(s)$ has a simple pole at $s=1$

$\longleftarrow$ Lemma 1·2

(*)

$|\zeta(\sigma+2it)| = O(\log|2t|)$, $\log|2t| = \log 2 + \log|t| \leq \log(|t|+2)$ $\Leftrightarrow$ $2|t| \leq |t|+$

or $\log 2|t| = \log(|t|+2) + O(1)$

Now observe that $|\zeta(\sigma+it)| \geq |\zeta(\sigma'+it)| - \int_\sigma^{\sigma'} |\zeta'(r+it)|\,dr$

$$= |\zeta(\sigma'+it)| + O(|\sigma'-\sigma|\log^2(|t|+2)) \text{ by Lemma 3·2}$$

If $\sigma \geq 1 + \dfrac{c}{\log^9(|t|+2)}$ then we can choose $\underline{\sigma' = \sigma}$, and

$$(*) \Rightarrow |\zeta(\sigma+it)| = |\zeta(\sigma'+it)| >> \frac{c^{3/4}}{\log^7(|t|+2)}$$

If instead $1 - \dfrac{c}{\log^9(|t|+2)} \leq \sigma < 1 + \dfrac{c}{\log^9(|t|+2)}$,

we choose $\sigma' = 1 + \dfrac{c}{\log^9(|t|+2)}$, and find that

$$\frac{(\sigma'-1)^{3/4}}{\log^{1/4}(|t|+2)} >> \frac{c^{3/4}}{\log^7(|t|+2)}, \quad |\sigma'-\sigma|\log^2(|t|+2) << \frac{c}{\log^7(|t|+2)}$$

So if c is sufficiently small, we again have

$\zeta(\sigma+it) >> c^{3/4}/\log^7(|t|+2)$ □

<u>Proof of Theorem 4·1</u>

Recall from Lemma 2·4 (and subsequent discussion) that if

x is large and $1 < T \leq x$ then we have

$$\psi(x) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds + O\left(\frac{x\log^2 x}{T}\right)$$

We will use Cauchy's Residue Theorem to estimate the integral, and then choose T to balance all the error terms.

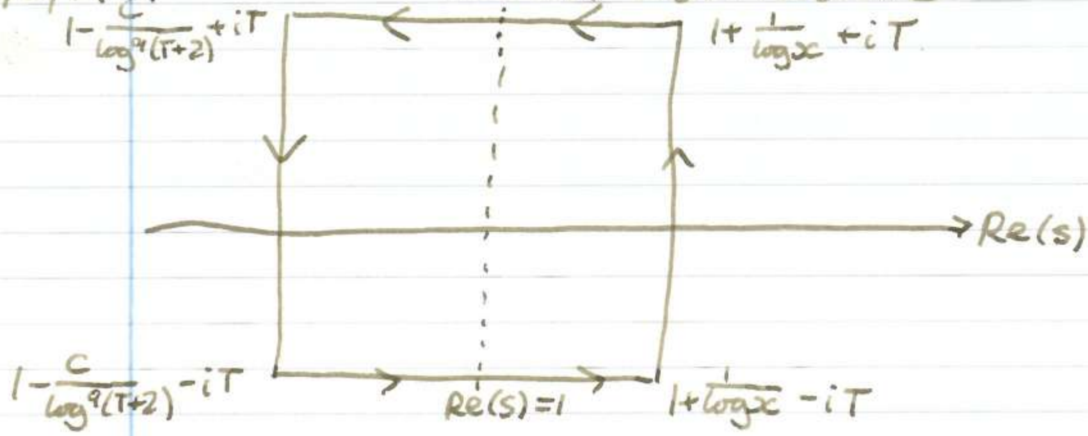
Indeed, we have $\psi(x) = \operatorname*{Res}_{s=1}\left(-\dfrac{\zeta'(s)}{\zeta(s)}\dfrac{x^s}{s}\right)$

$$+ \frac{1}{2\pi i}\int_{1-\frac{c}{\log^9(T+2)}-iT}^{1-\frac{c}{\log^9(T+2)}+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

$$+ \frac{1}{2\pi i}\int_{1-\frac{c}{\log^9(T+2)}+iT}^{1+\frac{1}{\log x}+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds - \frac{1}{2\pi i}\int_{1-\frac{c}{\log^9(T+2)}-iT}^{1+\frac{1}{\log x}-iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

$+O\left(\frac{x}{T}\log^2 x\right)$

30/01/14. Riemann-Zeta Function ⑦



since by Theorem 4·3, $\frac{\zeta'(s)}{\zeta(s)}$ has no poles except $s=1$ in the contour above.

You can check that $\operatorname{Res}_{s=1}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = x$

since $\zeta(s) = \frac{1}{s-1}$ + some error
$\zeta'(s) = -\frac{1}{(s-1)^2}$ + some error near $s=1$

$x^s = x \cdot x^{s-1} = \exp\left(x\log(s-1)\right) x$

x^s has residue x.

The rest have residue 1.

3

31/02/14

Riemann Zeta Function ⑧

First we bound the contribution from the short horizontal integrals.

On the line segment $\left[1-\frac{c}{\log^9(T+2)}+iT,\ 1+\frac{1}{\log x}+iT\right]$,

we have $\left|-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right| \leq \frac{1}{T}x^{1+\frac{1}{\log x}} \max_{s\in\left[1-\frac{c}{\log^9(T+2)}+iT,\ 1+\frac{1}{\log x}+iT\right]} |\zeta'(s)|$

$|s|\geq T$

Lemma 3.2

Lemma 4.3 $\times \max_{s\in\left[1-\frac{c}{\log^9(T+2)}+iT,\ 1+\frac{1}{\log x}+iT\right]} \frac{1}{|\zeta(s)|}$

$$\ll \frac{x}{T}\log^2 T\cdot\log^7(T) = \frac{x}{T}\log^9 T \leq \frac{x}{T}\log^9 x$$

So the same is true on the line segment $\left[1-\frac{c}{\log^9(T+2)}-iT,\ 1+\frac{1}{\log x}-iT\right]$.

Therefore, the contribution from both of these integrals is $O\left(\frac{x\log^9 x}{T}\right)$.

On the vertical line $\left[1-\frac{c}{\log^9(T+2)}-iT,\ 1-\frac{c}{\log^9(T+2)}+iT\right]$, we have

$$\left|-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right| \leq \frac{1}{|s|}x^{1-\frac{c}{\log^9(T+2)}} \max_{s\in\left[1-\frac{c}{\log^9(T+2)}-iT,\ 1-\frac{c}{\log^9(T+2)}+iT\right]} |\zeta'(s)|$$

$$\times \max_{s\in\left[1-\frac{c}{\log^9(T+2)}-iT,\ 1-\frac{c}{\log^9(T+2)}+iT\right]} 1/|\zeta(s)|$$

$$\ll \frac{1}{|s|}x^{1-\frac{c}{\log^9(T+2)}} \max|\zeta'(s)| \log^7 T$$ by Lemma 4.3

We must be careful when bounding $\max|\zeta'(s)|$, because (on the vertical line) s can be quite close to the pole at $s=1$.

However, we actually always have $|s-1| \geq \frac{c}{\log^9(T+2)}$, so we have

$\log^2 T$ away from pole — close to the pole $\log^{18} T$

$\max|\zeta'(s)| \ll \log^{18} T$ by Lemma 3.2 + considering the pole.

$$\psi(x) = x + O\left(x^{1-\frac{c}{\log^9(T+2)}}\log^{25}x\int_{-T}^{T}\frac{1}{1+|t|}dt\right) + O\left(\frac{x\log^9 x}{T}\right)$$

$$= x + O\left(x^{1-\frac{c}{\log^9(T+2)}}\log^{26}x\right) + O\left(\tfrac{1}{T}x\log^9 x\right)$$

$$= x + O\left(x \log^{26} x \left(e^{-\frac{c\log x}{\log^9(T+2)}} + e^{-\log T}\right)\right)$$

Choose $T = \exp(\log^{1/10} x)$ to balance the terms, we obtain

$$\Psi(x) = x + O(x \log^{26} x \cdot e^{-c\log^{1/10} x})$$

$'' \; e^{26 \log\log x}$

Finally, since $e^{c\log^{1/10} x}$ is much larger than $\log^{26} x$, we can remove the factor $\log^{26} x$ at the cost of replacing c by $c/2$. $\square$

Remark 4.4

From now on, we shall "absorb logarithmic factors" as we did in the last line of the above proof without much comment.

5 Widening the zero-free region

Theorem 4.1 (PNT) is a fundamental result, but the error term $O(x e^{-c\log^{1/10} x})$ is awkward and unsatisfactory. For example, suppose that we wanted to investigate the distribution of $\Lambda(n)$ in fairly long intervals $[x, x e^{-\log^{1/4} x} + x]$. The obvious approach is to note that

$$\sum_{x < n \leq x + xe^{-\log^{1/4} x}} \Lambda(n) = \Psi(x + xe^{-\log^{1/4} x}) - \Psi(x).$$

By PNT,
$$= (x + xe^{-\log^{1/4} x} + O(xe^{-c\log^{1/10} x})) - (x + O(xe^{-c\log^{1/10} x}))$$
$$= xe^{-\log^{1/4} x} + O(xe^{-c\log^{1/10} x})$$

But this is useless since the "main term" $xe^{-\log^{1/4} x}$ is much smaller than the O error term $O(xe^{-c\log^{1/10} x})$. So for all we know (at this point) they may cancel completely, so we cannot even guarantee that there will be a single prime in the interval.

31/02/14 Riemann-Zeta Function ⑧

Later in the course, we will develop a more sophisticated approach to primes in intervals. But as an immediate goal, we try to reduce th error term in Theorem 4.1 (PNT).

We saw, in the proof of Theorem 4.1, that the quality of the error term ~~decreases~~ depended on how far we could shift the line of integration to the left, which in turn depended on how wide a zero-free region we had.

To obtain a wider zero-free region, we shall prove an important technical theorem.

<u>Theorem 5.1</u> (Landau, 1924)

Let $\phi(t) \geq 1$, $w(t) \geq 1$, non-decreasing, and such that $\phi(t) \to \infty$ as $t \to \infty$. Also, let $t_0 \geq 0$ be any fixed constant.

Suppose that $w(t) = O(e^{\phi(t)/2})$, <u>and</u> that

$\zeta(\sigma+it) = O(e^{\phi(t)})$ $\forall\, \sigma$ with $1 - \frac{1}{w(t)} \leq \sigma \leq 2$, $\forall\, t \geq t_0$.

Then, $\exists$ a constant $c > 0$, depending only on the implicit constants in the conditions above, such that

$\zeta(\sigma+it) \neq 0 \;\; \forall\; 1 - \frac{c}{\phi(2t+1)\,w(2t+1)} \leq \sigma$, $t \geq t_0$

04/02/14 Riemann-Zeta Function (9)

Theorem 5·1 (Landau 1924)

may be redundant

$\phi(t)$, $w(t)$ non-decreasing, $\phi(t) \to \infty$ as $t \to \infty$. Let $t_0 \geq 0$

Suppose that $w(t) = O(e^{\phi(t)/2})$ and $\zeta(\sigma+it) = O(e^{\phi(t)})$

$\forall\ 1 - \frac{1}{w(t)} \leq \sigma \leq 2$, $\forall t \geq t_0$.

depends on implicit constants above

Then $\exists$ a constant $c > 0$ such that

$\zeta(\sigma+it) \neq 0 \quad \forall\ t \geq t_0,\ \sigma \geq 1 - \frac{c}{w(2t+1)\phi(2t+1)}$

Corollary 5·2 (Classical zero-free region)

There exists a constant $c > 0$ such that $\zeta(\sigma+it) \neq 0$ when

$\sigma \geq 1 - \frac{c}{\log(|t|+2)}$ – In Theorem 4·3 we had $\frac{c}{\log^9(|t|+2)}$

Proof

We apply Theorem 5·1, with $w(t) \equiv 2$, $\phi(t) = \log t$, $t_0 = 3$ say

We just need to check that $\zeta(\sigma+it) = O(t)$ $\forall \sigma \geq \frac{1}{2}$, $t \geq 3$

However, using Theorem 3·3 (with the choice $x = t$) we get

$$\zeta(\sigma+it) = \sum_{n \leq t} \frac{1}{n^{\sigma+it}} + \frac{t^{1-(\sigma+it)}}{\sigma+it-1} + O(t^{-\sigma})$$

$$= \sum_{n \leq t} \frac{1}{n^{\sigma+it}} + O(1) = O(\sqrt{t}) \quad \text{as } \sigma \geq \tfrac{1}{2}$$

So Theorem 5·1 $\Rightarrow \zeta(\sigma+it) \neq 0$ when $\sigma \geq 1 - \frac{c}{\log(2t+1)}$, $t \geq 3$

If $0 \leq t < 3$, the theorem follows from Theorem 4·3 (weak zero-free region), provided $c > 0$ is small.

Finally we can check in Definition 3·1 that

$\zeta(\sigma - it) = \overline{\zeta(\sigma+it)}$ so the result follows for $t < 0$ by symmetry.

Remark 5·3

i) It isn't really necessary to use Theorem 3·3 here; Definition 3·1 would suffice.

ii) By choosing to smaller and considering the pole at $s=1$, we can avoid using Theorem 4·3.

The proof of Theorem 5·1 has two ingredients. One is (more or less) the fact that $\zeta(\sigma')^3|\zeta(\sigma'+it)|^4|\zeta(\sigma'+2it)| \geq 1$ if $\sigma' > 1$, as we have seen before. The new ingredient is the following lemma:

Lemma 5·4

Let $r, M > 0$, and let $z_0 \in \mathbb{C}$. Suppose that $f(z)$ is a holomorphic function on the disc $|z-z_0| \leq r$, that $f(z_0) \neq 0$ and that $\left|\frac{f(z)}{f(z_0)}\right| \leq M \;\forall\, |z-z_0| \leq r$.

Then if $f(z) \neq 0$ in the right half of the disc $(\mathrm{Re}(z) > \mathrm{Re}(z_0))$ then

$$\underbrace{\mathrm{Re}\,\frac{f'(z_0)}{f(z_0)}}_{\left(\frac{d}{dz}\log f(z)\right)_{z=z_0}} \geq -\frac{8\log M}{r} + \mathrm{Re}\sum_{\substack{\rho : f(\rho)=0 \\ \mathrm{Re}(\rho)\leq \mathrm{Re}(z_0),\ (|\rho - z_0| \leq \frac{r}{2})}} \frac{1}{z_0-\rho}$$

Proof

We use some facts from Complex Analysis.

Let Z denote the multiset of all zeros of f in the small disc $|z-z_0| \leq \frac{r}{2}$, counted with multiplicity. (This must be a finite multiset since otherwise the zeroes of f would have a limit point, so we would have $f \equiv 0$ #)

Then we define a function $g(z)$ on the large disc $|z-z_0| \leq r$ by $g(z) = \begin{cases} f(z)\prod_{\rho\in Z}\frac{1}{z-\rho} & \text{if } z \notin Z \\ \lim g(z') & \text{if } z \in Z \end{cases}$ (N.B. $z' \notin Z$)

04/02/14 Riemann Zeta Function ⑨

Note that $g(z)$ is holomorphic on the large disc $|z-z_0| \leq r$ (and non-zero on the small disc $|z-z_0| \leq \frac{r}{2}$), so by the Maximum Modulus
because we have divided out all these zeroes
Principle,

$$\max_{|z-z_0|\leq r} \left|\frac{g(z)}{g(z_0)}\right| = \max_{|z-z_0|=r} \left|\frac{g(z)}{g(z_0)}\right| \leq \max_{|z-z_0|=r} \left|\frac{f(z)}{f(z_0)}\right| \max_{|z-z_0|=r} \prod_{\rho\in\mathcal{Z}} \left|\frac{z_0-}{z-\rho}\right.$$

$$\leq \underset{\text{by hypothesis}}{M} \cdot 1$$

since $|z-\rho| \geq \frac{r}{2}$ $\forall \rho \in \mathcal{Z}$, $|z-z_0|=r$, $|z_0-\rho| \leq \frac{r}{2}$ $\forall \rho \in \mathcal{Z}$

Next, we can define $h(z) := \log(g(z)/g(z_0))$ by taking the principal branch of log (since g does not vanish), a holomorphic function on the small disc $|z-z_0| \leq \frac{r}{2}$, such that

$h(z_0) = 0$, $\mathrm{Re}(h(z)) = \log\left|\frac{g(z)}{g(z_0)}\right| \leq \log M$.

