Chapter 1

Elementary Algebra: Equations and Formulas

1.1 The Quadratic Equation

$$ax^2 + bx + c = 0 \tag{1.1.1}$$

There are two main ways of solving: by factoring or by the quadratic formula.

1.1.1 Factoring quadratics

The following quadratic equation is solved by factoring and setting the factors to zero.

$$2x^2 - x - 6 = 0 \tag{1.1.2}$$

Factor	Set to 0; solve	Set to 0; solve
$(2x+3)(x-2) = 0$	$2x + 3 = 0$ $2x + 3 = 0$ $2x = -3$ $x = -\frac{2}{3}$	$x - 2 = 0$ $x = 2$

Other factorable quadratics are:

$$x^2 + 8x + 15 = 0 \tag{1.1.3}$$

Factor	Set to 0; solve	Set to 0; solve
$(x+3)(x+5) = 0$	$x + 3 = 0$ $x = -3$	$x + 5 = 0$ $x = -5$

$$4x^2 - 9x[\pm 0] = 0 \tag{1.1.4}$$

Notice that this quadratic's third term can be plus or minus zero, and also that it can be factored as the difference of two squares x^2 and 3^2.

Factor	Set to 0; solve	Set to 0; solve
$(2x+3)(2x-3)=0$	$2x+3=0$ $2x=3$ $x=\frac{3}{2}$	$2x-3=0$ $2x=-3$ $x=-\frac{3}{2}$

1.1.2 Quadratic Formula

A quadratic expression $ax^2+bx+c=0$ can also be solved by randomly assigning numerical values to the variables in the following quadratic formula; although true random assigment is not possible, as I make clear below.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{1.1.5}$$

For example, the equation $2x^2-x-6=0$; where a = 2, b = -1, and c = -6 yields the following:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2-4(2)(-6)}}{2(2)} = \frac{1 \pm \sqrt{49}}{4} = \frac{1 \pm 7}{4} \tag{1.1.6}$$

The $\pm$ solutions are then $x=\frac{1+7}{4}=2$ and $x=\frac{1-7}{4}=-\frac{3}{2}$. As usual with fractionals, apparent solutions that produce a zero denominator are undefined and incorrect. For example, $a \neq 0$. If so, the denominator is $2(0)=0$, giving $\frac{X}{0}$, which is undefined.

Lastly, most quadratics have at most two real solutions, but some only have one. For example, the following equation has one solution, x = -2

$$x^2+4x+4=0 \tag{1.1.7}$$

$$-2^2+4(-2)+4=0$$

$$4+-8+4=0$$

$$-4+4=0$$

Some quadratic problems have no solution.

$$x^2+x+5=0 \tag{1.1.8}$$

A simple way to find out how many solutions to a quadratic exist is to find the value of the b^2-4ac portion of the formula; this is called the discriminant and is symbolized by Δ.

Table 1.1: Discriminant value for number of solutions to quadratic

If $\Delta > 0$	there are two unique real solutions
If $\Delta = 0$	there is one unique real solution
If $\Delta < 0$	there are two unique, conjugate imaginary solutions
If Δ is perfect square	two solutions are rational, otherwise irrational conjugates

1.2 Two maxims for inequalities

Inequalities work like most equations. The two maxims below give the rules for direction of inequality when a combinatory operation is done on both sides. The maxims can also be defined in a different way by the the corollary that follows them.

Maxim 1. Additive Maxim
When the same constant is added to (or subtracted from) both sides of the inequality, the direction of inequality is preserved. And, the new inequality is equivalent to the old inequality.

Maxim 2. Multiplicative Maxim
When the same constant is multiplied to (or divided from) both sides of the inequality, *the direction of inequality is preserved iff the constant is positive, but reversed if the constant is negative*. In both cases, the new inequality is equivalent to the old inequality.

Corollary to Inequality Maxims. *An inequality can change direction iff both sides of the inequality are multiplied (or divided) by the same negative constant.*

1.3 Sequences

Definition 1. A **sequence** can be defined as a FUNCTION whose domain is the set of real numbers.

