Problems with worked out solutions for Regression and Correlation.

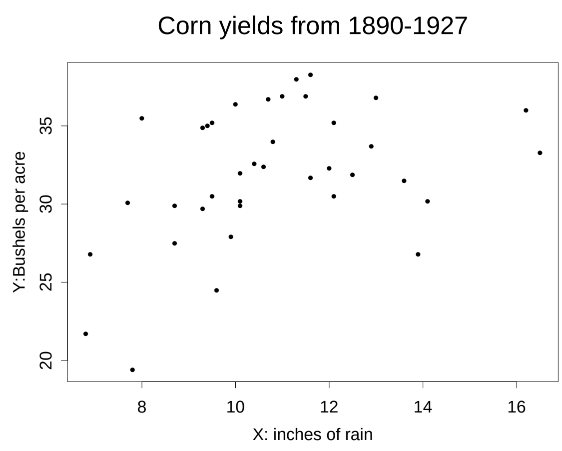
An example: The association between corn yield and rainfall:

Yields y (in bushels/acre) on corn raised in six midwestern states from 1890 to 1927 recorded with rainfall x (inches/yr).

$$y_1, \ldots, y_{38} \quad \text{and} \quad x_1, \ldots, x_{38}.$$

Year	1890	1891	1892	1893	1894	1895	1896	1897	1898	1899
Yield	24.5	33.7	27.9	27.5	21.7	31.9	36.8	29.9	30.2	32
Rainfall	9.6	12.9	9.9	8.7	6.8	12.5	13	10.1	10.1	10.1
Year	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909
Yield	34	19.4	36	30.2	32.4	36.4	36.9	31.5	30.5	32.3
Rainfall	10.8	7.8	16.2	14.1	10.6	10	11.5	13.6	12.1	12
Year	1910	1911	1912	1913	1914	1915	1916	1917	1918	1919
Yield	34.9	30.1	36.9	26.8	30.5	33.3	29.7	35	29.9	35.2
Rainfall	9.3	7.7	11	6.9	9.5	16.5	9.3	9.4	8.7	9.5
Year	1920	1921	1922	1923	1924	1925	1926	1927		
Yield	38.3	35.2	35.5	36.7	26.8	38	31.7	32.6		
Rainfall	11.6	12.1	8	10.7	13.9	11.3	11.6	10.4		

A *scatter plot* provides a visual for inspecting the association between Y and X.



From the scatter plot, the form of the association appears to be linear or slightly quadratic, the strength is weak to moderate, and the direction is positive.

A correlation analysis was done and a SLR model was fit using SAS yielding the following (partial) output:

```
proc corr data=corn cov;
var yield rain;
run;
```

```
proc reg data=corn;
model yield=rain;
run;
```

2 Variables:	yield rain

Covariance Matrix, DF = 37		
	yield	rain
yield	19.04190612	3.98025605
rain	3.98025605	5.13217639

Simple Statistics						
Variable	N	Mean	Std Dev	Sum	Minimum	Maximum
yield	38	31.91579	4.36370	1213	19.40000	38.30000
rain	38	10.78421	2.26543	409.80000	6.80000	16.50000

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	114.21474	114.21474	6.97	0.0122
Error	36	590.33578	16.39822		
Corrected Total	37	704.55053			

Use the output to

1. Find the value of the correlation coefficient.
2. Find the 95% confidence interval for the population correlation, ρ. Interpret the interval you found.
3. Without conducting a hypothesis test, use the confidence interval to make a conclusion about the hypotheses $H_0 : \rho = 0$ vs $H_A : \rho \neq 0$.
4. Find the fitted line for the SLR with yield as the response and rainfall as the predictor.
5. Use the summary statistics to create 95% confidence intervals for β_1 and β_0. Note: $t_{36,0.025} = 2.028$.
6. Find a 95% confidence interval for the true mean yield for corn from the six states for a rainfall of 14inches.
7. Suppose the rainfall collected for a year was 14 inches but the yield of the corn from the six states was not recorded. Find a 95% prediction interval for the yield of the corn from that year.

Solutions:

1. From the output we have

$$\bar{x} = 10.78, \quad s_X^2 = 5.13 \quad s_X = 2.27$$
$$\bar{y} = 31.92, \quad s_Y^2 = 19.04 \quad s_Y = 4.36$$
$$s_{XY} = 3.98$$

   Applying the formula for r, we get

$$r = \frac{s_{XY}}{s_X s_Y} = \frac{3.98}{\sqrt{5.13 \times 19.04}} = 0.40$$

2. With $r = 0.40$, $n = 38$, and $z_{\alpha/2} = 1.96$, a 95% interval is given by

$$\left(\frac{\frac{1+0.4}{1-0.4} e^{-2*1.96/\sqrt{38-3}} - 1}{\frac{1+0.4}{1-0.4} e^{-2*1.96/\sqrt{38-3}} + 1}, \frac{\frac{1+0.4}{1-0.4} e^{2*1.96/\sqrt{38-3}} - 1}{\frac{1+0.4}{1-0.4} e^{2*1.96/\sqrt{38-3}} + 1} \right) = (0.09, 0.64).$$

   We are 95% confident that the true correlation between corn yield and rainfall is between 0.09 and 0.64.