But one can bound the modulus of a holomorphic function at a point given a bound for its real part in a surrounding disc:

since $h(z_0) = 0$, the <u>Borel–Carathéodory Theorem</u> implies that for any $r' < \frac{r}{2}$, $\max_{|z-z_0|\leq r'} |h(z)| \leq \frac{2r'}{(\frac{r}{2})-r'} \log M$. $\overset{r'=r/4}{=} 2\log M$

Therefore $|h'(z_0)| = \left|\frac{1}{2\pi i}\int_{|z-z_0|=r/4} \frac{h(z)}{(z-z_0)^2}\,dz\right|$ Cauchy's Integral Formula

$\cancel{\leq \frac{8\log M}{r}}$

$$\leq \frac{1}{2\pi}\int_{|z-z_0|=r/4} \left|\frac{h(z)}{(z-z_0)^2}\right| dz$$

$$\leq \frac{1}{2\pi}\left(2\pi \tfrac{r}{4}\right)\left(\frac{4}{r}\right)^2 (2\log M)$$

$$= \frac{8\log M}{r}$$

2

Riemann Zeta Function (10)

Proof (Lemma 5.4)

Last time : $h(z) = \log\left(\frac{g(z)}{g(z_0)}\right)$

$|h'(z_0)| \leq \frac{8\log M}{r}$ (*)

Finally, $\frac{f'(z_0)}{f(z_0)} = \frac{d}{dz}\log f(z)\big|_{z=z_0}$

$= \frac{d}{dz}\log g(z)\big|_{z=z_0} + \sum_{\rho\in Z}\frac{1}{z_0-\rho}$

Check conclusion In online notes

$\mathrm{Re}\,\frac{f'(z_0)}{f(z_0)} = \mathrm{Re}\,h'(z_0) + \mathrm{Re}\sum_{\rho\in Z}\frac{1}{z_0-\rho} \geq -\frac{8\log M}{r} + \sum_{\rho\in Z}\frac{1}{z_0-\rho}$ by (*) □

Note that $\mathrm{Re}\,\frac{1}{z_0-\rho} = \mathrm{Re}\,\frac{(z_0-\rho)^*}{|z_0-\rho|^2} > 0$ $\forall \rho$ in the sum.

Thus we also have $\mathrm{Re}\,\frac{f'(z_0)}{f(z_0)} \geq -\frac{8\log M}{r}$

Now we are ready to prove Theorem 5.1 (Landau). The proof is quite similar to the proof of the weak zero-free region (Theorem 4.3), but using Lemma 5.4 instead of bounds for $\zeta'(s)$, $\zeta(s)$.

Therefore there are a few more technical details (selecting r, M).

Proof (of Theorem 5.1)

Let $t \geq t_0$ and $\sigma > 0$. We wish to prove that if $\zeta(\sigma+it)=0$ then we must have $\sigma < 1 - \frac{c}{w(2t+1)\phi(2t+1)}$ where $\sigma > 0$ is a constant that may depend (possibly) on the implicit constants in the hypotheses of the theorem.

weak zero-free region

We may assume that $\sigma < 1$ (because of Theorem 4.3), and we may assume that $|t| \geq 10$ (also due to Theorem 4.3).

Let $\sigma' > 1$ be a number, to be chosen later in terms of σ, t.

We will choose $\sigma'-1$ sufficiently small that $\zeta'(\sigma')/\zeta(\sigma')$ is under the influence of the pole at 1; more specifically, so that

$$\zeta'(\sigma')/\zeta(\sigma') \geq -\frac{5/4}{\sigma'-1}.$$

As calculated at the begining of Section 2 (and similarly in the proof of Theorem 4·3), since $\sigma' > 1$, the Euler Product implies that

$$-3\frac{\zeta'(\sigma')}{\zeta(\sigma')} - 4\,\mathrm{Re}\,\frac{\zeta'(\sigma'+it)}{\zeta(\sigma'+it)} - \mathrm{Re}\,\frac{\zeta'(\sigma'+2it)}{\zeta(\sigma'+2it)} \qquad \geq 0 \quad (*)$$

$$= 3\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma'}} + 4\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma'+it}} + \mathrm{Re}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma'+2it}}$$

$$= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma'}}\left(3 + 4\cos(t\log n) + \cos(2t\log n)\right)$$

$$= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}\, 2(1+\cos(t\log n))^2 \geq 0$$

Now let $0 < r \leq 1$ be another parameter to be chosen later, and let $M = M(r, \sigma', t)$ be such that

$$\left|\frac{\zeta(s)}{\zeta(\sigma'+it)}\right| \leq M \quad \forall\, |s-(\sigma'+it)| \leq r$$

and $\left|\frac{\zeta(s)}{\zeta(\sigma'+2it)}\right| \leq M \quad \forall\, |s-(\sigma'+2it)| \leq r$

Get from continuity of these functions.

WLOG, $M > 1$. Then, our assumption that $\frac{\zeta'(\sigma')}{\zeta(\sigma')} \geq -\frac{5/4}{\sigma'-1}$ together with Lemma 5·4 applied to $f = 3$, imply that

$$(*) \quad -\mathrm{Re}\,\frac{\zeta'(\sigma'+it)}{\zeta(\sigma'+it)} \geq \frac{3}{4}\zeta'(\sigma')/\zeta(\sigma') + \frac{1}{4}\mathrm{Re}\,\frac{\zeta'(\sigma'+2it)}{\zeta(\sigma'+2it)}$$

$$\geq -\frac{15/16}{\sigma'-1} - \frac{2\log M}{r}$$

Note that this number is <1.

discard terms of $\mathrm{Re}\sum\frac{1}{z_0-\rho}$

Lemma 5·4 also implies that $-\mathrm{Re}\frac{\zeta'(\sigma'+it)}{\zeta(\sigma'+it)} \leq \frac{8\log M}{r} - \mathrm{Re}\sum\frac{1}{\sigma'+it-\rho}$

(sum over $\rho : \zeta(\rho) = 0,\ |\rho-(\sigma'+it)| \leq \frac{r}{2},\ \mathrm{Re}\,\rho < \sigma'$)

In particular, if $\zeta(\sigma+it) = 0$, then

(i) <u>either</u> $\sigma < \sigma' - \frac{r}{2}$ close to pole

(ii) <u>or</u> $-\mathrm{Re}\,\frac{\zeta'(\sigma'+it)}{\zeta(\sigma'+it)} \leq \frac{8\log M}{r} - \frac{1}{\sigma'-\sigma}$ above estimates hold

(since all terms in $\sum_{\rho}$ are ≥ 0).

05/02/14 Riemann Zeta Function (10)

So we <u>either</u> have $\sigma < \sigma' - \frac{r}{2}$ (i)

$$\underline{\text{or}} \quad \frac{1}{\sigma'-\sigma} < \frac{15/16}{\sigma'-1} + \frac{10\log M}{r} \quad (*) \quad \text{(ii)}$$

Choose $\sigma' = 1 + c\min\{1, \frac{r}{\log M}\}$, where $c > 0$ is a small absolute constant.

Then the RHS of $(*)$ is $\leq \frac{31/32}{c\min\{1, \frac{r}{\log M}\}}$ ⟵ for small enough c, $10\frac{\log M}{r} \leq \frac{1/32}{c\min\{1, \frac{r}{\log M}\}}$

so <u>either</u> (i) $\sigma < \sigma' - \frac{r}{2} = 1 + c\min\{1, \frac{r}{\log M}\} - \frac{r}{2}$

<u>or</u> (ii) $\sigma < 1 - \frac{c}{100}\min\{1, \frac{r}{\log M}\}$ (plug σ' into $(*)$)

It remains to choose $0 < r \leq 1$, and to see what value of M is permissible. By the hypotheses of the theorem, if we choose $r = \frac{1}{w(2t+1)} \leq 1$, we have

$\zeta(s) = O(e^{\phi(2t+1)})$ in the discs $|s-(\sigma'+it)| \leq r$, $|s-(\sigma'+2it)| \leq r$.

3

08/02/14

Riemann Zeta Function 11

Proof (continued)

$|\zeta(s)| = O(e^{\phi(2t+1)})$ in the discs $|s-(\sigma'+it)| \le r$, $|s-(\sigma'+2it)| \le r$

In view of the Euler product $(\sigma' > 1)$

$$\frac{1}{|\zeta(\sigma'+it)|} = \prod_p \left|1 - \frac{1}{p^{\sigma'+it}}\right| \le \prod_p \left(1 + \frac{1}{p^{\sigma'}}\right)$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n^{\sigma'}} \le \zeta(\sigma') \ll \frac{1}{\sigma'-1} = \frac{1}{c}\max\left\{1, \frac{\log M}{r}\right\}$$

The same bound applies to $\frac{1}{|\zeta(\sigma'+2it)|}$

Now, we can choose $M = e^{c\phi(2t+1)}$ for a suitable large constant $c > 0$.

$\sigma < 1 + c\min\{1, \frac{r}{\log M}\} - \frac{r}{2}$

Inserting these values of r, M into $\sigma < 1 - \frac{c}{100}\min\{1, \frac{r}{\log M}\}$ finishes the proof. $\square$

Having established the classical zero-free region (Corollary 5·2) we are almost ready to prove the Prime Number Theorem with classical error term, $O(xe^{-c\sqrt{\log x}})$.

To do this, we need a bound for $\left|\frac{\zeta'(s)}{\zeta(s)}\right|$ inside the classical zero-free region (Recall that when we proved Theorem 4·3, we proved a bound for $\left|\frac{1}{\zeta(s)}\right|$ at the same time).

Lemma 5·5

There exists a small constant $c > 0$ such that:

For any $|t| \ge 1$, if $\sigma \ge 1 - \frac{c}{\log(|t|+2)}$ then

$$\left|\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right| \ll \log(|t|+2)$$

Proof

Like the proof of Lemma 5·4. Omitted. $\square$

Theorem 5·6 (PNT with Classical Error)

For all $x \geq 2$, we have $\Psi(x) = x + O(x e^{-c\sqrt{\log x}})$

Proof

(PNT with weak error term)

Exactly like Theorem 4·1, but moving the line of integration further to the left using the classical zero-free region.

Using the truncated Perron formula, Cauchy's residue theorem and the classical zero free region, we obtain

$$\Psi(x) = x + \frac{1}{2\pi i}\int_{1-\frac{c}{\log(T+2)} - iT}^{1-\frac{c}{\log(T+2)}+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

$$+ \frac{1}{2\pi i}\int_{1-\frac{c}{\log(T+2)}+iT}^{1+\frac{1}{\log x}+iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds - \frac{1}{2\pi i}\int_{1-\frac{c}{\log(T+2)}-iT}^{1+\frac{1}{\log x}-iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

$$+ O\left(\frac{x}{T}\log^2 x\right) \qquad \text{if } 1 < T \leq x.$$

On the short horizontal line, $\left[1-\frac{c}{\log(T+2)}+iT,\ 1+\frac{1}{\log x}+iT\right]$

we have $\left|-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right| \leq \frac{1}{T}x^{1+\frac{1}{\log x}} \max_{s\in\left[1-\frac{c}{\log(T+2)}+iT,\ 1-\frac{1}{\log x}+iT\right]}\left|\frac{\zeta'(s)}{\zeta(s)}\right|$.

$$\ll \frac{1}{T}x^{1+\frac{1}{\log x}}\log(T+2) \qquad \text{by Lemma 5·5}$$

$$\ll \frac{1}{T}x\log x$$

The same applies to $\left[1-\frac{c}{\log(T+2)}-iT,\ 1+\frac{1}{\log x}-iT\right]$.

On the vertical line, $\left[1-\frac{c}{\log(T+2)}-iT,\ 1-\frac{c}{\log(T+2)}+iT\right]$, we

have $\left|-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right| \leq \frac{1}{|s|}x^{1-\frac{c}{\log(T+2)}}\max\left|\frac{\zeta'(s)}{\zeta(s)}\right|$

by Lemma 5·5 if $|\mathrm{Im}(s)| \geq 1$ and since $\frac{\zeta'(s)}{\zeta(s)} \sim \frac{-1}{s-1}$ as $s \to 1$

$$\to \ll \frac{1}{|s|}x^{1-\frac{c}{\log(T+2)}}\log(T+2)$$

$$\ll \frac{1}{|s|}x^{1-\frac{c}{\log(T+2)}}\log x$$

28/02/14 Riemann - Zeta Function 11

So $\psi(x) = x + O\left(x^{1-\frac{c}{\log(T+2)}} \log x \int_{-T}^{T} \frac{dt}{1+|t|}\right) + O\left(\frac{x}{T}\log^2 x\right)$

$= x + O\left(x \log^2 x \left(e^{-\frac{c\log x}{\log(T+2)}} + e^{-\log T}\right)\right)$

Choosing $T = \exp(\sqrt{\log x})$ gives the result. □

6 A quick break

So far, we have only defined $\zeta(s)$ when $\mathrm{Re}(s) > 0$. But Riemann himself knew and proved that $\zeta(s)$ could be analytically continued to a meromorphic function on $\mathbb{C}$.

Theorem 6·1 (Functional equation, Riemann, 1859)

For all $s \in \mathbb{C}$, we have

$$\pi^{-s/2}\zeta(s)\Gamma\left(\tfrac{s}{2}\right) = \pi^{-\frac{1-s}{2}}\zeta(1-s)\Gamma\left(\tfrac{1-s}{2}\right)$$

where $\Gamma(z) := \int_0^\infty e^{-x} x^{z-1}\,dx \qquad \mathrm{Re}(z) > 0$

and defined elsewhere by analytic continuation (e.g. by using the equation $z\Gamma(z) = \Gamma(z+1)$ or using the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$).

Remark 6·2

When $0 < \mathrm{Re}(s) < 1$, all the terms in the functional equation have straightforward definitions, and it says that there is a certain symmetry around the critical line $\mathrm{Re}(s) = \frac{1}{2}$

But when $\mathrm{Re}(s) \leq 0$, we can use the functional equation to define $\zeta(s)$.

$\Gamma(z)$ is a meromorphic function, non-zero, with simple poles at $z = 0, -1, -2, \ldots$.

trivial zeros

Therefore, we must have $\zeta(s) = 0$ when $s = -2, -4, -6, \ldots$ to cancel the poles of $\Gamma(\frac{s}{2})$ on the left hand side of the functional equation.

Remark 6.3

Chapter 2, Titchmarsh, has seven different proofs of the functional equation.

Remark 6.4

It is believed that the reason that the Riemann Hypothesis is true is because the Euler Product forces all zeta zeroes to be close to the critical line, and the functional equation forces them onto the line.

11/02/14 Riemann-Zeta Function 12

As an alternative to repeatedly applying Perron's formula with the line of integration in different positions, one can use an explicit formula that directly links $\Psi(x)$ with the zeroes of $\zeta(s)$.

Theorem 6·5 (Explicit formula, von Mangoldt, 1895)

For any $2 \leq T \leq x$, we have

$$\Psi(x) = x - \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |Im(\rho)| \leq T}} \frac{x^\rho}{\rho} + O\left(\frac{x}{T} \log^2 x\right)$$

To prove Theorem 6·5, one applies Perron's formula for $\Psi(x)$, then moves the line of integration very far to the left (to pick up residues from the zeroes of $\zeta(s)$, as appear in $\sum_\rho$) and then estimate the contour integral using the functional equation.

Chapter 2

7 First Thoughts on Estimating ζ Sums

In Section 5 we proved Landau's Theorem (5·1) which showed that if we had a bound $\zeta(\sigma + it) = O(e^{\phi(t)})$ in some region to the left of the 1-line (with $\phi(t)$ hopefully not growing too fast), we could deduce a zero-free region for $\zeta(s)$.

In this chapter we will prove a very strong bound for $\zeta(\sigma + it)$ due to Vinogradov and Korobov in 1958, and use it to deduce the best known zero-free region.

