Dear Prof. Abels,

First let us express our gratitude to the referees for their very valuable comments and remarks on our paper. We are submitting the revised version, where we try to answer all questions and explain in more detail the relation of our work to other research. Our response to the reviewers' comments are stated below. Changes made to the manuscript are marked by blue color in the PDF file biot_fracture_changes.pdf. Thank you and all reviewers for considering the acceptance of our paper.

Sincerely,

Jan Březina and Jan Stebel

Answers to the reviewers' comments

Note: Reference to pages and lines correspond to the PDF file with marked changes.

<u>Reviewer: 1</u>

Points of criticism:

- ***The first lies in the validity of the assumptions, in particular the linear relationships at the fractures. Due to these assumptions, the model is not able to properly capture contact conditions, allowing the two sides to interpenetrate. Moreover, friction laws are ignored. These two processes are essential to modeling mechanics of fractured porous media. In fact, they form the main difficulty and the authors need to justify why these can be neglected and to which degree the resulting model is of interest to the reader.***

  We agree that the contact and friction conditions are essential to modeling realistic fractured porous media. The linearized model is valid for small deformations only. The reason why using this simplified assumption is to keep the subsequent analysis reasonably demanding. We provide a reference to our recent paper on numerical simulations which uses an extended model with nonlinear fracture mechanics. Hence, we have added the following explanation in the introduction (page 2, lines 76-83):

  „In order to keep the subsequent analysis reasonably demanding, we do not consider the nonlinear effects on fracture-matrix interface (such as contact, friction, and displacement-dependent conductivity). This makes the resulting model valid only in the case of small variations of fracture aperture. We also restrict ourselves to the case of a single fracture. We refer to [38] where we present a more complex numerical model with fracture network and nonlinear fracture poromechanics.“

- ***Second, the proof relies on a fixed-stress iterative scheme that depends on well-posed sub-problems for flow and mechanics. While this is a valid technique to use, it seems unnecessarily complicated in this linear case. The problem is given by a bilinear form in (5.2) concerning only two variables. Appendix B shows how to obtain an energy estimate with V = d_t U and Q = d_t P. Does the Babuska-Lax-Milgram theorem not immediately provide the result now for each time step? Another direct approach would be to recognize the system as an evolutionary equation and apply the theory of maximal monotone operators.***

  It is true that there are other methods for proving well-posedness of our problem, including the ones mentioned by the reviewer. We want to stress that the iterative splitting is chosen in order to analyze the convergence of the associated numerical strategy, which is our actual interest. We also note that irrespectively of the chosen method of proof, the well-posedness relies on the validity of the Korn inequality (see Lemma 5.3) which has to be proven in the mixed-dimensional case. We have therefore slightly modified the abstract and added the following paragraph to the introduction (p. 3, l. 92-103):

„We analyze the well-posedness of the DFM poroelasticity model using an iterative splitting technique, which provides explicit information about the convergence rate. The DFM model exhibits a similar parabolic-elliptic structure as the classical Biot system, enabling the use of monolithic methods such as in [37, 43]. However, our interest also extends to the numerical solution, where iterative splitting of flow and mechanics proves advantageous. The convergence of such splittings is a nontrivial issue that requires a careful analysis. Employing the optimized fixed-stress method [31], we prove the contractivity of the appropriate mapping, thereby demonstrating the well-posedness of the DFM poroelasticity. This result can later be used to design efficient numerical methods with a priori known convergence rate.“

(see also Remark 5.1 on p. 19)

Major comments:

- ***Are the Biot equations valid in the fracture zone Omega_f? If the fracture is treated as a poroelastic medium, then the model responds with a counteracting force in case of tension. These forces need to be explained, i.e. what physical process is keeping the fracture together instead of letting the fracture open?***

  Indeed, the fracture stiffness is used in many engineering models not only to prevent closing but also in connection with the surface roughness and healing of the fracture. This is now explained in the introduction (p. 2, l. 60-63):

  „Natural fractures are never smooth, and can be filled by solid material e.g. due to healing [39]. Hence fracture stiffness, non-penetration and friction conditions have to be taken into account. Fracture stiffness can be estimated using rock joint models [2], see also [35].“

- ***The equations look similar to those presented in:***
  "Girault, V., Wheeler, M. F., Ganis, B., & Mear, M. E. (2015). A lubrication fracture model in a poro-elastic medium. Mathematical Models and Methods in Applied Sciences, 25(04), 587-645."
  and
  "Girault, V., Wheeler, M. F., Kumar, K., & Singh, G. (2019). Mixed formulation of a linearized lubrication fracture model in a poro-elastic medium. In Contributions to Partial Differential Equations and Applications (pp. 171-219). Springer, Cham."
  Please clarify the contributions of this work with respect to the results presented therein.

