

# ROBUST DISCRETIZATION OF FLOW IN FRACTURED POROUS MEDIA

WIETSE M. BOON<sup>†</sup> AND JAN M. NORDBOTTEN<sup>†‡</sup>

**Abstract.** Flow in fractured porous media represent a challenge for discretization methods due to the disparate scales and complex geometry. Herein we propose a new discretization, based on the mixed finite element method and mortar methods. Our discretization handles complex, non-matching grids, and allows for fracture intersections and termination in a natural way. The discretization is applicable to both two and three spatial dimensions. Furthermore, we show how to obtain a finite volume variant in terms of cell-center pressures. Analysis shows the method to be first order convergent, which is sustained by the numerical examples.

**Key words.** mixed finite element, mortar finite element, nonconforming grids

**1. Introduction.** Fractures are ubiquitous in natural rocks, and in many cases have a leading order impact on the structure of fluid flow [1, 9]. Due to great differences in permeability, the fractures may either conduct the flow or act as blocking features. Due to their significant impact, detailed and robust modeling of coupled flow between fractures and a permeable rock is essential in applications spanning from enhanced geothermal systems, via CO<sub>2</sub> storage to petroleum extraction.

Because of the complex structure of natural fracture networks [9], it remains a challenge to provide robust and flexible discretization methods. Here, we identify a few distinct features which are attractive from the perspective of applications. The method formulated in this work is specifically designed to meet these goals.

First, we emphasize the importance of mass conserving discretizations. This is of particular significance when the flow field is coupled to transport (of heat, or composition), as transport schemes are typically very sensitive to non-conservative flow fields [14]. The second property of interest is grid flexibility. This is important both in order to accommodate the structure of the fracture network, but also in order to honour other properties of the problem, such as material heterogeneities or anthropogenic features such as wells [15]. Third, we prefer methods which are robust in the physically relevant limits. In the case of fractures, it is imperative to allow for arbitrarily large aspect ratios, that is to say, thin fractures.

A variety of discretizations exist, including upscaled methods and discrete fracture models. Because we aim to capture the flow characteristics influenced by fractures, the discrete fracture models form an interesting range of models on which we will focus.

Since their aspect ratios frequently range as high as 100-1000, it is appealing to consider fractures as lower-dimensional features [22]. In this setting, we consider a three-dimensional domain of permeable rock, within which (multiple) fractures will be represented by (multiple) two-dimensional manifolds. In the case where two or more fractures intersect, we will naturally also be interested in the intersection lines and points. Ultimately, we aim to handle such manifolds, lines, and points in a unified manner.

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<sup>†</sup>Department of Mathematics, University of Bergen, Allégaten 41, 5007 Bergen, Norway (wietse.boon@uib.no, jan.nordbotten@uib.no) This work was supported in part by Norwegian Research Council grants 233736 and 228832, and the second author is currently associated with the Norwegian Academy of Science and Letters through VISTA – a basic research program funded by Statoil.

<sup>‡</sup>Department of Civil and Environmental Engineering, Princeton University, Princeton, NJ 08544, USA.

Several methods have been proposed to discretize fractured porous media, some of which are reviewed below. However, to our knowledge, no method has been presented which fulfills the above properties, and which is amenable to rigorous analysis.

Examples of such methods include finite volume methods constructed for fracture networks (see e.g. [12, 21]). Here, the fractures are added as hybrid cells between the matrix cells. The small cells which are formed at the intersections are then excluded with the use of transformations in order to save condition numbers and computational cost. However, the formulation requires the grids to match along the fractures. Furthermore, while Sandve et al. present a formally consistent method [21], convergence analysis of these methods are lacking.

Alternatively, the extended finite element (XFEM) approach [8, 11] is a method in which the surroundings are meshed independently from the fractures. The fracture meshes are then added afterwards, crossing through the domain and cutting the elements. Consequently, the cut elements may become arbitrarily small such that special constructions are needed to ensure stability. Such constructions need to be introduced whenever multiple fractures, intersections, and fracture endings are considered in the model. Furthermore, the method is not easily extended to three dimensional problems.

Mortar methods, as introduced in [5, 7], form an appealing framework for fracture modeling, since both non-matching grids and intersections are naturally handled. Several methods (e.g. [3, 23, 27]) are based on its structure, yet the idea of conductive fractures was first exploited by Martin et al. [16, 10], where they allow for Darcy flow based on a pressure variable in the mortar space. However, as is common in most mortar methods, the choice of pressure variable in the mortar space does not allow for strong flux continuity. Moreover, the method of Martin et al. also displays oscillations in the case of thin fractures.

Herein we propose a new method, based on the structure of mortar methods. Our formulation is novel in that it uses a mixed formulation in the mortar space, consisting of tangential fluxes and pressure. These are then coupled to the surrounding domain via jumps in traces of the external fluxes which act as mortar variables.

We formulate the method hierarchically, which allows for a unified treatment of the permeable domain, the fractures, intersection lines and intersection points. We show through rigorous analysis that the method is robust independent of the aspect ratio. Furthermore, the mixed formulation ensures that the discretization is conservative and thus we also explicitly derive a finite volume variant of our method using the approach of Baranger et al. [4]. This variant delivers a significant improvement regarding the computational cost.

The paper is organized as follows. First, section 2 introduces the model in a continuous setting and explains the concept of composite function spaces formed by function spaces with different dimensions. Section 3 is devoted to the discretized problem and the analytical proofs of properties such as stability and convergence. Finally, results of numerical experiments confirming the theory are presented in section 4.

**2. Model Formulation.** In this section, we first describe the notion of working with subdomains with different dimensions and introduce the notation used in this paper. Next, the governing equations for the continuous problem are derived and presented. The section is concluded with the derivation of the weak formulation of the problem.

**2.1. Geometric Representation.** We consider an  $n$ -dimensional domain  $\Omega$ , which we decompose into subdomains with different dimensionalities. Setting the

ambient dimension of the problem to  $n$  equal to 2 or 3 will be sufficient for most practical purposes, but the theory allows for  $n$  to be arbitrary. The subdomains of dimension  $n - 1$  then represent fractures, while the lower-dimensional domains represent intersection lines and points.

We start by establishing notation. We denote each  $d$ -dimensional subdomain with  $0 \leq d \leq n$  and index  $i$  by  $\Omega_i^d$ . Let  $N^d$  denote the total number of  $d$ -dimensional subdomains and let the subdomains be indexed such that  $i \in \{1, 2, \dots, N^d\}$ . We define all intersections of  $d$ -dimensional subdomains as  $(d - 1)$ -dimensional features. For example, the point at the intersection between two lines becomes a new, lower-dimensional subdomain.

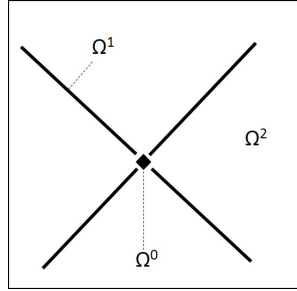


FIGURE 1. The domain is decomposed into subdomains where the dimensionality of each subdomain is given by the superscript. This decomposition allows us to model fractures and intersections as lower-dimensional features in the domain. In the particular illustration above, we have four fracture segments, thus  $N^1 = 4$ .

For notational simplicity, we define the union of all  $d$ -dimensional subdomains as

$$(2.1) \quad \Omega^d = \bigcup_{i=1}^{N^d} \Omega_i^d, \quad d = 0, 1 \dots n.$$

This is illustrated in Figure 1.

Physically, the flow between domains of different dimension (i.e. between fracture and matrix) is of particular importance. We are thus careful with the boundaries between the subdomains of successive dimension. Thus for each subdomain  $\Omega_i^d$  with  $d < n$ , let  $\Gamma_i^d$  be its  $d$  boundary with the neighboring  $(d+1)$ -dimensional domain(s).  $\Gamma_i^d$  coincides spatially with  $\Omega_i^d$ , but its importance lies in being a sub-set of the boundary of the adjacent  $(d+1)$ -dimensional domain(s). Following the same notation as before, we define the following unions as

$$(2.2) \quad \Gamma^d = \bigcup_{i=1}^{N^d} \Gamma_i^d, \quad \Gamma = \bigcup_{d=0}^{n-1} \Gamma^d.$$

For each  $\Gamma_i^d$ , a “left” and “right” side can be chosen. Then, each bordering, higher-dimensional subdomain is assigned to a side. The unit vectors normal to  $\Gamma$  pointing outward from the left and right sides will be denoted by  $\nu_L$  and  $\nu_R$ , respectively. It follows that  $\nu_L = -\nu_R$  and for notational convenience, we furthermore assign a  $\nu$  to  $\Gamma$  such that  $\nu = \nu_L$ .