If $t \geq 1$, $\sigma \geq 1$, then Hardy and Littlewood's approximation to $\zeta(s)$ (Theorem 3·3) with the choice $x = t$ yields that $\zeta(\sigma+it) = \sum_{n \le t} \frac{1}{n^{\sigma+it}} + \frac{t^{1-(\sigma+it)}}{\sigma+it-1} + O(t^{-\sigma})$

$$= \sum_{n \le t} \frac{1}{n^{\sigma+it}} + O(1)$$

So by partial summation, bounding $\zeta(\sigma+it)$ is basically equivalent to bounding sums $\sum_{N<n<N+M} n^{-it}$, where $M \le N \le t$.

These are sometimes called <u>zeta sums</u>.

Note that to bound the whole sum $\sum_{n \le t} \frac{1}{n^{\sigma+it}}$, we need to bound zeta sums with <u>N possibly much smaller than t</u>.

When we proved Theorem 3·3, we used Fourier analysis to show that $\sum_{N<n \le N+M} n^{-it}$ behaved like $\int_N^{N+M} w^{-it}\,dw$, when <u>$N >> t$</u>.

We don't know how to do that efficiently when N is smaller.

Instead, we will bound the zeta sums directly, using more combinatorial arguments.

The summands n^{-it} don't appear to have much useable structure, so the first step is to apply Taylor expansion to obtain some polynomial structure.

<u>Lemma 7·1</u>

Suppose that N is large, and $1 \le M \le N \le t$. powers are $\frac{4}{5}$

Set $r := \lfloor 5.01 \frac{\log t}{\log N} \rfloor$. Then $\sum_{N<n \le N+M} n^{-it} = O\left(M \max_{N<n \le 2N} \frac{|U(n)|}{N^{4/5}} + N^{4/5} + Mt^{-\frac{1}{500}}\right)$

where $U(n) := \sum_{x \le N^{2/5}} \sum_{y \le N^{2/5}} e(\alpha_1 xy + \alpha_2 x^2y^2 + \dots + \alpha_r x^r y^r)$

and $\alpha_j := \frac{(-1)^j t}{2\pi j n^j}$

$$\sum_{N<n \le N+M} n^{-it} = O\left(M \max_{N<n \le 2N} \frac{|U(n)|}{N^{4/5}} + N^{4/5} + Mt^{-\frac{1}{500}}\right)$$

11/02/14 Riemann Zeta Function ⑫ 1

Proof

Note that $\sum_{N<n\le N+M} n^{-it} = \underbrace{\frac{1}{\lfloor N^{2/5}\rfloor^2} \sum_{x\le N^{2/5}} \sum_{x\le N^{2/5}}}_{\text{trivial}} \sum_{N<n\le N+M} n^{-it}$

$$= \frac{1}{\lfloor N^{2/5}\rfloor^2} \sum_{x\le N^{2/5}} \sum_{y\le N^{2/5}} \left(\sum_{N<n\le N+M} (n+xy)^{-it} + O(N^{4/5}) \right)$$

$$= \sum_{N<n\le N+M} n^{-it} \frac{1}{\lfloor N^{2/5}\rfloor^2} \sum_{x\le N^{2/5}} \sum_{y\le N^{2/5}} \left(1+\frac{xy}{n}\right)^{-it} + O(N^{4/5})$$

Since the shift xy is much smaller than n, we can now apply a Taylor expansion efficiently. We have

$$\log\left(1+\frac{xy}{n}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}\left(\frac{xy}{n}\right)^j$$

$$= \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j}\left(\frac{xy}{n}\right)^j + O\left(\left(\frac{xy}{n}\right)^{r+1}\right)$$

$$= \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j}\left(\frac{xy}{n}\right)^j + O\left(t^{-(1+\frac{1}{500})}\right)$$

since $r+1 \geq 5.01 \frac{\log t}{\log N}$, $\frac{xy}{n} \le N^{-1/5}$

choice of r and xy

So $\sum_{x\le N^{2/5}} \sum_{y\le N^{2/5}} \left(1+\frac{xy}{n}\right)^{-it} = \sum_{x\le N^{2/5}} \sum_{y\le N^{2/5}} \underbrace{e\left(\sum_{j=1}^{r} \alpha_j x^j y^j\right) \times e\left(O(t^{-\frac{1}{500}})\right)}_{\text{apply exp to the above}}$

since $\left(1+\frac{xy}{n}\right)^{-it} = \exp\left(-it\log\left(1+\frac{xy}{n}\right)\right) = e\left(-\frac{t}{2\pi}\log\left(1+\frac{xy}{n}\right)\right)$

Now we have

$$\ll \sum_{x\le N^{2/5}} \sum_{y\le N^{2/5}} e\left(\sum_{j=1}^{r} \alpha_j x^j y^j\right) + O\left(N^{4/5} t^{-\frac{1}{500}}\right) \quad \square$$

Remark 7.2

The exact choices of many of the parameters (e.g. the exponents $\frac{2}{5}$) are not important. It is important that the degree r of the polynomial is $\asymp \frac{\log t}{\log n}$.

It will turn out that we can only handle the case

3

where r isn't "too big" compared with N. This will ultimately set the limit of the Vinogradov - Korobov method.

13/02/14 Riemann Zeta Function (13)

Remark 7·3

It may seem strange that we introduced two shift variables x, y in the proof, since we could have applied Taylor expansion with just one shift variable. However, it is very often a good idea to have two independent variables available.

8 Bilinear Forms

In this section we will think about the general problem of bounding $\sum_{\tilde{x}\in\mathcal{X}} \sum_{\tilde{y}\in\mathcal{Y}} e(\alpha\tilde{x}\cdot\tilde{y})$, where $\alpha\in\mathbb{R}$ and $\mathcal{X}, \mathcal{Y}$ are sets of r-vectors. First, we will consider a simple problem and then return to the sums $U(n)$ from Lemma 7·1 (where $\tilde{x} = (x, x^2, \dots, x^r)$)

Proposition 8·1 (Toy case)

Let $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$, where $q \geq 1$, $(a,q) = 1$, $|\theta| \leq 1$.

Let $N \in \mathbb{N}$ be a large number. Then

$$\sum_{p\leq N} \sum_{p'\leq N} e(\alpha pp') \ll N \max\left\{\frac{N}{\sqrt{q}}, \sqrt{q}\right\}\sqrt{\log(q+1)}$$

summing over primes p, p'.

Note that the trivial bound is $\pi(N)^2 \asymp \frac{N^2}{\log^2 N}$, so we beat the trivial bound provided that q isn't too big or too small.

To prove Proposition 8·1 we will need a small technical lemma (which we will need later).

Lemma 8·2

Let $\underset{\frac{a}{q}+\frac{\theta}{q^2}}{\alpha}$, $\underset{\text{large}}{N}$ be as in the statement of proposition 8·1.

Let $\beta, U \geq 0$ be arbitrary. Let $\|x\|$ denote the distance from $x\in\mathbb{R}$ to the nearest integer.

Then $\sum_{n\le N} \min\left\{U, \frac{1}{\|\alpha n+\beta\|}\right\} \ll \left(\frac{N}{q}+1\right)(U+q\log q)$

Proof of Proposition 8·1

The crucial first step is to complete one of the sums (using the bilinear structure) and then apply Cauchy-Schwarz.

$$\left|\sum_{p\le N}\sum_{p'\le N} e(\alpha p p')\right| \le \sum_{n\le N}\left|\sum_{p\le N} e(\alpha p n)\right|$$

"completion" of p' sum

$$\le \sqrt{N}\sqrt{\sum_{n\le N}\left|\sum_{p\le N} e(\alpha p n)\right|^2}$$

Now we have replaced a sum over primes by a sum over all integers which is much easier. We have

complex conjugates

$$\sum_{n\le N}\left|\sum_{p\le N} e(\alpha p n)\right|^2 = \sum_{n\le N}\left(\sum_{p\le N} e(\alpha p n)\right)\left(\sum_{p'\le N} e(-\alpha p' n)\right)$$

$$= \sum_{p\le N}\sum_{p'\le N}\sum_{n\le N} e(\alpha(p-p')n)$$

Sum geometric progression

$$= \sum_{p\le N}\sum_{p'\le N} \frac{e(\alpha(p-p')(N+1)) - e(\alpha(p-p'))}{e(\alpha(p-p')) - 1}$$

In general, we have

(6) $$\left|\sum_{n\le N} e(\beta n)\right| = \left|\frac{e(\beta(N+1)) - e(\beta)}{e(\beta)-1}\right| \le \frac{2}{|e(\beta/2) - e(-\beta/2)|}$$

$$= \frac{1}{|\sin(\pi\beta)|} \le \frac{1}{2\|\beta\|}$$

trivial bound

$$\text{So } \sum_{n\le N}\left|\sum_{p\le N} \cancel{\sum_{p'\le N}}\, e(\alpha p n)\right|^2 \ll \sum_{p\le N}\sum_{p'\le N} \min\left\{\frac{1}{2\|\alpha(p-p')\|}, N\right\}$$

$$\ll N \sum_{0\le n\le N} \min\left\{\frac{1}{2\|\alpha n\|}, N\right\}$$

since the numbers $p-p'$ range over integers between $-N$ and N hitting each integer at most N times (and $\|\alpha n\| = \|\alpha(-n)\|$)

Finally, Lemma 8·2 implies that $U=N$, $\frac{\alpha}{2}$, $\beta=0$

$$\sum_{0\le n\le N} \min\left\{\frac{1}{2\|\alpha n\|}, N\right\} \ll \left(\frac{N}{q}+1\right)(N+q\log q)$$

$$\ll \max\left\{q\,\cancel{\log(q+1)}, \frac{N^2}{q}\right\}\log(q+1) \qquad \square$$

13/02/14 Riemann Zeta Function (13)

Proof of Lemma 8.2

It will suffice to show that $\sum_{-\frac{q}{2} < n \leq \frac{q}{2}} \min\{U, \frac{1}{\|\alpha n+\beta\|}\} \ll U + q\log q$

since if $N \geq q$ we can break the sum into $\leq (\frac{N}{q}+1)$ sums each of length at most q (for some β).

We can also assume that $q \geq 2$, since otherwise the bound is trivial.

Note that for all $-\frac{q}{2} < n \leq \frac{q}{2}$, we have

(*) $|\alpha n - \frac{an}{q}| = \frac{|\theta n|}{q^2} \leq \frac{1}{2q}$ since $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$, $|\theta| \leq 1$.

Since $(a,q) = 1$, as n varies over $(-\frac{q}{2}, \frac{q}{2}]$, the numbers $an \pmod q$ hit each residue class $r \pmod q$ precisely once. So, as n varies, at most $O(1)$ of the numbers $\alpha n \pmod 1$ lie in each interval $[\frac{r-\frac{1}{2}}{q}, \frac{r+\frac{1}{2}}{q}]$, $-\frac{q}{2} \leq r \leq \frac{q}{2}$.

↑using (*)

On translating by β, we have that at most $O(1)$ of the numbers $\alpha n + \beta \pmod 1$ lie in each interval $[\frac{r-\frac{1}{2}}{q} + \beta, \frac{r+\frac{1}{2}}{q} + \beta]$

Finally, if $r = 0$, we use the bound

$\min\{U, \frac{1}{\|\alpha n+\beta\|}\} \leq U$ ← trivial

If $r \neq 0$, we use $\min\{U, \frac{1}{\|\alpha n+\beta\|}\} \ll \frac{q}{|r|}$ $\quad \|\alpha n+\beta\| \geq \frac{|r|-\frac{1}{2}}{q}$ or similar

So $\sum_{-\frac{q}{2} < n \leq \frac{q}{2}} \min\{U, \frac{1}{\|\alpha n+\beta\|}\} \ll U + \sum_{1 \leq r \leq \frac{q}{2}} \frac{q}{r} \ll U + q\log q$

□

15/02/14

Thursday 27th Feb 2.00pm

Riemann-Zeta Function (14)

In the proof of proposition 8·1, we lost a bit (a log factor.) when replacing sums over primes by sums over integers, because primes are a sparse set. But we didn't lose too much, because they aren't too sparse. $/\!/ \sum_{x \le N^{2/5}} \sum_{y \le N^{2/5}} e(\alpha_1 xy + \ldots + \alpha_r x^r y^r)$, $\alpha_j = \frac{(-1)^j t}{2\pi j n^j}$

For the sums $U(n)$ from Lemma 7·1, we ~~assign~~ are summing over vectors $(x, x^2, \ldots, x^r)$, $(y, y^2, \ldots, y^r)$, and these are a very sparse subset of the box that contains them.

To overcome this problem, we will need another idea (in addition to the bilinear structure).

Lemma 8·3 (Duplication of Variables)

Let $$U(n) = \sum_{x \le N^{2/5}} \sum_{y \le N^{2/5}} e(\alpha_1 xy + \alpha_2 x^2 y^2 + \ldots + \alpha_r x^r y^r)$$
be as in Lemma 7·1. Then for any natural number k, we have
$$|U(n)| \le N^{4/5} \left(\frac{1}{N^{8k/5}} (J_{k,r}(N^{2/5}))^2 \prod_{j=1}^{r} \sum_{-kN^{2j/5} \le \mu_j \le kN^{2j/5}} \min\left[3kN^{2j/5}, \frac{1}{\|\alpha_j \mu_j\|}\right] \right)^{\frac{1}{4k}}$$
$\|\cdot\|$ = distance to nearest integer

where $J_{k,r}(N^{2/5})$ is the number of solutions $(x_1, x_2, \ldots, x_{2k})$ of the simultaneous equations
$$\sum_{i=1}^{k} x_i^j = \sum_{i=k+1}^{2k} x_i^j \quad \forall 1 \le j \le r$$
with $1 \le x_i \le N^{2/5}$ integers.

Proof

(Like the proof of 8·1, but with the application of Cauchy-Schwarz replaced by two applications of Hölder's Inequality with exponent $2k$).

just like CS but with more sums i.e. $(\sum a_i b_i \ldots r_i)^{2k} \le (\sum a_i^{2k}) \ldots (\sum r_i^{2k})$

1

This has the effect of producing $2k$ duplicate copies of the summands $(x, x^2, \dots, x^r), (y, y^2, \dots, y^r)$ whose sums cover the containing box more uniformly.

To simplify the writing, set $Z := N^{\frac{3}{3}}$. By Hölder,

$|U(n)| \le \sum_{x\le Z} \left|\sum_{y \le Z} e(\alpha_1 xy + \dots)\right|$ $\leftarrow$ $$|U(n)|^{2k} \le Z^{2k-1} \sum_{x\le Z} \left|\sum_{y\le Z} e(\alpha_1 xy + \dots + \alpha_r x^r y^r)\right|^{2k}$$ $\leftarrow$ expand using complex conj.

