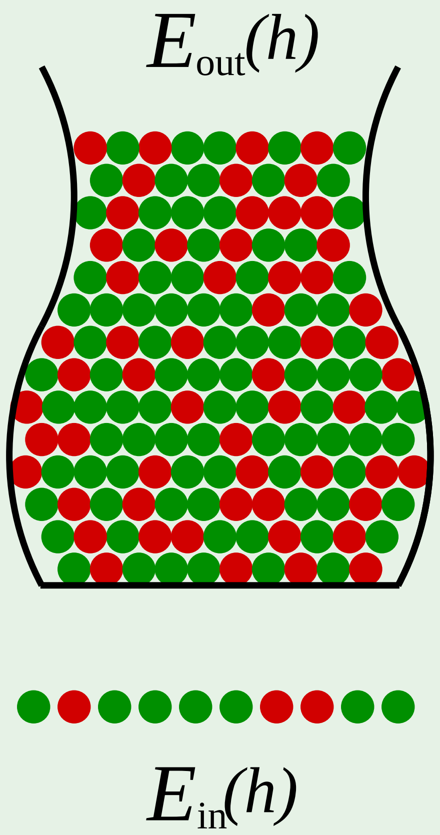
Review of Lecture 2

Is Learning feasible?

Yes, in a **probabilistic** sense.



$$\mathbb{P}\left[\ |E_{\text{in}}(h) - E_{\text{out}}(h)| > \epsilon\ \right] \ \le\ 2e^{-2\epsilon^2 N}$$

Since g has to be one of $h_1, h_2, \cdots, h_M$, we conclude that

If:

$|E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon$

Then:

$|E_{\text{in}}(h_1) - E_{\text{out}}(h_1)| > \epsilon$ **or**

$|E_{\text{in}}(h_2) - E_{\text{out}}(h_2)| > \epsilon$ **or**

$\cdots$

$|E_{\text{in}}(h_M) - E_{\text{out}}(h_M)| > \epsilon$

This gives us an added M factor.

Learning From Data

Yaser S. Abu-Mostafa
California Institute of Technology

Lecture 3: Linear Models I

Sponsored by Caltech's Provost Office, E&AS Division, and IST • Tuesday, April 10, 2012

Outline

- Input representation
- Linear Classification
- Linear Regression
- Nonlinear Transformation

A real data set

Input representation

'raw' input $\mathbf{x} = (x_0, x_1, x_2, \cdots, x_{256})$

linear model: $(w_0, w_1, w_2, \cdots, w_{256})$

Features: Extract useful information, e.g.,

intensity and symmetry $\mathbf{x} = (x_0, x_1, x_2)$

linear model: (w_0, w_1, w_2)

