Mathematical Formalizations: A brief study guide
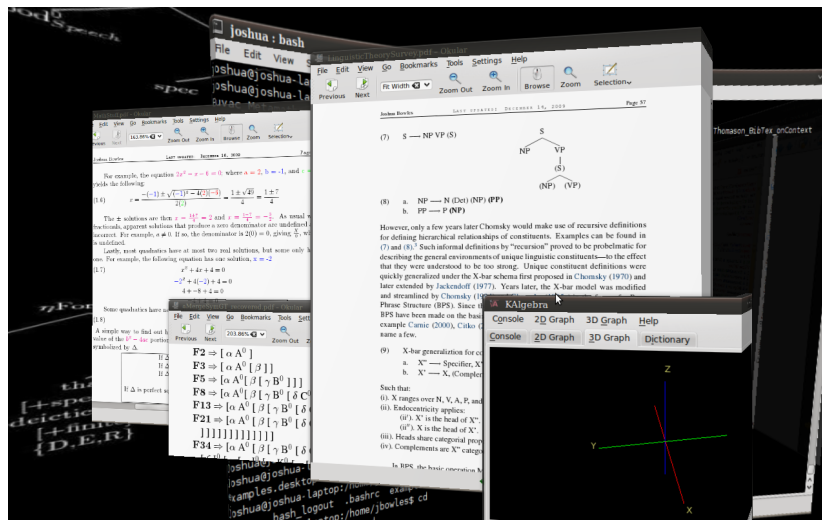
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Preface

> *Immature poets imitate; mature poets steal; bad poets deface what they take, and good poets make it into something better, or at least something different.*
> –T.S. Eliot

I would imagine the quote from Eliot could apply also to world of mathematics; one might addend it with the line "and genius creates directly from nature." I am far from a mature mathematician, and hope one day to be a good one; at least in terms of my respective field of application and theory of mathematical formaliztions to the study of natural language linguistics. If the quote above were truely accurate here then I would be doing more imitating then stealing..., but the truth is that I do more stealing than anything.

This book is intended as a study guide for myself; if you are reading it then I have made it public for whatever reason. **Please do not cite this reference guide**. I have taken liberally from multiple sources and have not been too cautious in making sure that I have not plagiarized. The reason for this is becuase I want to make sure the mathematics I present here is credible: proofs, examples, theorems, and so on. I have modified many examples and such, but only when I was sure of the outcome; if there was every any question on my part I simply copied freely in order to ensure that this personal study guide was true to the facts.

The following sources have both inspired and guided me, and represent sources that I have either borrowed from, or flatly taken taken material out of (without permission by the way): Frisk (2007), Korn & Korn (2000), Partee et al. (1990), Kleene (2002 [1967]), and .... Of particular note is Thompson (1986), originally written in 1910, and commendable for its Prologue which bears quoting in length.

> Considering how many fools can calculate, it is surprising tha tit should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.
>
> Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the text-books of advanced mathematics— and they are mostly clever fools—seldom take the trouble to show you how easy the calculations are. On the contrary, the see to desire

to impress you with their tremendous cleverness by going about it in the msot difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. . . . What one fool can do, another can.

Prologue

The quotation by Silvanus Thompson was written in time when a calculator (or better yet, a calculat-er) was an individual hired to plug variables into algorithms and perform other kinds of calculations. One could possibly see on a day to day basis, in Silvanus' day, that one did not need to be exceptionally bright in order to perform calculations. Today, we have sophisticated tools like scientitifc calculators, computational software, and computers to do the calculating for us. It is easy to be mistified by the process of brute calculation and from the depths of this mistification even easier to be lead into beleiving that calculation is an entirely difficult acitivty. Silvanus' humor, self-deprecation, and ease of exposition should be a model in both teaching and learning math. His all-around attitude is far too uncommon in a world that more resembles priests of some high-order who have secret knowledge of symbols and stand guard at the gates, waiting to dispel the weakest of underlings who dares try to join thier society than a mysterious world full of puzzles and beauty waiting to be explored by active young minds with rigorous tools developed thorugh the amazing history of human consciousness. One rarely hears about how mathematical developments rival the beauty and creativity of other human achievments in art, literature, music, and architecture. Silvanus' distinct attitude towards mathematics is conducive to the latter musings, and I hope to emulate his strength of character and appreciation for formalization.

I prefer to think of math like I often tell my daughters: doing math is like digging a ditch. It seems simple when you think about what needs to be done; that is, combining like properties or solving variables or employing various binary combinations on vaalues, *et cetera*. What is hard is the actual work involved – and the patience, discipline, and strength building that is necessitated by such work.

Chapter 1

Arithmetic

In this chapter I sketch the basic materials of arithmetic.