$\therefore |U(n)|^{2k} \le \left(\sum_{x\le Z} \left|\sum_{y\le Z} e(\alpha_1 xy + \dots)\right|\right)^{2k} = \left(\sum_{x\le Z} \underbrace{1\cdot 1 \cdot \dots \cdot 1}_{2k-1 \text{ times}} \cdot \left|\sum e(\alpha_1 x \dots)\right|\right)^{2k} \le Z^{2k-1}(\text{stuff})$ by Hölder as required

$$= Z^{2k-1} \sum_{x\le Z} \sum_{y_1, \dots, y_{2k} \le Z} e\left(\alpha_1 x\left(\sum_{i=1}^{k} y_i - \sum_{i=k+1}^{2k} y_i\right) + \dots + \alpha_r x^r\left(\sum_{i=1}^{k} y_i^r - \sum_{i=k+1}^{2k} y_i^r\right)\right)$$

If we let $J_{k,r}(\lambda_1, \dots, \lambda_r; Z)$ denote the number of solutions $(x_1, \dots, x_{2k})$ of $\sum_{i=1}^{k} x_i^j = \sum_{i=k+1}^{2k} x_i^j + \lambda_j \quad , \forall 1 \le j \le r, x_i \le Z$

then $$|U(n)|^{2k} \le Z^{2k-1} \sum_{x\le Z} \sum_{-kZ \le \lambda_1 \le kZ} \dots \sum_{-kZ^r \le \lambda_r \le kZ^r} J_{k,r}(\lambda_1, \dots, \lambda_r; Z) \times e(\alpha_1 x_1 \lambda_1 + \dots + \alpha_r x_j^r \lambda_r)$$

$e(\alpha_1 x \lambda_1 + \dots)$

min and max values $-kZ^i$ and kZ^i for $\sum_{j=1}^{k} y_j^i - \sum_{j=k+1}^{2k} y_j^i$

swap order of summation and add an absolute value sign $\rightarrow$ $$\le Z^{2k-1} \sum_{-kZ \le \lambda_1 \le kZ} \dots \sum_{-kZ^r \le \lambda_r \le kZ^r} \underbrace{J_{k,r}(\lambda_1, \dots, \lambda_r; Z)}\left|\sum_{x \le Z} e(\alpha_1 x_1 \lambda_r + \dots + \alpha_r x^r \lambda_r)\right|$$
$(*)$

To simplify the writing further, we will usually just write $\sum_{\lambda_j}$ (without a range) to mean $\sum_{-kZ^j \le \lambda_j \le kZ^j}$

In the proof of the Toy Proposition, we were now basically done because we could evaluate the inner sum. Here, we cannot do this, so we apply Hölder's Inequality again.

remove $J_{k,r}(\lambda_1, \dots, \lambda_r; Z)$ from inner sum and mods in $(*)$

Raise $(*)$ to power $2k$ $\rightarrow$ $$|U(n)|^{(2k)^2} \le Z^{2k(2k-1)}\left(\sum_{\lambda_1} \dots \sum_{\lambda_r} J_{k,r}(\lambda_1, \dots, \lambda_r; Z)\left|\sum_{x \le Z} e(\alpha_1 x \lambda_1 + \dots + \alpha_r x^r \lambda_r)\right|\right)^{2k}$$

$$\underset{\text{Hölder}}{\le} Z^{2k(2k-1)}\left(\sum_{\lambda_1} \dots \sum_{\lambda_r} J_{k,r}(\lambda_1, \dots, \lambda_r; Z)^{\frac{2k}{2k-1}}\right)^{2k-1}$$
$$\times \sum_{\lambda_1} \dots \sum_{\lambda_r} \left|\sum_{x\le Z} e(\alpha_1 x \lambda_1 + \dots + \alpha_r x^r \lambda_r)\right|^{2k} \qquad (\dagger)$$

apply over $\left(\sum_{\lambda_1, \dots, \lambda_r} \underbrace{(J_{k,r}(\dots))^{\frac{1}{2k-1}} \dots (J_{k,r}(\dots))^{\frac{1}{2k-1}}}_{2k-1 \text{ times}} \left|\sum_{x\le Z} \dots\right|\right)^{2k}$ with exponent $2k$

2

15/02/14

Riemann-Zeta Function (14)

To bound the first bracketed term, note that

$$\sum_{\lambda_1}\cdots\sum_{\lambda_r} J_{k,r}(\lambda_1,\ldots,\lambda_r)^{\frac{2k}{2k-1}} \le \max_{\lambda_1,\ldots,\lambda_r} J_{k,r}(\lambda_1,\ldots,\lambda_r)^{1/2k-1}$$

simply because $\sum A_i^{x+y} \le \max_i (A_i^x) \sum A_i^y$

$$\times \sum_{\lambda_1}\cdots\sum_{\lambda_r} J_{k,r}(\lambda_1,\ldots,\lambda_r;Z)$$

$$\le Z^{2k} \max_{(\lambda_1,\ldots,\lambda_r)} J_{k,r}(\lambda_1,\ldots,\lambda_r)^{\frac{1}{2k-1}}$$

~~exact, not just $\le$~~

since $\sum_{\lambda_1}\cdots\sum_{\lambda_r} J_{k,r}(\lambda_1,\ldots,\lambda_r;Z)$ counts all vectors with $1 \le x_i \le Z$ integers. ~~crudest possible, the trivial bound for $J_{k,r}$~~

Also, for any $(\lambda_1,\ldots,\lambda_r)$, we have

$$J_{k,r}(\lambda_1,\ldots,\lambda_r;Z) = \sum_{L_1,\ldots,L_r\in\mathbb{Z}} \Big(\underbrace{\#\{(x_1,\ldots,x_k): 1\le x_i\le Z, \textstyle\sum_{i=1}^k x_i^j = L_j \ \forall 1\le j\le r\}}_{a_1}$$

$$\times \underbrace{\#\{(x_{k+1},\ldots,x_{2k}): 1\le x_i\le Z, \textstyle\sum_{i=k+1}^{2k} x_i = L_j-\lambda_j \ \forall j\}}_{a_2}\Big),$$

$J_{k,r}$ counts solutions to $\sum x_i^j - \sum x_i^j = \lambda_j$

This is the explicit form in terms of the values of the left and right summations

$$\le \sum_{L_1,\ldots,L_r\in\mathbb{Z}} \#\{(x_1,\ldots,x_k): 1\le x_i\le Z, \textstyle\sum_{i=1}^k x_i^j = L_j \ \forall 1\le j\le r\}^2$$

Cauchy Schwarz

$$= J_{k,r}(0,0,\ldots,0;Z) = J_{k,r}(Z)$$

Therefore, we have $|u(n)|^{(2k)^2} \le Z^{4k(2k-1)} J_{k,r}(Z) \sum_{\lambda_1}\cdots\sum_{\lambda_r}$

$$\Big|\sum_{x\le Z} e(\alpha_1 x\lambda_1 + \ldots + \alpha_r x^r\lambda_r)\Big|^{2k}$$

simply by substituting in the above bound for the first bracketed term in (+)

$$\sum a_1 a_2 \le \left(\sum a_1^2\right)^{\frac{1}{2}}\left(\sum a_2^2\right)^{\frac{1}{2}} = \sum a_1^2$$

turn out to be the same

18/02/14 Riemann-Zeta Function (15)

Last time:

$Z = N^{3/5}$

$$|U(n)|^{(2k)^2} \leq Z^{4k(2k-1)} J_{k,r}(Z) \sum_{\lambda_1} \dots \sum_{\lambda_r} \left| \sum_{x \leq Z} e(\alpha_1 x \lambda_1 + \dots + \alpha_r x^r \lambda_r) \right|^{2k}$$

Expanding the $2k^{th}$ power as before, we get — with complex conjugates as in (*) last time

$$|U(n)|^{(2k)^2} \leq Z^{4k(2k-1)} J_{k,r}(Z) \sum_{\lambda_1} \dots \sum_{\lambda_r} \sum_{\mu_1} \dots \sum_{\mu_r} J_{k,r}(\mu_1, \dots, \mu_r; Z) \cdot e(\alpha_1 \mu_1 \lambda_1, \dots, \alpha_r \mu_r \lambda_r)$$

with $-kZ^i \leq \mu_i \leq kZ^i$

switch order of summation and add mod signs

$$\leq Z^{4k(2k-1)} J_{k,r}(Z) \sum_{\mu_1} \dots \sum_{\mu_r} J_{k,r}(\mu_1, \dots, \mu_r; Z) \cdot \left| \sum_{\lambda_1} e(\alpha_1 \lambda_1 \mu_1) \right| \dots \left| \sum_{\lambda_r} e(\alpha_r \lambda_r \mu_r) \right|$$

since $J_{k,r}(\mu_1, \dots, \mu_r; Z) \leq J_{k,r}(Z)$

$$\leq Z^{4k(2k-1)} (J_{k,r}(Z))^2 \sum_{\mu_1} \dots \sum_{\mu_r} \left| \sum_{\lambda_1} e(\alpha_1 \lambda_1 \mu_1) \right| \dots \left| \sum_{\lambda_r} e(\alpha_r \lambda_r \mu_r) \right|$$

since $J_{k,r}(\mu_1, \dots, \mu_r; Z) \leq J_{k,r}(Z) \quad \forall \mu_1, \dots, \mu_r$

$$\leq Z^{4k(2k-1)} (J_{k,r}(Z))^2 \sum_{\mu_1} \dots \sum_{\mu_r} \min\left\{3kZ, \frac{1}{\|\alpha_1 \mu_1\|}\right\} \dots \min\left\{3kZ^r, \frac{1}{\|\alpha_r \mu_r\|}\right\}$$

summing a geometric progression as in the proof of proposition 8·1.
See (⑥)
Then put $Z = N^{3/5}$ as before □

Remark

Note that in the proof of Lemma 8·3 we needed to switch the order of the sums more than once (as well as "duplicating variables"). This shows ~~that~~ the power of the simple idea of introducing two independent variables x, y.

9 Vinogradov's Mean Value Theorem

This section is devoted to the study of $J_{k,r}(Z)$, the number of solutions $(x_1, \dots, x_{2k})$ of the simultaneous equations

$$\sum_{i=1}^{k} x_i^j = \sum_{i=k+1}^{2k} x_i^j, \quad \forall 1 \leq j \leq r.$$

with $1 \leq x_i \leq Z$ (for Z large).

This quantity $J_{k,r}(Z)$ is called Vinogradov's Mean Value,
and is very widely studied (c.f. Waring's Problem).
We trivially have $J_{k,r}(Z) \geq \lfloor Z \rfloor^k$, since for any $(x_1, \ldots, x_k)$ we can choose $(x_{k+1}, \ldots, x_{2k})$. These trivial solutions are called diagonal solutions. Also, we saw last time that

$$\lfloor Z \rfloor^{2k} = \sum_{-kZ \leq \lambda_1 \leq kZ} \cdots \sum_{-kZ^r \leq \lambda_r \leq kZ^r} J_{k,r}(\lambda_1, \ldots, \lambda_r ; Z)$$

← since this just counts all vectors $(x_1, \ldots, x_k)$ with $x_i \leq K$

as $J_{k,r}(\lambda_1, \ldots) \leq J_{k,r}(Z)$ →

$$\leq J_{k,r}(Z) \sum_{\lambda_1} \cdots \sum_{\lambda_r} 1$$

So
$$J_{k,r}(Z) \geq \frac{\lfloor Z \rfloor^{2k}}{\prod_{j=1}^{r} (3kZ^j)} = (3k)^{-r} \lfloor Z \rfloor^{2k} Z^{-\frac{1}{2}r(r+1)} \qquad (*)$$

The bound $(*)$ would be close to the truth if all the differences $\sum_{i=1}^{k} x_i^j - \sum_{i=k+1}^{2k} x_i^j$ were roughly uniformly distributed as x_i vary.

It is conjectured that $J_{k,r}(Z)$ is never much bigger (as a function of Z) than the larger of our two lower bounds.

Conjecture 9.1

Let $k, r \in \mathbb{N}$, and $Z \geq 1$, $\varepsilon > 0$ be arbitrary.
Then $J_{k,r}(Z) \ll_{k,r,\varepsilon} Z^{k+\varepsilon} + Z^{2k - \frac{1}{2}r(r+1) + \varepsilon}$
where the implicit constant may depend on k, r, ε but not on Z.

Note in particular that if $k \geq \frac{1}{2} r(r+1)$ then the second term is larger so the conjecture says the behaviour is "roughly uniform". This is what we would want to substitute into Lemma 8.3.

18/02/14 Riemann-Zeta Function (15)

Recently (in 2012-13), Wooley has proved Conjecture 9·1 when $k \geq r^2-1$. Unfortunately, we shall not prove this great result, but we shall prove an older bound that seems just as good for our applications.

<u>Theorem 9·2</u> (Vinogradov's Mean Value Theorem)
Vinogradov, 1930s + later refinement

Suppose that Z is large, and let $k, r \in \mathbb{N}$ such that $k \geq r^2$. Let $F = F(k,r) := \lfloor \frac{k}{r} - r \rfloor$ and

$\delta = \delta(k,r) = (1-\frac{1}{r})^F$

(N.B. $F \geq 0$). Then

$J_{k,r}(Z) \leq (4r)^{4kF} Z^{2k-(1-\delta)\frac{1}{2}r(r+1)}$

<u>Remark 9·3</u>

Note that if k/r^2 is large, then δ is small.

powers of x_i involved

The proof of Theorem 9·2 works by fixing r and inducting on k. The inductive step is carried out by ~~contin~~ considering

variables involved

the system of equations $\sum_{i=1}^{k} x_i^j = \sum_{i=k+1}^{2k} x_i^j$ modulo a suitably chosen prime p, using a result called Linnik's Lemma. This will heavily exploit the polynomial structure.

20/02/14

Riemann Zeta Function (16)

Linnik's Lemma 9.4 (1942-3)

Let $r \in \mathbb{N}$, A and $m \geq 1$ be integers. Let $p > r$ be prime and $\lambda_1, \ldots, \lambda_r$ be integers. Then, the number of solutions $(x_1, \ldots, x_r)$ of the simultaneous congruences

$$\sum_{i=1}^{r} x_i^j \equiv \lambda_j \pmod{p^j} \quad \forall 1 \leq j \leq r$$

A is somewhat arbitrary

with $A \leq x_i \leq A + mp^r$ integers that are distinct mod p is $\leq (r!) m^r p^{\frac{1}{2}r(r-1)}$ (mod p, r variables, r equations)

Proof

Note first that for any given $(\lambda_1, \ldots, \lambda_r)$ there are $\prod_{j=1}^{r-1} p^{r-j} = p^{\frac{1}{2}r(r-1)}$ different vectors $(\mu_1, \ldots, \mu_r)$ mod p^r such that $\mu_j \equiv \lambda_j \pmod{p^j}$ $\forall 1 \leq j \leq r$. So it suffices to show that there are at most $(r!)m^r$ different solutions $(x_1, \ldots, x_r)$ with $\sum_{i=1}^{r} x_i^j \equiv \mu_j \pmod{p^r}$

Simplify by working to a single modulus p^r

Next, suppose that $\sum_{i=1}^{r} x_i^j = \sum_{i=1}^{r} y_i^j \equiv \mu_j \pmod{p^r}$ $\forall 1 \leq j \leq r$. Since $\underbrace{(r!, p) = 1}_{p > r}$, the elementary symmetric functions $\sum x_{(1)} \ldots x_{(j)}$ in the x_i are uniquely determined $\pmod{p^r}$ by the power sums $\sum_{i=1}^{r} x_i^j$ (by Newton's Identities). Therefore, the polynomials

These simply express the symmetric functions in terms of power sums

$P(Z) := \prod_{i=1}^{r} (Z - x_i)$, $Q(Z) := \prod_{i=1}^{r} (Z - y_i)$ are identically congruent $\pmod{p^r}$. However, we have $P(x_j) \equiv 0 \pmod{p^r}$, $\forall 1 \leq j \leq r$, so we must have $Q(x_j) = \prod_{i=1}^{r} (x_j - y_i)$ mod p^r. Since the y_i are distinct (mod p) by hypothesis, we must have $x_j \equiv y_i \pmod{p^r}$ for some i. Moreover, the x_j are

the alternative is say $x_i \equiv y_1 \bmod p^{\frac{1}{2}r}$ & $x_i \equiv y_2 \bmod p^{\frac{1}{2}r} \Rightarrow y_1 \equiv y_2 \bmod p$ #

1

so two x_i cannot be congruent to the same y_j mod p^r or $x_1 \equiv x_2$ mod p ※

also distinct (mod p) by hypothesis, so the $(x_i)_{1\le i\le r}$ must be a permutation of the $(y_i)_{1\le i\le r}$ (mod p^r).

So, there are $\le (r!)\, m^r$ solutions $A \le x_i \le A+mp^r$

perms — all solutions are congruent mod p^r. m choices for each of r variables — # copies of $\mathbb{Z}_{p^r}$ in the interval

□

We also need the following simple observation

Lemma 9.5 (Translation Invariance)

If $(x_1, \dots, x_{2k})$ is a solution of $\sum_{i=1}^{k} x_i^j = \sum_{i=k+1}^{2k} x_i^j$ $\forall 1 \le j \le r$ then so is $(x_1 - x, \dots, x_{2k} - x)$ for any x.

Proof

Note that $\sum_{i=1}^{k} (x_i - x)^j = \sum_{i=1}^{k} x_i^j + \sum_{t=1}^{j} \binom{j}{t} (-x)^t \sum_{i=1}^{k} x_i^{t-j}$ and similarly for $\sum_{i=k+1}^{2k} (x_i - x)^j$.

So if $\sum_{i=1}^{k} x_i^j = \sum_{i=k+1}^{2k} x_i^j$ then the same is true for $\sum_{i=1}^{k} (x_i - x)^j$, $\sum_{i=k+1}^{2k} (x_i - x)^j$. □

Now, we shall use Linnik's Lemma (9.4) to set up an induction on the number of variables k.

