Anisotropic Permeability of Fractured Media

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Abstract. The mathematical equivalents of parallel plate openings are used to simulate real fractures dispersed in orientation, distributed in aperture, and of arbitrary spacing. With this idealization of jointed pervious rock, the discharge of each conductor is a second rank tensor proportional to the cube of aperture and to the projection of a hydraulic gradient generally parallel to no conductor. One may add components of discharge through each intersecting conductor and components of discharge through intervening pervious blocks. The reciprocal of specific surface of the fracture system like spacing serves as a weighting factor for the tensor sum, which is the anisotropic Darcy's law permeability of an equivalent continuous medium.

Special cases of one, two, and three joint sets are modeled by applying Monte Carlo methods to pair orientations of individual planes sampled from Fisher dispersions with apertures sampled from skewed normal distributions. Statistics of the orientation of principal axes and of principal permeabilities are developed to show the relationship between joint geometry and anisotropy and to assess its variations.

INTRODUCTION

The permeability of fractured media has previously been modeled for two extreme cases seldom realized in nature: (1) when individual planar conductors, such as faults in rock, are so independent and infrequent that each may be treated as a separate channel or (2) when aggregates of fractures, as in fault breccia, so resemble sedimentary pores that the medium may be assumed continuous [*Muskat*, 1949, p. 267]. The present study models a wide variety of fractured media whose geometries are between the above extremes with any number of planar conductors of any orientation and any fine aperture. The models are inherently anisotropic because they simulate jointed rock masses whose nearly planar fluid conductors lie close to certain preferred directions.

The present objective is to establish the relationship between force and flow in idealized fractured media, describing it by the three-dimensional anisotropic Darcy's law:

$$v_j = -k_{ij} \frac{g}{\nu} I_i \tag{1}$$

where g is gravitational acceleration, ν is kinematic viscosity, I_i is the gradient of potential defined on a unit weight basis, v_j is the velocity vector, and k_{ij} is the linking coefficient, the directional permeability tensor. Versions of this formula have been published by *Vreedenburg* [1936], *Scheidegger* [1954], *Long* [1961], *Childs* [1957], and others.

Anisotropic permeability is one of the properties of jointed rock, applicable to foundation engineering, ground-water hydrology, petroleum technology, and other geotechnical disciplines. A completely general field test to determine anisotropic permeability is not likely to be developed because six unknowns must be determined.

One proposed method [*Snow*, 1966] of determining principal permeabilities employs arrangements of three orthogonal drill holes for water injection tests of the sort conventionally employed in damsite investigations. It requires foreknowledge of the principal axes along which drill holes are oriented. An objective of this paper is therefore to predict principal axes from field observations of joint systems.

To develop methods of interpreting anisotropy from joint orientation data, idealized model media are analyzed to establish the relationship between anisotropy and the geometry of conduit systems or conversely to estimate conduit geometry from measured anisotropy. Each model conduit is assumed to have smooth parallel plane walls of indefinite extent and an arbi-

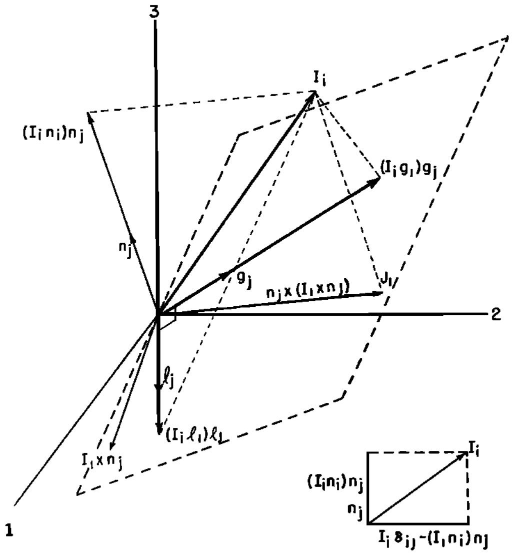

Fig. 1. Projections of an arbitrary gradient onto a planar conduit and (inset) projections of the gradient on the plane of the gradient and the conduit normal.

trary aperture. The average spacing between conduits (to be defined) is a constant for any model, whereas orientations and apertures are distributed variables. Much field, laboratory, and analytic study will be required to assess the departures of real fracture systems from the ideality of the model discussed here.

The conductivity of a single parallel plate opening must first be known, then the permeability due to parallel sets, to orientation dispersed sets with constant apertures, and finally to orientation dispersed sets with distributed apertures.

PARALLEL PLATE FLOW

From the Navier-Stokes equations [*Lamb*, 1932, p. 577; *Muskat*, 1937, p. 126] for slow nonturbulent single phase flow of an incompressible Newtonian fluid, equations have been derived [*Hele-Shaw*, 1898; *Lamb*, 1932, p. 583; *Long*, 1961, pp. 135–137; *Borg*, 1963, p. 264; and *DeWiest*, 1965, p. 323] for the average velocity and discharge per unit width of a parallel plate conduit. These are

$$\bar{v} = -\frac{b^2}{3}\frac{g}{\nu} I \quad \text{and} \quad q = -\frac{2}{3} b^3 \frac{g}{\nu} I \qquad (2)$$

where b is the half aperture and I is the projected gradient.

Figure 1 illustrates an arbitrarily oriented anisotropic plane conduit in a Cartesian coordinate system with axes 1, 2, 3. The orientation of the conduit is defined by its normal with direction cosines n_j or by two unit vectors in the plane g_j and l_j, the latter two being the directions of greatest and least conductivity. An arbitrary but uniform hydraulic gradient I_i is imposed across the medium. Flow along the conduit is proportional to the projection of I onto the plane, either the normal projection J_j or the two orthogonal components $(I_i g_i) g_j$ and $(I_i l_i) l_j$ as shown. The magnitudes of these orthogonal components are given by the dot products in parentheses, and the directions are given by the unit vectors g_j and l_j. The indicial notation of *Jeffreys and Jeffreys* [1956] is used here, wherein the repeated subscript indicates summation.

Discharge components are

$$G_j = -K_g (I_i g_i) g_j \quad \text{and} \qquad (3)$$

$$L_j = -K_l (I_i l_i) l_j \qquad (4)$$

per unit width measured normal to the resultant. The maximum and minimum conductivity coefficients are K_g and K_l, respectively. These might differ from the value

$$\frac{2}{3} b^3 \frac{g}{\nu}$$

indicated in equation 2 by reason of anisotropic fracture texture. The resultant discharge is

$$q_j = -(K_g g_i g_j + K_l l_i l_j) I_i \qquad (5)$$

The bracketed term is a symmetric tensor of second rank, relating the discharge to the gradient. It defines the discharge per unit length normal to the velocity by components for a gradient in any direction. A simpler equation can be derived for the discharge of an isotropic conduit. If $K_g = K_l = K$, then

$$q_j = -K[(I_i g_i) g_i + (I_i l_i) l_j] \qquad (6)$$

Since K is defined for all directions of an isotropic conduit, any pair of orthogonal projections of the hydraulic gradient or their resultant J will determine the discharge. In the plane containing n and J, it is (inset, Figure 1) the vector difference

$$J_j = I_i \delta_{ij} - (I_i n_i) n_j \qquad (7)$$

where δ_{ij} is the Kroneker delta, vanishing when $i \neq j$ and unity when $i = j$. Substitution of (7) into (2) gives

$$v_j = -\frac{b^2}{3}\frac{g}{\nu}(\delta_{ij} - n_i n_j) I_i \qquad (8)$$

The coefficient common to all terms of this equation may be considered the hydraulic conductivity for a unit width of conduit, but this equation cannot be applied as the permeability of a fractured medium until it is modified according to the area across which each conduit discharges. It is not useful to pursue the line of reasoning [*Muskat,* 1937, p. 246; *Amyx, Bass, and Whiting,* 1960, p. 84] that the permeability of a fracture is proportional to the discharge divided by the aperture, for such a procedure neglects the influence of spacing between fractures.