3. Since there is a one-to-one correspondence between a two-sided HT at the 0.05 level and a $100(1-\alpha)\%$ CI, we would reject $H_0 : \rho = 0$ as 0 is not in the interval.

4. To find the fitted least squares line:

$$\begin{aligned}
\hat{\beta}_1 &= \frac{s_{xy}}{s_x^2} \\
&= \frac{3.98}{5.13} = 0.776(\text{ bushels per acre } \div \text{ inches per year}) \\
or &= r_{xy}\frac{s_y}{s_x} \\
&= (0.40)\sqrt{\frac{19.04}{5.13}} \\
&= 0.771(\text{off due to rounding}) \\
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1\bar{x} \\
&= 31.92 - 0.776(10.78) \\
&= 23.555 \text{ bushels per acre}
\end{aligned}$$

   yielding the least squares line of

$$\hat{y} = 23.555 + (0.776)x.$$

5. To find confidence intervals for the regression parameters:
   For β_1, note that

$$S_{xx} = (n-1)s_x^2 = 5.13(38-1) = 189.81$$

   and we can estimate σ^2 using the $MS(E) = 16.40$. Thus, a 95% CI for β_1 is given by

$$0.776 \pm 2.028\sqrt{\frac{16.40}{189.81}} = (0.180, 1.372)$$

   We are 95% confident the true value of the slope lies in this interval. For $\hat{\beta}_0$,

$$23.555 \pm 2.028\sqrt{16.40\left(\frac{1}{38} + \frac{(10.78)^2}{189.81}\right)} = (16.992, 30.118)$$

6. First we can find the point estimate

$$\hat{\mu}(14) = 23.555 + 0.776 * 14 = 34.419$$

   The standard error of this mean estimate is

$$\sqrt{16.40\left(\frac{1}{38} + \frac{(14-10.78)^2}{189.81}\right)} = 1.125$$

   Thus, a 95% CI for the mean corn yield for 14 inches of rain is

$$34.419 \pm 2.028 * 1.125 = (32.138, 36.701)$$

   We are 95% confident that the true mean corn yield for all years with 14 inches of rain is between 32.138 and 36.701 bushels per acre.

7. A 95% prediction interval is then

$$34.419 \pm 2.028\sqrt{16.40\left(1 + \frac{1}{38} + \frac{(14-10.78)^2}{189.81}\right)} = (25.880, 42.958)$$

   We are 95% confident that a year that has 14 inches of rain will have a yield between 25.880 and 42.958 bushels per acre.

Looking at the scatter plot, there may be a quadratic relationship. We can fit a linear regression model with rainfall and rainfall squared as predictors to investigate this. Use the following SAS output to conduct a LOF test for the SLR model.

```
data corn;                proc reg data=corn;
set corn;                 model yield= rain rain2/ss1;
rain2=rain*rain;          run;
run;
```

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	209.02175	104.51087	7.38	0.0021
Error	35	495.52878	14.15797		
Corrected Total	37	704.55053			

Root MSE	3.76271	R-Square	0.2967
Dependent Mean	31.91579	Adj R-Sq	0.2565
Coeff Var	11.78948		

Parameter Estimates						
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Type I SS
Intercept	1	-5.01467	11.44158	-0.44	0.6639	38707
rain	1	6.00428	2.03895	2.94	0.0057	114.21474
rain2	1	-0.22936	0.08864	-2.59	0.0140	94.80700

The F statistic for the LOF test can be written as

$$F = \frac{\frac{SS(R)_f - SS(R)_r}{p-q}}{MS(E)_f} = \frac{\frac{209.02-114.21}{2-1}}{14.16} = 6.70$$

We compare this to the critical value from the appropriate F distribution: $F_{1,35,0.05} = 4.12$

Therefore, we reject $H_0 : \beta_2 = 0$ in favor of $H_A : \beta_2 \neq 0$.

Note: This test statistic of 6.70 is equivalent to the t-test squared $(-2.59)^2$ since we only have 1 numerator degree of freedom.

A couple random questions:

1. What type of data is needed to run a MLR model?

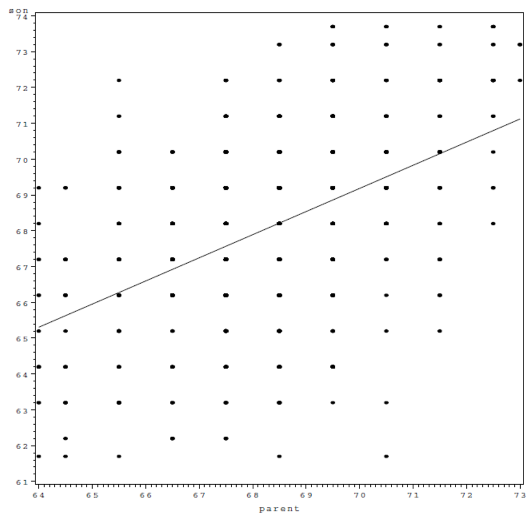
2. An industrial quality control expert takes 200 hourly measurements on an industrial furnace which is under control and finds that a 95% confidence interval for the mean temperature is (500.35, 531.36). As a result he tells management that the process should be declared out of control whenever hourly measurements fall outside this interval and, of course, is later fired for incompetence. (Why and what should he have done?)