Lemma 9.6 (Induction Lemma)

Let $r \ge 2$ and suppose that $Z \ge (2r)^{3r}$, $k \ge r^2 + r$. Then $J_{k,r}(Z) \le 4^{2k} \cdot Z^{\frac{2k}{r} + \frac{3r-5}{2}} J_{k-r,r}(4Z^{1-\frac{1}{r}})$

Proof

Let $\frac{1}{2} Z^{\frac{1}{r}} \le p \le Z^{\frac{1}{r}}$ be prime (exists by Chebyshev), and set $Z_1 := \lceil \frac{Z}{p} \rceil$. ①

We have $Z \le p Z_1$, so we certainly have $J_{k,r}(Z) \le J_{k,r}(pZ_1)$

We also have $Z_1 \le \frac{2Z}{p} \le 4Z^{1-\frac{1}{r}}$, so it will suffice to

20/02/14

Riemann Zeta Function (16)

①, ② Together give the result

show that $J_{k,r}(pZ_1) \leq 4^{2k} \cdot Z^{\frac{2k}{r} + \frac{3r-5}{2}} J_{k-r,r}(Z_1)$. ②

We also note that $p > r$, since $Z \geq (2r)^{3r}$ by $p \geq \frac{1}{2}Z^{\frac{1}{r}} \geq \frac{1}{2}(2r)^3 > r$

hypothesis, so later, we will be able to apply Linnik's Lemma.

Note that we choose $p \approx Z^{\frac{1}{r}}$ so that the ranges mp^r of the variables in Linnik's Lemma approximately match the ranges Z of our variables.

Next, let J_1 denote the number of solutions $(x_1, \ldots, x_{2k})$ counted by $J_{k,r}(pZ_1)$ such that at least r of the numbers $(x_1, \ldots, x_k)$ and at least r of the numbers $(x_{k+1}, \ldots, x_{2k})$ are distinct mod p.

Let J_2 denote the number of solutions not counted by J_1, so that $J_{k,r}(pZ_1) = J_1 + J_2$.

2

22/02/14 Riemann-Zeta Function 17

Last time:

Want to show that $J_{k,r}(Z,p) \le 4^{2k} Z^{\frac{2k}{r}+\frac{3r-5}{2}} J_{k-r,r}(Z_1, \ldots$

We have $J_{k,r}(pZ_1) = J_1 + J_2$.

similar to J_1 but only considers the first r coordinates

Let J_1' denote the number of solutions counted by $J_{k,r}(pZ_1)$ in which $(x_1, \ldots, x_r)$ are distinct mod p, and also $(x_{k+1}, \ldots, x_{k+r})$ are distinct mod p.

trivial

Then $J_1 \le k^{2r} J_1'$ (by permuting coordinates).

Bounding J_1':

We claim that $J_1' \le p^{2k-2r} \max_{1 \le x \le p} J_1'(x)$, where $J_1'(x)$ counts solutions (in J_1') with $x_{r+1}, \ldots, x_k \equiv x \pmod p$, $x_{k+r+1}, \ldots, x_{2k} \equiv x \pmod p$.

"Freezing the congruence class"

$J_1'(x)$ is J_1' with restricted congruence class

Assuming this claim for the present, we can use translation invariance (Lemma 9.5) to subtract x from all the components, obtaining that ~~$J_1'(x) = \#\{$~~

Have simply relabelled. The $\tilde{x}_i$ are the 'first r coordinates' while the py_i are the 'frozen coordinates'

$$J_1'(x) = \#\{(\tilde{x}_1, \ldots, \tilde{x}_r, y_1, \ldots, y_{k-r}, \tilde{x}_{r+1}, \ldots, \tilde{x}_{2r}, y_{k-r+1}, \ldots, y_{2k-2r})$$

$$\text{such that } 0 \le y_j \le Z_1 - 1 \quad \forall j \text{ with } 1 \le j \le 2k-2r$$
$$\text{and } 1-x \le \tilde{x}_j \le pZ_1 - x \quad \forall\, 1 \le j \le 2r$$
$$\text{and } \tilde{x}_1, \ldots, \tilde{x}_r \text{ distinct mod } p$$
$$\tilde{x}_{r+1}, \ldots, \tilde{x}_{2r} \text{ distinct mod } p$$
$$\text{with } \sum_{i=1}^{r} \tilde{x}_i^{\,j} = \sum_{i=r+1}^{2r} \tilde{x}_i^{\,j} - p^j \sum_{i=1}^{k-r} y_i^{\,j} + p^j \sum_{i=k-r+1}^{2k-2r} y_i^{\,j} \quad \forall\, 1 \le j \le r$$

components $r+1, \ldots, k$ and $k+r+1, \ldots, 2k$ of y were $\equiv 0 \bmod p$

So py_i are the 'frozen' coordinates of x above

Now for any fixed $\tilde{x}_{r+1}, \ldots, \tilde{x}_{2r}$, each vector $(\tilde{x}_1, \ldots, \tilde{x}_r)$ satisfies the conditions of Linnik's Lemma.

(with $A = 1 - x$ and $m = \lceil \frac{pZ_1}{p^r} \rceil \leq \lceil \frac{2Z}{p^r} \rceil \leq 2^{r+1}$ since $p \geq \frac{1}{2} Z^{\frac{1}{r}}$).

Also, for any choice of $\tilde{x}_{r+1}, \dots, \tilde{x}_{2r}$ and $\tilde{x}_1, \dots, \tilde{x}_r$, the number of possibilities for $(y_1, \dots, y_{2k-r})$ is $\leq J_{k-r,r}(Z_1)$.

So in total we have
$$J_1'(x) \leq \underbrace{(pZ_1)^r}_{\#\text{ choices of } (\tilde{x}_{r+1},\dots,\tilde{x}_{2r})} \underbrace{(r!)\, m^r p^{\frac{1}{2}r(r-1)}}_{\substack{\text{Linnik's Lemma} \\ \text{number of options for } \tilde{x}_1,\dots,\tilde{x}_r}} \underbrace{J_{k-r,r}(Z_1)}_{\substack{\text{Induction Hypothesis} \\ \text{number of options for } y_1,\dots,y_{2k-2}}}$$

$$J_1'(x) \leq (2Z)^r (r!)\, 2^{r(r+1)} Z^{\frac{1}{2}(r-1)} J_{k-r,r}(Z_1)$$

since $pZ_1 \leq 2Z$, $p \leq Z^{\frac{1}{r}}$, $m \leq 2^{r+1}$

$$J_1' \leq p^{2k-2r} (2Z)^r (r!)\, 2^{r(r+1)} Z^{\frac{1}{2}(r-1)} J_{k-r,r}(Z_1)$$
$$\leq Z^{\frac{2k}{r}-2} (2Z)^r (r!)\, Z^{\frac{1}{2}(r-1)} J_{k-r,r}(Z_1)\, 2^{r(r+1)}$$

$J_1' \leq p^{2k-2r} \max J_1'(x)$

using $p \leq Z^{\frac{1}{r}}$ again

$$J_1' \leq 2^{r(r+1)+r} (r!)\, Z^{\frac{2k}{r} + \frac{3r-5}{2}} J_{k-r,r}(Z_1) \leftarrow \text{just rearranged}$$
$$\leq 2^{2r(r+1)} Z^{\frac{2k}{r} + \frac{3r-5}{2}} J_{k-r,r}(Z_1)$$

since $r! \leq 2^{r^2}$

$$J_1' \leq Z^{2k} Z^{\frac{2k}{r} + \frac{3r-5}{2}} J_{k-r,r}(Z_1) \qquad (*)$$

because $k \geq r^2 + r$ by assumption.

We still need to prove the claim that $J_1' \leq p^{2k-2r} \max_{1 \leq x \leq p} J_1'(x)$

Let $S(x) := \sum_{\substack{z \leq pZ_1 \\ z \equiv x \pmod p}} e(\beta_1 z + \beta_2 z^2 + \dots + \beta_r z^r)$

Then $J_1' = \int_0^1 \cdots \int_0^1 \left| \sum_{\substack{x_1,\dots,x_r \\ \text{distinct}}} S(x_1) \dots S(x_r) \right|^2 \left| \sum_{x \leq p} S(x) \right|^{2k-2r} d\beta_1 \dots d\beta_r$

since $\int_0^1 e(\beta z)\, d\beta = \begin{cases} 1 & z = 0 \\ 0 & \text{otherwise} \end{cases}$ $\quad z \in \mathbb{Z}$

22/02/14

Riemann - Zeta Function (17)

$$\left|\sum_{x\le p} S(x)\right|^{2k-2r} \le \left(\sum_{x\le p} 1^{\frac{2k-2r}{2k-2r-1}}\right)^{2k-2r-1} \sum_{x\le p}|S(x)|^{2k-2r}$$

So by Hölder's Inequality,

$$J_1' \le \int_0^1 \dots \int_0^1 \left|\sum_{\substack{x_1,\dots,x_r \\ \text{distinct mod } p}} S(x_1)\dots S(x_r)\right|^2 \left(p^{2k-2r-1}\sum_{x\le p}|S(x)|^{2k-2r}\right) d\beta_1 \dots d\beta_r$$

Replace $\sum_{x\le p}|f(x)| \le p \max_{1\le x\le p}|f(x)|$

$$\le p^{2k-2r} \max_{1\le x\le p} \int_0^1 \dots \int_0^1 \left|\sum S(x_1)\dots S(x_r)\right|^2 |S(x)|^{2k-2r} d\beta_1 \dots d\beta_r$$

$$= p^{2k-2r} \max_{1\le x\le p} J_1'(x)$$

(we used Hölder like this: $\left|\sum_{x\le p} S\right|^{2k-2r} \le \left(\sum_{x\le p} 1\right)^{2k-2r-1} \sum_{x\le p} S^{2k-2r}$)

Bounding J_2 : = solutions not counted by J_1 i.e. at most $r-1$ of $x_1,\dots,x_k$ or $x_{k+1},\dots,x_{2k}$ are distinct mod p

Note that there are at most $2p^{r-1+k}r^k$ possibilities for $(x_1 \pmod p, \dots, x_{2k} \pmod p)$ if $(x_1,\dots,x_{2k})$ is counted by J_2.

(since we have p^{r-1} choices residue classes in $(x_1,\dots,x_k)$, and all k of $x_1,\dots,x_k$ are chosen from $<r$ values, p^k choices of residue classes in $(x_{k+1},\dots,x_{2k})$ and 2 if we switch the roles of $x_1,\dots,x_k$ and $x_{k+1},\dots,x_{2k}$)

Similarly to the above,

$$J_2 = \int_0^1 \dots \int_0^1 \sum_{(x_1,\dots,x_{2k}) \bmod p} S(x_1)\dots S(x_k)\overline{S(x_{k+1})}\dots\overline{S(x_{2k})}\, d\beta_1 \dots d\beta_r$$

counted by J_2, not just sum of all

$$\text{Hölder} \Rightarrow J_2 \le \int_0^1 \dots \int_0^1 \left(\sum_{x\le p}|S(x)|^{2k}\right)\left(\sum_{(x_1,\dots,x_{2k}) \bmod p} 1\right) d\beta_1 \dots d\beta_r$$

$$\left(\text{since } \left|\sum_{(x_1,\dots,x_{2k}) \bmod p} S(x_1)\dots S(x_k)\overline{S(x_{k+1})}\dots\overline{S(x_{2k})}\right| \le \left(\sum_{(x_1,\dots,x_{2k})}|S(x_1)|^{2k}\right)^{\frac{1}{2k}} \dots \left(\sum_{(x_1,\dots,x_{2k})}|S(x_{2k})|^{2k}\right)^{\frac{1}{2k}}\right)$$

$$\text{So } J_2 \le \sum_{(x_1,\dots,x_{2k})} 1 \cdot \sum_{x\le p}\int_0^1 \dots \int_0^1 |S(x)|^{2k} d\beta_1 \dots d\beta_r$$

Move $\sum$ / $\sum_{x\le p}$ outside

$$\le \underbrace{2p^{r-1+k}r^k}_{\text{by discussion higher up on this page}} \sum_{x\le p}\int_0^1 \dots \int_0^1 |S(x)|^{2k} d\beta_1 \dots d\beta_r$$

?

$\leq 2p^{r-\theta+k} r^{k} \rho J_{k,r}(Z_1)$ $\leftarrow$ next time

25/02/14

Riemann-Zeta Function (18)

We have:

$$J_{k,r}(pZ_1) \leq k^{2r} J_1' + J_2$$

from last time

where $J_1' \leq 2^{2k} Z^{\frac{2k}{r} + \frac{1}{2}(3r-5)} J_{k-r,r}(Z_1)$

and $J_2 \leq \int_0^1 \cdots \int_0^1 \left(\sum_{x \leq p} |S(x)|^{2k} \right) \left(\sum_{\substack{(x_1, \ldots, x_{2k}) \\ \bmod p}} 1 \right) d\beta_1 \ldots d\beta_r$

$$J_2 \leq 2p^{r-1+k} r^k \int_0^1 \cdots \int_0^1 \left(\sum_{x \leq p} |S(x)|^{2k} \right) d\beta_1 \ldots d\beta_r$$ ← from last time

$$\leq 2p^{r+k} r^k \max_{x \,(\bmod p)} \int_0^1 \cdots \int_0^1 |S(x)|^{2k} d\beta_1 \ldots d\beta_r$$

$$\leq 2p^{r+k} r^k J_{k,r}(Z_1)$$

since the integral counts solutions in $J_{k,r}(pZ_1)$, but in the fixed residue class $x \,(\bmod p)$.

$$\leq 2p^{r+k} r^k Z_1^{2r} J_{k-r,r}(Z_1)$$

trivial bound fixing $2r$ of the variables.

Note $p^{r+k} r^k Z_1^{2r} = p^{k-r} r^k (pZ_1)^{2r}$ ← trivial

$$\leq p^{k-r} r^k (2Z)^{2r} \qquad \text{since } pZ_1 \leq 2Z$$

$$\leq 2^{2r} r^k Z^{\frac{k}{r} + 2r - 1} \qquad \text{since } p \leq Z^{\frac{1}{r}}$$

$$= 2^{2r} r^k Z^{\frac{2k}{r} + \frac{1}{2}(3r-5)} Z^{-\frac{k}{r} + \frac{1}{2}(r+3)}$$

and since $k \geq r^2 + r$, $Z \geq (2r)^{3r}$, one can check that this is $\leq 2^{2r} 8^k Z^{\frac{2k}{r} + \frac{1}{2}(3r-5)}$ (consider the cases $r \leq 8$, $r \geq 9$ separately)

check online notes?

Finally, we have shown that

$$J_{k,r}(pZ_1) \leq \left(k^{2r} \cancel{Z^{2k}}\, 2^{2k} + 2^{2r+1} 8^k\right) Z^{\frac{2k}{r} + \frac{1}{2}(3r-5)} J_{k-r,r}(Z_1)$$

$$\leq 16^k Z^{\frac{2k}{r} + \frac{1}{2}(3r-5)} J_{k-r,r}(Z_1)$$

since $r \geq 2$, $k \geq r^2 + r$ $\square$

Now we have proved Lemma 9·6, it is a fairly straightforward book-keeping exercise to prove Vinogradov's Mean Value Theorem (9·2).

Proof (sketch, non examinable)

If $r=1$, we obviously have $J_{k,1}(Z) \leq Z^{2k-1}$ for all $Z \geq 1$ since if we fix $x_1, \dots, x_{2k-1}$, there is at most one possibility for x_{2k}. This bound is good enough.

For the main part of the proof, fix $r \geq 2$ and proceed by induction on $k \geq r^2$.

