

# Lecture 2 - Visualisations, density estimates and model choice

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See <https://github.com/jbrinchmann/MLD2023> as usual

Lectures/Lecture 2 has the PDF for today and

Lectures/Lecture 2/Notebooks for Jupyter notebooks used for this lecture

# The super-brief review of last lecture

**Manage your project using git:**

git add      git commit -m “I did something”      git push

**SQL is a language for querying databases - and it looks quite similar to English:**

```
SELECT Star, g, r FROM Stars  
WHERE r < 14.5
```

**When combining multiple tables we can JOIN them**

```
SELECT s.Star, o.Field, o.Date  
FROM Stars as s JOIN Observations as o  
    ON s.fieldID = o.ID  
WHERE s.Star = ‘S2’
```

# The super-brief review of last lecture

Parameters are called  $\theta$ , predicted values  $\hat{y}$

We always need a way to say how good a model is - usually composed of the residual of the model minus data:

$$\text{RSS} = \sum_i (y_i - \hat{y}_i)^2$$

Linear models are crucial building-blocks:

$$\hat{y}_i = \theta_0 + \sum_{j=1}^p \theta_j x_{ij} \quad \Leftrightarrow \quad Y = \mathbf{M}\theta$$

and if you have too few data you might want to regularise

$$\text{RSS} + \lambda \sum_{j=1}^p |\theta_j| \quad (\text{the LASSO})$$

# And now some words about Python & SQL

There are many ways to interact with databases in Python. I will cover two: an in-depth one and a simple one.

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Not covered, but maybe of relevance:

MySQL interface for Python: [MySQLdb](#)  
PostgresSQL with Python: [psycopg2](#)  
Oracle with Python: [cx\\_Oracle](#)

There are also many places you can learn much more about this topic:

I quite like the compilation of links on Full Stack Python:  
<https://www.fullstackpython.com/databases.html>

Google will give you many other options - it is a very popular topic...

# And now some words about Python & SQL

There are many ways to interact with databases in Python. I will cover two: an in-depth one and a simple one.

---

I will assume that you have a database file already created for the following slides. If you do not, then you can either get ProblemSets/MakeTables/MLD2019.db from the GitHub site, or you can play with e.g. the places.sqlite database in your Firefox profile directory.

# Python & sqlite3 the comprehensive way

```
import sqlite3 as lite
```

Load what is necessary

The database must be created first!

```
con = lite.connect(database)
```

Connect to database

```
with con:
```

```
# Get a cursor.
```

```
cur = con.cursor()
```

```
# Execute commands
```

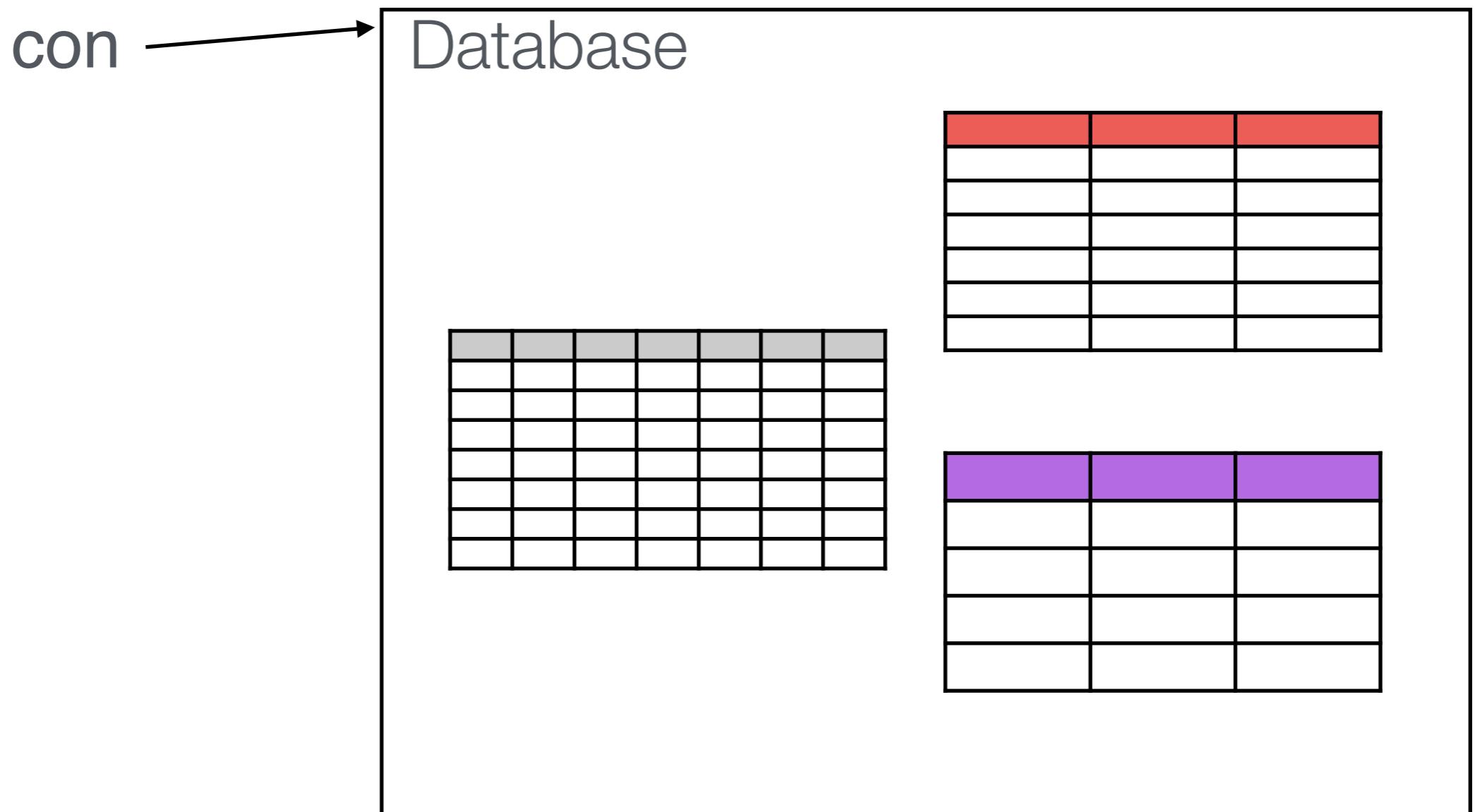
```
cur.execute(command)
```

The Cursor object is one way to pass SQL statements to the database - I like to think about it as the finger reading through a table.

This is frequently seen - but it is not strictly necessary. I would recommend not creating an explicit cursor object like this for simplicity.

# Python & SQL - the Cursor object

`con = lite.connect(database)`



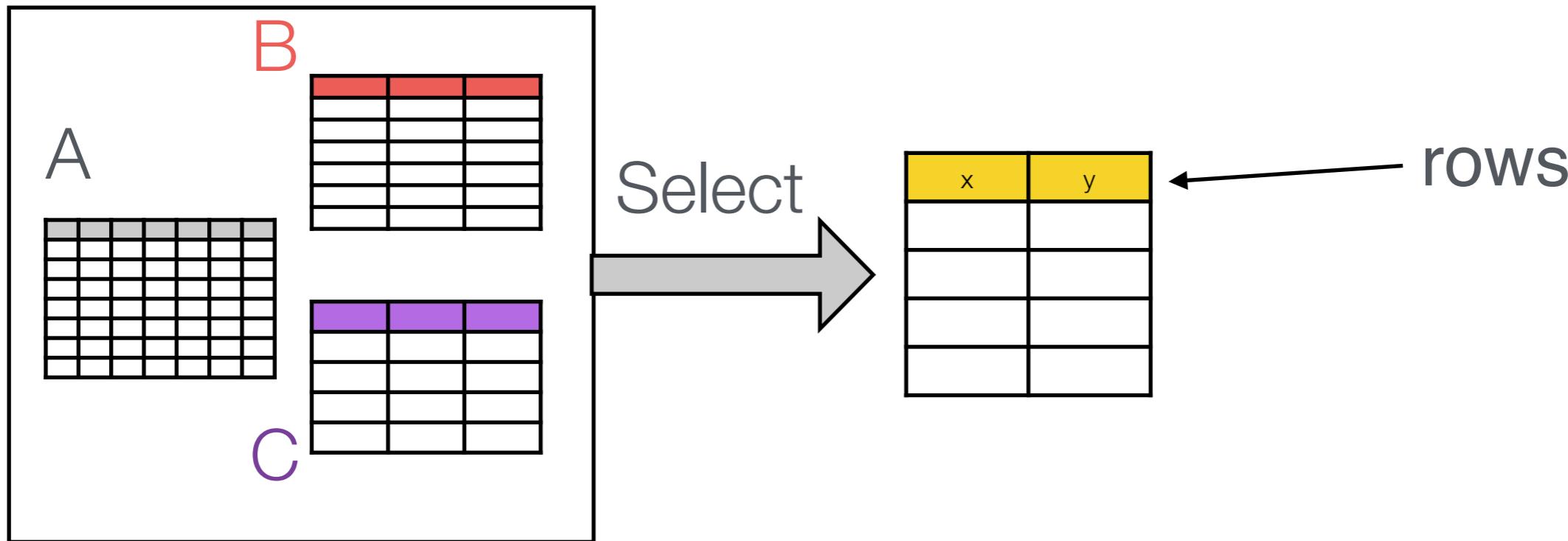
# Using python to query the database:

```
In [1]: import sqlite3 as lite;
```

```
In [2]: con = lite.connect("MLD2019.db")
```

```
In [4]: query = "select s.star, o.Date from stars as s JOIN observations as o ON  
s.fieldID=o.ID"
```

```
In [3]: rows = con.execute(query)
```



# Using python to query the database:

```
In [4]: row = rows.fetchone()
```

```
In [5]: print row
```

```
Out[57]: ('S1', 92.9885764)
```

x	y

← rows

# Using python to query the database:

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In [4]: row = rows.fetchone()
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In [5]: print row
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Out[5]: ('S1', 92.9885764)
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x	y

rows

```
In [6]: for row in rows:
```

```
....:     print(row)
```

```
....:
```

```
('S2', 92.9885764)
```

```
('S5', 93.5532134)
```

```
('S7', 97.3323764)
```

x	y

rows

# Using python to query the database:

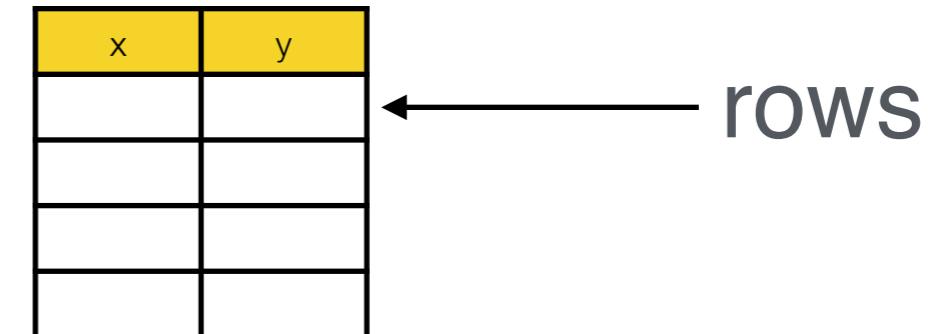
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In [4]: row = rows.fetchone()
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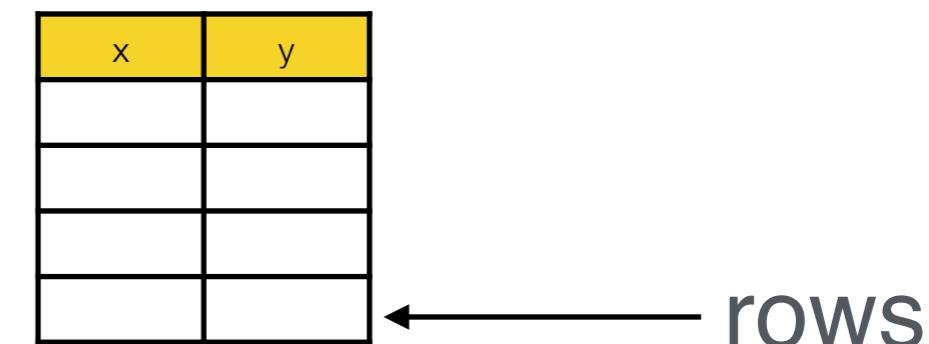
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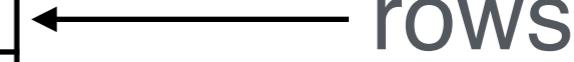
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In [5]: print row
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```
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('S2', 92.9885764)
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('S7', 97.3323764)
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x	y

rows



## Watch out!

```
In [7]: row = rows.fetchone()
```

```
In [8]: row
```

Because we used up all rows in the previous call! If you use cursors this would lead row to contain the last row - a bug waiting to happen.

# Using python to query the database - recommended:

```
In [1]: import sqlite3 as lite;
```

```
In [2]: con = lite.connect("MLD2019.db")
```

```
In [3]: rows = con.execute('SELECT ra, decl FROM Stars')
```

```
In [4]: for row in rows:
```

```
...:     print("Ra={0} Dec={1}".format(row[0], row[1]))
```

```
...:
```

```
Ra=198.8475 Dec=10.5034722
```

```
Ra=198.5654167 Dec=11.0231944
```

```
Ra=198.9370833 Dec=9.9168889
```

```
Ra=199.2516667 Dec=10.3486944
```

Another possibility (not good for large datasets!!)

```
cur = con.execute('SELECT ra, decl FROM Stars')
```

```
rows = cur.fetchall()
```

# The simplest(?) way of dealing with tables in Python

pandas:

```
In [35]: import pandas as pd
```

```
In [36]: con = lite.connect("MLD2019.db")
```

```
In [37]: t = pd.read_sql_query('Select ra, decl from Stars', con)
```

```
In [38]: t
```

```
Out[38]:
```

	ra	decl
0	198.847500	10.503472
1	198.565417	11.023194
2	198.937083	9.916889
3	199.251667	10.348694

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and it is easy to make new tables:

```
In [39]: t.to_sql('Stars2', con)
```

That's it.

# pandas vs normal sqlite

```
# Next, we create a connection to the database.  
con = lite.connect(database)  
  
with con:  
  
    table = "Stars"  
    # Create the command to create the table. I use a  
    # multiline string to ease readability here.  
    command = """CREATE TABLE IF NOT EXISTS {0} (StarID INT,  
                                         FieldID INT, Star varchar(10), ra DOUBLE,  
                                         decl DOUBLE, g FLOAT, r FLOAT,  
                                         UNIQUE(StarID), PRIMARY KEY(StarID),  
                                         FOREIGN KEY(FieldID) REFERENCES Observations(ID))""".format(table)  
  
    # Next, actually execute this command.  
    con.execute(command)  
  
    # Now that this is working, let us loop over the table entries  
    # and insert these into the table.  
    for row in cat:  
        command = "INSERT INTO Stars VALUES({0},{1},{2},{3},{4},{5},{6})".format(row[0], row[1], row[2], row[3], row[4], row[5], row[6])  
        print command  
        con.execute(command)
```

```
In [39]: t.to_sql('Stars2', con)
```

So why not only pandas?

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        con.execute(command)
```

```
In [39]: t.to_sql('Stars2', con)
```

So why not only pandas?

- ★ Control
- ★ Memory usage

# Visualising data

# Making good scientific plots

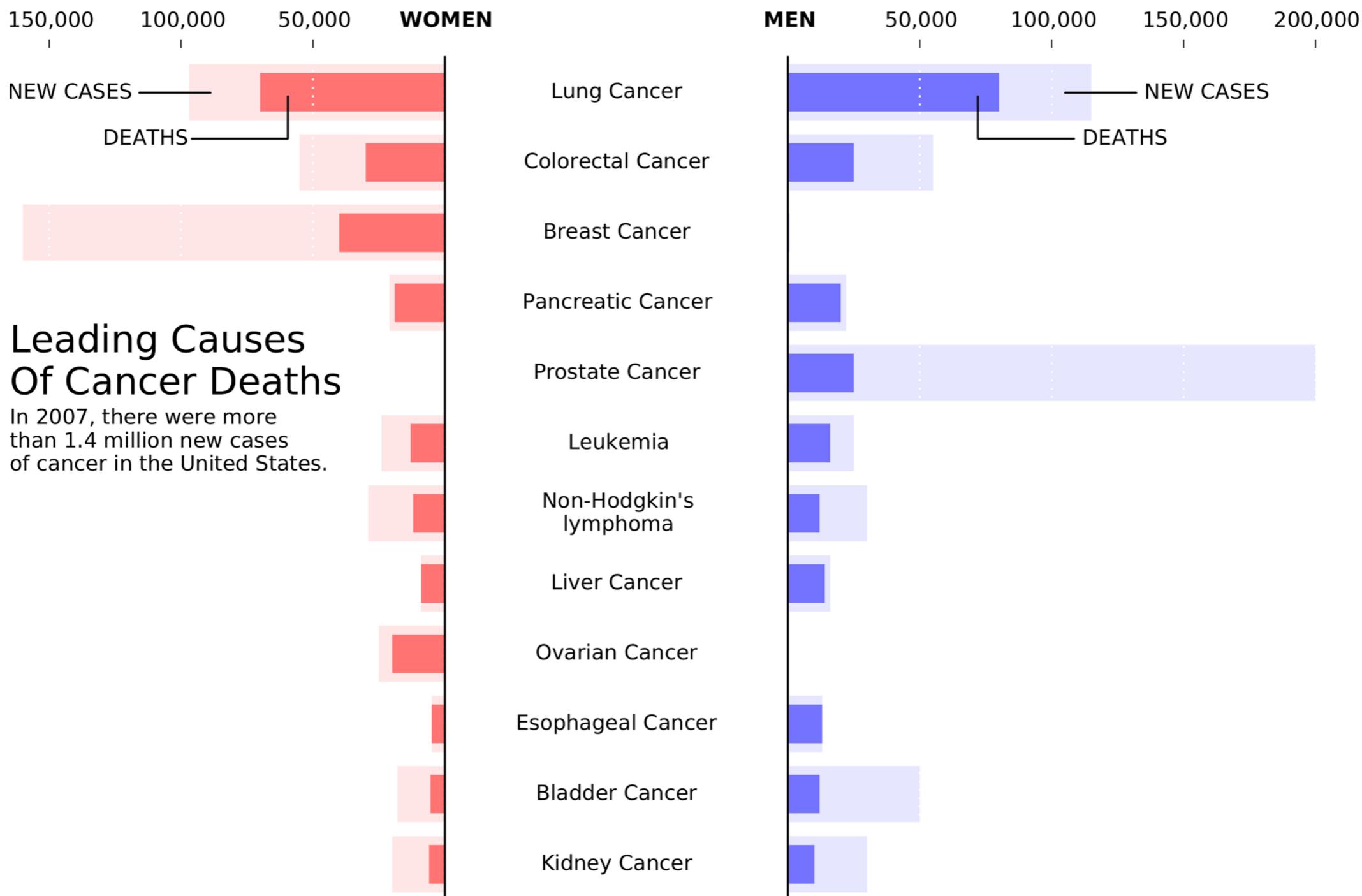
- Know the audience.
- Adapt to the situation.
- Be precise, include all information needed.
- Do not mislead.
- Do not use colour without thinking. Especially bad colour choices.
- Do not overload a figure - or include too little!
- Learn a couple of packages well.

Worth checking out: Rougier NP, Droettboom M, Bourne PE (2014) Ten Simple Rules for Better Figures. PLoS Comput Biol 10(9): e1003833.

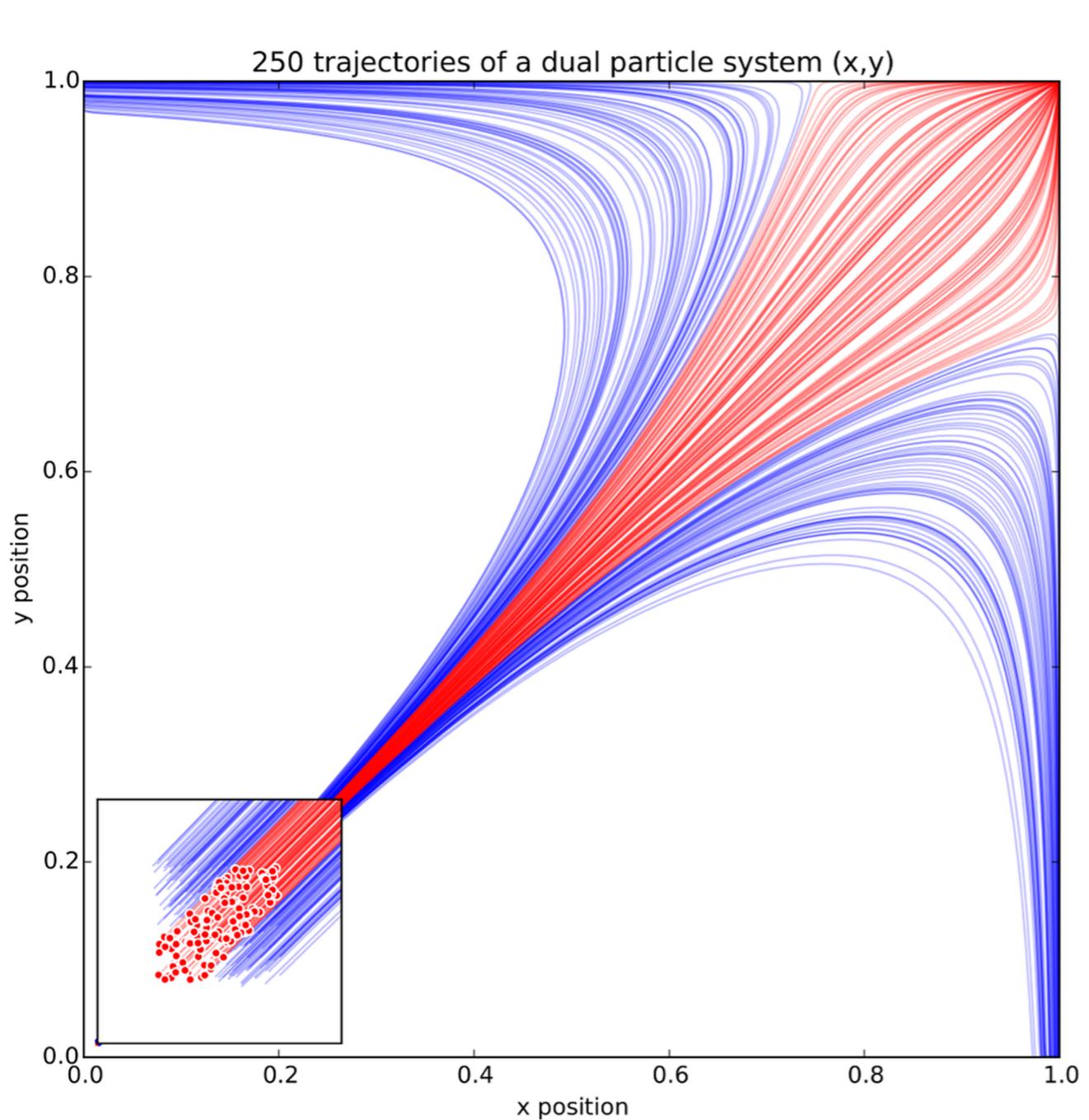
<http://journals.plos.org/ploscompbiol/article?id=10.1371/journal.pcbi.1003833>

# Adapting to the audience

from: Rougier NP, Droettboom M, Bourne PE (2014) - an example of a newspaper article



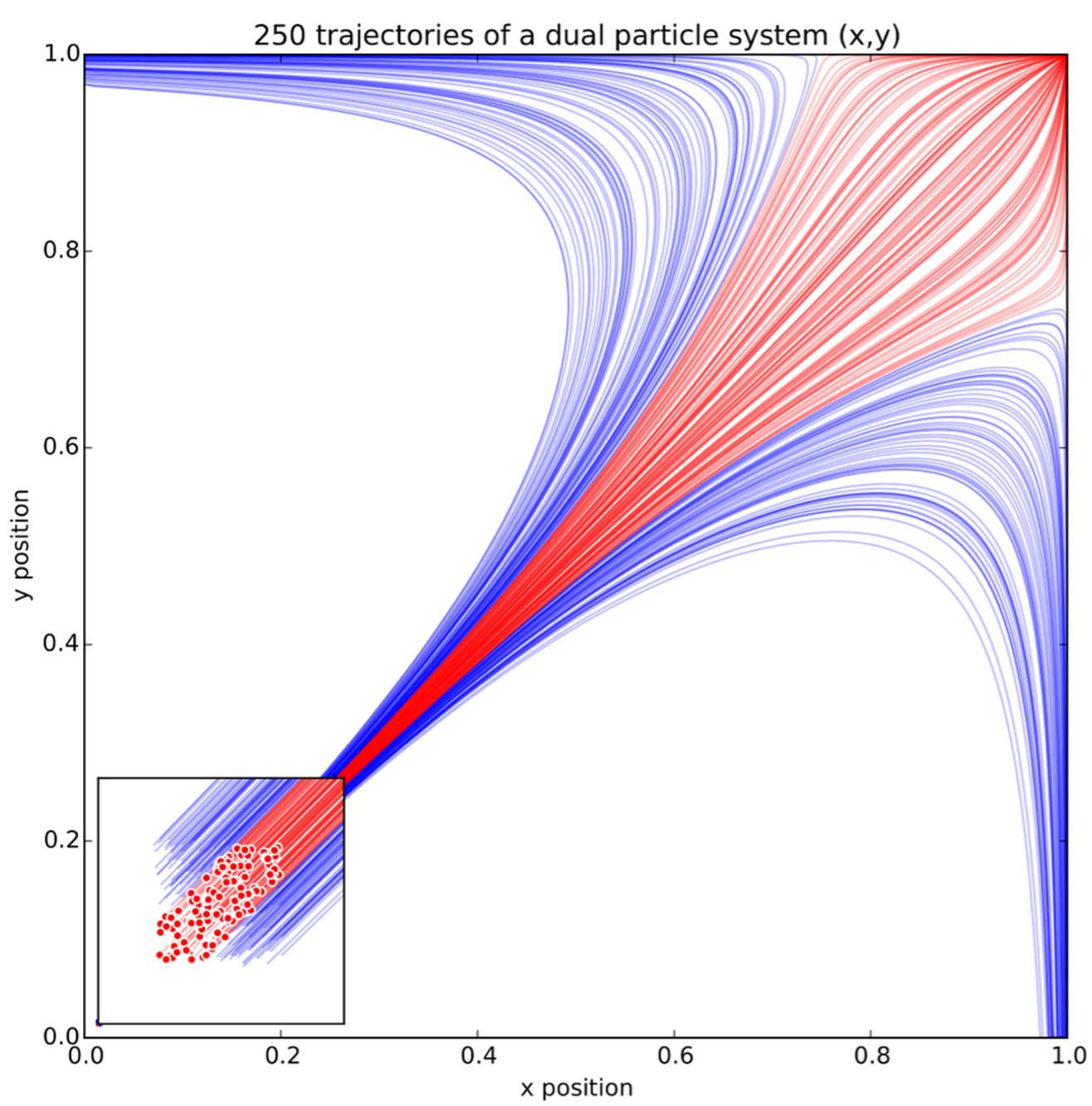
# Adapting to the situation



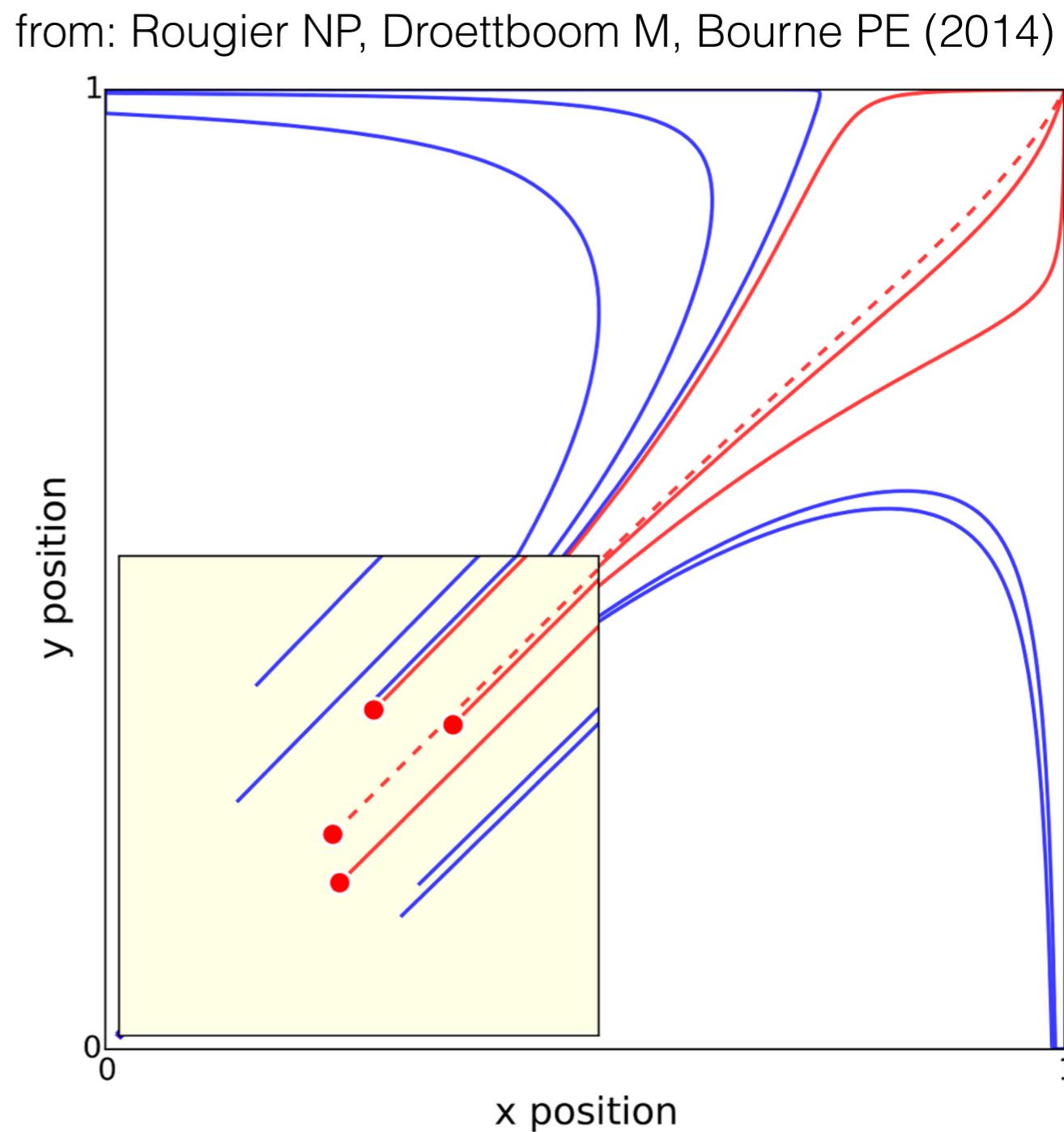
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Scientific paper

# Adapting to the situation



Scientific paper



Presentation

# Why do we create visualisations (plots)?

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To help our argumentation.

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To summarise a lot of data

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# Why do we create visualisations (plots)?

To help our argumentation.

To summarise a lot of data

To discover new relationships

To relate different pieces of information

To develop intuition

# A quick interlude - try out:

- Get the file Datafiles/several\_datasets.tsv from the course website.
- Read it in using e.g. astropy:

```
from astropy.table import Table  
t = Table().read("several_datasets.tsv",  
format="ascii.fast_tab")
```

There are 13 datasets there - t['x1'], t['y1'] contains the x & y values for dataset 1 etc.

Calculate summary statistics (mean, standard deviation, correlation coefficient, maybe a linear fit) for each dataset - what do you conclude?

# Looking at odd data

x	y	x	y	x	y	x	y
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58
8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
12.0	10.84	12.0	9.13	12.0	8.15	8.0	5.56
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89

# An example of *odd* data

x	y	x	y	x	y	x	y
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58
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11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
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14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
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$\text{mean}(x) = 9$ ,  $\text{mean}(y) = 7.5$

$\text{Var}(x) = 11$ ,  $\text{Var}(y) = 4.12$

The correlation coefficient: **0.816**

The best-fit line:  **$y = 3 + 0.5 x$**

# An example of *odd* data

x	y	x	y	x	y	x	y
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8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76
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14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
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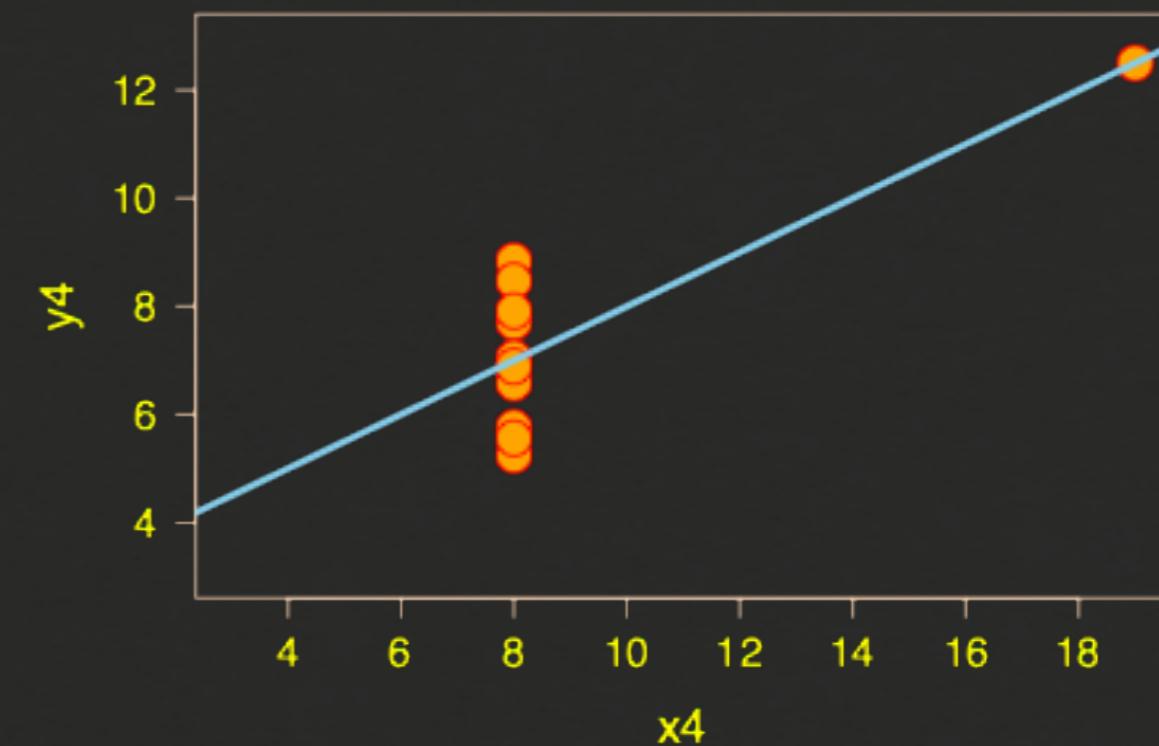
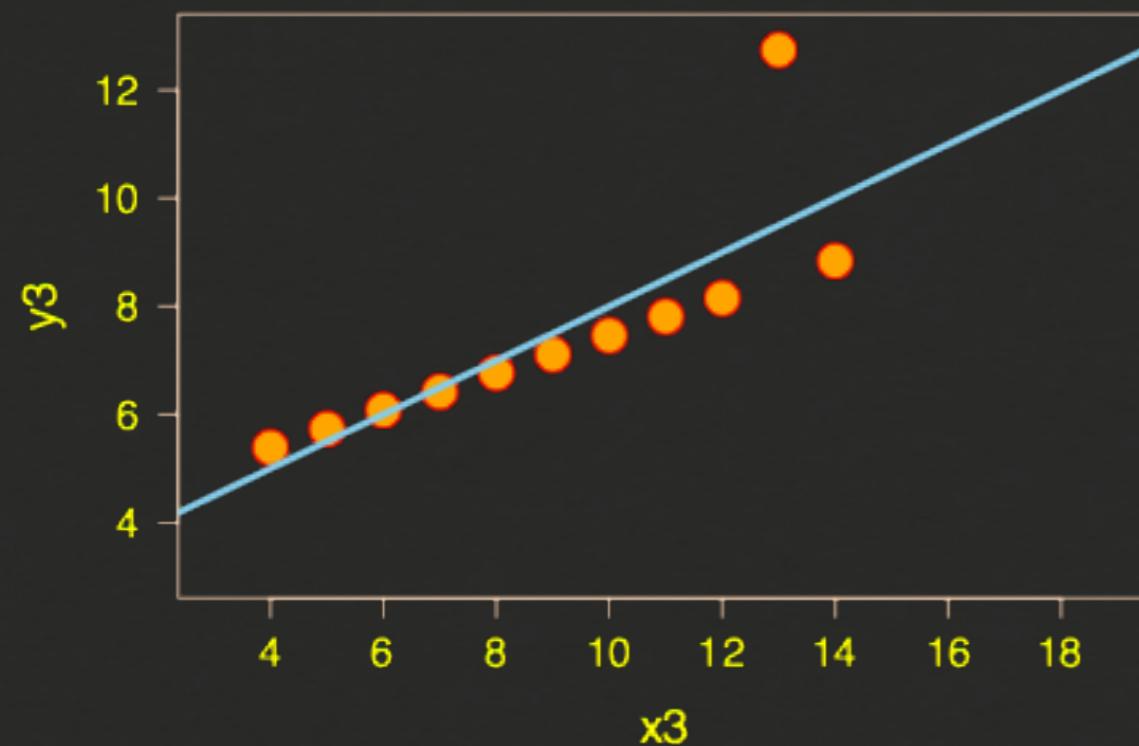
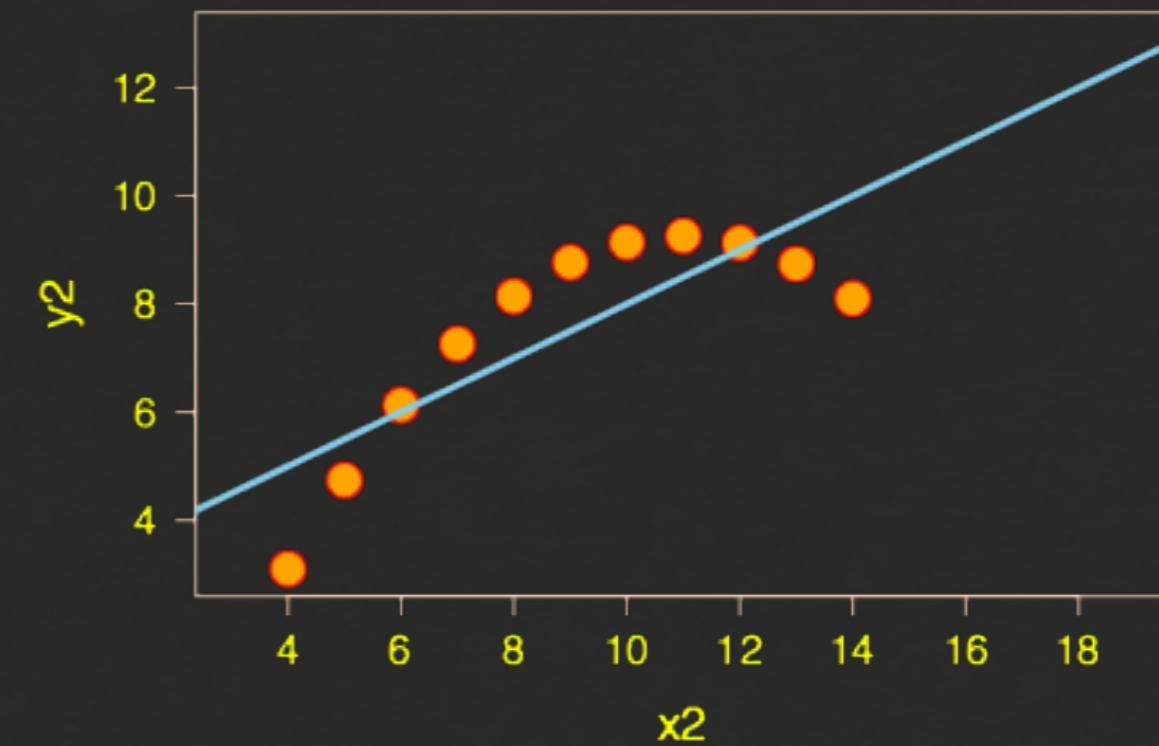
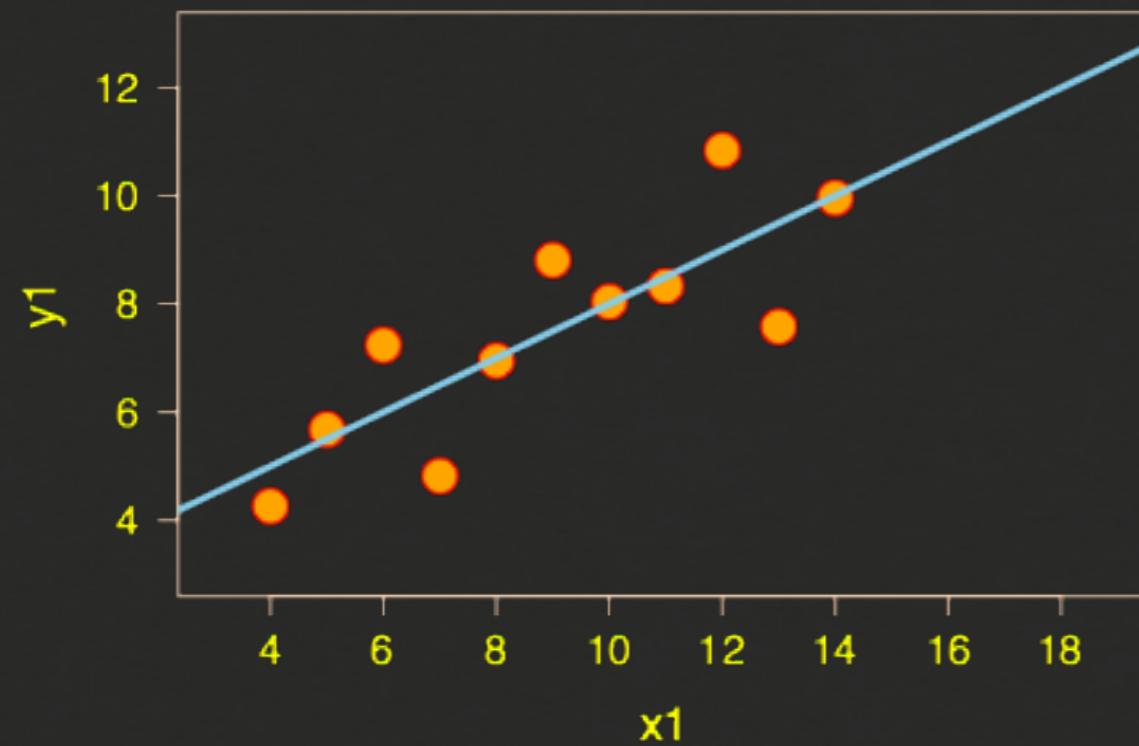
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The correlation coefficient: **0.816**

The best-fit line:  **$y = 3 + 0.5 x$**

**So they seem to be very similar!**



The bottom line: Always plot your data! (but you knew that already I hope!)  
But how would you deal with this in N dimensions?

# Graphical displays should:

- Show the data.
- Induce the viewer to think about the substance.
- Avoid distorting what the data have to say.
- Present many numbers in a small space.
- Make large data sets coherent.
- Encourage the eye to compare different pieces of data.
- Reveal data at several layers of detail.
- Be relevant.
- **Show units, show scales!**

Based on: Tufte: “The Visual Display of Quantitative Information”

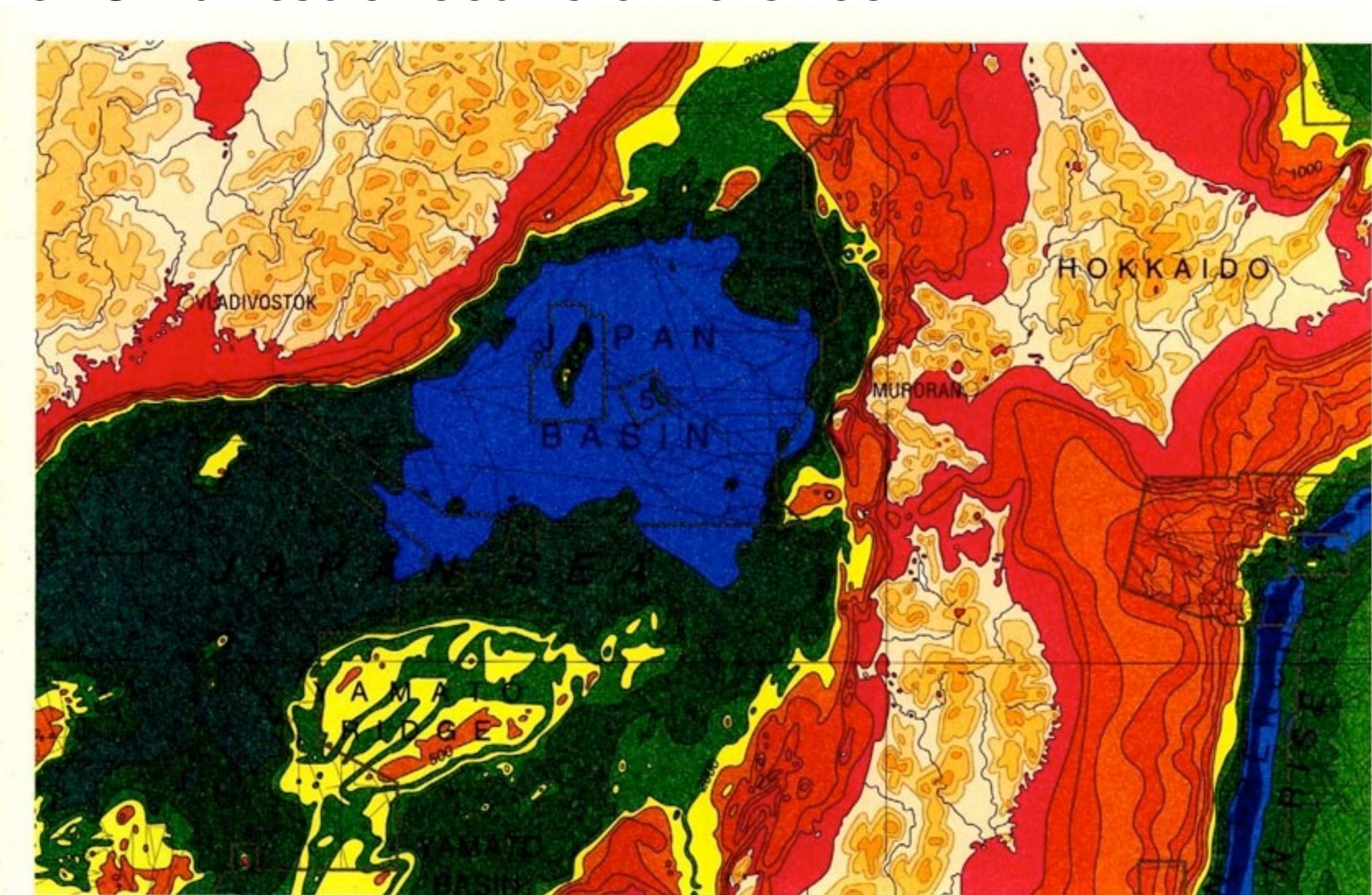
# Colour choice

The “Smallest effective difference”

From Edward Tufte (1997), “Visual explanations”

# Colour choice

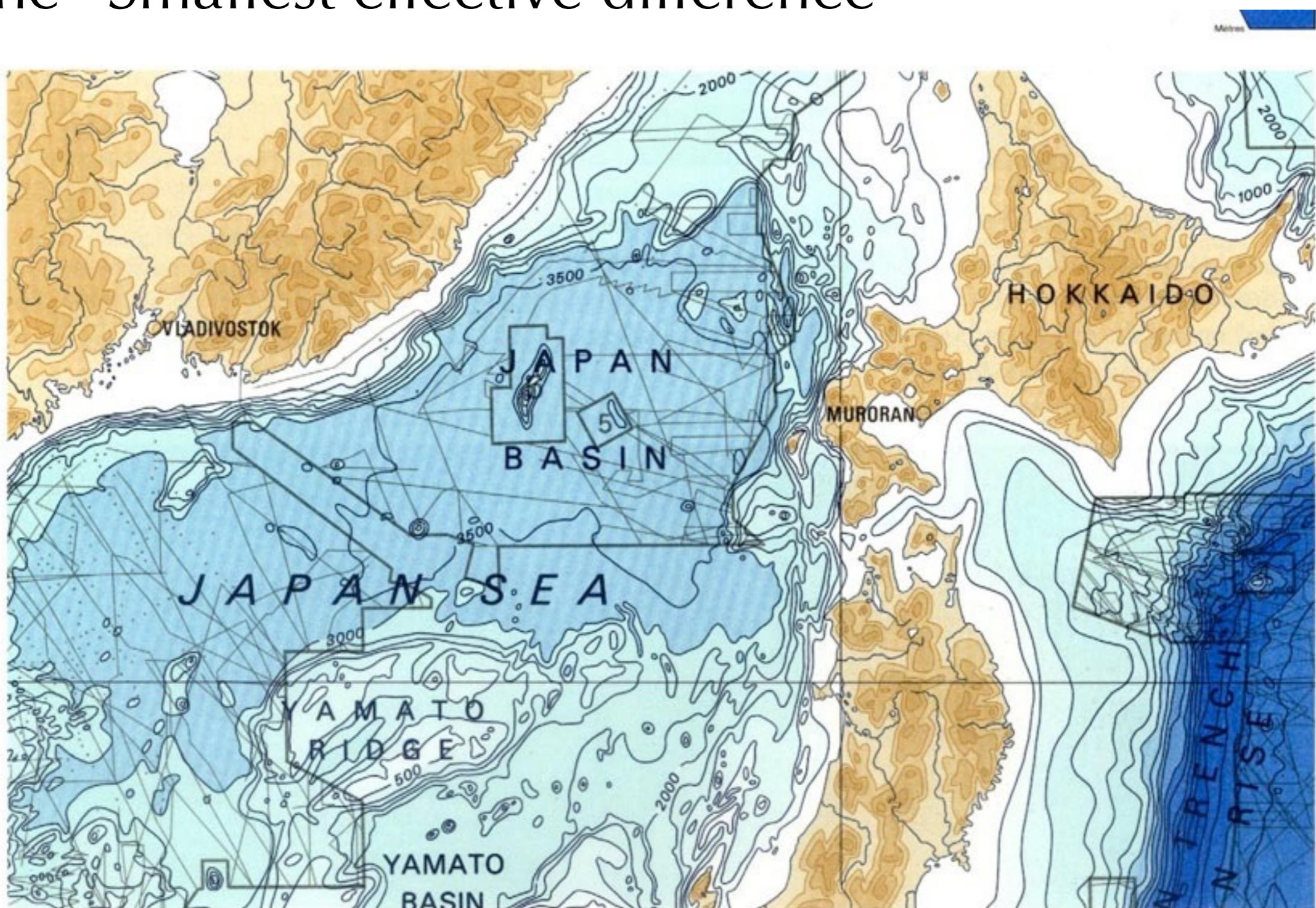
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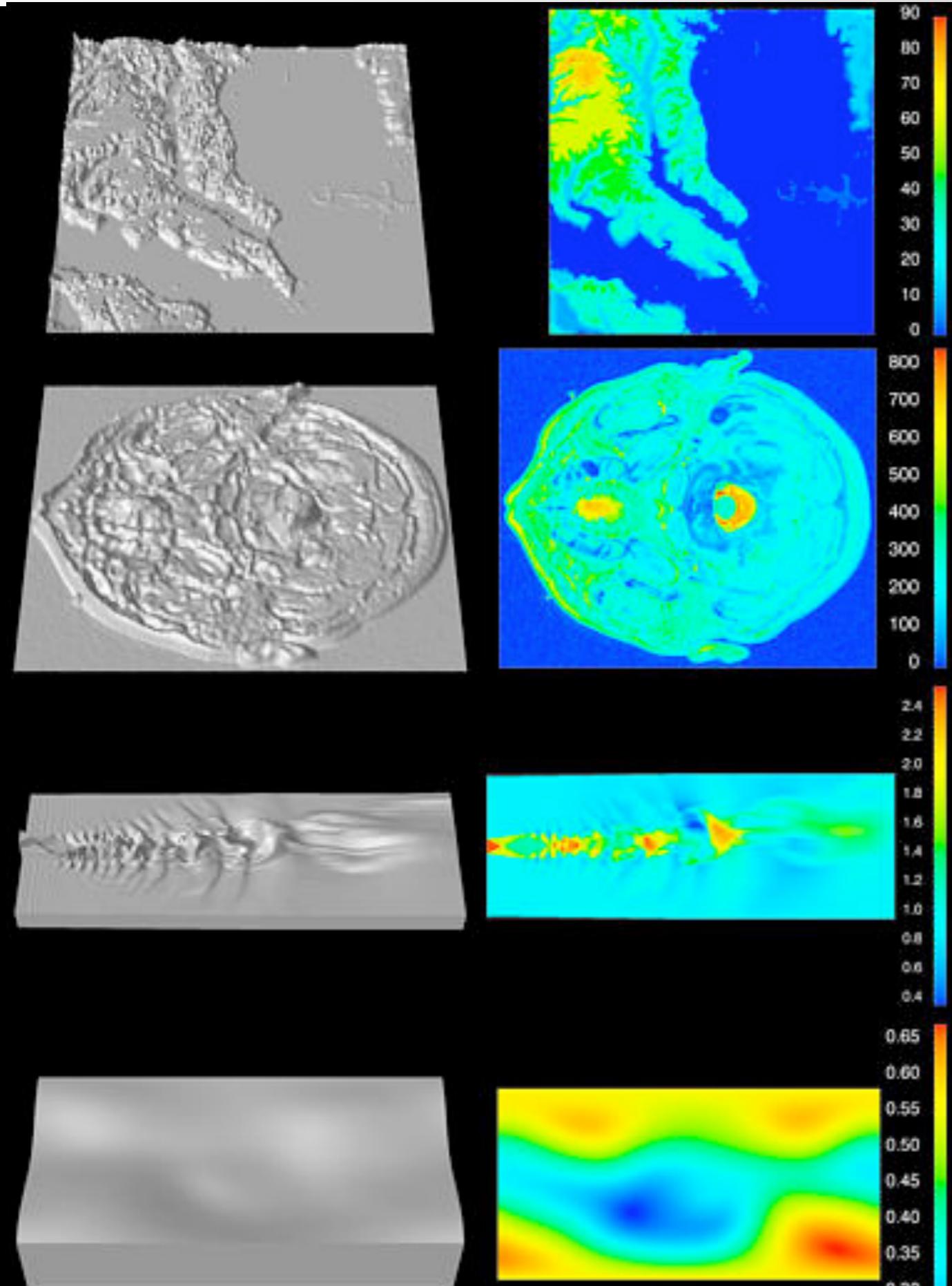
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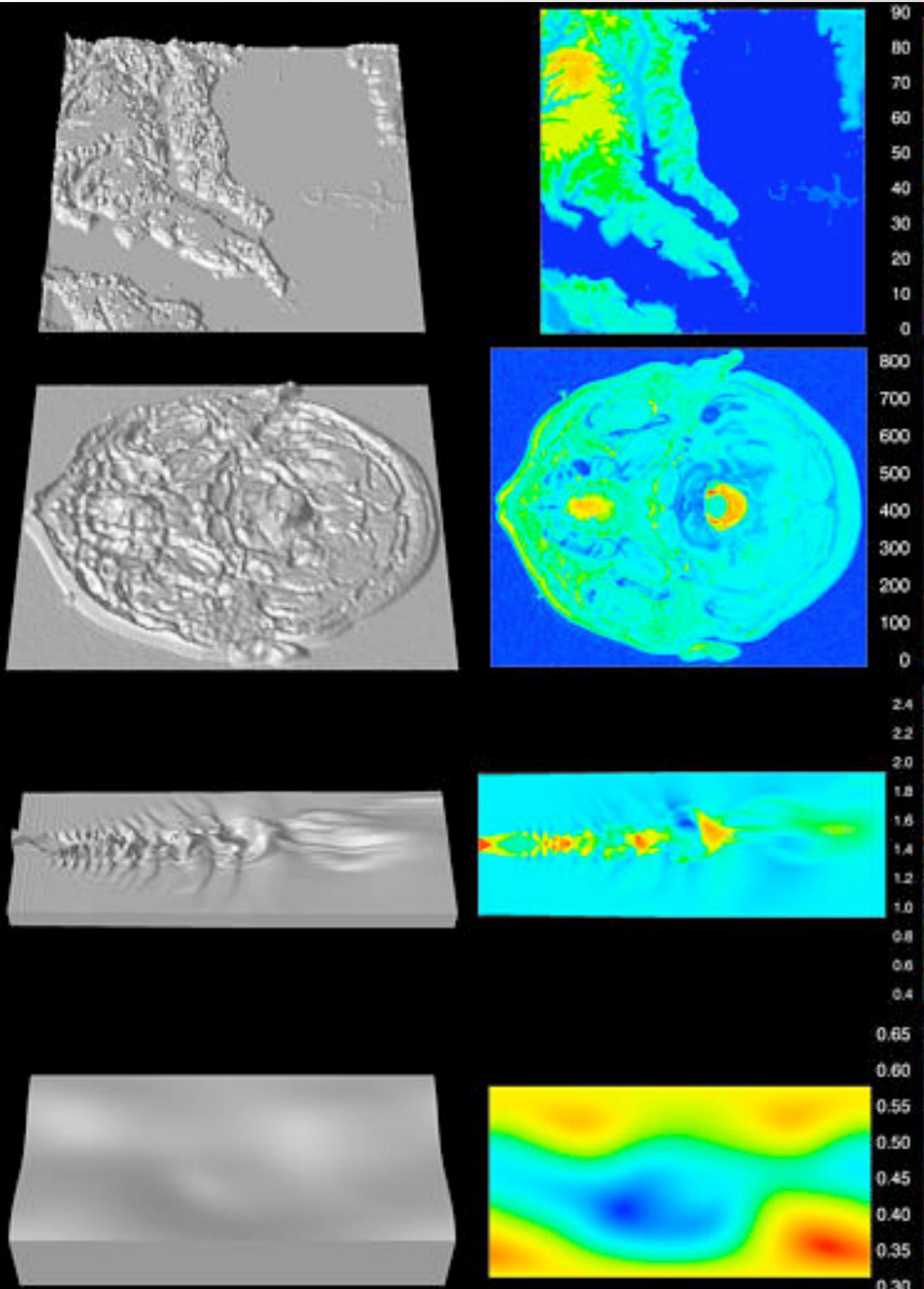
# The issue of colour & 2D plots

There is a tendency for astronomers to choose from a small set of colour schemes for their images and plots. Sometimes this leads you to create artificial trends where there are none.



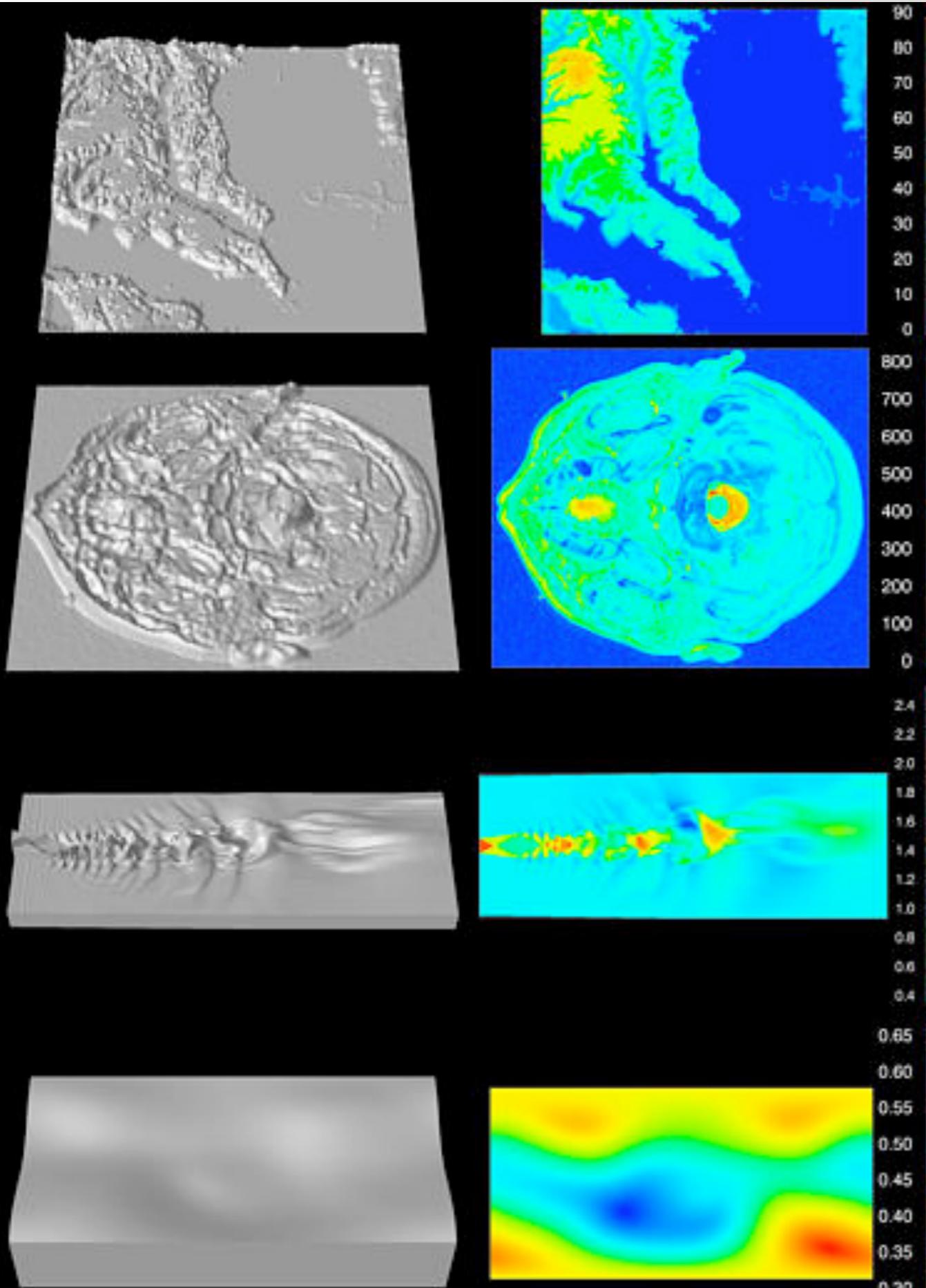
# Bad Habits

For more details: <http://www.research.ibm.com/people/l/lloyd/color/color.HTM>



Chesapeake Bay - note the artificial structure at higher altitude when it is actually quite gradual

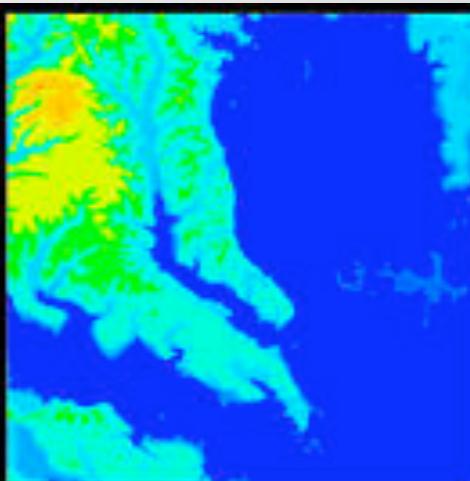
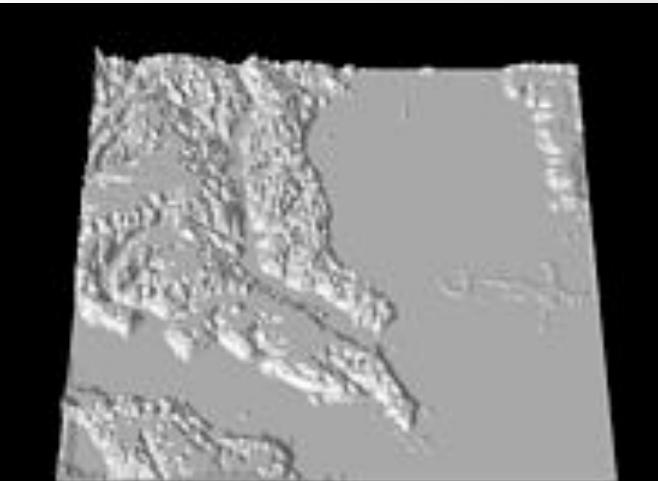
## Bad Habits



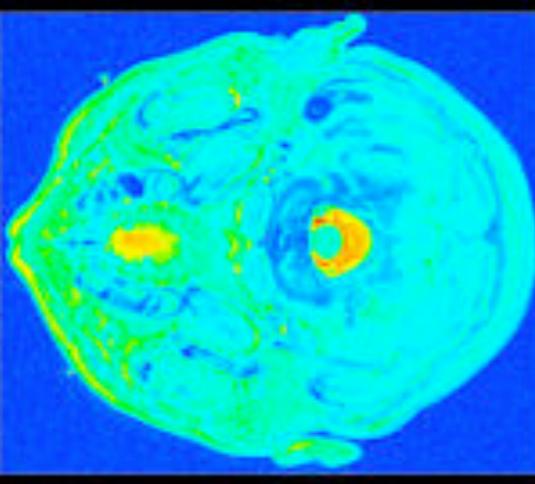
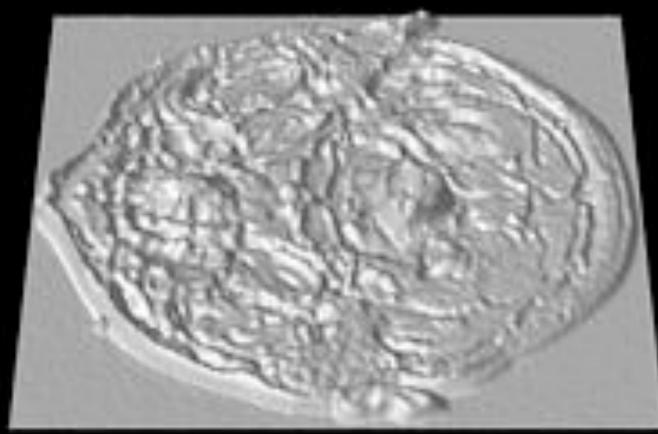
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An slice of an MRI scan of a human brain. Washing out of detail and artificial structure.

## Bad Habits

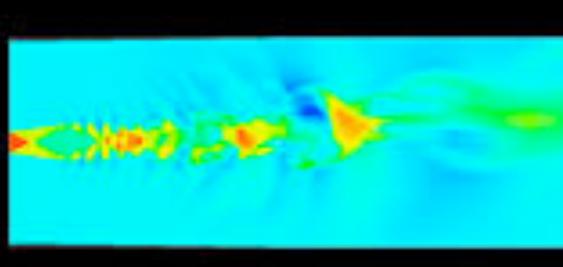


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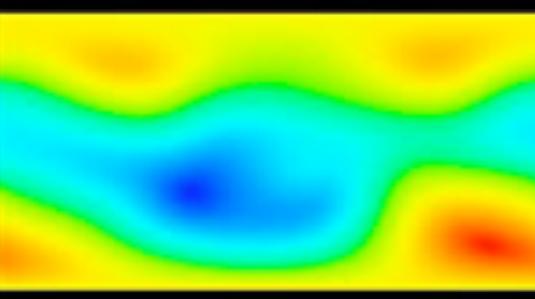


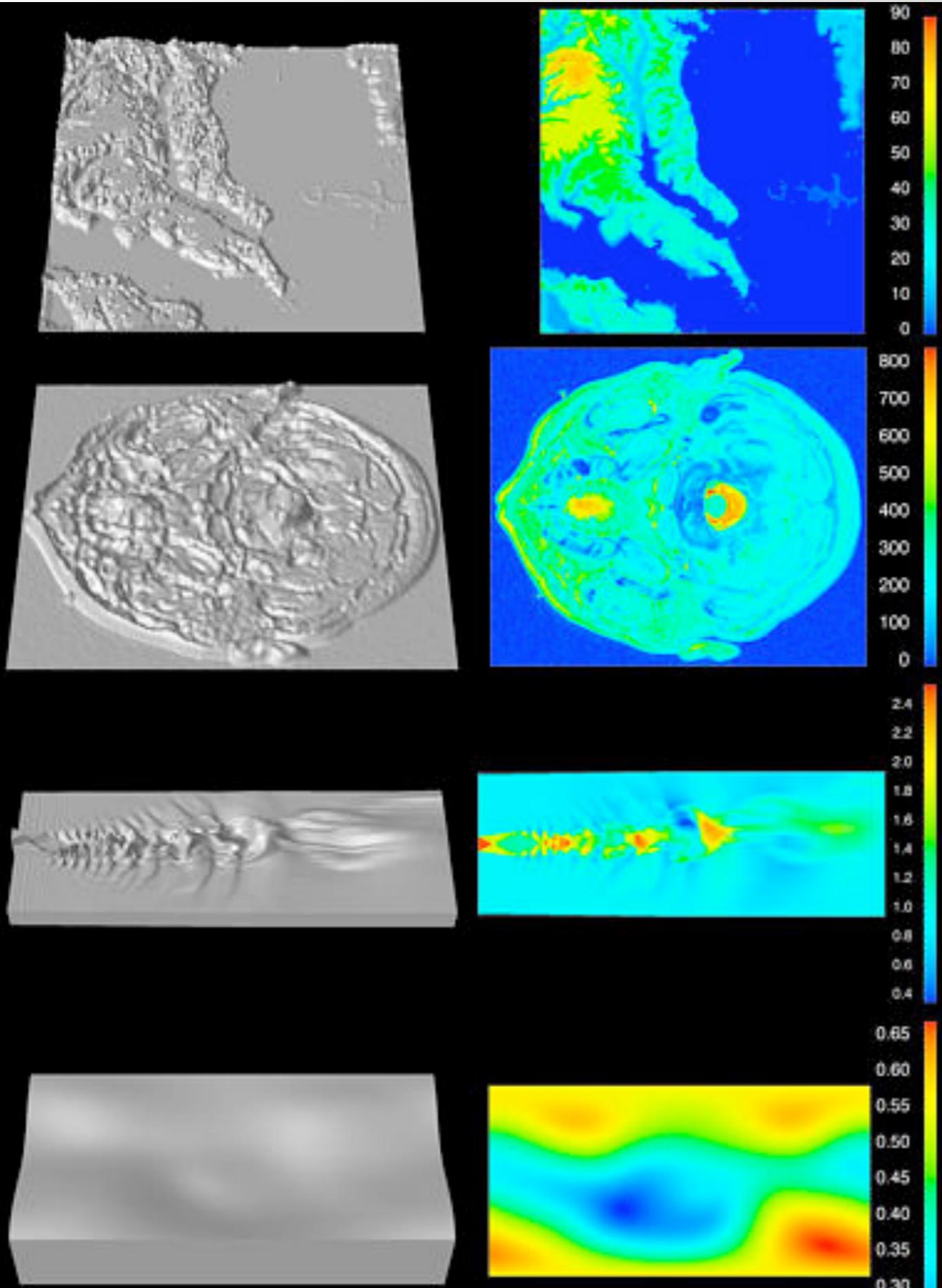
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## Bad Habits



Turbulent flow from a jet engine.





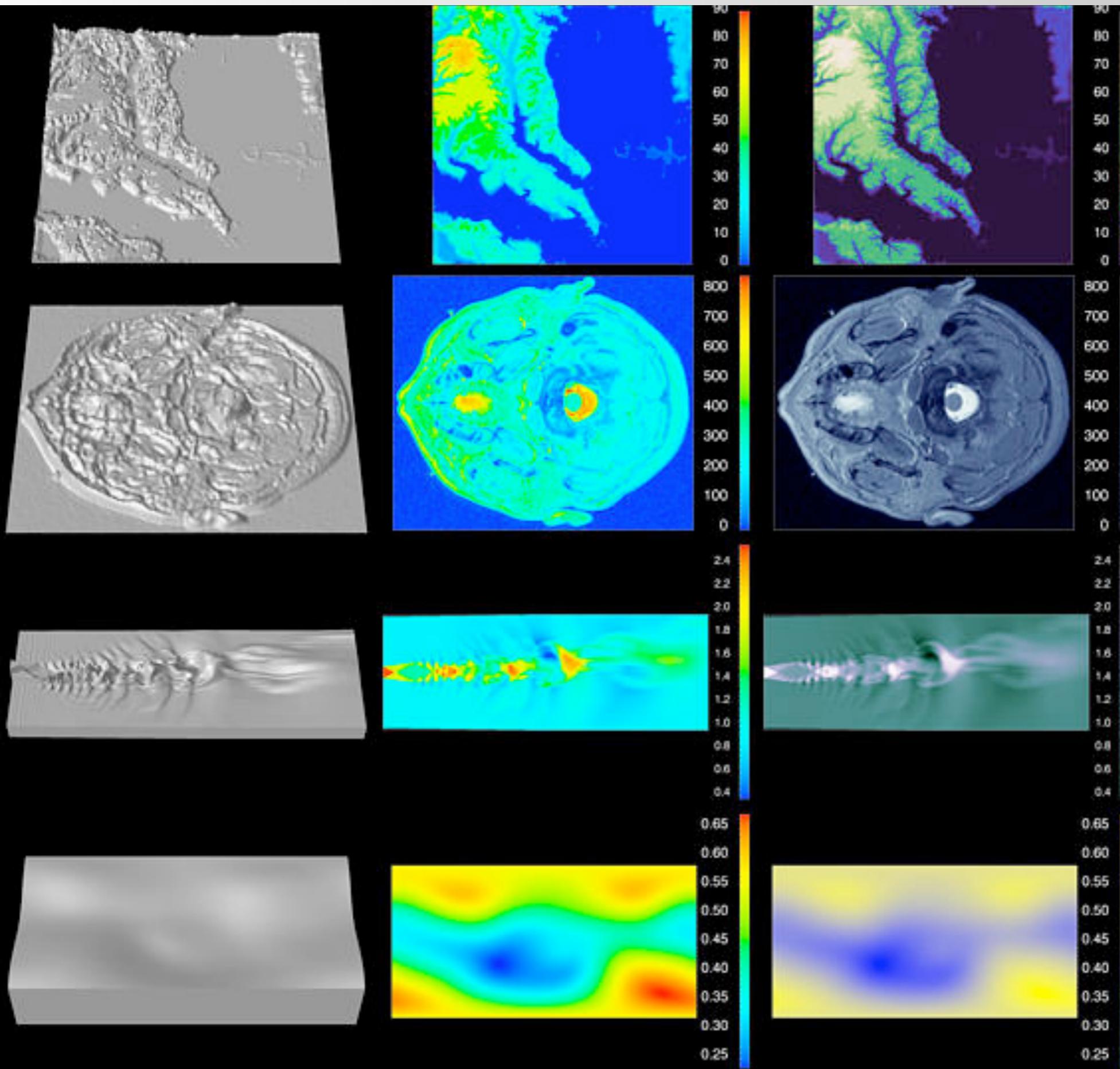
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## Bad Habits

Turbulent flow from a jet engine.

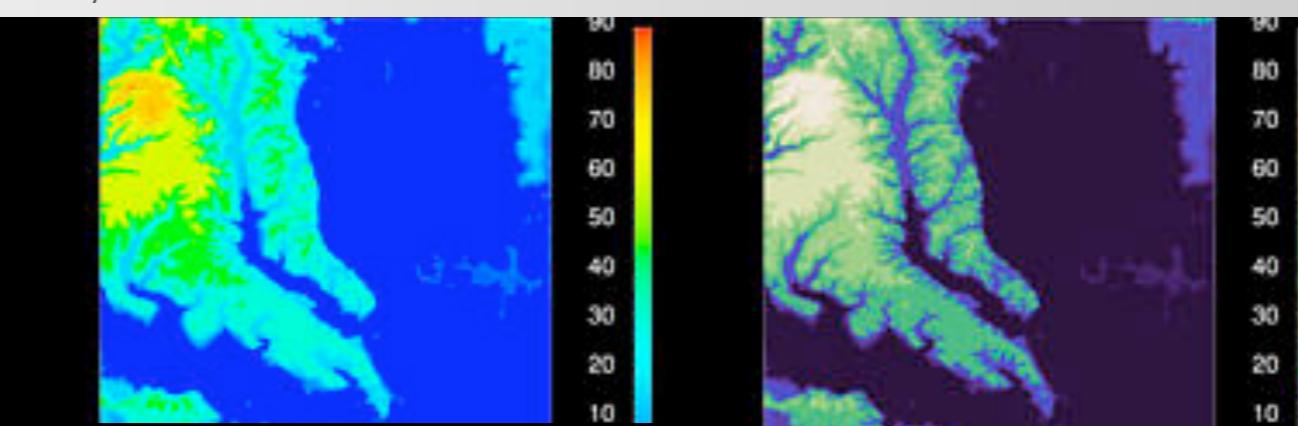
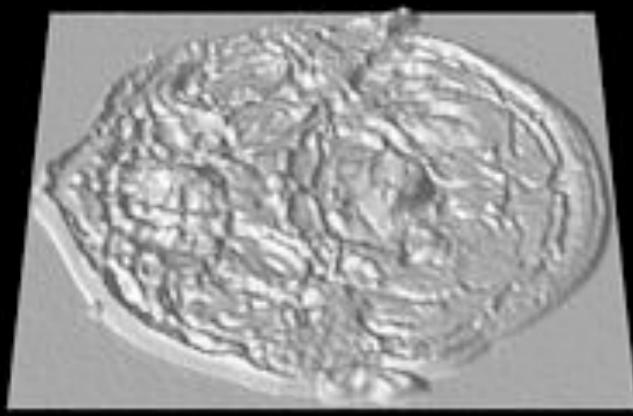
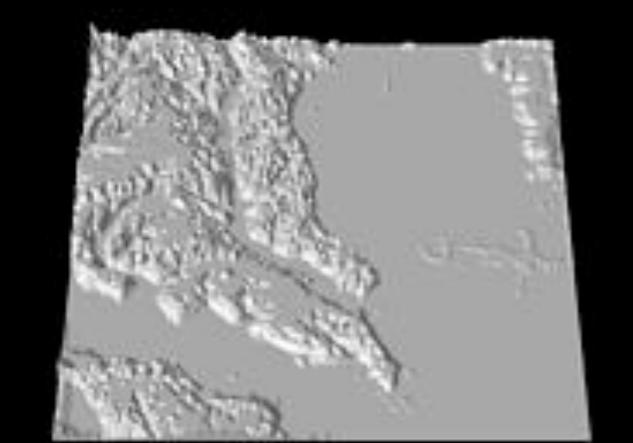
Earth's magnetic field in a Cartesian projection - note the smooth structure.



A natural zero

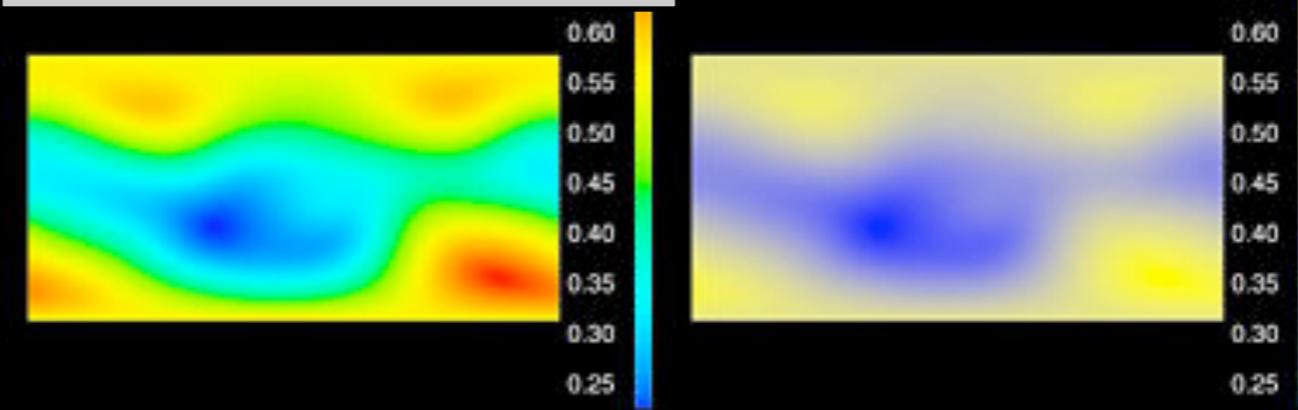
High frequency information - best with variation of luminance

Low frequency information - colour variation (saturation is good)



One key  
lesson:

Try different  
colour maps  
and do not use  
a default one  
mindlessly.

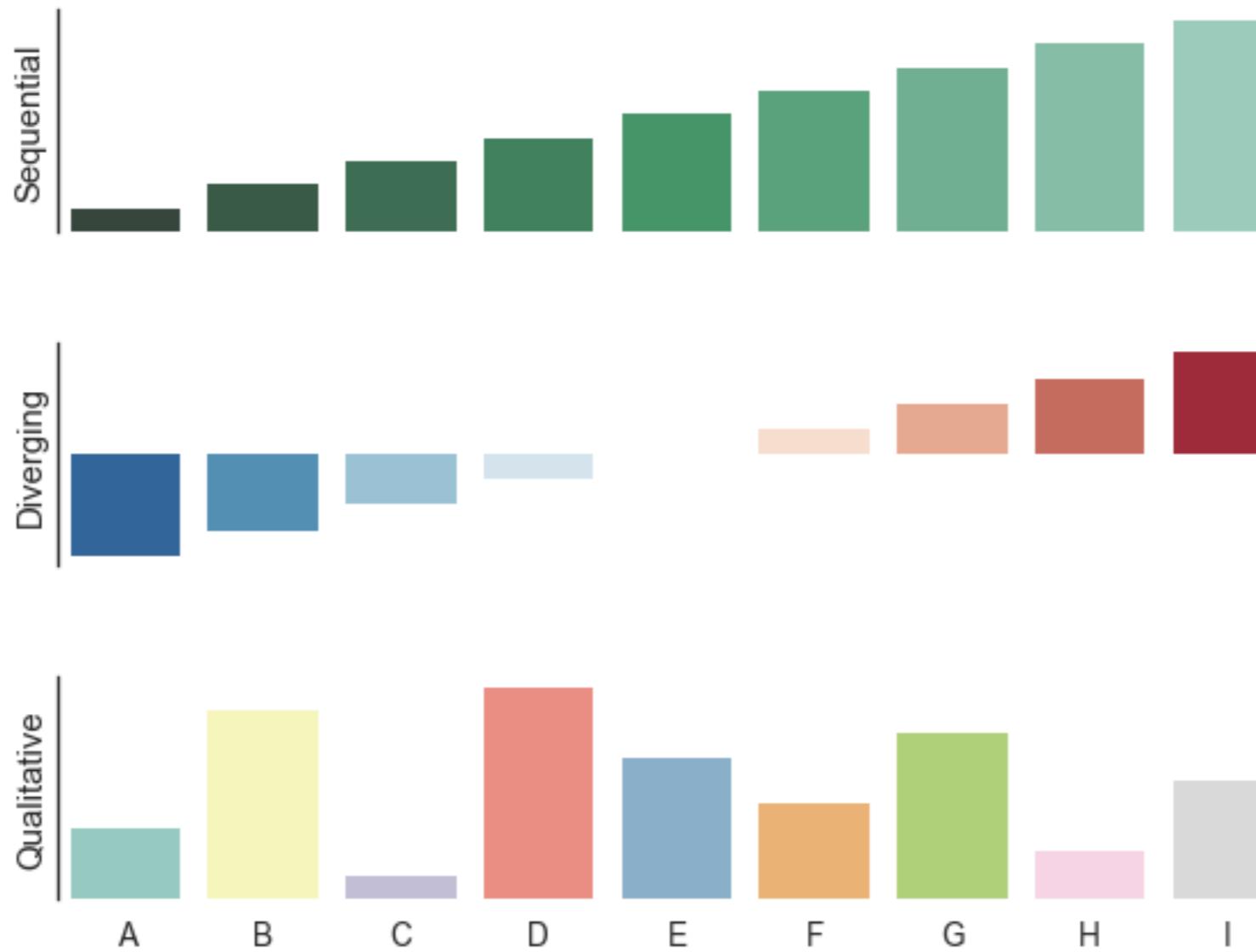


A natural zero

High  
frequency  
information  
- best with  
variation of  
luminance

Low frequency  
information -  
colour variation  
(saturation is  
good)

# Colour should provide information



From the seaborn gallery [but meant to illustrate colour choices]

# Colour choice - colour vision deficiency

This affects ~10% of the population so worth keeping in mind when you make plots for sharing with others.

Simplest: Do not use red & green to create contrast.

Better: use safe colour palettes (viridis/cividis in matplotlib):

In Seaborn:

```
import seaborn as sns  
sns.set_palette('colorblind')
```

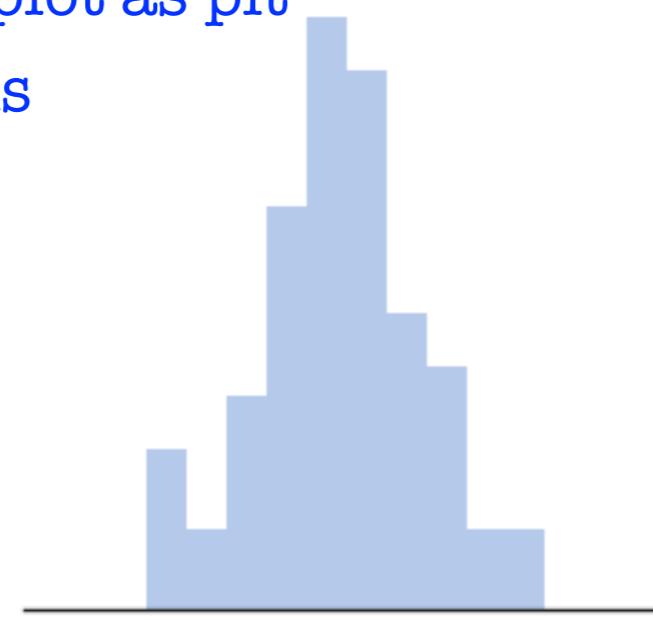
Paul Tol has a very useful page:

<https://personal.sron.nl/~pault/>

Checking: Photoshop/Illustrator have options, in Python, use Jörg Dietrich's Daltonize package: <https://github.com/joergdietrich/daltonize>

# Various ways to show 1D distributions

```
import matplotlib.pyplot as plt  
import seaborn as sns
```

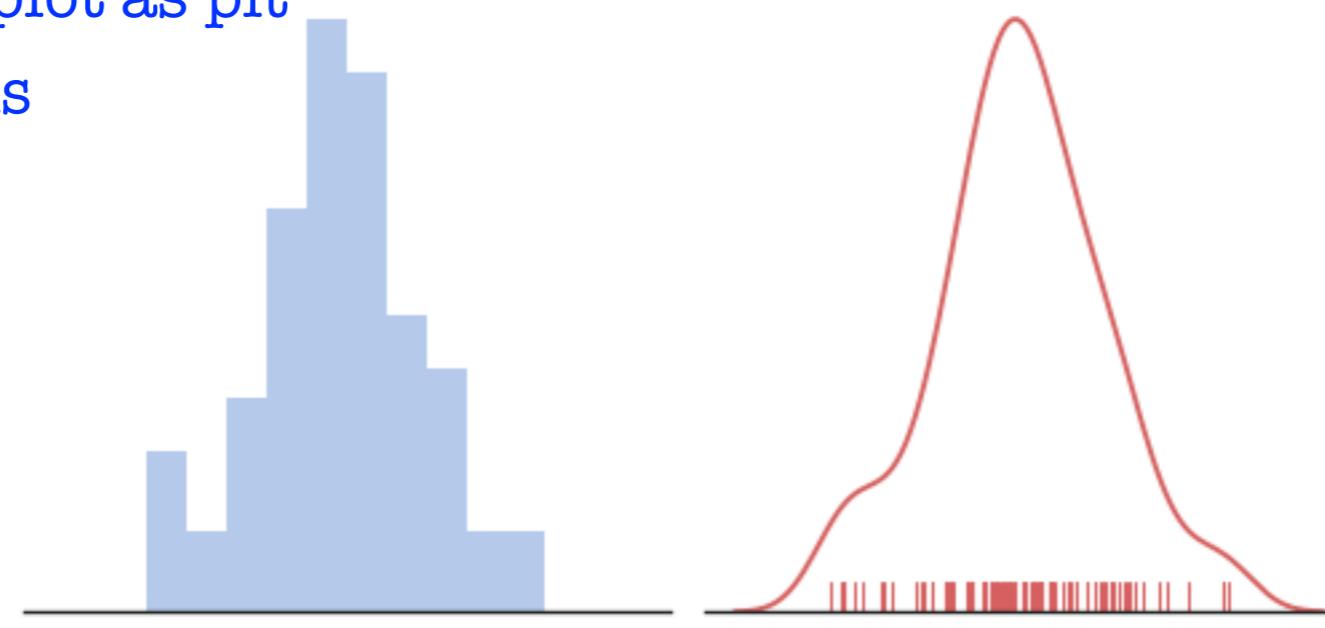


`sns.distplot(x, kde=False)`  
or  
`plt.hist(x)`

Histograms are convenient but bin the data and are not differentiable - we'll look at kernel density plots later. Rug plots complement histograms and kernel density plots by showing the data.

# Various ways to show 1D distributions

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import matplotlib.pyplot as plt  
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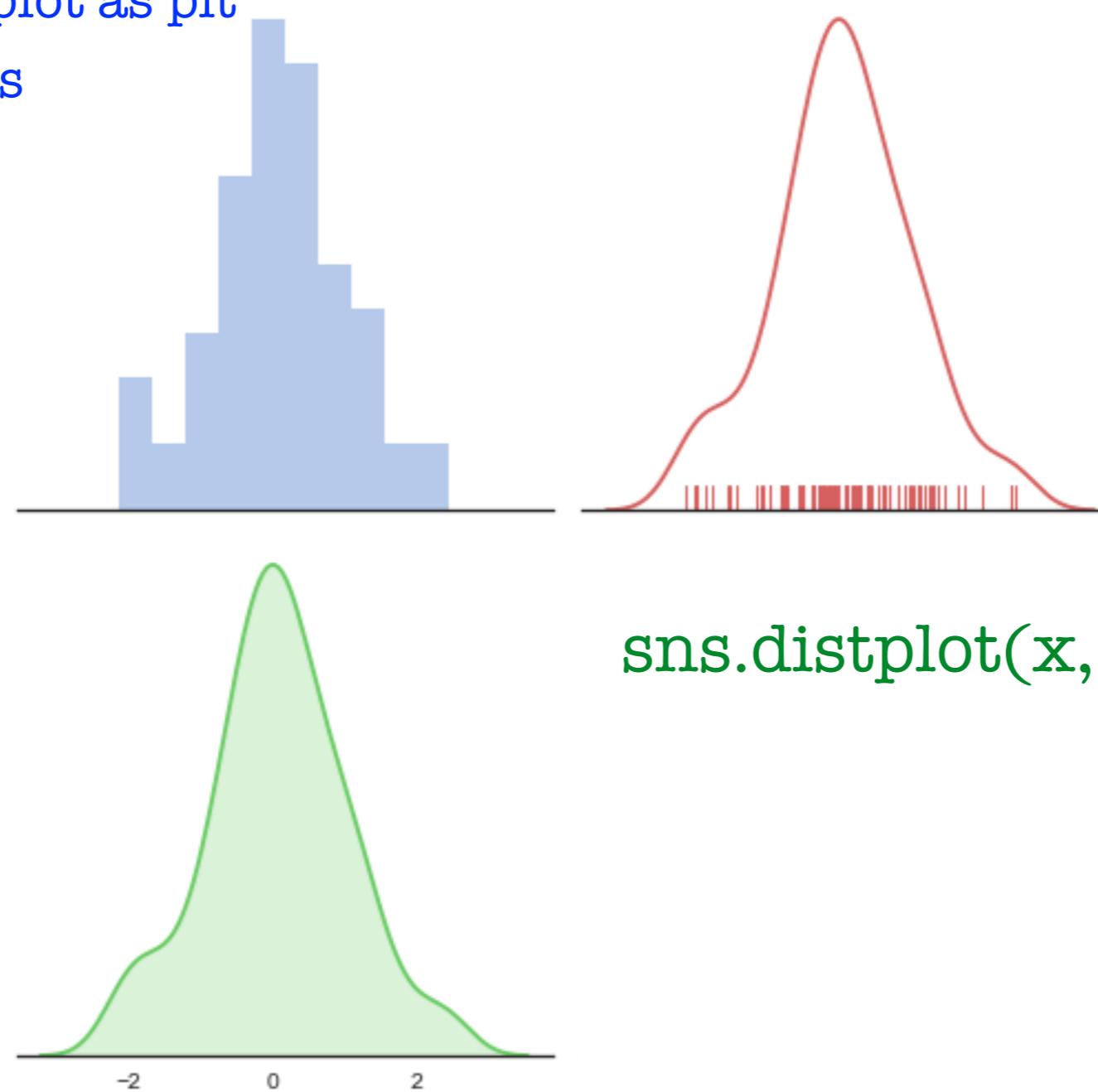


```
sns.distplot(x, kde=False,  
rug=True)
```

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# Various ways to show 1D distributions

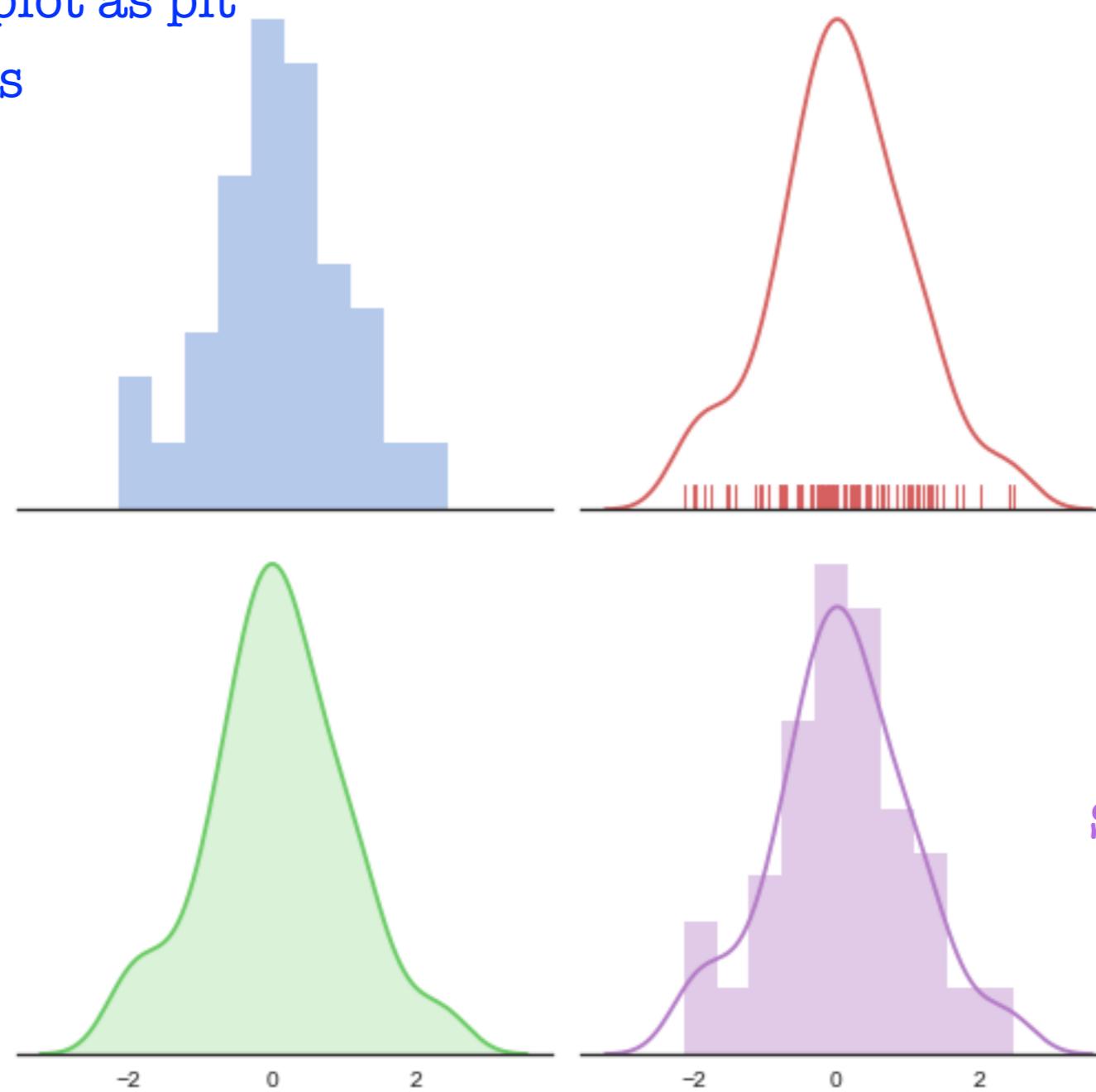
```
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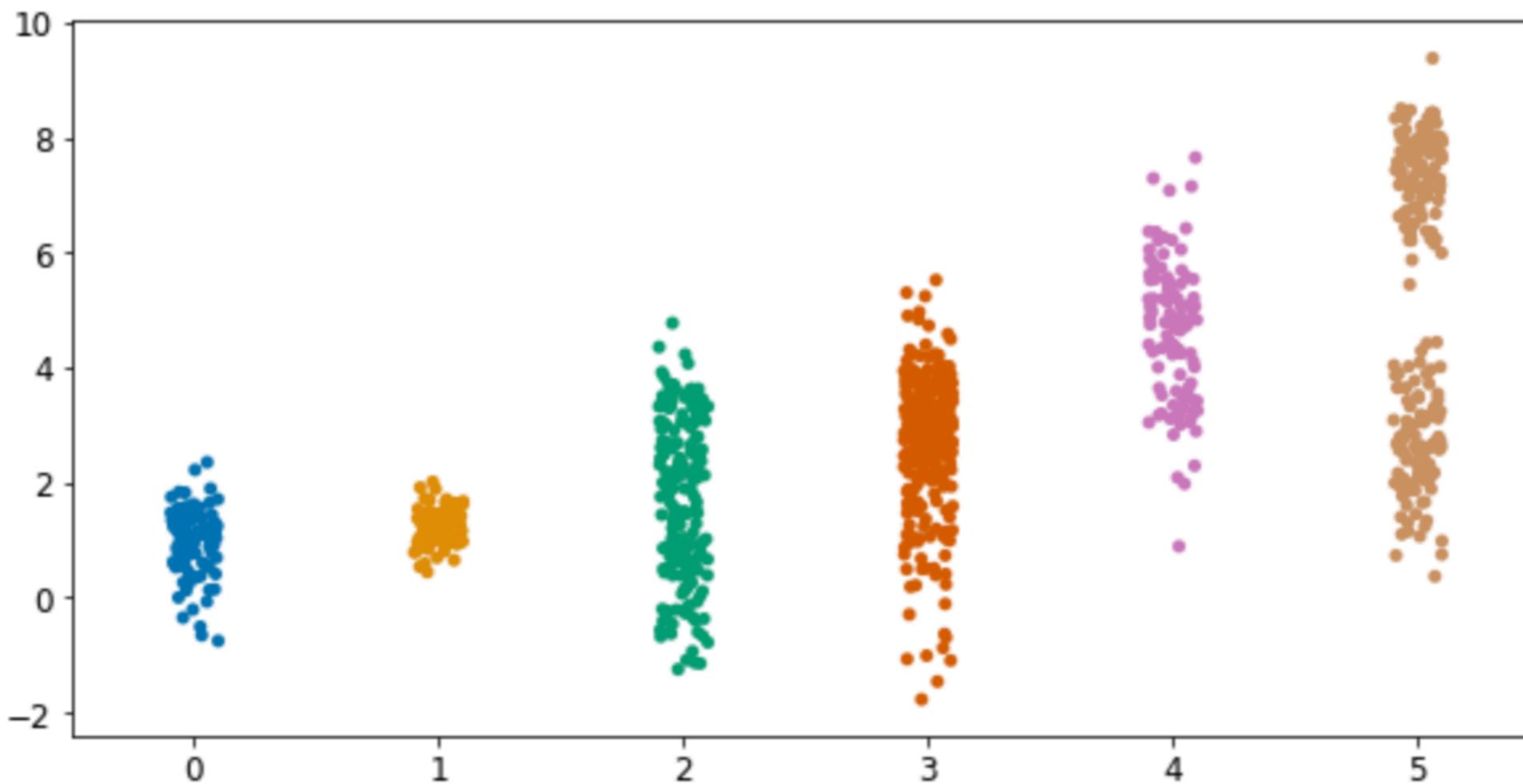


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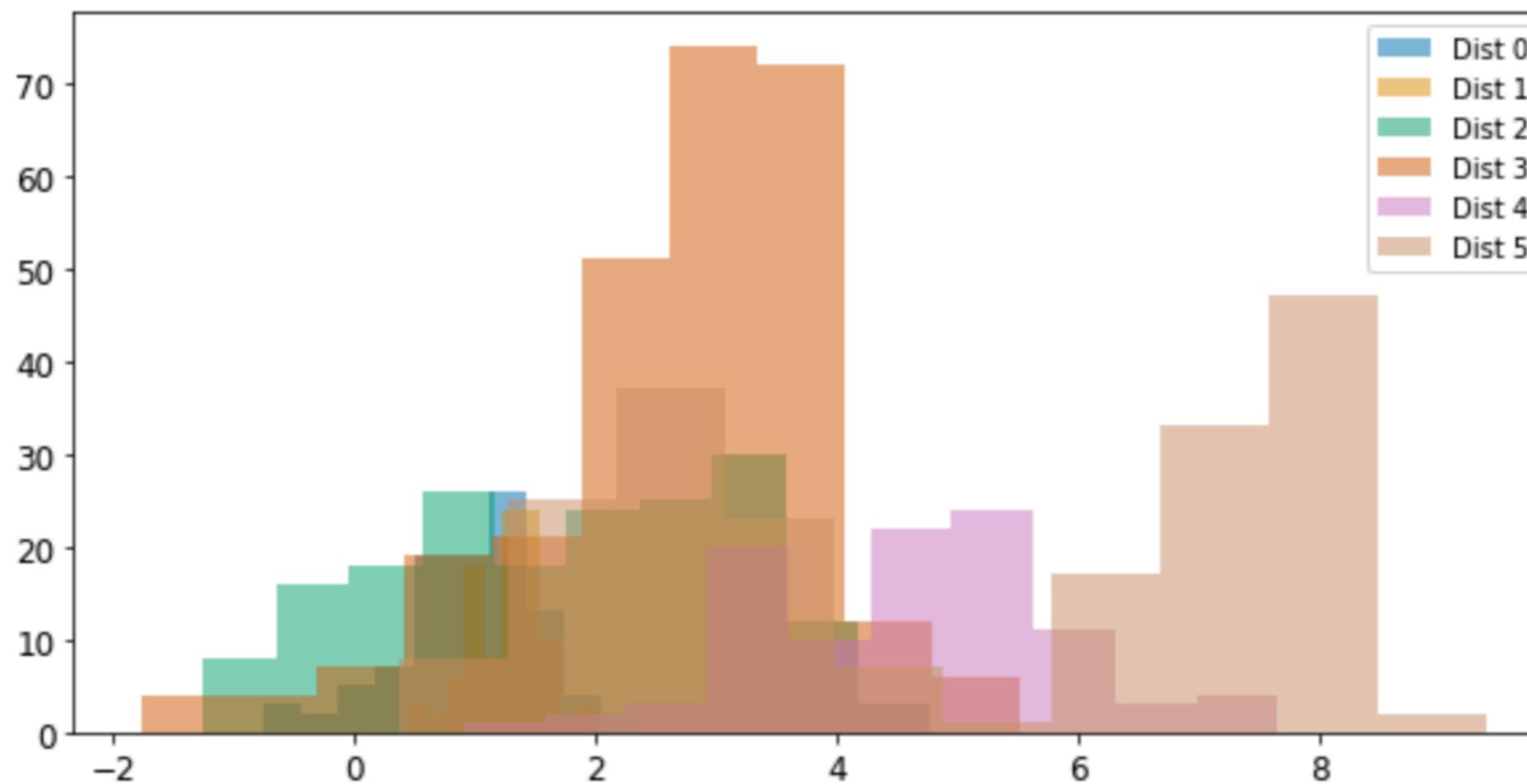
# Comparing multiple 1D distributions

Scenario: 6 classes with measurements -  
how do we compare them?



# Comparing multiple 1D distributions

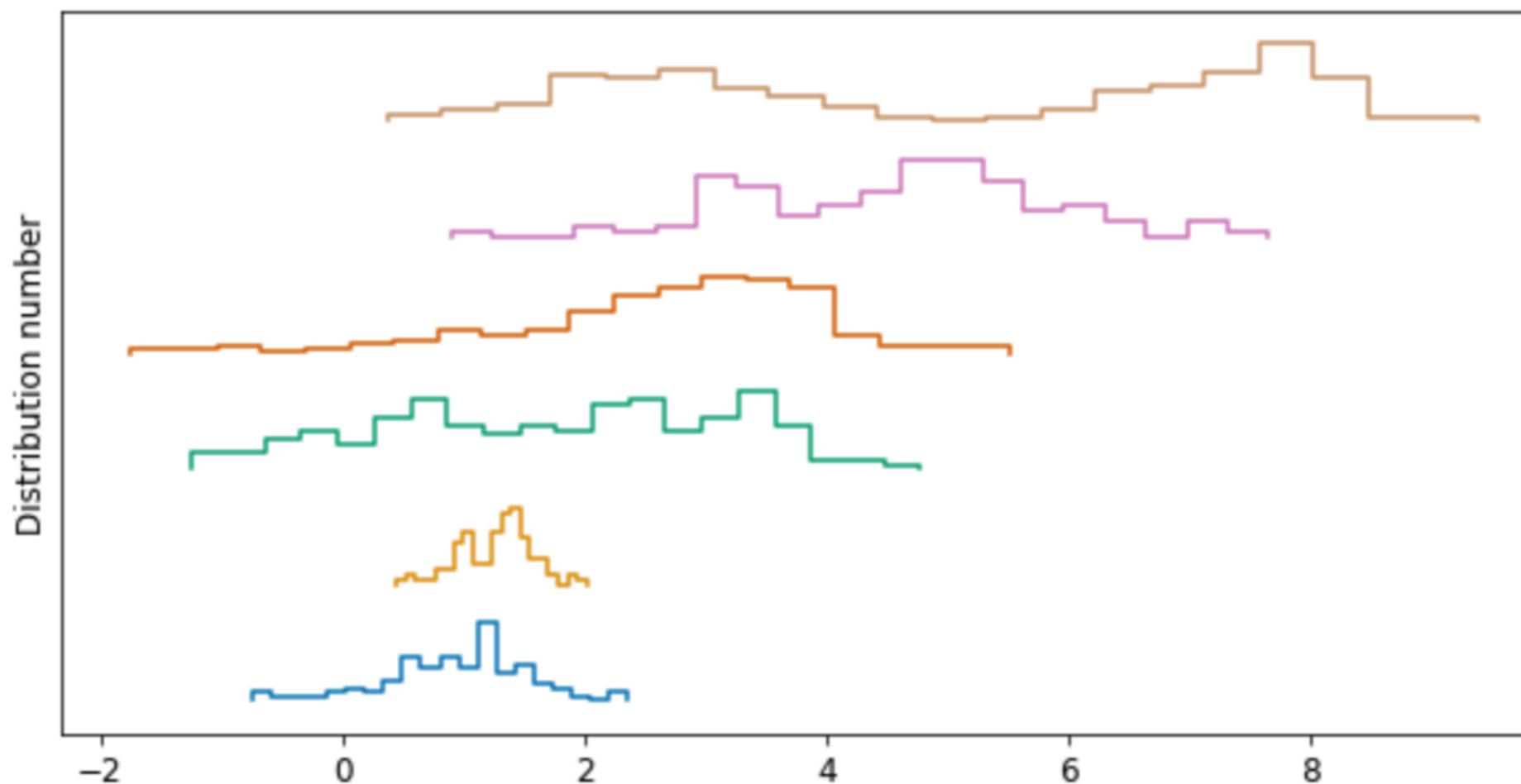
Scenario: 6 classes with measurements -  
how do we compare them?



over-plotting histograms does not work well

# Comparing multiple 1D distributions

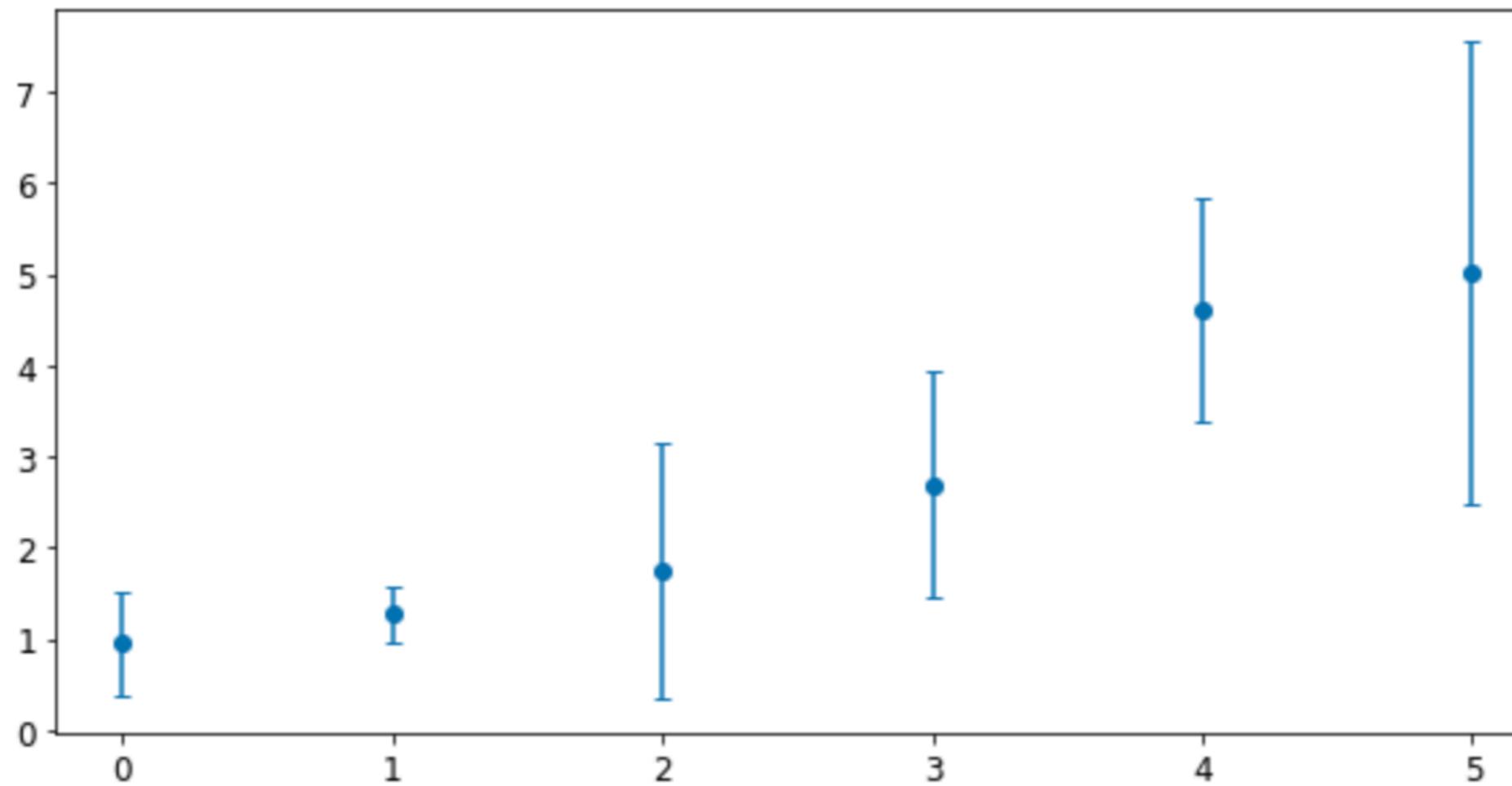
Scenario: 6 classes with measurements -  
how do we compare them?



offsetting histograms is a possibility

# Comparing multiple 1D distributions

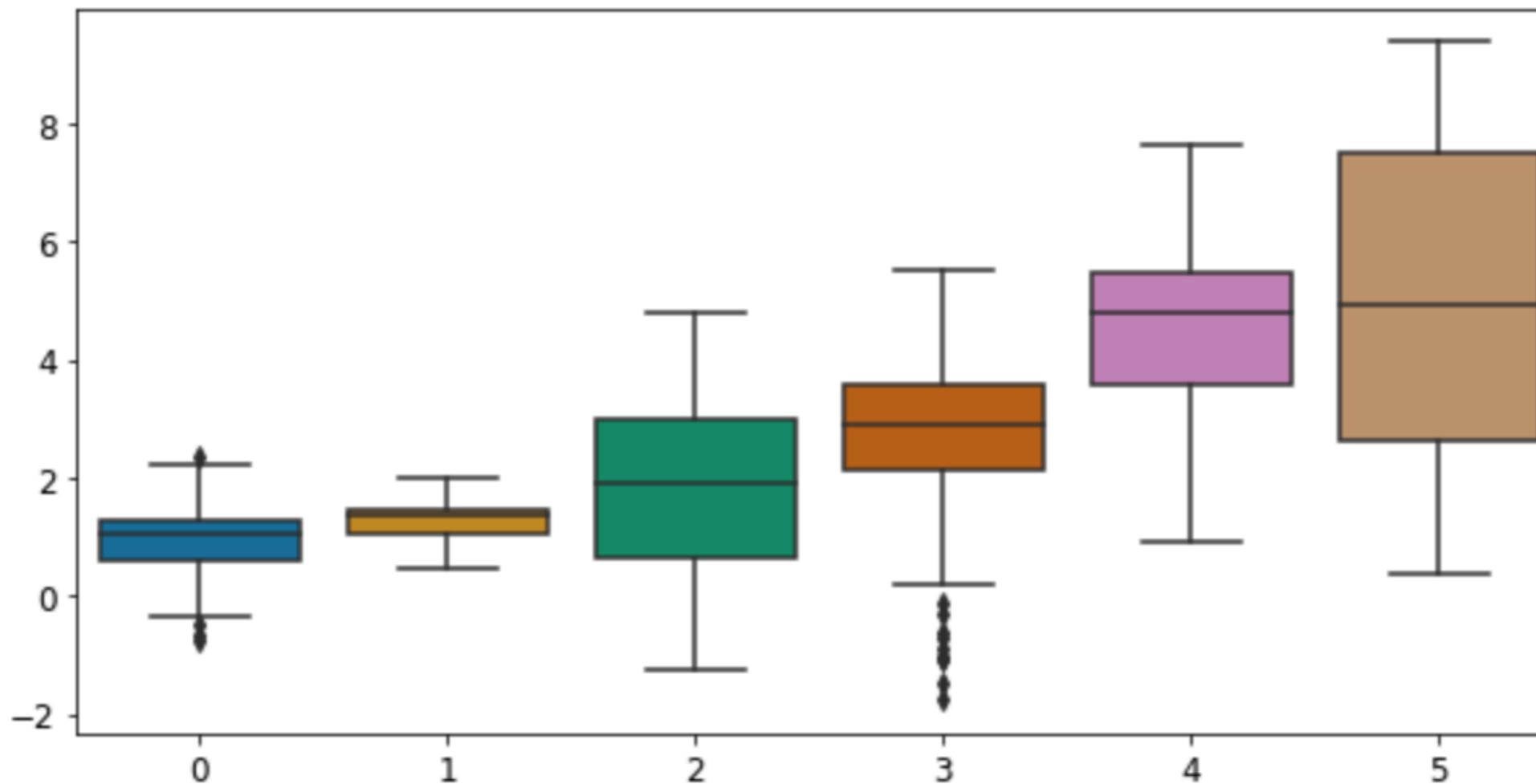
Scenario: 6 classes with measurements -  
how do we compare them?



errorbars give some illustration of spread but lacks detail

# Comparing multiple 1D distributions

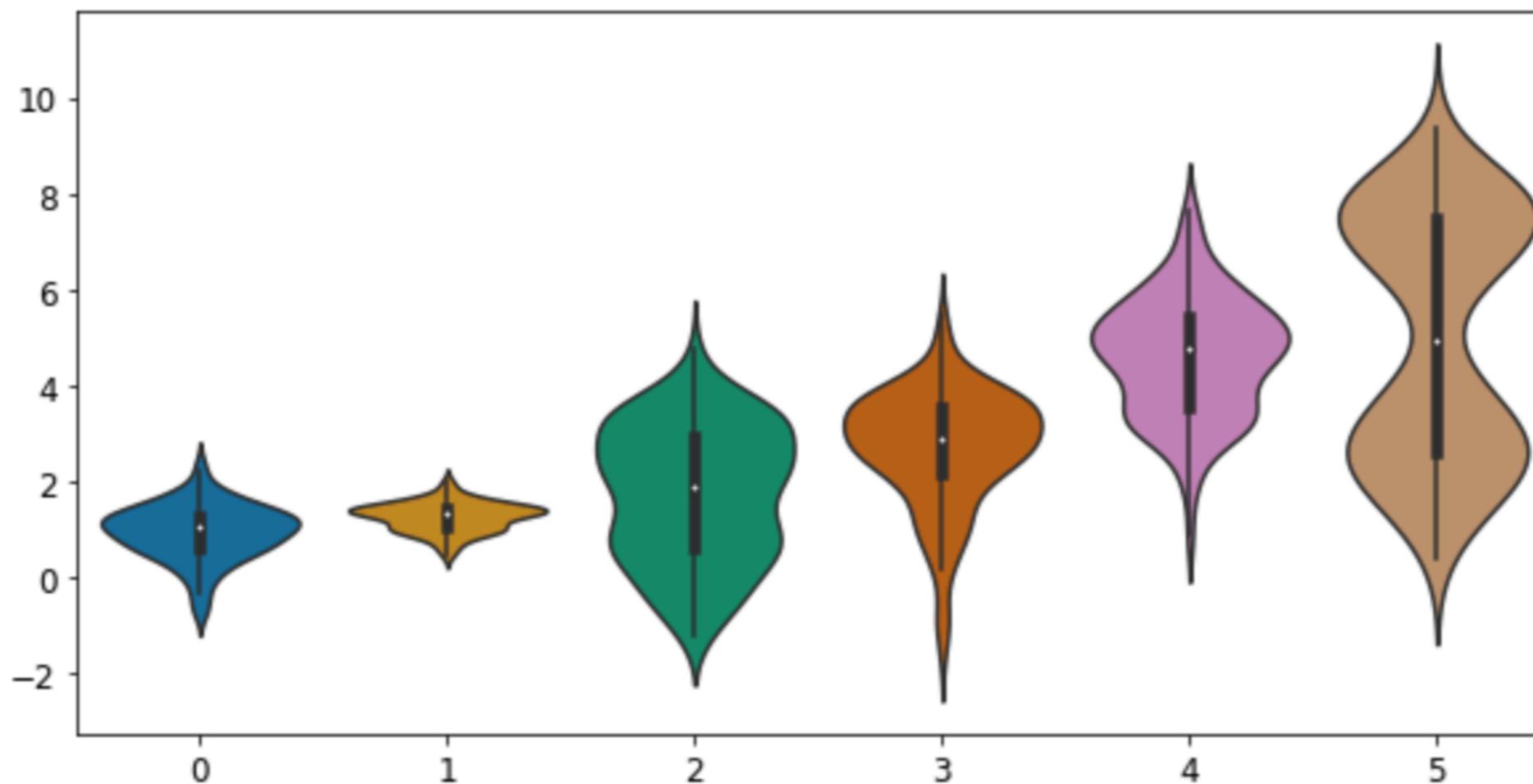
Scenario: 6 classes with measurements -  
how do we compare them?



box plots are better - but only for uni-modal data

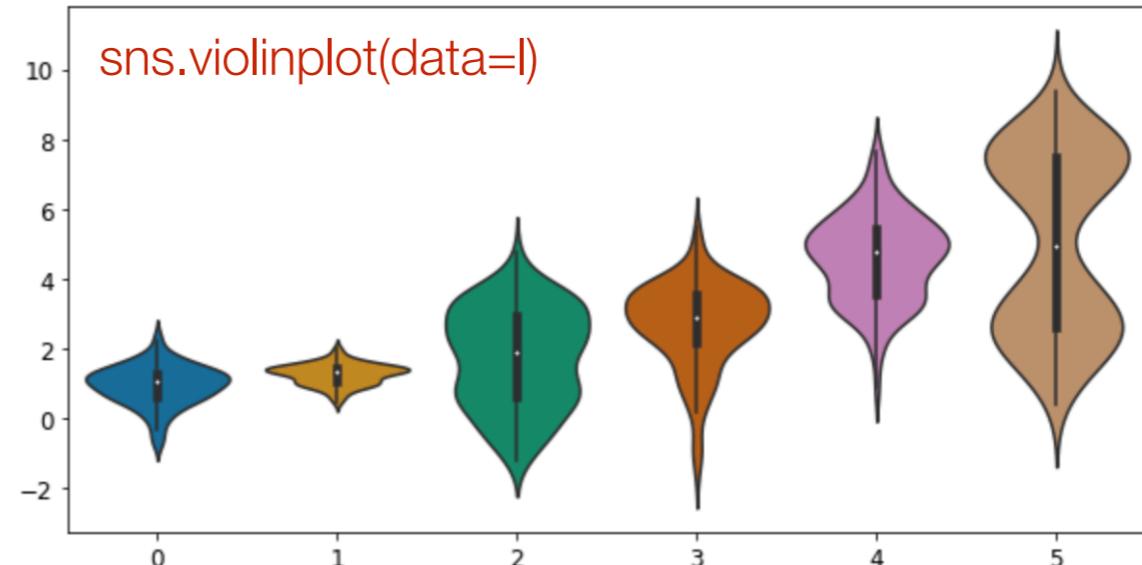
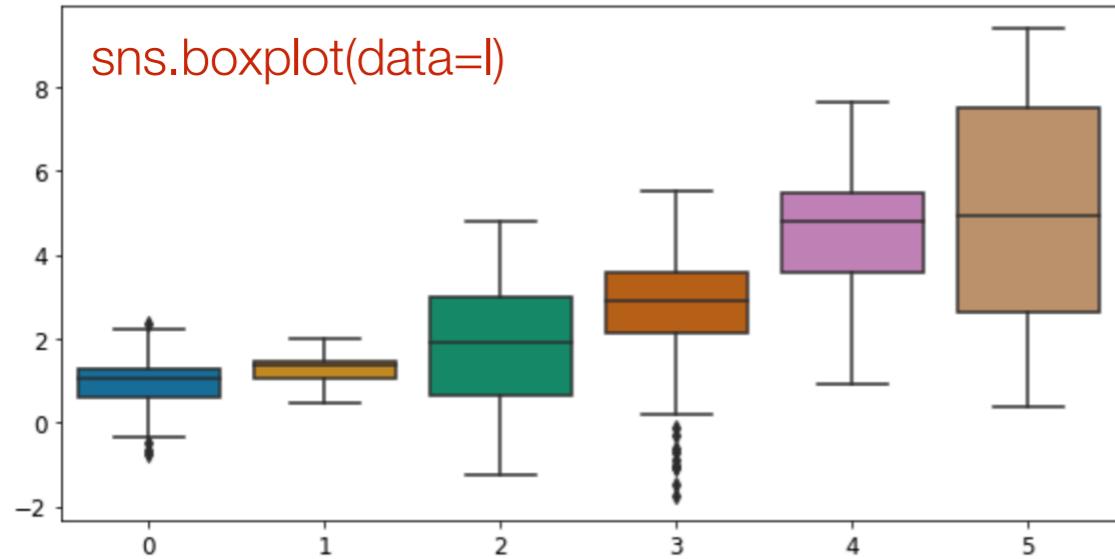
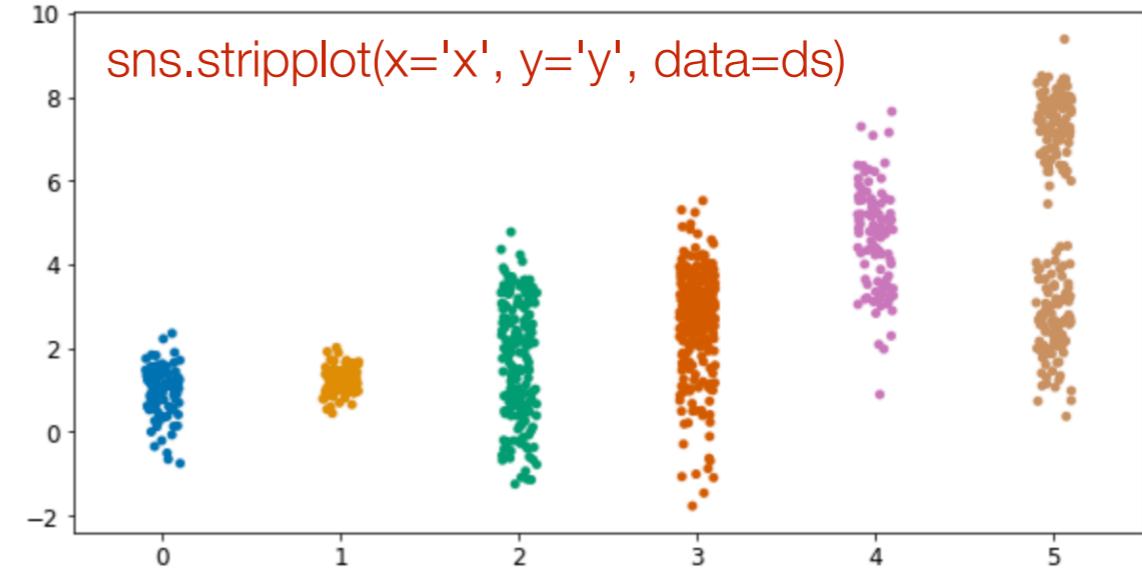
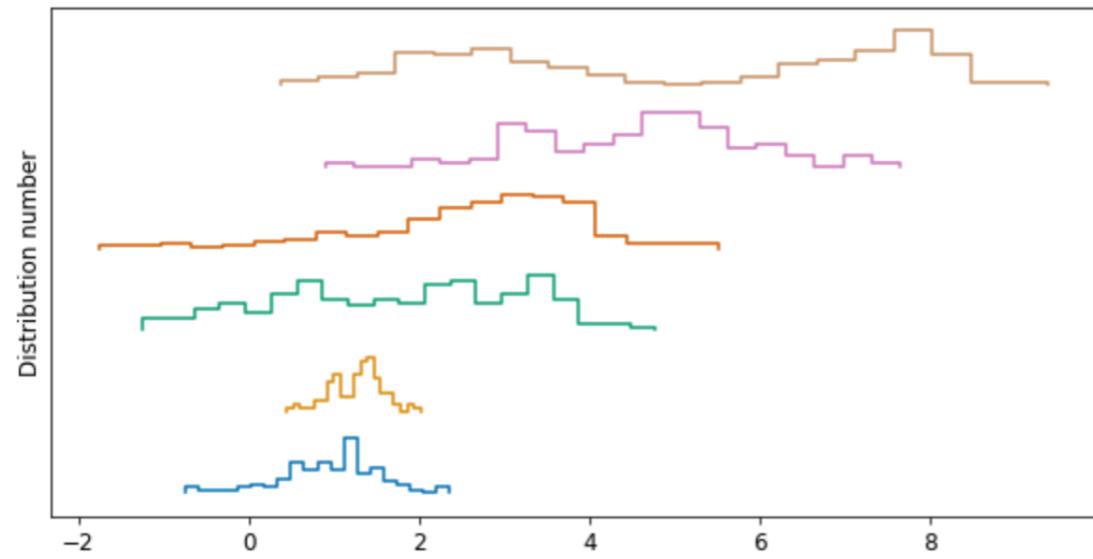
# Comparing multiple 1D distributions

Scenario: 6 classes with measurements -  
how do we compare them?