The boundary of the model domain will enter naturally with the governing equations below. We emphasize that subdomains of any dimensionality may contact the domain boundary.

As a minor comment we note that the geometric representation, as well as much of the analysis below, can be generalized to calculus on manifolds. However, while the framework of manifolds does increase the mathematical elegance, and in some places simplifies and makes the exposition more precise, we believe that the current presentation is accessible to a wider readership. As an immediate consequence of this choice, we will from here on assume that all domains  $\Omega_i^d$  are flat.

**2.2. Governing equations.** The model considered in this work is governed by two physical relationships, namely mass conservation and Darcy's law. In particular, it is assumed that Darcy's law holds not just in the porous material, but also in all lower-dimensional subdomain. This corresponds to the physical situation of either thin open fractures (Poiseuille flow), or fractures filled with some material. Similarly, for fracture intersections. The mathematical representations of these relationships have been well established and employed by several models [2, 3, 8, 10]. Here, we will introduce these relationships within the dimensional decomposition framework. Starting with the governing equations in the surrounding regions, we then continue with their analogues in lower-dimensional subdomains and finish with the coupling equations.

First, let us consider the surroundings  $\Omega^n$ . We aim to find the flux  $\mathbf{u}$  and pressure  $p$  satisfying

$$(2.3a) \quad K^{-1}\mathbf{u} = -\nabla p, \quad \text{in } \Omega^n,$$

$$(2.3b) \quad \nabla \cdot \mathbf{u} = f, \quad \text{in } \Omega^n,$$

$$(2.3c) \quad p = g, \quad \text{on } \partial\Omega_D,$$

$$(2.3d) \quad \mathbf{u} \cdot \nu = 0, \quad \text{on } \partial\Omega_N.$$

where we assume  $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ ,  $\partial\Omega_D \cap \partial\Omega_N = \emptyset$  and  $\partial\Omega_D$  with positive measure.  $K$  is a bounded, symmetric, positive definite,  $n \times n$  tensor representing the material permeability. Equation (2.3a) is known as Darcy's law and equation (2.3b) is conservation of mass in the case of incompressible fluids.

Next, we consider the lower-dimensional subdomains. Here, as the conservation equation is stated on a lower-dimensional surface, the fluxes entering through the boundary  $\Gamma$  must be accounted for [18]. Let  $\lambda$  denote the normal flux  $\mathbf{u} \cdot \nu$  over  $\Gamma$  where  $\nu$  is the unique normal vector associated with  $\Gamma$  as explained in section 2.1.

Mass may enter the fracture from one side and continue tangentially through the fracture creating a (pointwise) difference in normal fluxes. Thus,  $\lambda$  will consist of two components,  $\lambda_L$  and  $\lambda_R$ , which represent the normal trace of the flux on the left and right sides of  $\Gamma$ , respectively. Additionally, to evaluate the differences in normal fluxes, we introduce the jump operator  $[[\cdot]]$  for  $\lambda$  as

$$(2.4) \quad [[\lambda]] = \lambda_R - \lambda_L.$$

In the lower-dimensional domains a scaling is introduced which represents the aperture. For each subdomain  $\Omega_i^d$ , we introduce a characteristic aperture denoted by  $\epsilon_i^d$ . The spatial variability of aperture will be captured by  $K^d$  (see below). For notational convenience we will in general omit the indexes and simply write

$$(2.5) \quad \epsilon = \epsilon_i^d, \quad \text{on } \Omega_i^d \cup \Gamma_i^d.$$

The definitions described by (2.4) and (2.5) allow us to formulate the mass conservation equation analogous to [2, 10].

$$(2.6) \quad [[\lambda]] + \epsilon \nabla \cdot \mathbf{u} = f, \quad \text{in } \Omega^d.$$

Here,  $\mathbf{u}$  and  $f$  represent the flux and source terms within the fracture. Note that when in  $\Omega^d$ , the flux  $\mathbf{u}$  represents the tangential flux, which is  $d$ -dimensional.

Secondly, we consider Darcy's law in  $\Omega^d$  with  $d > 0$  which is described by the relationship

$$(2.7) \quad \mathbf{u} = -K^d \nabla p, \quad \text{in } \Omega^d.$$

Here,  $K^d$  represents the  $d \times d$  tensor of tangential permeability within the fracture. The permeability in the fracture scales with its aperture, thus we set  $K^d = \epsilon K$  for some positive definite permeability tensor  $K$  independent of  $\epsilon$ . Darcy's law is then rewritten to

$$(2.8) \quad K^{-1} \mathbf{u} = -\epsilon \nabla p, \quad \text{in } \Omega^d.$$

Naturally, boundary conditions are necessary for these pressure and flux variables. Due to the assumed, large ratio between the aperture and the fracture, it follows that the mass transfer through the sides of the fracture will be significant compared to the mass transfer through the fracture tips. Therefore, in accordance with [2], we impose a no-flux boundary condition on the boundary of  $\Omega^d$  whenever the fracture ends within the domain.

Alternatively, the fracture may cross the domain and end on the boundary  $\partial\Omega$ . In that case, the imposed boundary condition in  $\Omega^d$  is chosen to coincide with the boundary condition defined for the corresponding portion of  $\partial\Omega$ . In other words, if the fracture ends on  $\partial\Omega_N$ , a no-flux condition is imposed. Alternatively, if it ends on  $\partial\Omega_D$ , the pressure value is set according to  $g$  as given in (2.3c).

We continue with the coupling equation which is given by Darcy's law for flow in the direction normal to the fracture. For this, we introduce the normal permeability  $K_\nu^d$  in  $\Omega^d$ . For generality purposes, we allow it to be scalable with the aperture  $\epsilon$ . Thus, let  $K_\nu^d = \epsilon^\alpha K_\nu$  with  $K_\nu > 0$  and  $\alpha \in [0, 1]$  a parameter indicating the rate of dependency of the permeability on the aperture.

Analogous to [2, 16], Darcy's law for flow across the fracture gives the following relationship between the average normal flux and the pressure difference

$$(2.9) \quad (\lambda_L + \lambda_R) \frac{\epsilon}{2} = -\epsilon^\alpha K_\nu (p_R - p_L).$$

Here,  $p_L$  and  $p_R$  represent traces of the pressure from the surrounding domains on the left and right sides, respectively.

The above equations comprise our model problem for fractured porous media.

**2.3. Weak Formulation.** Let us continue by deriving the weak formulation of the problem. For this, we introduce the function spaces which are constructed using the dimensional decomposition as explained in section 2.1. For each value of  $d$  denoting the dimensionality, let the function space  $\mathbf{V}^d$  contain the (tangential) flux, let  $\Lambda^d$  contain the flux across intersubdomain interfaces, and let  $Q^d$  contain the pressure. For the continuous weak formulation, we define these function spaces as

$$(2.10) \quad \mathbf{V}^d = H(\text{div}; \Omega^d), \quad \Lambda^d = \left( H^{-1/2}(\Gamma^d) \right)^2, \quad Q^d = L^2(\Omega^d).$$

The key tool used to create a succinct method as proposed herein, is to create dimensionally structured function spaces by applying the direct sum over all different

dimensionalities. We thus introduce the composite function spaces

$$(2.11) \quad \mathcal{V} = \bigoplus_{d=1}^n \mathbf{V}^d, \quad \Lambda = \bigoplus_{d=0}^{n-1} \Lambda^d, \quad \mathcal{Q} = \bigoplus_{d=0}^n Q^d.$$

The dimensionally structured space  $\Lambda$  will contain the normal flux over  $\Gamma$  and therewith act as a mortar space. To rigorously impose this boundary condition, we introduce a linear extension operator  $\mathcal{R}$  for functions  $\Lambda$ . Again, this operator is constructed by using the dimensional decomposition. For  $d < n$ , let the operator  $\mathcal{R}^d : \Lambda^d \rightarrow \mathbf{V}^{d+1}$  be defined such that

$$(2.12) \quad \mathcal{R}^d \lambda^d \cdot \nu = \begin{cases} \lambda^d & \text{on } \Gamma^d \\ 0 & \text{on } \partial\Omega, \end{cases}$$

in which  $\nu$  represents the unit normal vector associated with  $\Gamma^d$ . Now, let us define the operator  $\mathcal{R} : \Lambda \rightarrow \mathcal{V}$  as

$$(2.13) \quad \mathcal{R}\lambda = \bigoplus_{d=0}^{n-1} \mathcal{R}^d \lambda^d.$$

REMARK 1. *Existence of  $\mathcal{R}$  is guaranteed since the trace operator from  $H(\text{div}, \Omega)$  to  $H^{-1/2}(\Gamma)$  is surjective [6]. The choice of  $\mathcal{R}$  is described in the discrete setting in Section 3.1. Uniqueness of  $\mathcal{R}$  is not necessary as we shall see in Section 3.2.*