Sequences are typically given as lists or series. For example, the function $f(n) = 2n-1$ will output a range over the real numbers. Notice that the function defines a sequence of the real numbers as a kind of subsequence that the function is supposed to use as the domain/input – this is why the definition explicitly mentions the set of real of numbers as the domain. But the presuppostion is stronger than simply assigning by definition the domain for the function; it in fact assumes that where one begins the subsequence is arbitrary. In other words, in the following example, I arbitrarily decided to start the subsequence at 0; also notice that it is possible to have my subsequence running at non-regular (perhaps even random) intervals such as $1, \frac{1}{2}, \frac{2}{3}, \sqrt{3}, \ldots$.

$$
\begin{aligned}
f(n) &= 2n - 1 \\
f(n) &= [2(1) - 1 = 1], [2(2) - 1 = 3], [2(3) - 1 = 5], [2(4) - 1 = 7], \ldots \\
&= 1, 3, 5, 7, 9, 11, \ldots \\
f(n) &= [2(1) - 1 = 1], [2(\frac{1}{2}) - 1 = 0], [2(\frac{2}{3}) - 1 = .\bar{3}], [2(\sqrt{3}) - 1 = 2.464], \ldots \\
&= 1, 0, .\bar{3}, 2.464, \ldots
\end{aligned}
$$

A Recursive Definition

Any sequence can be given a recursive definition by giving the first term and a rule that shows how to obtain further terms; i.e., how to get the $(n+1)th$ term from the nth term. Here is a general formulation followed by a specific example.

$$
\begin{aligned}
a_1 &= x \ \ \text{(}n\text{th term)} \\
a_n + 1 &= y \ \ \text{(recursive rule)}
\end{aligned} \tag{1.3.1}
$$

$$
\begin{aligned}
a_1 &= 5 \ \ \text{(}n\text{th term)} \\
a_n + 1 &= 3a_n - 2 \ \ \text{(recursive rule)} \\
&\text{therefore,} \\
a_1 &= 5 \\
a_2 &= 3(a_1(= 5)) - 2 = 13 \\
a_3 &= 3(a_2(= 13)) - 2 = 37 \\
a_4 &= 3(a_3(= 37)) - 2 = 109 \\
a_5 &= 3(a_4(= 109)) - 2 = 325
\end{aligned} \tag{1.3.2}
$$

Here we can see the details in the popular generative linguistic definition of recursiveness that points out that a certain kind of recursion (tail-recursion) provides input for the new function from the output of the previous function. For example, the new function a_2 is given the input $a_1 = 5$ (where 5 is the output of the function a_1) according to the recursive rule.

1.4 Arithmetic and Geometric Sequences

1.4.1 A-Sequence

An **arithmetic sequence** is in the form

$$
(a)_1, \quad (a+d)_2, \quad (a+2d)_3, \quad (a+3d)_4, \ldots, (a+(n-1)d)_n, \ldots \tag{1.4.1}
$$

where $a =$ first term of the sequence, $a + (n-1)d = n$th term of the sequence, n = number of terms, and $d =$ common difference of the terms. I have subscripted the seuqential terms in order to point out the fact that the first term a_1 has an addend of 0, the second term is d, or 1, the third term $2d$, or 2, the fourth term is $3d$ and so on. This is why the nth term is $n-1$. Some examples of adding sequences follow.

Example 1. *Write the first 6 terms and the 10th term of an arithmetic sequence with a first term of 7 and a common difference of 5.*

Solution 1. Since the first term is 7 and the common difference is 5, we can get the first 6 terms:

7, 7 + 5, 7 + 2(5), 7 + 3(5), 7 + 4(5), 7 + 5(5)

or

7, 12, 17, 22, 27, 32

Finding the 10th term requires substituting 10 for n in the formula for arithmetic sequence in 1.4.1:

$$\begin{aligned} n\text{th term} &= a + (n-1)d \\ 21\text{st term} &= 7 + (10-1)5 \\ &= 7 + (9)5 = 52 \end{aligned} \tag{1.4.2}$$

Of course, if given the first term a and the common difference d one can just simply add terms until getting to the nth term. The next example shows how this method would become tedious.

Example 2. *Find the 98th term of an arithmetic sequence whose first three terms are 2, 6, 10.*

Solution 2. First step is to find common difference $d : 10 - 6 = 4, 6 - 2 = 4$. $d = 4$. Then we can substitute variables in formula 1.4.1: $a = 2, n = 98$. Now plug in the solution.

$$\begin{aligned} n\text{th term} &= a + (n-1)d \\ 98\text{th term} &= 2 + (98-1)4 \\ &= 2 + (97)4 = 390 \end{aligned} \tag{1.4.3}$$

Adding terms by 4 for 97 consequentive terms would be a waste of time, even on a calculator.

