Deformations of Plane Curve Singularities

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Abstract

We study deformations of plane curve singularities over Artin local k-algebras. In particular, we classify deformations over the dual numbers and demonstrate that they have a natural k-vector space structure. We conclude by discussing the existence of miniversal deformations.

Contents

1 Introduction

A plane curve singularity is a polynomial $f(x,y) \in k[x,y]$ such that $\frac{\partial f}{\partial x}|_{(a,b)} = \frac{\partial f}{\partial y}|_{(a,b)} = 0$ for finitely many points $(a,b) \in k^2$. In this report, we will be interested in understanding deformations of $\frac{k[x,y]}{(f(x,y))}$ as a k-algebra. In particular, we will consider deformations over Artin local k-algebras. To this end, we will classify deformations over the dual numbers and show how they are related to $f(x,y)$ and its partial derivatives f_x, f_y. Furthermore, we will construct a functor of Artin local k-algebras. As a consequence, we will be able to deduce that first order deformations are naturally a k-vector space. We will conclude by discussing the existence of miniversal deformations as the inverse limit of deformations. Throughout k will denote an algebraically closed field.

To prepare this report, we have extensively consulted [Har09], [Ser07], [Sch68] and [Liu06].

2 First order deformations

Let $\mathbf{Art}_k$ denote the category of Artin local k-algebras. For any $A \in \mathbf{Art}_k$, we will denote m_A as its unique maximal ideal.

Definition 2.1. *Let $A \in \mathbf{Art}_k$, R an A-algebra and $f(x,y)$ a plane curve singularity. Given the data of the following commutative diagram*

$$\begin{tikzcd}
\frac{k[x,y]}{(f(x,y))} & R \arrow[l, "i"'] \\
k \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

we say that R is a deformation of $\frac{k[x,y]}{(f(x,y))}$ if R is flat over A and $i : R \to \frac{k[x,y]}{(xy)}$ induces an isomorphism $R \otimes_A k \cong \frac{k[x,y]}{(f(x,y))}$.

Often, we may just say R is a deformation. It will be implicit that we mean the pair (R,i) is a deformation over a fixed $A \in \mathbf{Art}_k$ of $\frac{k[x,y]}{(f(x,y))}$ for some plane curve singularity $f(x,y) \in k[x,y]$.

Definition 2.2. *Two deformations $(R_1, i_1), (R_2, i_2)$ over $A \in \mathbf{Art}_k$ are isomorphic if there exists an A-algebra isomorphism $\alpha : R_2 \to R_1$ such that the following diagram commutes*

$$\begin{tikzcd}
 & & R_2 \arrow[dll, bend right, "i_2"'] \arrow[dl, "\alpha"] \\
R_0 & R_1 \arrow[l, "i_1"] & \\
k \arrow[u] & A \arrow[l] \arrow[u] \arrow[uur, bend right] &
\end{tikzcd}$$

We will be particularly interested in the case when $A = \frac{k[t]}{(t^2)}$. In other words, when A is the dual numbers. We will call a deformation over the dual numbers a *first order* deformation. We start by determining a criteria for flatness over the dual numbers.

2.1 Flatness criteria

Recall that if M is an A-module, we say that M is flat over A if for every injective homomorphism of A-modules $N \xrightarrow{g} N'$, the induced map $N \otimes_A M \xrightarrow{g \otimes id} N' \otimes_A M$ is injective.

Proposition 2.3. *Given a ring A and an A-module M, M is A-flat if and only if for every ideal $I \subset A$, the natural map $I \otimes_A M \to M$ is injective.*

Proof. First, consider the case when M is A-flat. We may consider any ideal $I \subset A$ as an A-module and the inclusion map $g : I \to A$ as an injective homomorphism of A-modules. Then

$$M \otimes_A I \to M \otimes_A A \cong M$$

is injective. Now consider the converse direction. Assume that for every ideal $I \subset A$, the canonical inclusion map $I \hookrightarrow A$ induces an injection $I \otimes_A M \hookrightarrow M$ and let $N \xrightarrow{g} N'$ be an arbitrary injective homomorphism of A-modules. It suffices to prove that

$$N \otimes_A M \xrightarrow{g \otimes id} N' \otimes_A M$$

is injective. We will proceed in cases dependent on N' considering first the simpler case when N' is free with finite rank. In other words, for some $n \in \mathbb{N}$, we have $N' \cong \oplus_{1\leq i\leq n} A$. For $n = 1$, we have $N' \cong A$, hence $N \cong I \subset A$ for some ideal. Using our hypothesis it follows that $M \otimes_A N \xrightarrow{id\otimes g} M \otimes_A N'$ is injective. Assume that for $n \geq 2$, we preserve injectivity for all free modules of rank strictly less than n and let N' now be a free module of rank n. Then we may write $N' \cong A \oplus N'_1$ where $N'_1 = (\oplus_{i=1}^{n-1} A)$. Using the fact that g is injective, we may construct the following commutative diagram where the rows are exact and vertical morphisms are injective.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{i=1}^{n-1} A & \longrightarrow & N' & \longrightarrow & A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle g} & & \uparrow & & \\
0 & \longrightarrow & g^{-1}(\oplus_{i=1}^{n-1} A) & \longrightarrow & N & \longrightarrow & g(N)/(\oplus_{i=1}^{n-1} A) & \longrightarrow & 0
\end{array}$$

We may tensor this diagram with M yielding the following

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\oplus_{i=1}^{n-1} A)\otimes_A M \cong \oplus_{i=1}^{n-1} M & \underset{\beta}{\longrightarrow} & N'\otimes_A M \cong \oplus_{i=1}^{n} M & \longrightarrow & A\otimes_A M \cong M & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \alpha} & & {\scriptstyle g\otimes id}\uparrow & & \uparrow{\scriptstyle \gamma} & & \\
 & & g^{-1}(\oplus_{i=1}^{n-1} A)\otimes_A M & \longrightarrow & N\otimes_A M & \longrightarrow & g(N)/(\oplus_{i=1}^{n-1} A)\otimes_A M & \longrightarrow & 0
\end{array}$$

Our inductive hypothesis ensures that α and γ are injective and the injectivity of β is deduced from the distributive property of the tensor product.

We claim that $g \otimes id$ is injective. Let $x \in \ker(g \otimes id)$. Then by the commutativity of the diagram and injectivity of γ, x must be in the kernel of $N \otimes_A M \to g(N)/(\oplus_{i=1}^{n-1} A) \otimes_A M$. Hence, by the exactness of the sequence there exists some $x' \in g^{-1}(\oplus_{i=1}^{n-1} A) \otimes_A M$ mapping to x. Using the commutativity of the diagram and injectivity of α, β, it follows that $x = 0$. We conclude that $\ker(g \otimes id)$ is trivial. Therefore, $g \otimes id$ is injective and the statement holds when N' is a free module of finite rank.

