Derivation of ϵ

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1 Error Function

We derive the analytical form of the error ϵ

$$\epsilon = \frac{T_{ib} - (T_{ij} + T_{ik})}{T_{ib}} \tag{1}$$

for the case of the gravity model with an exponential deterrence function:

$$T_{ij} = k_{ij} m_j f(r_{ij}), \tag{2}$$

where

$$k_{ij}^{-1} = \sum_j m_j f(r_{ij}) \tag{3}$$

and

$$f(r_{ij}) = e^{-\gamma r_{ij}} \tag{4}$$

Substituting (2) into (1) gives

$$\epsilon = 1 - \left(\frac{k_{ij} m_j}{e^{\gamma r_{ij}}} + \frac{k_{ik} m_k}{e^{\gamma r_{ik}}} \right) \frac{e^{\gamma r_{ib}}}{k_{ib} m_b} \tag{5}$$

We obtain a continuous form of k_{ib} by approximating the sum to an integral. We then use a polar coordinates system with location k on the x-axis and perform an integral over the circular sector between the x-axis and $2\pi - \delta\theta$. Assuming $\delta\theta$ is small enough, this can approximate the area of interest sufficiently well.

$$\begin{aligned} k_{ib}^{-1} &= \int_A e^{-\gamma r_{ib}} dm \\ &= \int_{\theta=0}^{2\pi-\delta\theta} \int_{r_m in}^{r_{max}} \rho e^{-\gamma r_{ib}} r dr d\theta \\ &= \rho \frac{2\pi - \delta\theta}{\gamma^2} \left[e^{-\gamma r_{min}} (\gamma r_{min} + 1) - e^{-\gamma r_{max}} (\gamma r_{max} + 1) \right] \end{aligned} \tag{6}$$

[Note: we realised we might not need this form of k, since we can find an expression for ϵ by keeping k and decomposing the sum in the following way:]

$$k_{ib}^{-1} = \sum_{l \neq b} m_i e^{-\gamma r_{il}} + m_b e^{-\gamma r_{ib}} \tag{7a}$$

$$k_{ij}^{-1} = \sum_{l \neq j,k} m_i e^{-\gamma r_{il}} + m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}} \tag{7b}$$

Since $k_{ij} = k_{ik}$, substituting (7a) and (7b) into (5) gives

$$\epsilon = 1 - \frac{m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}}}{m_b e^{-\gamma r_{ib}}} \frac{k_{ib}}{k_{ij}} \tag{8}$$

Provided that $k_{ib} \simeq k_{ij}$, the expression for ϵ can be reduced to

$$\epsilon = 1 - \frac{m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}}}{m_b e^{-\gamma r_{ib}}} \tag{9}$$

From (7a) and (7a), we see that the assumption $k_{ib} \simeq k_{ij}$ is valid as long as

$$\frac{m_b e^{-\gamma r_{ib}}}{m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}}} \simeq 1. \tag{10}$$

Provided that $r_{ib} \gg r_{jk}$, we can then approximate $r_{ij} \simeq r_{ik} \simeq r_{ib}$. Since we know by definition that $m_b = m_i + m_j$, we can then assume the approximation in (10) holds.

Finally, noting that

$$r_{ij} \simeq r_{ik} \simeq \sqrt{r_{ib}^2 + \left(\frac{r_{jk}}{2}\right)^2} \tag{11}$$

(9) can be rewritten as

$$\epsilon\left(r_{ib}, r_{jk}\right)_e = 1 - e^{-\gamma(r_{ij} - r_{ib})} \tag{12}$$

The same derivation can be followed for the power law form of the deterrence function $f(r_{ij}) = r_{ij}^{-\gamma}$ to obtain

$$\epsilon(r_{ib}, r_{jk})_p = 1 - \frac{m_j r_{ij}^{-\gamma} + m_k r_{ik}^{-\gamma}}{m_b r_{ib}^{-\gamma}} \tag{13}$$