Chapter 4

ST 511 - Probability and Distributions

Readings: Chapter 4 - 4.1-4.4, 4.6-4.8 (ok to skip Poisson Distribution), 4.9-4.10, 4.12, 4.14

Recall: Our goal is to conduct inference. In order to do this, we need a firm understanding of probability.

Interpretation of Probability

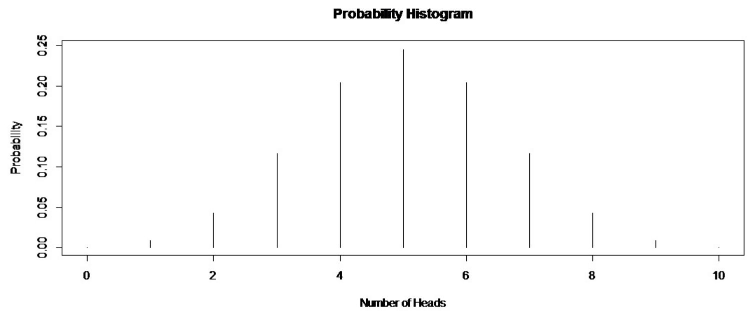
- ______________________________ of an outcome in repeated experiment = # of times outcome observed/# of times experiment was repeated
  - Ex: the chance of rolling snake eyes $((1,1))$ on two fair dice
  - Ex: the chance of getting a head on a flipped coin

Probability and Inference

- Ex: We want to see if a coin is fair. If we formulate the research question in terms of parameters, we want to test the hypothesis:

  Suppose the coin is tossed $n = 10$ times and yields $y = 10$ heads.

  - If hypothesis is true, how likely is the observed event?
  - With 10/10 heads, reasonable to conclude coin not fair. What about 9/10 heads? 7/10 heads?
  - To make a decision, need to know ___________________________



The above plot gives the probability of observing a given number of heads from a fair coin in 10 tosses.

Sets and Sample Spaces

A probability model is a mathematical representation of a random phenomenon. Defined by its

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-
-

Definitions:

- _______________: A collection of **elements**, $a_1, a_2, \ldots$
- ____________________________: the set of all outcomes under consideration
- _______________: Each possible distinct result of a random process (experiment)
- A is a _________________ of B if every element of A belongs to B ($A \subset B$)
- _______________: A collection of outcomes (a subset of $\mathbf{S}$)

Sample Space examples:
Define the sample space $\mathbf{S}$ for each situation below.
Of the parts manufactured today, randomly select and measure the thickness of **a single part**.

It is known that the thickness must be between 10 and 11 mm.

It is known that the thickness has only three values (low, medium or high).

Experiment asks, does the part thickness meet specifications?

Now two parts are randomly selected and measured.

Do the 2 parts conform to specifications?

Number of conforming parts from the two is measured.

Now, parts are randomly selected until a non-conforming part is found.

More Set definitions:

- ________________ $(A \cup B)$: the set of all points in A or B (including both)
- ________________ $(A \cap B)$: the set of all points in both A and B
- ________________ of A ($\bar{A}$ or A^c): contains elements in S but not in A
- A and B are ________________ or ________________ if $A \cap B = \emptyset$.

Gender of Children - Discrete Example: - A family has two children of different ages. Consider the possible genders of these children. Let a pair FM denote the element in which the younger child is female and the older is male.

1. Sample Space: $S = \{ \qquad\qquad\qquad\qquad \}$
2. Let A be the event in which both children are males, B the event in which there is at least one male, and C the event containing no males.
   List the elements of
   - $A = \{ \qquad\qquad\qquad \}$ $\qquad$ $A \cup C = \{ \qquad\qquad\qquad \}$
   - $B = \{ \qquad\qquad\qquad \}$ $\qquad$ $\bar{A} = \{ \qquad\qquad\qquad \}$
   - $C = \{ \qquad\qquad\qquad \}$ $\qquad$ $A \cup B = \{ \qquad\qquad\qquad \}$
   - $A \cap C = \{ \qquad\qquad\qquad \}$ $\qquad$ $B \cup \bar{A} = \{ \qquad\qquad\qquad \}$

Relating Set Theory to Probability

- An ________________ is any process that can be repeated (theoretically) and has a well-defined set of possible outcomes (sample space)

- An event corresponds to ____

- The Probability of the event is the likelihood or chance that a particular outcome or event from a random experiment will occur. We write

$$P(A) = \text{Probability the event A occurs}$$

- Probabilities are numbers between __________________________________

- May be written as proportion (0.15), percent (15%), or a fraction (3/20).