Next, consider when N' is a finitely generated A-module. Then there exists some $n \in \mathbb{N}$ and a surjective A-module homomorphism $\varphi : \oplus_{i=1}^{n} A \to N'$ inducing $N' \cong (\oplus_{i=1}^{n} A)/\ker(\varphi)$. This gives rise to the following commutative diagram where the horizontal morphisms are exact and vertical morphisms are injective.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ker(\varphi) & \longrightarrow & \oplus_{i=1}^{n} A & \xrightarrow{\varphi} & N' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle g} & & \\
 & & ker(\varphi) & \longrightarrow & \varphi^{-1}(N) & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Tensoring with M induces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ker(\varphi)\otimes_A M & \xrightarrow{\alpha} & \oplus_{i=1}^{n} A\otimes_A M & \xrightarrow{\varphi\otimes id} & N'\otimes_A M & \longrightarrow & 0 \\
 & & {\scriptstyle \sim}\uparrow & & {\scriptstyle \beta}\uparrow & & {\scriptstyle g\otimes id}\uparrow & & \\
 & & ker(\varphi)\otimes_A M & \xrightarrow{\theta} & \varphi^{-1}(N)\otimes_A M & \xrightarrow{\gamma} & N\otimes_A M & \longrightarrow & 0
\end{array}$$

where α, β are injective by our previous result when N' was free of finite rank. Again, we claim that $\ker(g \otimes id)$ is trivial. Let $x \in ker(g \otimes id)$. Then the surjectivity of γ implies that there exists $x' \in \varphi^{-1}(N) \otimes_A M$ such that $\gamma(x') = x$. By the injectivity of β and the commutativity of the diagram, we find that $\beta(x') \in ker(\varphi \otimes id)$. Hence, $\beta(x') \in im(\alpha)$. It follows that $x' \in im(\theta)$ and consequently $x' \in ker(\gamma)$. We conclude that $x = \gamma(x') = 0$ and consequently, $g \otimes id$ is injective.

Finally, when N' is an arbitrary A-module, we note that it is isomorphic to the direct limit of finitely generated submodules ordered by inclusion. In particular, since the tensor product commutes with direct limits, we conclude that $M \otimes_A N \xrightarrow{id\otimes g} M \otimes_A N'$ is injective for arbitrary injections $N \xrightarrow{g} N'$. □

This gives us a useful corollary for modules over the dual numbers.

Proposition 2.4. *A module over the dual numbers is flat if and only if the natural map*

$$(t) \otimes M \xrightarrow{t} M$$

is injective.

Proof. Let $I \subset \frac{k[t]}{(t^2)}$ be an ideal. Then by the correspondence theorem for rings, I corresponds uniquely with an ideal $J \subset k[t]$ containing (t^2) so that $I = J/(t^2)$. Since $k[t]$ is a principal ideal domain, $J = (f)$ for some $f \in k[t]$. It follows that $f|t^2$ and so $deg(f) \leq 2$. So we may write $f = a + bt + ct^2$ for $a, b, c \in k$. Reducing mod t^2, we see that $I = (a + bt)$. If $a \neq 0$, then $a + bt$ is a unit. It follows that the only non-trivial ideal of $\frac{k[t]}{(t^2)}$ is (t). Appealing to Proposition 2.3 gives the required result. □

2.2 First example

With this in mind, we may construct our first non-trivial example of a deformation over the dual numbers.

Example 2.5. *Consider the nodal singularity $xy \in k[x, y]$. Then $R := \frac{k[t,x,y]}{(t^2,xy-t)}$ is a deformation of $\frac{k[x,y]}{(xy)}$ over the dual numbers.*

Proof. From our definition of R, it is clear that $R \otimes_{\frac{k[t]}{(t^2)}} k \cong R/(tR) \cong \frac{k[x,y]}{(xy)}$. So it suffices to verify that R is flat over $\frac{k[t]}{(t^2)}$. Any element in R may be written as $f + gt + (t^2, xy - t)$ for $f, g \in k[x, y]$. Assume

$$f + gt + (t^2, xy - t) \otimes_{\frac{k[t]}{(t^2)}} t \mapsto ft + (t^2, xy - t) = 0$$

Then we must have $tf \in (t^2, xy - t)$. Hence, there exists $h_1, h_2 \in k[t, x, y]$ such that $tf = h_1t^2 + h_2(xy - t)$. Rearranging, it follows that $t(f - h_1t) = h_2(xy - t)$. Since $k[x, y, t]$ is a unique factorisation domain and $t \nmid xy - t$, we must have $t|h_2$. Consequently, we may write $f - h_1t = (\frac{h_2}{t})(xy - t)$ implying that $deg_t(h_1) + 1 = deg_t(h_2)$. However, this implies that $deg_t(f) > 0$, which is not possible as $f \in k[x, y]$. We conclude that $h_1 = h_2 = 0$ and so $f = 0$. Hence $R \otimes_{\frac{k[t]}{(t^2)}} (t) \xrightarrow{t} R$ is injective and we conclude that R is flat over $\frac{k[t]}{(t^2)}$. $\square$

We may also provide a non-example of a deformation. Consider again when $f(x, y) = xy$. However, let $R := \frac{k[t,x,y]}{(t^2,xy,tx)}$. Then $x \otimes_{\frac{k[t]}{(t^2)}} t \mapsto xt = 0$. But $x \neq 0$, so by our criterion established in Proposition 2.3, R is not flat over $\frac{k[t]}{(t^2)}$ and hence not a deformation.

2.3 Characterising first order deformations

Our goal in this section is characterise first-order deformations. More precisely, we will prove the following:

Theorem 2.6. *Let $f(x, y)$ be a plane curve singularity. Then deformations of $\frac{k[x,y]}{(f(x,y))}$ over the dual numbers up to isomorphism correspond bijectively to elements of the ring $\frac{k[x,y]}{(f,f_x,f_y)}$.*

To do so, we will first need to understand what shape a deformation can come in and how equivalence classes of deformations are related to f and in particular, its partial derivatives f_x, f_y.