1.1   Percent

Any percent can be transformed into a rational number fraction by dividing by 100, which also gives a decimal:

$$40\% = \frac{40}{100} = .4$$

Multiplying decimal values by other numerical values will yield a percentage relation. For example,

$$30\% \text{ of } 350 = (350)\left(\frac{30}{100}\right) = (350)\left(\frac{3}{10}\right) = \frac{1,050}{10} = 105$$

or,

$$30\% \text{ of } 350 = .3 \times 350 = 105$$

Increases (equation 1.1.1) and decreases (equation 1.1.2) of percentages are handled by the following; assume that x and y are positive numbers and that $x < y$:

$$y \text{ is } \left(\frac{y-x}{x}\right)(100) \text{ percent greater than } x \qquad (1.1.1)$$

$$x \text{ is } \left(\frac{y-x}{y}\right)(100) \text{ percent less than } y \qquad (1.1.2)$$

For example, an **increase** from 600 to 750 is found by dividing the difference of the increase (750 - 600 = 150) by the first or **smaller** of the two initial values, known as the *base*, or the denominator:

$$\left(\frac{150}{600}\right)(100) = 25\%$$

Whereas a **decrease** from 500 to 400 is found by dividing the difference of the decrease (500 - 400 = 100) by the first or **larger** of the two initial values, also known as the *base*, or the denominator:

$$\left(\frac{100}{500}\right)(100) = 20\%$$

1.2   Average and Arithmetic Mean

$$\frac{x_1 + y_2 + z_3}{3} = \text{average} \tag{1.2.1}$$

$$\frac{n_m}{m} = \text{average} \tag{1.2.2}$$

Example 1.

$$\frac{5_1 + 8_2 + 8_3 + 14_4 + 15_5 + 10_6}{6} = Average(x)$$

$$\frac{60}{6} = 10 \tag{1.2.3}$$

1.2.1   Range

Range is defined as the greatest measurement minus the least measurement, iff all numerical values are a discrete set.

$$5, 8, 8, 14, 15, 10 = Range(x) \tag{1.2.4}$$

$$15 - 5 = 10 \tag{1.2.5}$$

Notice that it is a coincidence that the *range* and the *average* are the same value. Range by itself is not very useful, but has been used as the starting intuition for developing the *standard deviation*.

Chapter 2

Basic Geometry

2.1 Area, Perimeter, Volume

2.1.1 Area

Triangle:
$$A = \frac{bh}{2} \qquad \text{b = base, h = height} \tag{2.1.1}$$

Square:
$$A = l^2 \qquad \text{l = length} \tag{2.1.2}$$

Rectangle:
$$A = bh \qquad \text{b = base, b = height} \tag{2.1.3}$$

Regular Polygons: Add this later...
Circle:
$$A = \pi r^2 \qquad \text{r = radius} \tag{2.1.4}$$

2.1.2 Perimeter

Triangle:
$$P = l_a + l_b + l_c \qquad \text{l = length} \tag{2.1.5}$$

Square:
$$P = 4l \qquad \text{l = length} \tag{2.1.6}$$

Rectangle:[1]
$$P = 2(b + h) \qquad \text{b = base, h = height} \tag{2.1.7}$$

Regular Polygons:

$$P = nl \qquad \text{l = length, n = number of sides} \tag{2.1.8}$$

---

[1]Perimeter for a rectangle can also be defined equivalently as $P = 2(l + w)$, where l = length and w = width, or as $P = 2(sideA + sideB)$

Circle (*circumfrence*):

$$C = 2 * \pi * r \qquad \text{C = circumfrence, r = radius} \tag{2.1.9}$$

2.1.3   Volume

Cone or Pyramid:

$$V = \frac{1}{3}(bh) \qquad \text{b = base, h = height} \tag{2.1.10}$$

Cube:

$$V = l^3 \qquad \text{l = length} \tag{2.1.11}$$

Rectangular prism:

$$V = bwh \qquad \text{b = base, w = width, h = height} \tag{2.1.12}$$

Sphere:

$$V = \frac{4}{3}\pi r^3 \qquad \text{r = radius} \tag{2.1.13}$$

Cylinder:

$$V = A_{base} * h \qquad A_{base} = \text{area of base, h = height} \tag{2.1.14}$$

2.2   The Circle

For a circle, the *radius* is measured from the midpoint to an edge, *diameter* is any line that meets two edges and passes through the center point; any line that touches two edges but does not pass through the mid point is called a *chord*.

The ration of circumference to diameter of a circle is

$$\frac{c}{d}\pi \tag{2.2.1}$$

Circumference, given above in 2.1.2, is repeated here:

$$c = 2\pi r \tag{2.2.2}$$

2.3   Pythagorean theorem

$$a^2 + b^2 = c^2 \tag{2.3.1}$$

2.4   Polygons

2.4.1   Sums for interior polygon angles

The sum of the interior angles for any n-sided polygon is:

$$(n - 2)(180°) \tag{2.4.1}$$

For example, where n equals any real value for triangle, quadrilateral, pentagon, hexagon, etc.:

$$
\begin{aligned}
(3-2)(180°) &= 180° \\
(4-2)(180°) &= 360° \\
(5-2)(180°) &= 540° \\
(6-2)(180°) &= 720° \\
(7-2)(180°) &= 900° \\
(8-2)(180°) &= 1080° \\
(9-2)(180°) &= 1260° \\
(10-2)(180°) &= 1440°
\end{aligned}
\tag{2.4.2}
$$

Notice the the difference between $n\pm1$ side for the sum of interior angles is $180°$, that is, adding or subtracting one side to a polygon is equivalent to increasing or decreasing the sum of the interior angles by $180°$. We can state this as a co-variation, where the number of sides varies directly with the value 180; where $k = 180$ and is the constant:

$$
y = kx \qquad \text{or} \qquad \frac{y}{x} = k
\tag{2.4.3}
$$

The following show the direct variation of the sum of interior angles for polygons; first a quadrilateral, then a pentagon:

$$
360 = (180)(4-2) \qquad \text{or} \qquad \frac{360}{4-2} = 180
\tag{2.4.4}
$$

and,

$$
540 = (180)(5-2) \qquad \text{or} \qquad \frac{540}{5-2} = 180
\tag{2.4.5}
$$

An interesting thing that falls out of this series for polygons is the definition of line.[2] Defining a line by this method could be used as a proof for why a line is not a polygon; I leave this up to imagination.