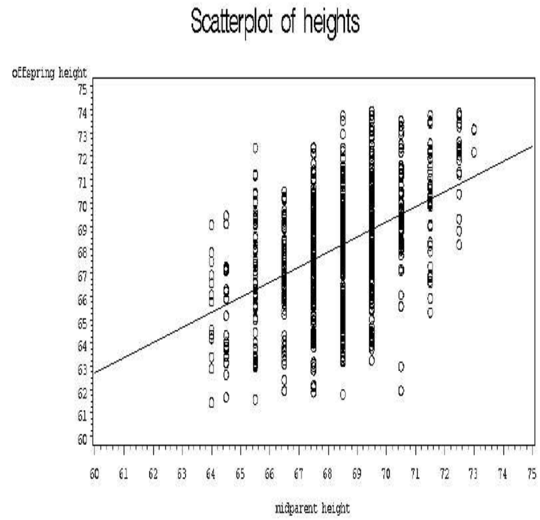
Solutions:

1. The data situation needed for a MLR model is p quantitative predictors and 1 quantitative response measured on the same individuals (i.e. units).

2. The interval found is for the **mean** temperature. The interval is not trying to capture a single new observation. The interval that should have been found is a prediction interval. This interval will be much wider as it is much more difficult to predict a new value as opposed to predicting the true mean.

The classical regression example - The association between height of adults and their parents






```
/* ----------------------------------------------------------------
|  Stigler , History of Statistics pg. 285 gives Galton's famous data |
| on heights of sons (columns,Y) and average parents' height (rows,X) |
| scaled to represent a male height (essentially sons' heights versus |
| fathers' heights).   Taken from Dickey's website.                   |
\ -----------------------------------------------------------------*/
```

	61.7	62.2	63.2	64.2	65.2	66.2	67.2	68.2	69.2	70.2	71.2	72.2	73.2	73.7
73.0	0	0	0	0	0	0	0	0	0	0	0	1	3	0
72.5	0	0	0	0	0	0	0	1	2	1	2	7	2	4
71.5	0	0	0	0	1	3	4	3	5	10	4	9	2	2
70.5	1	0	1	0	1	1	3	12	18	14	7	4	3	3
69.5	0	0	1	16	4	17	27	20	33	25	20	11	4	5
68.5	1	0	7	11	16	25	31	34	48	21	18	4	3	0
67.5	0	3	5	14	15	36	38	28	38	19	11	4	0	0
66.5	0	3	3	5	2	17	17	14	13	4	0	0	0	0
65.5	1	0	9	5	7	11	11	7	7	5	2	1	0	0
64.5	1	1	4	4	1	5	5	0	2	0	0	0	0	0
64.0	1	0	2	4	1	2	2	1	1	0	0	0	0	0

Many questions we could answer using this dataset: Note: $t_{927,0.025} \approx t_{926,0.025} = 1.963$

1. Suppose we ignore midparent height x. Consider estimating the mean $\mu_Y = E(Y)$. Use ST 511 knowledge to obtain a confidence interval for the mean height of the sons. (Use summary statistics in the output that follows to complete this naive analysis.)

For the rest of the problems, consider a linear regression between the sons' heights and the midparent height. Let $Y_1, \ldots, Y_n$ denote the sons' heights. Given their average parent height, $X = x_i$,

$$Y_i = \beta_0 + \beta_1 x_i + E_i \quad \text{for } i = 1, \ldots, n (n = 928).$$

where $E_1, \ldots, E_n$ are *iid* Normal with mean 0 and variance σ^2.

Output is given on the following pages, use it to answer the following:

2. What is the meaning, in words, of β_1?
3. True/false: (a) β_1 is a statistic (b) β_1 is a parameter (c) β_1 is unknown.
4. What is the observed value of $\hat{\beta}_1$?
5. True/false: (a) $\hat{\beta}_1$ is a statistic (b) $\hat{\beta}_1$ is a parameter (c) $\hat{\beta}_1$ is unknown.
6. Is $\hat{\beta}_1 = \beta_1$?
7. How much does $\hat{\beta}_1$ vary about β_1 from sample to sample? (Provide an estimate of the standard error, as well as an expression indicating how it was computed.)
8. What is a region of plausible values for β_1 suggested by the data? (i.e. a CI)
9. What is the line that best fits these data, using the criterion that smallest sum of squared residuals is "best?"
10. How much of the observed variation in the heights of sons (the y-axis) is explained by this "best" line?
11. Give an expression in terms of the parameters of the model for the true average height of sons with midparent height $x = 68$.
12. What is the estimated average height of sons whose midparent height is $x = 68$?
13. Is this the true average height in the whole population of sons whose midparent height is $x = 68$?
14. What is the estimated standard deviation among the population of sons whose parents have midparent height $x = 68$?
15. What is the estimated standard deviation among the population of sons whose parents have midparent height $x = 72$? Bigger, smaller, or the same as that for $x = 68$?
16. What is the estimated standard error of the estimated average height for sons with midparent height $x = 68$, i.e. $\hat{\mu}(68) = \hat{\beta}_0 + 68\hat{\beta}_1$? Provide an expression for this standard error.
17. Is the estimated standard error of $\hat{\mu}(72)$ bigger, smaller, or the same as that for $\hat{\mu}(68)$?

18. What quantity can you use to describe or characterize the linear association between height and midparent height in the whole population? Is this a parameter or a statistic?
19. Is the observed linear association between son's height and midparent height strong? To answer, report the value of r and the p-value from the appropriate test.
20. Define $\mu_Y, \sigma_Y, \mu_X, \sigma_X, \rho$. Parameters or statistics?
21. What are plausible values for ρ suggested by the data? (i.e. form a CI)
22. Is $\boxed{E_1, \ldots, E_{928} \overset{iid}{\sim} N(0, \sigma^2)}$ a reasonable assumption?
23. Based **solely** on this study, can we conclude that larger parents cause larger sons?

Galton Height Data Output

The CORR Procedure

2 Variables:	son parent

Covariance Matrix, DF = 927		
	son	parent
son	6.340028724	2.064614487
parent	2.064614487	3.194560689

Simple Statistics						
Variable	N	Mean	Std Dev	Sum	Minimum	Maximum
son	928	68.08847	2.51794	63186	61.70000	73.70000
parent	928	68.30819	1.78733	63390	64.00000	73.00000

Pearson Correlation Coefficients, N = 928 Prob > |r| under H0: Rho=0		
	son	parent
son	1.00000	0.45876 <.0001
parent	0.45876 <.0001	1.00000

Pearson Correlation Statistics (Fisher's z Transformation)							
Variable	With Variable	N	Sample Correlation	Fisher's z	95% Confidence Limits		p Value for H0:Rho=0
son	parent	928	0.45876	0.49574	0.406407	0.508115	<.0001

The REG Procedure
Model: MODEL1
Dependent Variable: son

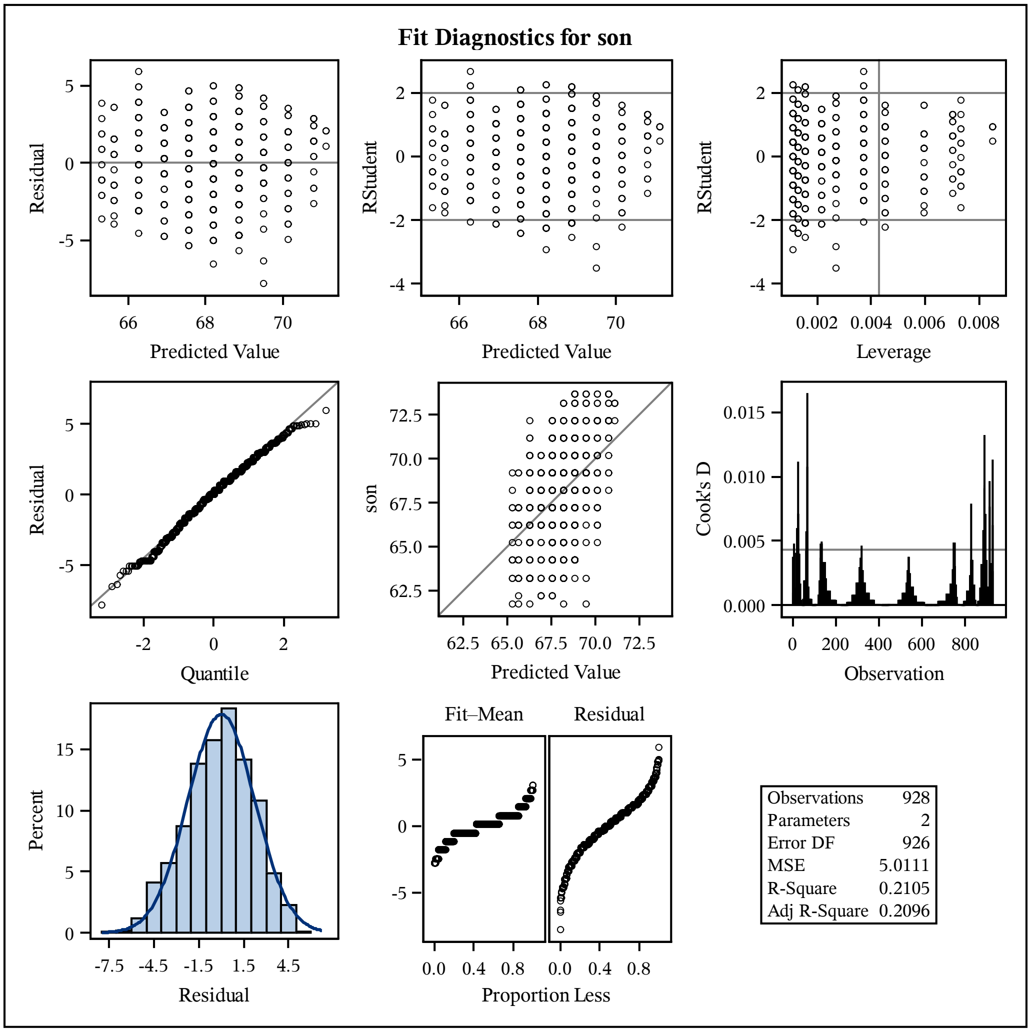
Number of Observations Read	930
Number of Observations Used	928
Number of Observations with Missing Values	2