MEDIA WITH PARALLEL CONDUITS

The permeability of a nonconducting solid cut by smooth parallel openings (Figure 2) is readily calculated. The discharge of each is

$$q = -\frac{b^2}{3}\frac{g}{\nu} W(2b) J \qquad (9)$$

where J is the projection of I on the conduit planes.

The total discharge of N equal conduits is

$$q = -\frac{2}{3} b^3 \frac{g}{\nu} N W J \qquad (10)$$

provided that each aperture is the same. If apertures differ from conduit to conduit but

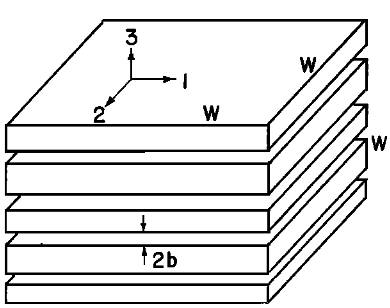

Fig. 2. A solid volume of dimensions W cut by parallel plane conduits.

remain constant in all directions along each, then

$$q = -\frac{2}{3}\frac{g}{\nu} W J \sum b^3 \qquad (11)$$

The flow through an equivalent continuous medium is given by Darcy's law:

$$q = -k \frac{g}{\nu} W^2 J \qquad (12)$$

Equating (11) and (12),

$$k = \frac{2}{3}\frac{1}{W} \sum b^3 \qquad (13)$$

Intrinsic permeability [*Richards,* 1952] measures the average cube of apertures in parallel systems when the average spacing Δ is known:

$$k = \frac{2}{3}\frac{1}{\Delta}\frac{\sum b^3}{N} \qquad (14)$$

Note that $k_{11} = k_{22} = k$, and $k_{33} = 0$ for parallel media.

Intrinsic permeability k_{ij} in equation 1 will be used in this model. It is assumed that there is no influence of the fluid on the medium. *Childs* [1957, p. 49] has pointed out the invalidity of this assumption for soils which contain colloidal or organic matter, whose structure is influenced by clay–water chemistry. Clay coatings and partial fillings are common in weathered joints in rock. These fillings may be observed in fine as well as large aperture conduits in crystalline rocks or in any fractured argillaceous rocks. Therefore the same limitations on intrinsic permeability apply to some jointed rock media as to soils. It is also assumed that the medium is rigid, so that permeability is independent of fluid pressure. This assumption is also violated in nature [*Snow,* 1968*b*].

SUPERPOSITION OF FLOWS

There can be no mutual interference of flows at the intersection of two or more smooth planar conduits if they are not sealed by dislocation or enlarged there by solution. Even if there is additional friction by mixing at intersections, it seems that in laminar flow the local energy losses cannot be more than a few percent of the losses between intersections. *Romm* *and Pozinenko* [1963] likewise assumed superposability.

Figure 3a shows a uniform isotropic conduit plane, with normal n_a cutting a solid medium. There is assumed to be a flow of fluid parallel to an arbitrary gradient vector I_a acting in the plane, and on another plane n_b shown in Figure 3b, flow parallel to an arbitrary gradient vector I_b generally not parallel to I_a. There is a unique vector I having I_a and I_b as projections; otherwise along an intersection of the two planes, as in Figure 3c, the gradient would not be single valued. Call I the field gradient, which may vary continuously from point to point throughout the space, existing everywhere if connected intergranular pores lace the solid blocks or existing only in the connected fracture pores if they do not.

Conversely a gradient of arbitrary orientation may be imposed across any mass transected by arbitrarily oriented uniform continuous conduits. A real gradient in each may be computed as a projection of the field gradient, and the corresponding flow may be computed according to the aperture. Since mutual interference may be assumed minimal, the total flow through the medium is the vector sum of the contributions of all individual conduits.

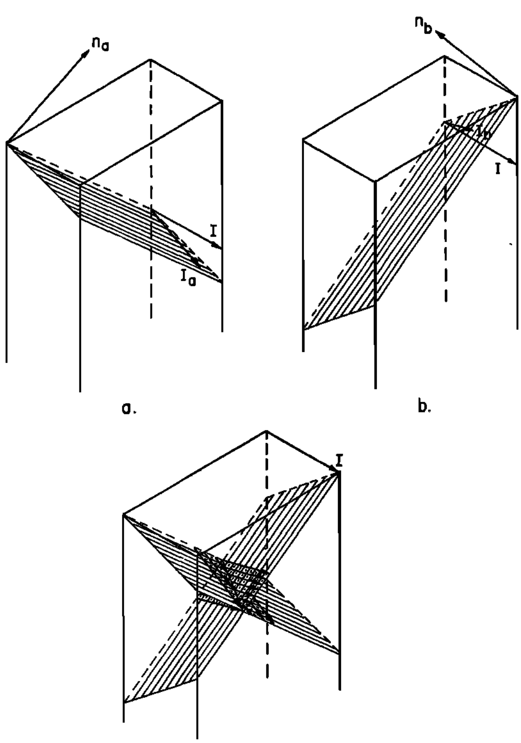

Fig. 3. Gradients on intersecting joints that cut a solid volume.

The foregoing does not say that the flow in one conduit is independent of all others when boundary conditions are specified, for an additional conduit alters the directional permeability of the medium and therefore the local field gradient. Rather it says that if a field gradient is given, on each conduit there acts a projected gradient independent of the gradient on any neighbors.

MEDIA WITH DISPERSED CONDUITS

Unlike parallel conduits which are characterized by a unique repetition unit, the average spacing, there is no obvious unit describing the spacing of conduits dispersed in orientation. It is shown below that a frequency measure serving the same purpose as the reciprocal of spacing for parallel systems is the specific surface of dispersed systems [*Snow*, 1967].

One practical method of sampling dispersions of joints in rock, measuring their orientations and positions in the field, entails line sampling or borehole photography along a length D. Each joint may be assigned to a set with the aid of a stereonet plot of normals. To avoid blindness to certain orientations, holes of differing orientation are needed [*Terzaghi*, 1965].

All joints of a set traversed by a sampling length D contribute to the permeability of the sample medium, and D is a repetition unit tantamount to some multiple of Δ for parallel media. No matter how large the sample, the unsampled rock beyond the hole must be assumed to have the same distribution of joints as the part traversed; thus each one measured is assumed repeated at all multiples of D.