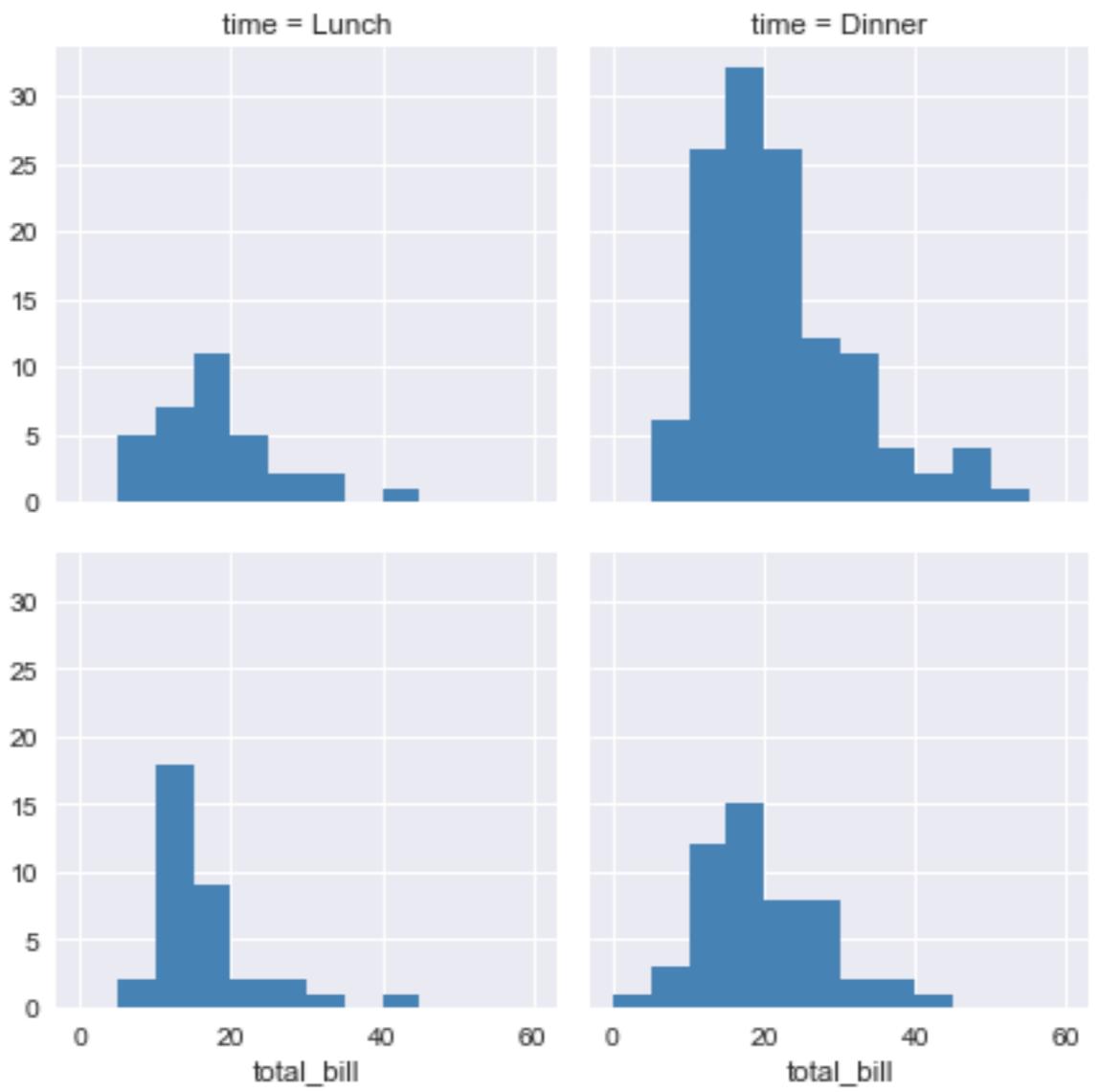


violin plots can be useful for this

# Comparing multiple 1D distributions

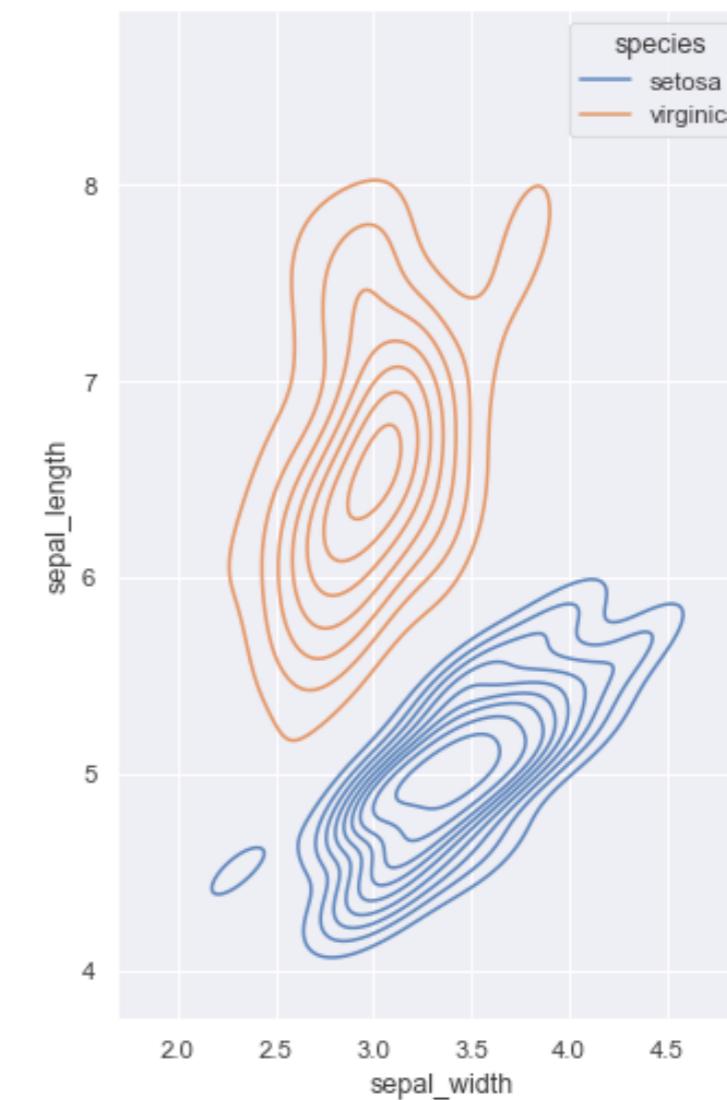
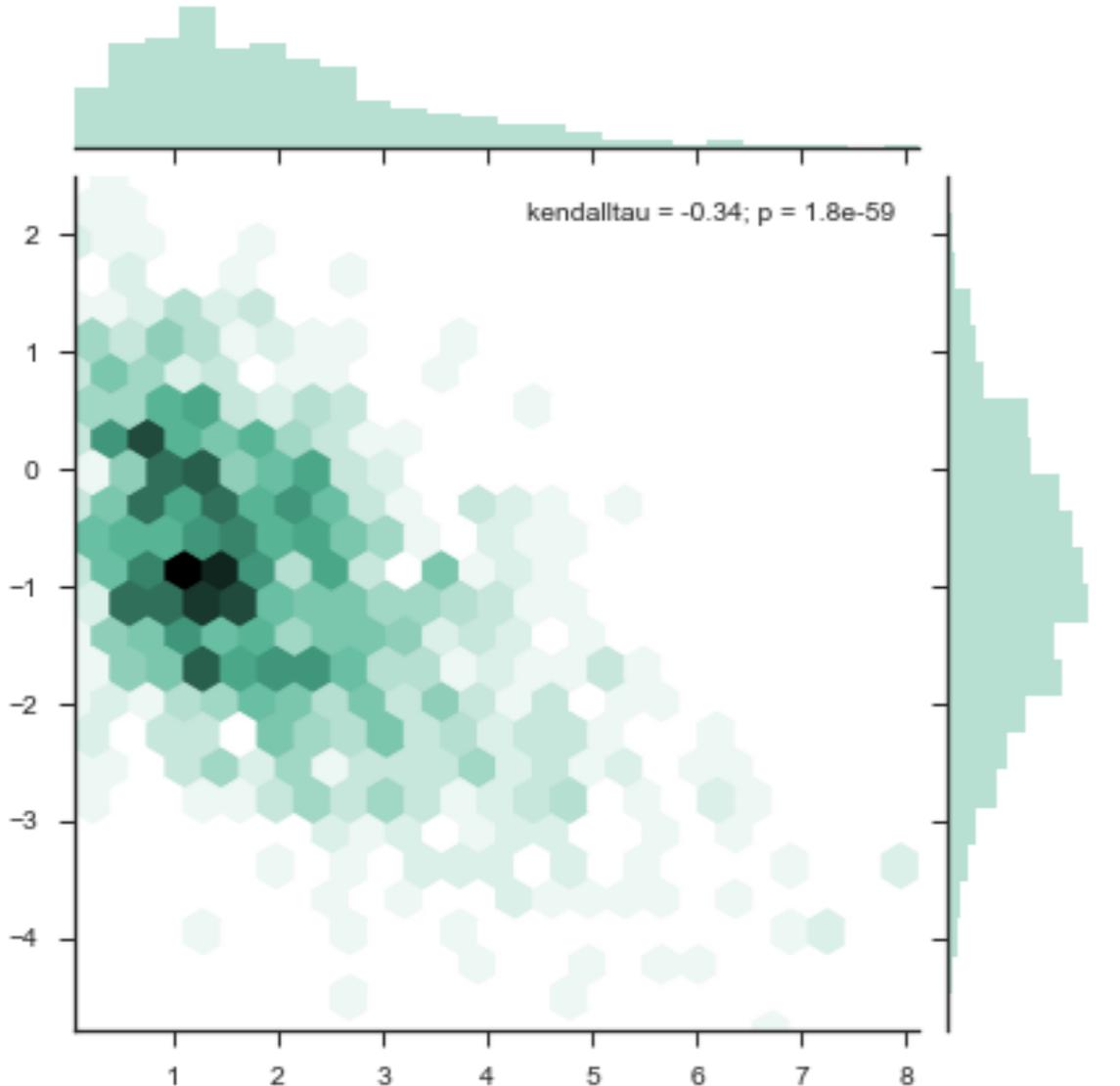


# Various ways to show 1D distributions



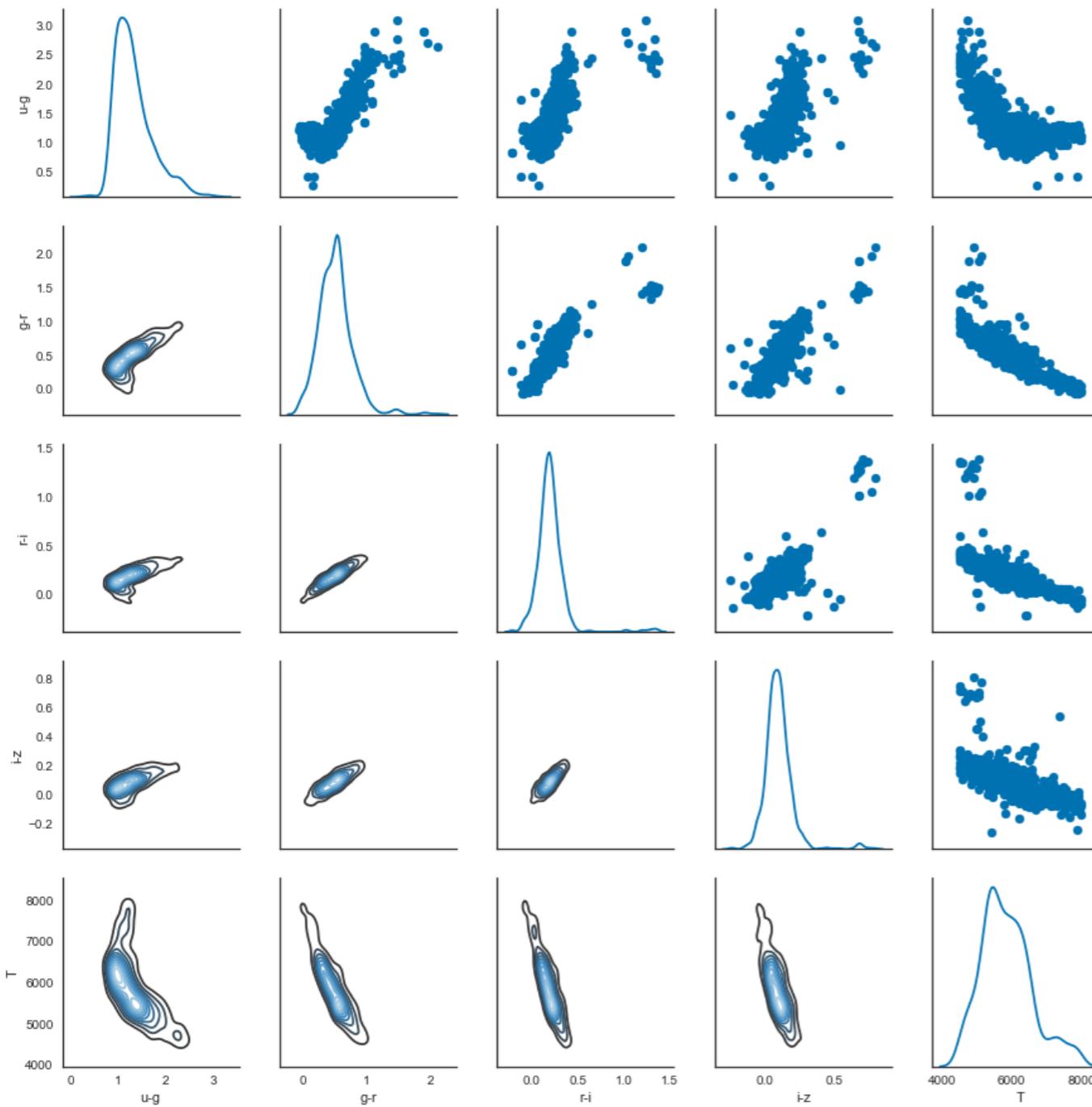
Conditional plots (called FacetGrids in seaborn - in R the lattice package provides these) are very powerful tools for exploring complex multi-dimensional data. While shown for histograms here they can be used for all kinds of plots.

# Various ways to show 2D distributions



Hex-bin plots (left), or kernel density maps (right) are powerful ways to view 2D data structures and should be in your toolbox.

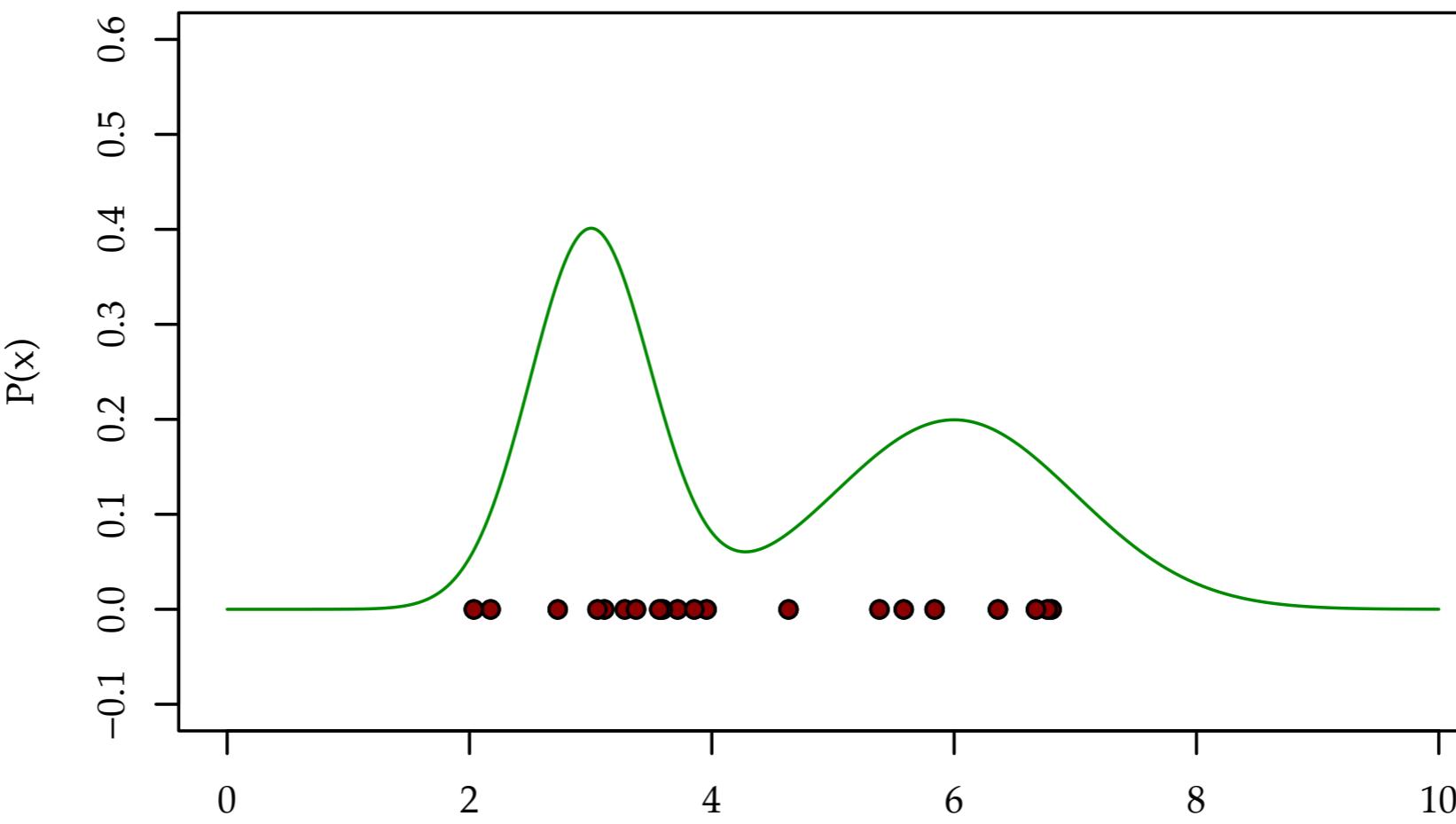
# Multi-dimensional data



Scatter-plot matrices or pairplots are very powerful ways to explore the inter-relationship of many variables quickly. But always keep in mind that the true correlation might be hidden in these projections.

# Density estimation

# How do you recover the “true” density distribution?



**Two main approaches:**

**Non-parametric** - where no assumptions about the functional form is made.

**Parametric** - where we do assume a particular functional form (two Gaussians for instance), and fit for the parameters of the function

# Density estimation - formally

Formally a density estimation problem is to find/  
learn a function

$$p_{\text{model}}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$$

In most cases it is natural to interpret  $p_{\text{model}}(\mathbf{x})$  as a  
probability density function.

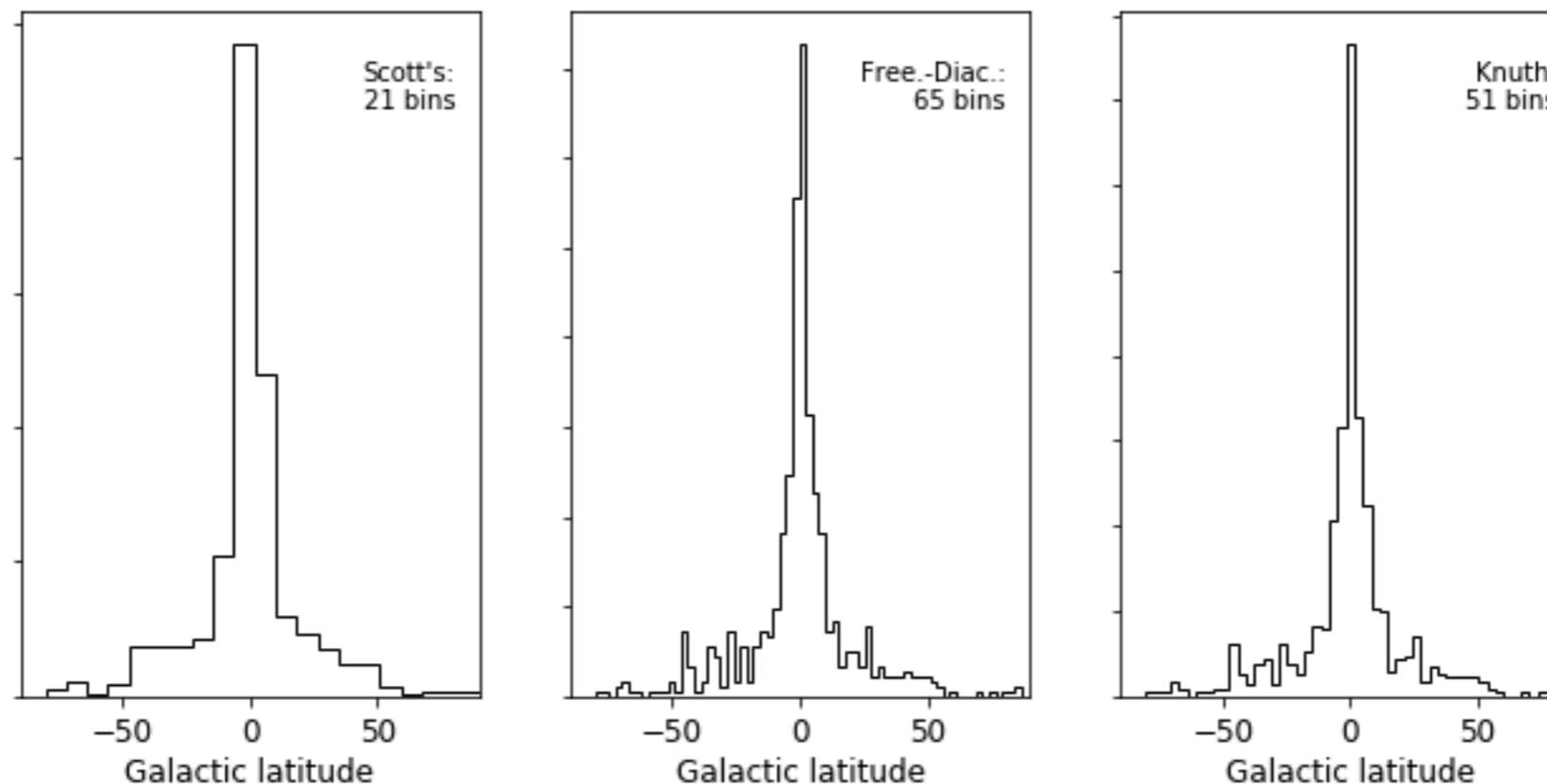
As soon as you have this function you could use it to  
predict missing data, take the derivative of it, create new  
data, or use it for visualisation - among other things.

# Simple Non-parametric estimators

## Histograms

Count objects in bins. Assume the bin size of bin  $i$  is  $\Delta_i$  and that there are  $n_i$  objects in this bin. The probability of the value  $x_i$  corresponding to that bin is then:

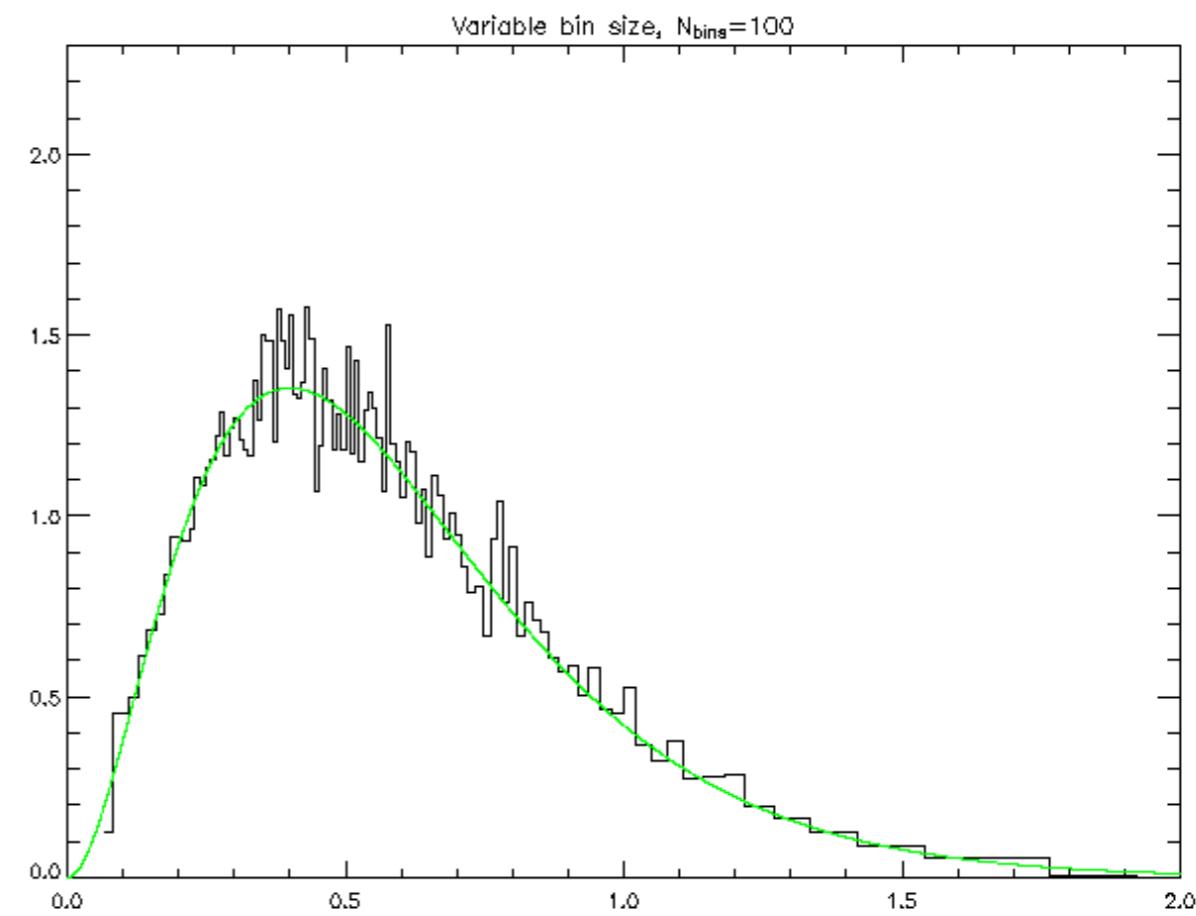
$$p_i(x_i) = \frac{n_i}{N\Delta_i}$$



# Histogram Problems

We can adapt the bin sizes to the data, but even so there are two major problems with the histogram:

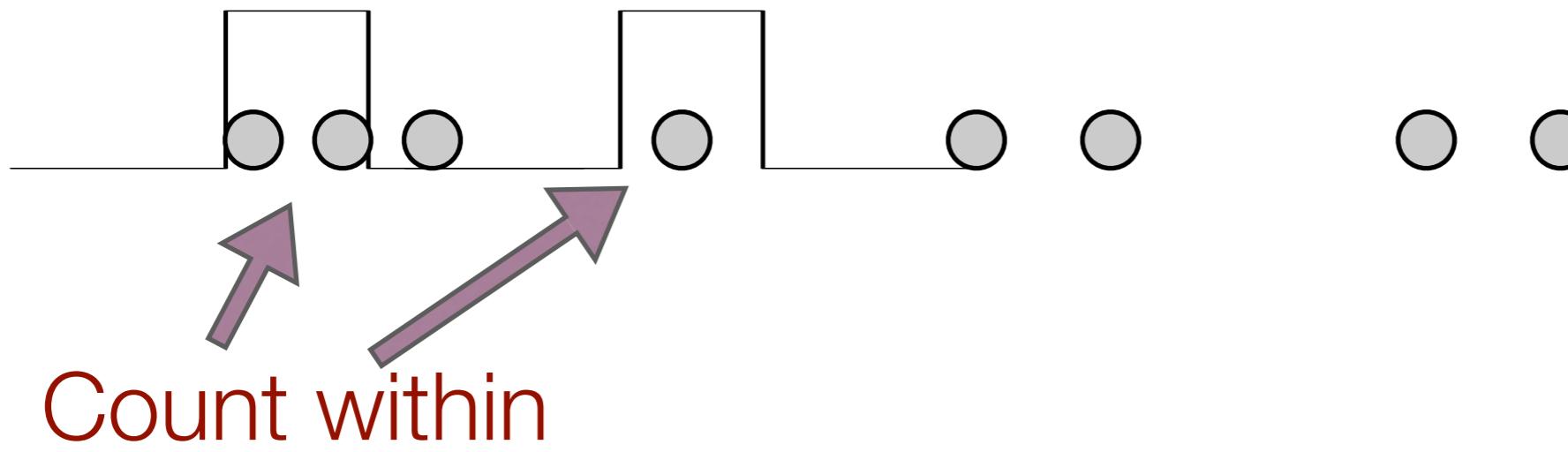
- a) It is discontinuous**
- b) It does not scale well to higher dimensions!**



But it is still very useful! [some authors disregard it unnecessarily]  
To do better we need to refine our density estimator.

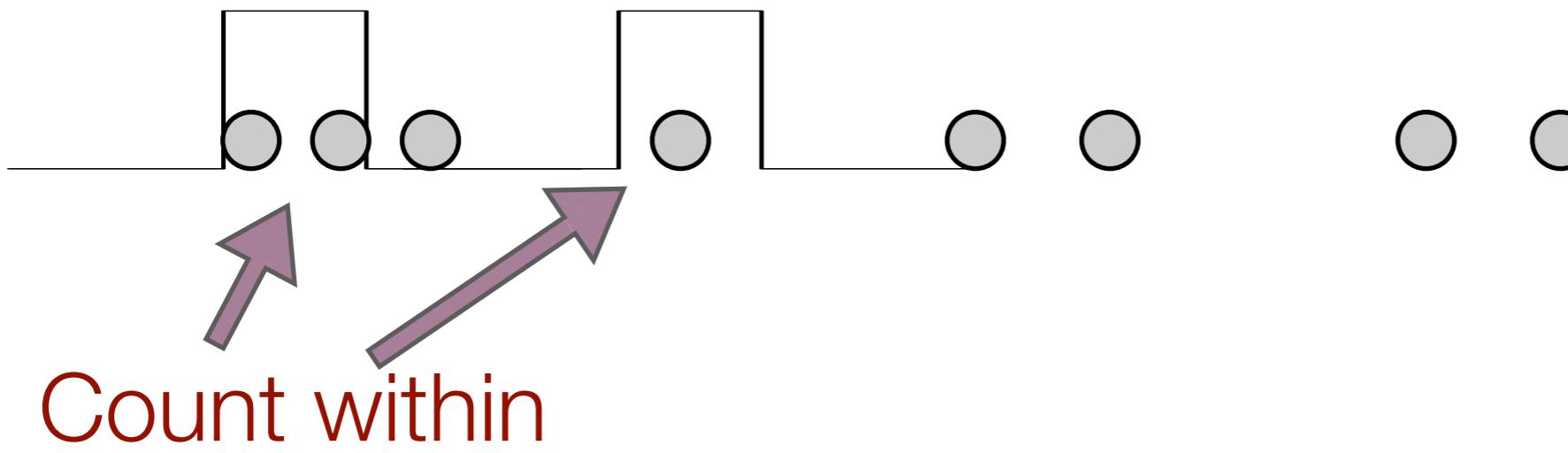
# Histograms & kernels

We want to add together effects from points close to **x**.



# Histograms & kernels

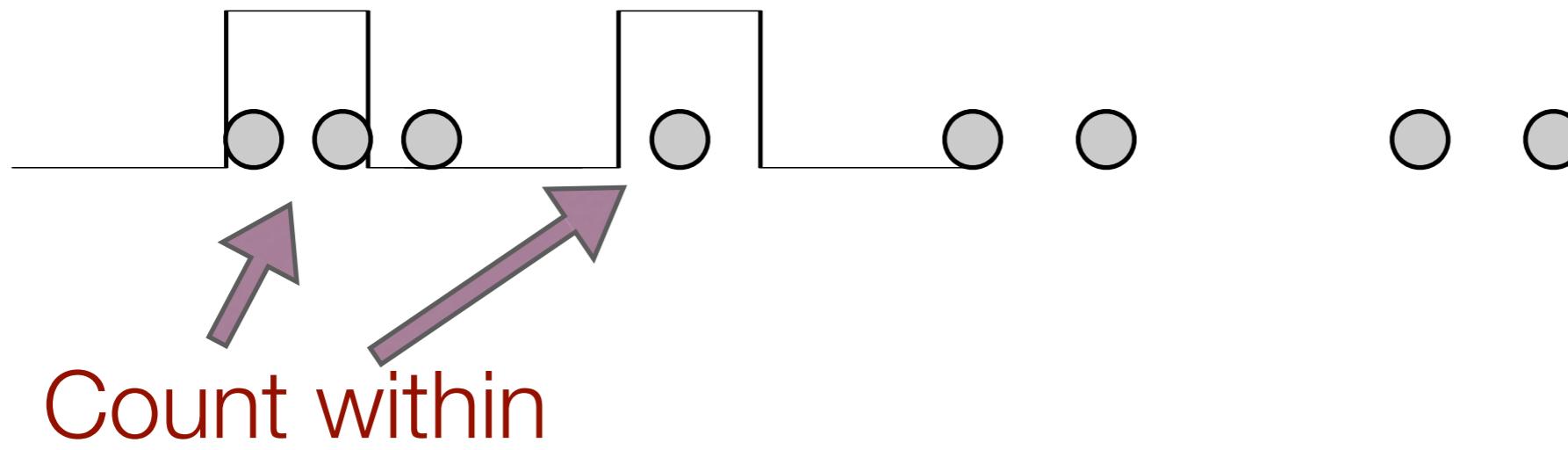
We want to add together effects from points close to  $\mathbf{x}$ .



Define a  $k(\mathbf{x})$  so that:  $k(\mathbf{u}) = 1$  iff  $\forall |u_i| < 1/2$

# Histograms & kernels

We want to add together effects from points close to  $\mathbf{x}$ .

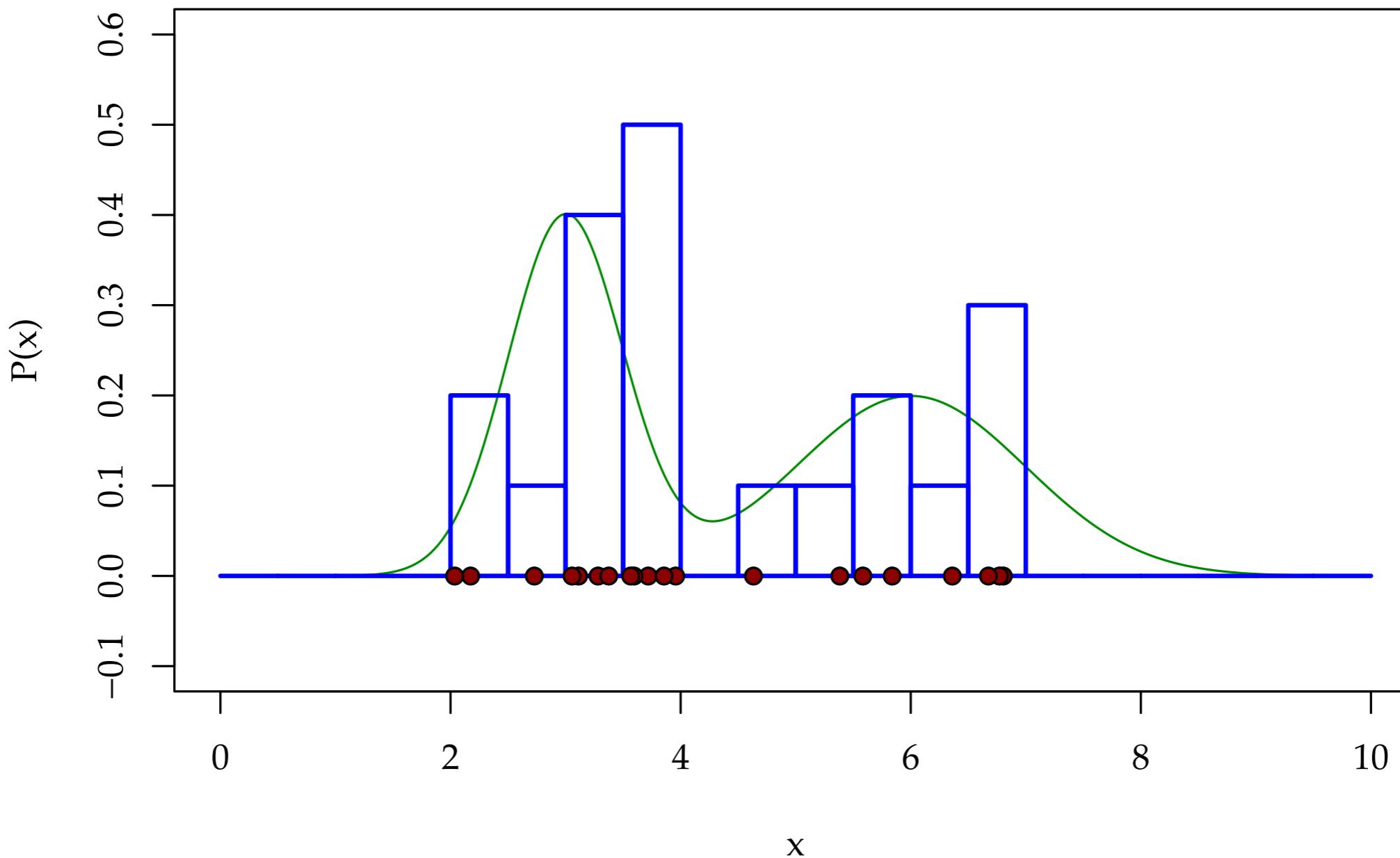


Define a  $k(\mathbf{x})$  so that:  $k(\mathbf{u}) = 1$  iff  $\forall |u_i| < 1/2$

Then the histogram (number of counts) can be defined as:

$$\#(\mathbf{x}) = \sum_{i=1}^N k\left(\frac{\mathbf{x} - \mathbf{x}_i}{h_i}\right)$$

# Histograms



What we are doing here is adding one box for each data point. This is our ‘kernel’.

# Histograms & kernels

The histogram (number of counts) can be defined as:

$$\#(\mathbf{x}) = \sum_{i=1}^N k \left( \frac{\mathbf{x} - \mathbf{x}_i}{h_i} \right)$$

Recasting this as a probability distribution we get:

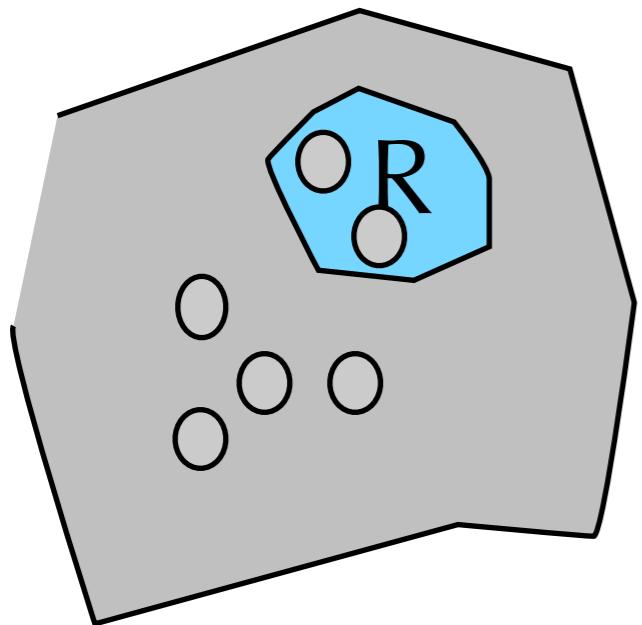
$$p_i(x_i) = \frac{1}{N} \sum_{j=1}^N \frac{1}{h} K\left(\frac{x_i - x_j}{h}\right) \quad K(x) = \begin{cases} 1, & |x| \leq \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$$

this is a good starting point for density estimators

# Simple Non-parametric estimators

## Kernel estimators

Like in the histogram method we want to use objects near to a point to help us estimate the density there. If we look at a small region:



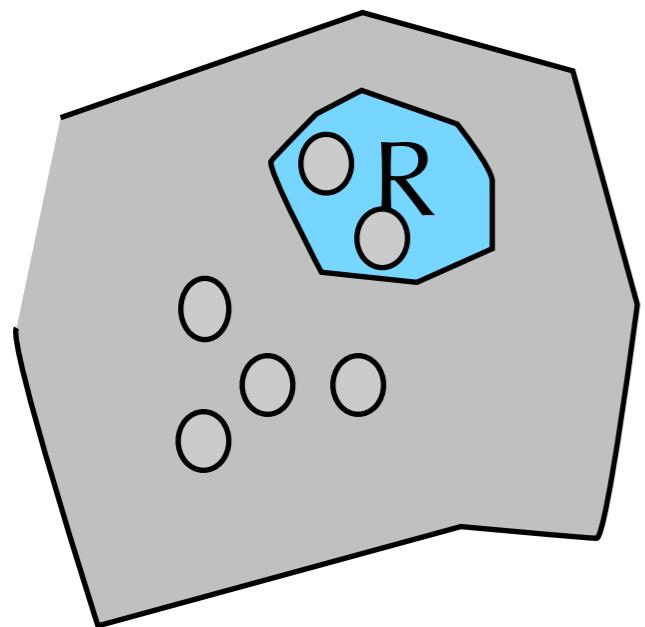
N observations in total  
K inside R  
V is volume of R

$$p(\mathbf{x}) = \frac{K}{NV}$$

# Simple Non-parametric estimators

## Kernel estimators

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N observations in total  
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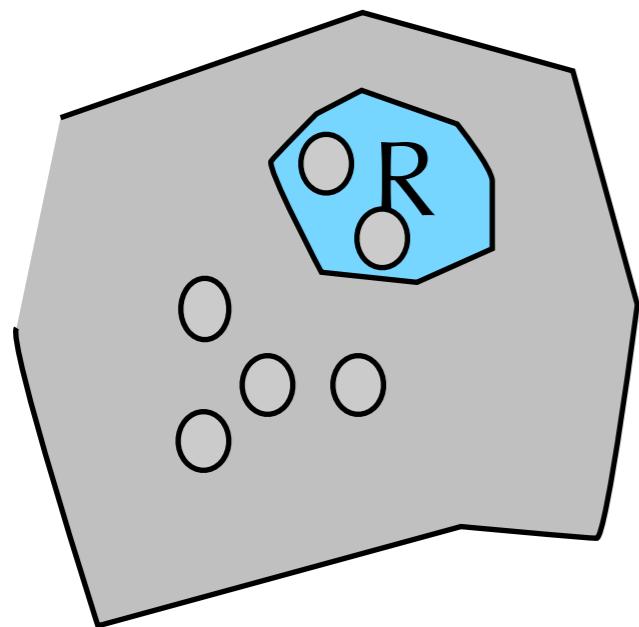
$$p(\mathbf{x}) = \frac{K}{NV}$$

So we can either fix K or fix V

# Simple Non-parametric estimators

## Kernel estimators

Like in the histogram method we want to use objects near to a point to help us estimate the density there. If we look at a small region:



N observations in total  
K inside R  
V is volume of R

$$p(\mathbf{x}) = \frac{K}{NV}$$

So we can either fix K or fix V

**Kernel estimators** fix V and the **k-nearest neighbours estimator** fixes K.

These are one type of instance-based learning methods

# Instance-based learning

Basically: methods that use all training data to evaluate new data.