Proposition 2.7. *(Nilpotent Nakayama) Let I be a nilpotent ideal of a ring A. Let M be an A-module. Consider a subset $S \subset M$ such that the image of S in M/IM generates M/IM. Then S generates M.*

Proof. Let $N \subset M$ be the submodule generated by S and for any $x \in M$, let $\bar{x}$ denote its image in M/IM. Consider an arbitrary $m \in M$. Then by assumption $\bar{m} = \sum_{i=1}^n a_i\bar{s}_i$ for $a_i \in A$ and $s_i \in S$. Pulling back to M, we let $n = \sum_{i=1}^n a_is_i \in N$ so that $m - n \in IM$. Consequently, we may write $m - n = \sum_{i=1}^k j_im'_i$ for $j_i \in I$ and $m'_i \in M$. However, we may repeat a similar argument again for each m_i deducing that there exists some $n_i \in N$ such that $m_i - n_i \in IM$. Consequently, $m = n' + x$ for some $n' \in N, x \in I^2M$. Repeating this step inductively for x means that for any $k \in \mathbb{N}$ we may find some $n_k \in N, x_k \in I^kM$ such that $m = n_k + x_k$. Therefore, since I is nilpotent, there exists sufficiently large k so that $I^k = 0$. Consequently, $m \in N$. $\square$

Proposition 2.8. *Let $A \in \boldsymbol{Art}_k$. Consider a short exact sequence of A-modules:*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

If M_3 is flat over A, then the following is also exact

$$0 \longrightarrow M_1 \otimes_A k \longrightarrow M_2 \otimes_A k \longrightarrow M_3 \otimes_A k \longrightarrow 0$$

Proof. Consider the following commutative diagram of exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & M_1 \otimes_A m_A \arrow[r, "j"] \arrow[d, "h"] & M_2 \otimes_A m_A \arrow[r, "h"] \arrow[d, "e"] & M_3 \otimes_A m_A \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "d"] \arrow[d, "b"'] & M_2 \arrow[r, "f"] \arrow[d, "c"'] & M_3 \arrow[r] \arrow[d] & 0 \\
 & M_1 \otimes_A k \arrow[r, "a"'] \arrow[d] & M_2 \otimes_A k \arrow[r] \arrow[d] & M_3 \otimes_A k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It suffices to prove that $\ker(a)$ is trivial. We will proceed in a diagram chasing argument. Let $x \in \ker(a)$. Then the surjectivity of b permits us to find some $x' \in M_1$ so $b(x') = x$. By the commutativity of the diagram $d(x') \in \ker(c)$ and consequently there exists some $y \in M_2 \otimes_A m_A$ mapping to $d(x')$. Then commutativity implies $g \circ h(y) = f \circ e(y) = f \circ d(x')$. Using the injectivity of g and the exactness of the middle row, we deduce that $y \in ker(h)$ and hence there exists some $y' \in M_1 \otimes_A m_A$ mapping to y under j. Injectivity of d implies that $x' = h(y')$. Finally, the exactness of the left column implies then that $x = b(x') = b(h(y')) = 0$. So, $\ker(a)$ is trivial and the bottom row is exact. ☐

Proposition 2.9. *Let $R_0 := \frac{k[x,y]}{(f(x,y))}$ for some plane curve singularity $f(x,y) \in k[x,y]$. Let $(R_1, i_1), (R_2, i_2)$ be two deformations over $A \in \boldsymbol{Art}_k$ of R_0 and $\alpha : R_2 \to R_1$ an A-algebra homomorphism such that the following diagram commutes*

$$\begin{tikzcd}
 & & R_2 \arrow[dll, bend right, "i_2"'] \arrow[dl, "\alpha"] \\
R_0 & R_1 \arrow[l, "i_1"] & \\
k \arrow[u] & A \arrow[l] \arrow[u] \arrow[uur, bend right] &
\end{tikzcd}$$

then α is an A-algebra isomorphism and $R_1 \cong R_2$ as deformations.

Proof. We may tensor the following exact sequence with k

$$R_2 \xrightarrow{\alpha} R_1 \longrightarrow coker(\alpha) \longrightarrow 0$$

yielding

$$R_2 \otimes_A k \xrightarrow{\alpha \otimes_A id} R_1 \otimes_A k \longrightarrow coker(\alpha) \otimes_A k \longrightarrow 0$$

Since $i_1 \circ \alpha = i_2$ and both i_1, i_2 induce respective isomorphisms $R_1 \otimes_A k \xrightarrow{\sim} R_0, R_2 \otimes_A k \xrightarrow{\sim} R_0$, we conclude that $\alpha \otimes id$ is an isomorphism. Therefore, $coker(\alpha) \otimes_A k = 0$. Appealing to Nilpotent Nakayama 2.7 we deduce that $coker(\alpha) = 0$ and therefore, α is surjective. Now, we have the following exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow R_2 \xrightarrow{\alpha} R_1 \longrightarrow 0$$

By Proposition 2.8, the following is also exact

$$0 \longrightarrow \ker(\alpha) \otimes_A k \longrightarrow R_2 \otimes_A k \cong R_0 \longrightarrow R_1 \otimes_A k \cong R_1 \longrightarrow 0$$

So $\ker(\alpha) \otimes_A k = 0$. By Nilpotent Nakayama, we conclude that $\ker(\alpha) = 0$ and so α is an isomorphism. ☐

Proposition 2.10. *Let $A \in \boldsymbol{Art}_k$. If R a deformation of $\frac{k[x,y]}{(f(x,y))}$ over A, then there exists some $g \in m_A[x,y]$ such that*

$$R \cong \frac{A[x,y]}{(f(x,y) + g)}$$

Proof. In the following diagram we may lift x, y in the image of i to some $r, s \in R_1$. Then we may define surjective A-algebra homomorphism $A[x,y] \to R_1$ sending x, y to r, s respectively ensuring that the diagram commutes.

$$\begin{tikzcd}
k[x,y] \arrow[r, "\pi"] & \frac{k[x,y]}{(f(x,y))} \\
A[x,y] \arrow[u] \arrow[r, dashed] & R_1 \arrow[u, "i"']
\end{tikzcd}$$

Consequently, for some ideal $J \subset A[x,y]$, we have the following exact sequence

$$0 \longrightarrow J \longrightarrow A[x,y] \longrightarrow R_1 \longrightarrow 0$$

Since R is flat over A, we use Proposition 2.8 to deduce that the following is exact

$$0 \longrightarrow J \otimes k \longrightarrow k[x,y] \longrightarrow \frac{k[x,y]}{(f(x,y))} \longrightarrow 0$$

Hence, $J/m_A J \cong (f(x,y))$. Therefore, since m_A is nilpotent, by Nilpotent Nakayama 2.7, we have $J \cong (f(x,y) + g)$ for some $g \in m_A[x,y]$. ☐

Applying Proposition 2.10 when $A = \frac{k[t]}{(t^2)}$ we have $m_A = (t)$ and consequently, every first order deformation is of the form $\frac{k[x,y]}{(f(x,y)+tg(x,y))}$ for some $g(x,y) \in k[x,y]$.