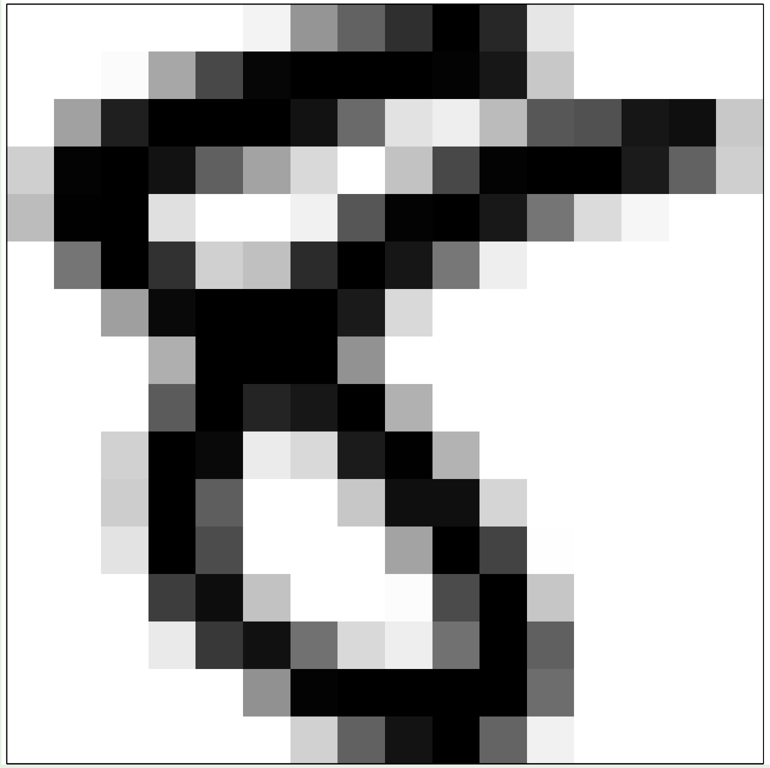
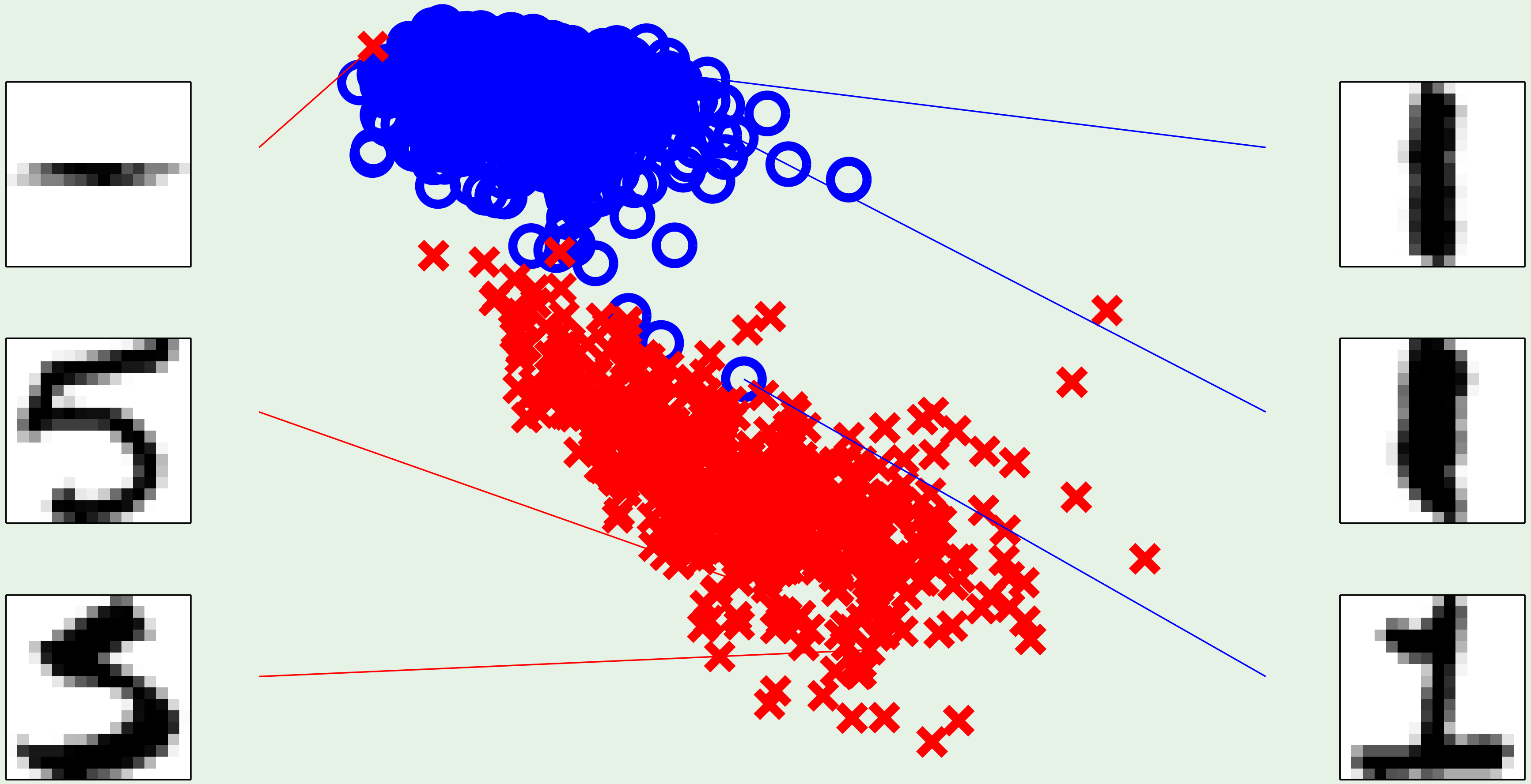