2.4.2   Area and Perimeter for Polygons

The *perimeter* of a polygon is the sum of the lengths of its sides. The *area* of a polygon is the measure of enclosed region.

Triangles

Maxim 1. No triangle can have a longest side that is greater than the sum to the two shortest sides.

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[2]According to the polygonal series and the direct variation defined above, a line, with 0 angles, ends up being $-360°$. Another way is to start with a line as defined as having 0 degree: by this path we have to give a line 2 sides so that $0 \times 180 = 0$.

Example 2. *If triangle $\triangle ABC$ has $A = 4$, $B = 7$, then $C \neq 12$. That is, $4 + 7 = 11$, and since no side of a triangle can be greater than the sum of two other sides, C cannot be greater than 11. In other words, take the longest side of $\triangle ABC$, the sum of the other sides cannot be greater.*

Definition 1. Equilateral Triangle
Has all three sides equal

Example 3. *For equilateral $\triangle ABC$, all angles must equal $60°$. If any angle is more or less than 60, then $\triangle ABC \neq 180$.*

Definition 2. Isosceles Triangle
Has two sides of equal length.

Example 4. *For isosceles $\triangle ABC$, angle ABC and angle $BCA = 50°$, then*

$$50 + 50 + x = 180$$
$$100 + x = 180$$
$$x = 80$$

Definition 3. Right Triangle
Has one interior angle that equals $90°$. For right $\triangle DEF$, say DF is hypotenuse (side oposite of the right angle), then Pythogorean theorem says that $(DF)^2 = (DE)^2 + (EF)^2$

Example 5. *For right $\triangle DEF$, $DE = 5$, $DF = 8$ and is the hypotenuse, and $EF = x$, then*

$$8^2 = 5^2 + x^2$$
$$64 = 25 + x^2$$
$$39 = x^2$$

Chapter 3

Elementary Algebra: Equations and Formulas

3.1 The Quadratic Equation

$$ax^2 + bx + c = 0 \tag{3.1.1}$$

There are two main ways of solving: by factoring or by the quadratic formula.

3.1.1 Factoring quadratics

The following quadratic equation is solved by factoring and setting the factors to zero.

$$2x^2 - x - 6 = 0 \tag{3.1.2}$$

Factor	Set to 0; solve	Set to 0; solve
$(2x+3)(x-2) = 0$	$2x+3 = 0$	$x-2 = 0$
	$2x+3 = 0$	$x = 2$
	$2x = -3$	
	$x = -\dfrac{2}{3}$	

Other factorable quadratics are:

$$x^2 + 8x + 15 = 0 \tag{3.1.3}$$

Factor	Set to 0; solve	Set to 0; solve
$(x+3)(x+5) = 0$	$x+3 = 0$	$x+5 = 0$
	$x = -3$	$x = -5$

$$4x^2 - 9x[\pm 0] = 0 \tag{3.1.4}$$

Notice that this quadratic's third term can be plus or minus zero, and also that it can be factored as the difference of two squares x^2 and 3^2.

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Factor	Set to 0; solve	Set to 0; solve
$(2x+3)(2x-3)=0$	$2x+3=0$ $2x=3$ $x=\dfrac{3}{2}$	$2x-3=0$ $2x=-3$ $x=-\dfrac{3}{2}$

3.1.2   Quadratic Formula

A quadratic expression $ax^2+bx+c=0$ can also be solved by randomly assigning numerical values to the variables in the following quadratic formula; although true random assigment is not possible, as I make clear below.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.1.5}$$

For example, the equation $2x^2 - x - 6 = 0$; where $a = 2$, $b = $ -1, and $c = $ -6 yields the following:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)} = \frac{1 \pm \sqrt{49}}{4} = \frac{1 \pm 7}{4} \tag{3.1.6}$$

The $\pm$ solutions are then $x = \frac{1+7}{4} = 2$ and $x = \frac{1-7}{4} = -\frac{3}{2}$. As usual with fractionals, apparent solutions that produce a zero denominator are undefined and incorrect. For example, $a \neq 0$. If so, the denominator is $2(0) = 0$, giving $\frac{X}{0}$, which is undefined.

Lastly, most quadratics have at most two real solutions, but some only have one. For example, the following equation has one solution, $x = $ -2

$$x^2 + 4x + 4 = 0 \tag{3.1.7}$$

$$-2^2 + 4(-2) + 4 = 0$$

$$4 + -8 + 4 = 0$$

$$-4 + 4 = 0$$

Some quadratic problems have no solution.

$$x^2 + x + 5 = 0 \tag{3.1.8}$$

A simple way to find out how many solutions to a quadratic exist is to find the value of the $b^2 - 4ac$ portion of the formula; this is called the discriminant and is symbolized by Δ.