At this point, we have let the normal fluxes across  $\Gamma$  be represented by functions in  $\Lambda$ . To avoid doubly defining these normal fluxes with functions from  $\mathcal{V}$ , we introduce one final function space containing functions with zero normal flux across  $\Gamma$ . Let

$$(2.14) \quad \mathbf{V}_0^d = \{ \mathbf{v} \in \mathbf{V}^d : \mathbf{v} \cdot \nu = 0 \text{ on } \Gamma^{d-1} \cup \partial\Omega_N \}, \quad d = 1, 2, \dots, n,$$

$$(2.15) \quad \mathcal{V}_0 = \bigoplus_{d=1}^n \mathbf{V}_0^d.$$

Now that the function spaces are defined, we continue with the derivation of the weak form of the problem. The derivation is standard for all equations yet the relationship described by equation (2.9) requires some additional attention. As usual, the inner product is taken with a test function  $\mu \in \Lambda$ . After some simple manipulation, this results in

$$(2.16) \quad \epsilon^{1-\alpha} \frac{K_n^{-1}}{2} \langle \lambda, \mu \rangle_\Gamma = \langle p_L - p_R, \mu \rangle_\Gamma + \langle p, \llbracket \mu \rrbracket \rangle_\Gamma,$$

in which we naturally define the inner product  $\langle \lambda, \mu \rangle_\Gamma = \langle \lambda_L, \mu_L \rangle_\Gamma + \langle \lambda_R, \mu_R \rangle_\Gamma$ . This relationship allows us to formulate the weak problem.

The weak solution  $(\mathbf{u}_0, \lambda, p) \in \mathcal{V}_0 \times \Lambda \times \mathcal{Q}$  satisfies

$$(2.17a) \quad (K^{-1}(\mathbf{u}_0 + \mathcal{R}\lambda), \mathbf{v}_0)_\Omega - (\epsilon p, \nabla \cdot \mathbf{v}_0)_\Omega = -\langle g, \mathbf{v}_0 \cdot \nu \rangle_{\partial\Omega_D}, \quad \forall \mathbf{v}_0 \in \mathcal{V}_0,$$

$$(K^{-1}(\mathbf{u}_0 + \mathcal{R}\lambda), \mathcal{R}\mu)_\Omega - (\epsilon p, \nabla \cdot \mathcal{R}\mu)_\Omega +$$

$$(2.17b) \quad \epsilon^{1-\alpha} \frac{K_n^{-1}}{2} \langle \lambda, \mu \rangle_\Gamma - \langle p, \llbracket \mu \rrbracket \rangle_\Gamma = 0, \quad \forall \mu \in \Lambda,$$

$$(2.17c) \quad -(\nabla \cdot (\mathbf{u}_0 + \mathcal{R}\lambda), \epsilon q)_\Omega - \langle \llbracket \lambda \rrbracket, q \rangle_\Gamma = -(f, q)_\Omega, \quad \forall q \in \mathcal{Q}.$$

REMARK 2. In the system (2.17), the function  $p$  in the inner product  $\langle p, \llbracket \mu \rrbracket \rangle_\Gamma$  represents the pressure in the subdomain corresponding to  $\Gamma$ . This should not be confused with the trace of  $p$  from a neighboring subdomain. To avoid such confusions, we only define the inner products between spaces with the same value for  $d$ .

Next, we note that the system (2.17) can be classified as a saddle point problem. For this purpose, we rewrite the problem into a different format by introducing the bilinear operators  $a$  and  $b$  as

$$(2.18) \quad a(\mathbf{u}_0, \lambda, \mathbf{v}_0, \mu) = (K^{-1}(\mathbf{u}_0 + \mathcal{R}\lambda), \mathbf{v}_0 + \mathcal{R}\mu)_\Omega + \epsilon^{1-\alpha} \frac{K_\nu^{-1}}{2} \langle \lambda, \mu \rangle_\Gamma,$$

$$(2.19) \quad b(\mathbf{v}_0, \mu, p) = -(\epsilon p, \nabla \cdot (\mathbf{v}_0 + \mathcal{R}\mu))_\Omega - \langle p, \llbracket \mu \rrbracket \rangle_\Gamma,$$

Additionally, let the functionals  $\mathcal{B}$  and  $F$  be defined as

$$(2.20) \quad \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q)) = a(\mathbf{u}_0, \lambda, \mathbf{v}_0, \mu) + b(\mathbf{v}_0, \mu, p) - b(\mathbf{u}_0, \lambda, q)$$

$$(2.21) \quad F(\mathbf{v}_0, \mu, q) = -\langle g, \mathbf{v}_0 \cdot \nu \rangle_{\partial\Omega^D} + (f, q)_\Omega.$$

Collecting these definitions allows us to rewrite the system (2.17) to the following, equivalent problem:

Find the functions  $(\mathbf{u}_0, \lambda, p) \in \mathcal{V}_0 \times \Lambda \times \mathcal{Q}$  such that

$$(2.22) \quad \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q)) = F(\mathbf{v}_0, \mu, q),$$

for all  $(\mathbf{v}_0, \mu, q) \in \mathcal{V}_0 \times \Lambda \times \mathcal{Q}$ .

**3. Discretization.** In this section, the discretized version of the problem is considered. First, the restrictions on the discrete function spaces are stated. We then continue by showing uniqueness and stability for the discrete problem which leads to the a priori error estimates.

**3.1. Discrete Spaces.** For the discretized equivalent of (2.22), we introduce the discrete function spaces  $\mathcal{V}_h$ ,  $\mathcal{V}_{0,h}$ ,  $\Lambda_h$ , and  $\mathcal{Q}_h$  such that

$$(3.1) \quad \mathcal{V}_h \subset \mathcal{V}, \quad \mathcal{V}_{0,h} \subset \mathcal{V}_0, \quad \Lambda_h \subset \Lambda, \quad \mathcal{Q}_h \subset \mathcal{Q}.$$

In order to derive the analysis shown in section 3.2, we present a total of four conditions on the discrete function spaces. The first two are necessary, while the second two represent attractive features of the numerical method.

First, to ensure that mass is conserved, the space  $\mathcal{V}_h$  is chosen such that the fluxes have normal continuity over internal faces in  $\Omega$ . Furthermore, it is essential that the pair  $(\mathcal{V}_h, \mathcal{Q}_h)$  has the property

$$(3.2) \quad Q_h^d \subseteq \nabla \cdot \mathcal{V}_h^d, \quad d = 1, 2, \dots, n.$$

This can be satisfied by choosing any of the usual mixed finite element methods [3].

Secondly, in accordance with [3, 10], we require that the mortar space should be sufficiently coarse compared to the bordering meshes. This implies the existence of a constant  $C_{\mathcal{R}} > 0$  such that

$$(3.3) \quad \|\mathcal{R}\lambda\|_{\mathcal{V}_h}^2 \geq C_{\mathcal{R}} \|\lambda\|_{\Lambda_h}^2.$$

As a rule of thumb, the mortar grid is required to be a coarsening by two of the trace of either one of the bordering grids [3].

As a third condition, let us consider the space  $\Lambda_h = (\Lambda_{h,L}, \Lambda_{h,R})$  which contains a pair of normal fluxes across  $\Gamma$ . For simplicity, we assume that the function spaces on opposing boundaries are the same, i.e.

$$(3.4) \quad \Lambda_{h,L} = \Lambda_{h,R}.$$

As a fourth and final condition, it is attractive to counteract the coarsening implied by the second condition, since we may wish to have a high fidelity of the solution within the fractures and their intersections: A finer mesh allows us to accurately model the flow inside the fracture. Thus, we let the pressure space  $\mathcal{Q}_h$  be richer than  $\Lambda_h$ . In other words,

$$(3.5) \quad \Lambda_{h,L} \subseteq \mathcal{Q}_h, \quad \Lambda_{h,R} \subseteq \mathcal{Q}_h.$$

This implies that a weaker restriction is placed on the mesh for the pressures in the fractures than in conventional mortar methods.

Next, we introduce the mesh in each of the subdomains. Let  $\mathcal{T}_\Omega^d$  be a finite element partition of  $\Omega^d$  made up of  $d$ -dimensional elements. Furthermore, let  $\mathcal{T}_\Gamma^d$  represent the mortar mesh which means that  $\mathcal{T}_\Gamma^d$  is a sufficiently coarse finite element partition of  $\Gamma^d$  made up of  $d$ -dimensional elements.