Arithmetic means

These are numbers inserted between a first and last term to form a segment of an a-sequence. The method here is to assume a last term l as the nth term of the arithemtic sequence in 1.4.1

$$l = a + (n-1)d \tag{1.4.4}$$

Example 3. *Insert three means between -3 and 12.*

Solution 3. To insert three terms between the two we already have gives a total of 5 terms such that $n = 5$. We are given the first and last terms: $a = 3, l = 12$. What we do not know is the common difference d:

$$\begin{aligned} l &= a + (n-1)d \\ 12 &= -3 + (5-1)d \\ 15 &= 4d \quad \text{(Add 3 to both sides)} \\ \frac{15}{4} &= d \quad \text{(Divide both sides by 4)} \end{aligned} \tag{1.4.5}$$

Now with $d = \frac{15}{4}$ we plug in variables for finding the three middle terms which is really just part of the formula for arithmetic sequences in 1.4.1 and can be represented as $(a+d)_2,\ (a+2d)_3,\ (a+3d)_4$, which is the sequence minus the first and nth (last) term.

$$\begin{aligned} a + d &= -3 + \frac{15}{4} = \frac{3}{4} \\ a + 2d &= -3 + 2\frac{15}{4} = -3 + \frac{30}{4} = 4\frac{1}{2} \\ a + 3d &= -3 + 3\frac{15}{4} = -3 + \frac{45}{4} = 8\frac{1}{4} \end{aligned} \tag{1.4.6}$$

The whole sequence, then, is -3, $\frac{3}{4}, 4\frac{1}{2}, 8\frac{1}{4}$, 12

Sum of first n a-sequence terms

The first n terms of an arithmetical sequence can also be calculated.

$$S_n = \frac{n(a+l)}{2} \tag{1.4.7}$$

where, as above, $a =$ first term of the sequence, $a + (n-1)d = n$th term (or $l = [a + (n-1)d] =$ last term) of the sequence, and $n =$ number of terms.

Example 4. *Find the sum of the first 30 terms of the arithmetic sequence 5, 8, 11, ...*

Solution 4. Plug-in the variables: $a = 5,\ n = 30,\ d = 3,\ l = [5+(30-1)3] = 92$ and substitute into the formula from 1.4.7 to get

$$\begin{aligned} S_n &= \frac{n(a+l)}{2} \\ S_{30} &= \frac{30(5+92)}{2} \\ &= 15(97) = 1,455 \end{aligned} \tag{1.4.8}$$

1.4.2 G-Sequence

A **geometric sequence** is in the form

$$(a)_1, \quad (ar)_2, \quad (ar^2)_3, \quad (ar^3)_4, \ldots, (ar^{n-1})_n, \ldots \tag{1.4.9}$$

where a = first term of the sequence, $ar^{n-1} = nth$ term of sequence, and r = common ratio. Just as arithmetical sequences, I have numbered the terms by subscript to highlight the fact that the first term a_1 has no power, the second term ar_2 has power of 1, the third has power of 2, and so on. This is why the nth term power is $n - 1$.

Example 5. *Write the first 6 terms and the 25th term of the geometric sequence whose first term = 3 and common ratio = 2.*

Solution 5. Plugin the variables:

$3, \ 3(2), \ 3(2)^2, \ 3(2)^3, \ 3(2)^4, \ 3(2)^5$

or

3, 6, 12, 24, 48, 96

For the 25th term simple computation is a waste of time. Instead, substitute 25 for n; as well as the other variables in the formula for the nth term that is at the end of the formula in 1.4.9.

$$\begin{aligned} n\text{th term} &= ar^{n-1} \\ 25\text{th term} &= 3(2)^{25-1} \\ 3(2)^{24} &= 50,331,648 \end{aligned} \tag{1.4.10}$$

Geometric means

Geometric means are similar to arithmetical means.

Example 6. *Insert two geometric means between 4 and 256.*

Solution 6. Inserting two means within two terms give us a total of 4; $n = 4$. Plug-in the rest of the variables: first term $a = 4$; last term is assumed to be the nth term. What we need is r:

$$\begin{aligned} ar^{n-1} &= l \\ 4r^3 &= 256 \\ r^3 &= 64 \text{ (Divide both sides by 4)} \\ r &= 4 \text{ (Find cube root on both sides)} \end{aligned} \tag{1.4.11}$$

Now that we have r, just plug-in variables for the second and third terms in the geometric sequence in 1.4.9.

$$\begin{aligned} (ar)_2 &= 4 * 4 = 16 \\ (ar^2)_3 &= 4 * 4^2 = 4 * 16 = 64 \end{aligned} \tag{1.4.12}$$

The final sequence is 4, 16, 64, 256.