Table 3.1: Discriminant value for number of solutions to quadratic

If $\Delta > 0$	there are two unique real solutions
If $\Delta = 0$	there is one unique real solution
If $\Delta < 0$	there are two unique, conjugate imaginary solutions
If Δ is perfect square	two solutions are rational, otherwise irrational conjugates

3.2   Two maxims for inequalities

Inequalities work like most equations. The two maxims below give the rules for direction of inequality when a combinatory operation is done on both sides. The maxims can also be defined in a different way by the the corollary that follows them.

Maxim 2. Additive Maxim
When the same constant is added to (or subtracted from) both sides of the inequality, the direction of inequality is preserved. And, the new inequality is equivalent to the old inequality.

Maxim 3. Multiplicative Maxim
When the same constant is multiplied to (or divided from) both sides of the inequality, *the direction of inequality is preserved iff the constant is positive, but reversed if the constant is negative.* In both cases, the new inequality is equivalent to the old inequality.

Corollary to Inequality Maxims. *An inequality can change direction iff both sides of the inequality are multiplied (or divided) by the same negative constant.*

3.3   Sequences

Definition 4. A **sequence** can be defined as a FUNCTION whose domain is the set of real numbers.

Sequences are typically given as lists or series. For example, the function $f(n) = 2n-1$ will output a range over the real numbers. Notice that the function defines a sequence of the real numbers as a kind of subsequence that the function is supposed to use as the domain/input – this is why the definition explicitly mentions the set of real of numbers as the domain. But the presuppostion is stronger than simply assigning by definition the domain for the function; it in fact assumes that where one begins the subsequence is arbitrary. In other words, in the following example, I arbitrarily decided to start the subsequence at 0; also notice that it is possible to have my subsequence running at non-regular

(perhaps even random) intervals such as $1, \frac{1}{2}, \frac{2}{3}, \sqrt{3}, \ldots$.

$$f(n) = 2n - 1$$
$$f(n) = [2(1) - 1 = 1], [2(2) - 1 = 3], [2(3) - 1 = 5], [2(4) - 1 = 7], \ldots$$
$$= 1, 3, 5, 7, 9, 11, \ldots$$
$$f(n) = [2(1) - 1 = 1], [2(\frac{1}{2}) - 1 = 0], [2(\frac{2}{3}) - 1 = .\bar{3}], [2(\sqrt{3}) - 1 = 2.464], \ldots$$
$$= 1, 0, .\bar{3}, 2.464, \ldots$$

A Recursive Definition

Any sequence can be given a recursive definition by giving the first term and a rule that shows how to obtain further terms; i.e., how to get the $(n+1)th$ term from the nth term. Here is a general formulation followed by a specific example.

$$a_1 = x \quad (n\text{th term})$$
$$a_n + 1 = y \quad (\text{recursive rule}) \qquad (3.3.1)$$

$$a_1 = 5 \quad (n\text{th term})$$
$$a_n + 1 = 3a_n - 2 \quad (\text{recursive rule})$$
$$\text{therefore,}$$
$$a_1 = 5$$
$$a_2 = 3(a_1(= 5)) - 2 = 13$$
$$a_3 = 3(a_2(= 13)) - 2 = 37$$
$$a_4 = 3(a_3(= 37)) - 2 = 109$$
$$a_5 = 3(a_4(= 109)) - 2 = 325 \qquad (3.3.2)$$

Here we can see the details in the popular generative linguistic definition of recursiveness that points out that a certain kind of recursion (tail-recursion) provides input for the new function from the output of the previous function. For example, the new function a_2 is given the input $a_1 = 5$ (where 5 is the output of the function a_1) according to the recursive rule.

3.4   Arithmetic and Geometric Sequences

3.4.1   A-Sequence

An **arithmetic sequence** is in the form

$$(a)_1, \quad (a + d)_2, \quad (a + 2d)_3, \quad (a + 3d)_4, \ldots, (a + (n - 1)d)_n, \ldots \qquad (3.4.1)$$

where a = first term of the sequence, $a + (n - 1)d$ = nth term of the sequence, n = number of terms, and d = common difference of the terms. I have subscripted

the seuqential terms in order to point out the fact that the first term a_1 has an addend of 0, the second term is d, or 1, the third term $2d$, or 2, the fourth term is $3d$ and so on. This is why the nth term is $n - 1$. Some examples of adding sequences follow.

Example 6. *Write the first 6 terms and the 10th term of an arithmetic sequence with a first term of 7 and a common difference of 5.*

Solution 1. Since the first term is 7 and the common difference is 5, we can get the first 6 terms:

$7,\ 7 + 5,\ 7 + 2(5),\ 7 + 3(5),\ 7 + 4(5),\ 7 + 5(5)$

or

$7,\ 12,\ 17,\ 22,\ 27,\ 32$

Finding the 10th term requires substituting 10 for n in the formula for arithmetic sequence in 3.4.1:

$$n\text{th term} = a + (n - 1)d$$
$$21\text{st term} = 7 + (10 - 1)5$$
$$= 7 + (9)5 = 52 \tag{3.4.2}$$

Of course, if given the first term a and the common difference d one can just simply add terms until getting to the nth term. The next example shows how this method would become tedious.