With the defined mesh, we can explicitly state a choice of function spaces satisfying all conditions described earlier. Let  $P_k(e)$  be the space of polynomials of degree less than or equal to  $k$  on element  $e$ . Then, after a choice of  $k$ , let the function spaces be defined element by element such that for  $e \in \mathcal{T}_\Omega^d$  and  $e_\Gamma \in \mathcal{T}_\Gamma^d$

$$(3.6) \quad \mathbf{V}_h^d|_e = (P_{k+1}(e))^d, \quad \mathcal{Q}_h^d|_e = P_k(e), \quad \Lambda_h^d|_{e_\Gamma} = P_k(e_\Gamma).$$

The dimensionally structured spaces  $\mathcal{V}_h$ ,  $\Lambda_h$ , and  $\mathcal{Q}_h$  can now be constructed by applying the direct sum over the dimensionalities analogous to (2.11). To ensure mass conservation, normal continuity of the functions in  $\mathcal{V}_h$  is enforced over the internal faces of  $\mathcal{T}_\Omega$ . The choice of discrete function spaces is concluded by choosing  $\mathcal{V}_{0,h}$  as the subset of  $\mathcal{V}_h$  with zero normal flux on  $\Gamma$ .

Finally, we construct the extension operator  $\mathcal{R}$  for a given function  $\lambda$ . First, an  $L^2$ -projection is performed from  $\Gamma$  onto the trace of the bordering mesh. The function  $\mathcal{R}\lambda \in \mathcal{V}_h$  is then created such that its normal trace is equal to the projection in accordance with (2.12). The choice is finalized by choosing the unique  $\mathcal{R}\lambda$  with a support consisting only of the elements bordering on the fractures.

To finish the section, we explicitly state choice of function spaces or lowest order. When choosing  $k = 0$ , the following choice of spaces satisfies all conditions on simplicial elements

$$(3.7) \quad \mathcal{V}_h = \mathbb{RT}_0(\mathcal{T}_\Omega), \quad \mathcal{Q}_h = \mathbb{P}_0(\mathcal{T}_\Omega), \quad \Lambda_h = \mathbb{P}_0(\mathcal{T}_\Gamma).$$

Here,  $\mathbb{RT}_0$  represents the lowest order Raviart-Thomas(-Nedelec) space [20, 17] and  $\mathbb{P}_0$  is the space of piecewise constants. As is required, we choose  $\mathcal{V}_{0,h} \subset \mathbb{RT}_0(\mathcal{T}_\Omega)$  such that the degrees of freedom for the normal fluxes on  $\Gamma$  are zero.

At this point, the discrete spaces are chosen and the finite element problem associated with (2.22) can be formulated as follows:

Find  $(\mathbf{u}_0, \lambda, p) \in \mathcal{V}_{0,h} \times \Lambda_h \times \mathcal{Q}_h$  such that

$$(3.8) \quad \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q)) = F(\mathbf{v}_0, \mu, q),$$

for all  $(\mathbf{v}_0, \mu, q) \in \mathcal{V}_{0,h} \times \Lambda_h \times \mathcal{Q}_h$ .



**3.2. A Priori Estimates.** With the choice of discrete function spaces and the formulation of the finite element problem in section 3.1, we continue with the stability study. For this purpose, let us define the following norms

$$(3.9) \quad \|\mathbf{v}\|_{\mathcal{V}_h}^2 = \|K^{-1/2}\mathbf{v}\|_{L^2(\Omega)}^2 + \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)}^2, \quad \|q\|_{\mathcal{Q}_h}^2 = \|q\|_{L^2(\Omega)}^2,$$

$$(3.10) \quad \|\lambda\|_{\Lambda_h}^2 = \frac{1}{2} \|K_\nu^{-1/2}\lambda\|_{L^2(\Gamma)}^2.$$

In addition to these norms, we will require a norm which evaluates a pressure function  $q \in \mathcal{Q}_h$  on the coarser mortar mesh. Again, the dimensional decomposition is used to construct the necessary projection operator. Let  $\Pi^d : Q_h^d \rightarrow \Lambda_h^d$  be an  $L^2$ -projection for  $0 \leq d < n$ . The functional  $\Pi$  and the norm  $\|\cdot\|_\Pi$  are then defined as

$$(3.11) \quad \Pi q = \bigoplus_{d=0}^{n-1} \Pi^d q, \quad \|q\|_\Pi = \|\Pi q\|_{L^2(\Gamma)}.$$

Now, let energy norm be formed by combining the norms defined by (3.9), (3.10), and (3.11) as follows

$$(3.12) \quad \|(\mathbf{u}_0, \lambda, p)\|^2 = \|\mathbf{u}_0 + \mathcal{R}\lambda\|_{\mathcal{V}_h}^2 + \|\lambda\|_{\Lambda_h}^2 + \|\epsilon p\|_{\mathcal{Q}_h}^2 + \|p\|_\Pi^2.$$

In what follows next, the energy norm will be used to prove two lemmas which are necessary for the main result. First, we derive the boundedness of the form  $\mathcal{B}$  and second, we show an inf-sup condition on the functional  $b$ .

LEMMA 3.1. (*Boundedness*)  $\mathcal{B}$  is bounded with respect to the energy norm (3.12), i.e. a constant  $C_1 > 0$  exists such that

$$(3.13) \quad \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q)) \leq C_1 \|(\mathbf{u}_0, \lambda, p)\| \cdot \|(\mathbf{v}_0, \mu, q)\|.$$

*Proof.* The main term that requires consideration for this proof is the coupling term between  $\mathcal{Q}_h$  and  $\Lambda_h$ . The boundedness with respect to the relevant norms is shown using the Cauchy-Schwartz inequality.

$$(3.14) \quad \langle \llbracket \lambda \rrbracket, q \rangle_\Gamma \leq |\langle \llbracket \lambda \rrbracket, q \rangle_\Gamma| \leq \langle |\lambda|, |q| \rangle_\Gamma = \langle |\lambda|, \Pi |q| \rangle_\Gamma \leq \sqrt{2K_\nu} \|\lambda\|_{\Lambda_h} \|q\|_\Pi.$$

Next, we apply the Cauchy-Schwartz inequality to each of the remaining inner products in the form  $\mathcal{B}$ . Together with inequality (3.14), this yields

$$(3.15) \quad \begin{aligned} \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q)) &\leq C_1 (\|\mathbf{u}_0 + \mathcal{R}\lambda\|_{\mathcal{V}_h}^2 + \|\lambda\|_{\Lambda_h}^2 + \|\epsilon p\|_{\mathcal{Q}_h}^2 + \|p\|_\Pi^2)^{1/2} \\ &\quad \cdot (\|\mathbf{v}_0 + \mathcal{R}\mu\|_{\mathcal{V}_h}^2 + \|\mu\|_{\Lambda_h}^2 + \|\epsilon q\|_{\mathcal{Q}_h}^2 + \|q\|_\Pi^2)^{1/2} \\ &= C_1 \|(\mathbf{u}_0, \lambda, p)\| \cdot \|(\mathbf{v}_0, \mu, q)\|. \end{aligned}$$

which shows that  $\mathcal{B}$  is bounded with respect to the energy norm, as desired.  $\square$

We continue with the second lemma which concerns an inf-sup condition on the coupling term  $b$ .

LEMMA 3.2. Let the bilinear form  $b$  be defined by equation (2.19). Then there exists a constant  $C_2 > 0$  such that for any given function  $p \in \mathcal{Q}_h$ ,

$$(3.16) \quad \sup_{\substack{\mathbf{v}_0 \in \mathcal{V}_{0,h} \\ \mu \in \Lambda_h}} \frac{b(\mathbf{v}_0, \mu, p)}{\|(\mathbf{v}_0, \mu, 0)\|} \geq C_2 \|(\mathbf{0}, 0, p)\|.$$

*Proof.* First, we recall the definitions of  $b$  and the energy norm and substitute them as follows

$$(3.17) \quad \begin{aligned} \sup_{\substack{\mathbf{v}_0 \in \mathcal{V}_{0,h} \\ \mu \in \Lambda_h}} \frac{b(\mathbf{v}_0, \mu, p)}{\|(\mathbf{v}_0, \mu, 0)\|} &= \sup_{\substack{\mathbf{v}_0 \in \mathcal{V}_{0,h} \\ \mu \in \Lambda_h}} \frac{(\epsilon p, \nabla \cdot (\mathbf{v}_0 + \mathcal{R}\mu))_\Omega + \langle p, \llbracket \mu \rrbracket \rangle_\Gamma}{(\|\mathbf{v}_0 + \mathcal{R}\mu\|_{\mathcal{V}_h}^2 + \|\mu\|_{\Lambda_h}^2)^{1/2}} \\ &\geq \sup_{\substack{\mathbf{v}_0 \in \mathcal{V}_{0,h} \\ \mu \in \Lambda_h}} \frac{(\epsilon p, \nabla \cdot (\mathbf{v}_0 + \mathcal{R}\mu))_\Omega + \langle p, \llbracket \mu \rrbracket \rangle_\Gamma}{\|\mathbf{v}_0 + \mathcal{R}\mu\|_{\mathcal{V}_h} + \|\mu\|_{\Lambda_h}}. \end{aligned}$$