Sum of first n g-sequence terms

The sum of the first n terms of a geometric sequence can be found by the following formula:

$$S_n = \frac{a - ar^n}{1 - r} \quad r \neq 1 \tag{1.4.13}$$

The usual variables for sequences applies: where $a =$ first term of the sequence, $a + (n-1)d = n$th term (or $l = [a + (n-1)d] =$ last term) of the sequence, and $n =$ number of terms.

1.4.3 Infinite Geometric Sequences

Under certain conditions, we can find the sum of all the terms in an **infinite g-sequence**. To define this sum, consider the following g-sequence $a,\ ar,\ ar^2, \ldots$

$$\begin{aligned} a &= S_1 && \text{The first partial sum of the sequence} \\ a + ar &= S_2 && \text{The second partial sum of the sequence} \\ a + ar + ar^2 + \cdots + ar^{n-1} &= S_n && \text{The } n\text{th partial sum of the sequence} \end{aligned}$$

Definition 2 (SUM OF INFINITE GEOMETRIC SEQUENCE)**.** If S_n of an infinite g-sequence approaches some number S as n approaches ∞, then S is called the **sum of the infinite geometric sequence**.

$$S = \sum_{n=1}^{\infty} ar^{n-1} \tag{1.4.14}$$

To develop a formula for finding the sum of all the terms in an infinite g-sequence, consider the formula used for finding the sum of the first n terms in a g-sequence in 1.4.13

$$\boxed{S_n = \frac{a - ar^n}{1 - r} \quad r \neq 1}$$

Remark 1. If $|r| < l$ and a is a constant, then as n approaches ∞, ar^n approaches 0, and the term ar^n above can be dropped:

$$S = \frac{a}{1 - r} \quad |r| < 1 \tag{1.4.15}$$

where, as just noted, $|r| < l$; also, as previously stated, $a =$ first term and $r =$ common ratio.

Remark 2. If $|r| \geq l$, the terms get larger and the sum S does not approach a number. The result: theorem 1.4.15 does not apply.

Some examples will be helpful here.

Example 7. *Change* $0.\bar{3}$ *to a common fraction*

Solution 7. Write the decimal as an infinite geometric series,

$$S = \frac{3}{10} + \frac{3}{100} + \frac{3}{1,000} + \frac{3}{10,000} + \dots$$
$$S = \frac{3}{10} + \frac{3}{10}\left(\frac{1}{10}\right) + \frac{3}{10}\left(\frac{1}{10}\right)^2 + \frac{3}{10}\left(\frac{1}{10}\right)^3 + \dots$$

Since the common ratio $r = \frac{1}{10} : |\frac{1}{10}| < 1$, formula 1.4.15 is relevant for finding the sum of the infinite geometric series:

$$S = \frac{a}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{9} = \frac{1}{3} \tag{1.4.16}$$

Example 8. *A town with a population of 3500 has a predicted growth rate of 6% per year for the next 20 years. How many people are expected to live in the town 20 years from now?*

Solution 8. Let p_0 be the initial population. After 1 year the population p_1 will be the initial population (p_0) plus the growth (the product of p_0 and the rate of growth r).

$$\begin{aligned} p_1 &= p_0 + p_0 r \\ &= p_0(1+r) \quad \text{Factor out } p_0 \end{aligned} \tag{1.4.17}$$

The population p_2 at the end of 2 years will be

$$\begin{aligned} p_2 &= p_1 + p_1 r \\ p_2 &= p_1(1+r) \quad (\text{Factor out } p_1) \\ p_2 &= p_0(1+r)(1+r) \quad (\text{Sub for } p_1) \\ p_2 &= p_0(1+r)^2 \end{aligned} \tag{1.4.18}$$

The population at the end of year 3 will be $p_3 = p_0(1+r)^3$. Writing the terms in a sequence gives

$$\boxed{p_0,\ p_o(1+r),\ p_0(1+r)^2,\ p_0(1+r)^3, p_0(1+r)^4, \dots} \tag{1.4.19}$$

This geometric sequence with $a = [p_0] =$ first term, and $r = [1+r] =$ common ratio. In other words, $p_0 = a = 3500$, 1 + r = $r = 1.06$, and $n = 21$ (for the total number of terms to reach 20). To find the last term we assume the nth term (ar^{n-1}) is l such that $l = ar^{n-1}$.

$$\begin{aligned} l &= ar^{n-1} \\ l &= 3500(1.06)^{21-1} \\ l &= 3500(1.06)r^{20} \\ l &= ar^{n-1} \\ l &\approx 11,224.97415 \end{aligned} \tag{1.4.20}$$

The population after 20 years of annual growth of 6% is approximately 11,225.