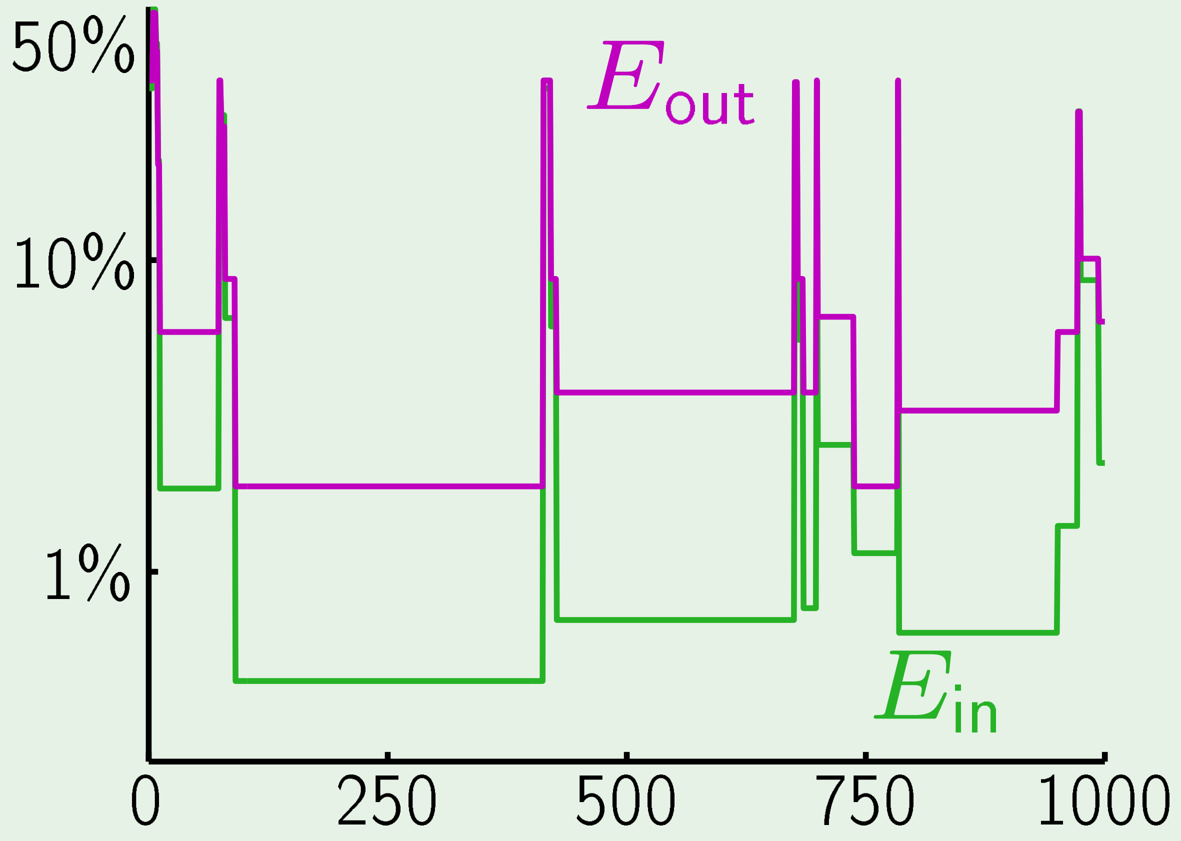
Illustration of features

$\mathbf{x} = (x_0, x_1, x_2)$ $\qquad$ x_1: intensity $\qquad$ x_2: symmetry

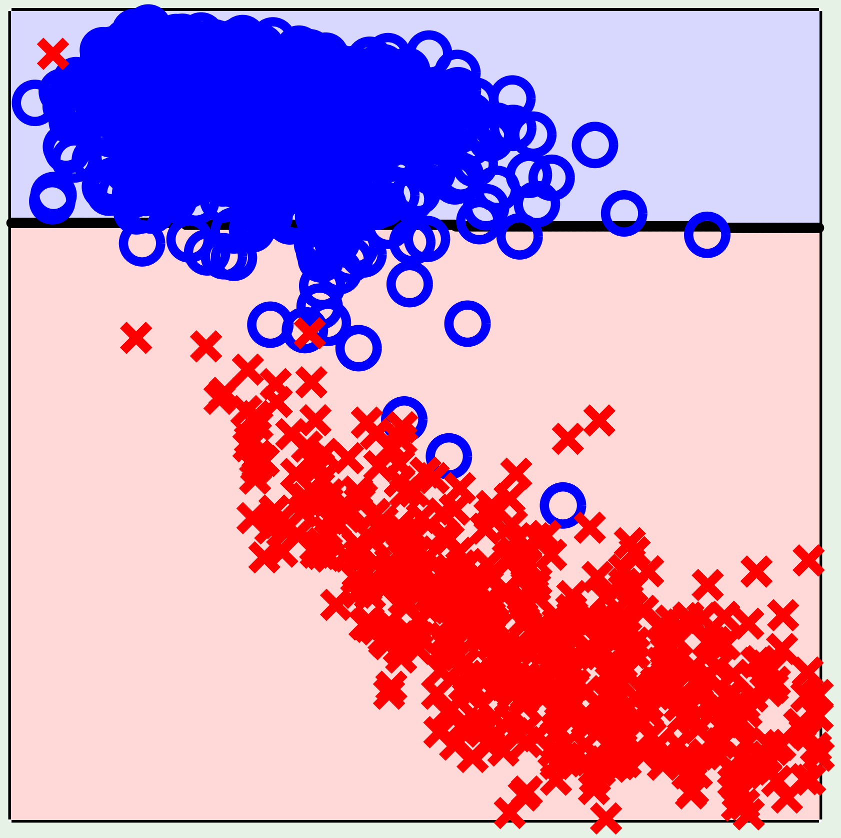