Figure 4 shows a sampled joint with orientation n_i and a second joint identical and parallel to it (both shaded). They are separated by an oblique distance D_i, a vector representing the sampling line. If a potential gradient is given, the direction of fluid flow can be computed. Let each joint lie on the bisector of a cubic element whose edges form a second coordinate system, designated below by primed variables: normal to the joint and, in the plane of the joint, parallel and normal to v, respectively. The

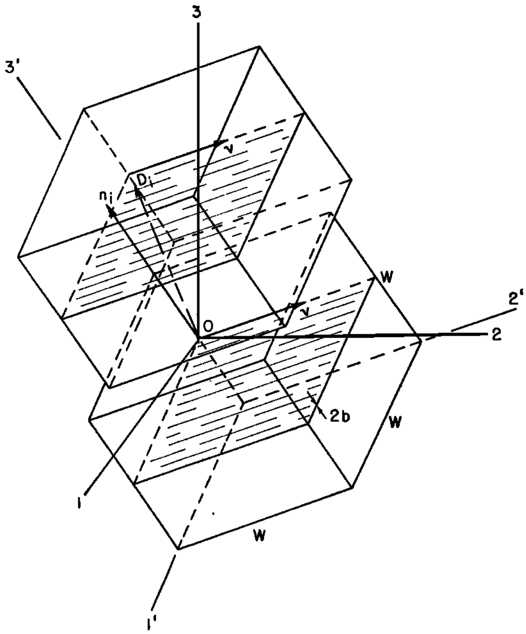

Fig. 4. A joint conductor with normal n_i and its image, distant D_i, the sampling length.

cube's dimension W depends upon D_i and the absolute value of the cosine of the angle between the sampling line and the normal:

$$W = |n_i D_i| \tag{15}$$

Components of discharge from a joint are

$$q_j' = (2bW)v_j' \tag{16}$$

The same volume element can be evaluated as a continuum by Darcy's law, using the most general form of the coefficient, the conductivity tensor in the primed system:

$$q_j' = -W^2 \frac{g}{\nu} k_{ij}' I_i' \tag{17}$$

where I' is the potential gradient in the primed system. The right-hand sides of the last two equations may be equated:

$$v_j' = -\frac{W}{2b} \frac{g}{\nu} k_{ij}' I_i' \tag{18}$$

then transformed to the unprimed system common to all joint conduits. A is a coordinate transformation matrix.

$$Av_j' = -\frac{W}{2b} \frac{g}{\nu} (Ak_{ij}' A^{-1}) AI_i' \tag{19}$$

$$v_j = -\frac{W}{2b} \frac{g}{\nu} k_{ij} I_i \tag{20}$$

We can now substitute for W and the velocity vector derived previously, namely,

$$v_j = -\frac{1}{3} b^2 \frac{g}{\nu} (\delta_{ij} - m_{ij}) I_i \tag{8}$$

where $m_{ij} = n_i n_j$.

$$\tfrac{2}{3} b^3 (\delta_{ij} - m_{ij}) I_i = |n_i D_i| \; k_{ij} I_i \tag{21}$$

The conductivity tensor for each joint is symmetric:

$$\begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} = \frac{2b^3}{3\,|n_i D_i|} \cdot \begin{vmatrix} (1 - m_{11}) & -m_{12} & -m_{13} \\ -m_{21} & (1 - m_{22}) & -m_{23} \\ -m_{31} & -m_{32} & (1 - m_{33}) \end{vmatrix} \tag{22}$$

The facial area of the repetition cube for a given joint, as illustrated in Figure 4, will differ from that for other joints having different orientations. The dimensions of each cube are proportional to the absolute value of the cosine of the angle defined, a value which acts as a weighting factor. The disparate facial areas and discharges for different joints are translated into permeability for an equivalent continuum, a property independent of the areas. By repeating these cubic elements along the trace of a joint and along parallel sampling lines across the trace, an entire plane may be covered with stepped faces lying normal to the velocity. The tensor permeability contributions of all joints may be added to find the total, since the independence of flows at the intersection of joints has been assumed. The location of any specific member of a joint set is ignored for present purposes, the presumption being that inhomogeneities within the sample are duplicated in successive samples and averaged out over the region within problem boundaries distant several D. The permeability tensor for a joint set is

$$k_{ij} = \frac{2}{3} \sum \frac{b^3}{|n_i D_i|} (\delta_{ij} - m_{ij}) \tag{23}$$

where the summation is taken over all members of a joint set intersected by a sampling line D_i, each of which has a particular aperture $2b$, uniform over its extensive surface and constant in orientation n_i.

Irmay [1958, p. 706] has already noted that three inclined or different sets of straight seams lead to a second rank tensor permeability. *Romm and Pozinenko* [1963] have derived equation 23 without the weighting factor $|n_i D_i|$. Theirs is adequate for any system of conduits whose sets are of parallel individuals. It cannot be applied in the usual more general cases of dispersed orientations.

More than one set of joints may be synthesized in the model. Sampling lines of different length, $\overset{1}{D}$, $\overset{2}{D}$, intersecting sets with different fracture frequency generally imply different numbers of members in each joint set. The coefficient $1/|n_i D_i|$ gives proper weighting to the different sized samples of each set. Permeability contributions of several sets can therefore be added to obtain the permeability of the medium. It is convenient to use the central tendency of each set as a sampling line and to simplify further using the same D for each set.

To analyze permeability from fracture orientations and apertures obtained in drill holes in real rock a different procedure from that required to synthesize a medium from artificial fractures [*Bianchi*, 1968] is necessary. Each real sampling line generally intersects some of the fractures from all sets. Those at an acute angle to the hole have low probability of interception and so are weighted heavily when encountered. The measures from M different holes cannot be added as they are in synthesis; rather they must be averaged.

$$k_{ij} = \frac{2}{3M} \sum \frac{b^3}{|n_i D_i|} (\delta_{ij} - m_{ij}) \tag{24}$$

The number of joints in each set, $\overset{1}{N}$, $\overset{2}{N}$, combined for synthesis of directional permeability may be proportioned according to assigned measures of specific surface and coefficients of orientation dispersion. Designating specific surface as S the sampling lengths would be:

$$\overset{1}{D} : \overset{2}{D} : \overset{3}{D} = \frac{2}{S_1} \sum_{i=1}^{\overset{1}{N}} \frac{1}{n_i} : \frac{2}{S_2} \sum_{i=1}^{\overset{2}{N}} \frac{1}{n_i} : \frac{2}{S_3} \sum_{i=1}^{\overset{3}{N}} \frac{1}{n_i} \tag{25}$$

If $\overset{1}{D} = \overset{2}{D} = \overset{3}{D}$, then

$$\frac{2}{S_1} \overset{1}{N} \left(\frac{1}{n}\right)_{av_1} = \frac{2}{S_2} \overset{2}{N} \left(\frac{1}{n}\right)_{av_2} = \frac{2}{S_3} \overset{3}{N} \left(\frac{1}{n}\right)_{av_3} \tag{26}$$

The bracketed coefficient, the average inverse cosine of the angles between normals and the central tendency, is readily computed for any given joint set:

$$c = \left(\frac{1}{n}\right)_{av} = \frac{1}{N} \sum_{i=1}^{N} (1/\cos\theta)_i \tag{27}$$

where N is the total number of joints in the set. For model joint set orientations we may use *Fisher's* [1953] dispersion of errors on a sphere, whose density is proportional to $e^{K_f \cos\theta}$ and whose total number or flux is

$$N = \frac{2\pi}{K_f} (e^{K_f} - e^{-K_f}) \tag{28}$$

K_f is a dispersion coefficient.