E.g. compare a linear method:

$$\hat{y}(x_i) = \theta_0 + \sum_{j=1}^N \theta_j \phi_j(x_i)$$

with using the nearest neighbours:

$$\hat{y}(x_i) = \frac{1}{k} \sum_{j:x_j \in N_k(x_i)} y_j$$

# Instance-based learning

These are very useful techniques and are often overlooked.

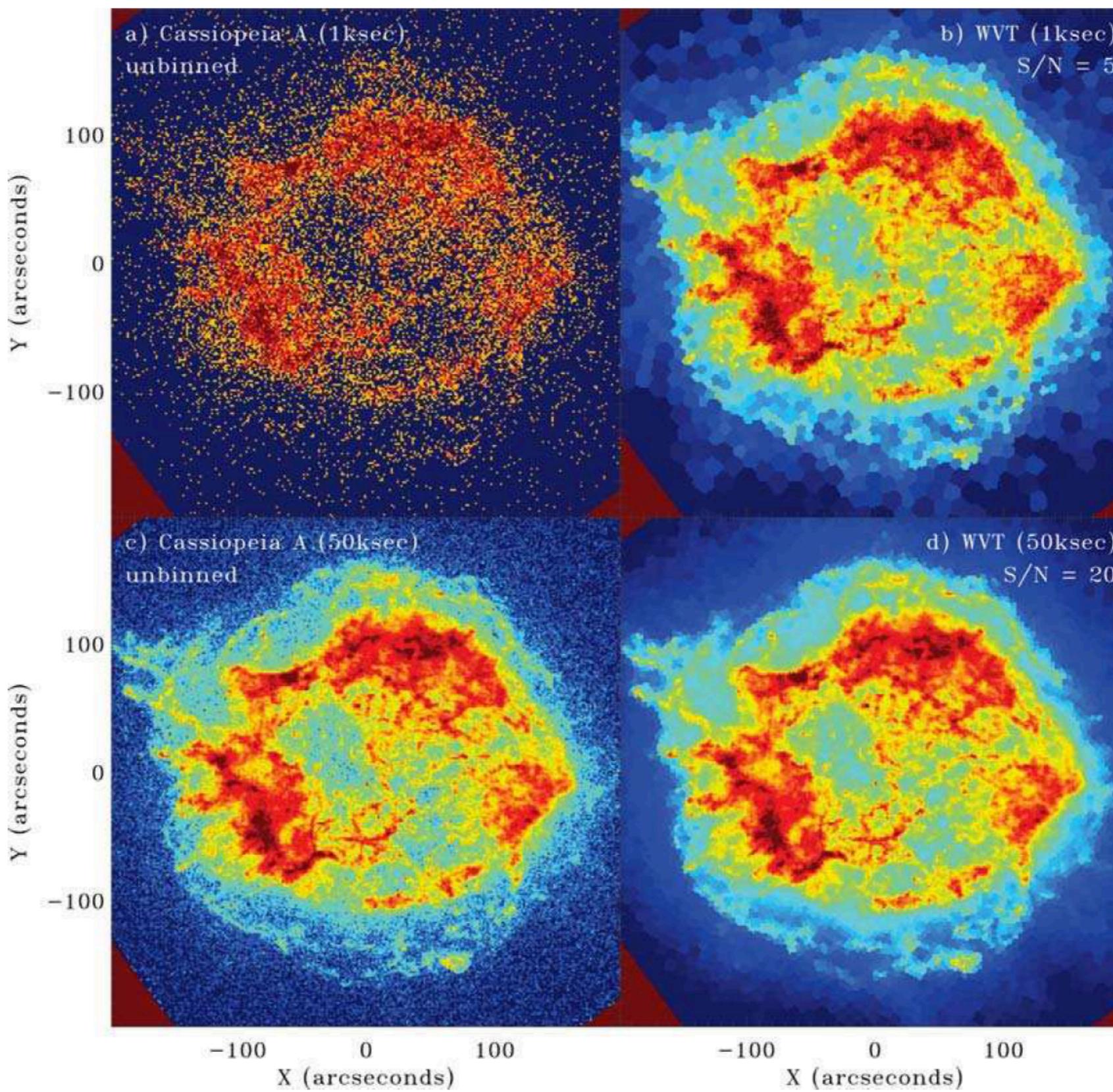
Pros:

- **Flexible** - can approximate very complex functions fairly easily
- **Can adapt to new data** - and throw away old
- **Simple**
- **Scales to large data**

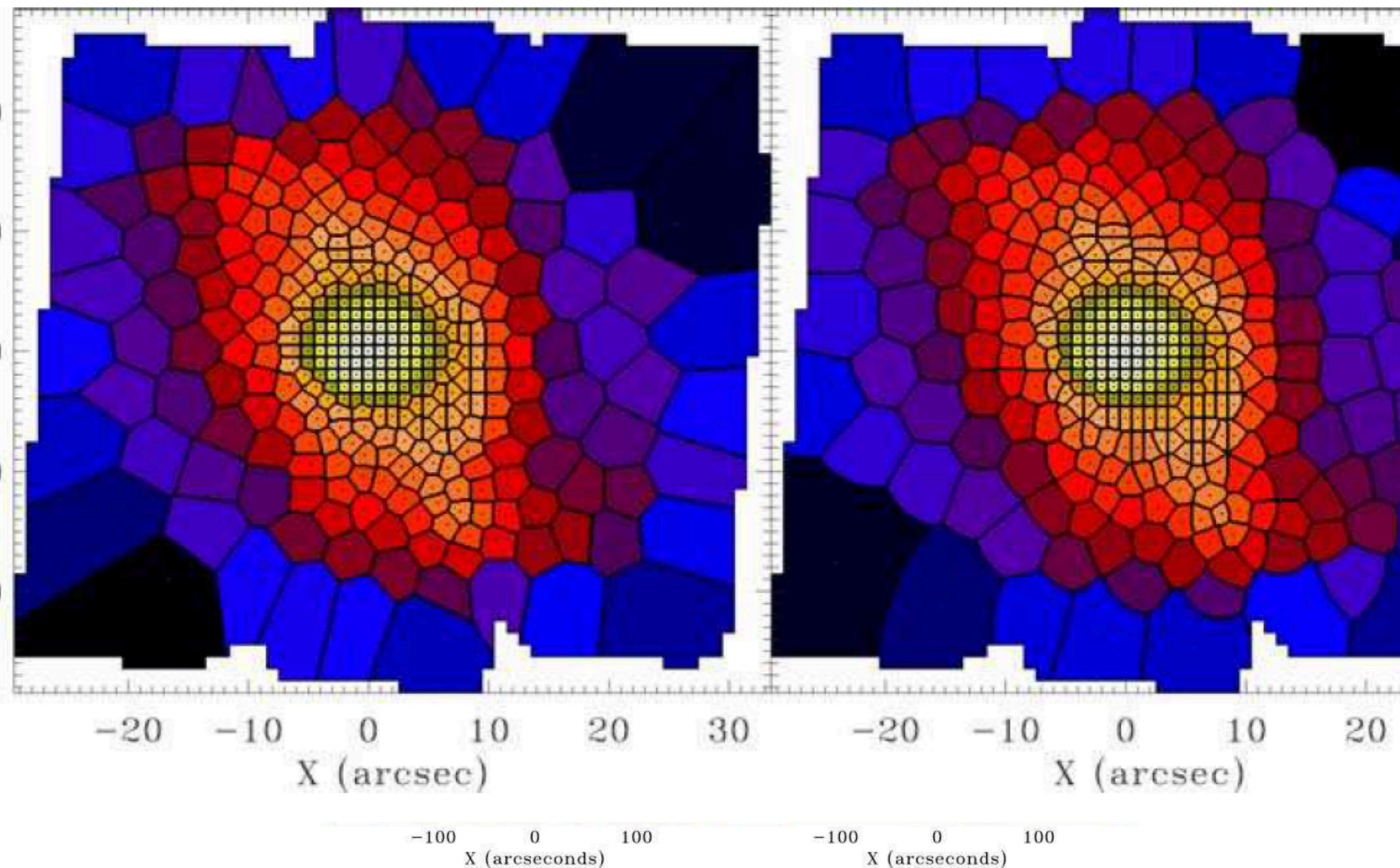
Cons:

- **Memory expensive** - all data needs to be in memory
- **Time expensive** - naïve implementations need comparison to all data when a new example is found
- **Opaque** - it is difficult to gain understanding from these

Diehl &amp; Statler (2006)



Dichi &amp; Stotler (2006)



# Kernel methods

We want to add together effects from points close to  $\mathbf{x}$ .

$$p(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \frac{1}{V_i} k\left(\frac{\mathbf{x} - \mathbf{x}_i}{h_i}\right)$$

This is the basic equation for [kernel estimation](#)

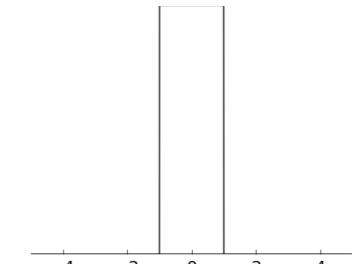
Different kernels lead to somewhat different final distributions. The Epanchenikov kernel is the one that theoretically gives the minimum variance.

# Kernel methods

We want to add together effects from points close to  $\mathbf{x}$ .

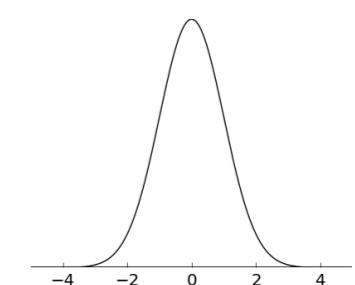
**Square kernel:**

$$k(\mathbf{u}) = 1 \text{ iff } \forall |u_i| < 1/2$$



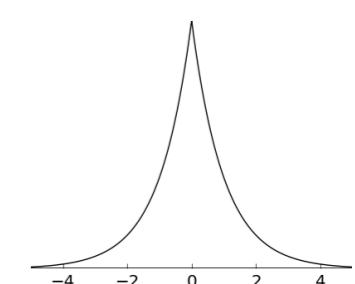
**Gaussian kernel:**

$$k(\mathbf{u}) = \frac{1}{(2\pi)^{1/D}} e^{-\|\mathbf{u}\|^2/2}$$



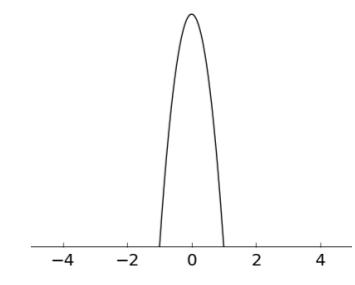
**Exponential kernel:**

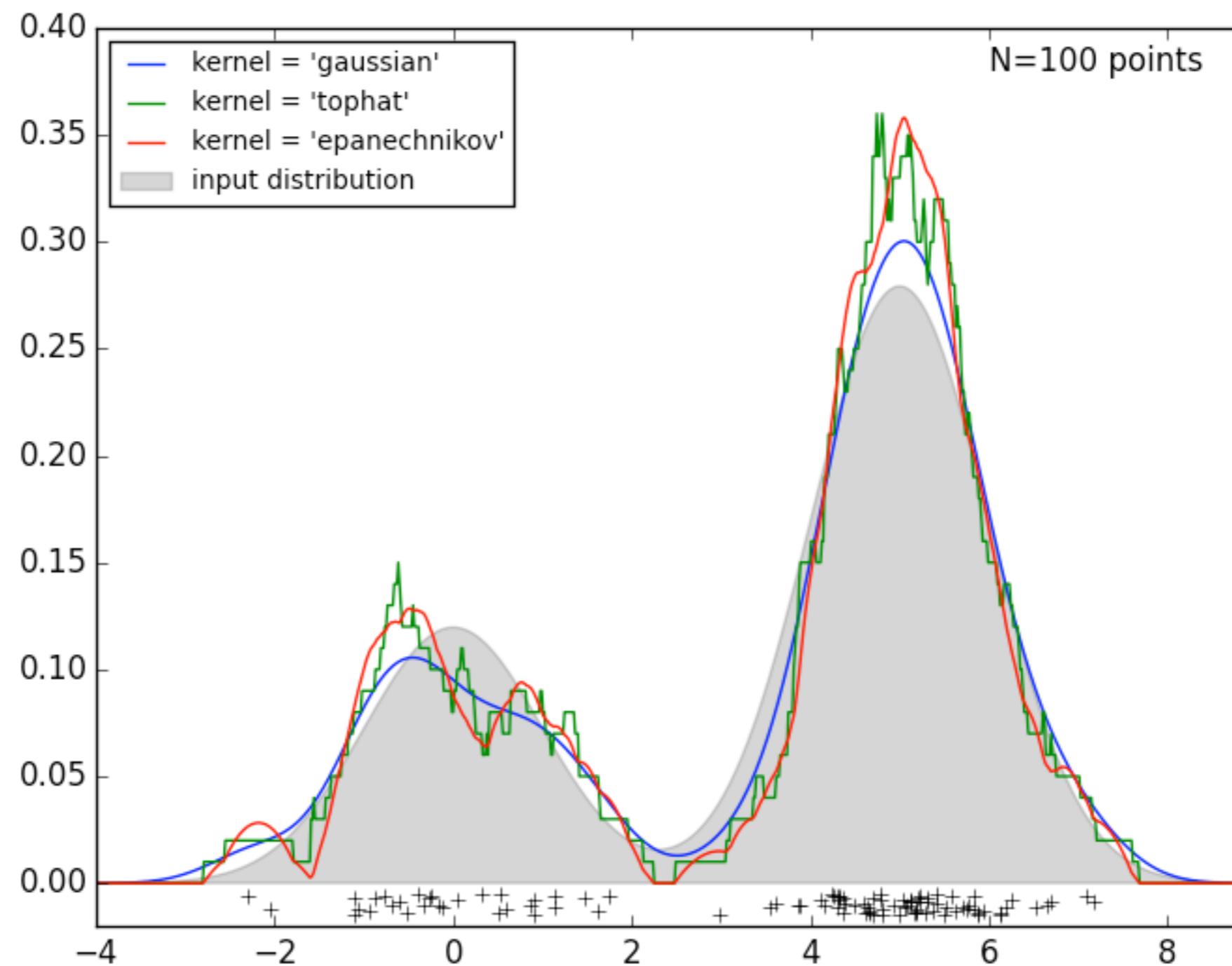
$$k(u) = \frac{1}{D! V_D(1)} e^{-|u|}$$



**Epanechnikov kernel:**

$$k(u) = \begin{cases} \frac{3}{4} (1 - u^2), & \text{if } |u| < 1 \\ 0 & \text{otherwise} \end{cases}$$





Taken from: [http://scikit-learn.org/stable/auto\\_examples/neighbors/plot\\_kde\\_1d.html](http://scikit-learn.org/stable/auto_examples/neighbors/plot_kde_1d.html)

# Kernel methods

We want to add together effects from points close to  $\mathbf{x}$ .

$$p(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \frac{1}{V} K\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right)$$

This is the basic equation for **kernel density estimation (KDE)**

Note that this **has** a parameter: the bandwidth - changing this changes the results.

# Kernel methods

We want to add together effects from points close to  $\mathbf{x}$ .

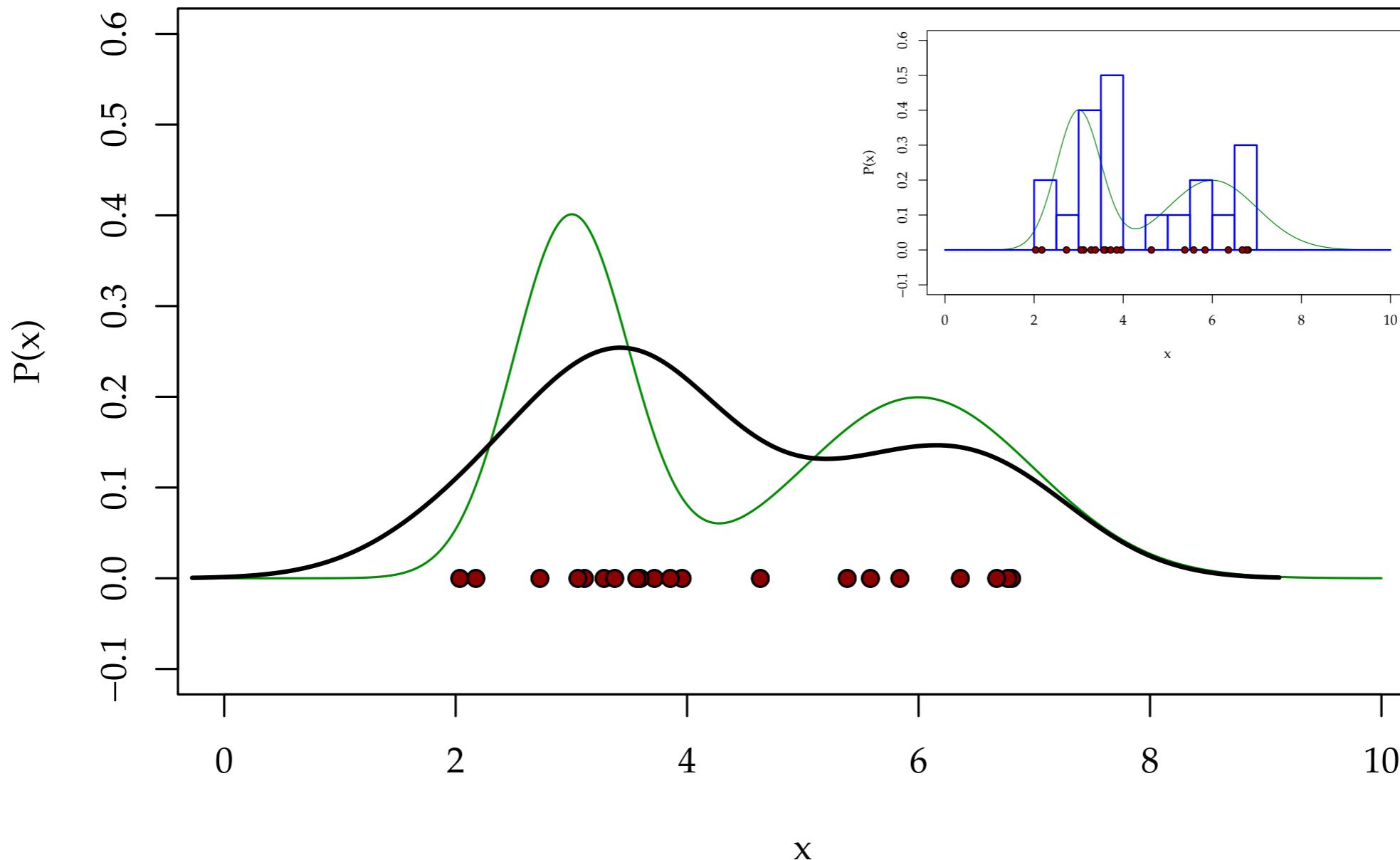
$$p(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \frac{1}{V} K\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right)$$

Bandwidth

This is the basic equation for **kernel density estimation (KDE)**

Note that this **has** a parameter: the bandwidth - changing this changes the results.

# Count the number weighted by a kernel



# Practical use:

```
from sklearn.neighbors import KernelDensity
kde = KernelDensity(<bandwidth<, kernel='gaussian').fit(X)
Xgrid = <same shape as X: Nsamples x Nfeatures>
log_dens = kde.score_samples(Xgrid)
```

# Try it out

Load the Datafiles/gaia-stars.vot file from the course website.

Create a kernel density estimate for the parallax - can you find any structure here?

There is a big area in ra, dec that has fewer stars than the rest - why?

**How do you do it?**

# Kernels to expansions - radial basis functions

Inspired by this we can expand the function  $p(x)$  using the kernel functions

$$p(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \frac{1}{V} K\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right)$$

This gives radial basis function expansion. Very useful for interpolation.

$$f(\mathbf{x}) = \sum_{i=1}^N \beta_i K\left(\frac{\mathbf{x} - \xi_i}{\lambda_i}\right)$$

[see e.g. chap. 6.7 in the Elements of Statistical Learning]

By making this a bit more flexible we get a very useful density estimation technique.

# Gaussian Mixture Modeling

# Gaussian mixture modelling

Let us assume that our unknown distribution can be written as a sum of Gaussians:

$$p(x_i|\theta) = \sum_{j=1}^M \alpha_j N(x_i; \mu_j, \sigma_j)$$

We need to estimate the parameters and can write the log likelihood of all data as

$$\ln L = \sum_{i=1}^N \ln \left[ \sum_{j=1}^M \alpha_j N(x_i; \mu_j, \sigma_j) \right]$$

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And this now needs to be maximised to determine the  $3M-1$  parameters [why -1?]

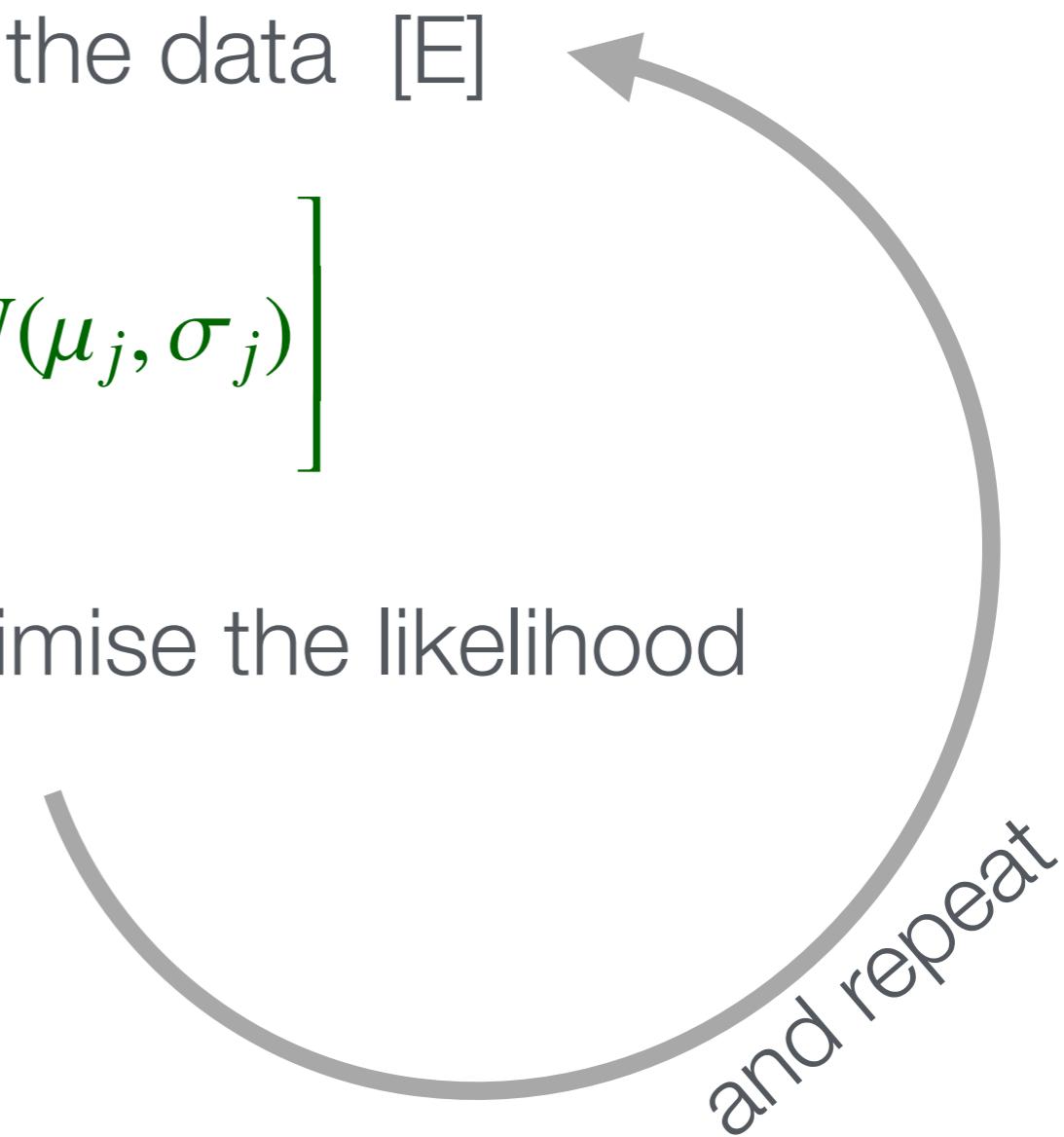
# Gaussian mixture modelling

In this case, the parameters are found using the Expectation-Maximisation algorithm:

1. Calculate the log likelihood of the data [E]

$$\ln L = \sum_{i=1}^N \ln \left[ \sum_{j=1}^M \alpha_j N(\mu_j, \sigma_j) \right]$$

2. Find the parameters that maximise the likelihood



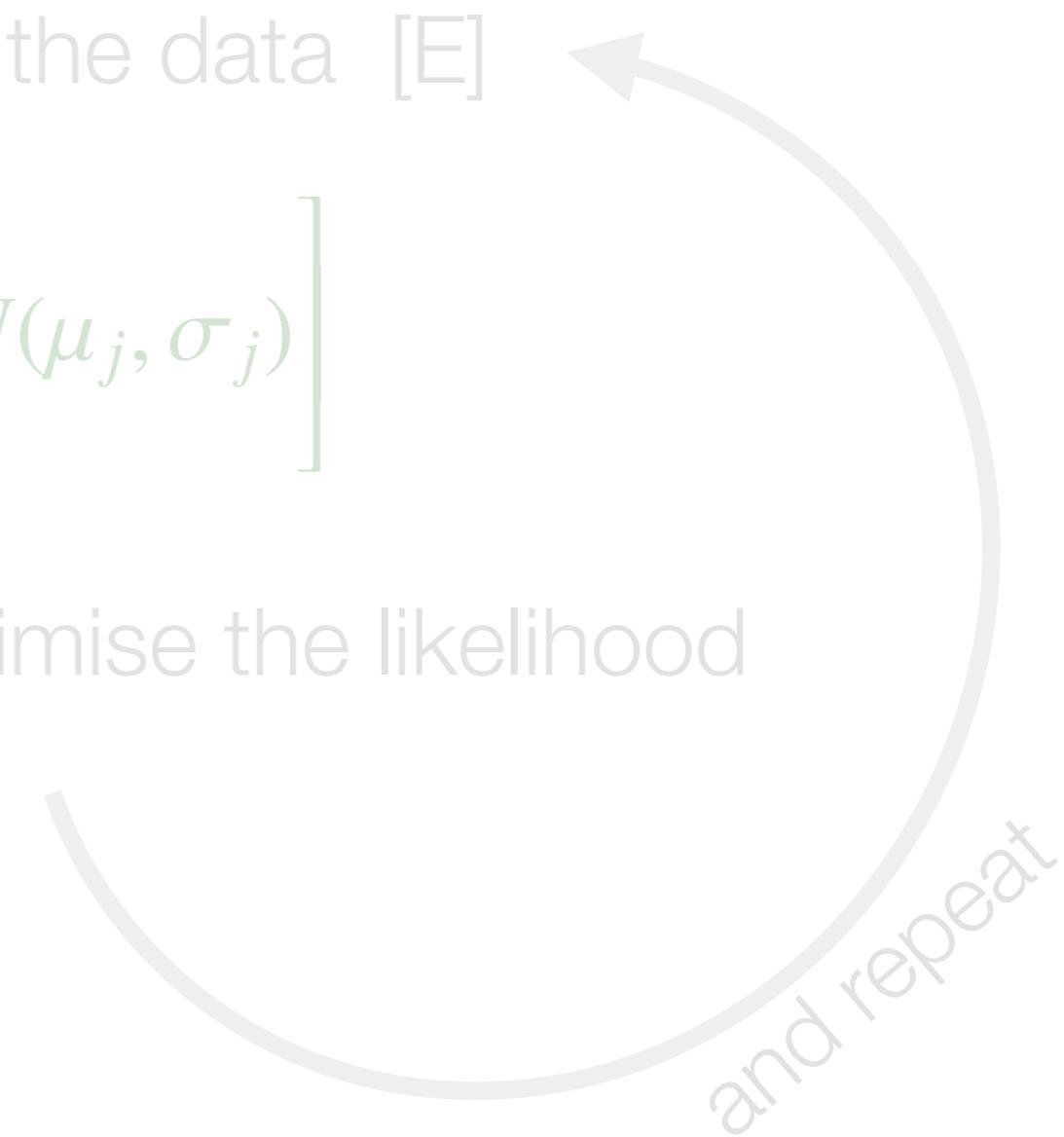
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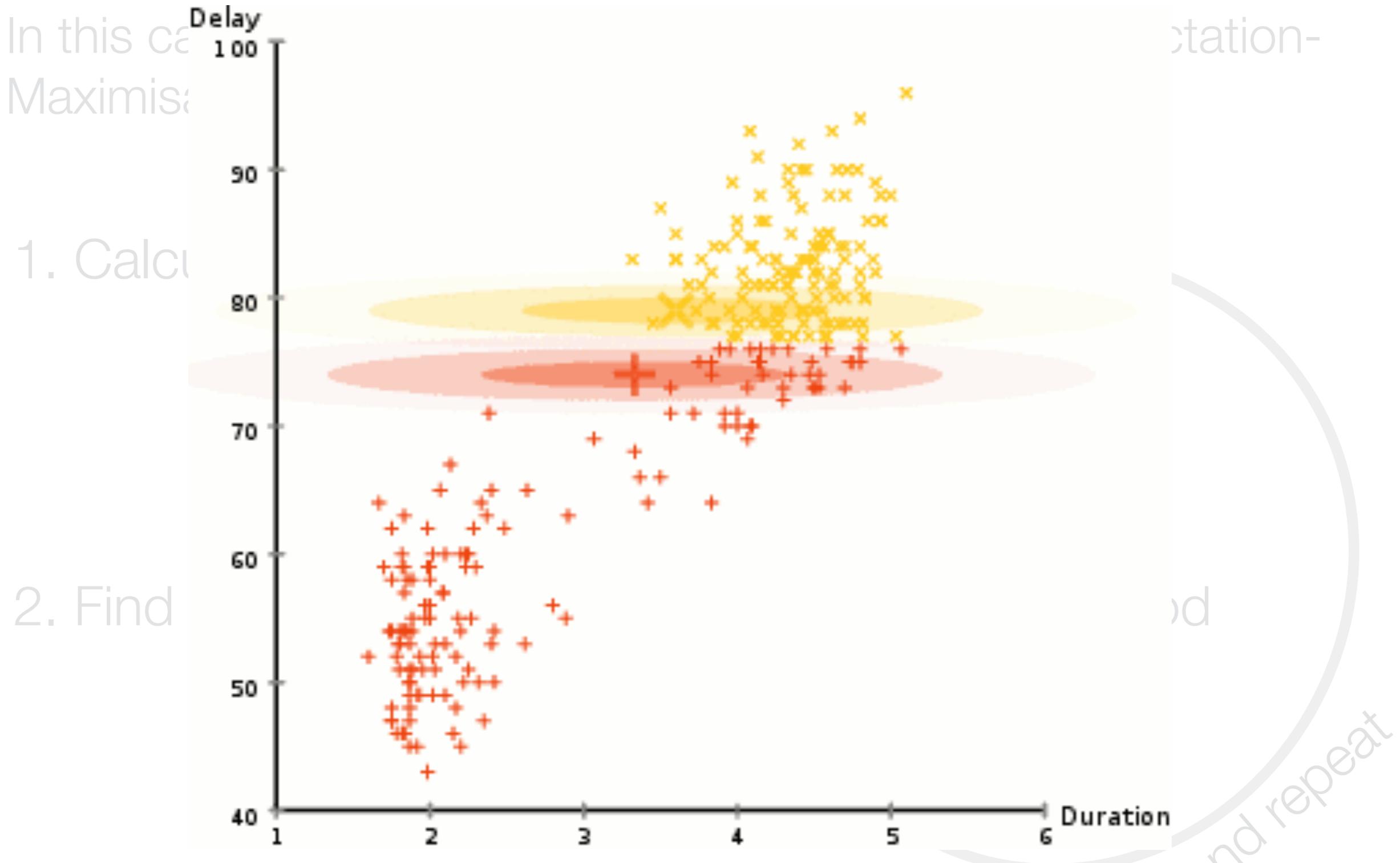
1. Calculate the log likelihood of the data [E]

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# Gaussian mixture modelling



# Gaussian mixture modelling - in practice

(in older versions GMM)

`from sklearn.mixture import GaussianMixture`

(you can make mixtures of other things than Gaussians but they are not implemented in `sklearn.mixture`)

`model = GaussianMixture(2)`

Create a model with a given number of components - here two - this number can be known *a priori*, or will have to be determined as the kernel bandwidth.

`res = model.fit(X)`

fit the data - just as with regression or kernel density estimation

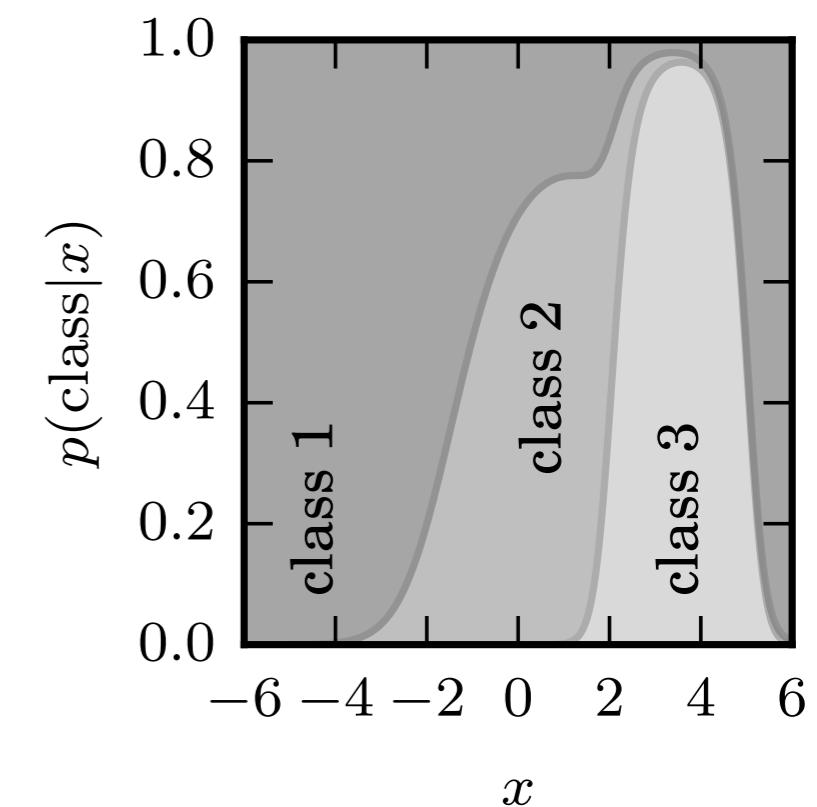
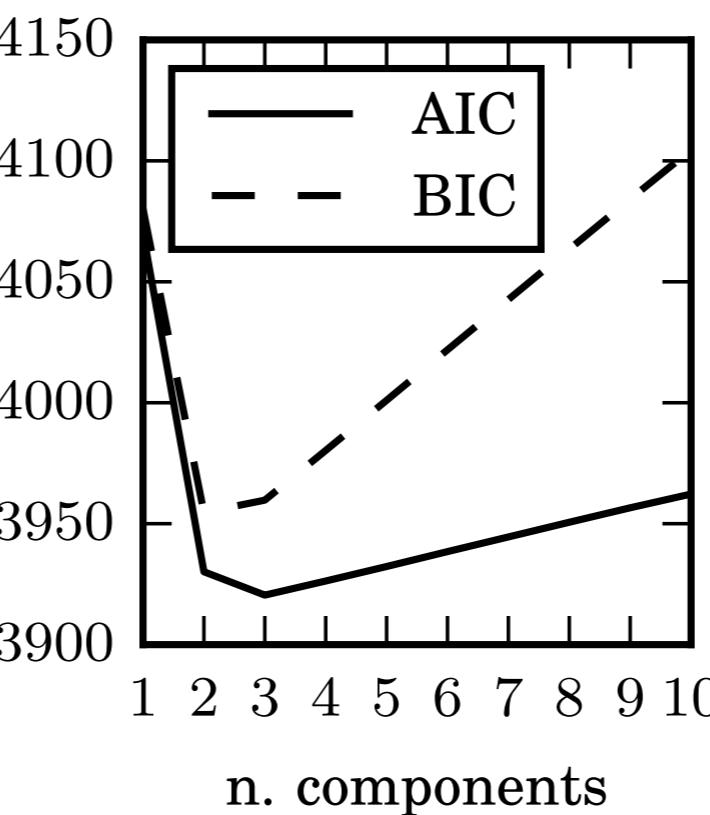
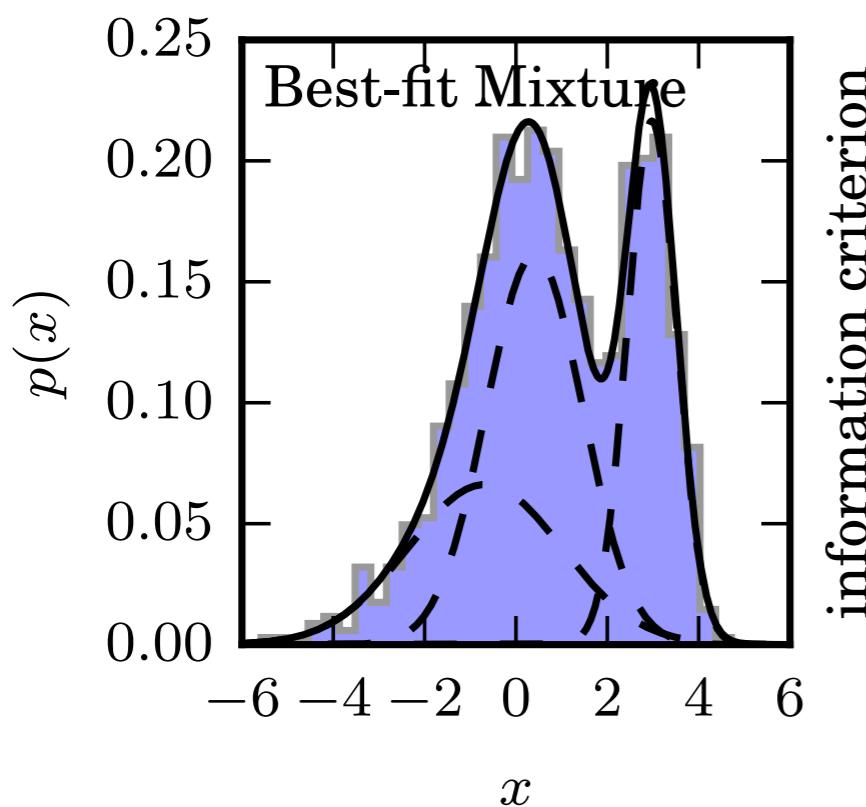
`res.bic(X)`

`print model.means_`

You can get the BIC or AIC easily - these are qualities of fit.

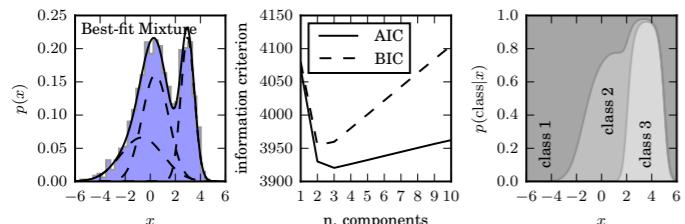
# Gaussian mixture modelling

Using the BIC or AIC to find the best number of Gaussians



# Gaussian mixture modelling

Use of AIC/BIC in practice:



**X = X[:, np.newaxis]**

```
from sklearn.mixture import GaussianMixture
```

```
# fit models with 1-10 components
```

```
N = range(1, 11)
```

```
models = [None for i in N]
```

```
for i in range(len(N)):
```

```
    models[i] = GaussianMixture(N[i]).fit(X)
```

```
# compute the AIC and the BIC
```

```
AIC = [m.aic(X) for m in models]
```

```
BIC = [m.bic(X) for m in models]
```

# Gaussian mixture modelling

Getting fit results back:

```
logprob, p_component = M_best.score_samples(u)
```

logprob: The log  $p(x)$  for the overall model. So how likely data  $\mathbf{u}$  is under this model.

p\_component: The likelihood of the data  $\mathbf{u}$  for each individual component.

If  $\mathbf{u}$  has 10 elements and the GMM has 3 components, p\_component will be 10x3.

# Gaussian mixture modelling

When do we use GMM?

- When you think your distributions are reasonably described by Gaussians.
- As a way to describe distributions.
- When you want a simple approach that works often.
- For classification/assignment.