What PLA does

Evolution of E_{in} and E_{out}

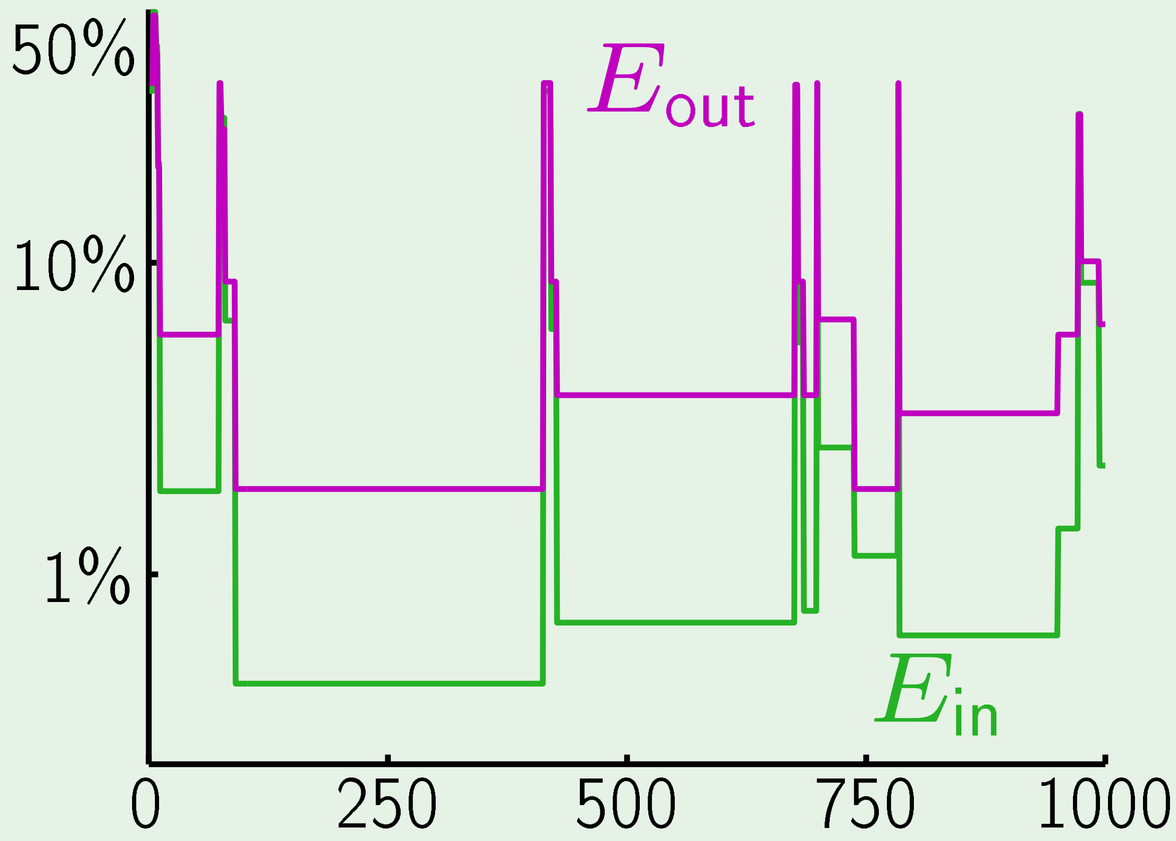


Final perceptron boundary

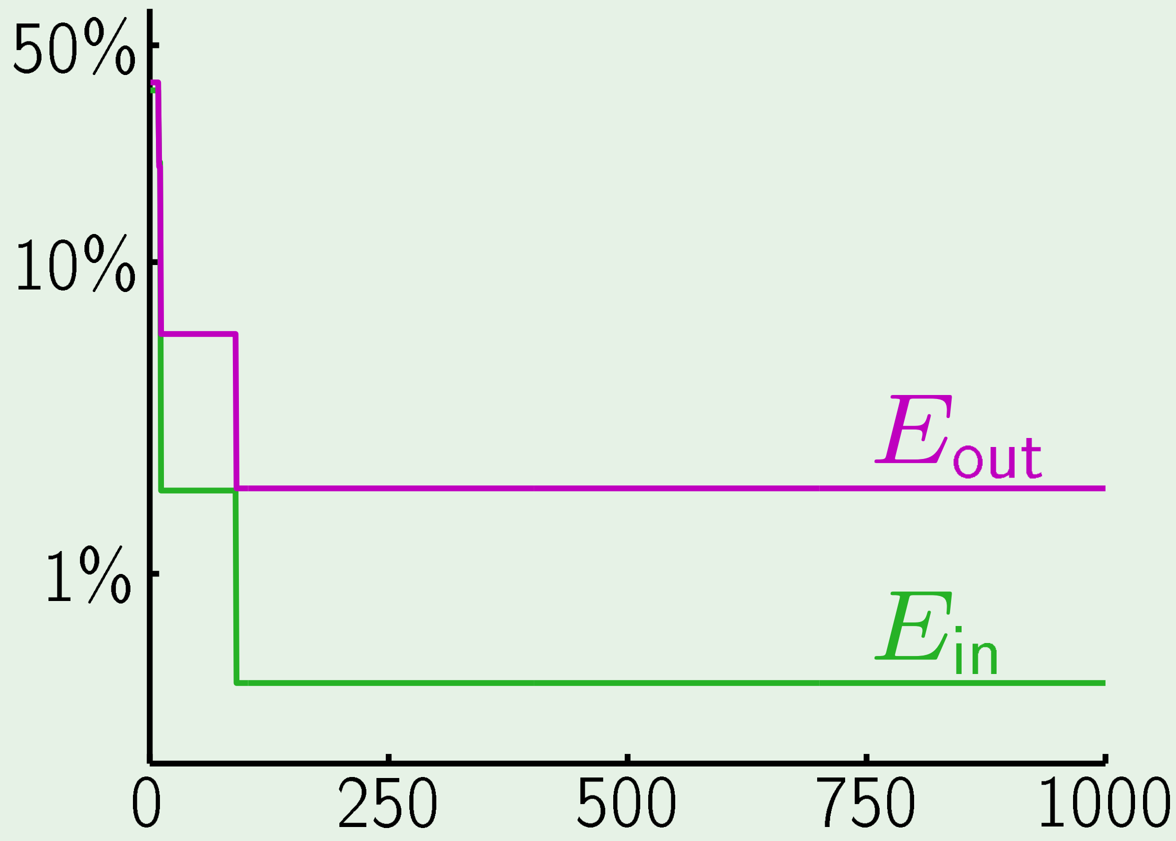


The 'pocket' algorithm

PLA:

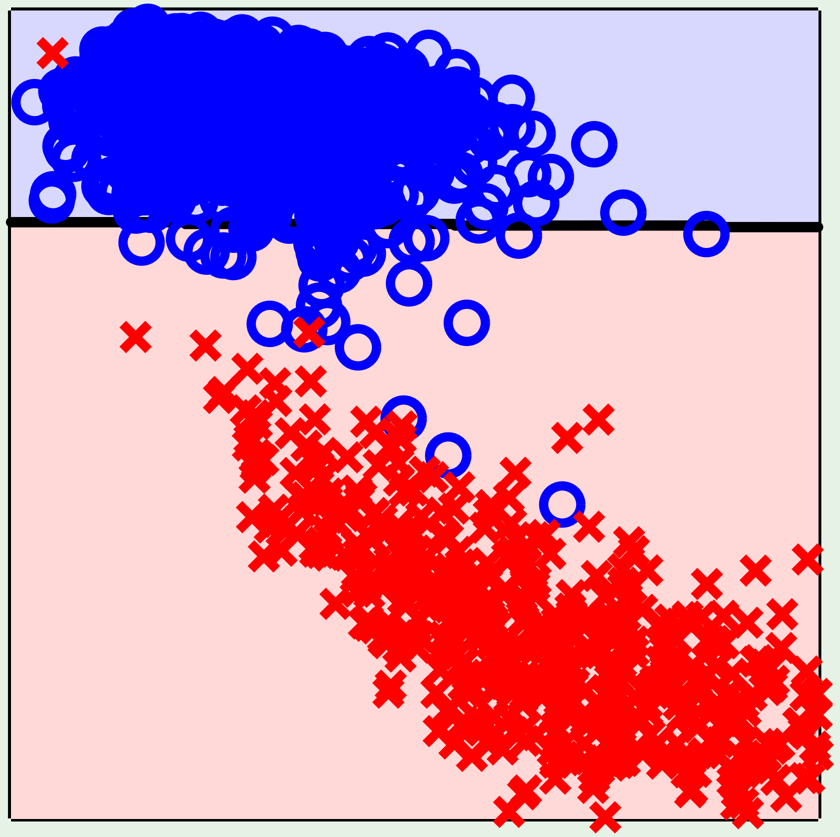


Pocket:

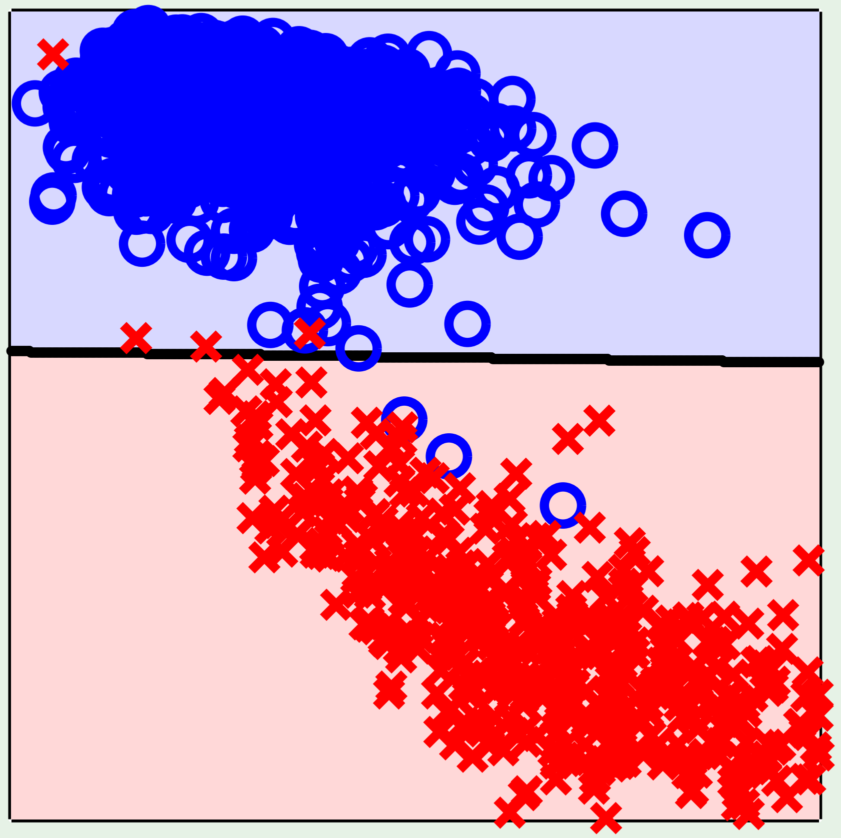


Classification boundary - PLA versus Pocket

PLA:



Pocket:



Outline

- Input representation
- Linear Classification
- Linear Regression **regression $\equiv$ real-valued output**
- Nonlinear Transformation

Credit again

Classification: Credit approval (yes/no)

Regression: Credit line (dollar amount)

Input: $\mathbf{x}$ =

age	23 years
annual salary	\$30,000
years in residence	1 year
years in job	1 year
current debt	\$15,000
$\cdots$	$\cdots$

Linear regression output: $h(\mathbf{x}) = \displaystyle\sum_{i=0}^{d} w_i\, x_i = \mathbf{w}^{\mathsf{T}}\mathbf{x}$

The data set

Credit officers decide on credit lines:

$$(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \cdots, (\mathbf{x}_N, y_N)$$

$y_n \in \mathbb{R}$ is the credit line for customer $\mathbf{x}_n$.

Linear regression tries to replicate that.