- If $r^2 \leq k \leq r^2+r-1$, then $F(k,r) = \lfloor 1-\frac{1}{r} \rfloor = 0$ and $\delta(k,r) = (1-\frac{1}{r})^F = 1$ so the bound we need to prove is $J_{k,r}(Z) \leq Z^{2k}$. But this is (worse than) trivial for all $Z \geq 1$ (Base case)
- For the inductive step, suppose that $r^2+fr \leq r^2+(f+1)r-1$ for some integer $f \geq 1$, and suppose we have proved the theorem when $k \leq r^2+fr-1$.
- If $Z < (2r)^{3r}$, we cannot use Lemma 9·6, so we use the trivial bound $J_{k,r}(Z) \leq Z^{2r} J_{k-r,r}(Z)$

(apply induction hypothesis and check that it works)

If $Z \geq (2r)^{3r}$, then Lemma 9·6 implies that

$J_{k,r}(Z) \leq 4^{2k} Z^{\frac{2k}{r}+\frac{1}{2}(3r-5)} J_{k-r,r}(4Z^{1-\frac{1}{r}})$

25/02/14 Riemann-Zeta Function (18)

10 Second Thoughts on Estimating Zeta Sums

Recall that our goal in Chapter 2 was to bound zeta sums $\sum_{N<n\leq N+M} n^{-it}$ and use the estimates to obtain bounds for $\zeta(s)$, and a wider zero-free region. We can combine our results so far (Lemma 7.1, Lemma 8.3, Vinogradov's Mean Value Theorem 9.2) to prove the following:

bounds $|U(n)|$ in terms of $J_{k,r}(Z)$ bounds $J_{k,r}(Z)$

bounds $\sum_{1\leq n\leq N+M} n^{-it}$ in terms of $U(n)$

<u>Theorem 10.1</u> (Zeta-Sum Estimate, Vinogradov, Korobov, 1958)

There exists a small constant $c>0$ such that the following holds

For any $1\leq M\leq N\leq t$

$$\left|\sum_{N<n\leq N+M} n^{-it}\right| \ll M\exp\left(-c\frac{\log^3 N}{\log^2(t+2)}\right) + \left(N^{4/5}\right)$$

<u>Proof:</u>

but $N\leq t$ also

We may assume that t is large and $N\geq \exp(\log^{2/3} t)$, since otherwise the bound is trivial by adjusting the $\ll$ constant appropriately.

By Lemma 7.1, we have

$$\left|\sum_{N<n\leq N+M} n^{-it}\right| \ll M \max_{N\leq n\leq 2N} \frac{|U(n)|}{N^{4/5}} + N^{4/5} + Mt^{-\frac{1}{100}}$$

where $U(n) = \sum_{x\leq N^{2/5}}\sum_{y\leq N^{2/5}} e(\alpha_1 xy + \ldots + \alpha_r x^r y^r)$

$$\alpha_j = \frac{(-1)^j t}{2\pi j n^j}, \quad r = \left\lfloor \frac{5.01\log t}{\log N}\right\rfloor$$

Note that $t^{-\frac{1}{500}} = e^{-\frac{1}{500}\log t} \leq e^{-\frac{1}{500}\frac{\log^3 N}{\log^2 t}}$ since $N\leq t$

By Lemma 8.3, for any $N\leq n\leq 2N$ and any parameter $k\in\mathbb{N}$, we have

$$\frac{|U(n)|}{N^{4/5}} \leq \left(\frac{1}{N^{8k/5}}\left(J_{k,r}(N^{2/5})\right)^2 \prod_{j=1}^{r} \sum_{-kN^{2j/5}\leq \mu_j\leq kN^{2j/5}} \min\left\{3kN^{2j/5}, \frac{1}{\|\alpha_j\mu_j\|}\right\}\right)^{\frac{1}{4k}}$$

If we choose $k = Cr^2$ for some constant $c \geq 1$ (to be chosen later) then Vinogradov's Mean Value Theorem implies:

$$\left(J_{k,r}(N^{2/5}) \right)^2 \leq (4r)^{8kF} N^{\frac{4}{5}(2k - \frac{1}{2}r(r+1)(1-\delta))}$$

where $F = \lfloor (c-1) r \rfloor$, $\delta = (1 - \frac{1}{r})^F$

So we have

$$\frac{|u(n)|}{N^{4/5}} \leq \left((4r)^{8Ckr} N^{-\frac{4}{5}(1-\delta)\frac{1}{2}r(r+1)} \prod_{j=1}^{r} \sum_{\mu_j} \min\left\{ 3kN^{\frac{2j}{5}}, \frac{1}{\|\alpha_j \mu_j\|} \right\} \right)^{\frac{1}{4}}$$

It remains to bound the sums over μ_j

27/02/14

Riemann-Zeta Function (19)

$k = Cr^2$, $r = \lfloor 5.01 \frac{\log t}{\log N} \rfloor$

We have

$$\frac{|U(n)|}{N^{4/5}} \leq \left((4r)^{8Ckr} N^{-\frac{4}{5}(1-\delta)\frac{1}{2}r(r+1)} \prod_{j=1}^{r} \sum_{-kN^{2j/5} \leq \mu_j \leq kN^{2j/5}} \min\left\{3kN^{2j/5}, \frac{1}{\|\alpha_j \mu_j\|}\right\} \right)^{1/4k}$$

It remains to bound the sums over μ_j. We always have the trivial bound $\ll k^2 N^{4j/5}$. But if $\alpha_j = \frac{a_j}{q_j} + \frac{\theta_j}{q_j^2}$ for some $q_j \geq 1$, $(a_j, q_j) = 1$, $|\theta_j| \leq 1$,

Then by Lemma 8.2 we have

$$\sum_{\mu_j} \min\left\{3kN^{2j/5}, \frac{1}{\|\alpha_j\mu_j\|}\right\} \ll \left(\frac{kN^{2j/5}}{q_j} + 1\right)\left(kN^{2j/5} + q_j \log q_j\right)$$

$$\ll \max\left\{\frac{(kN^{2j/5})^2}{q_j}, q_j\right\} \log(q_j + 1) \quad (*)$$

If $j \geq \frac{\log t}{\log N}$ then $\alpha_j = \frac{(-1)^j t}{2\pi j n^j} = \frac{(-1)^j}{q_j} + \frac{\theta_j}{q_j}$

where $q_j = \left\lfloor \frac{2\pi j n^j}{t} \right\rfloor \geq 1$ $\qquad \frac{(-1)^j}{q_j + \{\cdot\}} = \frac{(-1)^j}{q_j} + \frac{(-1)^{j+1}\left\{\frac{2\pi j n^j}{t}\right\}}{\left(\frac{2\pi j n^j}{t}\right)^2}$

$\lfloor |\alpha_j| \rfloor$ $\quad$ (for $j \geq \frac{\log t}{\log N}$, $n \geq N$) $\qquad$ take logs and prove $\frac{\log 2\pi e}{\log N} \geq 1$

In particular, if $2\frac{\log t}{\log N} \leq j \leq 3\frac{\log t}{\log N}$ then $q_j \geq n^{j - \frac{\log t}{\log N}} \underbrace{\geq}_{\text{easy}} n^{j/2}$

and also $q_j \ll j n^j N^{-j/3} \ll j 2^j N^{2j/3}$ (since $N \leq n \leq 2N$)

$\frac{j}{3}\log N \leq \log t \Rightarrow \frac{1}{t} \leq N^{-j/3}$

So $\sum_{\mu_j} \min\left\{3kN^{2j/5}, \frac{1}{\|\alpha_j \mu_j\|}\right\} \ll \frac{k^2 N^{4j/5}}{N^{j/10}}$ $\qquad$ Put the bounds for q_j into $(*)$

Putting everything together, we get (for some constant $D > 0$)

$$\frac{|U(n)|}{N^{4/5}} \leq \left((4r)^{8Ckr} N^{-\frac{4}{5}(1-\delta)\frac{1}{2}r(r+1)} \prod_{j=1}^{r} \left(Dk^2 N^{4j/5}\right) \prod \frac{1}{N^{j/10}} \right)^{\frac{1}{4k}}$$

the same $\qquad$ sum exponent over j $\qquad$ $\frac{2\log t}{\log N} \leq j \leq \frac{3\log t}{\log N}$

$$\leq \left((4r)^{8Ckr} N^{-\frac{4}{5}(1-\delta)\frac{1}{2}r(r+1)} (Dk^2)^r N^{\frac{4}{5}\left(\frac{1}{2}r(r+1)\right)} \underbrace{N^{-\frac{1}{10}\left(\frac{\log t}{\log N}\right)^2}}_{j \geq \frac{\log t}{\log N} \text{ along interval of length } \frac{\log t}{\log N}} \right)^{\frac{1}{4k^2}}$$

If we choose C large enough (with $k = Cr^2$) we have $\delta \leq \frac{1}{280}$ and therefore $\frac{4}{5}\delta\frac{1}{2}r(r+1)$ $\qquad$ $F = \lfloor (k-1)r \rfloor$, $\delta = (1 - \frac{1}{r})^F$

$$\leq 14\delta\left(\frac{\log t}{\log N}\right)^2 \leq \frac{1}{20}\left(\frac{\log t}{\log N}\right)^2$$

since $r \lesssim 5.01 \frac{\log t}{\log N}$.

So $\frac{|u(n)|}{N^{ks}} \lesssim \left((4r)^{8ckr} (Dk^2)^r N^{-\frac{1}{20}\left(\frac{\log t}{\log N}\right)^2} \right)^{\frac{1}{4k^2}}$

$(k = Cr^2)$ $= \left((4r)^{8cCr^3} (DC^2r^4)^r N^{-\frac{1}{20}\left(\frac{\log t}{\log N}\right)^2} \right)^{\frac{1}{4C^2r^4}}$

since $r \asymp \frac{\log t}{\log N}$

But we have the trivial bound anyway unless $N \geq e^{\log^{2/3} t}$

So the dominant term in the bracket is $N^{-\frac{1}{20}\left(\frac{\log t}{\log N}\right)^2}$, so we finally get

e.g. dominating behaviour NOT largest size

different c from before

$$\frac{|u(n)|}{N^{ks}} \ll N^{-c/\left(\frac{\log t}{\log N}\right)^2} = e^{-\frac{c\log^3 N}{\log^2 t}} \qquad \square$$

If $M = N = t^\Theta$, some $0 < \Theta \leq 1$, then the bound in Theorem 10.1 is

$$\Big|\sum_{N<n\leq 2N} n^{-it}\Big| \ll N e^{-c\Theta^2 \log N} = N^{1-c\Theta^2}$$

This is a <u>power-saving</u> in N, which we usually think of as a good result.

The best bound we might hope for is $\ll N^{\frac{1}{2}+o(1)}$, which would say that the n^{-it} behave "like random", "like the standard deviation".

If $N \geq e^{\log^{2/3} t}$ (but smaller than any fixed power of t) then we don't get a power saving, but we still get a non-trivial bound.

11 Some Spectacular Consequences

Using our zeta sum estimate (Theorem 10·1), we can deduce the promised upper bound for the size of $\zeta(s)$.

Theorem 11·1 (Richert, 1967, building on Vinogradov + Korobov)

There exists a large constant $c>0$ such that the following is true:

For any large t, and any $0<\sigma\leq 1$, we have

$$\zeta(\sigma+it) \ll t^{c(1-\sigma)^{3/2}}\log^{2/3} t$$

In particular, if $\sigma \geq 1-\frac{1}{\log^{2/3} t}$ then $\zeta(\sigma+it)\ll \log^{2/3} t$

Remarks

- Note that by the Hardy-Littlewood approximation to $\zeta(s)$ (Theorem 3·3) we have

$$\zeta(\sigma+it) = \sum_{n\leq t}\frac{1}{n^{\sigma+it}} + O(1)$$

- The trivial bound for the sum is $\sum_{n\leq t}\frac{1}{n^{\sigma}} \ll \frac{t^{1-\sigma}}{1-\sigma}$, which is fine if σ is far from 1, but is much weaker than Theorem 11·1 if σ is close to 1.
- In that case, we can bound part of the sum more efficiently using Theorem 10·1.

Proof (of Theorem 11·1)

Suppose first that $1-\frac{1}{\log^{2/3} t} \leq \sigma \leq 1$. Then

Hardy Littlewood

$$\zeta(\sigma+it) = \sum_{n\leq e^{\log^{2/3} t}}\frac{1}{n^{\sigma+it}} + \sum_{e^{\log^{2/3} t}<n\leq t}\frac{1}{n^{\sigma+it}} + O(1)$$

$$\ll \sum_{n\leq e^{\log^{2/3} t}}\frac{1}{n} + \sum_{\lfloor \log^{2/3} t\rfloor \leq j \leq \log t}\left|\sum_{e^{j}<n\leq e^{j+1}}\frac{1}{n^{\sigma+it}}\right|$$

$n^{\sigma} \asymp n$ on this page

simply split the sum

$$\ll \log^{2/3} t + \sum_{\lfloor \log^{2/3} t \rfloor \le j \le \log t} \left| \sum_{e^j < n \le e^{j+1}} \frac{1}{n^{\sigma + it}} \right|$$

The sequence $\frac{1}{n^\sigma}$ is monotone decreasing, so by Abel's Summation Lemma (in the proof of Lemma 3·4) we have

$$\left| \sum_{e^j < n < e^{j+1}} \frac{1}{n^{\sigma+it}} \right| \ll \frac{1}{e^{j\sigma}} \max_{e^j < n' \le e^{j+1}} \left| \sum_{e^j < n \le n'} n^{-it} \right|$$

Proof (Theorem 11·1, continued)

In the case where $\sigma \geq 1 - \frac{1}{\log^{2/3} t}$, we had shown that

$$\zeta(\sigma+it) \ll \log^{2/3} t + \sum_{\lfloor \log^{2/3} t \rfloor \leq j \leq \log t} \frac{1}{e^{j\sigma}} \max_{e^j < n' \leq e^{j+1}} \left| \sum_{e^j < n \leq n'} n^{-it} \right|$$

(By 10·1) Zeta sum estimate

$$\ll \log^{2/3} t + \sum_{\lfloor \log^{2/3} t \rfloor \leq j \leq \log t} \frac{1}{e^{j\sigma}} e^j e^{-cj^3/\log^2 t}$$

$$= \log^{2/3} t + \sum_{\lfloor \log^{2/3} t \rfloor \leq j \leq \log t} e^{j(1-\sigma) - cj^3/\log^2 t}$$

Since $1-\sigma \leq \frac{1}{\log^{2/3} t}$, this is

$$\ll \log^{2/3} t + \sum_j e^{\frac{j}{\log^{2/3} t} - c\left(\frac{j}{\log^{2/3} t}\right)^3}$$

sum over $\frac{j}{\log^{2/3} t} = r$ instead of j

$$= \log^{2/3} t + \sum_{r=1}^{\log^{1/3} t} \sum_{r\lfloor \log^{2/3} t \rfloor \leq j \leq (r+1)\lfloor \log^{2/3} t \rfloor} e^{r - cr^3}$$

$$\ll \log^{2/3} t + \log^{2/3} t \sum_{r=1}^{\log^{1/3} t} e^{r - cr^3} \ll \log^{2/3} t$$

If instead, $\sigma < 1 - \frac{1}{\log^{2/3} t}$, then one can proceed in a similar way, but breaking the sum over n at a different place to obtain a saving: ——— see Sheet 3 from here onwards:

If d is a large constant, then

$$\zeta(\sigma+it) = \sum_{n \leq t} \frac{1}{n^{\sigma+it}} + O(1) \quad \text{(Hardy-Littlewood)}$$

$$= \sum_{n \leq \exp(d \log t \sqrt{1-\sigma})} \frac{1}{n^{\sigma+it}} + \sum_{\exp(d\log t \sqrt{1-\sigma}) < n \leq t} \frac{1}{n^{\sigma+it}} + O(1)$$

$$\ll \frac{e^{d \log t\,(1-\sigma)^{3/2}}}{1-\sigma} + \sum_{\lfloor d\log t \sqrt{1-\sigma} \rfloor \leq j \leq \log t} \left| \sum_{e^j < n \leq e^{j+1}} \frac{1}{n^{\sigma+it}} \right|$$

$$\ll t^{d(1-\sigma)^{3/2}} \log^{2/3} t + \sum_{\lfloor d\log t \sqrt{1-\sigma} \rfloor \leq j \leq \log t} e^{j(1-\sigma) - \frac{cj^3}{\log^2 t}}$$

since $1-\sigma \geq \frac{1}{\log^{2/3} t}$

using Abel's Lemma and Theorem 10·1

$$\ll t^{C_1(1-\sigma)^{3/2}} \log^{2/3} t \quad + \sum_{\lfloor C_1 \log t \sqrt{1-\sigma} \rfloor \leq j \leq \log t} e^{j(1-\sigma) - cj(1-\sigma)C_1^2}$$

since $j^2 \geq C_1^2 \log^2 t\,(1-\sigma)$ $\square$

<u>Remark 11·2</u>

We stated Theorem 11·1 for $0 < \sigma \leq 1$, but it is easy to see that if $\sigma > 1$ then the first part of the proof shows that

$\zeta(\sigma + it) \ll \log^{2/3} t, \quad t \geq t_0$.