Note that one of the conditions was that the spaces  $\mathcal{Q}_h$  and  $\mathcal{V}_h$  are chosen such that they satisfy (3.2). From this, it follows that a constant  $C_{2,u} > 0$  exists such that for every  $p \in \mathcal{Q}_h$

$$(3.18) \quad \sup_{0 \neq \mathbf{v} \in \mathcal{V}_h} \frac{(p, \nabla \cdot \mathbf{v})_\Omega}{\|\mathbf{v}\|_{\mathcal{V}_h}} \geq C_{2,u} \|p\|_{\mathcal{Q}_h}.$$

This shows that the lemma holds for  $\epsilon > 0$ . On the other hand, if  $\epsilon = 0$ , the remaining coupling term is used to control  $p$ . Therefore, we require the existence of  $C_{2,\lambda} > 0$  such that for every  $p \in \mathcal{Q}_h$

$$(3.19) \quad \sup_{0 \neq \mu \in \Lambda_h} \frac{\langle p, \llbracket \mu \rrbracket \rangle_\Gamma}{\|\mu\|_{\Lambda_h}} \geq C_{2,\lambda} \|p\|_\Pi.$$

Proving the existence of this constant is relatively straightforward, as shown by the following

$$(3.20) \quad \sup_{0 \neq \mu \in \Lambda_h} \frac{\langle p, \llbracket \mu \rrbracket \rangle_\Gamma}{\|\mu\|_{\Lambda_h}} = \sup_{0 \neq \mu \in \Lambda_h} \frac{\langle p, \mu \rangle_\Gamma}{\|\mu\|_{\Lambda_h}} = \sqrt{2K_\nu} \|p\|_\Pi.$$

Combining (3.18) and (3.19) in inequality (3.17), we conclude that a  $C_2 > 0$  exists such that

$$(3.21) \quad \begin{aligned} \sup_{\substack{\mathbf{v}_0 \in \mathcal{V}_{0,h} \\ \mu \in \Lambda_h}} \frac{b(\mathbf{v}_0, \mu, p)}{\|(\mathbf{v}_0, \mu, 0)\|} &\geq C_2 (\| \epsilon p \|_{\mathcal{Q}_h} + \|p\|_\Pi) \\ &\geq C_2 (\| \epsilon p \|_{\mathcal{Q}_h}^2 + \|p\|_\Pi^2)^{1/2} \\ &= C_2 \|(\mathbf{v}_0, \mu, p)\|. \quad \square \end{aligned}$$

At this point, the lemmas 3.1 and 3.2 provide us with sufficient basis to prove the main result given by the following theorem.

**THEOREM 3.3 (Inf-Sup).** *Let  $\mathcal{B}$  be defined as in section 2.3 and let the function spaces  $\mathcal{V}_{0,h}$ ,  $\Lambda_h$ , and  $\mathcal{Q}_h$  be chosen such that they comply with the restrictions from section 3.1. Then a constant  $C > 0$  exists such that*

$$(3.22) \quad \inf_{\substack{\mathbf{u}_0 \in \mathcal{V}_{0,h} \\ \lambda \in \Lambda_h \\ p \in \mathcal{Q}_h}} \sup_{\substack{\mathbf{v}_0 \in \mathcal{V}_{0,h} \\ \mu \in \Lambda_h \\ q \in \mathcal{Q}_h}} \frac{\mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q))}{\|(\mathbf{u}_0, \lambda, p)\| \|(\mathbf{v}_0, \mu, q)\|} \geq C.$$

*Proof.* First, we introduce the functions  $\mathbf{w}_0 \in \mathcal{V}_{0,h}$  and  $\lambda_w \in \Lambda_h$  as functions for which the supremum of (3.16) is attained. Since we can scale these functions to our liking, we choose  $\mathbf{w}_0$  and  $\lambda_w$  to have the following norms.

$$(3.23) \quad \|\mathbf{w}_0 + \mathcal{R}\lambda_w\|_{\mathcal{V}_h} = \| \epsilon p \|_{\mathcal{Q}_h}, \quad \|\lambda_w\|_{\Lambda_h} = \|p\|_\Pi.$$

This choice of functions allows us to substitute the corresponding energy norm with an energy norm containing the pressure  $p$ .

$$(3.24) \quad \|\!(\mathbf{w}_0, \lambda_w, 0)\!\|^2 = \|\!(0, 0, p)\!\|^2.$$

The next step is to substitute the chosen test functions in  $\mathcal{B}$ . By using the linearity of  $\mathcal{B}$ , the boundedness result from Lemma 3.1, and the inf-sup condition from Lemma 3.2, we derive

$$\begin{aligned} \mathcal{B}((\mathbf{u}_0, \lambda, p); (-\mathbf{w}_0, -\lambda_w, 0)) &= \mathcal{B}((\mathbf{u}_0, \lambda, 0); (-\mathbf{w}_0, -\lambda_w, 0)) + \mathcal{B}((0, 0, p); (-\mathbf{w}_0, -\lambda_w, 0)) \\ &\geq -C_1 \|\!(\mathbf{u}_0, \lambda, 0)\!\| \cdot \|\!(\mathbf{w}_0, \lambda_w, 0)\!\| + C_2 \|\!(0, 0, p)\!\| \cdot \|\!(\mathbf{w}_0, \lambda_w, 0)\!\| \\ &= -C_1 \|\!(\mathbf{u}_0, \lambda, 0)\!\| \cdot \|\!(0, 0, p)\!\| + C_2 \|\!(0, 0, p)\!\|^2. \end{aligned}$$

We continue by applying the inequality of arithmetic and geometric means

$$\begin{aligned} \mathcal{B}((\mathbf{u}_0, \lambda, p); (-\mathbf{w}_0, -\lambda_w, 0)) &\geq -\frac{1}{\gamma} C_1 \|\!(\mathbf{u}_0, \lambda, 0)\!\|^2 + (C_2 - \gamma C_1) \|\!(0, 0, p)\!\|^2 \\ (3.25) \quad &= -C_3 \|\!(\mathbf{u}_0, \lambda, 0)\!\|^2 + C_4 \|\!(0, 0, p)\!\|^2. \end{aligned}$$

By choosing  $0 < \gamma < C_2 C_1^{-1}$ , we ensure that  $C_3$  is bounded and  $C_4$  is positive. This concludes one of two inequalities required later in the proof. The second inequality is derived by considering the following

$$\begin{aligned} \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{u}_0, \lambda, p)) &= a(\mathbf{u}_0, \lambda, \mathbf{u}_0, \lambda) \\ &= \|\mathbf{u}_0 + \mathcal{R}\lambda\|_{\mathcal{V}_h}^2 + \|\epsilon^{\frac{1}{2}(1-\alpha)} \lambda\|_{A_h}^2 \\ (3.26) \quad &\geq \|\mathbf{u}_0 + \mathcal{R}\lambda\|_{\mathcal{V}_h}^2. \end{aligned}$$

Next, we note that the functions  $\mathbf{u}_0$  and  $\mathcal{R}\lambda$  have  $H(\text{div})$  regularity. Furthermore,  $\mathbf{u}_0$  has zero normal trace on  $\Gamma$  which implies that the  $\mathcal{V}_h$ -norm of  $\mathbf{u}_0 + \mathcal{R}\lambda$  can be bounded from below by the sum of the  $\mathcal{V}_h$ -norms of the two individual functions. Moreover,  $\|\mathcal{R}\lambda\|$  is bounded from below by  $C_{\mathcal{R}}\|\lambda\|$  as stated by (3.3). This allows us to obtain the second inequality.

$$(3.27) \quad \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{u}_0, \lambda, p)) \geq C_5 (\|\mathbf{u}_0 + \mathcal{R}\lambda\|_{\mathcal{V}_h}^2 + C_{\mathcal{R}} \|\lambda\|_{A_h}^2) \geq C_6 \|\!(\mathbf{u}_0, \lambda, 0)\!\|^2.$$

In the next step, we choose the test functions  $(\mathbf{v}_0, \mu, q)$  in such a way that the inequalities (3.25) and (3.27) can be exploited. Let  $\delta \in \mathbb{R}$  and

$$(3.28) \quad (\mathbf{v}_0, \mu, q) = (\mathbf{u}_0 - \delta \mathbf{w}_0, \lambda - \delta \lambda_w, p).$$