Proposition 2.11. *Let $f(x,y) \in k[x,y]$ and $B \in \boldsymbol{Art}_k$. Then for any $G, H \in B[x,y]$*

$$f(x+G, y+H) = f(x,y) + Gf_x + Hf_y \mod (G,H)^2$$

Proof. Consider the monomial $x^n y^m$ for some $n, m \in \mathbb{N}$. Then

$$\begin{aligned}
(x+G)^n(y+H)^m &= (\sum_{i=0}^{n} \binom{n}{i} x^{n-i} G^i)(\sum_{k=0}^{m} \binom{m}{k} y^{m-k} H^k) \\
&= (x^n + nx^{n-1}G + o(G^2))(y^m + my^{m-1}H + o(H^2)) \\
&= x^n y^m + nx^{n-1}Gy^m + my^{m-1}Hx^n + K(x,y) \text{ (where } K(x,y) \in (G,H)^2) \\
&= x^n y^m + G\frac{\partial}{\partial x}(x^n y^m) + H\frac{\partial}{\partial y}(x^n y^m) + K(x,y)
\end{aligned}$$

Hence,

$$\begin{aligned}
f(x+G, y+H) &= \sum_{i,j}^{n,m} a_{i,j}(x+G)^n(y+H)^m \\
&= \sum_{i,j}^{n,m} a_{i,j}x^i y^j + \sum_{i,j}^{n,m} a_{i,j}G\frac{\partial}{\partial x}(a_{i,j}x^i y^j) + \sum_{i,j}^{n,m} a_{i,j}H\frac{\partial}{\partial y}(a_{i,j}x^i y^j) + K(x,y) \text{ (where } K(x,y) \in (G,H)^2) \\
&= f(x,y) + Gf_x(x,y) + Hf_y(x,y) + K(x,y)
\end{aligned}$$

□

Remark 2.12. *Consider $B = \frac{k[t]}{(t^2)}$. An important observation is that if $f(x,y) \in k[x,y]$ is arbitrary, then $tf(x,y)$ is invariant under a coordinate change of the form*

$$x \mapsto x + tG$$

$$y \mapsto y + tH$$

Now, we prove Theorem 2.6. Let $T(A)$ denote the set of equivalence classes of deformations over A.

Proof. Consider the map of sets

$$\begin{aligned}
\Delta : \frac{k[x,y]}{(f, f_x, f_y)} &\to T\left(\frac{k[t]}{(t^2)}\right) \\
g(x,y) + (f, f_x, f_y) &\mapsto \frac{k[t,x,y]}{(t^2, f(x,y) + tg(x,y))}
\end{aligned}$$

First, we prove that Δ is well defined. Let $g_1(x,y), g_2(x,y) \in k[x,y]$ be such that $g_1(x,y) - g_2(x,y) \in (f, f_x, f_y)$. Then for some $h_1, h_2, h_3 \in k[x,y]$, we may write $g_1 = g_2 + h_1 f + h_2 f_x + h_3 f_y$. Since $1 - th_1$ is a unit and $t^2 = 0$, the following ideals are equal

$$\begin{aligned}
(f + t(g_2 + h_1 f + h_2 f_x + h_3 f_y)) &= ((1 - th_1)(f + t(g_2 + h_1 f + h_2 f_x + h_3 f_y)) \\
&= (f(1 + th_1 - th_1) + t(g_2 + h_2 f_x + h_3 f_y)) \\
&= (f + t(g_2 + h_2 f_x + h_3 f_y))
\end{aligned}$$

Now, consider the $\frac{k[t]}{(t^2)}$-algebra homomorphism α induced by the change of coordinates

$$x \mapsto x - th_2$$

$$y \mapsto y - th_3$$

Using Proposition 2.9, we conclude that this is induces an isomorphism of deformations. Appealing to Proposition 2.11, we may compute

$$\begin{aligned}
f + t(g_2 + h_2 f_x + h_3 f_y) &\mapsto f - th_2 f_x - th_3 f_y + t(g_2 + h_2 f_x + h_3 f_y) \\
&= f + tg_2
\end{aligned}$$

It follows that
$$\frac{(k[t,x,y])}{(t^2, f+tg_2)} \cong \frac{k[t,x,y]}{(t^2, f+t(g_2+h_1f+h_2f_x+h_3f_y))}$$
Therefore, $T(g_1) = T(g_2)$ and so Δ is well defined. By Proposition 2.10, Δ is surjective. So it remains to verify that Δ is injective. Assume that we have $g_1, g_2 \in k[x,y]$ so that $\Delta(g_1) = \Delta(g_2)$. Consequently, there exists a $\frac{k[t]}{(t^2)}$-algebra isomorphism ρ
$$\rho : \frac{k[t,x,y]}{(t^2, f+tg_1)} \to \frac{k[t,x,y]}{(t^2, f+tg_2)}$$
such that ρ reduces to the identity mod t. Therefore, we must have $\rho(x) = x + tu_1, \rho(y) = y + tu_2$ for $u_1, u_2 \in k[x,y]$. Consequently, $\rho(f) + t\rho(g_1) = f + tg_2 \mod t^2$. Again, using Proposition 2.11, we see that
$$\rho(f) + t\rho(g_1) = f + tu_1f_x + tu_2f_y + tg_1 \mod t^2$$
$$\implies g_2 - g_1 = u_1f_x + u_2f_y$$
We conclude that $g_1 - g_2 \in (f, f_x, f_y)$. Therefore, Δ is injective. ☐

3 The deformation functor

In the previous section, fixing a plane curve singularity $f(x,y) \in k[x,y]$, we associated with each $A \in \mathbf{Art}_k$, $T(A)$ the set of deformations of f over A up to isomorphism. We may verify that this association is actually functorial. Given any $\gamma \in Hom(A,B)$ for $A, B \in \mathbf{Art}_k$, we may induce a change of base map $T(A) \to T(B)$. In particular, any $R_1 \in T(A)$ is mapped to $R_1 \otimes_A B$. It is a basic property of flat modules that flatness is preserved under base change. Additionally, for morphisms $\gamma_1 : A \to B$, $\gamma_2 : B \to C$ the associativity property of base change ensures
$$(R_1 \otimes_A B) \otimes_B C \cong R_1 \otimes_A (B \otimes_B C) \cong R_1 \otimes_A C$$
This property ensures, $T(\gamma)(R_1) \in T(B)$ and $T(\gamma_1 \circ \gamma_2) = T(\gamma_1) \circ T(\gamma_2)$. Hence, T defines a covariant functor $\mathbf{Art}_k \to \mathbf{Set}$.

3.1 The vector space of first-order deformations

In this section, our goal is to show that there is a natural k-vector space structure associated to $T\left(\frac{k[t]}{(t^2)}\right)$, the set of equivalence classes of first order deformations.