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	1	1236.93401	1236.93401	246.84	<.0001
Error	926	4640.27261	5.01109		
Corrected Total	927	5877.20663			

Root MSE	2.23855	**R-Square**	0.2105
Dependent Mean	68.08847	**Adj R-Sq**	0.2096
Coeff Var	3.28770		

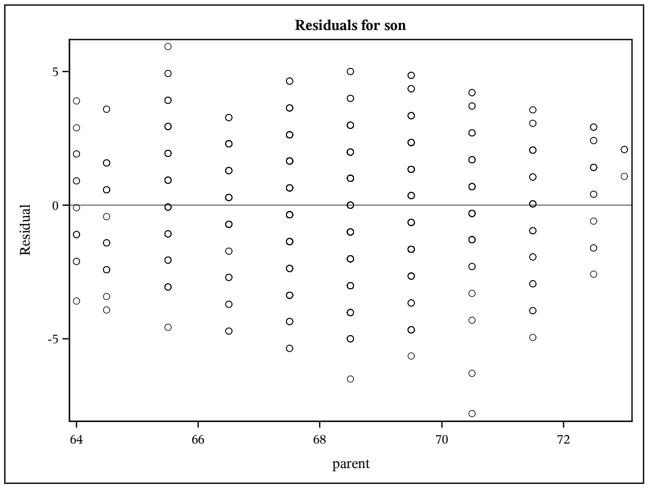
Parameter Estimates							
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**	**95% Confidence Limits**	
Intercept	1	23.94153	2.81088	8.52	<.0001	18.42510	29.45796
parent	1	0.64629	0.04114	15.71	<.0001	0.56556	0.72702

The REG Procedure
Model: MODEL1
Dependent Variable: son



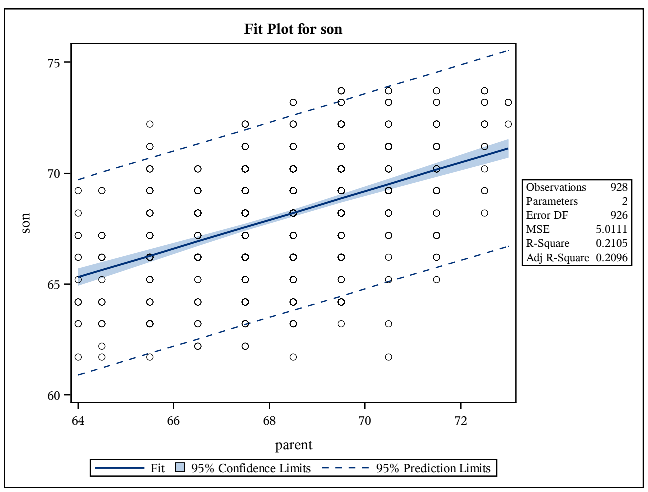
Galton Height Data Output 4

The REG Procedure
Model: MODEL1
Dependent Variable: son



Galton Height Data Output 5

The REG Procedure
Model: MODEL1
Dependent Variable: son



Galton Height Data Output 6

Obs	parent	son	yhat	stdmean	cilow	cihigh	pilow	pihigh	r
1	68	.	67.8893	0.07457	67.7429	68.0356	63.4936	72.2849	.
2	72	.	70.4745	0.16871	70.1434	70.8056	66.0688	74.8801	.