These test functions  $(\mathbf{v}_0, \mu, q)$  are then substituted in  $\mathcal{B}$ . With the of inequalities (3.25) and (3.27), this yields

$$\begin{aligned} \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q)) &= \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{u}_0, \lambda, p)) + \mathcal{B}((\mathbf{u}_0, \lambda, p); (-\delta \mathbf{w}_0, -\delta \lambda_w, 0)) \\ (3.29) \quad &\geq (C_6 - \delta C_3) \|\!(\mathbf{u}_0, \lambda, 0)\!\|^2 + \delta C_4 \|\!(0, 0, p)\!\|^2. \end{aligned}$$

The choice of  $0 < \delta < C_6 C_3^{-1}$  lets us obtain a lower bound for the numerator in equation (3.22) in terms of the energy norm.

$$\begin{aligned} \mathcal{B}((\mathbf{u}_0, \lambda, p); (\mathbf{v}_0, \mu, q)) &\geq C_7 (\|\!(\mathbf{u}_0, \lambda, 0)\!\|^2 + \|\!(0, 0, p)\!\|^2) \\ (3.30) \quad &= C_7 \|\!(\mathbf{u}_0, \lambda, p)\!\|^2. \end{aligned}$$

On the other hand, an upper bound for the denominator in equation (3.22) can be found by simply considering the energy norm of the test functions from (3.28)

$$\begin{aligned} \|\!(\mathbf{v}_0, \mu, q)\!\|^2 &\leq \|\!(\mathbf{u}_0, \lambda, p)\!\|^2 + \delta^2 \|\!(\mathbf{w}_0, \lambda_w, 0)\!\|^2 \\ (3.31) \quad &\leq (1 + \delta^2) \|\!(\mathbf{u}_0, \lambda, p)\!\|^2. \end{aligned}$$

Together, (3.30) and (3.31) give us the desired result.  $\square$

With inf-sup condition from theorem 3.3, we continue with the basic error estimates. The true solution, i.e. the unique solution to (2.22), will be denoted by  $(\mathbf{u}_0, \lambda, p)$  and the finite element solution will be called  $(\mathbf{u}_{0,h}, \lambda_h, p_h)$ . The approximation errors are then defined as

$$(3.32) \quad E_u = \inf_{\substack{\mathbf{v}_{0,h} \in \mathcal{V}_{0,h} \\ \mu_h \in \Lambda_h}} \|\mathbf{u} - (\mathbf{v}_{0,h} + \mathcal{R}\mu_h)\|_{\mathcal{V}},$$

$$(3.33) \quad E_\lambda = \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda, \quad E_p = \inf_{q_h \in \mathcal{Q}_h} \|p - q_h\|_{\mathcal{Q}}.$$

Note that the approximation error  $E_u$  differs from the conventional error due to the decomposition in  $\mathbf{u}_{0,h}$  and  $\lambda_h$ . With this minor change, the analysis as described in [6] can be followed using Theorem 3.3. This immediately results in the following basic error estimates

$$(3.34) \quad \|\mathbf{u} - (\mathbf{u}_{0,h} + \mathcal{R}\lambda_h)\|_{\mathcal{V}} + \|\lambda - \lambda_h\|_\Lambda \leq C(E_u + E_\lambda),$$

$$(3.35) \quad \|p - p_h\|_{\mathcal{Q}} \leq C(E_u + E_\lambda + E_p).$$

Since the upper bound is formed by the infima over the corresponding spaces, we can choose any functions from  $\mathcal{V}_{0,h} \times \Lambda_h \times \mathcal{Q}_h$  and use the inequalities (3.34) and (3.35). The key is to use the solution to the continuous problem and project it to the discrete spaces. For this, we introduce the necessary three projection operators.

Let  $\mathcal{P}^{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}_h$  be the standard interpolation operator for  $\mathcal{V}_h$ . Additionally, let  $\mathcal{P}^\Lambda : \Lambda \rightarrow \Lambda_h$  and  $\mathcal{P}^{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathcal{Q}_h$  be  $L^2$ -projection operators to the corresponding discrete spaces.

Let  $k$  represent the order of the polynomials in the pressure space. It is well known [6, 3] that the following interpolation estimates hold for the operators  $\mathcal{P}^{\mathcal{V}}$ ,  $\mathcal{P}^\Lambda$ , and  $\mathcal{P}^{\mathcal{Q}}$  and a chosen value of  $k$ .

$$(3.36a) \quad \|\mathbf{u} - \mathcal{P}^{\mathcal{V}}\mathbf{u}\|_{0,\Omega} \leq C\|\mathbf{u}\|_{r,\Omega} h^r, \quad 1 \leq r \leq k+1,$$

$$(3.36b) \quad \|\nabla \cdot (\mathbf{u} - \mathcal{P}^{\mathcal{V}}\mathbf{u})\|_{0,\Omega} \leq C\|\nabla \cdot \mathbf{u}\|_{r,\Omega} h^r, \quad 0 \leq r \leq k+1,$$

$$(3.36c) \quad \|\lambda - \mathcal{P}^\Lambda\lambda\|_{-s,\Gamma} \leq C\|\lambda\|_{r,\Gamma} h^{r+s}, \quad 1 \leq r \leq k+2, \quad 0 \leq s \leq k+2,$$

$$(3.36d) \quad \|p - \mathcal{P}^{\mathcal{Q}}p\|_{0,\Omega} \leq C\|p\|_{r,\Omega} h^r, \quad 1 \leq r \leq k+1.$$

Here,  $\|\cdot\|_{r,\Sigma}$  represents the  $H^r(\Sigma)$ -norm. From (3.34) and (3.36a)–(3.36c), we obtain

$$\begin{aligned} \|\mathbf{u} - (\mathbf{u}_{0,h} + \mathcal{R}\lambda_h)\|_{\mathcal{V}} + \|\lambda - \lambda_h\|_\Lambda &\leq C(\|\mathbf{u} - \mathcal{P}^{\mathcal{V}}\mathbf{u}\|_{\mathcal{V}} + \|\lambda - \mathcal{P}^\Lambda\lambda\|_\Lambda) \\ (3.37) \quad &\leq Ch^{k+1}(\|\mathbf{u}\|_{k+1,\Omega} + \|\nabla \cdot \mathbf{u}\|_{k+1,\Omega} + \|\lambda\|_{k+1,\Gamma}) \end{aligned}$$

Similarly, the estimate for the pressure is formulated using (3.35) and (3.36d)

$$(3.38) \quad \|p - p_h\|_{\mathcal{Q}} \leq Ch^{k+1}(\|\mathbf{u}\|_{k+1,\Omega} + \|\nabla \cdot \mathbf{u}\|_{k+1,\Omega} + \|\lambda\|_{k+1,\Gamma} + \|p\|_{k+1,\Omega})$$

The lowest order choice described by (3.7) corresponds to choosing  $k = 0$ . In that case, we have  $\mathcal{O}(h)$  convergence in all variables wherever the solution is sufficiently regular.

**3.3. Finite Volume Variant.** By construction, the method allows for the derivation of a finite volume variant. Having this possibility is a noteworthy advantage of this method since finite volume methods may be preferred in certain applications for several reasons. First, due to the fewer degrees of freedom and resulting definite system, finite volume methods are sometimes computationally favourable [19]. Furthermore, the finite volume structure allows for the incorporation of upstream weighting, which is important in order to capture accurately flow and transport phenomena for coupled problems (see e.g. [13]).

There are several ways of generating finite volume methods which vary greatly in complexity. Comprehensive analysis of these methods is outside the scope of the current paper. In particular, we note the possibility of using the framework of multipoint flux mixed finite element methods [25, 26], which may be advantageous for problems with anisotropy in the permeability coefficients, as well as for more complex grids (e.g. quadrilateral grids).

As an example of how to create finite volume variants of our scheme, we consider the technique described by [4, 6] in which a finite volume method is generated with the use of mass lumping. During this procedure, the form  $a$  is replaced with a simpler form denoted by  $a_L$  which allows the elimination of the flux variables.

In order to describe the mass lumping procedure, we first introduce a few necessary definitions. For each element  $e \in \Omega^d$ , let  $\mathbf{C}_e$  be defined as its circumcenter. In other words,  $\mathbf{C}_e$  is the center of the unique  $d$ -sphere which passes through the vertices of  $e$ . Even though the following procedure will be valid if  $\mathbf{C}_e$  is outside the element, it seems unnatural to define the center of an element on its exterior. Such situations are avoided by only using meshes in which the centers are inside the corresponding elements. In 2D, for example, this implies that we only consider acute triangles.