Lemma 3.1. *Let $A \in \boldsymbol{Art}_k$ and M be a flat A-module. Then M is a free A-module.*

Proof. Observe that $M \otimes_A k \cong M/m_AM$. Hence $M \otimes_A k$ is a k-vector space. Consequently, there exists some basis $(\bar{x}_i)_{i\in I}$. Choosing x_i to represent $\bar{x}_i$, we can use Nilpotent Nakayama 2.7 to deduce that $(x_i)_{i\in I}$ generates M. Consequently, we can define a A-module homomorphism $\alpha : \oplus_{i\in I} A \to M$ so that the following sequence is exact
$$0 \longrightarrow ker(\alpha) \longrightarrow \oplus_{i\in I} A \longrightarrow M \longrightarrow 0$$
However, M is flat. Then using Proposition 2.8, the following is a short exact sequence.
$$0 \longrightarrow ker(\alpha) \otimes_A k \longrightarrow \oplus_{i\in I} k \longrightarrow M \otimes_A k \longrightarrow 0$$
Hence, $ker(\alpha) \otimes_A k = 0$ which implies, by Nilpotent Nakayama 2.7, that $ker(\alpha) = 0$. We conclude that M is free. ☐

Proposition 3.2. *Let $A, B \in \boldsymbol{Art}_k$. There exists a bijection of sets*
$$T(A \times_k B) \xrightarrow{\sim} T(A) \times T(B)$$

Proof. By functorality, the natural projection maps π_A, π_B of $A \times_k B$ induce a map of sets
$$\beta : T(A \times_k B) \to T(A) \times_{T(k)} T(B)$$
However, $T(k)$ consists only of the trivial deformation. So, $T(A) \times_{T(k)} T(B) = T(A) \times T(B)$. We aim to construct an inverse to this map. Consider a pair $(R_1, R_2) \in T(A) \times T(B)$ and let $R_0 := \frac{k[x,y]}{(f(x,y))}$. Then we may construct the following commutative diagram:
$$\begin{tikzcd} R_1 \times_{R_0} R_2 \arrow[r] \arrow[d] & R_2 \arrow[d, two heads, "i_2"] \\ R_1 \arrow[r, two heads, "i_1"'] & R_0 \end{tikzcd}$$

By Lemma 3.1, R_2 is a free B-algebra with some basis $\{e_i\}_{i\in I}$. This gives a collection $\{i_2(e_i)\}_{i\in I} \subset R_0$. However, since i_1 is a surjection, we may find a collection of elements $\{e'_i\}_{i\in I}$ such that $i_1(e'_i) = i_2(e_i)$. Moreover, it follows from Nilpotent Nakayama 2.7 that since $\{i_1(e'_i)\}_{i\in I}$ generates $R_1 \otimes_A k \cong R_0$, then $\{i_1(e'_i)\}_{i\in I}$ generates R_1. Thus, we may define surjective A-module homomorphism $\oplus_{i\in I} A \to R_1$. By writing the corresponding exact sequence and tensoring with k, we use Proposition 2.8 to deduce that R_1 is free with basis $\{i_1(e'_i)\}_{i\in I}$. Since both the bases of R_1, R_2 agree under i_1, i_2 respectively, it follows that $R_1 \times_{R_0} R_2$ is a free $A \times_k B$ algebra with basis $\{(e'_i, e_i)\}_{i\in I}$. Consequently, $R_1 \times_{R_0} R_2$ is then flat over $A \times_k B$. Furthermore, π_A induces

$$(R_1 \times_{R_0} R_2) \otimes_{A\times_k B} A \cong (\oplus_{i\in I} A \times_k B) \otimes_{A\times_k B} A \cong \oplus_{i\in I} A \cong R_1$$

Similarly, for π_B we will have $(R_1 \times_{R_0} R_2) \otimes_{A\times_k B} A \cong R_2$. Therefore,

$$(R_1 \times_{R_0} R_2) \otimes_{A\times_k B} k \cong R_1 \otimes_A k \cong R_0$$

We conclude that $R_1 \times_{R_0} R_2 \in T(A \times_k B)$ and so β is surjective. Now assume that we have $M \in T(A \times_k B)$ mapping to the pair (R_1, R_2). Then there exists a $q : A \times_k B$-algebra homomorphism making the following commute

$$\begin{tikzcd}
M \arrow[dr, "q"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R_1 \times_{R_o} R_2 \arrow[r] \arrow[d] & R_2 \arrow[d, two heads] \\
& R_1 \arrow[r, two heads] & R_0
\end{tikzcd}$$

We see that

$$M \otimes_{A\times_k B} A \cong R_1 \cong (R_1 \times_{R_0} R_2) \otimes_{A\times_k B} A$$

Therefore, using Proposition 2.7 we see q is a surjection. Finally, proceeding similarly to the injectivity case of Proposition 2.9 we conclude that q is injective. Hence, q is an isomorphism. Consequently, β must also be injective. We conclude that β is a bijection. ☐

Theorem 3.3. *The set of equivalence classes of deformations over the dual numbers, $T\left(\frac{k[t]}{(t^2)}\right)$, is a k-vector space.*

Proof. Consider $\frac{k[t_1]}{(t_1^2)} \times_k \frac{k[t_2]}{(t_2^2)} \cong \frac{k[t_1,t_2]}{(t_1^2, t_1 t_2, t_2^2)}$. There is a natural addition map

$$\Sigma : \frac{k[t_1]}{(t_1^2)} \times_k \frac{k[t_2]}{(t_2^2)} \to \frac{k[t]}{(t^2)}$$

$$a + bt_1 + ct_2 \mapsto a + (b + c)t$$

and multiplication map

$$\lambda_c : \frac{k[t]}{(t^2)} \to \frac{k[t]}{(t^2)}$$

$$t \mapsto ct$$

Hence, we may define scalar multiplication as $T(\lambda_c)$ for any $c \in k$ and addition as the composition

$$\begin{tikzcd}
T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[r, "\beta^{-1}"] \arrow[rr, bend right, "+"'] & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[r, "T(\Sigma)"] & T(\frac{k[t]}{(t^2)})
\end{tikzcd}$$

It suffices to verify the vector space axioms. We note that compatibility of scalar multiplication with field multiplication and the identity element of scalar multiplication follow easily from functorality. We explicitly verify the remaining axioms.