- P(Event)=1 implies ____

- P(Event)=0 implies ____

Simplified axioms of probability

- The probability of an event, $P(A)$, a function, must satisfy:

  ____

  ____

  If A and B are disjoint (mutually exclusive) then

  ____

Probability example (Recall Gender of Children ex) - The sample space for this experiment was

$$\mathbf{S} = \{MM, MF, FM, FF\}$$

1. What would be reasonable probabilities for each outcome in S?

2. For A, B, and C, defined earlier, find $P(A)$, $P(B)$, $P(C)$, $P(A \cup C)$, and $P(S)$

Conditional Probability, Independence, and Other Probability Rules

Often we will have knowledge of one event's occurrence. How does that change the probability of another event?

For example, the probability of getting a 1 in the toss of a six-sided die is __________.

If we know that an odd number has fallen, then the probability of occurrence of a 1 is _________.

The __________________________________ of an event A given that an event B has occurred is equal to

provided $P(B) > 0$.

Independent Events
Two events and are said to be _________________________ if and only if **any** one of the following 3 conditions hold:

$$\begin{aligned} P(A|B) &= P(A) \\ P(B|A) &= P(B) \\ P(A \cap B) &= P(A)P(B). \end{aligned}$$

Otherwise, the events are said to be ___________________________________

Independence Examples

- Consider again the "Gender of Children" example. Let A be the event that the younger child is female, and B be the event that the older child is male. Are A and B dependent?

- (Credit Card Example) - The proportion of NCSU students with a VISA card is 0.48, the proportion with a MasterCard is 0.64, the proportion with both is 0.35.

  1. Calculate the conditional probability that a randomly sampled student has a VISA given he/she has a MasterCard.

  2. Are the events 'having a VISA' and 'having a MasterCard' independent?

Laws of Probability
Sometimes probabilities of events can be obtained by using multiplicative and additive rules.

____:

$$\boxed{P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)}$$

Notice that if A and B are ________________________, then

$$\boxed{P(A \cap B) = P(A)P(B)}$$

Using the Multiplicative Law
(Urn Example) - An urn contains 10 marbles, 4 are red (R) and 6 are black (B).

1. If 2 are randomly chosen from the urn, what is the probability that both are black?

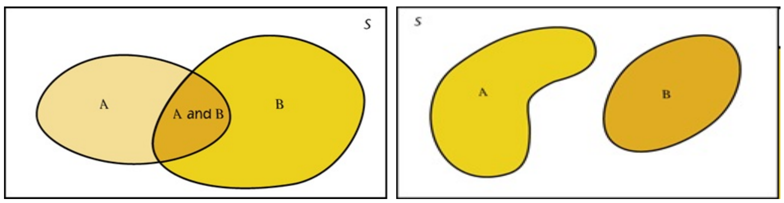
2. If 1 is randomly chosen, then replaced, and then another randomly chosen (making the selections independent events), what is the probability of selecting a red then a black?

3. Flip a fair coin 3 times, find the probability of observing 3 heads (HHH) assuming independent flips.

____:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Recall 3rd axiom: if A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.



Using the Additive Law

- (Credit Card Example) - The proportion of NCSU students with a VISA card is 0.48, the proportion with a MasterCard is 0.64, the proportion with both is 0.35.
  Find the probability that a randomly sampled student has a VISA or MasterCard (or both).

- (Axiom Example) - Can A and B be mutually exclusive if $P(A) = 0.4$ and $P(B) = 0.7$? What if $P(B) = 0.3$?

______________________________:
A special case of additive law is obtained by taking $B = A^c$, then

$$\boxed{P(A) + P(A^c) = 1 \text{ implies } P(A) = 1 - P(A^c)}$$

Ex: In 17th century De M'er'e asked Pascal, which is more likely:
A: rolling at least one six in four throws of a single dice, or
B: rolling at least one double six in 24 throws of a pair of dice?
Find $P(A)$ and $P(B)$.

(See example 4.1 on page 149.)