# Derivation of $\epsilon$

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15/01/18

## 1  Error Function

We derive the analytical form of the error $\epsilon$

$$\epsilon = \frac{T_{ib} - (T_{ij} + T_{ik})}{T_{ib}} \tag{1}$$

for the case of the gravity model with an exponential deterrence function:

$$T_{ij} = k_{ij}m_j f(r_{ij}), \tag{2}$$

where

$$k_{ij}^{-1} = \sum_j m_j f(r_{ij}) \tag{3}$$

and

$$f(r_{ij}) = e^{-\gamma r_{ij}} \tag{4}$$

Substituting (2) into (1) gives

$$\epsilon = 1 - \left( \frac{k_{ij}m_j}{e^{\gamma r_{ij}}} + \frac{k_{ik}m_k}{e^{\gamma r_{ik}}} \right) \frac{e^{\gamma r_{ib}}}{k_{ib}m_b} \tag{5}$$

We obtain a continuous form of $k_{ib}$ by approximating the sum to an integral. We then use a polar coordinates system with location $k$ on the x-axis and perform an integral over the circular sector between the x-axis and $2\pi - \delta\theta$. Assuming $\delta\theta$ is small enough, this can approximate the area of interest sufficiently well.

$$
\begin{aligned}
k_{ib}^{-1} &= \int_A e^{-\gamma r_{ib}} dm \\
&= \int_{\theta=0}^{2\pi - \delta\theta} \int_{r_{min}}^{r_{max}} \rho e^{-\gamma r_{ib}} r \, dr \, d\theta \\
&= \rho \frac{2\pi - \delta\theta}{\gamma^2} \left[ e^{-\gamma r_{min}} \left( \gamma r_{min} + 1 \right) - e^{-\gamma r_{max}} \left( \gamma r_{max} + 1 \right) \right]
\end{aligned}
\tag{6}
$$

[Note: we realised we might not need this form of $k$, since we can find an expression for $\epsilon$ by keeping $k$ and decomposing the sum in the following way:]

$$k_{ib}^{-1} = \sum_{l \neq b} m_i e^{-\gamma r_{il}} + m_b e^{-\gamma r_{ib}} \tag{7a}$$

$$k_{ij}^{-1} = \sum_{l \neq j,k} m_i e^{-\gamma r_{il}} + m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}} \tag{7b}$$

Since $k_{ij} = k_{ik}$, substituting (7a) and (7b) into (5) gives

$$\epsilon = 1 - \frac{m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}}}{m_b e^{-\gamma r_{ib}}} \frac{k_{ib}}{k_{ij}} \tag{8}$$

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Provided that $k_{ib} \simeq k_{ij}$, the expression for $\epsilon$ can be reduced to

$$\epsilon = 1 - \frac{m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}}}{m_b e^{-\gamma r_{ib}}} \tag{9}$$

From (7a) and (7a), we see that the assumption $k_{ib} \simeq k_{ij}$ is valid as long as

$$m_b e^{-\gamma r_{ib}} \ll \sum_{l \neq b} m_i e^{-\gamma r_{il}} \tag{10a}$$

and

$$m_j e^{-\gamma r_{ij}} + m_k e^{-\gamma r_{ik}} \ll \sum_{l \neq j,k} m_i e^{-\gamma r_{il}}. \tag{10b}$$

Provided that $N = \sum_j 1$ is sufficiently large, the expressions above hold and we can then assume (9) is a valid approximation.

Finally, noting that

$$r_{ij} \simeq r_{ik} \simeq \sqrt{r_{ib}^2 + \left(\frac{r_{jk}}{2}\right)^2} \tag{11}$$

(9) can be rewritten as

$$\epsilon \left(r_{ib}, r_{jk}\right)_e = 1 - e^{-\gamma(r_{ij} - r_{ib})} \tag{12}$$

The same derivation can be followed for the power law form of the deterrence function $f(r_{ij}) = r_{ij}^{-\gamma}$ to obtain

$$\epsilon(r_{ib}, r_{jk})_p = 1 - \frac{m_j r_{ij}^{-\gamma} + m_k r_{ik}^{-\gamma}}{m_b r_{ib}^{-\gamma}} \tag{13}$$