Homework #5

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Problem 1. The energy required to dissociate the H_2 gas is equal to

$$E_d = \epsilon_d \frac{M}{2m_H}, \tag{1}$$

where $\frac{M}{2m_H}$ gives the number of H_2 molecules in the gas. Similarly, once the hydrogen has been dissociated, it will take an additional energy equal to

$$E_i = \epsilon_i \frac{M}{m_H} \tag{2}$$

to ionize the monatomic hydrogen. Assuming that the gravitational potential energy is what provides this dissociation and ionization energy, we can find that for a given gravitational potential energy

$$\begin{aligned} \Omega &= -\int_0^M \frac{GM_r}{r} dM \\ &\approx -\frac{GM^2}{R}, \end{aligned} \tag{3}$$

the required radius of the cloud to achieve complete ionization is

$$\begin{aligned} |\Omega| &= |E_d + E_i| \\ \frac{GM^2}{R} &= \epsilon_i \frac{M}{m_H} + \epsilon_d \frac{M}{2m_H} \\ R &= \frac{GMm_H}{\epsilon_i + \frac{1}{2}\epsilon_d} \\ &= 8.75 \times 10^{10}\,\mathrm{m} \\ &\approx 125.7\,\mathrm{R}_\odot \\ &\approx 0.58\,\mathrm{AU}. \end{aligned} \tag{4}$$

Problem 2. The Virial Theorem relates the kinetic energy of the particles KE with the gravitational potential energy Ω by

$$2KE + \Omega = 0. \tag{5}$$

For a given number density N of particles at temperature T, the average kinetic energy is given by

$$KE = \frac{3}{2} N k_B T. \tag{6}$$

Since our gas consists of ionized hydrogen, that is, two particles for every hydrogen atom, we have

$$KE_{tot} = \frac{3}{2} \frac{M}{\frac{1}{2} m_H} k_B T. \tag{7}$$

Plugging equations (7) and (3) into Virial's Theorem, we get

$$\begin{aligned} 2\left(3 \frac{M}{m_H} k_B T\right) &= \frac{GM^2}{R} \\ T &= \frac{GMm_H}{6k_B R} \\ &\approx 30700\,\mathrm{K}. \end{aligned} \tag{8}$$

Now, if we combine equations (8) and (4), we find

$$\begin{aligned} T &= \frac{GMm_H}{6k_B} \frac{\epsilon_i + \frac{1}{2}\epsilon_d}{GMm_H} \\ &= \frac{\epsilon_i + \frac{1}{2}\epsilon_d}{6k_B}, \end{aligned} \tag{9}$$

and thus the temperature has no dependence on the mass.