  In the introduction, we have cited these papers and mentioned that (in contrast to the present paper), they consider void fractures without an intrinsic stiffness (p. 2, l. 53-55):

  „Poroelasticity models coupled to discrete fracture flow, neglecting the mechanical response of the fractures, were studied e.g., in […, 22, 23, ...].“

- ***It is not clear whether the proposed model belongs to the class of models introduced in: "Boon, W. M., & Nordbotten, J. M. (2022). Mixed-dimensional poromechanical models of fractured porous media. Acta Mechanica, 1-48." The authors are asked to include a***

statement to emphasize the difference between their work and the well-posedness theory presented therein.

Comments to the difference have been added to the introduction (p. 2, l. 66-72) and Remark 4.1 (p. 9). In principle, the model of Boon & Nordbotten is obtained by a different approach, starting already from the reduced geometry of fractures. The constitutive and balance laws that they generalize to the mixed-dimensional setting are designed in order to preserve a structure suitable for the analysis. We do not see any fundamental argument why this approach should yield exactly the same result as the semi-discretization used in our paper, which starts from the classical continuum mechanical relations on the full-dimensional fracture domain. While certain differences between the two models are mentioned, another study might be useful to further investigate their impact on the qualitative as well as quantitative properties of solutions.

- ***A clarification needs to be made whether the fracture aperture delta is independent of the displacement U and why this is assumed to be the case or not. With a delta of 1e-5 in the numerical examples, how is it guaranteed that the fracture does not close? What happens if the jump in displacements is larger than delta?***

  In the introduction (p. 3, l. 79-80) we mention that the model is valid under small variations of fracture aperture. On p. 4, l. 136-137 we explicitly state that we consider the aperture independent of the displacement field. Concerning the numerical example, the setting leads naturally to opening of the fracture, hence the problem with possibly closed fracture does not appear here.

- ***Please include the limitations of the proposed model in the introduction. For example, it needs to emphasize that the analysis concerns a single fracture, that the model is linear and the non-physical effects this introduces, and that the numerical examples only concern a 2D example.***

  Done, see l. 76-81 and l. 112 in the introduction.

Minor comment:

- ***Similar equations were recently considered in "Bonaldi, F., Droniou, J., & Masson, R. (2022). Numerical analysis of a mixed-dimensional poromechanical model with frictionless contact at matrix-fracture interfaces. arXiv preprint arXiv:2201.09646."***
  I would suggest the authors to note the similarities and differences.

  Done, see l. 63-66 in the introduction.

<u>Reviewer: 2</u>

1. ***There is a recent paper on poroelasticity with fractures***
   https://link.springer.com/arti
   The authors should compare their (theoretical) results with the one in this paper.

   (The same answer as to the 3rd major comment of Referee 1.) Comments to the difference have been added to the introduction (p. 2, l. 66-72) and Remark 4.1 (p. 9). In principle, the model of Boon & Nordbotten is obtained by a different approach, starting already from the reduced geometry of fractures. The constitutive and balance laws that they generalize to the mixed-dimensional setting are designed in order to preserve a structure suitable for the analysis. We do not see any fundamental argument why this approach should yield exactly the same result as the semi-discretization used in our paper, which starts from the classical continuum mechanical relations on the full-dimensional fracture domain. While certain differences between the two models are mentioned, another study might be useful to further investigate their impact on the qualitative as well as quantitative properties of solutions.

2. ***The authors considered linear, quasi-static Biot. Would this work for non-linear extensions as well? For example for the models in***
https://link.springer.com/arti
or/and
https://link.springer.com/arti
I suggest that the authors add a comment regarding non-linear models.

A comment on possible generalization to nonlinear poroelasticity such as in the proposed papers has been added to the conclusions, p. 22, l. 397-399.