Example 7. *Find the 98th term of an arithmetic sequence whose first three terms are 2, 6, 10.*

Solution 2. First step is to find common difference $d : 10 - 6 = 4, 6 - 2 = 4$. $d = 4$. Then we can substitute variables in formula 3.4.1: $a = 2, n = 98$. Now plug in the solution.

$$n\text{th term} = a + (n - 1)d$$
$$98\text{th term} = 2 + (98 - 1)4$$
$$= 2 + (97)4 = 390 \tag{3.4.3}$$

Adding terms by 4 for 97 consequentive terms would be a waste of time, even on a calculator.

Arithmetic means

These are numbers inserted between a first and last term to form a segment of an a-sequence. The method here is to assume a last term l as the nth term of the arithemtic sequence in 3.4.1

$$l = a + (n - 1)d \tag{3.4.4}$$

Example 8. *Insert three means between -3 and 12.*

Solution 3. To insert three terms between the two we already have gives a total of 5 terms such that $n = 5$. We are given the first and last terms: $a = 3, l = 12$. What we do not know is the common difference d:

$$l = a + (n-1)d$$
$$12 = -3 + (5-1)d$$
$$15 = 4d \quad \text{(Add 3 to both sides)}$$
$$\frac{15}{4} = d \quad \text{(Divide both sides by 4)} \tag{3.4.5}$$

Now with $d = \frac{15}{4}$ we plug in variables for finding the three middle terms which is really just part of the formula for arithmetic sequences in 3.4.1 and can be represented as $(a + d)_2$, $(a + 2d)_3$, $(a + 3d)_4$, which is the sequence minus the first and nth (last) term.

$$a + d = -3 + \frac{15}{4} = \frac{3}{4}$$
$$a + 2d = -3 + 2\frac{15}{4} = -3 + \frac{30}{4} = 4\frac{1}{2}$$
$$a + 3d = -3 + 3\frac{15}{4} = -3 + \frac{45}{4} = 8\frac{1}{4} \tag{3.4.6}$$

The whole sequence, then, is -3, $\frac{3}{4}, 4\frac{1}{2}, 8\frac{1}{4}$, 12

Sum of first n a-sequence terms

The first n terms of an arithmetical sequence can also be calculated.

$$S_n = \frac{n(a + l)}{2} \tag{3.4.7}$$

where, as above, a = first term of the sequence, $a + (n-1)d = n$th term (or $l = [a + (n-1)d] =$ last term) of the sequence, and $n =$ number of terms.

Example 9. *Find the sum of the first 30 terms of the arithmetic sequence 5, 8, 11, ...*

Solution 4. Plug-in the variables: $a = 5$, $n = 30$, $d = 3$, $l = [5 + (30-1)3] = 92$ and substitute into the formula from 3.4.7 to get

$$S_n = \frac{n(a + l)}{2}$$
$$S_{30} = \frac{30(5 + 92)}{2}$$
$$= 15(97) = 1,455 \tag{3.4.8}$$

3.4.2   G-Sequence

A **geometric sequence** is in the form

$$(a)_1, \quad (ar)_2, \quad (ar^2)_3, \quad (ar^3)_4, \ldots, (ar^{n-1})_n, \ldots \tag{3.4.9}$$

where a = first term of the sequence, $ar^{n-1} = nth$ term of sequence, and $r =$ common ratio. Just as arithmetical sequences, I have numbered the terms by subscript to highlight the fact that the first term a_1 has no power, the second term ar_2 has power of 1, the third has power of 2, and so on. This is why the nth term power is $n - 1$.

Example 10. *Write the first 6 terms and the 25th term of the geometric sequence whose first term = 3 and common ratio = 2.*

Solution 5. Plugin the variables:
$$3, \; 3(2), \; 3(2)^2, \; 3(2)^3, \; 3(2)^4, \; 3(2)^5$$
or
$$3, \, 6, \, 12, \, 24, \, 48, \, 96$$
For the 25th term simple computation is a waste of time. Instead, substitute 25 for n; as well as the other variables in the formula for the nth term that is at the end of the formula in 3.4.9.

$$n\text{th term} = ar^{n-1}$$
$$25\text{th term} = 3(2)^{25-1}$$
$$3(2)^{24} = 50,331,648 \tag{3.4.10}$$

Geometric means

Geometric means are similar to arithmetical means.

Example 11. *Insert two geometric means between 4 and 256.*

Solution 6. Inserting two means within two terms give us a total of 4; $n = 4$. Plug-in the rest of the variables: first term $a = 4$; last term is assumed to be the nth term. What we need is r:

$$ar^{n-1} = l$$
$$4r^3 = 256$$
$$r^3 = 64 \text{ (Divide both sides by 4)}$$
$$r = 4 \text{ (Find cube root on both sides)} \tag{3.4.11}$$

Now that we have r, just plug-in variables for the second and third terms in the geometric sequence in 3.4.9.

$$(ar)_2 = 4 * 4 = 16$$
$$(ar^2)_3 = 4 * 4^2 = 4 * 16 = 64 \tag{3.4.12}$$

The final sequence is 4, 16, 64, 256.

