Jboy's Solution to [some] Exercises in The Art of Problem Solving Volume 1

Jboy Flaga

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Chapter 1: Exponents and Logarithms

EXERCISE 1-1 Evaluate each of the following.

(August 5, 2016 about 12:10 AM)

i. $3^4 = 3 \times 27 = 81$

ii. $2^5 2^2 = 2^7 = 128$

iii. $5^{-3}5^5 5^{-1}5^1 = 5$

iv. $4^3/4 = 4^2 = 16$

v. $2^7/2^2 = 2^5 = 32$

vi. $(3^4 3^{-2})/(3^5 3^{-2}) = 3^2/3^3 = \dfrac{1}{3}$

vii. $2^5 3^2 2^{-3} = 2^2 3^2 = 4 \times 9 = 36$

viii. $5^2 3^{-1} 2^4 5^{-1} 2^{-2} = (5 \times 2^2)/3 = \dfrac{20}{3} = 6\dfrac{2}{3}$

EXERCISE 1-2 Try these.

i. $9^{3/2} = (9^{1/2})^3 = 3^3 = 9$

ii. $\left(\sqrt[3]{81}\right)^{3/2} = (81^{1/3})^{3/2} = 81^{(1/3)(3/2)} = 81^{1/2} = 9$

iii. $64^{-4/3} = \left(64^{1/3}\right)^{-4} = 4^{-4} = \dfrac{1}{4^4} = \dfrac{1}{64 \times 4}$

iv.

v.

vi.

EXERCISE 1-3 Find all real x in each of the following.

i. $x = (-2)^5 \qquad x = -32$

ii. $x = \sqrt[3]{-1/8} \qquad x = \Big(-1/8\Big)^{1/3}$

$$x = \Big(\frac{-1^{1/3}}{8^{1/3}}\Big)$$

$$x = \Big(\frac{-1^{1/3}}{8^{1/3}}\Big)$$

$$x = -\frac{1}{2}$$

iii. $x^6 = 64 \qquad x^6 = 2^6$

$x = 2$

iv. $x^3 = 64 \qquad x^3 = 4^3$

$x = 4$

v.

vi. $x^{5/3} = 243 \qquad \Big(x^{1/3}\Big)^5 = \Big(27^{1/3}\Big)^5$

$x = 27$

EXAMPLE 1-11

(August 5, 2016 11:37 PM)

Here, I will try to answer #3 of Example 1-11 and see if my answer is the same with that in the book

#3. Simplify $\sqrt[6]{6912}$

I will first create a table of some prime numbers with exponents:

$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$

$3^2 = 9$
$3^3 = 27$
$3^4 = 81$

Now I will do some trial and error to find the prime factorization of 6912

$81 \times 128 = 10,368$
$81 \times 64 = 5,184$
$27 \times 128 = 3,456$

Let's try to use 2^8

$2^8 = 256$

$256 \times 27 = 6,912$
$2^8 \times 3^3 = 6,912$

I found it!

$\sqrt[6]{6912} = \sqrt[6]{2^8 3^3} = 2^{8/6} 3^{3/6} = 2^{4/3} 3^{1/2} = \left(2^{1/3}\right)^4 3^{1/2}$

I'm stuck! I'm going to look for a hint.

Ahh! He did $2 \times 2^{1/3}$.

I think he went throught this step:

$... = 2^{4/3} 3^{1/2} = (\sqrt[3]{2^4}) 3^{1/2} = (\sqrt[3]{2^3 2^1}) 3^{1/2} = 2(\sqrt[3]{2}) 3^{1/2} = 2(\sqrt[3]{2})(\sqrt{3})$

The answer from the book is $2\sqrt[6]{108}$

But I used http://web2.0calc.com/ to verify if my answer is also correct.

It is correct!

$\sqrt[6]{6912} \quad = 4.3644945438868856$
$2(\sqrt[3]{2})(\sqrt{3}) = 4.3644945438868856$
$2\sqrt[6]{108} \quad = 4.3644945438868856$

Yeeey!

EXERCISE 1-4 Find the following

I will not be doing Exercise 1-4 anymore. I can use a calculator to do this. Hahahaha.

i. $\sqrt{27} =$

...

vi. $\sqrt{\frac{56}{126}} =$

EXAMPLE 1-12 Rationalize the denominator of $\frac{2\sqrt{5}}{3\sqrt[4]{72}}$

(Aug 9, 2016)

I will try to answer EXAMPLE 1-12 and see if my answer is the same with that in the book

$$\frac{2\sqrt{5}}{3\sqrt[4]{72}} = \frac{2\sqrt{5}}{3\sqrt[4]{9}\sqrt[4]{8}} = \frac{2\sqrt{5}}{3\sqrt[4]{3^2}\sqrt[4]{2^3}}$$