In an element $d(\cos\theta)$ about the central tendency, there are

$$dF = -2\pi e^{K_f \cos\theta}\, d(\cos\theta) \tag{29}$$

members, so c may be obtained by integrating over the surface of the sphere,

$$c = \frac{1}{N} \int dF \frac{1}{\cos\theta} = \frac{\displaystyle\int_{-1}^{+1} 2\pi e^{K_f \cos\theta} \frac{1}{\cos\theta}\, d(\cos\theta)}{\displaystyle\frac{2\pi}{K_f} (e^{K_f} - e^{-K_f})} \tag{30}$$

Unlike the vectors of a Fisher distribution the normals to planes in space are two-headed. Within the region $\pi/2 < \theta \leq \pi$ we must represent a vector by its negative, directed into the region $0 \leq \theta < \pi/2$. If a significant portion of a Fisher distribution lies outside the hemisphere having the central tendency as polar axis, reflection of such extreme vectors would produce an abnormal concentration near $\theta = \pi/2$. Most natural joint sets are reasonably concentrated (Fisher's $K_f > 10$, Plate 2), and if joints lie outside of $\theta = \pi/2$, they would be identified with another set. The probability of a vector's lying in the region $\theta > \pi/2$ is only 0.0007 for dispersions of $K_f = 5.0$ and is less for larger K_f. If we limit the definition of a joint set to members lying within a 120-degree cone ($\theta = \pi/3$), and if $K_f = 5.0$ or greater, we find a maximum probability of 0.006 that a vector generated by Fisher's equation will lie outside these limits.

The improper integral given above can therefore be evaluated over a practical range $1 \geq \cos \theta \geq \frac{1}{2}$ for all values of $K_f > 5.0$:

$$c = \frac{K_f}{e^{K_f} - e^{-K_f}} \cdot \left[\ln \cos \theta + \frac{K_f \cos \theta}{1} + \frac{K_f^2 \cos^2 \theta}{2(2!)} + \frac{K_f^3 \cos^3 \theta}{3(3!)} + \cdots \right]_{1/2}^{1} \tag{31}$$

The integral is graphed in Figure 5 as a function of K_f.

Once the number of elements in a given joint set is established, the tensor permeability contribution of each is determined by equation 23. Similar tensors for each of the other joint sets are added term by term to form the permeability tensor for the fractured medium.

In any boundary problem, jointed rock may be treated as a homogeneous continuous anisotropic medium with velocity components given by

$$v_i = -k_{ij} \frac{g}{\nu} I_i \tag{1}$$

In some cases where boundary transformations are inconvenient, for instance if two or more adjoining regions have different anisotropy, the fully expanded form may be required for each region. Such a problem should be solved in its

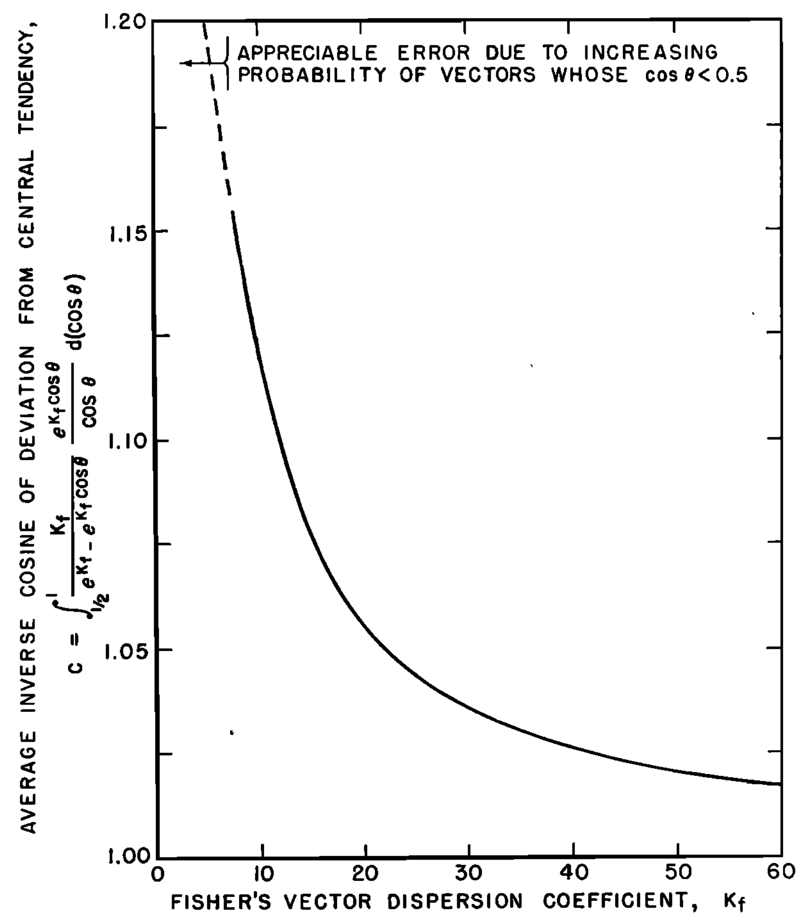

Fig. 5. The average reciprocal of the cosine of angles between joint plane normals and their central tendency as a function of dispersion used for proportioning size of sets and determining repetition length: $m_1 : m_2 : m_3 = S_1/C_1 : S_2/C_2 : S_3/C_3$.

original coordinate system, utilizing all nine terms of the tensor.

Computer programs, such as those developed by *Warren et al.* [1960] and by *Freeze and Witherspoon* [1966] to solve boundary problems, could satisfy continuity of flow through a cubic or radial volume element when each discharge component is a function of all three gradient components, i.e.,

$$V_1 = -\frac{g}{\nu}[k_{11}I_1 + k_{12}I_2 + k_{13}I_3], \text{ etc.} \quad (1)$$

More commonly, problems are solved isotropically after transforming coordinates. To obtain the transformation factors and the effective conductivity of the fictitious medium, principal axes and permeabilities are required. To do so, it remains only to diagonalize the summary tensor of equation 23, finding the principal axes as eigenvectors and principal permeabilities as eigenvalues. Equation 1 then becomes

$$\begin{vmatrix} V_1' \\ V_2' \\ V_3' \end{vmatrix} = -\frac{g}{\nu}\begin{vmatrix} k_{11}' & 0 & 0 \\ 0 & k_{22}' & 0 \\ 0 & 0 & k_{33}' \end{vmatrix}\begin{vmatrix} I_1' \\ I_2' \\ I_3' \end{vmatrix} \quad (32)$$

the primes signifying reference to a coordinate system parallel to the principal axes of the tensor. Equation 32 is equivalent to the familiar equations

$$\begin{aligned} V_x &= -\frac{g}{\nu}k_x\frac{\partial\varphi}{\partial x} \\ V_y &= -\frac{g}{\nu}k_y\frac{\partial\varphi}{\partial y} \\ V_z &= -\frac{g}{\nu}k_z\frac{\partial\varphi}{\partial z} \end{aligned} \quad (33)$$

given by *Muskat* [1937, p. 226], *Childs* [1957, p. 63], and others.