Sum of first n g-sequence terms

The sum of the first n terms of a geometric sequence can be found by the following formula:

$$S_n = \frac{a - ar^n}{1 - r} \quad r \neq 1 \tag{3.4.13}$$

The usual variables for sequences applies: where $a =$ first term of the sequence, $a + (n-1)d = n$th term (or $l = [a + (n-1)d] =$ last term) of the sequence, and $n =$ number of terms.

3.4.3 Infinite Geometric Sequences

Under certain conditions, we can find the sum of all the terms in an **infinite g-sequence**. To define this sum, consider the following g-sequence $a,\ ar,\ ar^2, \ldots$

$$
\begin{aligned}
a &= S_1 &&\text{The first partial sum of the sequence}\\
a + ar &= S_2 &&\text{The second partial sum of the sequence}\\
a + ar + ar^2 + \cdots + ar^{n-1} &= S_n &&\text{The nth partial sum of the sequence}
\end{aligned}
$$

Definition 5 (SUM OF INFINITE GEOMETRIC SEQUENCE). If S_n of an infinite g-sequence approaches some number S as n approaches ∞, then S is called the **sum of the infinite geometric sequence**.

$$S = \sum_{n=1}^{\infty} ar^{n-1} \tag{3.4.14}$$

To develop a formula for finding the sum of all the terms in an infinite g-sequence, consider the formula used for finding the sum of the first n terms in a g-sequence in 3.4.13

$$\boxed{S_n = \frac{a - ar^n}{1 - r} \quad r \neq 1}$$

Remark 1. If $|r| < l$ and a is a constant, then as n approaches ∞, ar^n approaches 0, and the term ar^n above can be dropped:

$$S = \frac{a}{1 - r} \quad |r| < 1 \tag{3.4.15}$$

where, as just noted, $|r| < l$; also, as previously stated, $a =$ first term and $r =$ common ratio.

Remark 2. If $|r| \geq l$, the terms get larger and the sum S does not approach a number. The result: theorem 3.4.15 does not apply.

Some examples will be helpful here.

Example 12. *Change $0.\bar{3}$ to a common fraction*

Solution 7. Write the decimal as an infinite geometric series,

$$S = \frac{3}{10} + \frac{3}{100} + \frac{3}{1,000} + \frac{3}{10,000} + \dots$$

$$S = \frac{3}{10} + \frac{3}{10}\left(\frac{1}{10}\right) + \frac{3}{10}\left(\frac{1}{10}\right)^2 + \frac{3}{10}\left(\frac{1}{10}\right)^3 + \dots$$

Since the common ratio $r = \frac{1}{10} : |\frac{1}{10}| < 1$, formula 3.4.15 is relevant for finding the sum of the infinite geometric series:

$$S = \frac{a}{1-r} = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{9} = \frac{1}{3} \tag{3.4.16}$$

Example 13. *A town with a population of 3500 has a predicted growth rate of 6% per year for the next 20 years. How many people are expected to live in the town 20 years from now?*

Solution 8. Let p_0 be the initial population. After 1 year the population p_1 will be the initial population (p_0) plus the growth (the product of p_0 and the rate of growth r).

$$p_1 = p_0 + p_0 r$$
$$= p_0(1+r) \quad \text{Factor out } p_0 \tag{3.4.17}$$

The population p_2 at the end of 2 years will be

$$p_2 = p_1 + p_1 r$$
$$p_2 = p_1(1+r) \quad \text{(Factor out } p_1\text{)}$$
$$p_2 = p_0(1+r)(1+r) \quad \text{(Sub for } p_1\text{)}$$
$$p_2 = p_0(1+r)^2 \tag{3.4.18}$$

The population at the end of year 3 will be $p_3 = p_0(1+r)^3$. Writing the terms in a sequence gives

$$\boxed{p_0, \ p_o(1+r), \ p_0(1+r)^2, \ p_0(1+r)^3, p_0(1+r)^4, \dots} \tag{3.4.19}$$

This geometric sequence with $a = [p_0] = \text{first term}$, and $r = [1+r] = \text{common ratio}$. In other words, $p_0 = a = 3500$, $1 + \text{r} = r = 1.06$, and $n = 21$ (for the total number of terms to reach 20). To find the last term we assume the nth term (ar^{n-1}) is l such that $l = ar^{n-1}$.

$$l = ar^{n-1}$$
$$l = 3500(1.06)^{21-1}$$
$$l = 3500(1.06)r^{20}$$
$$l = ar^{n-1}$$
$$l \approx 11,224.97415 \tag{3.4.20}$$

The population after 20 years of annual growth of 6% is approximately 11,225.

3.5   Binomials and the Theorem

Expanding a binomial is integral to working with probability and statistics. The following section shows how to expand a binomial, $(x+y)^n$, and fleshes out some interesting properties.