1.5 Binomials and the Theorem

Expanding a binomial is integral to working with probability and statistics. The following section shows how to expand a binomial, $(x+y)^n$, and fleshes out some interesting properties.

1.5.1 Expanding binomials

If we take a binomial and expand its power, $(x+y)^{n \to n+1 \cdots}$, some interesting patterns develop. The expansion is as follows:

$$\begin{aligned}
(a+b)^0 =& 1 \ \text{(riase to zero is 1)} \\
(a+b)^1 =& a+b \ \text{(riase to 1 is identity)} \\
(a+b)^2 =& a^2+2ab+b^2_3 \\
=& (a+b)_1(a+b)_2 \\
(a+b)^3 =& a^3+3a^2b+3ab^2+b^3_4 \\
=& (a+b)_1(a+b)_2(a+b)_3 \\
(a+b)^4 =& a^4+4a^3b+6a^2b^2+4ab^3+b^4_5 \\
=& (a+b)_1(a+b)_2(a+b)_3(a+b)_4 \\
(a+b)^5 =& a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5_6 \\
=& (a+b)_1(a+b)_2(a+b)_3(a+b)_4(a+b)_5 \\
(a+b)^6 =& a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6_7 \\
=& (a+b)_1(a+b)_2(a+b)_3(a+b)_4(a+b)_5(a+b)_6
\end{aligned}$$

There are at least four patterns that can be found in this sequence:

1. After the power of 2: Each expansion has one more term than the power of the binomial (e.g., the b^n_{n+1} term). As an array, each expansion has one more row and column than the number of terms.

2. The degree of each term in each expansion equals the exponent of the binomial.

3. The first term in each expansion is a raised to the power of the binomial.

4. The exponents of a decrease by 1 in each successive term, and the exponents on b, beginning with b^0 in the first term, increase by 1 in each successive term.

Another pattern can be found by making an array out the coefficent's successive expansion. This pattern produces the following:

Column / *Row*	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	2	1				
4	1	3	3	1			
5	1	4	6	4	1		
6	1	5	10	10	5	1	
7	1	6	15	20	15	6	1

A more visually friendly image of what is known as Pascal's triangle is shown in figure 1.1. This pretty image shows (and implies) some of the stunning patterns that arise from expanding binomials. It is easy, perhaps, to see why such systematic patterned expansion of numerical values would be useful for modeling systems and patterns in nature and society.

Factorials If n is a natural number, then $n!$ is defined as

$$\boxed{n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)} \tag{1.5.1}$$

Two properties of factorials include

Property 1. Identity By definition, $0! = 1$.

Property 2. Equivalence If n is a natural number, then $n(n-1)! = n!$.

A couple examples should suffice:

$$\begin{aligned} 3! &= 3*2*1 = 6 \\ 6! &= 6*5*4*3*2*1 = 720 \\ 3(3-1)! &= 3*2 = 6 = 3! \end{aligned} \tag{1.5.2}$$

1.5.2 Binomial Theorem

If n is any positive number, then

$$\begin{aligned} &\text{(a+b)}^n = \\ &a^n + \frac{n!}{1!(n-1)!)}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + \\ &\frac{n!}{3!(n-3)!)}a^{n-3}b^3 + \cdots + b^n \end{aligned} \tag{1.5.3}$$