It has been noted already that intergranular flow may be superposed upon fracture flow, so that the permeability of jointed granular porous media may be determined. If the solid of the medium has permeability k_s, this value may be added to each of the principal permeabilities determined for the joint system. If the solid is itself anisotropic, described in the manner of equation 1, then one may transform its tensor to the coordinate system used to orient the joints, add each term to the tensor for the joint system, and then diagonalize the sum.

Equation 14 for sets of parallel joints is a special case deduced from the tensor form. When all have the same orientation n_i, then

$$k_{ij} = \frac{2}{3\,|n_i D_i|}(\delta_{ij} - m_{ij})\sum b^3 \quad (34)$$

If n_i is also a coordinate axis, say the 3-axis, then

$$n_1 = n_2 = 0 \quad \text{and} \quad n_3 = 1$$

so,

$$\begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} = \frac{2}{3D}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}\sum b^3 \quad (35)$$

or

$$k_{11} = k_{22} = \frac{2}{3D}\sum b^3, \qquad k_{33} = 0 \quad (36)$$

as previously derived.

MEDIA WITH DISPERSED CONDUITS AND DISTRIBUTED APERTURES

The permeability tensor for a fractured medium may be evaluated by the above method if all conductors are parallel plate openings of the same aperature $2b$ or alternatively if the aperture of each conductor of different orientation is known. Equal apertures do not occur in jointed rocks, and measurement of individual joint apertures in the undisturbed state is currently no more possible than the measurement of intergranular pore dimensions. However, like pore size distributions, aperture distributions can be obtained indirectly [*Snow,* 1969]. Evaluation of the permeability tensor then requires a method of pairing an aperture with each orientation. In the absence of established dependence between the two variables, the evaluation may proceed statistically, recognizing that given distributions of apertures and orientations may be combined in many ways to yield tensors that differ within a range of orientations of principal axes and principal permeabilities.

One method of determining the distribution of a variable, such as permeability, dependent upon two other distributed variables such as orientation and aperture, is to employ Monte Carlo sampling [*U. S. National Bureau of*

Standards, 1951]. The method of obtaining dependent distributed properties is to compute successive samples in batches, each sample having random input data taken from large discrete or continuous populations of the independent variables. If the samples are large, the central limit theorem [*Mood and Graybill,* 1963, pp. 149, 403] justifies the application of normal error statistics for analyzing the resulting distributions of answers. The central limit theorem operates on a lower level also, for each run accumulates (equation 23) the contributions N conductors make to the sample permeability. If N is large, permeabilities are normally distributed.

If batches of runs are made with smaller and smaller numbers of conductors, their permeability distributions approach something proportional to the distribution of apertures cubed. Almost any distribution cubed is highly skewed to the right; thus any distribution of small sample permeabilities is also skewed. Natural distributions obtained from pressure tests in damsite drill holes [*Snow,* 1968*a*, p. 75] are skewed because the samples are of small numbers of fractures. Plates 1 to 5 suggest normally distributed permeabilities computed from large samples. But the distributions become skewed when small samples are taken. As a result, it was found that permeabilities of model fractured media increase with sample size [*Snow,* 1965]. Intuitively, the permeability of a 10-foot cube of jointed rock should model the same as a statistically homogeneous 100-foot cube containing it, yet it can be shown to increase when the dimensions increase. *Davis and Turk* [1964] reached an equivalent conclusion that N wells in a crystalline rock should produce more than N times the expectation of a single well. Presently, incomplete studies [*Snow,* 1968*c*] of dependence upon sample size cannot be included here, since the emphasis is on orientation of permeability axes and not on magnitudes. The reader interested in the mechanics of the computer programs implementing this model may consult the source [*Snow,* 1965].

There is a need for model studies to guide the field worker in determining the orientation of principal axes of any geometric system of joints, faults, or other planar conductors. Improved testing methods [*Snow,* 1966] may then be tried. While the model joint systems reported below may not fit any real joint system exactly, the variety of special cases should serve as guides to the approximate orientations of axes for nearly all cases. Alternatively, field orientation data and assumed or determined aperture distributions may be used to compute distributions of the permeability tensor.

COMPUTED ANISOTROPY OF MODELS OF ONE, TWO, AND THREE SETS

Application of equation 23 to a sampled set of fracture apertures paired with sampled orientations gives a permeability tensor for the model. Another sampling from the same populations yields a different tensor. All possible ones describe the sampling heterogeneity of the medium. The conductors modeled may be faults, joints, foliation, sand seams, or saw cuts. The assumption that they are uniform and isotropic throughout their infinite extent departs considerably from the reality of discontinuous, nonuniform, anisotropic, or elastic individuals; yet the model is useful for estimating the principal permeability axes of all such media. Models may be characterized (1) by parameters describing the variation of the apertures of conductors, (2) by parameters specifying the frequency of occurrence in a volume, and (3) by parameters specifying orientations.

One distribution of apertures has been used here and is shown in Plate 1, Figure 7, cumulatively and as a density distribution of half apertures for 10-micron classes. Apertures are distributed according to the absolute value of a normal distribution with mean 0.025 cm and standard deviation 0.035 cm. By changing the sign of negative apertures, a skewed distribution is formed resembling the log normal form found later to be appropriate [*Snow,* 1969].

Frequency variations have been included in unpublished parameter studies [*Snow,* 1965].

The *Fisher* [1953] distribution was used to generate a variety of synthetic joint orientation distributions. They are symmetric about a central or average orientation, whereas natural distributions are not. Any natural distribution portrayed in stereographic projection can be compared to the synthetic distributions shown in Plate 2 to estimate its Fisher coefficient. Another comparison is by the computed vector strength [*Arnold,* 1941; *Pincus,* 1953].

Fourteen systems of one, two, or three sets of

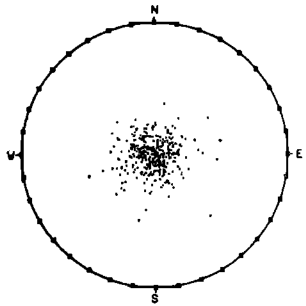

Fig. 1. Normals to a set of joints dispersed about 0.0, 0.0, 1.0, K_f 15.

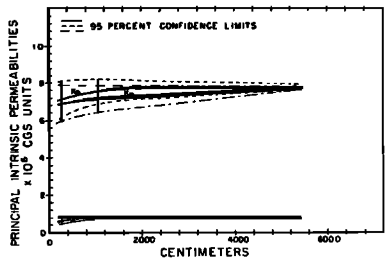

Fig. 2. Median principal permeabilities as funciton of sample size. Single set, (Fig. 1.) spaced $2/S = 10$ cm.

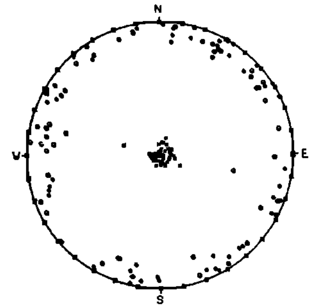

Fig. 3. Principal axes of permeability. Samples of 100 joints each solution.