3.5.1   Expanding binomials

If we take a binomial and expand its power, $(x + y)^{n \to n+1 \cdots}$, some interesting patterns develop. The expansion is as follows:

$$
\begin{aligned}
(a + b)^0 &= 1 \;\; \text{(riase to zero is 1)} \\
(a + b)^1 &= a + b \;\; \text{(riase to 1 is identity)} \\
(a + b)^2 &= a^2 + 2ab + b_3^2 \\
&= (a + b)_1 (a + b)_2 \\
(a + b)^3 &= a^3 + 3a^2 b + 3ab^2 + b_4^3 \\
&= (a + b)_1 (a + b)_2 (a + b)_3 \\
(a + b)^4 &= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b_5^4 \\
&= (a + b)_1 (a + b)_2 (a + b)_3 (a + b)_4 \\
(a + b)^5 &= a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b_6^5 \\
&= (a + b)_1 (a + b)_2 (a + b)_3 (a + b)_4 (a + b)_5 \\
(a + b)^6 &= a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b_7^6 \\
&= (a + b)_1 (a + b)_2 (a + b)_3 (a + b)_4 (a + b)_5 (a + b)_6
\end{aligned}
$$

There are at least four patterns that can be found in this sequence:

1. After the power of 2: Each expansion has one more term than the power of the binomial (e.g., the b_{n+1}^n term). As an array, each expansion has one more row and column than the number of terms.

2. The degree of each term in each expansion equals the exponent of the binomial.

3. The first term in each expansion is a raised to the power of the binomial.

4. The exponents of a decrease by 1 in each successive term, and the exponents on b, beginning with b^0 in the first term, increase by 1 in each successive term.

Another pattern can be found by making an array out the coefficent's successive expansion. This pattern produces the following:

Column / Row	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	2	1				
4	1	3	3	1			
5	1	4	6	4	1		
6	1	5	10	10	5	1	
7	1	6	15	20	15	6	1

A more visually friendly image of what is known as Pascal's triangle is shown in figure 3.1. This pretty image shows (and implies) some of the stunning patterns that arise from expanding binomials. It is easy, perhaps, to see why such systematic patterned expansion of numerical values would be useful for modeling systems and patterns in nature and society.

Factorials If n is a natural number, then $n!$ is defined as

$$n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)$$ (3.5.1)

Two properties of factorials include

Property 1. Identity By definition, $0! = 1$.

Property 2. Equivalence If n is a natural number, then $n(n-1)! = n!$.

A couple examples should suffice:

$$3! = 3*2*1 = 6$$
$$6! = 6*5*4*3*2*1 = 720$$ (3.5.2)
$$3(3-1)! = 3*2 = 6 = 3!$$

3.5.2 Binomial Theorem

If n is any positive number, then

$$(a+b)^n =$$

$$a^n + \frac{n!}{1!(n-1)!}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 +$$
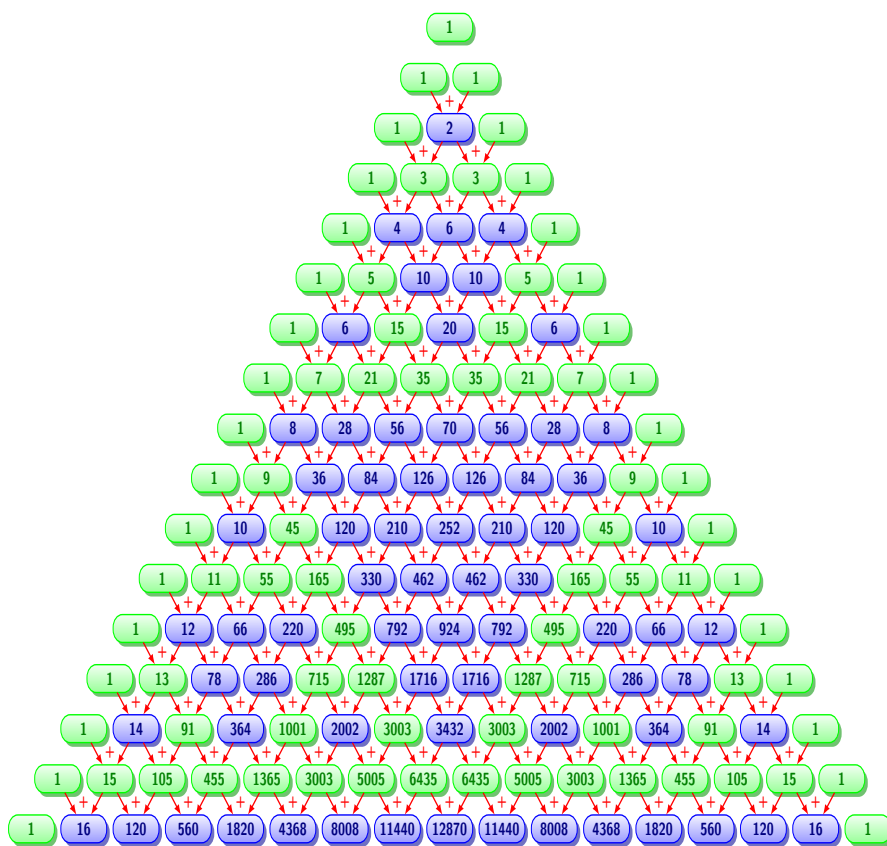$$\frac{n!}{3!(n-3)!)}a^{n-3}b^3 + \cdots + b^n$$ (3.5.3)

Figure 3.1: Pascal and Sierpinski Triangle

There is a more compact way to write the theorem using summation notation. I give the compact version here but lead up to it in the next section; see equation 3.6.2.