We denote the faces of  $e \in \Omega^d$  by  $T_i^e$ ,  $i \in \{1, 2, \dots, d+1\}$ . The key is then to divide each element into  $d+1$  subelements such that each subelement is a  $d$ -simplex formed by the circumcenter  $\mathbf{C}_e$  and a face  $T_i^e$ . If we then consider the entire mesh, we note that each face  $T_k$  (with  $k = 1, 2, \dots, N_T$  and  $N_T$  the total number of faces) borders on either one or two of such subelements. The patch  $L_k$  is then defined as the union of all subelements bordering on  $T_k$ .

Next, we introduce the function space  $\mathcal{V}_L$  consisting of particular constant functions on the patches  $L_k$ .

$$(3.39) \quad \mathcal{V}_L = \{\mathbf{v}_h : \mathbf{v}_h|_{L_k} = c \nu_k \text{ with } c \in \mathbb{R}, k = 1, 2, \dots, N_T\},$$

in which  $\nu_k$  denotes the normal vector associated with face  $T_k$ . With this space, let the projection operator  $\mathcal{P}^L : \mathbb{RT}_0 \rightarrow \mathcal{V}_L$  be defined such that

$$(3.40) \quad (\mathcal{P}^L \mathbf{v}_h)|_{L_k} = (\mathbf{v}_h \cdot \nu_k)|_{T_k} \nu_k, \quad k = 1, 2, \dots, N_T.$$

We are now ready to replace the form  $a$  by the simpler form  $a_L$  in which the flux variables are first projected to the newly introduced space  $\mathcal{V}_L$ . To be more precise, with the lowest order choice of  $(\mathcal{V}_{0,h}, \Lambda_h, \mathcal{Q}_h)$  described by (3.7), let  $a_L$  be defined as

$$(3.41) \quad a_L(\mathbf{u}_0, \lambda, \mathbf{v}_0, \mu) = (K^{-1} \mathcal{P}^L(\mathbf{u}_0 + \mathcal{R}\lambda), \mathcal{P}^L(\mathbf{v}_0 + \mathcal{R}\mu))_\Omega + \epsilon^{1-\alpha} \frac{K_\nu^{-1}}{2} \langle \lambda, \mu \rangle_\Gamma.$$

Next, we exploit the disjointness of the created patches in the associated matrix  $A_L$ . Note that  $A_L$  will not be completely diagonal due to the possible mismatch between the coarse mortar grid and the surrounding grids. However, it will be block

diagonal in which each block corresponds to a side of a fracture. Therefore  $A_L$  can be inverted locally and this enables the elimination of the flux variables  $\mathbf{u}_0$  and  $\lambda$  creating a finite volume scheme consisting of only pressure variables.

To construct the matrix  $A_L$ , we simply integrate over each patch  $L_k$  and use the properties of  $\mathcal{V}_L$  to derive

$$(3.42) \quad A_{L,kk} = \nu_k^T \int_{L_k} K^{-1} dx \nu_k,$$

in which  $\nu_k$  is once again the normal vector associated with  $T_k$ .

It is known that the linear systems arising from this can be resolved efficiently with direct solvers due to symmetry and positive definiteness [24]. Furthermore, it has been proven [4] that the replacement of  $a$  with  $a_L$  introduces a consistency error of the same order as the approximation error in  $\mathbb{RT}_0$ . Therefore, the a priori estimates from Section 3.2 are maintained.

**4. Numerical Results.** To confirm the theory derived in the previous sections, we show the numerical results using a test case designed to highlight some of the typical challenges associated with fracture simulation. First, we will introduce the considered domain and describe its features. The different variants and the parameters therein are discussed afterwards. The section is concluded with an evaluation of the results.

**4.1. Setup.** The domain  $\Omega$  is the unit square. A unit pressure drop is simulated by imposing a Dirichlet boundary condition for the pressure at the top and bottom boundaries of  $\Omega$ . On the remaining sides, a no-flow boundary condition is imposed. For simplicity, the source function  $f$  is set to zero.

Multiple fractures with different properties are added to  $\Omega$  to show the extent of the capabilities of the method. Figure 2 gives an illustration of these fractures. First and foremost, the domain contains a fracture intersection. Modeling intersections forms a challenge for a variety of methods including XFEM [8, 11] and finite volume methods [21]. However, for the method we present, the implementation of an intersection follows naturally due to the dimensional decomposition. The test case contains only one intersection yet this can easily be extended to multiple intersections.

In addition to the intersections, a second aspect is the termination of fractures. The method is designed to handle these endings well, which is investigated by introducing immersed as well as half immersed fractures and fractures crossing through the boundary as shown in Figure 2. In existing methods, the modeling of fracture endings is typically avoided by creating virtual extensions from the fracture towards the boundary of the domain. The method presented here does not require such virtual extensions and handles these fracture endings naturally.

Creating a virtual extension or a completely virtual fracture may be desired for computational reasons. Due to the close relation to mortar methods, a virtual fracture can serve as an interface between two subdomains with non-matching grids, or for the purpose of creating a domain decomposition method. By letting the aperture  $\epsilon$  decrease to zero, tangential flow will not occur. The method then simplifies to a standard mortar with the normal flux as the mortar variable [5]. While we do not exploit these computational aspects here, we emphasize the flexibility by including a virtual interface, as is represented by the dashed line in Figure 2.

Furthermore, since the construction of the method is based on mortar methods, it is expected to handle non-matching grids well. Therefore, we have not enforced matching nodes in the mesh generation.

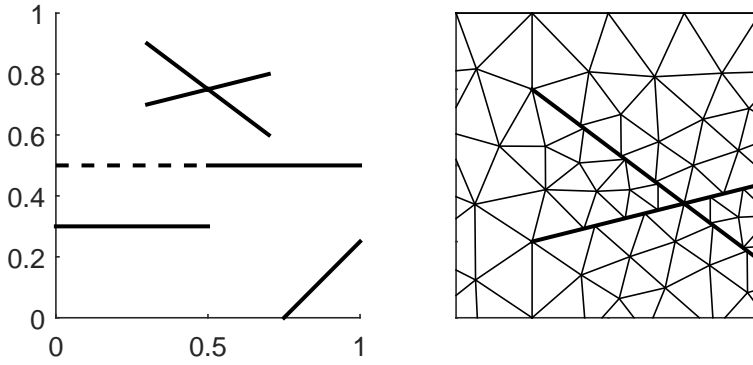


FIGURE 2. (Left) The domain contains an intersection, multiple fracture endings, a fracture passing through the domain and a virtual extension of a fracture represented by the dashed line. (Right) The grid is non-matching along all (virtual) fractures.

Let us continue by defining the parameters for this test case. We assume isotropic permeability in  $\Omega^2$  which is implemented by setting  $K$  as the  $2 \times 2$  identity tensor. Fracture flow is stimulated by setting  $K = 10^4$  in  $\Omega^1$ . Additionally, we set the aperture  $\epsilon = 10^{-2}$  in  $\Omega^1$  and  $\epsilon = 10^{-4}$  in the intersection  $\Omega^0$ . By definition,  $\epsilon = 0$  is imposed in the virtual feature and  $\epsilon = 1$  in  $\Omega^2$ .

**4.2. Channel Flow and Blocking Features.** Two cases are considered by varying the two remaining parameters, namely the normal permeability  $K_\nu$  and the rate of dependency of  $K_\nu$  on the aperture,  $\alpha$ . The values corresponding to each case are given by Table 1.

TABLE 1

The parameters used in each case. Case 1 represents stimulated flow through the fractures whereas Case 2 changes the fractures to blocking features.

	$K_\nu$	$\alpha$
Case 1	1	0
Case 2	$\epsilon^1$	0

Case 1 represents the situation in which the fractures act as channels for the flow. The permeability normal to the fractures is equal to the surrounding permeability so that the flow will easily enter and leave the lower-dimensional domains. The results are shown in Figure 3, where a more gradual pressure distribution can be seen inside the fractures compared to the surroundings. This is due to the increased permeability there, allowing for more flow. As expected, the results show no oscillations and the fracture endings and intersection form no challenge for the method.

In Case 2, we lower the normal permeability significantly to the order of magnitude of the aperture size  $\epsilon^1$ . This restricts the flow in and out of the fractures which makes them act as blocking features. As a result, pressure discontinuities are formed and the flow is forced around the fractures. Both the pressure discontinuities and the flow's reaction to the lowered permeability are clearly shown by the solution.