1. Associativity:
   We exploit the fact that $\sum$ is associative in $\mathbf{Art}_k$ and that β is induced by the natural projection maps

to conclude the following commutes.

$$\begin{tikzcd}
 & & T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[dll, "\sim"'] \arrow[d, "\sim"] \arrow[drr, "\sim"] & & \\
T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[rr, "\sim"] \arrow[dd, bend right, "id \times T(\sum)"'] & & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[dll, "T(\sum \times_k id)"] \arrow[drr, "T(id \times_k \sum)"'] & & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[ll, "\sim"'] \arrow[dd, bend left, "T(\sum) \times id"] \\
T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[drr, "T(\sum)"] & & & & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[dll, "T(\sum)"'] \\
T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[u, "\sim"] & & T(\frac{k[t]}{(t^2)}) & & T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[u, "\sim"']
\end{tikzcd}$$

2. Commutativity of addition:
Similarly, we use the fact that $\sum$ is commutative in $\mathbf{Art}_k$ to deduce that the following commutes

$$\begin{tikzcd}
T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[r, "\text{swap}"] \arrow[d, "\sim"] & T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[d, "\sim"'] \\
T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[r, "T(s)"] \arrow[dr, "T(\sum)"'] & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[d, "T(\sum)"] \\
 & T(\frac{k[t]}{(t^2)})
\end{tikzcd}$$

where

$$\begin{aligned} s : \frac{k[t_1]}{(t_1^2)} \times_k \frac{k[t_2]}{(t_2^2)} &\to \frac{k[t_1]}{(t_1^2)} \times_k \frac{k[t_2]}{(t_2^2)} \\ t_1 &\mapsto t_2 \\ t_2 &\mapsto t_1 \end{aligned}$$

3. Identity element:
Noting that k is initial in $\mathbf{Art}_k$ and $T(k)$ is a singleton consisting of the trivial deformation, we define $\mathbf{0}$ as the image of $\frac{k[x,y]}{(f(x,y))} \in T(k)$ in $T\left(\frac{k[t]}{(t^2)}\right)$. In particular, $\mathbf{0} = \frac{k[t,x,y]}{(t^2, f(x,y))}$. Fix an arbitrary $R \in T(\frac{k[t]}{(t^2)})$. Then by Proposition 2.10, there exists some $g(x,y) \in k[x,y]$, so that $R \cong \frac{k[t,x,y]}{(t^2, f(x,y)+tg(x,y))}$. We see that

$$\frac{k[t,x,y]}{(t^2, f(x,y))} \times_{\frac{k[x,y]}{(f)}} R \cong \frac{k[t_1,t_2,x,y]}{(t_1^2, t_2^2, t_1t_2, f(x,y) + t_1 g(x,y))}$$

It follows then that $T(\sum)(\mathbf{0} \times_{\frac{k[x,y]}{(f)}} R) = R$.

4. Additive inverses:
Consider arbitrary $R \in T(\frac{k[t]}{(t^2)})$. As before, by Proposition 2.10, there exists some $g \in k[x,y]$ so that $R \cong \frac{k[t,x,y]}{(t^2, f(x,y)+tg)}$. Consider $-R := \frac{k[t,x,y]}{(t^2, f(x,y)-tg)}$. Then we may compute

$$R \times_{\frac{k[x,y]}{(f)}} (-R) = \frac{k[t_1,t_2,x,y]}{(t_1^2, t_2^2, t_1t_2, f(x,y) + (t_1 - t_2)g)}$$

Consequently, $T(\sum)(R \times_{\frac{k[x,y]}{(f)}} (-R)) = \mathbf{0}$

5. Distributivity of scalar multiplication with respect to addition:

Using the fact that $\sum \circ (\lambda_c \times_k \lambda_c) = \lambda_c \circ \sum$, we verify that the following diagram commutes:

$$\begin{tikzcd}
 & & T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[dll, "\sim"'] \arrow[drr, "T(\lambda_c)\times T(\lambda_c)"] & & \\
T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[rr, "T(\lambda_c \times_k \lambda_c)"'] \arrow[d, "T(\sum)"'] & & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[dd, "T(\sum)"'] & & T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[ll, "\sim"] \arrow[d, "\sim"] \\
T(\frac{k[t]}{(t^2)}) \arrow[drr, "T(\lambda_c)"'] & & & & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[dll, "T(\sum)"] \\
 & & T(\frac{k[t]}{(t^2)}) & &
\end{tikzcd}$$

6. Distributivity of scalar multiplication with respect to field addition:
   Similarly, the following commutes

$$\begin{tikzcd}
 & T(\frac{k[t]}{(t^2)}) \arrow[r] \arrow[d] \arrow[dddl, bend right, "T(\lambda_{a+b})"'] & T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[d, "T(\lambda_a)\times T(\lambda_b)"] \\
 & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[d, "T(\lambda_a \times_k \lambda_b)"'] & T(\frac{k[t]}{(t^2)}) \times T(\frac{k[t]}{(t^2)}) \arrow[d, "\sim"] \\
 & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[dl] & T(\frac{k[t]}{(t^2)} \times_k \frac{k[t]}{(t^2)}) \arrow[l] \arrow[dll, bend left, "T(\sum)"] \\
T(\frac{k[t]}{(t^2)}) & &
\end{tikzcd}$$

□

Now, we may prove that $T(\frac{k[t]}{(t^2)})$ is in fact finite dimensional.

Proposition 3.4. *Let $f(x,y) \in k[x,y]$ be a plane curve singularity, then $\frac{k[x,y]}{(f,f_x,f_y)}$ is a finite dimensional k-vector space.*

Consider the variety $V(f,f_x,f_y) = \{(a,b) \in k^2 : f(a,b) = f_x(a,b) = f_y(a,b) = 0\}$. Then by assumption $V(f,f_x,f_y)$ consists of finitely many points $\{(a_1,b_1),...,(a_r,b_r)\} \subset k^2$. Consider the polynomials

$$F(x) = (x-a_1)...(x-a_r), G(y) = (y-b_1)...(y-b_r)$$

Both polynomials vanish at every $(a_i,b_i) \in V(f,f_x,f_y)$ and so $F,G \in I(V(f,f_x,f_y))$. By Hilbert's Nullstellensatz, $F,G \in \sqrt{(f,f_x,f_y)}$. Consequently, there exists $n,m \in \mathbb{N}$ so that $F^n(x), G^m(y) \in (f,f_x,f_y)$. Hence, we may define a surjective k-linear map

$$\frac{k[x,y]}{(F^n(x),G^m(y))} \to \frac{k[x,y]}{(f,f_x,f_y)}$$

Since $\frac{k[x,y]}{(F^n(x),G^m(y))}$ is a finite dimensional k-vector space we conclude that $\frac{k[x,y]}{(f,f_x,f_y)}$ must also be finite dimensional.