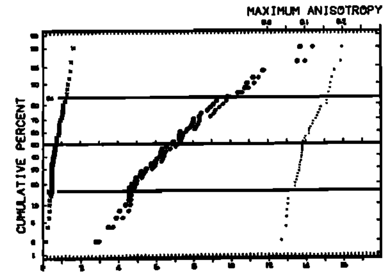

Fig. 4. Principal permeability distributions, cgs units $\times 10^6$. Samples of 100 joints.

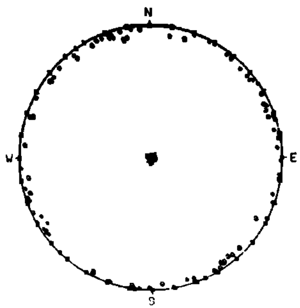

Fig. 5. Principal axes of permeability. Samples of 500 joints each solution.

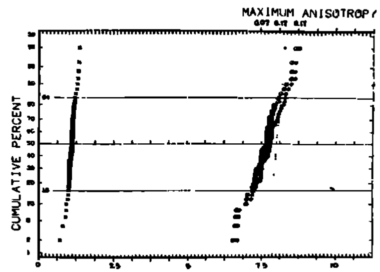

Fig. 6. Principal permeability distributions, cgs units $\times 10^6$. Samples of 500 joints.

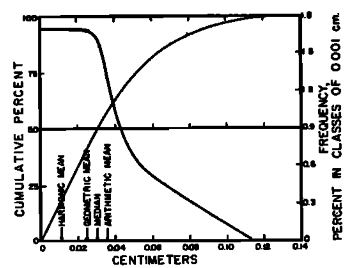

Fig. 7. Half apertures generated by absolute value of normal distribution ($\mu = 0.025$, $\sigma = 0.035$).

PLATE 1. Permeability versus sample size, one set of joints. $\diamondsuit$ = maximum, 0 = intermediate, x = minimum.

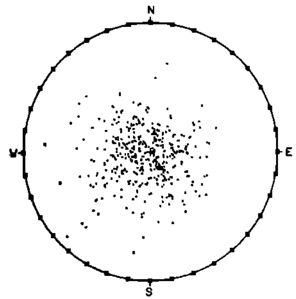

Fig. 1. Joints dispersed about 0.0, 0.0, 1.0, $K_f = 6.0$.

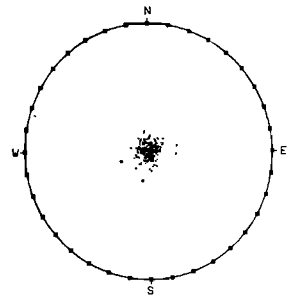

Fig. 2. Joints dispersed about 0.0, 0.0, 1.0, $K_f = 60.0$.

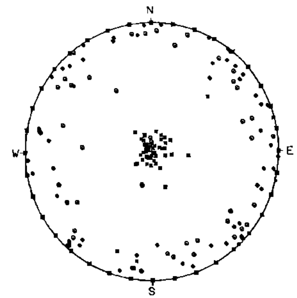

Fig. 3. Principal axes, $K_f = 6.0$, 92 joints/sample, $2/S = 10$ cm.

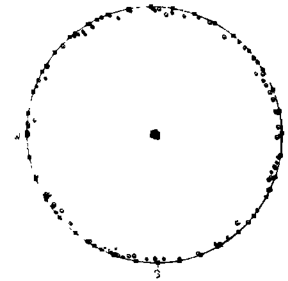

Fig. 4. Principal axes, $K_f = 60.0$. 106 joints/sample, $2/S = 10$ cm.

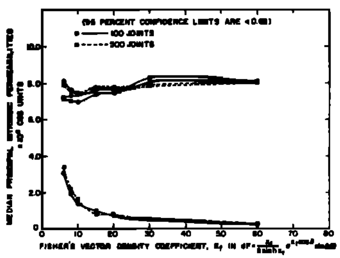

Fig. 5. Principal permeabilities versus dispersion of single joint sets.

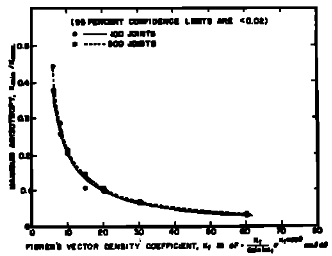

Fig. 6. Maximum permeability anisotropy versus dispersion of single joint sets.

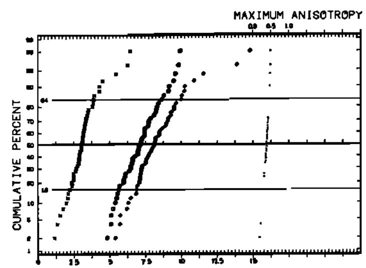

Fig. 7. Principal permeabilities from above model.

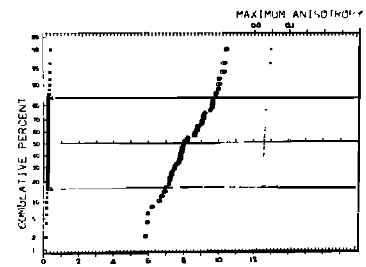

Fig. 8. Principal permeabilities from above model.

PLATE 2. Permeability versus orientation dispersion, single set of joints.

dispersed joints have been modeled [*Snow*, 1965], a few examples of which are shown here in stereograhic projections. Systems of any desired complexity can be generated by combining individual dispersions differing in direction of central tendency, dispersion coefficient, or spacing. Plate 1 shows the change of anisotropy for a single set having a given dispersion as the sample size changes. Plate 2 shows the effect of decreasing orientation dispersion from Fisher's $K_f = 6$ to 60. Plate 3 treats two orthogonal sets, Plate 4 two nonorthogonal sets, and Plate 5 three orthogonal sets.

In any one model, spacing has been maintained by proportioning the sample size of each set according to equation 26 and Figure 5.

Also displayed in stereographic projection are the principal permeability axes for 49 samplings, disclosing the range of possible solutions. Orthogonal triplets of diamonds, circles, and crosses correspond to maximum, intermediate, and minimum permeability axes. The scatter simulates the variation in principal permeability axes that must exist from one place to another in jointed rock. The axes close on mean directions as the sample size or volume increases. Dispersion of the axes would be concealed by mean values, using only large samples. Note that aberrant axial orientations are possible from a random model, as illustrated in Plate 1, Figure 3, by the minor axis (cross) oriented N 78°W, 30° from the vertical. If a sample contains one large opening it will dominate the directional permeability just as an open fault will dominate the flow in a region.

Distribution curves of principal permeabilities are plotted. Diamonds, circles, and crosses maintain correspondence of principal axes and permeabilities. Since the maximum, intermediate, and minimum permeabilities were ranked separately for cumulatve plotting, the three values found at any one ordinate do not constitute a particular solution. To preserve individual relationships, on the right of the permeability plots the cumulative distribution of maximum anisotropies has been plotted, each being the ratio of minimum permeability to maximum permeability for a particular solution.