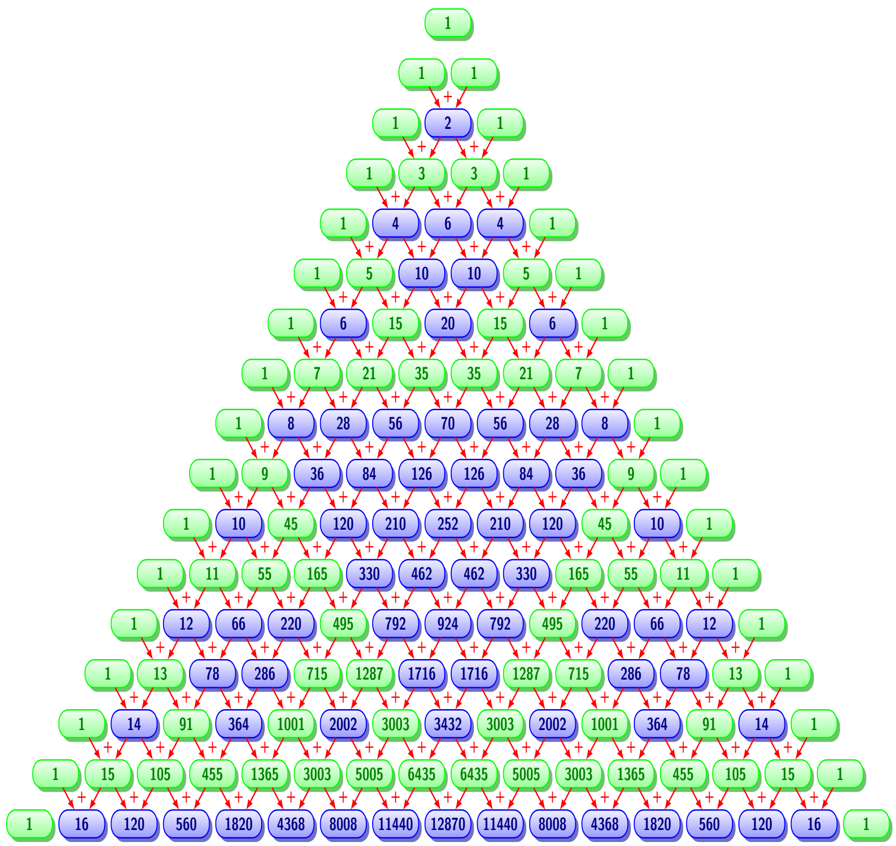

Figure 1.1: Pascal and Sierpinski Triangle

There is a more compact way to write the theorem using summation notation. I give the compact version here but lead up to it in the next section; see equation 1.6.2.

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r!)} a^{n-r} b^r$$

1.6 Summation

A shorthand way to indicate the sum of the first n terms, or the nth partial term, of a sequence. Notice that because summation shows which nth terms are to be used the arbitrariness of subsequences noted above no longer applies. That is, in $\sum_n^m$ the n value tells us where to start the numerical sequence and the m term tells us how far to go in the sequence.

$$\begin{aligned} &\sum_{n=1}^{3}(2n^2+1) \\ &= [2(1)^2+1]+[2(2)^2+1]+[2(3)^2+1] \\ &= 3+9+19 \\ &= 31 \end{aligned} \tag{1.6.1}$$

In this example there is no arbitrariness about where to begin the subsequence and how far to go in the sequence; nor about the systematicity of the interval partitions – at least not in this example (i.e., by numerical values of 1). The bottom domain tells us where to start (e.g., $n = 1$: start at 1), and the upper domain tells us how far to go (e.g., numerical value 3). One more example should suffice; notice that the upper domain is 5 and the lower one is 3. This does not mean start at 3 and continue 5 sequence steps, such as a_5. Instead, it means start at 3 and work our way up to 5, which is three sequential steps a_3; or more acurately $a_3 \rightarrow a_4 \rightarrow a_5$.

$$\sum_{n=3}^{5}(3n+2) \tag{1.6.2}$$

$$\begin{aligned} &= [3(3)+2]+[3(4)+2]+[3(5)+2] \\ &= 11+14+7 \\ &= 42 \end{aligned}$$

1.6.1 Three Basic Properties of Summation

Property 3. Summation of a constant. The summation of constant (c) as the value k runs from 1 to n is n times the constant.

$$\boxed{\text{If } c \text{ is constant, then } \sum_{k=1}^{n} c = nc.} \tag{1.6.3}$$

For example,

$$\boxed{\sum_{k=1}^{5} 13 = 13_1 + 13_2 + 13_3 + 13_4 + 13_5 = 5(13) = 65} \tag{1.6.4}$$

Proof. Because c is a constant, each term c is constant for each value k as k runs from 1 to n.

$$\sum_{k=1}^{n} c = \overbrace{c + c + c + c + c + \cdots + c}^{n \text{ number of } c_k} = nc$$

□

Property 4. Summation of a product. A constant factor can be brought outside a summation sign.

$$\boxed{\text{If } c \text{ is constant, then} \sum_{k=1}^{n} cf(k) = c\sum_{k=1}^{n} f(k).} \tag{1.6.5}$$