Finally, by combining Theorem 11·1 with Landau's Theorem (5·1), we obtain the best (i.e. widest) zero free region known.

<u>Corollary 11·3</u> (Vinogradov - Korobov Zero-free Region)

There exists a small absolute constant $c > 0$ such that $\zeta(s)$ has no zeros $s = \sigma + it$ in the region:

$$\left\{ s : \sigma \geq 1 - \frac{c}{\log^{2/3}(|t|+2)\,(\log\log(|t|+3))^{1/3}} \right\}$$

<u>Proof</u>

Theorem 11·1 tells us that if $t \geq t_0$ (a large constant) then

$$\zeta(\sigma+it) \ll t^{c(1-\sigma)^{3/2}} \log^{2/3} t$$
$$= \exp\left(c \log t\,(1-\sigma)^{3/2} + \tfrac{2}{3}\log\log t\right)$$

In particular, if $\sigma \geq 1 - \left(\frac{\log\log t}{\log t}\right)^{2/3}$, then

$$\zeta(\sigma+it) \ll e^{(c+\frac{2}{3})\log\log t}$$

So we can apply Landau's Theorem (5·1) with the choices

$$\phi(t) = (c + \tfrac{2}{3})\log\log t, \quad W(t) = \left(\frac{\log t}{\log\log t}\right)^{2/3},$$

obtaining that $\zeta(\sigma+it) \neq 0$ in the region

$$\sigma \geq 1 - \frac{c}{\phi(2t+1)\,W(2t+1)} = 1 - \frac{c}{(c+\frac{2}{3})\log^{2/3}(2t+1)(\log\log(2t+1))^{1/3}}$$

for $t \geq t_0$

By relabelling the constants, this yields Corollary 11·3 when $t \geq t_0$. For $t < t_0$, use the existing zero-free regions and symmetry (as in the proof of Corollary 5·2) $\square$

By repeating the proof of the Prime Number Theorem with the line of integration shifted into the Vinogradov - Korobov zero-free region (and obtaining a bound for $\frac{\zeta'(s)}{\zeta(s)}$ in that region as in Lemma 5·5), we get the best known error term for the Prime Number Theorem.

Corollary 11·4 (PNT with Vinogradov - Korobov error term)

For all $x \geq 3$, we have

$$\psi(x) = x + O\left(x \exp\left(-c \frac{\log^{3/5} x}{(\log\log x)^{\frac{1}{5}}}\right)\right)$$

Proof

Omitted. At the end you get

$$\psi(x) = x + O\left(x \log^2 x \left(\exp\left(-c \frac{\log x}{\log^{2/3} T (\log\log T)^{\frac{1}{3}}}\right) + e^{-\log T}\right)\right)$$

Choosing $T = \exp\left\{(\log^{3/5} x) / (\log\log x)^{\frac{1}{5}}\right\}$ is optimal and gives corollary 11·4.

Remark 11·5

Is $O\left(x \exp\left(-c \frac{\log^{3/5} x}{(\log\log x)^{\frac{1}{5}}}\right)\right)$ really much better than $O(x \exp(-c \log^{1/2} x))$?

i) In some problems, the exact size of the error term is qualitatively important. For example, if we had a zero-free region of the form $\sigma \geq 1 - \frac{c}{\sqrt{\log(|t|+2)}}$, this would solve some problems about numbers with only small prime factors ("smooth numbers").

ii) We know that the Vinogradov - Korobov error term isn't just a technical improvement because it took a lot of ideas to prove it!

04/03/14 Riemann Zeta Function (21)

Chapter 3

12 Primes in Short Intervals

If the Riemann Hypothesis is true, then $\psi(x) = x + O(\sqrt{x}\log^2 x)$

So $\psi(x + \sqrt{x}\log^3 x) - \psi(x) = (x + \sqrt{x}\log^3 x + O(\sqrt{x}\log^2 x)$

$- (x + O(\sqrt{x}\log^2 x)$

$= \sqrt{x}\log^3 x + O(\sqrt{x}\log^2 x)$

$= (1 + O(1))\sqrt{x}\log^3 x \quad$ as $x \to \infty$

Therefore there are prime powers in the short interval $[x, x+\sqrt{x}\log^3 x$

Actually, since $\psi(x) = \sum_{p \le x} \log p + O(\sqrt{x}\log^2 x)$

there must be primes in the interval.

We don't know how to prove the Prime Number Theorem with a power saving error term, but we might still hope to show that short intervals contain primes (e.g. $(x, x + x^{0.99}]$ contains primes). This is because we might hope that the error term in the Prime Number Theorem changes slowly, so that when we take the difference $\psi(x+y) - \psi(x)$ the error terms partially cancel.

To explore this, we will use von Mangoldt's explicit formula (Theorem 6.5) which asserts that for any $2 \le T \le x$,

$$\psi(x) = x - \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\mathrm{Im}(\rho)| \le T}} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}\log^2 x\right)$$

We may assume that the sum is restricted to ρ with $0 < \mathrm{Re}(\rho) < 1$, because we know from our zero-free regions that there are no zeroes with real part ≥ 1, and we know from the functional equation (Theorem 6·1) that the only zeroes with real part ≤ 0 are the "trivial zeroes" at $s = -2, -4, -6, \ldots$ whose contribution is

$$\sum_{k=1}^{\infty} \frac{1}{(-2k)x^{2k}} = O\left(\frac{1}{x^2}\right) = O\left(\frac{x}{T}\log^2 x\right)$$

So the sum over ρ is certainly finite (since otherwise the zeroes would have a limit point and $\zeta(s)$ would be identically zero).

Remark 12·1

Note that the sum over ρ forms part of the error term in the Prime Number Theorem, but we see that it is actually quite structured.

For any $1 \leq y \leq x$, the explicit formula yields

$$\psi(x+y) - \psi(x) = y - \sum_{\substack{\rho : \zeta(\rho)=0 \\ |Im(\rho)| \leq T}} \frac{(x+y)^\rho - x^\rho}{\rho} + O\left(\frac{x}{T}\log^2 x\right)$$

$$= y - \sum_{\substack{\rho : \zeta(\rho)=0 \\ |Im(\rho)| \leq T}} x^\rho \frac{(1+y/x)^\rho - 1}{\rho} + O\left(\frac{x}{T}\log^2 x\right)$$

$$= y + O\Big(\frac{y}{x} \sum_{\substack{\rho : \zeta(\rho)=0 \\ |Im(\rho)| \leq T}} x^{Re(\rho)}\Big) + O\left(\frac{x}{T}\log^2 x\right)$$

If $y/x \leq \frac{1}{|\rho|}$ then $\frac{(1+y/x)^\rho - 1}{\rho} = \frac{e^{O(y/x\,|\rho|)} - 1}{\rho}$

$$= O(y/x) \quad \text{(Taylor Expansion)}$$

If $y/x > \frac{1}{|\rho|}$, then $\frac{(1+y/x)^\rho - 1}{\rho} \ll \frac{1}{|\rho|} < y/x$

Here, we have already gained something because the first "O" term depends on the interval length y.

The Vinogradov-Korobov zero-free region (Corollary 11·3)

24/03/14 Riemann Zeta Function (21)

implies that $Re(\rho) \leq 1 - \frac{c}{\log^{2/3} T (\log\log T)^{1/3}}$ for every ρ in the sum (if T is large), but this will not be good enough if there are too many terms in the sum with $Re(\rho)$ large.

Let $N(T)$ denote the number of zeroes ρ of $\zeta(s)$ counted with multiplicity, such that $0 < Re(\rho) < 1$ and $0 \leq Im(\rho) \leq T$.

For any $0 < \sigma < 1$, let $N(\sigma, T)$ denote the number of zeroes ρ of $\zeta(s)$ counted with multiplicity, such that $\sigma \leq Re(\rho) < 1$ and $0 \leq Im(\rho) \leq T$.

Lemma 12·2

Let $0 < \varepsilon \leq \frac{1}{4}$ be a small parameter. Let T be large. With notation as above, for any $x \geq T$ and any $1 \leq y \leq x$, we have

See 12·1

$$\Psi(x+y) - \Psi(x) = y + O\Big(y \sum_{0 \leq j \leq \frac{1}{2\varepsilon}} \min\{x^{-j\varepsilon}, x^{-\frac{c}{\log^{2/3} T (\log\log T)^{1/3}}}\} \times N(1-(j+1)\varepsilon, T)\Big)$$
$$+ O(y x^{-\frac{1}{2}} N(T)) + O\big(\tfrac{x}{T} \log^2 T\big)$$

See 12·1

Proof

The Lemma just collects together the previous discussion; it only remains to show that

remaining two terms in the lemma

$$\frac{y}{x} \sum_{\substack{\rho: \zeta(\rho)=0 \\ |Im(\rho)| \leq T}} x^{Re(\rho)} = O\Big(y \sum_{0 \leq j \leq \frac{1}{2\varepsilon}} \min\{\ldots\} N(1-(j+1)\varepsilon, T) + y x^{-\frac{1}{2}} N(T)\Big),$$

or equivalently it remains to check that

$$\sum_{\substack{\rho: \zeta(\rho)=0 \\ |Im(\rho)| \leq T}} x^{Re(\rho)} \ll \sum_{0 \leq j \leq \frac{1}{2\varepsilon}} \min\{x^{1-j\varepsilon}, x^{1-\frac{c}{\log^{2/3} T (\log\log T)^{1/3}}}\} N(1-(j+1)\varepsilon, T) + \sqrt{x}\, N(T)$$

(multiplying both sides by $\frac{x}{y}$)

06/03/14 Riemann-Zeta Function (22)

Proof (Lemma 12·2, continued)

We needed to show that

$$\sum_{\substack{\rho:\zeta(\rho)=0 \\ |Im(\rho)|\le T}} x^{Re(\rho)} \ll \sum_{0\le j\le \frac{1}{2\varepsilon}} \min\{x^{1-j\varepsilon}, x^{1-\frac{c}{\log^{2/3}T(\log\log T)^{1/3}}}\} N(1-(j+1)\varepsilon, T) + \sqrt{x}N(T)$$

Note that

$$\sum_{\substack{\rho:\zeta(\rho)=0 \\ |Im(\rho)|\le T}} x^{Re(\rho)} = \underbrace{\sum_{0\le j\le\frac{1}{2\varepsilon}} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |Im(\rho)|\le T \\ 1-(j+1)\varepsilon\le Re(\rho) < 1-j\varepsilon}} x^{Re(\rho)}}_{\text{Covers } Re(\rho)>\frac{1}{2}} + O\left(\sum_{\substack{\rho:\zeta(\rho)=0 \\ |Im(\rho)|\le T \\ Re(\rho)\le\frac{1}{2}}} x^{Re(\rho)}\right)$$

①, ②

due to VK Zero-Free Region

$$\ll \sum_{0\le j\le\frac{1}{2\varepsilon}} \min\{x^{1-j\varepsilon}, x^{1-\frac{c}{\log^{2/3}T(\log\log T)^{1/3}}}\} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |Im\rho|\le T \\ 1-(j+1)\varepsilon\le Re(\rho)<1-j\varepsilon}} 1 \quad ①$$

$$+ \sqrt{x}\sum_{\substack{\rho:\zeta(\rho)=0 \\ |Im(\rho)|\le T}} 1 \quad ②$$

From the definition of $\zeta(s)$, we always have $\zeta(\bar{s}) = \overline{\zeta(s)}$, so the sums over ρ are $\le 2N(1-(j+1)\varepsilon, T)$ and $\le 2N(T)$ respectively. □

In the rest of the chapter we will obtain bounds for $N(T)$ and $N(\sigma, T)$. Bounding $N(T)$ doesn't require much "arithmetic" information about $\zeta(s)$, but bounding $N(\sigma, T)$ non-trivially is much more interesting. Such bounds are called <u>zero-density estimates</u>.

13 Counting all the Zeroes

In this section we will bound the function $N(T)$. In fact, we will prove a more precise result about the number of zeroes in horizontal strips of width 1.

Lemma 13·1

For any $t \geq 2$, $N(t+1) - N(t) = O(\log t)$

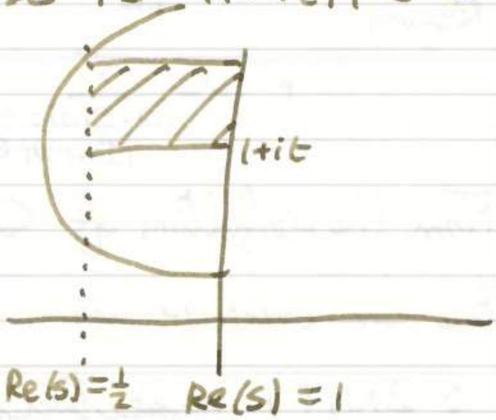
There are only finitely many zeroes ρ with $0 < Re(\rho) < 1$ and $0 \leq Im(\rho) \leq 2$, so we get:

Corollary 13·2

For any $T \geq 2$, $N(T) = O(T \log T)$.

Proof (of Lemma 13·1)

By Q4, Ex. Sheet 2, for any $t \geq 2$, the number of zeroes of $\zeta(s)$ in the disc $|s - (1+it)| \leq 0{\cdot}99$ is $O(\log t)$.



In particular, the number of zeroes ρ in the box

$\frac{1}{2} \leq Re(\rho) < 1$, $t \leq Im(\rho) \leq t + \frac{1}{2}$

is $O(\log t)$, since the box is contained in the disc. The same is true in the box where $t + \frac{1}{2} \leq Im(\rho) \leq t+1$, so in total there are $O(\log t)$ zeroes with $\frac{1}{2} \leq Re(\rho) < 1$, $t \leq Im(\rho) \leq t+1$.

Finally, if $\rho = \sigma + i\tau$ is a zero of the zeta function with $0 < \underset{\text{"}Re(\rho)\text{"}}{\sigma} \leq \frac{1}{2}$, then by the functional equation (Theorem 6·1) we see that

$$\zeta(1-\sigma-i\tau) = \zeta(1-\rho) = \frac{\pi^{-\rho/2}\Gamma(\frac{\rho}{2})\zeta(\rho)}{\pi^{-\frac{1-\rho}{2}}\Gamma(\frac{1-\rho}{2})} = 0$$

Symmetry in $Re(s) = \frac{1}{2}$ Sort of

since $\zeta(\rho) = 0$.

Also, we always have $\zeta(\bar{s}) = \overline{\zeta(s)}$, so actually

PTO

06/03/14

$\zeta(1-\sigma+i\tau) = \zeta(1-\sigma-i\tau) = 0$

Since $\frac{1}{2} \leq 1-\sigma < 1$, we see that the number of zeroes ρ with $0 < Re(\rho) \leq \frac{1}{2}$ and $t \leq Im(\rho) \leq t+1$ is the same as the number with $\frac{1}{2} \leq Re(\rho) < 1$ and $t \leq Im(\rho) \leq t+1$, so is $O(\log t)$. $\square$

(N.B. $N(T) \sim \frac{1}{2\pi} T \log T$)

14 Counting the zeroes with large real part

When we proved zero-free regions for the zeta-function, we used the fact that if $\zeta(s) = 0$ for some s just left of the 1-line, then $\zeta(s')$ is small for some s' just right of the 1-line. Then we used the Euler product to show that there must be another point to the right where ζ is large, which is impossible.

The bound for $N(\sigma, T)$ will also use the switch from counting zeroes to counting "large" values of other Dirichlet polynomials $\sum_{n \leq N} a_n/n^s$.

Because we no longer have access to the Euler product, we cannot show that this never happens, but we can show that it can't happen at many points s.