# Gaussian mixture modelling - degeneracies

Consider the model

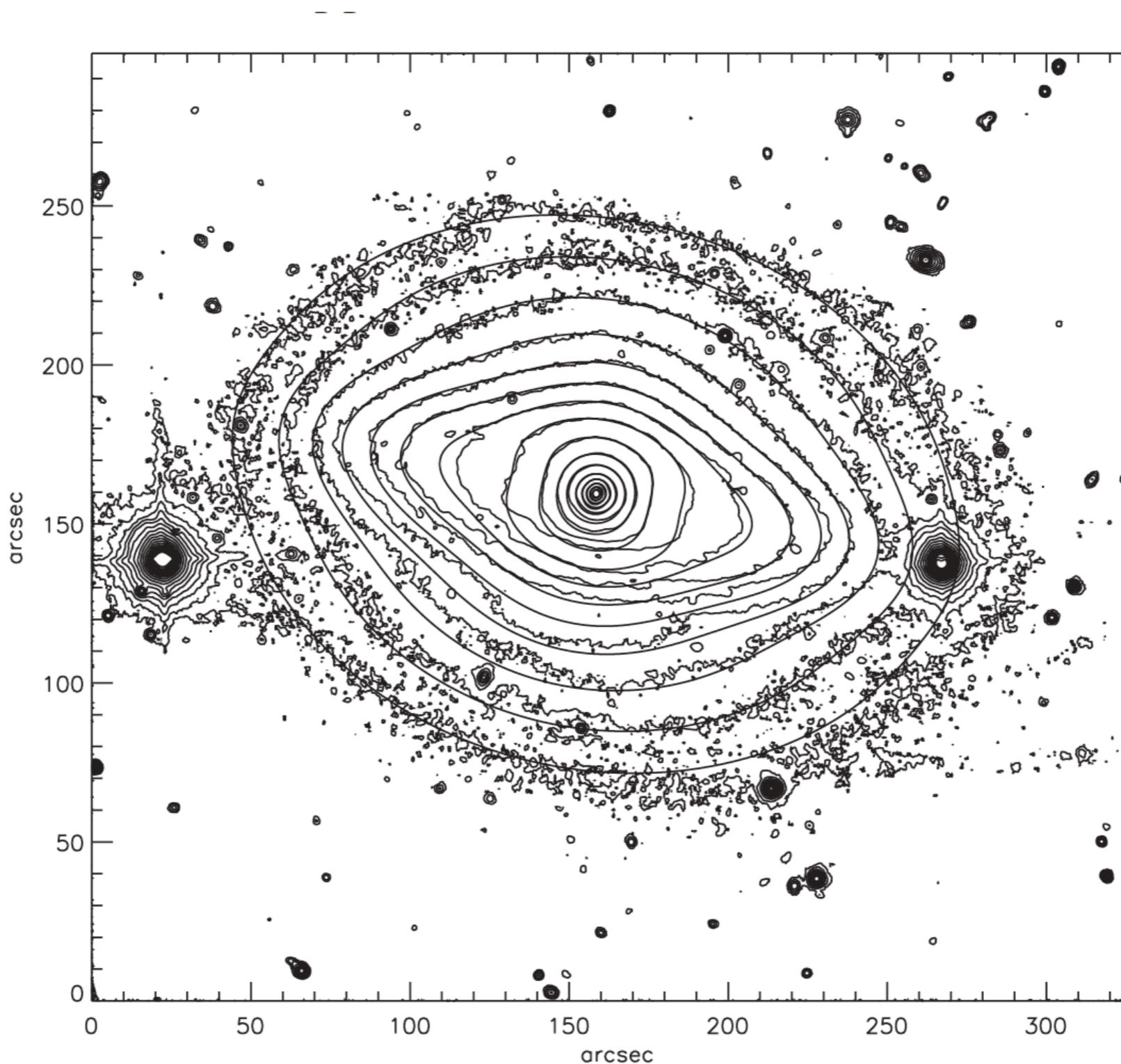
$$\ln L = \sum_{i=1}^N \ln \left[ \sum_{j=1}^M \alpha_j N(x_i; \mu_j, \sigma_j) \right]$$

Assume now that  $\mu_k = x_i$  for some  $i, k$ . In that case we have a term

$$p_k = \frac{1}{\sqrt{2\pi}\sigma_k}$$

which diverges for  $\sigma_k \rightarrow 0$  - in which case that component takes all power.

# Multi-gaussian expansion of galaxy images:



Cappellari (2002)

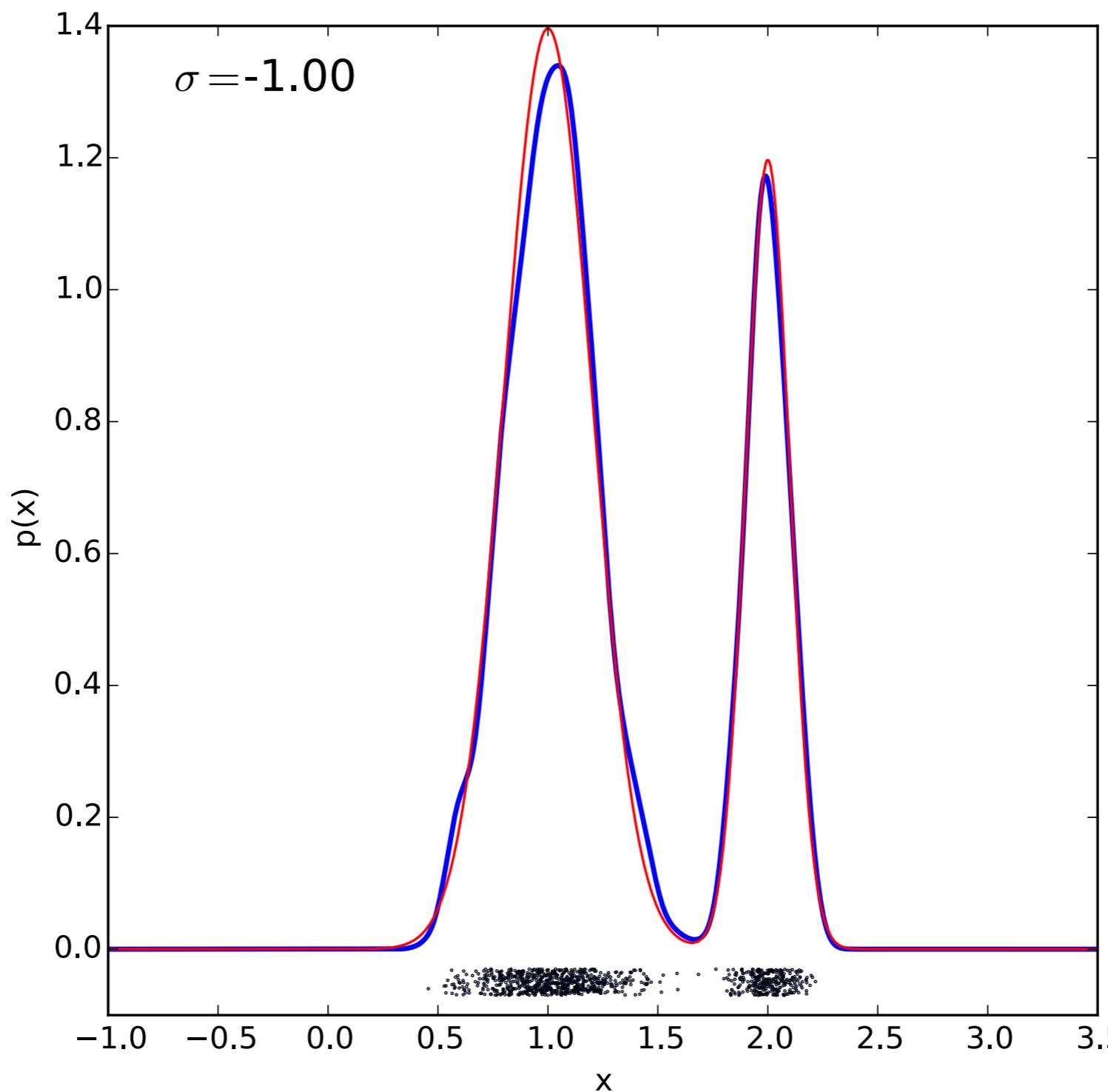
# The effect of noise

# Uncertainties in the data

True

KDE

1000 points - no noise

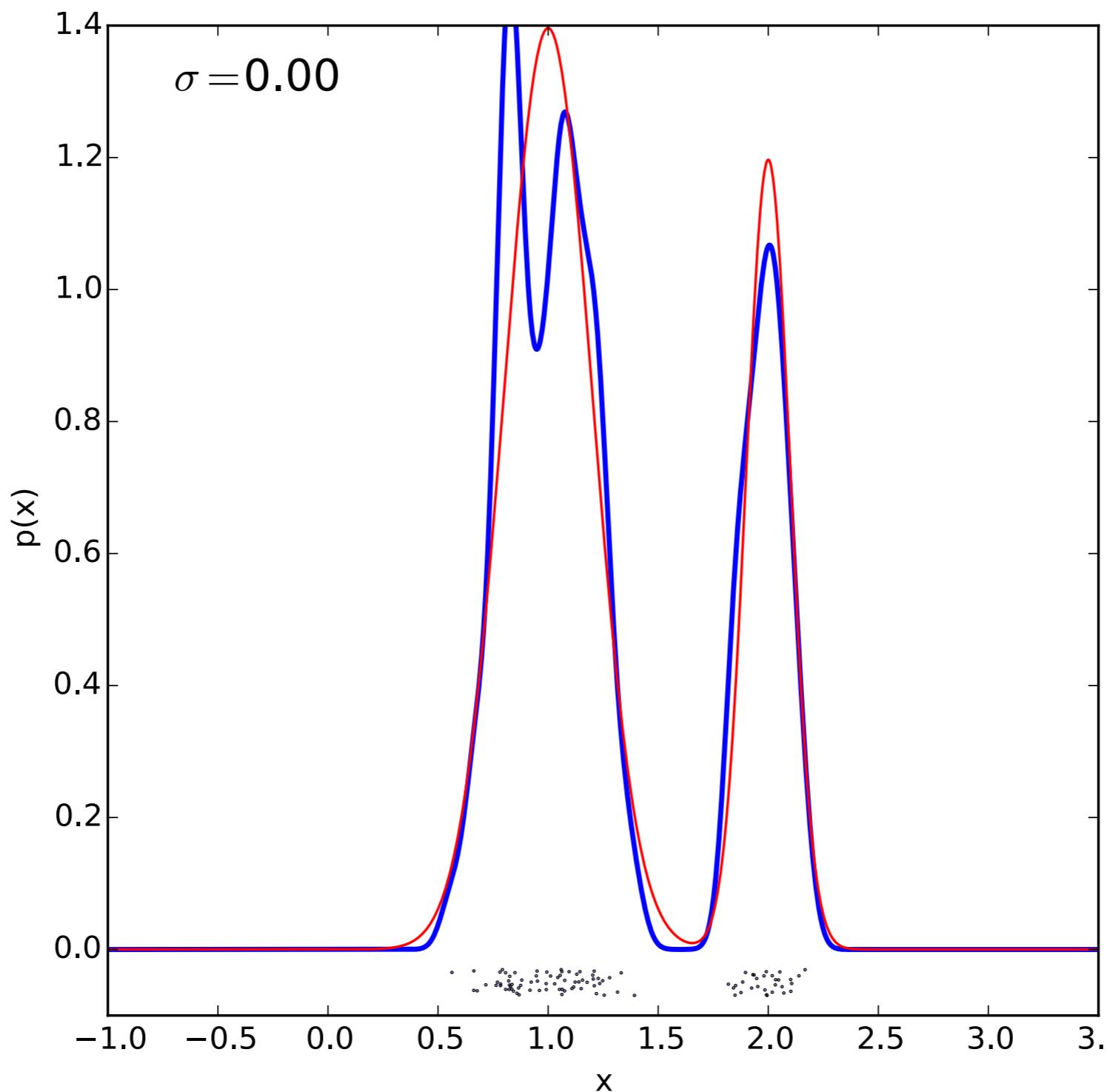


# Uncertainties in the data

True

KDE

100 points - no noise

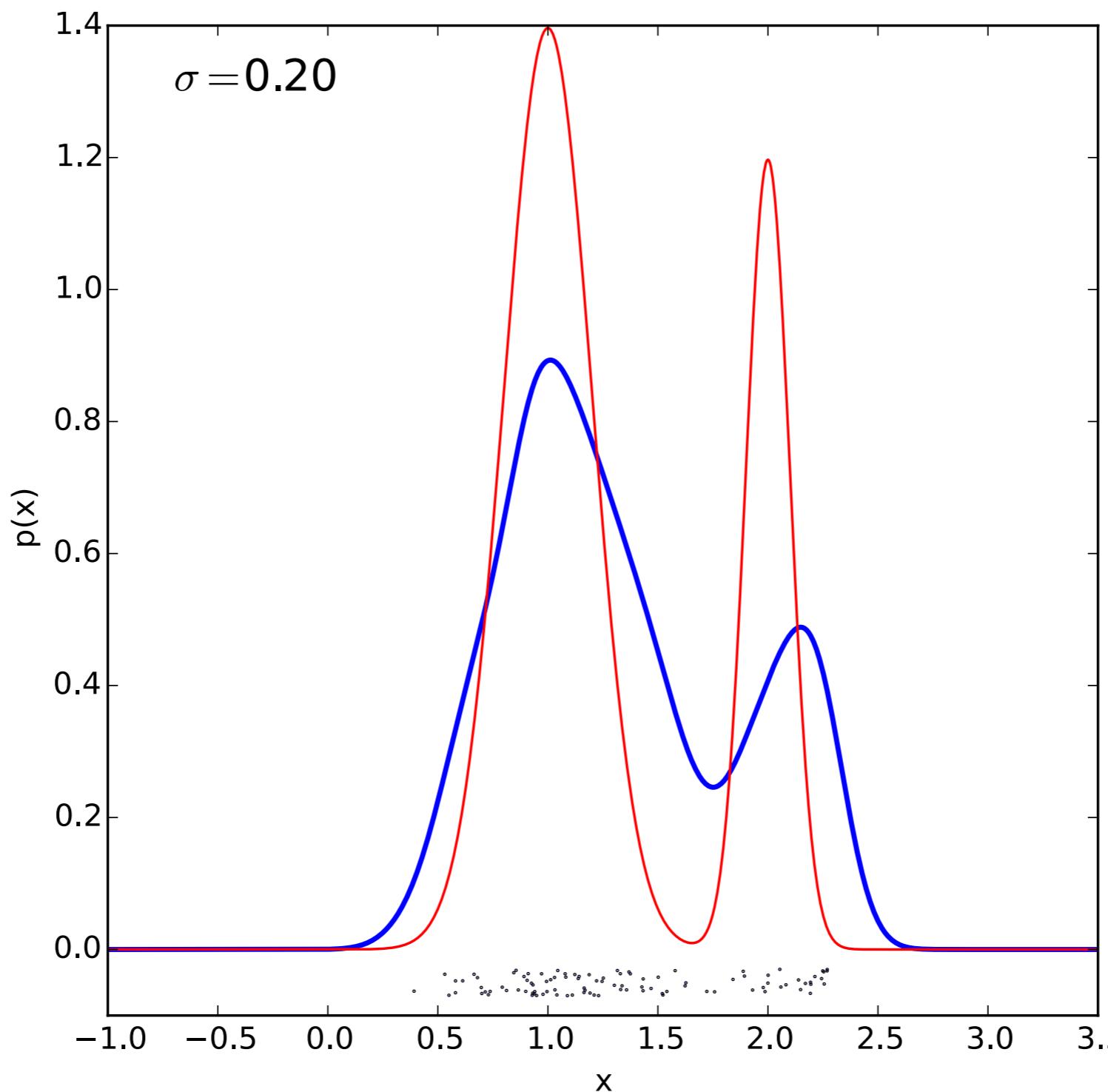


# Uncertainties in the data

True

KDE

100 points

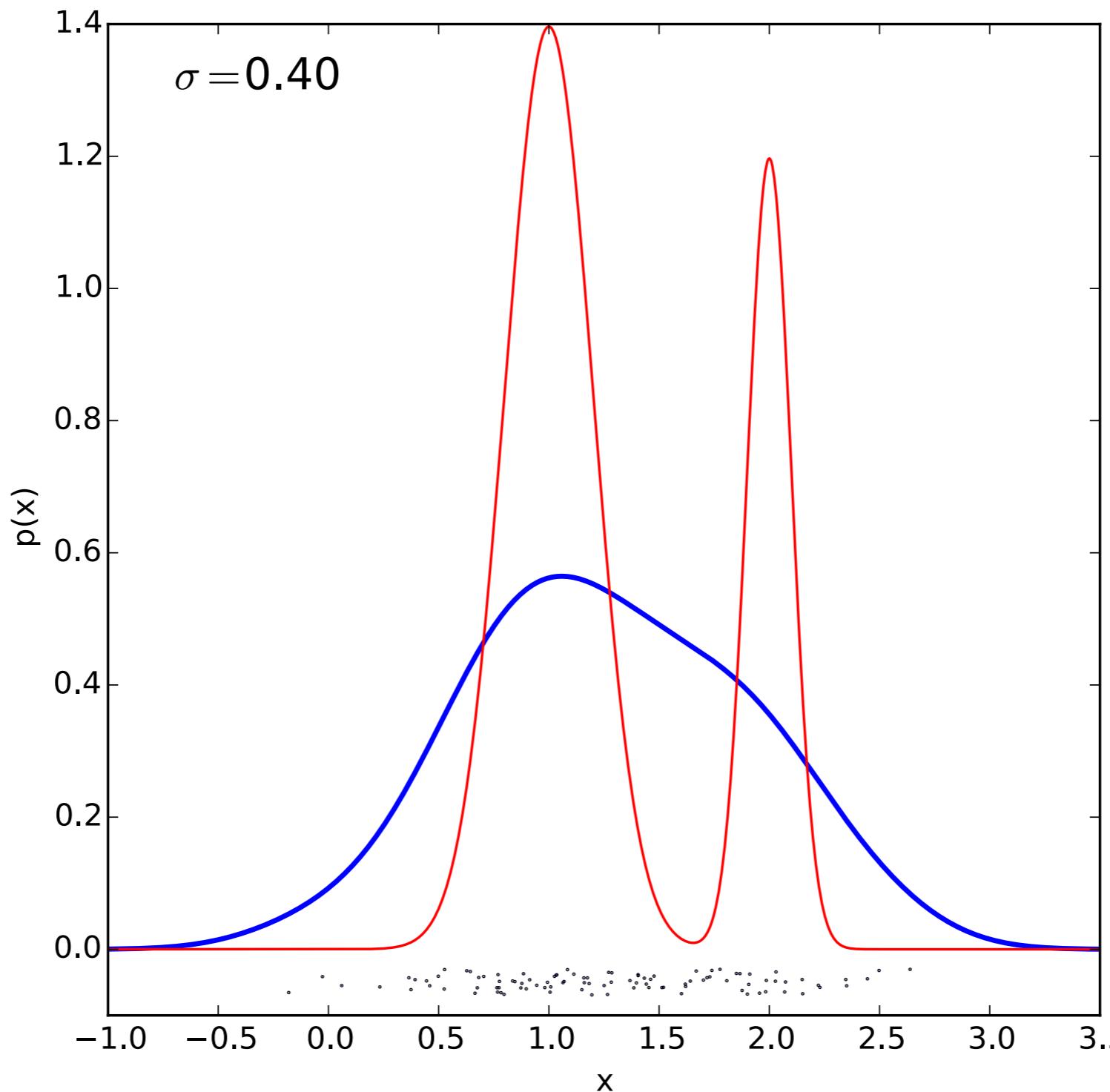


# Uncertainties in the data

True

KDE

100 points



# Uncertainties in the data - KDE

Your observations are (presumably) a random realisation of the underlying true value - this noise acts to broaden the recovered distribution. For equal uncertainties,  $\sigma$ , we have:

$$h_{\text{obs}} = (h_{\text{true}} * g_\sigma)(x) = \int_{-\infty}^{\infty} h(u)g_\sigma(x - u) du$$

So can we recover a better estimate of the “true” distribution?

# Uncertainties in the data - KDE

Your observations are (presumably) a random realisation of the underlying true value - this noise acts to broaden the recovered distribution. For equal uncertainties,  $\sigma$ , we have:

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So can we recover a better estimate of the “true” distribution?

Yes: Deconvoluting KDE

(Delaigle & Meister (2008, Bernoulli, 14, 2, 562) - no Python implementation, but Matlab & R available at <http://www.ms.unimelb.edu.au/~aurored/links.html#Code>

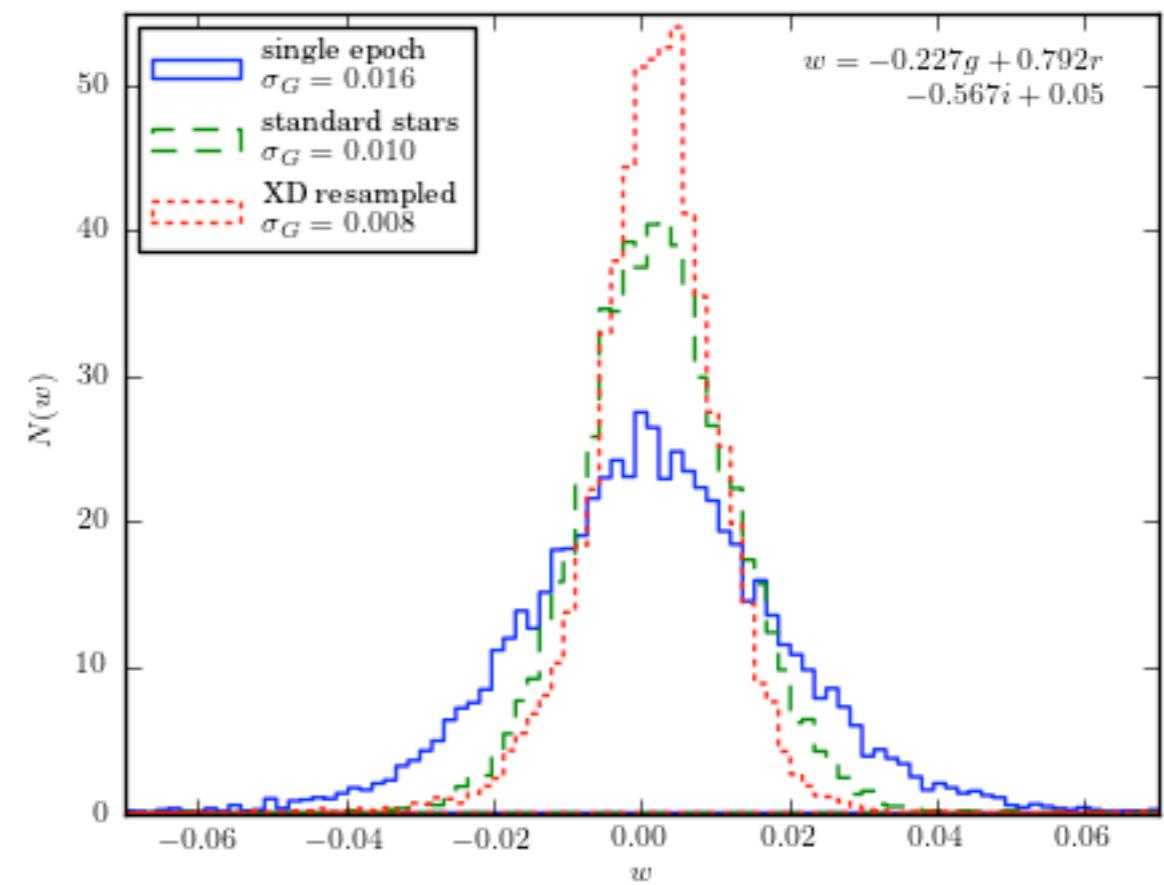
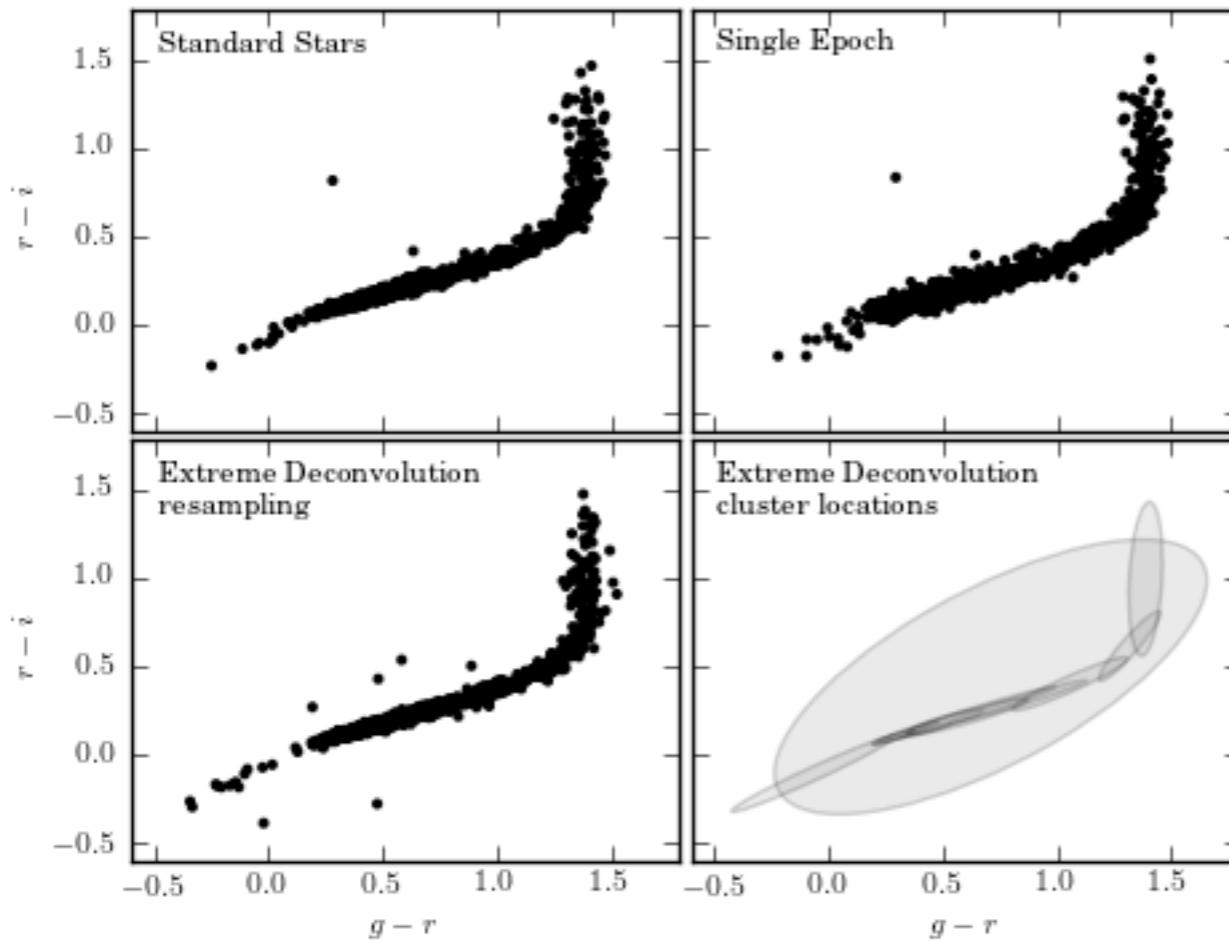
$$h_{\text{est}} = \mathcal{F}^{-1} \left[ \frac{\mathcal{F}(h_{\text{obs}})}{\mathcal{F}(g)} \right] \quad \mathcal{F} \text{ being the Fourier transform}$$

# Uncertainties in the data - GMM

Extreme-deconvolution

(Bovy, Hogg & Roweis 2011)

$$p(\mathbf{x}) + \sum_j \alpha_j N(\mathbf{x} | \mu_j, \Sigma_j) \quad + \quad \mathbf{x}_i = \mathbf{R}_i \mathbf{v}_i + \epsilon_i$$



# Uncertainties in the data - GMM

Trying it out (for after the lecture!):

```
from astroML.density_estimation import XDGMM

@pickle_results("XD_lgm-d4000.pkl")
def compute_XD_results(X, Xerr, n_components=10,
n_iter=500):

    clf = XDGMM(n_components, n_iter=n_iter)
    clf.fit(X, Xerr)
    return clf
```

X:  $N_{\text{obs}} \times N_{\text{features}}$

Data - e.g. x & y

Xerr:  $N_{\text{obs}} \times N_{\text{features}} \times N_{\text{features}}$

Covariances

# Uncertainties in the data - GMM

Working with the result:

`sample = clf.sample(1000)`

Draw from the fitted distribution

`clf.mu`

Position of each component

`clf.V`

Covariance matrix of each component

# When is good, good enough?

Model choice/complexity decision

# Key ingredients of machine learning

Reminder

## Accuracy/error-rate

How well is our algorithm doing? A common measure - the mean squared error:

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (y - y_{\text{pred}})^2$$

# Key ingredients of machine learning

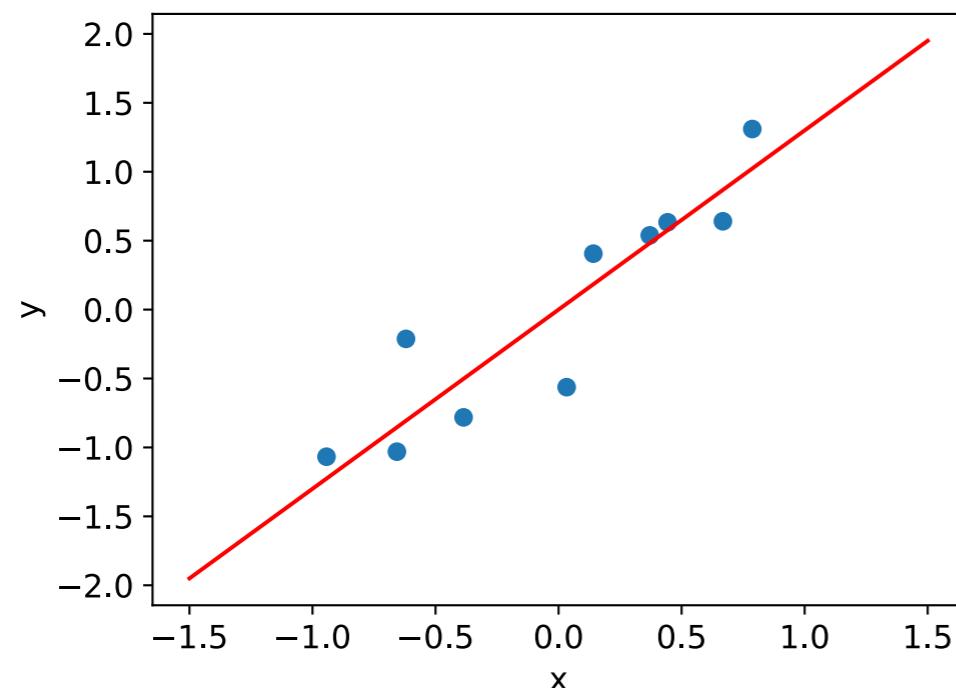
Reminder

## Accuracy/error-rate

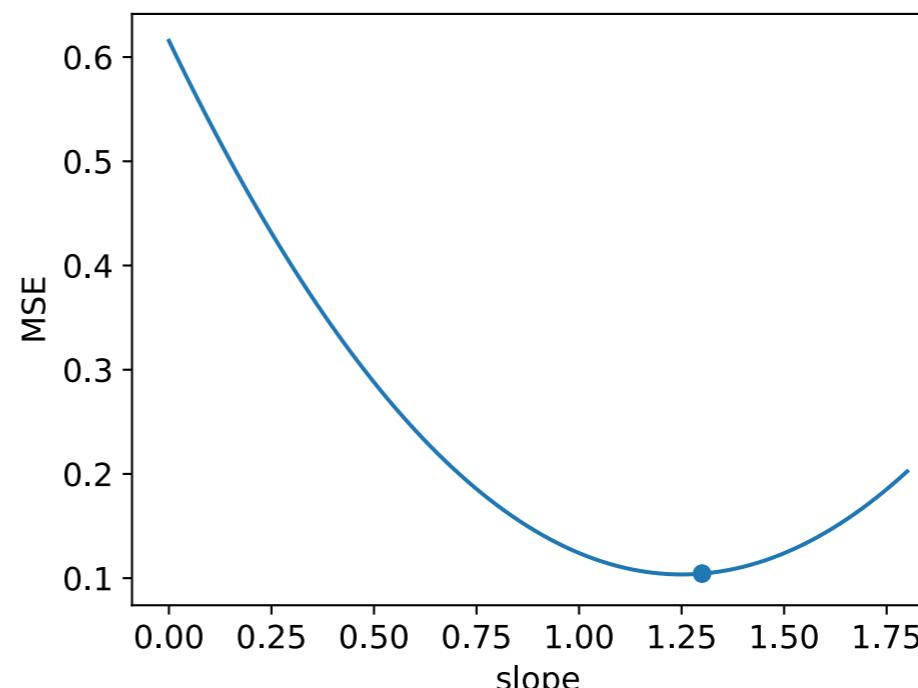
How well is our algorithm doing? A common measure - the mean squared error:

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (y - y_{\text{pred}})^2$$

Try this on linear regression



$$y = \text{slope} \times x$$

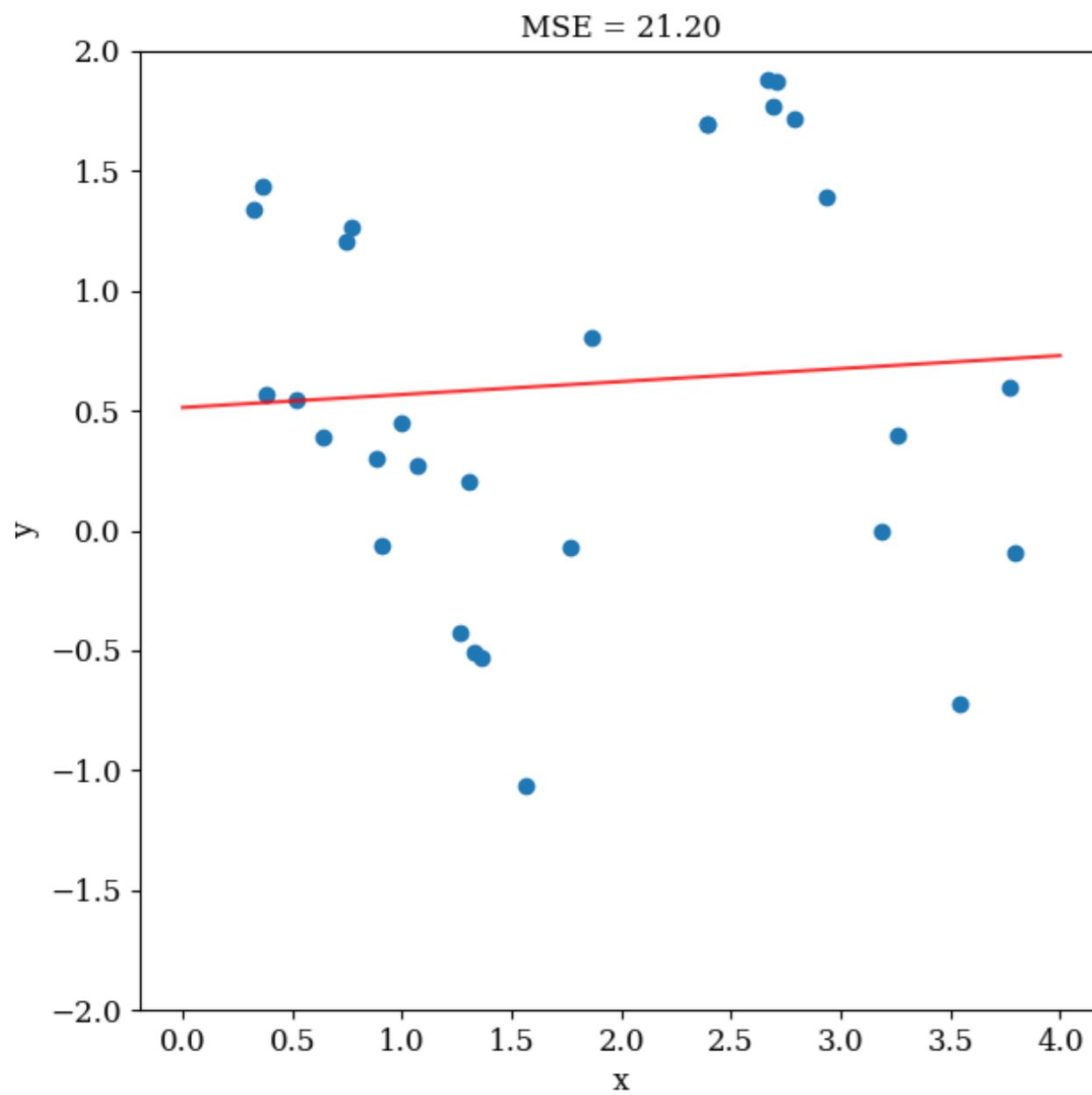


# A simple example - fitting a polynomial

Reminder

## Overfitting, underfitting

We want our method to **generalise** well



High bias!

# A simple example - fitting a polynomial

```
import numpy as np  
from numpy.polynomial import Polynomial
```

```
<... Get data x & y ...>
```

```
p = Polynomial.fit(x, y, degree)  
y_pred = p(x)
```

And we can also easily calculate the MSE:

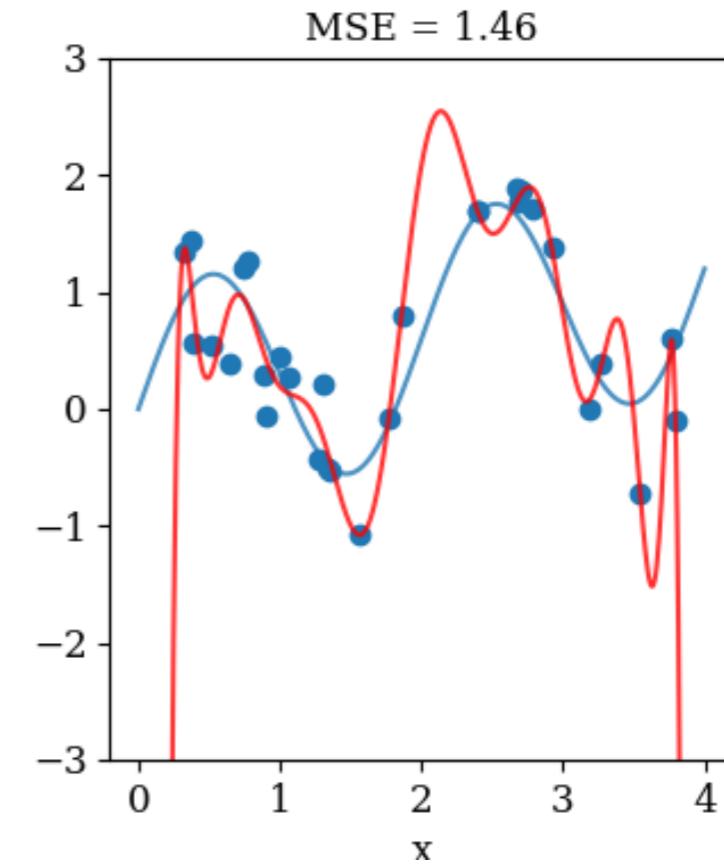
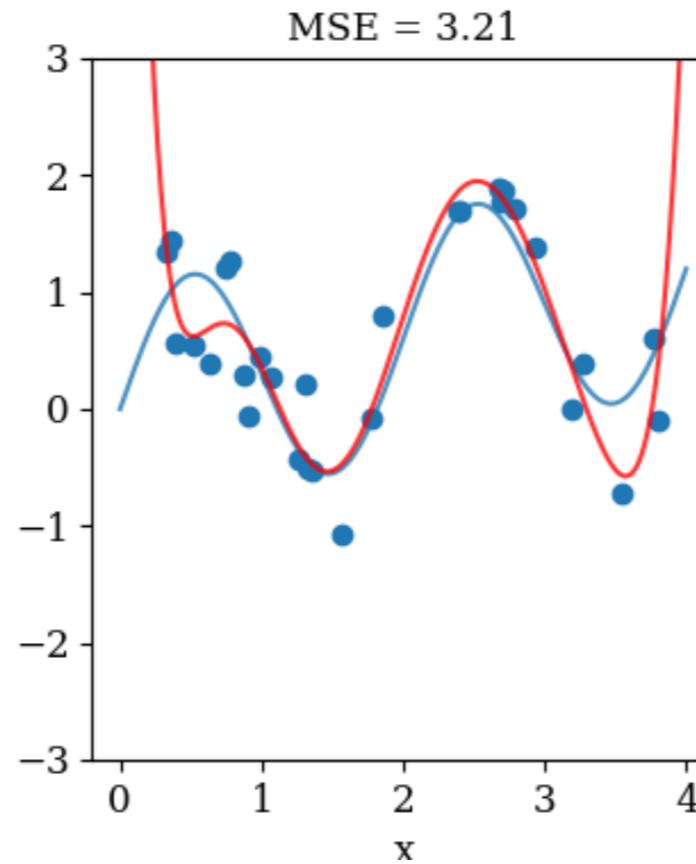
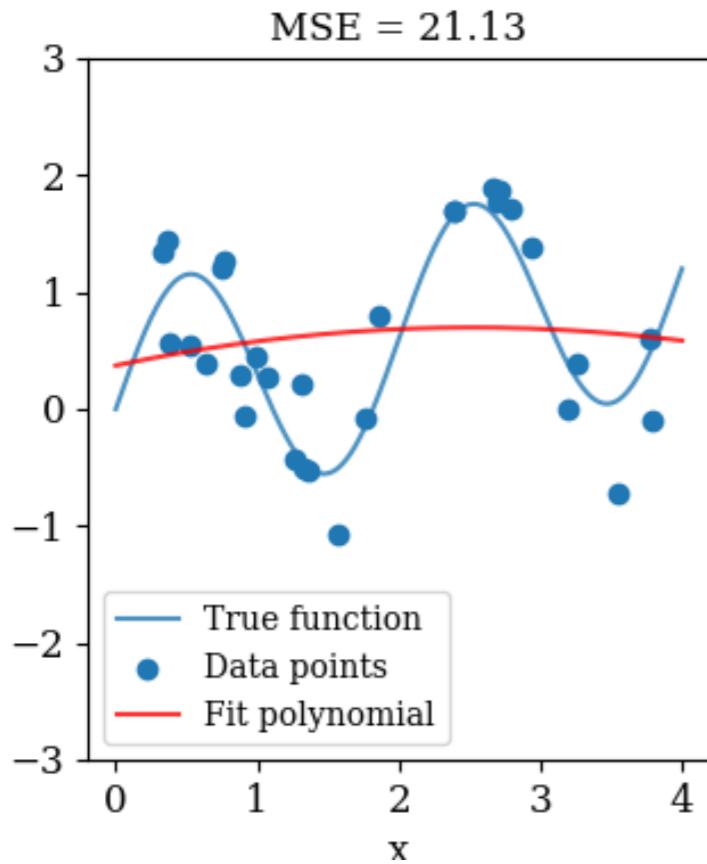
```
mse = np.sum((y_pred-y)**2)
```

Reminder

How do you do it?