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r!)} a^{n-r} b^r$$

3.6 Summation

A shorthand way to indicate the sum of the first n terms, or the nth partial term, of a sequence. Notice that because summation shows which nth terms are to be used the arbitrariness of subsequences noted above no longer applies. That is, in $\sum_{n}^{m}$ the n value tells us where to start the numerical sequence and the m term tells us how far to go in the sequence.

$$\sum_{n=1}^{3} (2n^2 + 1)$$

$$= [2(1)^2 + 1] + [2(2)^2 + 1] + [2(3)^2 + 1]$$
$$= 3 + 9 + 19$$
$$= 31 \tag{3.6.1}$$

In this example there is no arbitrariness about where to begin the subsequence and how far to go in the sequence; nor about the systematicity of the interval partitions – at least not in this example (i.e., by numerical values of 1). The bottom domain tells us where to start (e.g., $n = 1$: start at 1), and the upper domain tells us how far to go (e.g., numerical value 3). One more example should suffice; notice that the upper domain is 5 and the lower one is 3. This does not mean start at 3 and continue 5 sequence steps, such as a_5. Instead, it means start at 3 and work our way up to 5, which is three sequential steps a_3; or more acurately $a_3 \rightarrow a_4 \rightarrow a_5$.

$$\sum_{n=3}^{5} (3n + 2) \tag{3.6.2}$$

$$= [3(3) + 2] + [3(4) + 2] + [3(5) + 2]$$
$$= 11 + 14 + 7$$
$$= 42$$

3.6.1 Three Basic Properties of Summation

Property 3. Summation of a constant. The summation of constant (c) as the value k runs from 1 to n is n times the constant.

$$\boxed{\text{If } c \text{ is constant, then} \sum_{k=1}^{n} c = nc.} \tag{3.6.3}$$

For example,

$$\sum_{k=1}^{5} 13 = 13_1 + 13_2 + 13_3 + 13_4 + 13_5 = 5(13) = 65 \qquad (3.6.4)$$

Proof. Because c is a constant, each term c is constant for each value k as k runs from 1 to n.

$$\sum_{k=1}^{n} c = \overbrace{c + c + c + c + c + \cdots + c}^{n \text{ number of } c_k} = nc$$

$\square$

Property 4. Summation of a product. A constant factor can be brought outside a summation sign.

$$\text{If } c \text{ is constant, then } \sum_{k=1}^{n} cf(k) = c \sum_{k=1}^{n} f(k). \qquad (3.6.5)$$

For example,

$$\text{Show that } \sum_{k=1}^{3} 5k^2 = 5 \sum_{k=1}^{3} k^2$$

$$\sum_{k=1}^{3} 5k^2 = 5(1)^2 + 5(2)^2 + 5(3)^2 \qquad\qquad 5\sum_{k=1}^{3} k^2 = 5[(1)^2 + (2)^2 + (3)^2]$$
$$= 5 + 20 + 45 \qquad\qquad\qquad\qquad = 5[1 + 4 + 9]$$
$$= \mathbf{70} \qquad\qquad\qquad\qquad\qquad = 5(14) = \mathbf{70} \qquad (3.6.6)$$

Proof.

$$\sum_{k=1}^{n} cf(k) = cf(1) + cf(2) + cf(3) + \cdots + cf(n)$$
$$= c[f(1) + f(2) + f(3) + \cdots + f(n)] \quad \text{(Factor out } c\text{)}$$
$$= c \sum_{k=1}^{n} f(k)$$

$\square$

Property 5. Summation of a sum. The summation of a sum is equal to the sum of the summations.

$$\boxed{\sum_{k=1}^{n} [f(k) + g(k)] = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k).} \qquad (3.6.7)$$

For example,

$$\boxed{\text{Show that } \sum_{k=1}^{3} (k + k^2) = \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2}$$

$$\sum_{k=1}^{3} (k + k^2) = (1 + 1^2) + (2 + 2^2) + (3 + 2^2)$$

$$= 2 + 6 + 12 = \mathbf{20}$$

$$\sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2 = (1 + 2 + 3) + (1^2 + 2^2 + 3^2)$$

$$= 6 + 14 = \mathbf{20} \qquad (3.6.8)$$

Proof.

$$\sum_{k=1}^{n} [f(k) + g(k)]$$

$$= [f(1) + g(1)] + [f(2) + g(2)] + \cdots + [f(n) + g(n)]$$
$$= [f(1) + f(2) + \cdots + f(n)] + [g(1) + g(2) + \cdots + g(n)]$$
$$= \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$$

$$\square$$

3.6.2   Binomial theorem as summation

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r!)} a^{n-r} b^r \qquad (3.6.9)$$

3.7   Induction

A general definition for mathematical induction is based on the successor function.

Chapter 4

Probability

The basic expression for showing a relationship for the probability P between two (or more) events is

$$P(Event) = \frac{\text{The number of outcomes involving E}}{\text{The total number of possible outcomes}} \qquad (4.0.1)$$

Law 1. ADDITION LAW FOR PROBABILITIES

$$P(E \ or \ F) = P(E) + P(F) - P(E * F)$$

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