Theorem 3.5. *For a plane curve singularity f, there is a k-vector space isomorphism $T\left(\frac{k[t]}{(t^2)}\right) \xrightarrow{\sim} \frac{k[x,y]}{(f,f_x,fy)}$*

Proof. We verify that our map Δ defined in Theorem 2.6 is additive and k-linear. Let $g_1, g_2 \in \frac{k[x,y]}{(f,f_x,f_y}$ be arbitrary. Then

$$\Delta(g_1+g_2) = \frac{k[t,x,y]}{f+t(g_1+g_2)}$$

To compute $\Delta(g_1)+\Delta(g_2)$, we make use of our proof in Proposition 3.2 to first compute

$$\frac{k[t,x,y]}{(t^2,f+tg_1)} \times_{\frac{k[x,y]}{(f)}} \frac{k[t,x,y]}{(t^2,f+tg_2)} \cong \frac{k[t_1,t_2,x,y]}{(t_1^2,t_2^2,t_1t_2,f+t_1g_1+t_2g_2)}$$

noting that $\frac{k[t_1]}{(t_1^2)} \times_k \frac{k[t^2]}{(t_2^2)} \cong \frac{k[t_1,t_2]}{(t_1^2,t_2^2,t_1t_2)}$. Then applying $T(\sum)$ as defined in Theorem 3.3, we see that

$$T(\sum)(\frac{k[t_1,t_2,x,y]}{(t_1^2,t_2^2,t_1t_2,f+t_1g_1+t_2g_2)}) = \frac{k[t,x,y]}{(f+t(g_1+g_2))} = \Delta(g_1+g_2)$$

Next, let $c \in k$ be arbitrary. Then it follows immediately from our definition of scalar multiplication in $T(\frac{k[t]}{(t^2)})$ that

$$\begin{aligned} c\Delta(g_1) = T(\lambda_c)(\frac{k[t,x,y]}{f+tg_1}) &= \frac{k[t,x,y]}{(f+ctg_1))} \\ &= \Delta(cg_1) \end{aligned}$$

We conclude that Δ defines a k-vector space isomorphism. □

Example 3.6. *Consider the nodal singularity defined by $f(x,y) = xy$. Then $f_x = y, f_y = x$ and consequently*

$$\frac{k[x,y]}{(f,f_x,f_y)} = \frac{k[x,y]}{(xy,y,x)} \cong k$$

We see that first order deformations of the nodal singularity form a 1 dimensional k-vector space.

4 Miniversal deformations

In this final section, we will consider consider the existence of miniversal deformations.

4.1 Versal and miniversal deformations

Definition 4.1. *Let $\widehat{\boldsymbol{Art}_k}$ denote the category of local Noetherian k-algebras with the property that for any $A \in \widehat{\boldsymbol{Art}_k}$ and $n \in \mathbb{N}$, $A/(m_A)^n \in \boldsymbol{Art}_k$.*

For example, the power series ring $k[[t]] \in \widehat{\mathbf{Art}_k}$. Also note that any local Artin k-algebra is necessarily a local Noetherian k-algebra and so $\mathbf{Art}_k$ is a subcategory of $\widehat{\mathbf{Art}_k}$. With this in mind, we may extend our deformation functor to a functor of the form $\widehat{T} : \widehat{\mathbf{Art}_k} \to \mathbf{Set}$ by demanding that $\widehat{T(A)} = \varprojlim T(A/m_A^n)$ where $\varprojlim$ denotes the inverse limit.

Definition 4.2. *Let $A \in \widehat{\boldsymbol{Art}_k}$ and $\xi \in \varprojlim T(A/m_A^n)$. We call the pair (A,ξ) a couple.*

Given $A \in \widehat{Art_k}$, the Yoneda lemma asserts that there is a bijective correspondence between natural transformations of the form $Hom(A,-) \to \widehat{T}$ and the set $\widehat{T}(A)$. Hence, a couple (A,ξ) uniquely determines the data of such a natural transformation. If the couple induces an isomorphism of functors, we may call the couple a universal deformation. However, universal deformations are, in general, unlikely to exist. So, we will consider a weaker notion of universality.

Definition 4.3. *Let $F, G : \boldsymbol{Art}_k \to \boldsymbol{Set}$ be functors. A natural transformation is said to be smooth if the following conditions hold.*

1. *For $A \in \boldsymbol{Art}_k$, the corresponding map $F(A) \to G(A)$ is surjective.*
2. *For $A, B \in \boldsymbol{Art}_k$ such that $B \to A$ is a surjection, the natural map*
$$F(B) \to F(A) \times_{G(A)} G(B)$$
is surjective.

Definition 4.4. *Let (A,ξ) be a couple. If the induced morphism of functors $Hom(A,-) \to T$ is smooth, we will call (A,ξ) a versal couple. Moreover, if (A,ξ) is versal and induces a bijection $Hom(A, \frac{k[t]}{(t^2)}) \xrightarrow{\sim} T(\frac{k[t]}{(t^2)})$, we will call (A,ξ) a miniversal couple.*

Remark 4.5. *To justify the usefulness of this definition as a weaker notion of universality, we defer to the following result found in [Har09, Ch 3 §15 Proposition 15.2]:*

1. *Let (S,ξ) be a miniversal deformation. Then for any couple (A,η), there exists a homomorphism $S \to A$ such that $\xi \mapsto \eta$ under the induced map $\widehat{T}(S) \to \widehat{T}(A)$. Furthermore, the induced homomorphism $S/m_S^2 \to A/m_A^2$ is unique.*
2. *Let (S,ξ) be a universal deformation. Then for any (A,η) as in 1., the corresponding homomorphism $S \to A$ is unique.*

4.2 Miniversal deformations for plane curve singularities

Before proceeding we require a small lemma.

Definition 4.6. *Let $\alpha : B \to A$ be a surjective homomorphism in $\boldsymbol{Art}_k$. We say that α is a principal small extension if $ker(\alpha) = (b)$ for some $b \in B$ and $(b)m_B = (0)$.*

Lemma 4.7. *Any surjection $B \to A$ in $\boldsymbol{Art}_k$ can be factored into a sequence of principal small extensions.*

Proof. Let $s : B \to A$ be a surjection in $\mathbf{Art}_k$. We assume that s is not an isomorphism so that $ker(s) = I \subset B$ is non-trivial. Choose $p \in I$ non-zero then there is minimal $n \in \mathbb{N}$ such that $pm_B^n = (0)$. Choose non-zero $q \in pm_B^{n-1}$ (setting $q = p$ if $n = 1$) then the following diagram commutes.

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend right, "s"'] & B/(q) \arrow[r, two heads] & A \end{tikzcd}$$

If $B/(q) \cong A$ we are done. Suppose not, then may we may factor $B/(q) \to A$ similarly. Furthermore, this process must terminate after finitely many steps for else we have an infinite chain of descending ideals of B contradicting the descending chain property of Artin rings. ∎

Theorem 4.8. *Let $f(x, y)$ be a plane curve singularity. Choose $S = \{h_1, ..., h_m\} \subset k[x, y]$ so that the image of S in $\frac{k[x,y]}{(f,f_x,f_y)}$ forms a k-vector space basis. Then the couple*

$$(k[[t_1, ..., t_m]], \frac{k[[t_1, ..., t_m]][x, y]}{(f(x, y) + \sum_{i=1}^m t_i h_i)})$$

is a miniversal deformation.