Answers to questions from simple linear regression:

1. A CI for a mean when the true standard deviation is unknown is

$$\bar{y} \pm t_{n-1,\alpha/2} s/\sqrt{n} = 68.09 \pm 1.963 * 2.52/\sqrt{928} = (67.928, 68.252)$$

2. Change in average son's height (inches) per one inch increase in midparent height.
3. β_1 is an unknown parameter.
4. $\hat{\beta}_1 = 0.65$ son inches/midparent inch.
5. $\hat{\beta}_1 = 0.65$ is an observed value of a statistic.
6. β_1 is the slope of the population mean, $\hat{\beta}_1$ is the slope from the SLR of the observed data. $\hat{\beta}_1 = \beta_1$ is very unlikely.
7. $\widehat{SE}(\hat{\beta}_1) = \sqrt{MS[E]/S_{xx}} = 0.041$.
8. Add and subtract 1.963 times the SE to get $(0.566, 0.727)$
9. $\hat{y} = 23.942 + 0.646x$
10. $r^2 = 0.2105$
11. $\mu(68) = \beta_0 + 68\beta_1$
12. $\hat{\mu}(68) = \hat{\beta}_0 + 68\hat{\beta}_1 = 67.889$
13. Probably not! $\mu(68) = \beta_0 + 68\beta_1$ is unknown and $\hat{\mu}(68)$ is only an estimate.
14. This question is asking for the square root of the estimate of variation due to experimental error or $\hat{\sigma} = \sqrt{MS[E]} = 2.24$.
15. Same as the previous question as an assumption on our model is that the errors have the same variance (and hence square root) at every point on the line. (Assumption of homoskedasticity.)
16. $\hat{SE}(\hat{\beta}_0 + 68\hat{\beta}_1) = 0.075$. Expressions given by

$$\begin{aligned}\widehat{SE}(\hat{\mu}(68)) &= \sqrt{MS[E]\left(\frac{1}{n} + \frac{(68-\bar{x})^2}{\sum(x_i - \bar{x})^2}\right)} \\ &= \sqrt{(1,68)' MS[E] (X'X)^{-1} (1,68)}\end{aligned}$$

X a (928×2) *design matrix.*

17. $\widehat{SE}(\hat{\mu}(72)) > \widehat{SE}(\hat{\mu}(68))$ as it is further from $\bar{x}$, where we have the most information about our data.
18. ρ, which is the population correlation coefficient (a parameter).

19. $r = 0.459$, moderate, positive. P-value < 0.0001, there is a significant linear relationship.

20. These are all parameters and describe the mean and standard deviation of the sons' heights, the mean and standard deviation of the midparent's heights, and the correlation between them.

21. The confidence interval is

$$\left(\frac{\frac{1+r}{1-r} e^{-2z/\sqrt{n-3}} - 1}{\frac{1+r}{1-r} e^{-2z/\sqrt{n-3}} + 1}, \frac{\frac{1+r}{1-r} e^{2z/\sqrt{n-3}} - 1}{\frac{1+r}{1-r} e^{2z/\sqrt{n-3}} + 1} \right)$$

or $(0.406, 0.508)$.

22. Residuals reasonably symmetric, no heavy tails. Model fit is ok.

23. Based on this study alone, no. This is an observational study as no midparent heights were assigned by the researchers. However, if science and genetics are brought in, a causal relationship might be inferred.

Recall the example about a random sample of students taking the same exam:

IQ	TIME	GRADE
105	10	75
110	12	79
120	6	68
116	13	85
122	16	91
130	8	79
114	20	98
102	15	76

Consider the additive regression model for the GRADE of subject i, Y_i, in which the mean of Y_i is a linear function of IQ and Time ($X_{i1} = \text{IQ}$ and $X_{i2} = \text{TIME}$) for subjects $i = 1, \ldots, 8$:

$$Y = \beta_0 + \beta_1 \text{IQ} + \beta_2 \text{TIME} + \text{error}$$

or

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + E_i$$

Let's write out the model in matrix form:

$$\mathbf{y} = \begin{pmatrix} 75 \\ 79 \\ 68 \\ 85 \\ 91 \\ 79 \\ 98 \\ 76 \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} 1 & 105 & 10 \\ 1 & 110 & 12 \\ 1 & 120 & 6 \\ 1 & 116 & 13 \\ 1 & 122 & 16 \\ 1 & 130 & 8 \\ 1 & 114 & 20 \\ 1 & 102 & 15 \end{pmatrix} \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 8 & 919 & 100 \\ 919 & 106165 & 11400 \\ 100 & 11400 & 1394 \end{pmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 28.90 & -0.23 & -0.22 \\ -0.23 & 0.0018 & 0.0011 \\ -0.22 & 0.0011 & 0.0076 \end{pmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \begin{pmatrix} 0.74 \\ 0.47 \\ 2.10 \end{pmatrix}$$

$$SS(E) = \mathbf{e}'\mathbf{e} = (\mathbf{Y} - \hat{\mathbf{Y}})'(\mathbf{Y} - \hat{\mathbf{Y}}) = 45.8, \quad \mathbf{e}'\mathbf{e}/df = 9.15$$

$$\widehat{\boldsymbol{\Sigma}} = MS(E)(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 264.45 & -2.07 & -2.05 \\ -2.07 & 0.017 & 0.010 \\ -2.05 & 0.010 & 0.070 \end{pmatrix}$$

Some questions to answer:

1. What is the estimate for β_1? Interpretation?
2. What is the standard error of $\hat{\beta}_1$?
3. Is $\beta_1 = 0$ plausible, while controlling for possible linear associations between Test Score and Study time? ($t_{0.025,5} = 2.57$)
4. Estimate the mean grade among the population of ALL students with $IQ = 113$ who study $TIME = 14$ hours.
5. Report a standard error for this mean.
6. Report a 95% confidence interval for this mean.
7. What is the estimate of the error variance?