The dispersion of principal permeabilities is portrayed in Plate 1. The large dispersions computed are consistent with the large observed variations in measured permeabilities in jointed rocks, reflecting dependence upon the cube of variable aperture. The dispersions of principal axes (Plate 1, Figures 3 and 5) and permeabilities (Plate 1, Figures 4 and 6) decrease for samples of increasing size (100 and 500 joints, respectively), as anticipated from the central limit theorem. Principal axes are concentrated within about 10° and 5° of arc for the 100 and 500 conduit samples, respectively. Median permeability values increase slightly with sample size as shown in the summary plot (Figure 2). Studies have indicated that model permeability is well-defined for 100 or more joints; otherwise it is a function of size.

The stereographic projections of principal axes display three of the quantities required to define a second rank tensor, and the cumulative permeability plots display the other three. Thus each pair of plots should be read together. Certain mutual relationships are evident. If two principal permeabilities are nearly the same each time a sample is taken, then their axial orientations are sensitive to differences in the conductors in the samples. Since any two equal orthogonal permeability vectors define a plane of isotropy, we find that successive solutions for a single set of dispersed joint orientations scatter orientations throughout a great circle of the stereonet, such as the equatorial planes of Figures 3 and 5 of Plate 1. The normal to the plane of isotropy is the only unique principal axis, as is the optic axis of a uniaxial mineral. Figures 4 and 6 of Plate 1 show slight differences between principal permeabilities on the isotropic plane (maximum and intermediate permeabilities), whereas the intermingling of diamonds and circles on the girdle in the stereogram indicates statistical isotropy, especially for large samples.

Sheeted granite is a prototype of a single set of joints. Plate 2 illustrates the effect of decreasing orientation dispersion of a single set of joints. The joint populations for Fisher's $K_f = 6$ to 60 are shown, as well as the principal axes and permeabilities for samples of size 92 to 106, varying according to equation 26 to maintain a sample volume of 1035 cm cubed for a jointed medium of inverse specific surface 10 cm. As orientation dispersion decreases, there is progressive reduction of the dispersion of axes and permeabilities, diminution of the minor perme-

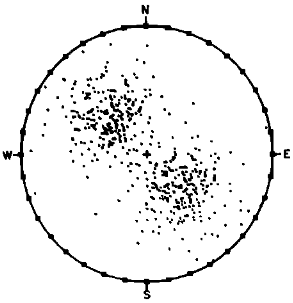

Fig. 1. Two equal normal joint sets, K_f = 10, dip 45° NW, SE.

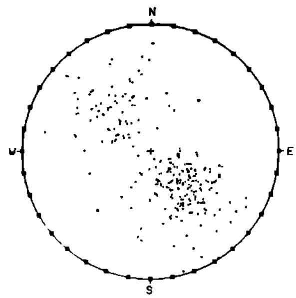

Fig. 4. Two normal joint sets, K_f = 10, dip 45° SE spaced 10 cm and 45° NW 20 cm.

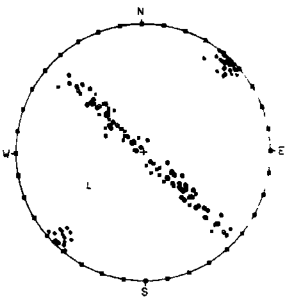

Fig. 2. Principal axes, two equal normal sets. Each sample, 200 joints per set, spaced $2/S$ = 10 cm.

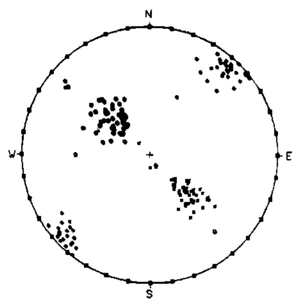

Fig. 5. Principal axes, two nonequal normal sets. Each sample, 100 joints 45° SE, $2/S$ = 10 cm, and 50 joints 45° NW, $2/S$ = 20 cm.

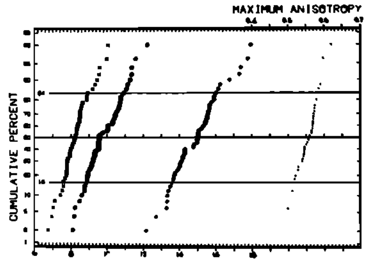

Fig. 3. Principal permeabilities, cgs units $\times 10^6$, from above model.

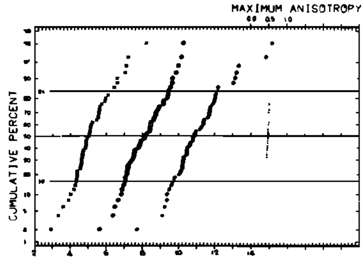

Fig. 6. Principal permeabilities, cgs units $\times 10^6$, from above model.

PLATE 3. Anisotropic permeability of two sets of fractures.

ability, convergence and increase of the two highest permeabilities, and a marked increase of anisotropy. Figures 5 and 6 of Plate 2 illustrate the dependence on orientation dispersion.

It is rarer to find only two sets of joints in rock than to find one or three sets. Two orthogonal joint sets of equal properties are represented in Plate 3, Figure 1. Principal axes shown in Plate 3, Figure 2, indicate that the central tendencies of normals to two sets of orthogonal joints of like properties lie on a plane of near isotropy, even for small samples. The unique major axis contains the central plane of each set, in other words, lies parallel to the predominant direction of intersections. Permeability in the isotropic plane is half the value in the direction of the intersection.

When the two orthogonal sets differ, as in Plate 3, Figure 4, three unique axes appear, one parallel to the central normal of each set. The

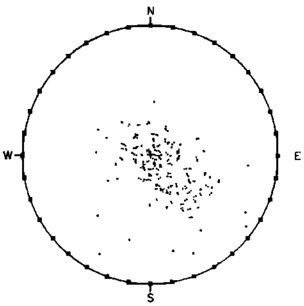

Fig. 1. Two equal sets, $K_f = 10$, dip 45° SE and horizontal.

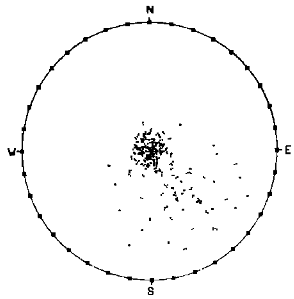

Fig. 4. Two sets, $K_f = 10$. 45° SE, $K_f = 30$. horizontal.

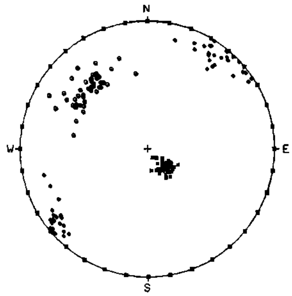

Fig. 2. Principal axes, samples of 100 joints from each above set. $2/S = 10$ cm.

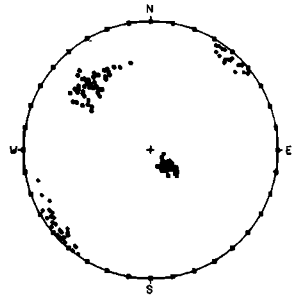

Fig. 5. Principal axes, 100 joints from 45° SE set, 108 from horizontal set. $2/S = 10$.

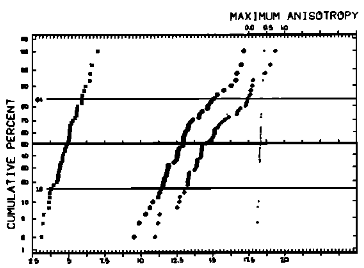

Fig. 3. Principal permeabilities, cgs units $\times 10^6$, from above model.