Proof. First, we will prove that the couple is versal. Let $\alpha : B \to A$ be a surjection in $\mathbf{Art}_k$. By Lemma 4.7, it suffices to consider the case when α is a principal small extension so that $A \cong B/(b)$ for some $b \in B$ and $m_B(b) = (0)$. Consider the pair $(\gamma, R) \in Hom(k[[t_1, ..., t_m]], A) \times_{T(A)} T(B)$ both mapping to some $R' \in T(A)$. Using Proposition 2.10, we may write $R = \frac{B[x,y]}{(f+g)}$ and $R' = \frac{A[x,y]}{(f+g')}$ for some $g \in m_B[x, y]$ and $g' \in m_A[x, y]$. By assumption, $\gamma : k[[t_1, ..., t_m]] \to A$ induces

$$\frac{k[[t_1, ..., t_m]][x, y]}{(f + \sum_{i=1}^m t_i h_i)} \mapsto \frac{k[[t_1, ..., t_m]][x, y]}{(f + \sum_{i=1}^m t_i h_i)} \otimes A \cong \frac{A[x, y]}{(f + g')}$$

Hence, we may write

$$g' = \sum_{i=1}^m \gamma(t_i) h_i$$

Additionally, α induces

$$\frac{B[x, y]}{(f + g)} \otimes_B A \cong \frac{A[x, y]}{(f + g')}$$

However, noting that $A \cong B/(b)$, we must have $g - g' \in (b)[x, y]$. As a result, there exists some $H(x, y) \in k[x, y]$ such that

$$f + g = f + g' + bh(x, y) = f + \sum_{i=1}^m \gamma(t_i) h_i + bH(x, y)$$

Consider the projection of $H(x, y)$ into $\frac{k[x,y]}{(f,f_x,f_y)}$. Then we may find $\alpha_1, ..., \alpha_m \in k$ such that $H - \sum_{i=1}^m \alpha_i h_i \in (f, f_x, f_y)$. Consequently, there exists $P_1, P_2, P_3 \in k[x, y]$ so that $H = \sum_{i=1} \alpha_i h_i + P_1 f + P_2 f_x + P_3 f_y$. We may then write

$$f + g = f + \sum_{i=1}^m (\gamma(t_i) + b\alpha_i) h_i + bP_1 f + bP_2 f_x + bP_3 f_y$$

To bring the ideal $(f + g)$ into a more suitable form, we will proceed similarly as in the proof of Theorem 2.6. Firstly, $1 - bP_1$ is a unit. We find that

$$\begin{aligned}
(f + g) &= ((1 - bP_1)(f + g)) \\
&= ((1 - bP_1)(f + \sum_{i=1}^m (\gamma(t_i) + b\alpha_i) h_i + bP_1 f + bP_2 f_x + bP_3 f_y)) \\
&= (f(1 - bP_1) + bP_1 f + \sum_{i=1}^m (\gamma(t_i) + b\alpha_i) h_i + bP_2 f_x + bP_3 f_y) \\
&= (f + \sum_{i=1}^m (\gamma(t_i) + b\alpha_i) h_i + bP_2 f_x + bP_3 f_y)
\end{aligned}$$

Consider the B-algebra homomorphism induced by the change of coordinates

$$x \mapsto x - bP_2$$

$$y \mapsto y - bP_3$$

From Proposition 2.9, this defines an isomorphism of deformations. Using Proposition 2.11 and the fact that $m_B(b) = (0)$, we see that

$$f + \sum_{i=1}^{m}(\gamma(t_i) + b\alpha_i)h_i + bP_2 f_x + bP_3 f_y \mapsto f + \sum_{i=1}^{m}(\gamma(t_i) + b\alpha_i)h_i$$

Implying that

$$R \cong \frac{B[x,y]}{(f + \sum_{i=1}^{m}(\gamma(t_i) + b\alpha_i)h_i)}$$

With this in mind, consider the map $\eta \in Hom(k[[t_1, ..., t_m]], B)$ defined by $t_i \mapsto \gamma(t_i) + b\alpha_i$. It follows that $\eta \mapsto \gamma$ under $Hom(k[[t_1, ..., t_m]], B) \to Hom(k[[t_1, ..., t_m]], A)$ and $\eta \mapsto R$ under $Hom(k[[t_1, ..., t_m]], B) \to T(B)$. We conclude that that

$$Hom(k[[t_1, ..., t_m]], B) \to Hom(k[[t_1, ..., t_m]], A) \times_{T(A)} T(B)$$

is surjective. Next, consider the special case when B is arbitrary and $A = k$. Since both $Hom(k[[t_1, ..., t_m]], k)$ and $T(k)$ are singletons, we conclude that

$$Hom(k[[t_1, ..., t_m]], B) \to T(B)$$

is surjective. It follows that the couple is versal.

It remains to prove that the induced map $Hom(k[[t_1, ..., t_m]], \frac{k[t]}{(t^2)}) \to T(\frac{k[t]}{(t^2)})$ is injective. Indeed, if $\gamma_1, \gamma_2 \in Hom(k[[t_1, ..., t_m]], \frac{k[t]}{(t^2)})$ induce some deformation $\frac{k[t,x,y]}{(t^2, f+gt)}$. Then following our proof of injectivity in Theorem 2.6, we must have $\sum_{i=1}^{m}\gamma_1(t_i)h_i - \sum_{i=1}^{m}\gamma_2(t_i)h_i \in (f, f_x, f_y)$. However, by assumption $\{h_1, ..., h_m\}$ is a basis for $\frac{k[x,y]}{(f, f_x, f_y)}$. Consequently, $\gamma_1(t_i) = \gamma_2(t_i)$ and so $\gamma_1 = \gamma_2$. We conclude that the couple is a miniversal deformation. $\square$

We apply Theorem 4.8 to a familiar example.

Example 4.9. *Again we will consider the case when $f(x, y) = xy$, the nodal singularity. We have established that in this case $T(\frac{k[t]}{(t^2)}) \cong \frac{k[x,y]}{(xy,x,y)} \cong k$. So, any $c \in k/\{0\}$ forms a basis for the space of first-order deformations. Consequently, using the proof of 4.8, we see that the couple*

$$(k[[t]], \frac{k[[t]][x,y]}{(xy - ct)})$$

is miniversal.

5 Further exploration

We conclude by stating further areas of exploration that may be of interest following this report. Firstly, we restricted our interest to deformations of $\frac{k[x,y]}{(f(x,y))}$ for a single plane curve singularity. In particular, we found that deformations were closely related to the possible images of $f(x, y)$ under a change of coordinates which were, in turn, controlled by the first order partial derivatives of f. Something we have not considered is how our discussion of deformations changes when we consider a richer class of k-algebras. For example, we may consider k-algebras of the form $\frac{k[x,y]}{(f_1,...,f_r)}$ for $f_1, ..., f_r \in k[x, y]$.

Finally, in this report it was necessary to construct a deformation functor. In general, we may ask what conditions are necessary and sufficient for an arbitrary covariant functor $\mathbf{Art}_k \to \mathbf{Set}$ to have a miniversal or universal couple. This is answered in [Sch68].

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