Multiply both numerator and denominator by $\sqrt[4]{3^2}$

$$= \Big(\frac{2\sqrt{5}}{3\sqrt[4]{3^2}\sqrt[4]{2^3}}\Big)\Big(\frac{\sqrt[4]{3^2}}{\sqrt[4]{3^2}}\Big) = \frac{2\sqrt{5}\sqrt[4]{3^2}}{3\sqrt[4]{3^4}\sqrt[4]{2^3}} = \frac{2\sqrt{5}\sqrt[4]{3^2}}{9\sqrt[4]{2^3}}$$

Multiply both numerator and denominator by $\sqrt[4]{2}$

$$= \Big(\frac{2\sqrt{5}\sqrt[4]{3^2}}{9\sqrt[4]{2^3}}\Big)\Big(\frac{\sqrt[4]{2}}{\sqrt[4]{2}}\Big) = \frac{2\sqrt{5}\sqrt[4]{3^2}\sqrt[4]{2}}{9\sqrt[4]{2^4}} = \frac{2\sqrt{5}\sqrt[4]{18}}{18}$$

The book's answer is more simplified. But it's the same with mine... I'm ok with my answer for now. :)

EXERCISE 1-5 Express the following as fractions with rational denominators.

EXERCISE 1-6 Rationalize the denominators of each of the following expressions.

// TODO_JBOY: EXERCISE 1-5 & 1-6

I will answer these Exercise 1-5 & 1-6 later because I have to go to logarithms. I need it because I want to be able to understand the first chapter of "Building Blocks of Theoretical Computer Science"

EXERCISE 1-7 Convert the following exponential equations to logarithmic equations.

(Aug 9, 2016)

i. $3^3 = 27 \quad \rightarrow \quad \log_3 27 = 3$

ii. $16^{1/4} = 2 \quad \rightarrow \quad \log_{16} 2 = 1/4$

iii. $x^z = y \quad \rightarrow \quad \log_x y = z$

EXERCISE 1-8 Convert the following logarithmic equations to exponential equations.

i. $\log_{36} 6 = 1/2 \quad \rightarrow \quad 36^{1/2} = 6$

ii. $\log_3 1/9 = -2 \quad \rightarrow \quad 3^{-2} = 1/9$

iii. $\log_x y = z \quad \rightarrow \quad x^z = y$

EXAMPLE 1-15 Evaluate $\log_8 2$.

$$\begin{aligned} x &= \log_8 2 \\ 8^x &= 2 \\ x &= 3 \end{aligned}$$

Oh! The book has a better method.

$$\begin{aligned} x &= \log_8 2 \\ 2^{3x} &= 2^1 \end{aligned}$$

Two expressions with the same base are equal only if their exponents are equal

$$\begin{aligned} 3x &= 1 \\ x &= 1/3 \end{aligned}$$

Oh! hahaha.. I was thinking cube root.. Why did I write $x = 3$ above? haha

EXERCISE 1-9 Find each of the following.

i. $\log_5 625$

$$\begin{aligned} x &= \log_5 625 \\ 5^x &= 625 \\ 5^x &= 5^4 \\ x &= 4 \end{aligned}$$

ii. $\log_{1/2} 2$

$$\begin{aligned} x &= \log_{1/2} 2 \\ \left(\frac{1}{2}\right)^x &= 2 \\ \frac{1}{2^x} &= 2 \\ 1 &= 2(2^x) \\ 1 &= 2^{x+1} \end{aligned}$$

Anything raised to zero is equal to one.

$$\begin{aligned} 1 = 2^0 &= 2^{x+1} \\ 0 &= x + 1 \\ x &= -1 \end{aligned}$$

iii. $\log_9 \sqrt{3}$

$$\begin{aligned} x &= \log_9 \sqrt{3} \\ 9^x &= \sqrt{3} \\ \left(3^2\right)^x &= 3^{1/2} \\ 3^{2x} &= 3^{1/2} \\ 2x &= \frac{1}{2} \\ x &= \frac{1}{4} \end{aligned}$$

Observation: $9^{\frac{1}{4}} = \sqrt{\sqrt{9}} = \sqrt{3}$

iv. TODO_JBOY:

Problems to Solve for Chapter 1

1. Find the value of $\log_5 \frac{(125)(625)}{25}$. (AHSME 1950)

$$\begin{aligned} x &= \log_5 \frac{(125)(625)}{25} \\ 5^x &= \frac{(125)(625)}{25} \\ 5^x &= \frac{(5)(25)(625)}{25} \\ 5^x &= (5)(625) \\ 5^x &= (5)(5^4) \\ 5^x &= 5^{4+1} \\ 5^x &= 5^5 \\ x &= 5 \end{aligned}$$

2. TODO_JBOY

3.