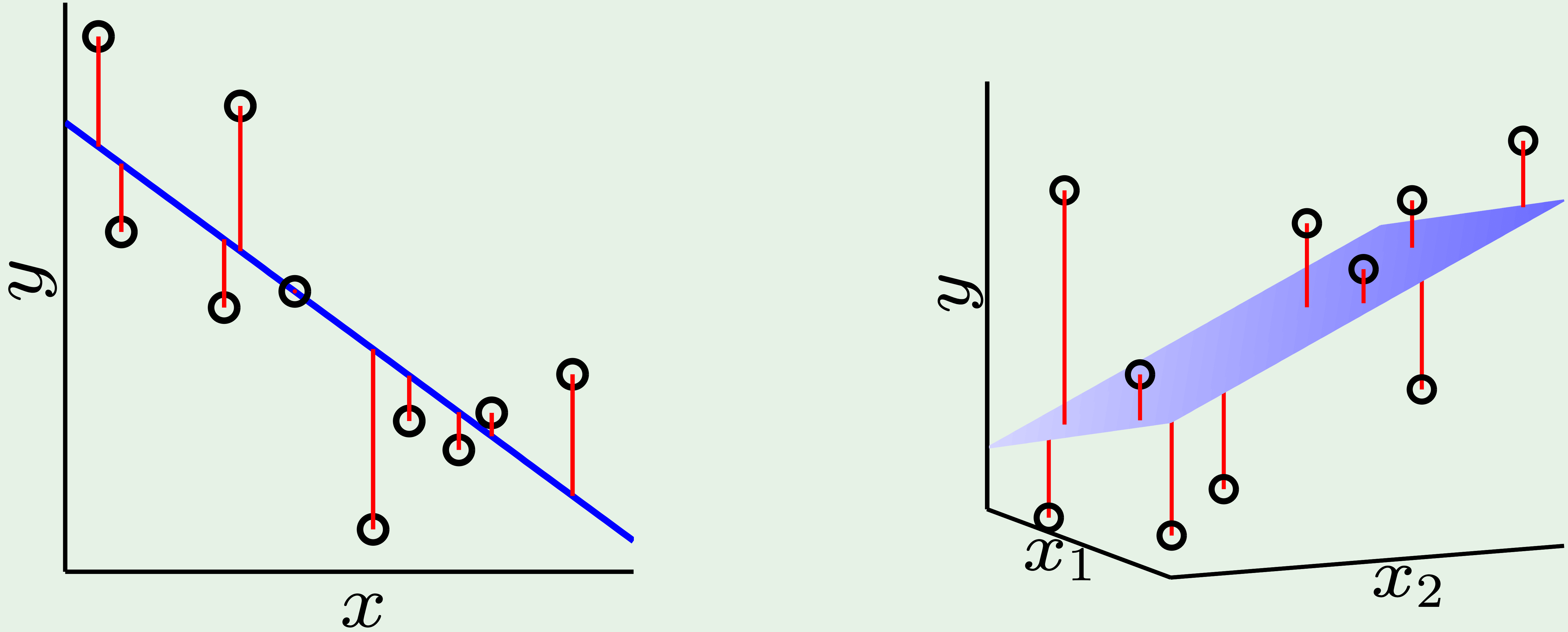
How to measure the error

How well does $h(\mathbf{x}) = \mathbf{w}^{\intercal}\mathbf{x}$ approximate $f(\mathbf{x})$?

In linear regression, we use squared error $(h(\mathbf{x}) - f(\mathbf{x}))^2$

in-sample error: $E_{\text{in}}(h) = \dfrac{1}{N}\displaystyle\sum_{n=1}^{N}(h(\mathbf{x}_n) - y_n)^2$

Illustration of linear regression



The expression for E_{in}

$$
\begin{aligned}
E_{\text{in}}(\mathbf{w}) &= \frac{1}{N}\sum_{n=1}^{N}\left(\mathbf{w}^{\intercal}\mathbf{x}_n - y_n\right)^2 \\
&= \frac{1}{N}\|\mathrm{X}\mathbf{w} - \mathbf{y}\|^2
\end{aligned}
$$

where $\quad \mathrm{X} = \begin{bmatrix} \text{—}\mathbf{x}_1{}^{\intercal}\text{—} \\ \text{—}\mathbf{x}_2{}^{\intercal}\text{—} \\ \vdots \\ \text{—}\mathbf{x}_N{}^{\intercal}\text{—} \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$

Minimizing E_{in}

$$E_{\text{in}}(\mathbf{w}) = \frac{1}{N}\|\mathrm{X}\mathbf{w} - \mathbf{y}\|^2$$

$$\nabla E_{\text{in}}(\mathbf{w}) = \frac{2}{N}\mathrm{X}^{\intercal}(\mathrm{X}\mathbf{w} - \mathbf{y}) = \mathbf{0}$$

$$\mathrm{X}^{\intercal}\mathrm{X}\mathbf{w} = \mathrm{X}^{\intercal}\mathbf{y}$$

$$\mathbf{w} = \mathrm{X}^{\dagger}\mathbf{y} \quad \text{where} \quad \mathrm{X}^{\dagger} = (\mathrm{X}^{\intercal}\mathrm{X})^{-1}\mathrm{X}^{\intercal}$$

$\mathrm{X}^{\dagger}$ is the '**pseudo-inverse**' of X

The pseudo-inverse

$$\mathrm{X}^{\dagger} = (\mathrm{X}^{\intercal}\mathrm{X})^{-1}\mathrm{X}^{\intercal}$$

$$\underbrace{\left(\underbrace{\begin{bmatrix} \quad & \quad \\ & \end{bmatrix}}_{d+1\ \times\ d+1}\right)^{-1}\underbrace{\begin{bmatrix} \qquad\qquad & \qquad\qquad \\ & \end{bmatrix}}_{d+1\ \times\ N}}_{d+1\ \times\ N}$$