# Increasing the flexibility:

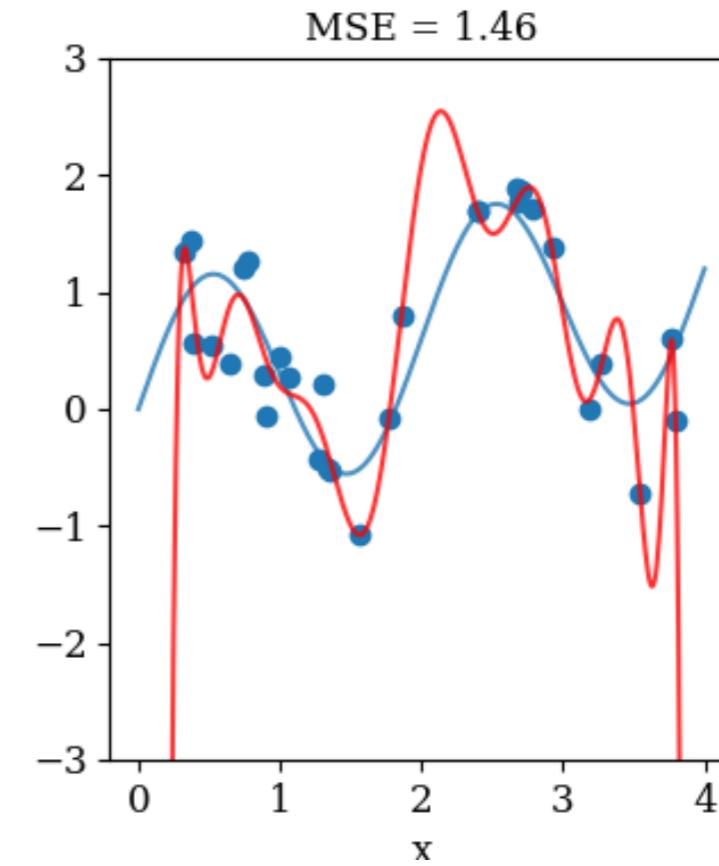
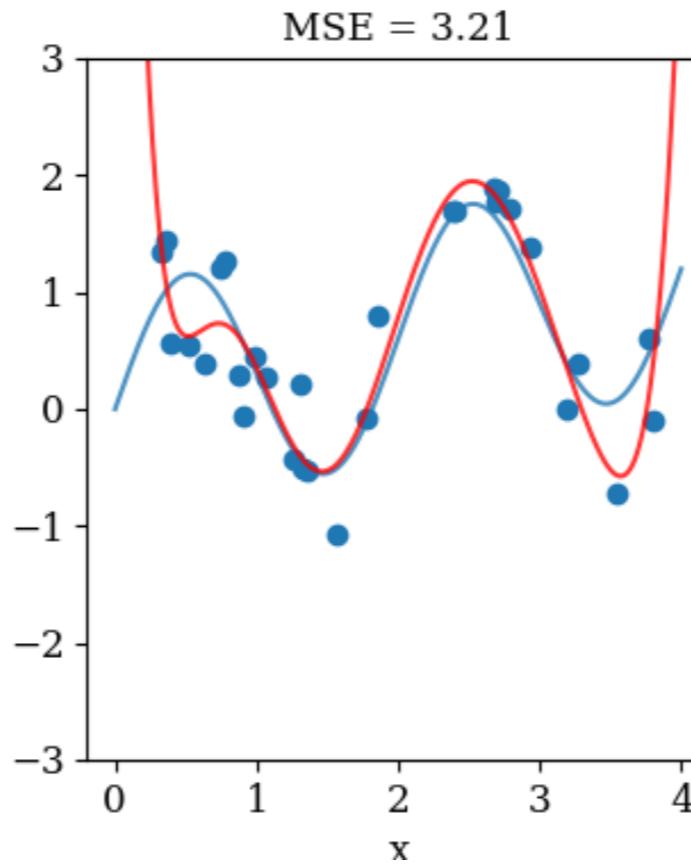
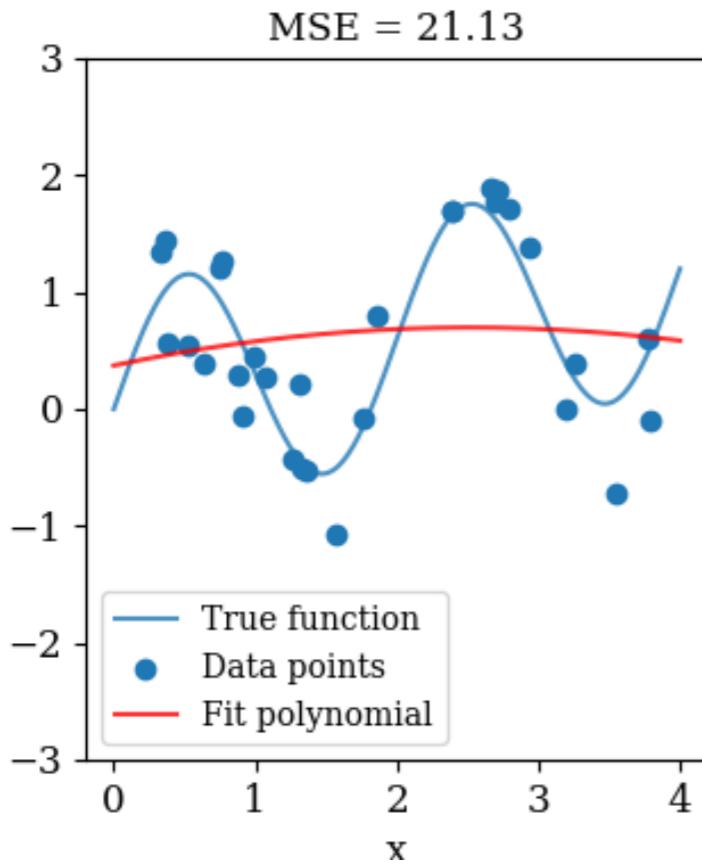
Reminder



$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (y - y_{\text{pred}})^2$$

# Increasing the flexibility:

Reminder



Underfitting

Overfitting

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (y - y_{\text{pred}})^2$$

# Key ingredients of machine learning

Reminder

## Parameters, hyper-parameters

Consider a polynomial:

$$y = \sum_{i=0}^M a_i x^i$$

Here  $a_i$  are the parameters of the model and what our machine learning algorithm will give us.

But M - the maximum polynomial power is also a parameter. It is a different kind of parameter and we call this a **hyperparameter**.

# How do you decide

We will return to this but we should think of dividing our data in three:

Reminder

## The training set

This is what we use to find the best-fitting parameters and we minimise the **training error** (e.g. MSE) on this.

## The validation set

We use this to determine the optimal settings of the hyper-parameters.

## The test set

We evaluate the generalisation error of our algorithm on this sample. These data are *not* used for the fitting. We refer to the error achieved on this set as the **test error**.

# Taking this for our polynomial example

1. For each polynomial order  $M$ , fit a polynomial to the test data.
2. Use this fitted polynomial on the validation set and calculate an MSE for this.
3. Repeat for the next  $M$
4. Choose the  $M$  that gives the smallest MSE on the validation set.
5. Calculate the MSE on the test set to get a quantitative estimate of how well the calculation did go.

How do you do it?

# Evaluating the generalisation error

What we want:

How well does a fitted function or trained Machine  
Learning method work on new data? **Generalisation error**

We can't quite have that - but this is where we use the test set/validation set.

Data

One possibility

This is fine if we have 100,000s of data (usually), but can leave too few points to train on.

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One possibility

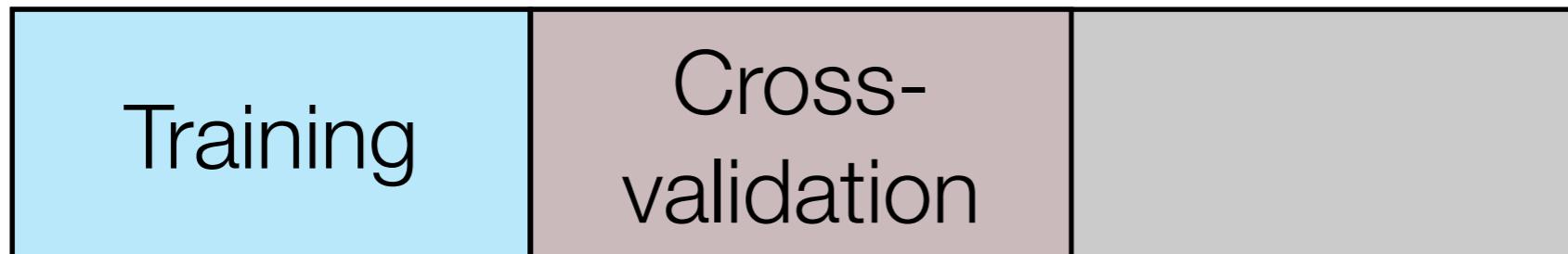
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One possibility

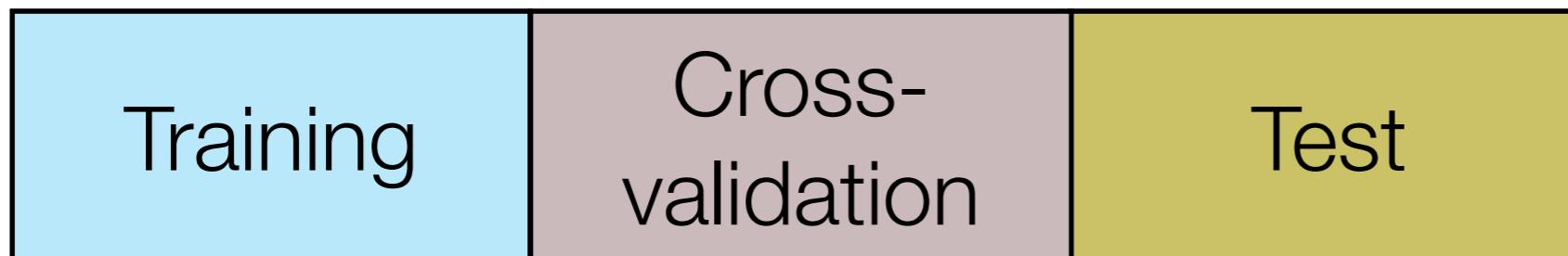
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# Evaluating the generalisation error

What we want:

How well does a fitted function or trained Machine Learning method work on new data? **Generalisation error**

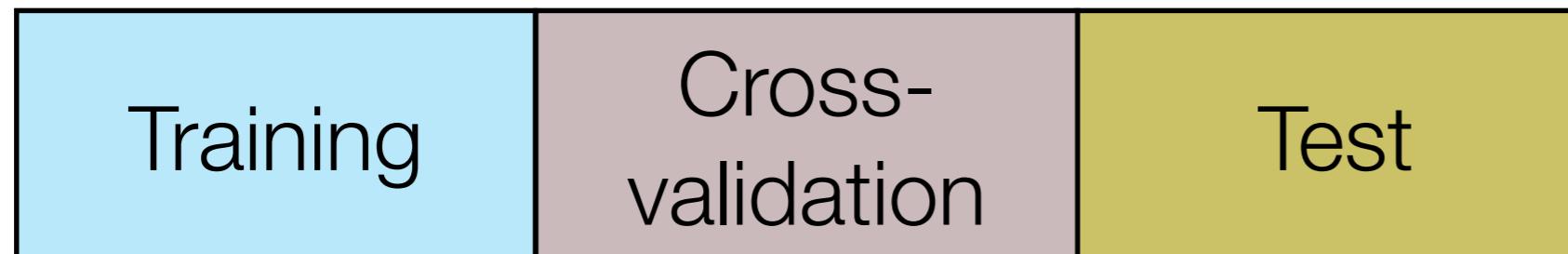
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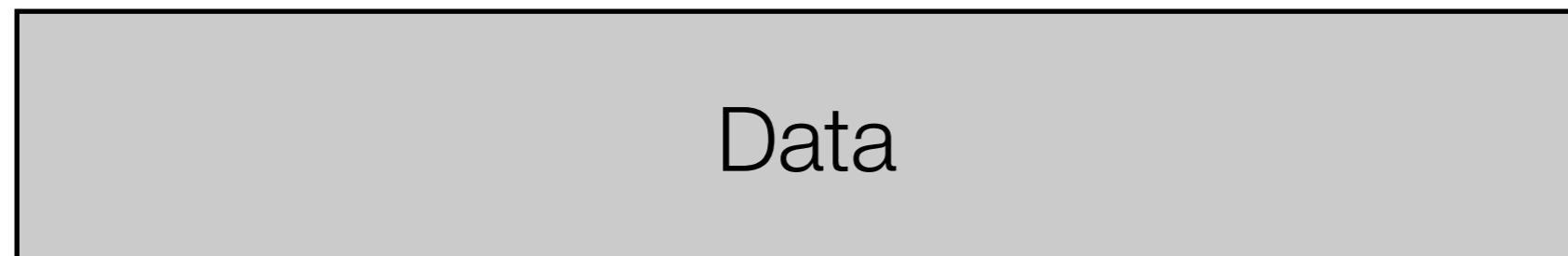
One possibility

This is fine if we have 100,000s of data (usually), but can leave too few points to train on.

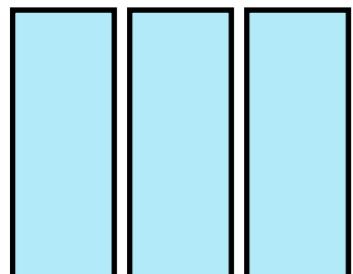
# Cross-validation to the rescue



One possibility



k-fold CV



Training

k-copies



Test

Do this splitting  
k times.

It is generally recommended to use 5- or 10- fold cross-validation.

# Example: determining the optimal bandwidth for KDE

KDE allows us to construct a distribution function.

$$p_x(\mathbf{x})$$

This should tell me how likely a value  $z$  is, so I can evaluate  $p_x(z)$  for  $z$  in my **test set**. This gives the likelihood of that test set.

Do this for all  $k$  folds.

$$\ln L_{CV} = \frac{1}{k} \sum_{i=1}^k \ln p_{x_i}(z_i)$$

Then do this for all band-widths and select the best (highest  $L_{cv}$  in this case)

# Practical steps

You can do CV completely manually - but I recommend using the `sklearn.cross_validation` package:

```
from sklearn.cross_validation import KFold

# Create folds for N data points in n_folds:
kf = KFold(N, n_folds=n_folds)

for train, test in kf:
    x_train = x[train, :]
    x_test = x[test, :]

<do calculations & calculate score using x_test>
```

# Short-cutting CV - BIC & AIC

A full CV evaluation can be very costly time-wise - in that case you might want a cheaper way to assess quality of fit.

Enter the information criteria:

The Bayesian and the Akaike, BIC & AIC

$$\text{BIC} = -2 \ln L_{\max} + k \ln N$$

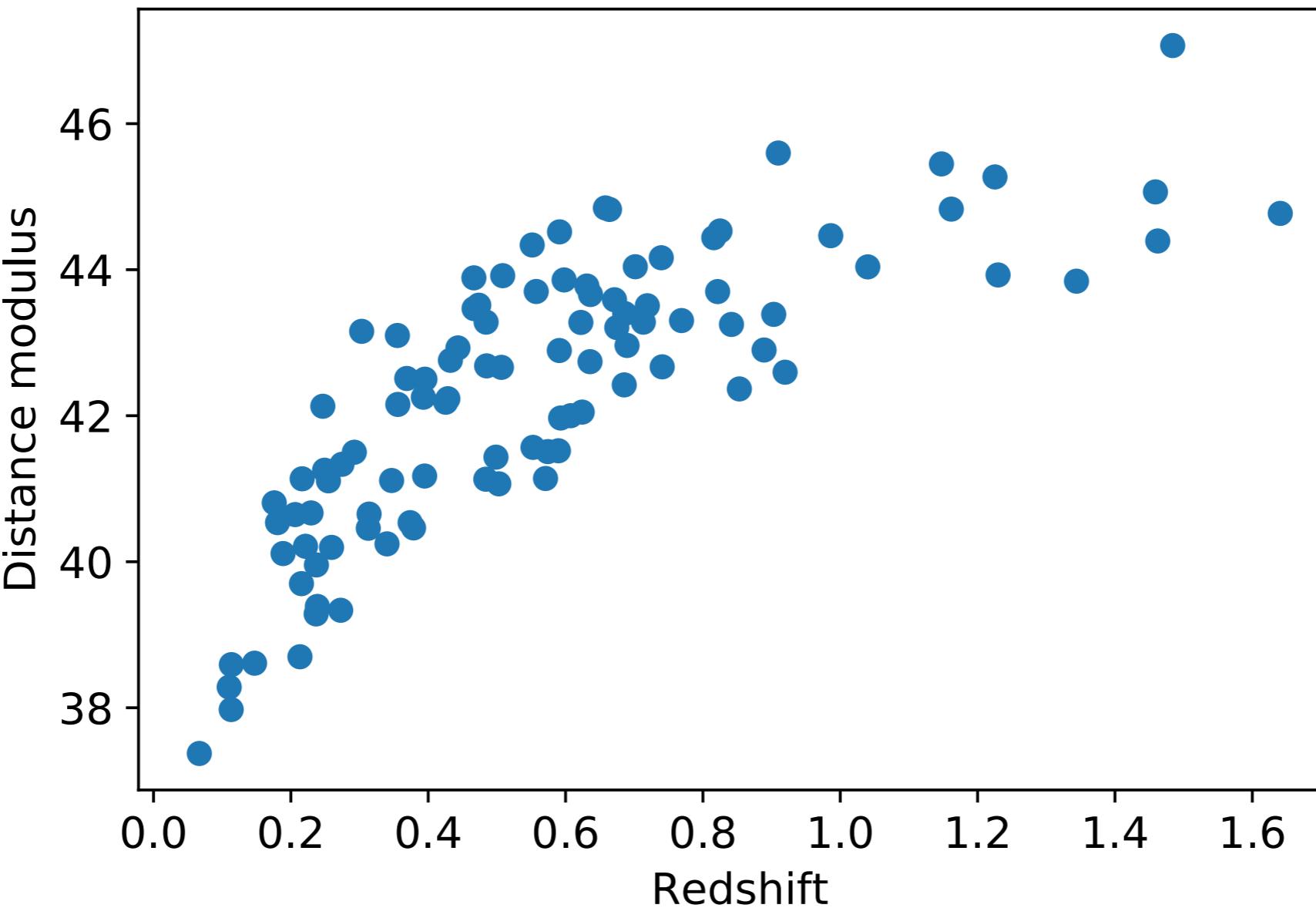
$k$ : number of parameters

$$\text{AIC} = -2 \ln L_{\max} + 2k + \frac{2k(k+1)}{N-k-1}$$

$N$ : number of data points

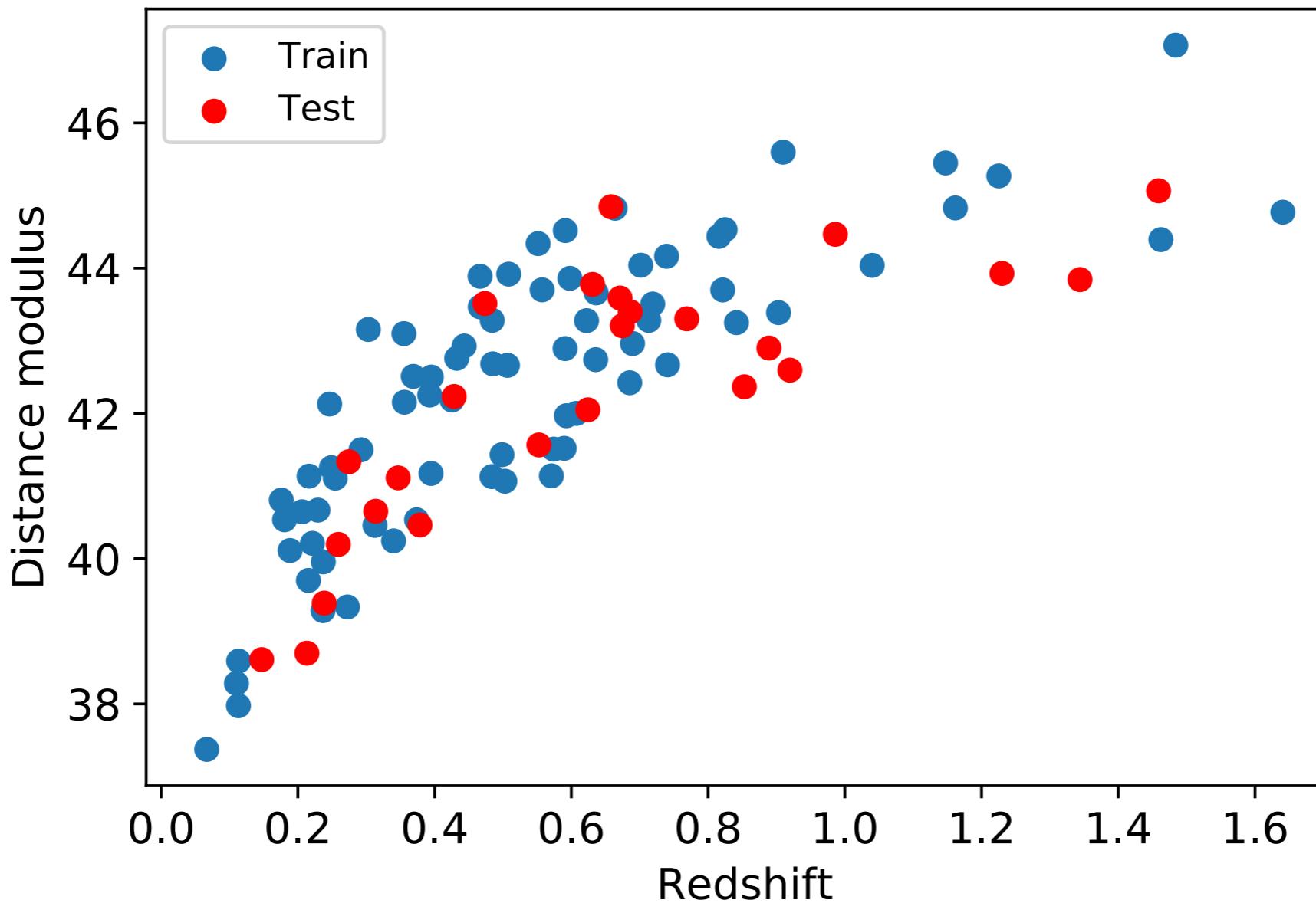
$L_{\max}$ : max likelihood

# A simple example



```
from sklearn.model_selection import train_test_split  
  
indices = np.arange(n_sample, dtype=np.int)  
i_train, i_test = train_test_split(indices)
```

# A simple example



```
plt.scatter(z_sample[i_train], mu_sample[i_train],  
           label='Train')  
plt.scatter(z_sample[i_test], mu_sample[i_test],  
           color='red', label='Test')
```

# A simple example

# A simple example

## Fit the training sample:

```
orders = np.arange(n_orders)
for i, order in enumerate(orders):
    # Fit the training sample using polyfit
    p = np.polyfit(z_sample[i_train], mu_sample[i_train], order)
```

# A simple example

## Fit the training sample:

```
orders = np.arange(n_orders)
for i, order in enumerate(orders):
    # Fit the training sample using polyfit
    p = np.polyfit(z_sample[i_train], mu_sample[i_train], order)
```

## Calculate quality of fit on training sample:

```
mu_fit_train = np.polyval(p, z_sample[i_train])
mu_train = mu_sample[i_train]
MSE_train[i] = np.sum((mu_train-mu_fit_train)**2)/n_train
```

# A simple example

## Fit the training sample:

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orders = np.arange(n_orders)
for i, order in enumerate(orders):
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mu_train = mu_sample[i_train]
MSE_train[i] = np.sum((mu_train-mu_fit_train)**2)/n_train
```

## Calculate quality of fit on test sample:

```
mu_fit_test = np.polyval(p, z_sample[i_test])
mu_test = mu_sample[i_test]
MSE_train[i] = np.sum((mu_test-mu_fit_test)**2)/n_test
```

# A simple example

## Fit the training sample:

```
orders = np.arange(n_orders)
for i, order in enumerate(orders):
    # Fit the training sample using polyfit
    p = np.polyfit(z_sample[i_train], mu_sample[i_train], order)
```

## Calculate quality of fit on training sample:

```
mu_fit_train = np.polyval(p, z_sample[i_train])
mu_train = mu_sample[i_train]
MSE_train[i] = np.sum((mu_train-mu_fit_train)**2)/n_train
```

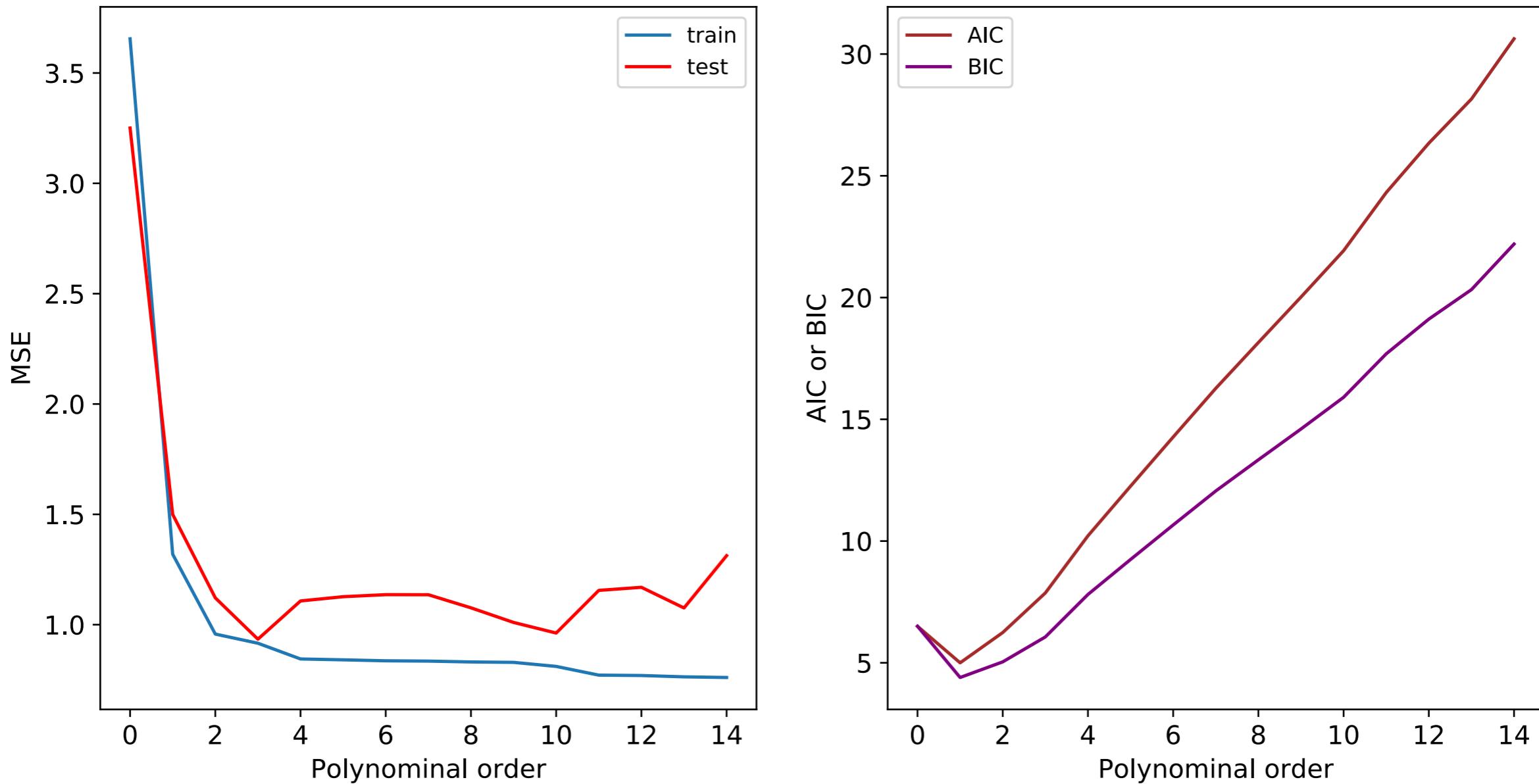
## Calculate quality of fit on test sample:

```
mu_fit_test = np.polyval(p, z_sample[i_test])
mu_test = mu_sample[i_test]
MSE_train[i] = np.sum((mu_test-mu_fit_test)**2)/n_test
```

## and for completeness, calculate BIC & AIC:

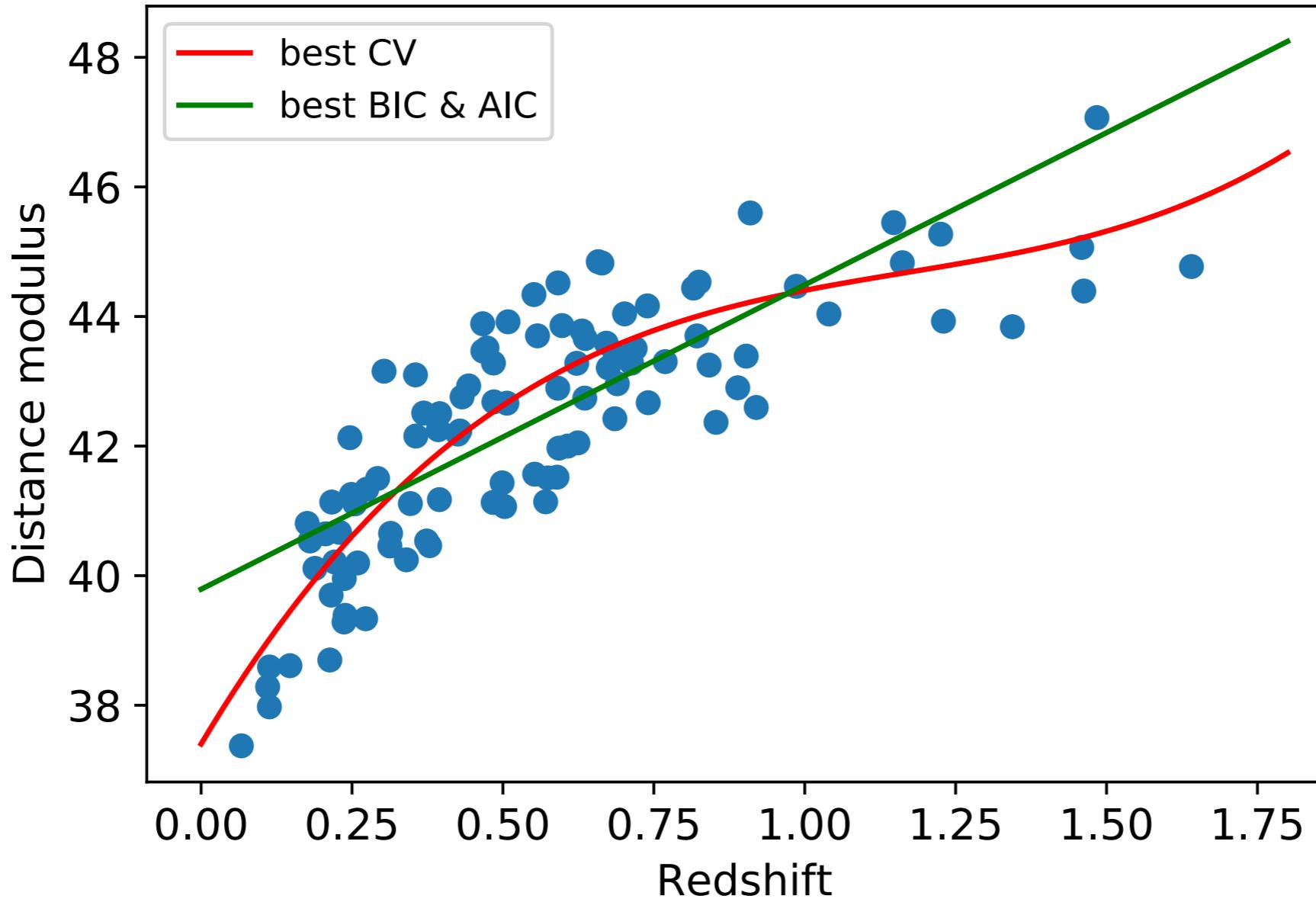
```
BIC[i] = 2*MSE_test[i] + order*np.log10(len(i_test))
AIC[i] = 2*order + 2*MSE_test[i]
```

# A simple example - result



So CV gives 3 as the best order, and AIC & BIC gives 1...

# A simple example - result



Illustrates the challenge of “best fit” - it is not always obvious. The BIC and AIC often want simple models, while the data might support more complexity.

# **Why does it matter - the bias-variance trade-off**

# Complexity versus simplicity - the bias variance trade-off

Assume that the true relation between  $x$  &  $y$  is

$$y = f(x) + \epsilon \quad \epsilon \sim N(0, \sigma_\epsilon)$$

And write the predicted value of  $y$  at  $x_0$   $\hat{y}$

Let us ask what the expected mean square error in our prediction at  $x=x_0$  is:

$$\mathbb{E}[(f(x_0) - \hat{y})^2]$$

# Small versus big bins - the bias variance trade-

Expected mean square error at  $x=x_0$ :

$$\mathbb{E}[(f(x_0) - \hat{y})^2]$$

$$y = f(x) + \epsilon$$

$$\epsilon \sim N(0, \sigma_\epsilon)$$

We can expand this and add and subtract  $(\mathbb{E}[\hat{y}])^2$  to get:

$$\text{Err}(x_0) = \sigma_\epsilon^2 + (\mathbb{E}[\hat{y}] - f(x_0))^2 + \mathbb{E}[(\hat{y} - \mathbb{E}[\hat{y}])^2]$$

# Small versus big bins - the bias variance trade-

Expected mean square error at  $x=x_0$ :

$$\mathbb{E}[(f(x_0) - \hat{y})^2]$$

$$y = f(x) + \epsilon$$

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We can expand this and add and subtract  $(\mathbb{E}[\hat{y}])^2$   
to get:

$$\text{Err}(x_0) = \sigma_\epsilon^2 + (\mathbb{E}[\hat{y}] - f(x_0))^2 + \mathbb{E}[(\hat{y} - \mathbb{E}[\hat{y}])^2]$$

Bias<sup>2</sup>

# Small versus big bins - the bias variance trade-

## Expected mean square error at $x=x_0$ :

$$\mathbb{E} \left[ (f(x_0) - \hat{y})^2 \right] \quad \epsilon \sim N(0, \sigma_\epsilon)$$

We can expand this and add and subtract  $(E[\hat{y}])^2$  to get:

$$\text{Err}(x_0) = \sigma_\epsilon^2 + (E[\hat{y}] - f(x_0))^2 + E[(\hat{y} - E[\hat{y}])^2]$$

Bias<sup>2</sup>                              Variance

# Small versus big bins - the bias variance trade-

Expected mean square error at  $x=x_0$ :

$$\mathbb{E}[(f(x_0) - \hat{y})^2]$$

$$y = f(x) + \epsilon$$

$$\epsilon \sim N(0, \sigma_\epsilon)$$

We can expand this and add and subtract  $(\mathbb{E}[\hat{y}])^2$  to get:

$$\text{Err}(x_0) = \sigma_\epsilon^2 + (\mathbb{E}[\hat{y}] - f(x_0))^2 + \mathbb{E}[(\hat{y} - \mathbb{E}[\hat{y}])^2]$$

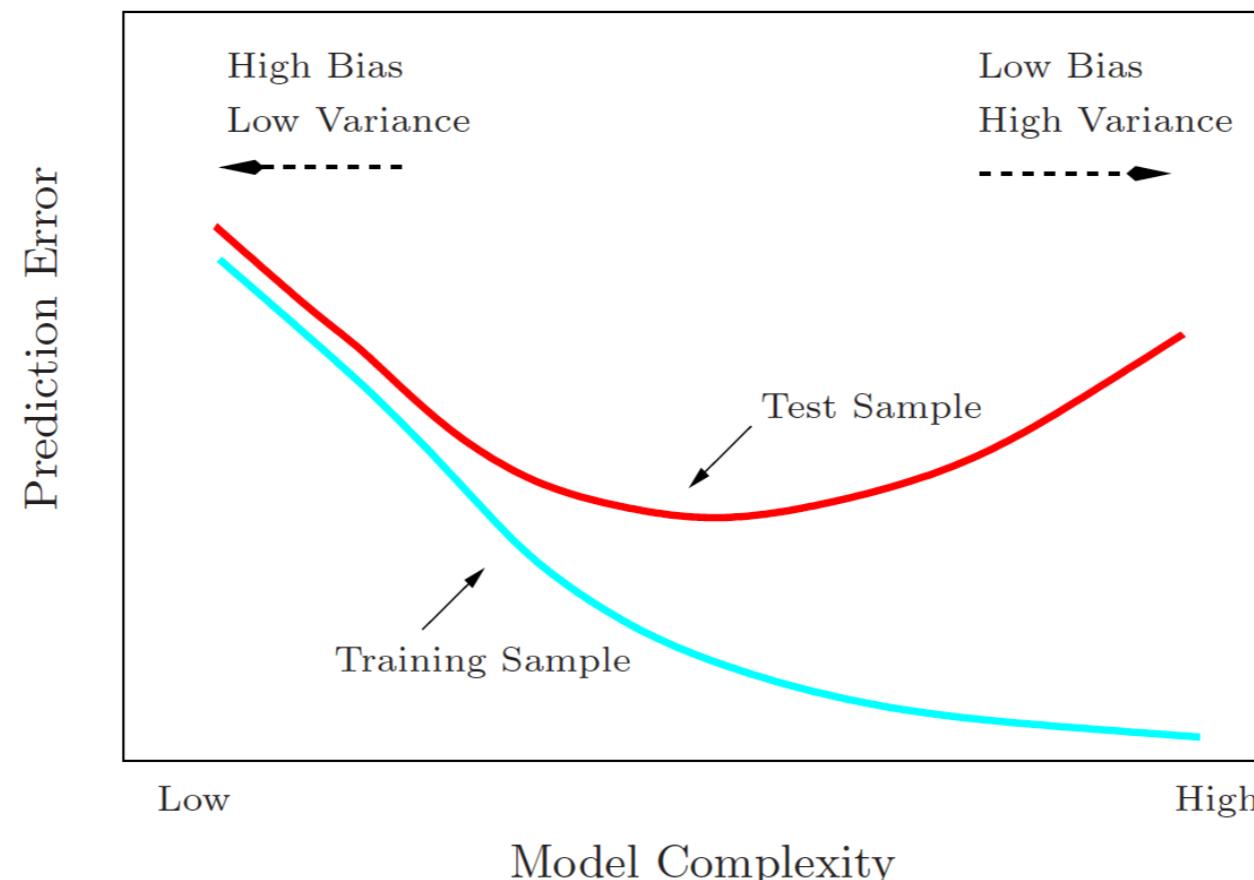
Irreducible	Bias <sup>2</sup>	Variance
Error		

# The bias variance trade-off

$$\text{Err}(x_0) = (E[\hat{y}] - f(x_0))^2 + E[(\hat{y} - E[\hat{y}])^2]$$

Bias<sup>2</sup>                              Variance

This decomposition into two positive terms is a general result and usually we have to choose our method to have low bias or low variance but not both.



# Inference

# Degree of belief

**f(x)** - the degree of belief about the value of a quantity

## Example:

We observe X photons from a source, as long as the number of photons is large we might say that we observed a flux of  $X \pm \sqrt{X}$

What we mean then is that our “degree of belief” about the value of X is given by (in this case):

$$f(x) \propto e^{-(x-\hat{\mu})^2/2\hat{\sigma}^2} = e^{-(x-X)^2/2X}$$

This can be formalised (Jaynes 1998) using probabilities

# Statistical context

Assume:

$$\mathbf{x} \sim p(\mathbf{x}; \theta)$$

If you know  $p(\mathbf{x}; \theta)$  you might want to **estimate** its parameters  $\Theta$ .

**Inference/  
estimation**

If you know  $p(\mathbf{x}; \Theta)$  you might want to draw random variables from it.

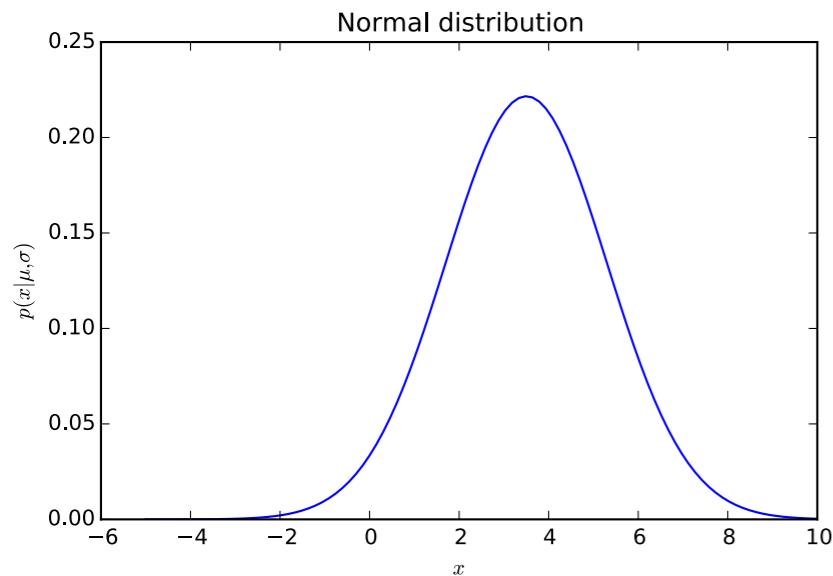
**Simulation**

If you know  $\mathbf{x}$  but not  $p(\mathbf{x})$ , you might want to estimate  $p(\mathbf{x})$ .

**Density  
estimation**

See GitHub for a IPython notebook on distributions.

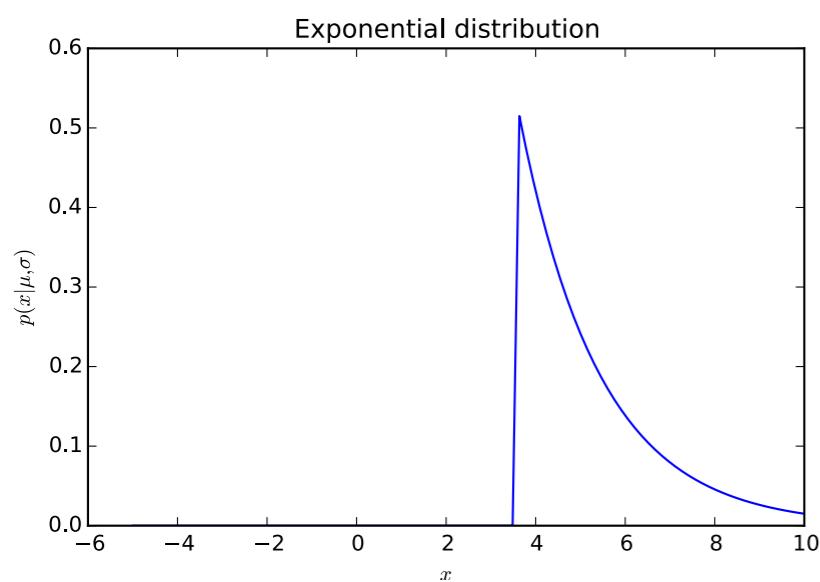
# Summarising distributions/data



$$p(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[x] = \mu$$

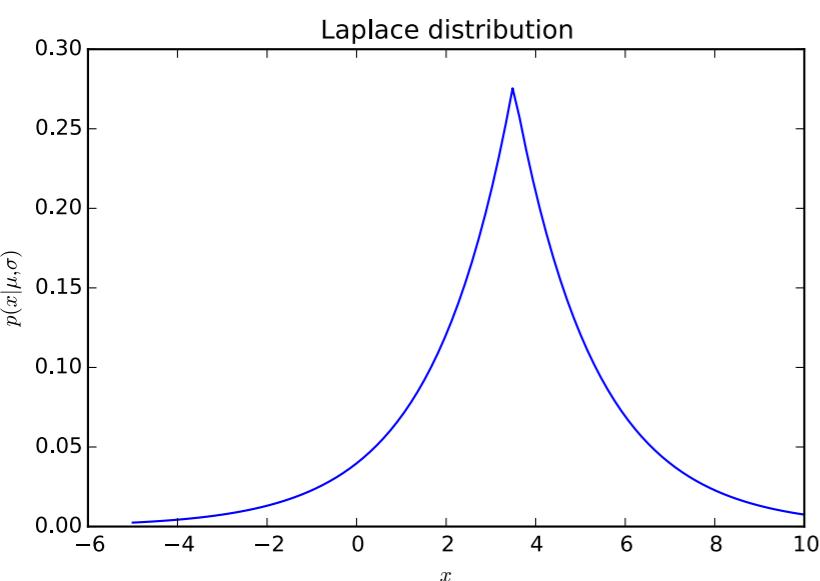
$$V = E[(x - \mu)^2] = \sigma^2$$



$$p(x|\mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma} H(x - \mu)$$

$$E[x] = \mu + \sigma$$

$$V = \sigma^2$$



$$p(x|\mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}$$

$$E[x] = \mu$$

$$V[x] = 2\sigma^2$$

# The Normal distribution

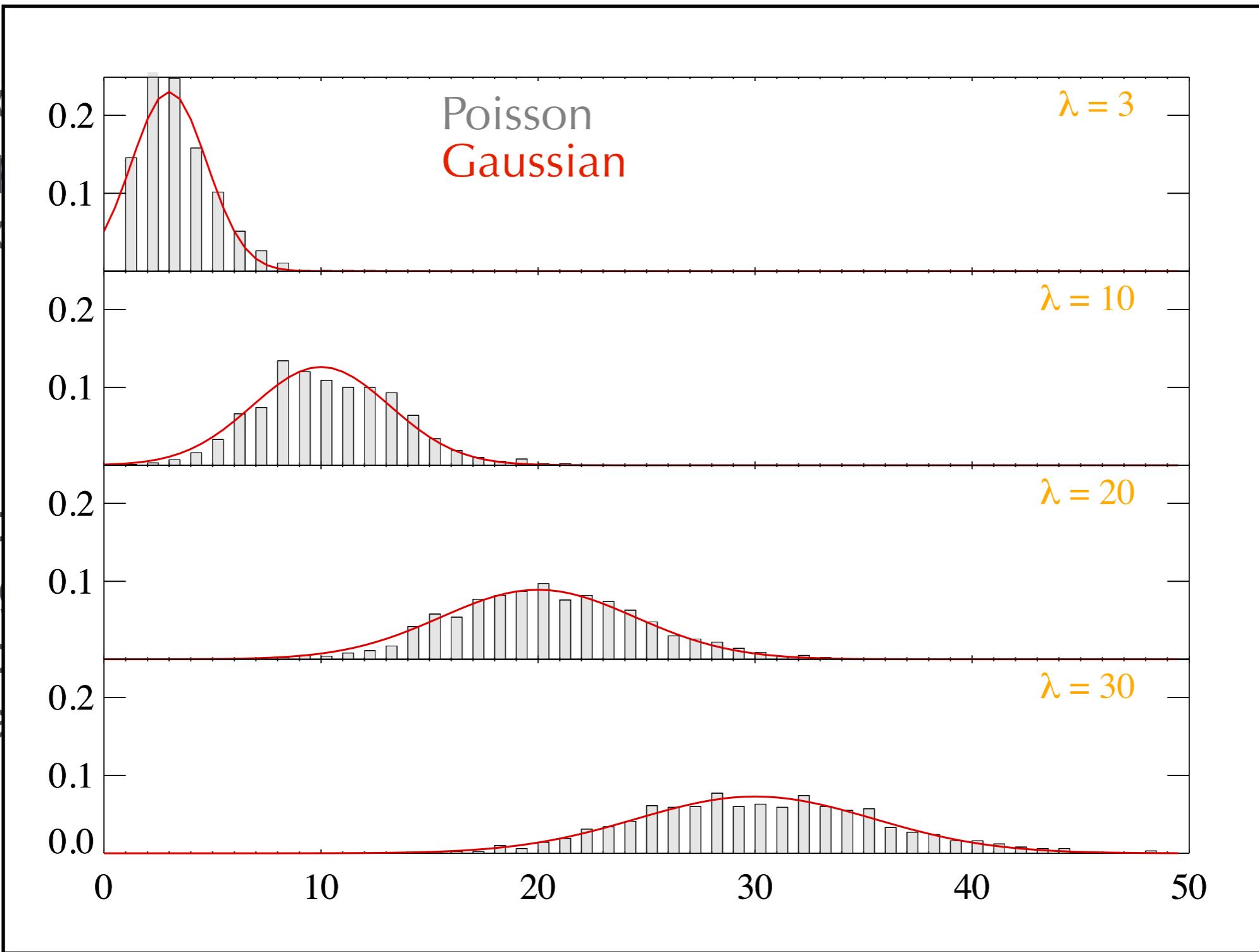
A particularly important distribution is the normal distribution. It is important in data mining because it is easy to deal with analytically and because the mean of large numbers of random numbers is distributed as a normal distribution.

In astronomy, the key reason is that if you observe a source emitting  $N$  photons you will detect  $n$ , where  $n$  is distributed as a Poisson distribution. However as soon as  $n > 10$  (or so), that is essentially a Gaussian distribution.

# The Normal distribution

A particular  
It is important  
analytic  
number

In astronomy,  
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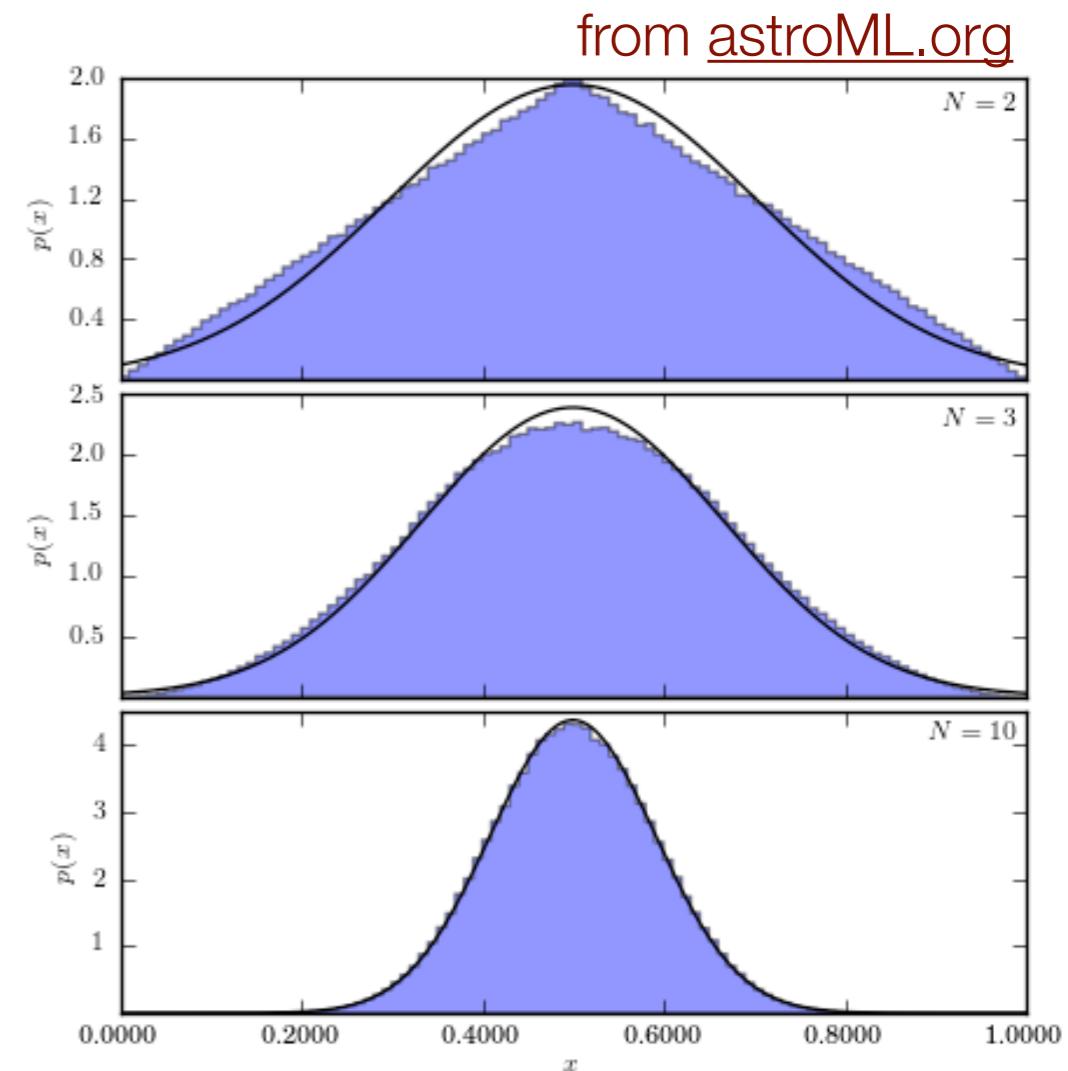
# The Central Limit Theorem

Arbitrary distribution  $h(x)$  with mean,  $\mu$ , and standard deviation  $\sigma$

Draw M values  $x$  from this distribution - the mean of  $x$  will follow

$$\bar{x} \sim N(\mu, \sigma/\sqrt{M})$$

Illustration for the mean of  $N = 2$ ,  
3 & 10 values drawn from a  
uniform distribution.



# The normal distribution - multi-dimensional

$$P(\mathbf{x}) = \frac{1}{(2\pi)^{1/D}} \frac{1}{|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

Where  $\Sigma$  is the covariance matrix:

$$\Sigma = E [(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$$

This is a useful distribution in many cases and underlies a lot of the techniques we will be using.

# Drawing from a distribution

This is a common task. Most computer languages provide you with a (more or less useful) random number generator:

**R:**

