Phy 981 Assignment 4

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Exercise 9

(a) The two states $|\Phi_i^a\rangle$ and $\left|\Phi_{ij}^{ab}\right\rangle$ can be written in terms of the new reference vacuum $|\Phi_0\rangle$ as follows:

$$\boxed{|\Phi_i^a\rangle = a_a^\dagger a_i |\Phi_0\rangle}$$

$$\boxed{\left|\Phi_{ij}^{ab}\right\rangle = a_a^\dagger a_b^\dagger a_j a_i |\Phi_0\rangle}$$

(b) The one-body expectation value is

$$\langle\Phi_0|\hat{F}_N|\Phi_0\rangle = \sum_{pq} \langle p|f|q\rangle \langle\Phi_0|\{a_p^\dagger a_q\}|\Phi_0\rangle = \boxed{0}$$

since the expectation value of a normal-ordered string of operators is zero by definition.

Similarly, for the two-body operator,

$$\langle\Phi_0|\hat{G}_N|\Phi_0\rangle = \frac{1}{4}\sum_{pqrs} \langle pq|g|rs\rangle_{\mathrm{AS}} \langle\Phi_0|\{a_p^\dagger a_q^\dagger a_s a_r\}|\Phi_0\rangle = \boxed{0.}$$

(c) For the one-body operator and the one-particle–one-hole excitation,

$$\begin{aligned}
\langle\Phi_0|\hat{F}_N|\Phi_i^a\rangle &= \sum_{pq} \langle p|f|q\rangle \langle\Phi_0|\{a_p^\dagger a_q\}a_a^\dagger a_i|\Phi_0\rangle \\
&= \sum_{pq} \langle p|f|q\rangle \langle\Phi_0|\{a_p^\dagger a_q\}a_a^\dagger a_i|\Phi_0\rangle \\
&= \sum_{pq} \langle p|f|q\rangle \delta_{ip}\delta_{aq} = \boxed{\langle i|f|a\rangle\,.}
\end{aligned}$$

For the two-body operator,

$$\langle\Phi_0|\hat{G}_N\left|\Phi_{ij}^{ab}\right\rangle = \frac{1}{4}\sum_{pqrs} \langle pq|g|rs\rangle_{\mathrm{AS}} \langle\Phi_0|\{a_p^\dagger a_q^\dagger a_s a_r\}a_a^\dagger a_i|\Phi_0\rangle = \boxed{0}$$

since there is no way to make a fully contracted term without contracting two operators that are both inside the normal-ordered string.

(d) For the one-body operator and the two-particle–two-hole excitation,

$$\langle \Phi_0 | \hat{F}_N | \Phi_i^a \rangle = \sum_{pq} \langle p|f|q \rangle \langle \Phi_0 | \{a_p^\dagger a_q\} a_a^\dagger a_b^\dagger a_j a_i | \Phi_0 \rangle = \boxed{0}$$

for the same reason as above.

For the two-body operator,

$$\langle \Phi_0 | \hat{G}_N \big| \Phi_{ij}^{ab} \rangle = \frac{1}{4} \sum_{pqrs} \langle pq|g|rs \rangle_{\mathrm{AS}} \langle \Phi_0 | \{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_b^\dagger a_j a_i | \Phi_0 \rangle$$

Here, there are four non-zero contractions:

$\{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_b^\dagger a_j a_i$	$\delta_{pi}\delta_{qj}\delta_{sb}\delta_{ra}$	$\frac{1}{4} \langle ij|g|ab \rangle_{\mathrm{AS}}$
$\{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_b^\dagger a_j a_i$	$-\delta_{pj}\delta_{qi}\delta_{sb}\delta_{ra}$	$-\frac{1}{4} \langle ji|g|ab \rangle_{\mathrm{AS}}$
$\{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_b^\dagger a_j a_i$	$-\delta_{pi}\delta_{qj}\delta_{sa}\delta_{rb}$	$-\frac{1}{4} \langle ij|g|ba \rangle_{\mathrm{AS}}$
$\{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_b^\dagger a_j a_i$	$\delta_{pj}\delta_{qi}\delta_{sa}\delta_{rb}$	$\frac{1}{4} \langle ji|g|ba \rangle_{\mathrm{AS}}$

Due to the antisymmetry, these four matrix elements are equal. Thus,

$$\langle \Phi_0 | \hat{G}_N \big| \Phi_{ij}^{ab} \rangle = \boxed{\langle ij|g|ab \rangle_{\mathrm{AS}}}$$

By extension of the fact that a one-body operator produces a result of zero when acting on a two-particle–two-hole state, I would expect that a two-body operator would produce zero when acting on $|\Phi_{ijk}^{abc}\rangle$, a three-particle–three-hole excitation.