Again, no oscillations occur and the results show various qualitatively sound prop-

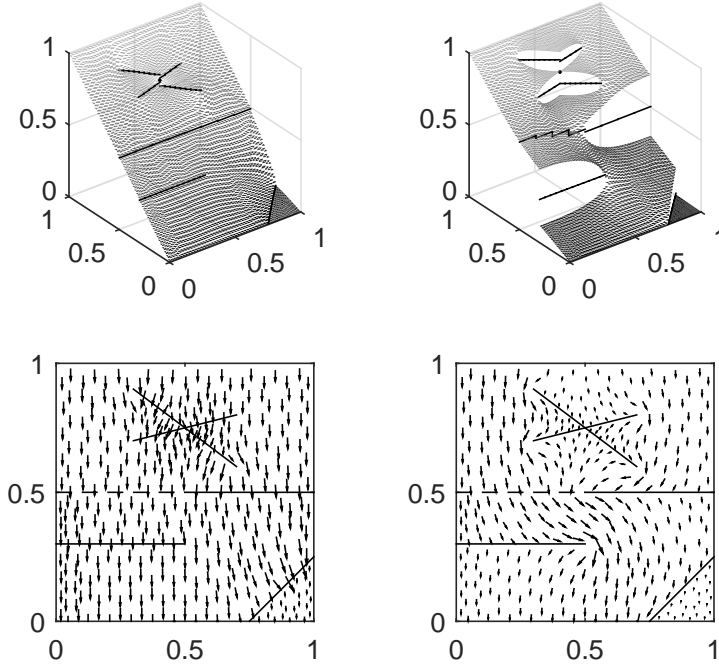


FIGURE 3. (Top) The pressure distributions for the two considered cases given by the full method. Case 1 (left) gives a more regular solution while Case 2 (right) shows the expected pressure discontinuity due to the blocking fractures. (Bottom) The fluxes clearly react to the different properties of the features.

erties. First of all, the virtual extension acts as a typical mortar space, enforcing continuity of the solution without introducing flow inside the feature. This can be seen by the larger pressure differences inside the fracture compared to the other fractures. Secondly, the fracture pressures are shown to be a weighted average of the traces of the surrounding pressures in accordance with the theory. This also holds for the pressure in the intersection which is represented by the dot placed there. Finally, no direct influence of the non-matching grids can be seen in the solution which is n

Additionally, we can alter the parameter  $\alpha$ . By setting this to one, the normal permeability will scale linearly with the aperture. Due to this, the virtual fracture also becomes a blocking feature. Since the results are similar to that of Case 2, with an added blockage in the middle, these results were omitted.

**4.3. Convergence and Computational Cost.** According to the theory, we expect to see linear convergence in all variables for the lowest order choice of spaces described by (3.7). To show this, numerical experiments were performed for the described test cases on 6 consecutively refined grids. All solutions were then compared to the solution on the finest grid.

Let us continue by describing the norms used in this comparison, starting with the flux variables. These fluxes are expected to have irregular behaviour around the fracture tips resulting in a loss of convergence rates in these regions. In order to save



the convergence rate, let us introduce  $\mathfrak{B}_\rho$  which represent the balls with radius  $\rho > 0$  centered at the fracture tips. The fluxes are then compared to the solution from the finest grid in the norm

$$(4.1) \quad |\mathbf{v}|_{\mathcal{V}}^2 = \|K^{-1/2}\mathbf{v}\|_{L^2(\Omega \setminus \mathfrak{B}_\rho)}^2, \quad \mathbf{v} \in \mathcal{V}.$$

For this test case, it has been found sufficient to set  $\rho = 4 \cdot 10^{-2}$  in order to obtain the convergence rate. Note that the flux variable is given by  $\mathbf{u}_h = \mathbf{u}_{0,h} + \mathcal{R}\lambda_h$ , which means that the full flux is compared in accordance with the theory from Section 3.2.

Second, the normal traces denoted by  $\lambda$  are subject to the same irregular behaviour as the flux  $\mathbf{u}$ . Thus, the norm on which these are compared is defined as

$$(4.2) \quad |\lambda|_A^2 = \frac{1}{2} \|K_\nu^{-1/2}\lambda\|_{L^2(\Gamma \setminus \mathfrak{B}_\rho)}^2, \quad \lambda \in A.$$

Third and finally, the pressures comparison is done by using the following norm on the grid with the finest resolution.

$$(4.3) \quad |q|_{\mathcal{Q}}^2 = \|\epsilon q\|_{L^2(\Omega)}^2 + \|\Pi q\|_{L^2(\Gamma)}^2, \quad q \in \mathcal{Q}.$$

Here,  $\Pi$  is once again the  $L^2$ -projection onto the mortar space as defined by (3.11).

Next, the adjusted energy norm is constructed with these norms analogous to (3.12). With this energy norm, the comparisons are made and the results from each case are shown in Figure 4. Here, linear convergence is indicated and this is confirmed by the calculated rates in Table 2.

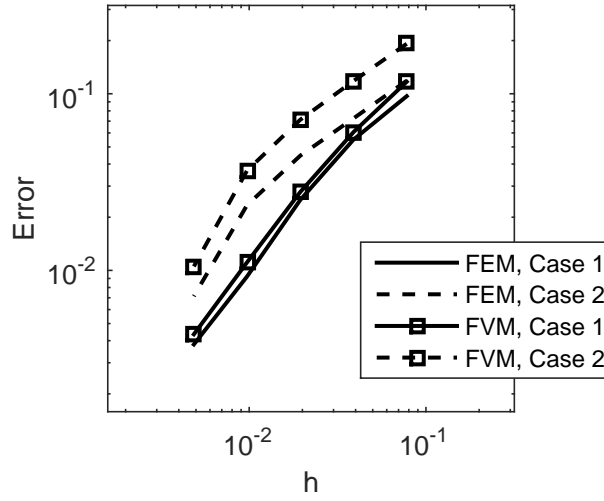


FIGURE 4. The error in the energy norm decreases linearly with the mesh size for both the full method and the finite volume variant.

TABLE 2  
Convergence rates.

	FEM			FVM		
	$ \mathbf{u} - \mathbf{u}_h _{\mathcal{V}}$	$ \lambda - \lambda_h _{\Lambda}$	$ p - p_h _{\mathcal{Q}}$	$ \mathbf{u} - \mathbf{u}_h _{\mathcal{V}}$	$ \lambda - \lambda_h _{\Lambda}$	$ p - p_h _{\mathcal{Q}}$
Case 1	1.21	1.12	1.07	1.23	1.04	1.05
Case 2	0.99	0.91	1.06	1.02	0.92	1.14

The advantage of using a finite volume variant becomes apparent when we compare the computational cost of the linear solve. As described by [24], the method of mass lumping creates a linear system which can be solved efficiently by direct solvers. Therefore, we evaluate the time necessary for a direct solve of the system and compare the results as shown in Figure 5. The computations were performed on a notebook with Intel Core Duo 1.6GHz processor using Matlab R2014b. The results show that the finite volume variant gives a linear system which can be resolved approximately three times faster than that of the mixed finite element scheme.

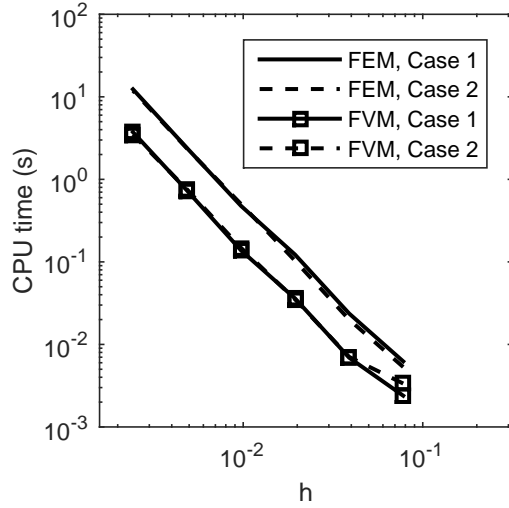


FIGURE 5. The computational time necessary for the linear solve. The finite volume variant improves this time with a factor of 3.

However, we note that although the computational time decreases, a larger error is made when using this finite volume variant as shown in Figure 4. In this context, the finite volume variant is competitive with the full method since it can produce a solution with similar error at an equivalent computational cost, but on a finer grid.

**5. Conclusion.** In this work, we proposed a mixed finite element method for Darcy flow problems in fractured porous media. The method is mass conservative and is able to handle non-matching grids due to its close relation to mortar methods. The key novel components of the method is the hierarchical approach obtained after subdividing the domain in a dimensional manner, as well as the use of mixed finite element spaces for flow embedded within a finer grid connected to flux mortar variables. We have shown how this naturally allows for rigorous analysis which was then used to obtain stability results and consequently, a priori error estimates. Moreover,

it is robust with respect to both permeable and blocking fractures. Numerical results confirm the theory, showing the expected convergence rates and stability.

Furthermore, the method allows for the construction of a finite volume variant which is shown to perform significantly faster while maintaining first order convergence rates.

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