For example,

$$\boxed{\text{Show that} \sum_{k=1}^{3} 5k^2 = 5\sum_{k=1}^{3} k^2}$$

$$\begin{aligned} \sum_{k=1}^{3} 5k^2 &= 5(1)^2 + 5(2)^2 + 5(3)^2 \\ &= 5 + 20 + 45 \\ &= \mathbf{70} \end{aligned} \qquad \begin{aligned} 5\sum_{k=1}^{3} k^2 &= 5[(1)^2 + (2)^2 + (3)^2] \\ &= 5[1 + 4 + 9] \\ &= 5(14) = \mathbf{70} \end{aligned} \tag{1.6.6}$$

Proof.

$$\begin{aligned} \sum_{k=1}^{n} cf(k) &= cf(1) + cf(2) + cf(3) + \cdots + cf(n) \\ &= c[f(1) + f(2) + f(3) + \cdots + f(n)] \quad \text{(Factor out } c\text{)} \\ &= c\sum_{k=1}^{n} f(k) \end{aligned}$$

□

Property 5. Summation of a sum. The summation of a sum is equal to the sum of the summations.

$$\boxed{\sum_{k=1}^{n}[f(k)+g(k)] = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k).} \tag{1.6.7}$$

For example,

$$\boxed{\text{Show that} \sum_{k=1}^{3} (k+k^2) = \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2}$$

$$\begin{aligned}
\sum_{k=1}^{3} (k+k^2) &= (1+1^2)+(2+2^2)+(3+2^2) \\
&= 2+6+12 = \mathbf{20} \\
\sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2 &= (1+2+3)+(1^2+2^2+3^2) \\
&= 6+14 = \mathbf{20}
\end{aligned} \tag{1.6.8}$$

Proof.

$$\begin{aligned}
&\sum_{k=1}^{n}[f(k)+g(k)] \\
&\qquad = [f(1)+g(1)]+[f(2)+g(2)]+\cdots+[f(n)+g(n)] \\
&\qquad = [f(1)+f(2)+\cdots+f(n)]+[g(1)+g(2)+\cdots+g(n)] \\
&\qquad = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)
\end{aligned}$$

□

1.6.2 Binomial theorem as summation

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r!)} a^{n-r} b^r \tag{1.6.9}$$

1.7 Induction

A general definition for mathematical induction is based on the successor function. This is a now classic method made standard by Giuseppe Peano around the beginning of the 20th century; in which the arithmetic of cardinal numbers of axiomatized. Typically, we define a primitive and some kind of binary combinatoric operation (i.e., addition and/or multiplication), and then simply iterate the primtive and combining operation.[1]

[1] It is hard to overemphasize the importance of the use of induction to contemporary formalization.

A very common model for successor functions and induction is the domain of the whole numbers. The primitive is zero and the combinatory operation is addition (or multiplication). To this model we apply the axioms of Peano. Begin with three primitive (undefined and unmotivated) terms: **number**, **zero**, and **immediate successor of**. The following axioms can be given:

Peano Axiom 1. *Zero is a number*

Peano Axiom 2. *The immediate successor of a number is a number.*

Peano Axiom 3. *Zero is not the immediate successor of a number.*

Peano Axiom 4. *No two numbers have the same immediate successor.*

Peano Axiom 5. *Any property belonging to zero, and also to the immediate successor for every number that has the property, belongs to all numbers.*

The last axiom is referred to as the PRINCIPLE OF MATHEMATICAL INDUCTION; this reflects that understanding that the last 'axiom' is not technically an axiom; it may also be a called an INDUCTION AXIOM SCHEME. Peano's axioms can be formally represented as follows.

1.7.1 A Formalized Induction Scheme for Peano's Axioms

$$\text{Peano Axiom 3} \quad \neg 0 = sx \tag{1.7.1}$$

$$\text{Peano Axiom 4} \quad sx = sy \rightarrow x = y \tag{1.7.2}$$

1.7.1 says that the negation of zero is a number with a successor, or conversely, if a number has no successor, then it must be zero, or that zero is not the successor of any number. 1.7.2 says that if the successor of x equals the successor of y, then x equals y, or in other words, given any number, its predecessor is unique. Now we can provide representations for axioms of the binary operations addition, $+$, and multiplication, $*$,

$$x + 0 = x \tag{1.7.3}$$

$$x + sy = s(x + y) \tag{1.7.4}$$

$$x * 0 = 0 \tag{1.7.5}$$

$$x * sy = x * y + x \tag{1.7.6}$$

1.7.3 is the identity of addition (any number x added to zero sums to that number x); 1.7.4 says that any number x plus the successor of y is equivalent to the successor of $x + y$; 1.7.5 is straightforward, and 1.7.6 says that any number x times the sucessor of the number y is equivalent to $x * y$ plus the number x.