Definition 14.1

We define the Möbius function $\mu(n) = \begin{cases} (-1)^w & n \text{ has } w \text{ distinct prime factors} \\ 0 & n \text{ has a repeated factor} \end{cases}$

~~Lemma~~

Lemma 14.2

Let T, M be large, and suppose that $0 < \sigma \leq 1$, $\frac{T}{2} \leq t \leq T$.

Then $$\zeta(\sigma+it)\sum_{m \leq M} \frac{\mu(m)}{m^{\sigma+it}} = 1 + \sum_{\min\{M,T\} < n \leq MT} \frac{a_n}{n^{\sigma+it}} + O\left(\frac{M \log M}{M^{\sigma}T^{\sigma}}\right)$$

where $$a_n = \sum_{\substack{m \mid n \\ m \leq M \\ (n/m) \leq T}} \mu(m)$$

08/03/14 Riemann-Zeta Function (23)

Lemma 14·2

Let T, M be large. Suppose $0 < \sigma \leq 1$ and $\frac{T}{2} \leq t \leq T$

Then $\zeta(\sigma+it)\sum_{m\leq M}\frac{\mu(m)}{m^{\sigma+it}} = 1 + \sum_{\min\{M,T\}\leq n\leq MT}\frac{a_n}{n^{\sigma+it}} + O\left(\frac{M\log M}{(MT)^{\sigma}}\right)$

where $a_n := \sum_{m|n,\, m\leq M,\, \frac{n}{m}\leq T} \mu(m)$

Proof

As we have seen before, the Hardy-Littlewood approximation to $\zeta(s)$ (Theorem 3·3) yields that

$$\zeta(\sigma+it) = \sum_{n\leq T}\frac{1}{n^{\sigma+it}} + \frac{T^{1-\sigma-it}}{\sigma+it-1} + O(T^{-\sigma}) \quad t \asymp T,\ \frac{1}{t} \asymp \frac{1}{T}$$

$$= \sum_{n\leq T}\frac{1}{n^{\sigma+it}} + O(T^{-\sigma}) \text{ since } \frac{T}{2}\leq t\leq T.$$

So multiplying out term by term, we get

$$\zeta(\sigma+it)\sum_{m\leq M}\frac{\mu(m)}{m^{\sigma+it}} = \sum_{n\leq T}\frac{1}{n^{\sigma+it}}\sum_{m\leq M}\frac{\mu(m)}{m^{\sigma+it}} + O\left(T^{-\sigma}\sum_{m\leq M}\frac{1}{m^{\sigma}}\right)$$

$$= \sum_{\substack{nm\leq MT \\ n\leq T,\, m\leq M}}\frac{\mu(m)}{(nm)^{\sigma+it}} + O(T^{-\sigma}M^{1-\sigma}\log M)$$

by comparison with an integral, considering cases $1-\sigma < \frac{1}{\log M}$, $1-\sigma \geq \frac{1}{\log M}$

$$= \sum_{n\leq MT}\frac{a_n}{n^{\sigma+it}} + O(T^{-\sigma}M^{1-\sigma}\log M)$$

$M^{1-\sigma} = O(1)$; $\sum m^{-\sigma} < \sum m^{\frac{1}{\log M}-1}$ Compare with $\int dm$ | $\log M \geq \frac{1}{1-\sigma}$; $\frac{1}{1-\sigma}M^{1-\sigma} \ll M^{1-\sigma}\log M$

by definition of a_n (and relabelling nm as n).

Finally, if $n \leq \mathrm{Min}\{M, T\}$ then $a_n = \sum_{m|n}\mu(m)$ so by definition of $\mu(n)$, we see that

$n \leq T$ so $\frac{n}{m} \leq T$ automatically

$$\sum_{m|n}\mu(m) = \sum_{\substack{m|n \\ m \text{ squarefree}}}\mu(m) = \prod_{p|n}\underbrace{(1+\mu(p))}_{=\sum_{d|p}\mu(d)} = 0$$

because $\mu(M)=0$ if M not squarefree

since $\mu(p) = -1$.

Therefore $\sum_{m|n}\mu(m) = \begin{cases} 1 & n=1 \\ 0 & n\geq 2\end{cases}$ □

Lemma 14.3 (Zero-Detection Lemma)

Let T be large, and let $0 < \sigma < 1$, and M be large, satisfying $M^{1-\sigma} \log^2 M \leq T^{\sigma}$. $(*)$

Then there exist real numbers b_n, depending only on M and T, such that $|b_n| \leq \sum_{d|n} 1$ for all n, and such that

$$\sum_{0 \leq K \leq \max\left\{\frac{\log M}{\log 2}, \frac{\log T}{\log 2}\right\}} \left| \sum_{2^K \min\{M,T\} < n \leq 2^{K+1} \min\{M,T\}} \frac{b_n}{n^s} \right| \geq \frac{1}{2}$$

wherever s is a zero of the zeta function with $Re(s) \geq \sigma$ and $\frac{T}{2} \leq Im(s) \leq T$.

Proof

Note that if $Re(s) \geq \sigma$, then $\frac{M \log M}{(MT)^{Re(s)}} \leq \frac{M \log M}{(MT)^{\sigma}} \overset{\text{by } (*)}{\leq} \frac{1}{\log M}$ by hypothesis for M. Therefore, if $\frac{T}{2} \leq Im(s) \leq T$ and if $\zeta(s) = 0$, Lemma 14.2 implies that

$$\left| \sum_{\min\{M,T\} < n \leq MT} \frac{a_n}{n^s} \right| = 1 + O\left(\frac{1}{\log M}\right) \overset{M \text{ large}}{\geq} \frac{1}{2}$$

By the Δ-inequality, we have

$$\left| \sum_{\min\{M,T\} < n \leq MT} \frac{a_n}{n^s} \right| \leq \sum_{0 \leq K \leq \max\left\{\frac{\log M}{\log 2}, \frac{\log T}{\log 2}\right\}} \left| \sum_{\substack{2^K \min\{M,T\} \\ < n \leq 2^{K+1} \min\{M,T\}}} \frac{b_n}{n^s} \right|$$

where $b_n := a_n$ if $n \leq MT$, $b_n := 0$ otherwise.

$|b_n| \leq |a_n| \leq \sum_{m|n} 1$ $\square$

Remark 14.4

The point of multiplying $\zeta(\sigma+it)$ by $\sum_{m \leq M} \frac{\mu(m)}{m^{\sigma+it}}$ was that it removed all terms in the product with $n \leq \min\{M,T\}$, except for the constant term 1. This is not surprising since we

08/03/14 **Riemann-Zeta Function** 23

Prove that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$ (when $Re(s) > 1$).

The remaining terms with $n > \min\{M, T\}$ have denominators of relatively large size $n^{Re(s)} \geq n^{\sigma} > \min\{M^{\sigma}, T^{\sigma}\}$, which (as we shall see) makes it rare for the sum in the conclusion of Lemma 14·3 to be large.

In order to exploit Lemma 14·3 we will need a new tool, a simple but extended tool of Halász. The following special case will suffice:

<u>Lemma 14·5</u> (L^1 version of Halasz's Inequality, 1969-70)

Let $(c_n)_{N \leq n \leq 2N}$ be any complex numbers, and let (σ_1, t_1), ..., (σ_R, t_R) be any pairs of real numbers. Then

$$\sum_{1 \leq r \leq R} \left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| \leq \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2} \sqrt{\sum_{r,s \leq R} \left| \sum_{N \leq n \leq 2N} \frac{1}{n^{\sigma_r + \sigma_s + i(t_r - t_s)}} \right|}$$

<u>Proof</u>

The proof is just a clever application of the Cauchy-Schwartz Inequality. For each $1 \leq r \leq R$, let η_r be a complex number of absolute value 1, such that

$$\left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| = \eta_r \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}}$$ i.e. η_r is the argument

Then
$$\sum_{1 \leq r \leq R} \left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| = \sum_{1 \leq r \leq R} \eta_r \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}}$$
$$= \sum_{N \leq n \leq 2N} \left(c_n \sum_{1 \leq r \leq R} \eta_r \frac{1}{n^{\sigma_r + it_r}} \right)$$
$$\leq \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2} \sqrt{\sum_{N \leq n \leq 2N} \left| \sum_{r \leq R} \eta_r \frac{1}{n^{\sigma_r + it_r}} \right|^2}$$

Cauchy-Schwarz

$$= \sqrt{\sum_{N \le n \le 2N} |c_n|^2} \sqrt{\sum_{r,s \le R} \eta_r \overline{\eta_s} \sum_{N \le n \le 2N} \frac{1}{n^{\sigma_r + it_r}} \overline{\frac{1}{n^{\sigma_s - it_s}}}}$$

note: complex conjugate

Then easy to see the result

$\square$

11/03/14

Riemann-Zeta Function (24)

<u>Lemma 14.6</u>

Let $\sigma > 0$ and T be large. Also, let $(\sigma_1, t_1), \ldots, (\sigma_R, t_R)$ be pairs of real numbers such that $\sigma_r \geq \sigma$ and $|t_r| \leq T$ $\forall r \leq R$ <u>and</u> such that $|t_r - t_s| \geq 1$ if $r \neq s$. Finally, <u>let $N \geq T$</u> and let $(c_n)_{N \leq n \leq 2N}$ be any complex numbers.

Then $\displaystyle\sum_{1 \leq r \leq R} \left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| \ll \frac{\sqrt{RN\log T}}{N^{\sigma}} \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2}$

<u>Proof</u>

For any $r \leq R$ and $s \leq S$, we have $\quad c_n = n^{\sigma_r + \sigma_s} \quad d_n = n^{i(t_r - t_s)}$

$$\left| \sum_{N \leq n \leq 2N} \frac{1}{n^{\sigma_r + \sigma_s + i(t_r - t_s)}} \right| \underset{\substack{\text{Abel's} \\ \text{Lemma (as in 3.4)}}}{\ll} \frac{1}{N^{\sigma_r + \sigma_s}} \max_{N \leq N' \leq 2N} \left| \sum_{N \leq n \leq N'} \frac{1}{n^{i(t_r - t_s)}} \right|$$

since $\sigma_r, \sigma_s \geq \sigma$

$$\ll \frac{1}{N^{2\sigma}} \max_{N \leq N' \leq 2N} \left| \sum_{N \leq n \leq N'} \frac{1}{n^{i(t_r - t_s)}} \right| \quad (1)$$

Note that $n^{-i(t_r - t_s)} = e(f(n))$, where $f(x) = -\frac{1}{2\pi}(t_r - t_s)\log x$ is continuous, monotonic and satisfies

$$|f'(x)| = \frac{1}{2\pi} \frac{|t_r - t_s|}{x} \leq \frac{2T}{2\pi N} \underset{N \geq T}{\leq} \frac{1}{\pi}$$

Therefore, by van der Corput's Lemma (3.4) we have

$$\sum_{N \leq n \leq N'} \frac{1}{n^{i(t_r - t_s)}} = \int_N^{N'} x^{-i(t_r - t_s)}\,dx + O(1)$$

Get $\dfrac{x^{1 - i(t_r - t_s)}}{1 - i(t_r + t_s)}$ Simply integrate $\quad N' \asymp N$

$$= O\left(\frac{N}{1 + |t_r - t_s|}\right) \quad (2)$$

Substituting all of this into Lemma 14.5, we obtain that

$$\sum_{1 \leq r \leq R} \left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| \ll \frac{\sqrt{N}}{N^{\sigma}} \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2} \sqrt{\sum_{r,s \leq R} \frac{1}{1 + |t_r - t_s|}}$$

Use 14.5 then (1), then (2)

For any fixed $r \leq R$, the numbers $t_r - t_s$ are at least 1 apart as s varies, since we assumed that for the numbers t_s.

Moreover, they are all $\leq 2T$ in absolute value. See hypothesis

Therefore $\displaystyle\sum_{1 \le r, s \le R} \frac{1}{1+|t_r - t_s|} \ll \overbrace{\sum_{r \le R}}^{R} \overbrace{\sum_{1 \le s \le 2T}}^{\log T} \frac{1}{s} \ll R \log T$ $\square$

Finally, by combining Lemma 14.6 with our zero-detection result (Lemma 14.3) we can show that very few zeroes have large real part.

<u>Theorem 14.7</u> (Basic Zero-Density Estimate)

For any large T, and any $\frac{3}{4} \le \sigma \le 1$, we have

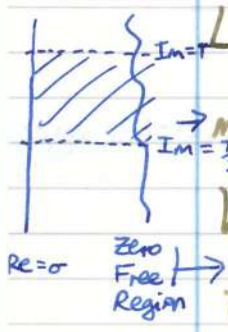
$N(\sigma, T) = O(T^{4(1-\sigma)} \log^8 T)$

In fact, the theorem is trivially true when $\sigma < \frac{3}{4}$ as well, since we always have $N(\sigma, T) \le N(T) = O(T \log T)$.

$\sigma = \frac{3}{4}$, $4(1-\sigma) = 1$

$T^{4(1-\sigma)} \log^8 T = T \log^8 T$

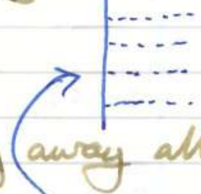
<u>Proof</u>



Let $\mathcal{N}$ denote the multiset of all zeroes ρ, counted with multiplicity, such that $Re(\rho) \ge \sigma$ and $\frac{T}{2} \le Im(\rho) \le T$.

We will show that $\# \mathcal{N} = O(T^{4(1-\sigma)} \log^7 T)$ and then the theorem will follow by applying this with T replaced by $\frac{T}{2^k}$, $0 \le k \le \frac{\log T}{\log 2}$

Let $\mathcal{N}_{even} \subseteq \mathcal{N}$ denote the subset consisting of those zeroes ρ such that $\lfloor Im(\rho) \rfloor$ is even, and $\mathcal{N}_{odd}$ similarly. We will show that $\# \mathcal{N}_{even} = O(T^{4(1-\sigma)} \log^7 T)$, and $\mathcal{N}_{odd}$ can be handled similarly.



Next, choose a subset $\mathcal{N}' \subseteq \mathcal{N}_{even}$ by throwing away all but one zero with imaginary part in each strip $2k \le Im(\rho) \le 2k+1$ (and if there are no zeroes with imaginary part in a given strip, doing nothing)

?

Q4, Sheet 2

By Lemma 13·1, in each of these strips, there were at most $O(\log T)$ zeroes counted with multiplicity so it will suffice to show that $\# \mathcal{N}' = O(T^{4(1-\sigma)} \log^6 T)$

Zero detection Lemma $\quad \min\{M, T\} = M = T$

Now we can apply Lemma 14·3 with the choice ~~m~~ $M = T$, obtaining that $\displaystyle\sum_{\rho \in \mathcal{N}'} \sum_{0 \leq k \leq \frac{\log T}{\log 2}} \left| \sum_{2^k T < n \leq 2^{k+1} T} \frac{b_n}{n^\rho} \right| \geq \left(\tfrac{1}{2}\right) \# \mathcal{N}'$

On the other hand, since the points in $\mathcal{N}'$ have imaginary parts differing by at least 1, we can apply Lemma 14·6 to obtain that

$$\sum_{\rho \in \mathcal{N}'} \left| \sum_{2^k T < n \leq 2^{k+1} T} \frac{b_n}{n^\rho} \right| \ll \sqrt{\frac{(\#\mathcal{N}') \, 2^k T \log T}{(2^k T)^\sigma}} \sqrt{\sum_{2^k T < n \leq 2^{k+1} T} |b_n|^2}$$

$$\forall\ 0 \leq k \leq \frac{\log T}{\log 2}$$

Check that (since $|b_n| \leq \sum_{d \mid n} 1$) that

$$\sum_{2^k T < n \leq 2^{k+1} T} |b_n|^2 \ll 2^k T \log^3 T$$

Therefore summing over k, we get

$$\# \mathcal{N}' \leq 2 \sum_{\rho \in \mathcal{N}'} \sum_{0 \leq k \leq \frac{\log T}{\log 2}} \left| \sum_{2^k T < n \leq 2^{k+1} T} \frac{b_n}{n^\rho} \right|$$

$$\ll \sqrt{\# \mathcal{N}'} \, (\log^3 T) \, T^{2(1-\sigma)} \qquad \square$$

Exam

- Answer (at most) 3 questions of 5
- Each question out of 30, and rescale.
- Mostly book-work, but with some problem elements based on lectures and examples sheets