The linear regression algorithm

Construct the matrix X and the vector $\mathbf{y}$ from the data set $(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_N, y_N)$ as follows

$$\mathrm{X} = \underbrace{\begin{bmatrix} \text{—}\mathbf{x}_1^\top\text{—} \\ \text{—}\mathbf{x}_2^\top\text{—} \\ \vdots \\ \text{—}\mathbf{x}_N^\top\text{—} \end{bmatrix}}_{\text{input data matrix}}, \qquad \mathbf{y} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\text{target vector}}.$$

Compute the pseudo-inverse $\mathrm{X}^\dagger = (\mathrm{X}^\top\mathrm{X})^{-1}\mathrm{X}^\top$.

Return $\mathbf{w} = \mathrm{X}^\dagger\mathbf{y}$.

Linear regression for classification

Linear regression learns a real-valued function $y = f(\mathbf{x}) \in \mathbb{R}$

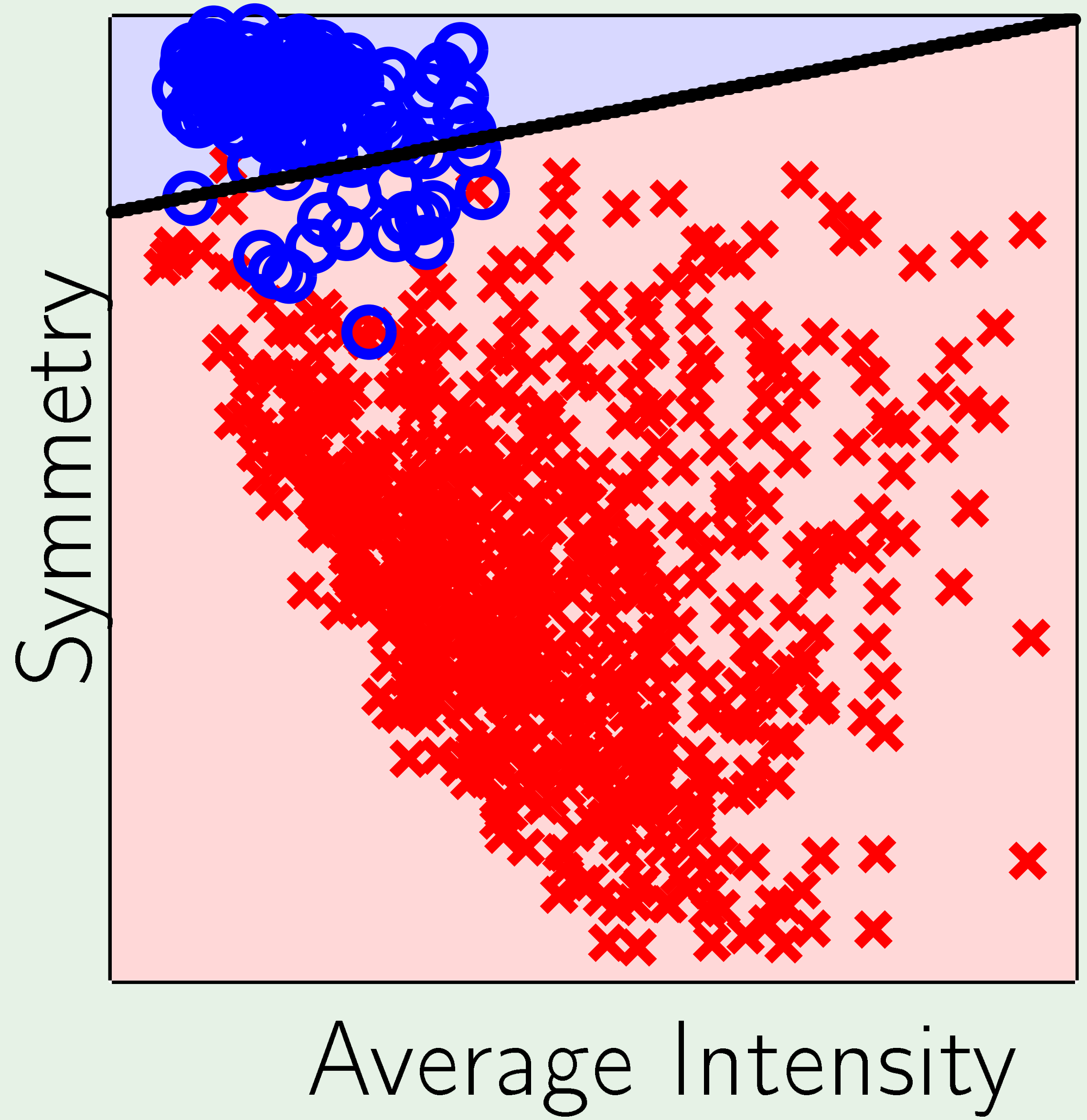
Binary-valued functions are also real-valued! $\pm 1 \in \mathbb{R}$

Use linear regression to get $\mathbf{w}$ where $\mathbf{w}^\intercal \mathbf{x}_n \approx y_n = \pm 1$

In this case, $\text{sign}(\mathbf{w}^\intercal \mathbf{x}_n)$ is likely to agree with $y_n = \pm 1$