To the representations in 1.7.1–1.7.6 we can add the following instance of the induction scheme, or principle of mathematical induction, from 5.

Given any formula with a free variable x, $P(x)$, we can get the following instance of induction

$$(P(0)\&\forall x(P(x)) \rightarrow P(s(x))) \longrightarrow \forall x P(x) \tag{1.7.7}$$

What this says is that for the property P that belongs to 0, and for all x with that property, then the successor of x, sx, also has that property, $P(s(x))$; If the latter is the case, then this will hold for all x.

Writing numerals inductively

The common way of writing the Arabic numerals in the set of whole numbers, $0, 1, 2, 3, 4, \ldots, n$, implies the inductive principle and is an efficient way to compact the information in that principle. A more explicit way to write the set whole numbers is the following (I also show the correlation with Arabic numerals):

$$\begin{aligned} 0 &= 0 \\ 0s &= 1 \\ 0ss &= 2 \\ 0sss &= 3 \end{aligned} \tag{1.7.8}$$

Addition of $2 + 2 = 4$ looks like this (the second representation is in Polish notation and is equivalent with the first):

$$s(s(0)) + s(s(0)) = s(s(s(s(0)))).$$
or,
$$+(s(s(0)),\ s(s(0)) = s(s(s(s(0))))).$$

Division in a formalized induction scheme Induction is defined for binary combinatoric operations multiplication and addition. But we can also define division as (assume the definitions 1.7.3–1.7.6):

$$\exists z(x * z = y) \tag{1.7.9}$$

1.8 Induction and Sequences

Another way to get at induction is through arithmetical sequences (a-sequences); see section 1.4. As a transition I show an alternate way to write the addition of numerals as an inductive scheme:

$$\begin{aligned} A^3_0 &= 3 + 0 = 3 \\ A^3_1 &= 3 + 1 = 4 \\ A^3_2 &= 3 + 2 = 5 \end{aligned} \tag{1.8.1}$$

Another way to write this would be as

$$\begin{aligned} A^3_0 &= 3 + 0 = 3 \\ A^3_1 &= (3 + 1) = (3 + 0s) = (3 + 0)s = 4 \\ A^3_2 &= (3 + 2) = (3 + 0ss) = (3 + 0s)s = ((3 + 0)s)s = 5 \end{aligned}$$

The first line needs no explanation. The second line says that three plus one is equal to three plus the successor of zero, which itself is equal to the successor of three plus zero, which is 4. The third line says that three plus two is equal to three plus the successor of the successor of 0, which itself is equal to the successor of three plus the successor of zero, which is itself equal to the successor of the successor of three plus 0, which is 5. If ever there was a motivation for using Arabic numerals, this would be one: Arabic numerals implicitly provide the inductive information contained in their respective numerical values. In fact, the information is so implicit and assumed, that most college educated people are unable to provide a proof of why simple addition works. This is because we don't need to, the additive operation already contains the information needed for proof of the mathematical inductive principle.

A still yet more explicit way of writing this is

$$\begin{aligned}
A_0^{0sss} &= (0sss + 0) = (0sss) = 3 \\
A_{0s}^{0sss} &= (0sss + 0s) = (0ssss) = 4 \\
A_{0ss}^{0sss} &= 0sss + 0ss = (0sssss) = 5
\end{aligned}$$

What this suggests is that we can think of numerals defined through an induction scheme, or induction principle, as similar to a-sequences. That is, adding the sequence of successors defines the cardinality of the numerical value. This is an interesting point of relation between induction and a-sequences: a numerical value itself, say 4, can be derived through adding all the successors together.[2]

[2]Multiplication can be shown in the same way, except that instead of counting successors by a value of 1, we count successors by the value of the multiplicand. For example, if the mulitplicand is 3, then we count successive values by increments of 3: $3 * 2$ is equivalent to counting two times by increments of 3, $= (3, 6) = (3 + 3) = 6$. Another example, $5 * 4$ is equivalent to counting four times by increments of 5, $= (5, 10, 15, 20) = (5 + 5 + 5 + 5) = 20$; So on and so on.