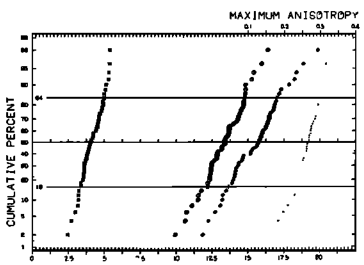

Fig. 6. Principal permeabilities, cgs units $\times 10^6$, from above model.

PLATE 4. Permeability of two nonorthogonal joint sets.

intersection direction is the major axis. Since the NW set is only half as frequent as the SE set, the higher permeability lies in the plane of the more frequent set.

When two equal sets are not orthogonal, as in Plate 4, Figure 1, the principal axes coincide with the axes of symmetry of the system: the major axis is on the intersection of the sets, the minor bisects the obtuse angle, and the intermediate bisects the acute angle between fractures.

If one of the two nonorthogonal sets is less dispersed than the other, as shown in Plate 4, Figure 4, the minor principal axis shifts closer to the normal of the less dispersed set.

Most rock formations contain three sets of joints, as represented in Plate 5. Three equal sets in a dispersed cubic pattern, Plate 5, Figure 1, produce isotropic permeability for all sample sizes, with axes randomly scattered over all orientations.

If two orthogonal sets are equal, their normals

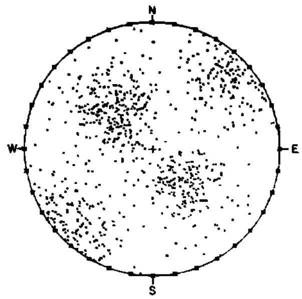

Fig. 1. Three equal normal sets, K_f = 15, dip 45° SE and NW and 90° SW.

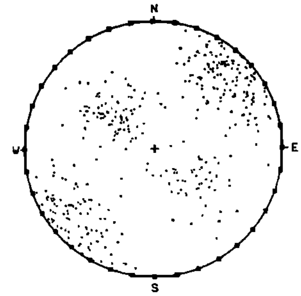

Fig. 4. Three normal sets, K_f = 15, different spacings: 20 cm dip 45° SE, 10 cm 45° NW, and 5 cm 90° SW.

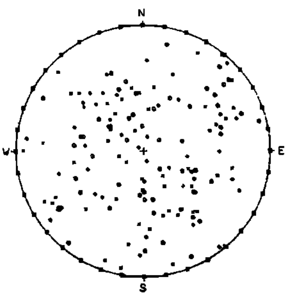

Fig. 2. Principal axes, samples of 50 from each of above sets.

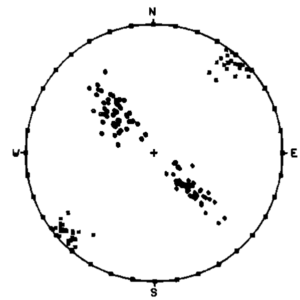

Fig. 5. Principal axes, samples of 50 45°SE, 100 45°NW and 200 90°SW.

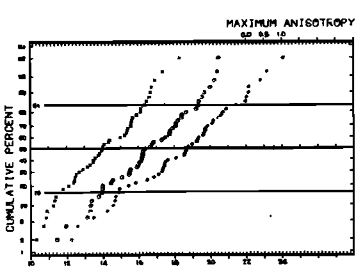

Fig. 3. Principal permeabilities, cgs units × 10^6 from above model.

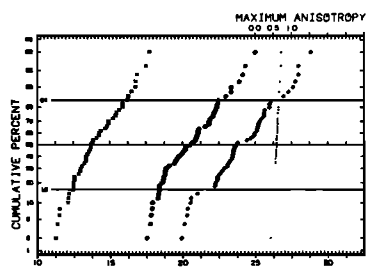

Fig. 6. Principal permeabilities, cgs units × 10^6 from above model.

PLATE 5. Anisotropic permeability of three sets of fractures.

lie on a plane of isotropy even though a third orthogonal set exists. If that third set is weaker than the other two, for instance with greater spacing, then the central tendency of normals to the weaker set is the major axis. If the third set is stronger by reason of closer spacing or less dispersion, then that axis is the minor permeability direction. Directional permeability can be depicted by a prolate or oblate spheroid in these respective cases.

If all three orthogonal sets are different by reason of different dispersions or by different spacings, as in Plate 5, Figure 4, the principal axes are parallel to the central tendencies of the sets. The major axis is parallel to the normal of the weakest set of joints, and the minor axis is parallel to the normal of the strongest. If each set has fractures of similar aperture distribution, spacing is more important than orientation dispersion in determining anisotropy.

When three sets are not orthogonal, then the major axis lies closest to the greatest number of joint intersections. The position of the three principal axes may be estimated roughly according to the findings of this study of special cases.

CONCLUSIONS AND RECOMMENDATIONS

Some properties of permeable fractured media can be anticipated from model studies of idealized systems of planar conduits:

1. The discharge of a single parallel plate opening can be expressed as a symmetric second rank tensor, and if the conduit is itself anisotropic, the tensor has two symmetric components.
2. If there is flow on each of two or more intersecting parallel plate openings, there is a unique hydraulic gradient generally not lying in either plane, whose projections onto the planes cause the flows. The flow through an intergranular porous medium lying between the fractures is proportional to such a field gradient.
3. The tensor permeability of jointed granular porous media may be obtained by superposition of contributions due to the fractures and due to the permeable solids.
4. A medium cut by parallel fractures has infinite anisotropy. The permeability parallel to the conductors is proportional to the average of cubes of apertures and inversely to the average spacing between conductors. When both coarse and fine openings are present, the coarse ones dominate.
5. The inverse of specific surface serves to define the spacing of plane conductors dispersed in orientation.
6. The permeability of a dispersed set of plane conduits is a symmetric second rank tensor, the contributing terms from each individual conduit weighted inversely to the absolute value of the cosine of its inclination from the average direction.
7. If several sets exist in a volume, the number of individuals of each set is related to its specific surface and orientation dispersion.

Since real joints in rock are not parallel plate conduits, their areal nonuniformities and roughnesses need to be assessed. Mathematical or physical models could assess the extent-to-spacing ratio. A distribution of apertures could describe areal variations of a single fracture.

On the assumption of uniformity it is found that:

1. The anisotropic permeability ellipsoid of a single dispersed set is an oblate spheroid, flattening as fracture alignment improves, with a plane of isotropy parallel to the average conductor plane.
2. Two orthogonal sets with equal properties have the anisotropic permeability of a prolate spheroid, with maximum parallel to the intersection of the sets twice that on an isotropic plane normal to both sets.
3. Three equal orthogonal sets are statistically isotropic.
4. Three unique principal axes occur in all cases of lower conduit orientation symmetry, with the maximum along the most frequent direction of intersections, the least tending to lie normal to the set of greatest conductivity.
5. The principal axes of any arbitrary system of plane conductors can be estimated from inspection of a stereonet plot of normals but cannot be specified unless apertures are measured.
6. Variations in permeability measures, as in drill holes, are consequences of sampling heterogeneity, each test reflecting the properties of a few intersected joint conductors in the large population contained in a formation.

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