Good initial weights for classification

Linear regression boundary

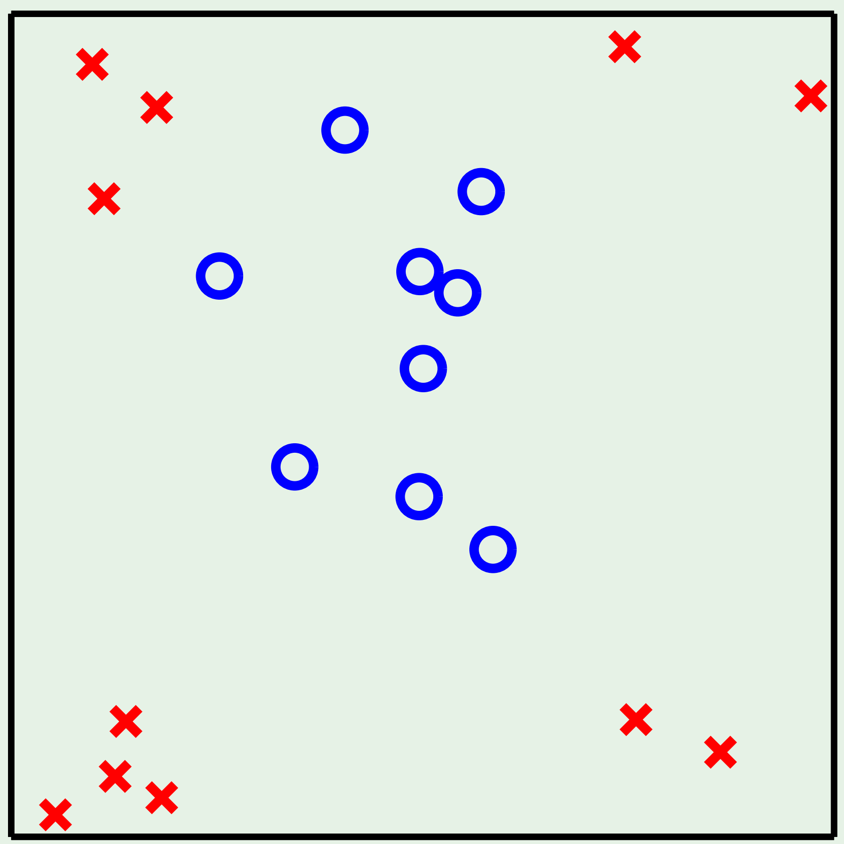


Outline

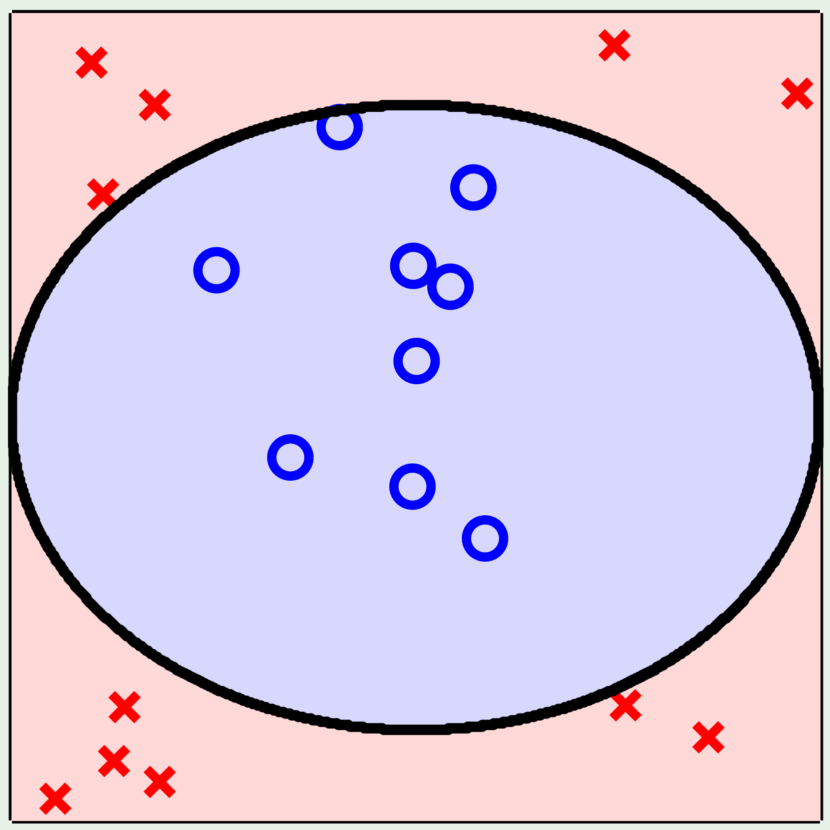
- Input representation
- Linear Classification
- Linear Regression
- Nonlinear Transformation

Linear is limited

Data:



Hypothesis:



Another example

Credit line is affected by ‘years in residence’

but **not** in a linear way!

Nonlinear $[[x_i < 1]]$ and $[[x_i > 5]]$ are better.

Can we do that with linear models?

Linear in what?

Linear regression implements

$$\sum_{i=0}^{d} w_i \; x_i$$

Linear classification implements

$$\text{sign}\left(\sum_{i=0}^{d} w_i \; x_i\right)$$

Algorithms work because of **linearity in the weights**

Transform the data nonlinearly

$$(x_1, x_2) \xrightarrow{\ \Phi\ } (x_1^2, x_2^2)$$