Some answers:

1. $\hat{\beta}_1 = 0.47$ (second element of $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$, estimated average exam points per IQ point increase for students studying the same amount)
2. $\sqrt{0.017} = 0.13$ (square root of middle element of $\widehat{\mathbf{\Sigma}}$)
3. $H_0 : \beta_1 = 0 \quad vs \quad H_A : \beta_1 \neq 0$, T-statistic: $t = (\hat{\beta}_1 - 0)/\hat{SE}(\hat{\beta}_1)$
   Observed value is $t = .47/\sqrt{.017} = .47/.13 = 3.6 > 2.57$ ("$\hat{\beta}_1$ differs significantly from 0.")
4. Unknown population mean: $\mu(113, 14) = \beta_0 + \beta_1(113) + \beta_1(14)$
   Estimate : $\hat{\mu}(113, 14) = (1, 113, 14) * \hat{\beta} = 83.6$
5. $\hat{Var}((1, 113, 14) * \hat{\boldsymbol{\beta}}) = (1, 113, 14)\widehat{\text{Var}}(\hat{\boldsymbol{\beta}})(1, 113, 14)'$
   or $(1, 113, 14)\widehat{\mathbf{\Sigma}}(1, 113, 14)' = 1.3$ or $SE(\hat{\theta}) = \sqrt{1.3} = 1.14$
6. $\hat{\mu}(113, 14) \pm t(0.025, 5)\hat{SE}(\hat{\mu}(113, 14))$ or $83.6 \pm 2.57(1.14)$ or $(80.7, 86.6)$
7. The estimate of the error variance is the $MS(E) = 9.15$

Continuing this example, consider this sequence of analyses:

1. Regress GRADE on IQ.
2. Regress GRADE on IQ and TIME.
3. Regress GRADE on TIME IQ TI where TI = TIME*IQ.

ANOVA (Grade on IQ)

SOURCE	DF	SS	MS	F	p-value
IQ	1	15.9393	15.9393	0.153	0.71
Error	6	625.935	104.32		

It appears that IQ has nothing to do with grade, but we did not look at study time.

Looking at the *multiple* regression we get

The REG Procedure

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	596.11512	298.05756	32.57	0.0014
Error	5	45.75988	9.15198		
Corrected Total	7	641.87500			

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	0.73655	16.26280	0.05	0.9656
IQ	1	0.47308	0.12998	3.64	0.0149
Time	1	2.10344	0.26418	7.96	0.0005

Now the test for dependence on IQ is significant $p = 0.0149$. Why? This is a slightly different test. This is testing if IQ is important after taking into account the linear relationship between Grade and Time.

Now recall when we fit the interaction model we found the following (here type I SS are included, ignore the Type I SS for the intercept):

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Type I SS
Intercept	1	72.20608	54.07278	1.34	0.2527	52975
IQ	1	-0.13117	0.45530	-0.29	0.7876	15.93930
Time	1	-4.11107	4.52430	-0.91	0.4149	580.17582
TI	1	0.05307	0.03858	1.38	0.2410	14.69521

This model now appears to be over-fit as the type III tests (tests done after accounting for all other variables in the model) are all non-significant.

We can perform a LOF test to see if the interaction model or the MLR model is preferred. We can find $SS(R)_f$ by summing the type I SS,

$$SS(R)_f = 15.939 + 580.176 + 14.695 = 610.810$$

The $SS(R)_r$ can be found by subtracting off the TI type I SS from $SS(R)_f$ or by adding all type I SS except TI giving

$$SS(R)_r = 610.810 - 14.695 = 596.115$$

Now the numerator of our LOF statistic is

$$\frac{610.810 - 596.115}{3 - 2} = 14.7$$

(Note this is the type I SS for TI!). Our LOF stat is then

$$F = 14.7/7.766 = 1.893$$

(MSE found from output in earlier notes.) Comparing this to $F_{1,4,0.05} = 7.709$ we fail to reject H_0 in favor of H_A. That is, the additive model is adequate.

A random sample of $n = 31$ trees is drawn from a population of trees. On each tree, indexed by i, three variables are measured:

- x_{i1}: "girth", tree diameter in inches
- x_{i2}: "height" (in feet)
- Y_i: volume of timber, in cubic feet.

Given x_1 and x_2, a MLR model for these data is given by

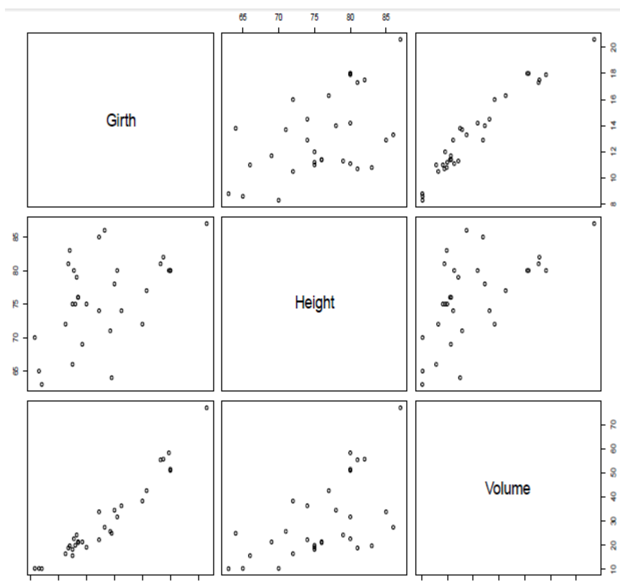
$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + E_i \text{ for } i = 1, \ldots, n$$

where errors are assumed iid normal w/ constant variance σ^2.

For trees with x_1, x_2 the model for mean volume is

$$\mu(x_1, x_2) = E(Y|x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2.$$

A scatterplot matrix



Consider all trees with girth $x_{01} = 15$ *in* and height $x_{02} = 80$ ft .

1. Estimate the mean volume among these trees, along with a standard error and 95% confidence interval. $Note : t_{28,0.025} = 2.048$
2. Obtain a 95% prediction interval of y_0, the volume from an individual tree sampled from this population of 80 footers, with girth 15 inches.

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	7684.16251	3842.08126	254.97	<.0001
Error	28	421.92136	15.06862		
Corrected Total	30	8106.08387			

Root MSE	3.88183	R-Square	0.9480
Dependent Mean	30.17097	Adj R-Sq	0.9442
Coeff Var	12.86612		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	-57.98766	8.63823	-6.71	<.0001
Girth	1	4.70816	0.26426	17.82	<.0001
Height	1	0.33925	0.13015	2.61	0.0145

Covariance of Estimates

Variable	Intercept	Girth	Height
Intercept	74.6189461	0.4321713812	-1.050768886
Girth	0.4321713812	0.0698357838	-0.017860301
Height	-1.050768886	-0.017860301	0.0169393298

Recall: Let $\mathbf{W}$ denote a $p \times 1$ random vector with mean $\boldsymbol{\mu}_{\mathbf{W}}$ and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{W}}$. Suppose $\mathbf{a}$ is a $p \times 1$ (fixed) vector of coefficients. Then

$$\boxed{\begin{array}{rcl} E(\mathbf{a}'\mathbf{W}) & = & \mathbf{a}'\boldsymbol{\mu}_{\mathbf{W}} \\ \mathrm{Var}(\mathbf{a}'\mathbf{W}) & = & \mathbf{a}'\boldsymbol{\Sigma}_{\mathbf{W}}\mathbf{a}. \end{array}}$$

1. Consider all trees with Girth 15 and Height 80 To estimate mean volume among these trees, along with an estimated standard error, take $\mathbf{x}_0' = (1, 15, 80)$ and consider

$$\hat{\mu}(\mathbf{x}_0) = \mathbf{x}_0'\hat{\boldsymbol{\beta}}$$

$$E(\mathbf{x}_0'\hat{\boldsymbol{\beta}}) = \mathbf{x}_0'\boldsymbol{\beta}$$

$$Var(\mathbf{x}_0'\hat{\boldsymbol{\beta}}) = \mathbf{x}_0'\hat{\boldsymbol{\Sigma}}\mathbf{x}_0$$

Substitution of $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\Sigma}} = MSE(\mathbf{X}'\mathbf{X})^{-1}$ gives the estimates:

$$\begin{aligned}
\hat{\mu}(\mathbf{x}_0) &= (1, 15, 80)\begin{pmatrix} -58.0 \\ 4.71 \\ 0.34 \end{pmatrix} \\
&= 39.8 \\
\widehat{\mathrm{Var}}(\hat{\mu}(\mathbf{x}_0)) &= (1, 15, 80)\begin{pmatrix} 74.62 & 0.43 & -1.05 \\ 0.43 & 0.070 & -0.018 \\ -1.05 & -0.018 & 0.017 \end{pmatrix}\begin{pmatrix} 1 \\ 15 \\ 80 \end{pmatrix} \\
&= 0.72 \\
\widehat{SE}(\hat{\mu}(\mathbf{x}_0)) &= \sqrt{.72} = 0.849
\end{aligned}$$

Thus, a 95% CI is given by

$$39.8 \pm 2.048 * 0.849 = (38.061, 41.539)$$

2. 95% Prediction limits? Same idea but we add and subtract $t(.025, 28)\sqrt{.72 + MS(E)}$.