```
x = runif(N, [min=0, max=1])
```

```
x = runif(1000, min=-1, max=1)
```

**IDL:**

```
x = randomu(seed, N)
```

```
x = 2*(randomu(ss, 1000)-0.5)
```

## Python (after doing: import numpy as np)

```
x = np.random.uniform(a, b, N)
```

```
x = np.random.uniform(-1, 1, 1000)
```

## More complex distributions:

$$r = \text{CDF}^{-1}(x)$$

where  $x$  is uniformly distributed

In many-D this is more complex and other algorithms might be used.

# A brief intermezzo - storing and caching

Some of the calculations you will do can be time-consuming. When nothing has changed there is no reason to redo the calculation - but of course you need to store the result.

The general programming technique is called memoizing, in case you cared, but we will typically want to store the results of calculations to disk.

# Pickling

```
import cPickle
```

```
def pickle_to_file(data, fname):  
    fh = open(fname, 'w')  
    cPickle.dump(data, fh)  
    fh.close()
```

```
def pickle_from_file(fname):  
    fh = open(fname, 'r')  
    data = cPickle.load(fh)  
    fh.close()  
    return data
```

A standard way to store data in Python - can store complex data (but in general use other formats for this).

# Decorating

This is a standard Pythonic way to modify functions.

```
def get_text(name):  
    return "I ({0}) am a decorator".format(name)
```

```
def p_decorate(func):  
    def wrapper(name):  
        return "*** {0} ***".format(func(name))  
    return wrapper
```

```
my_get_text = p_decorate(get_text)
```

“Nicer” way (more Pythonic):

```
@p_decorate  
def get_text_d(name):  
    return "I ({0}) am a decorator".format(name)
```

# Why should you care?

astroML has a very convenient way to store results of heavy calculations:

```
from astroML.decorators import pickle_results

@pickle_results("GMM_three_Gaussians.pkl")
def compute_GMM(data, N, n_iter=1000):
    # Setup the output array
    models = [None for n in N]
    for i in range(len(N)):
        models[i] = GMM(n_components=N[i], n_iter=n_iter,
                        covariance_type=covariance_type)
        models[i].fit(data)

    return models
```

When the function is called with the same arguments -  
the results are read from the file & not redone!.

# Classical inference

# The broad idea

We want to compare data,  $\{y_i\}$ , to a model  $M(a, b, \dots)$

To do so we need:

A way to compare our data to the model (cost functions)

e.g.:  $|f(x) - f_{\text{true}(x)}|$   $\int (f(x) - f_{\text{true}(x)})^2 dx$

A way to rank/compare different models

Likelihood ratios, information criteria (AIC) etc.

A way to assess whether the fit we obtained is “good”

e.g.  $\chi^2$  tests

This is arguably the approach most widely used in Machine Learning/Data Mining.

# Maximum Likelihood

How were the data generated?

1. Define a likelihood of the data given a model

$$p(D|M)$$

I will write the parameters of the model  $\boldsymbol{\theta}$  and the model  $M(\boldsymbol{\theta})$   
sometimes

2. Find  $\boldsymbol{\theta} = \boldsymbol{\theta}^0$  that maximise  $p(D|M)$

**point estimates**

For this we use a minimization routine (e.g. `scipy.optimize`)

3. Find confidence levels for  $\boldsymbol{\theta}^0$

$$\sigma_{j,k}^2 = - \left. \frac{d^2 \ln L}{d\theta_j d\theta_k} \right|_{\theta=\theta_0}$$

# ML - Simple example - Gaussian errors

$$\theta = (\alpha, \beta)$$

$$\epsilon_i \sim N(0, \sigma_i^2)$$

Model y as

$$y_i = \alpha + \beta x_i + \epsilon_i$$

Likelihood of data:

$$P(x_i, y_i, \sigma_i | \alpha, \beta) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-[y_i - (\alpha + \beta x_i)]^2 / 2\sigma_i^2}$$

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Full Likelihood

$$L(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma} | \alpha, \beta) = \prod_i P(x_i, y_i, \sigma_i | \alpha, \beta)$$

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So log likelihood:

$$\ln L = -\frac{1}{2} \ln 2\pi - \frac{1}{2} \sum_i \frac{[y_i - (\alpha + \beta x_i)]^2}{\sigma_i^2}$$

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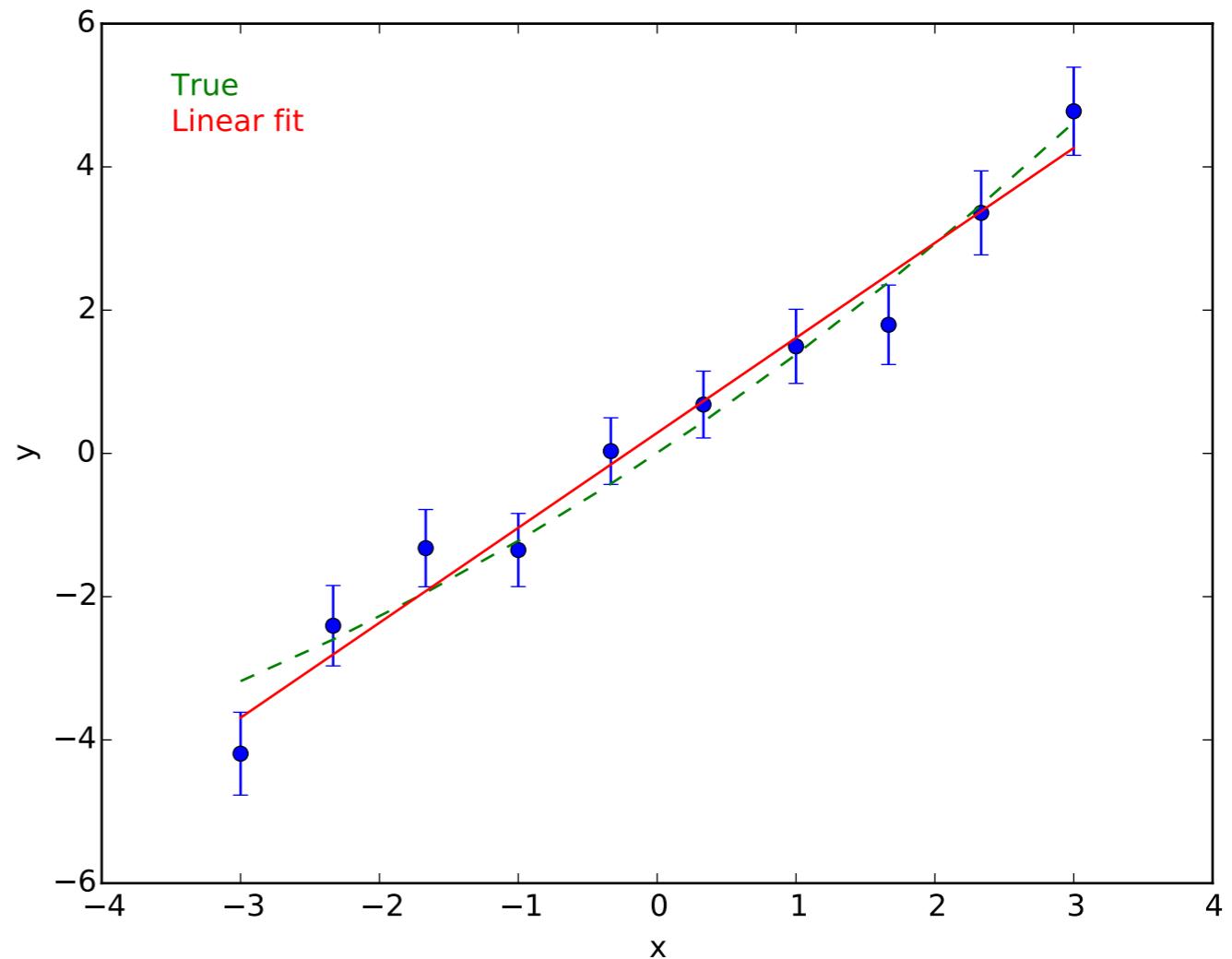
Minimisation of this is the same as least squares minimisation.

# ML - Simple example - Gaussian errors, standard function

```
from scipy.optimize import curve_fit

def func_line(x, a, b):
    return a + b*x

pars, cov = curve_fit(func_line, x, y_obs)
```



An alternative approach with more flexibility in the regression function:

```
from astroML.linear_model import LinearRegression
m = LinearRegression()
m.fit(x[:, None], y_obs, sigma)
a, b = m.coef_
```

# ML - Simple example - explicit likelihood

```
def negLnL(theta, x, y, yerr):  
    a, b = theta  
    model = b * x + a  
    inv_sigma2 = 1.0/(yerr**2)  
  
    return 0.5*(np.sum((y-model)**2*inv_sigma2))
```

$$-\ln L = \frac{1}{2} \sum_i \frac{[y_i - (\alpha + \beta x_i)]^2}{\sigma_i^2}$$

```
import scipy.optimize as op  
result = op.minimize(negLnL, [1.0, 0.0], args=(x, y_obs, sigma))  
a_ml, b_ml = result["x"]
```

More work in this case - but a much more general approach.

# Bayesian inference

# Bayesian statistics

This allows us to calculate the likelihood distribution of parameters

$$p(\text{Model}|\text{Data}) = p(M|D) = \frac{p(D|M)p(M)}{p(D)}$$

The key part here is that we need to specify what **prior** information we have on the model parameters.

$$p(M, \boldsymbol{\theta}|D, I) = \frac{p(D|M, \boldsymbol{\theta}, I)p(M, \boldsymbol{\theta}|I)}{p(D|I)}$$

# The overall plan of attack

- Define the likelihood - the best is a generative one that mimic how you think the data were created.  $p(D|\theta, M, I)$
- Decide on the prior - ie. what range of model parameters are likely.  $p(\theta, M|I)$
- Use Bayes' theorem to calculate the posterior likelihood distribution.  $p(M|D, I)$
- To calculate the posterior distribution we often use Markov Chain Monte Carlo calculations. MCMC

The result of the calculation can be summarised - the maximum of  $p(M|D,I)$  gives the maximum a posteriori (**MAP**) estimate - means, medians are also reasonable options.

# A practical example - line fitting - model

Generative model:  $y_i = \alpha + \beta x_i + \epsilon_i$        $\epsilon_i \sim N(0, \sigma_i^2)$

# A practical example - line fitting - model

Generative model:  $y_i = \alpha + \beta x_i + \epsilon_i$        $\epsilon_i \sim N(0, \sigma_i^2)$

So log likelihood:  $\ln L = -\frac{1}{2} \ln 2\pi - \sum_i \ln \sigma_i - \frac{1}{2} \sum_i \frac{[y_i - (\alpha + \beta x_i)]^2}{\sigma_i^2}$

(note that since  $\sigma_i$  are known, the first two terms are constants and can be ignored for maximisation)

```
def lnL(theta, x, y, yerr):
    a, b = theta
    model = b * x + a
    inv_sigma2 = 1.0/(yerr**2)

    return -0.5*(np.sum((y-model)**2*inv_sigma2))
```

# A practical example - line fitting - prior

Prior:

$$p(a, b, M|I) = \begin{cases} \text{const}, & \text{if } a \in [-5, 5] \text{ and } b \in [-10, 10]; \\ 0 & \text{otherwise} \end{cases}$$

```
def lnprior(theta):
    a, b = theta
    if -5.0 < a < 5.0 and -10.0 < b < 10.0:
        return 0.0
    return -np.inf
```

(the -np.inf is because p=0 means  $\ln p = -\infty$ )

# A practical example - line fitting - posterior

Putting it together:

$$p(D|a, b, M, I)p(a, b, M|I)$$

```
def Inprob(theta, x, y, yerr):
    """
    The likelihood to include in the MCMC.
    """

    lp = Inprior(theta)
    if not np.isfinite(lp):
        return -np.inf
    return lp + lnL(theta, x, y, yerr)
```

Note that I ignore the normalisation  $p(D|I)$

# A practical example - line fitting

```
import emcee
```

```
# Use ML to get a starting point.  
# result = run_ml()  
p_init = np.array([ 0.28233725, 1.31299656])
```

(result["x"])

```
# Set up the properties of the problem.  
ndim, nwalkers = 2, 100
```

```
# Setup a number of initial positions.  
pos = [p_init + 1e-4*np.random.randn(ndim) for i in range(nwalkers)]
```

```
# Create the sampler.  
sampler = emcee.EnsembleSampler(nwalkers, ndim, lnprob, args=(x, y_obs,  
sigma))
```

```
# Run the process.  
sampler.run_mcmc(pos, 500)
```

# A practical example - importing the package

```
import emcee
```

We will use the MCMC Hammer library (**emcee**) -

<http://dan.iel.fm/emcee/current/>

it is a pure Python implementation (as pyMC), flexible and reasonably fast - for challenging problems other packages (MultiNest, BUGS, JAGS, STAN) might be better choices but it is well worth starting with this as the learning curve is less steep.

It is installed on STRW computers - on your laptop you can install it with:

**pip install emcee**  
**pip install corner**

# A practical example - starting position

```
# Use ML to get a starting point.  
# result = run_ml()  
p_init = np.array([ 0.28233725, 1.31299656])  
                                (result["x"])
```

The MCMC process will randomly (but cleverly) step around in your parameter space. To do this well you need a good starting position. A maximum-likelihood solution gives a good starting point.

I used `scipy.optimize.minimize` here. I set the result to a variable `p_init`.

# A practical example - line fitting

```
# Set up the properties of the problem.  
ndim, nwalkers = 2, 100
```

We have two parameters - so the dimensionality is 2

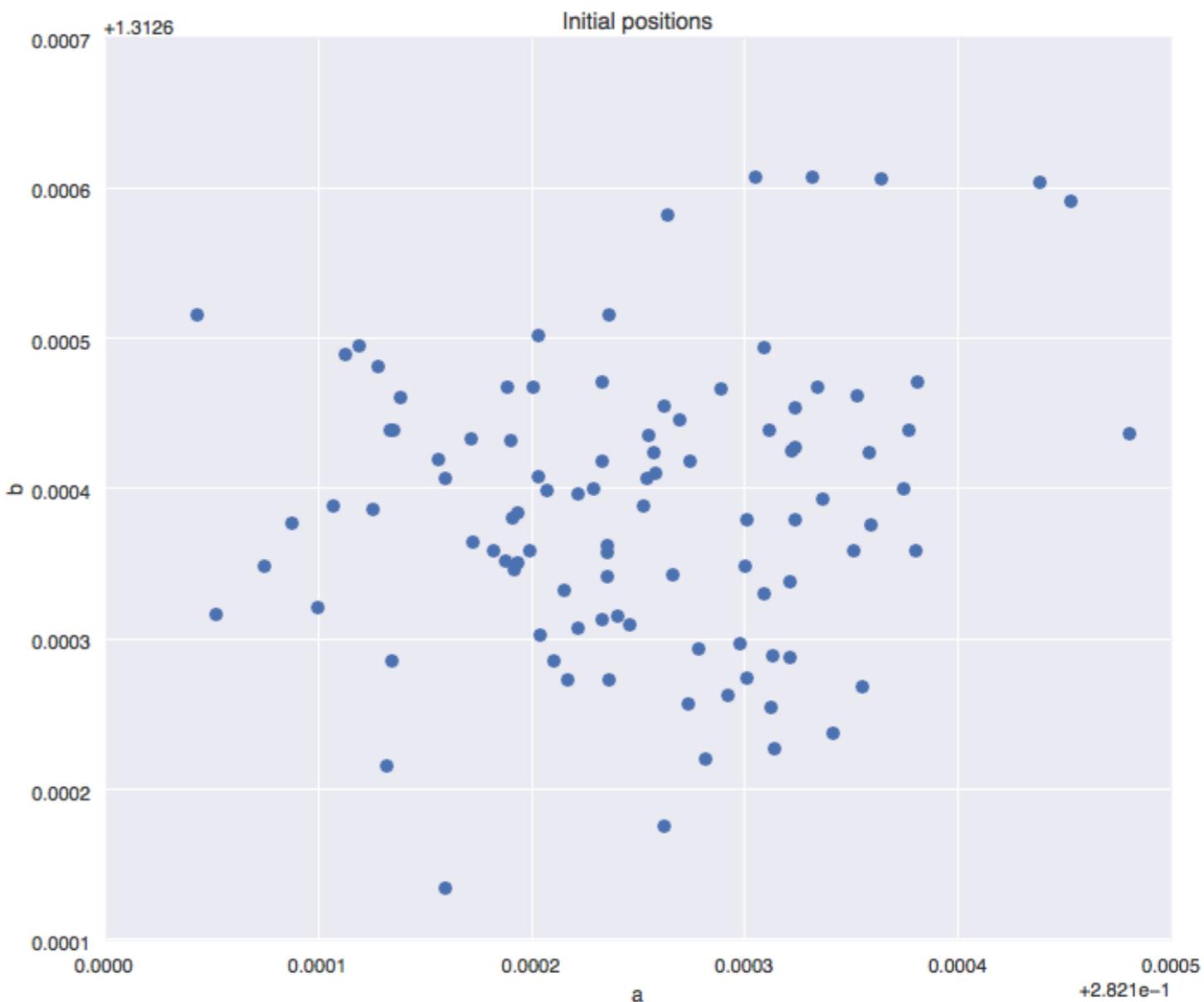
And we also need “walkers” - view these as random processes that explore your parameter space along different paths. The [emcee](#) documentation recommends using as many as you can get away with.

# A practical example - line fitting

```
# Setup a number of initial positions.
```

```
pos = [p_init + 1e-4*np.random.randn(ndim) for i in range(nwalkers)]
```

This creates a set of different starting positions - one for each walker.



# A practical example - line fitting

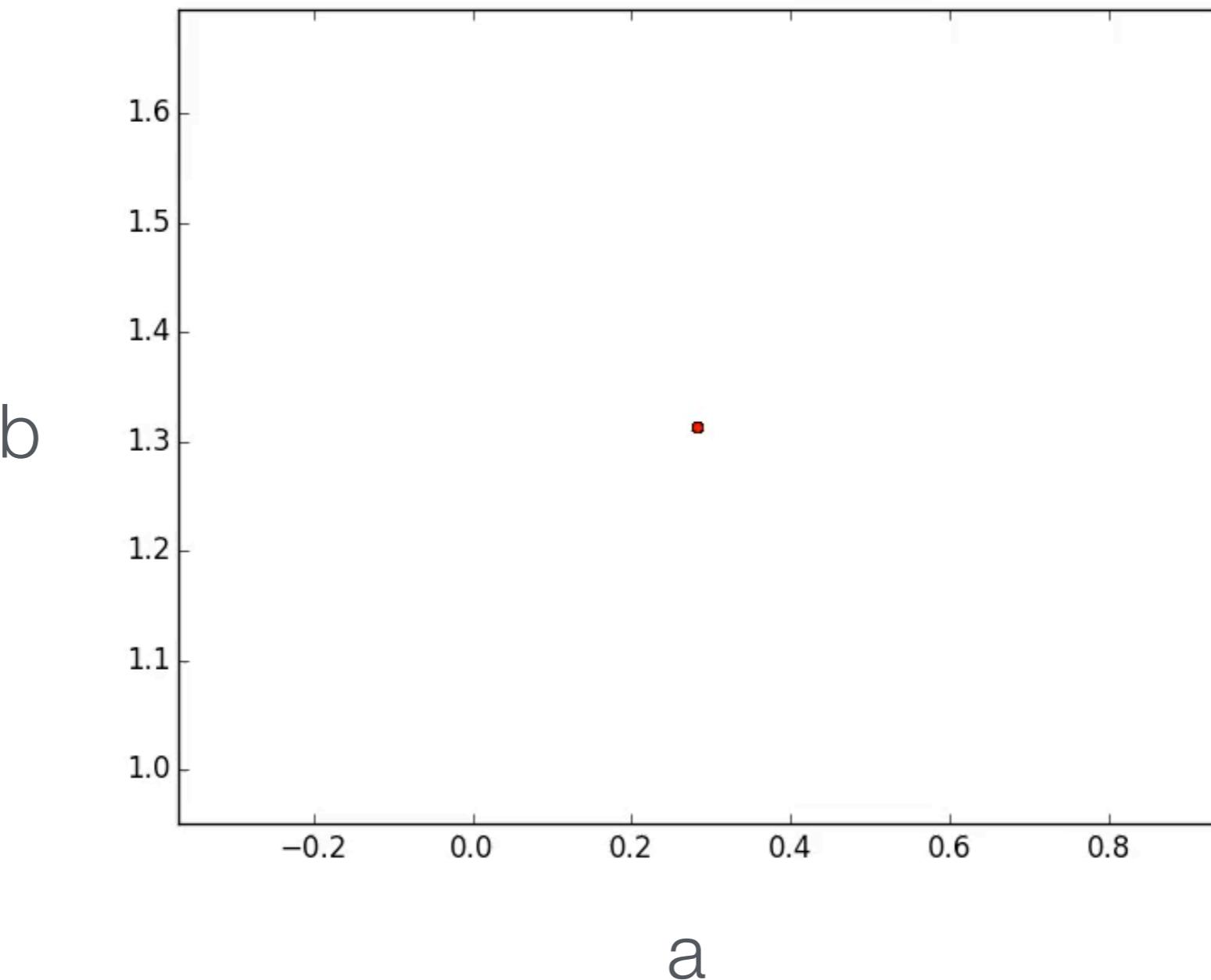
```
# Create the sampler.  
sampler = emcee.EnsembleSampler(nwalkers, ndim, lnprob, args=(x, y_obs,  
sigma))
```

This sets up the MCMC process - we give the function lnprob and the data as a tuple.

# A practical example - line fitting

```
# Run the process.
```

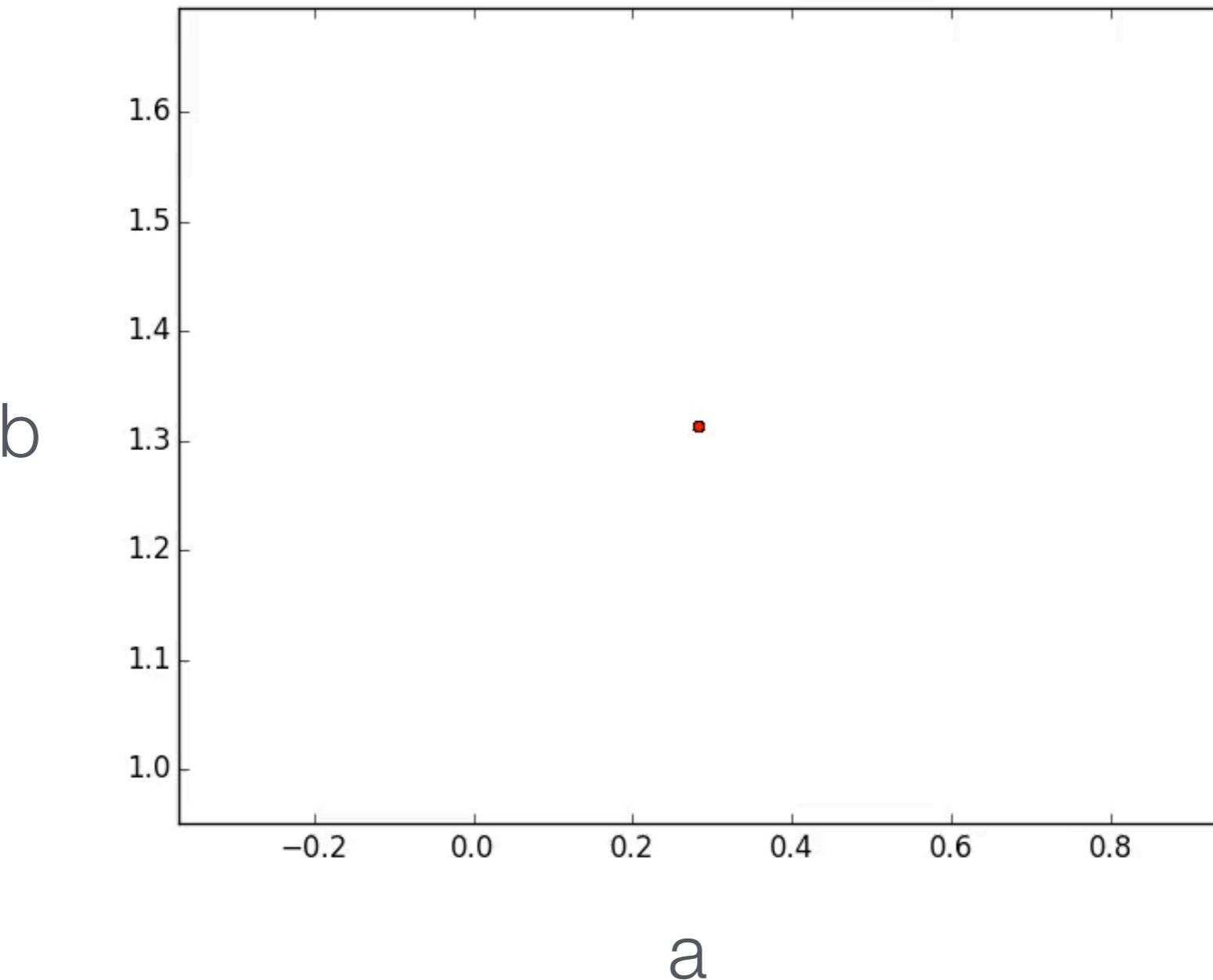
```
sampler.run_mcmc(pos, 500)
```



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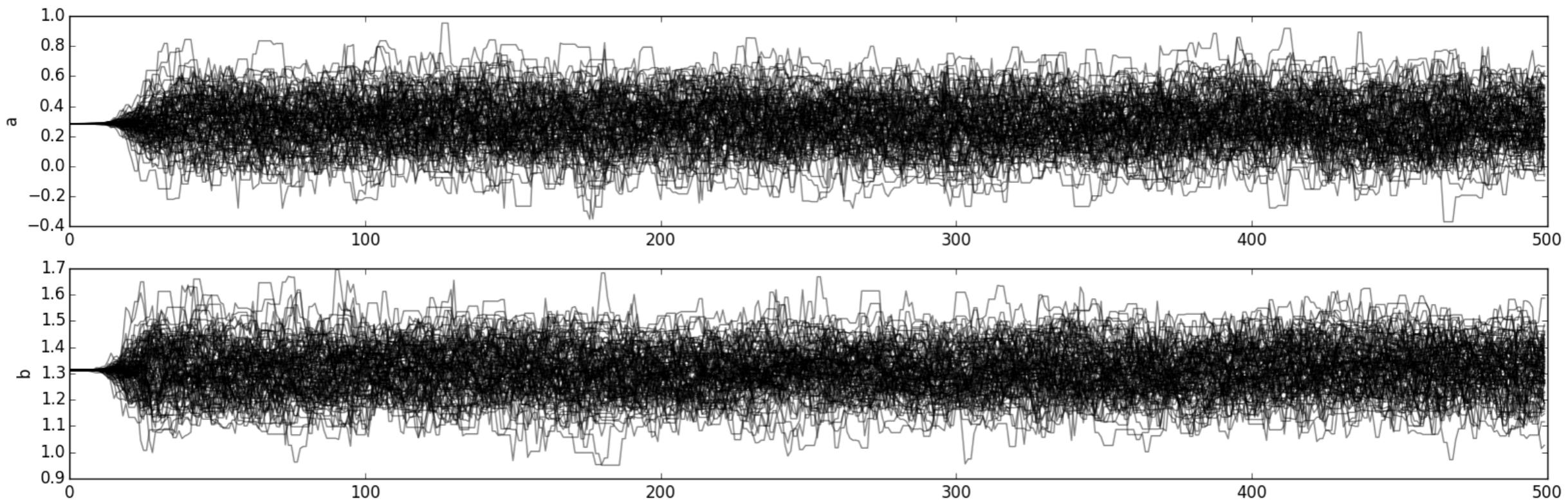
```
sampler.run_mcmc(pos, 500)
```



# Look at chain behaviour

```
labels = ['a', 'b']
chain = sampler.chain
for i_dim in range(2):
    plt.subplot(2,1,i_dim+1)
    plt.ylabel(labels[i_dim])

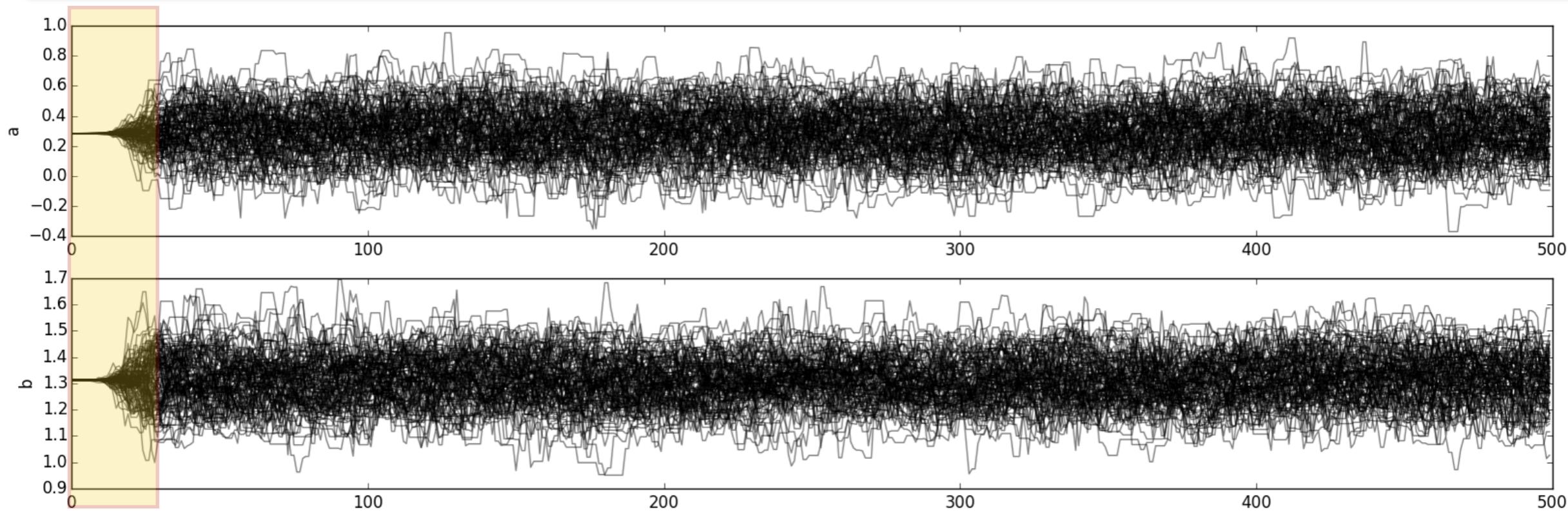
    for i in range(100):
        plt.plot(chain[i,:,:i_dim],color='black', alpha=0.5)
```



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for i_dim in range(2):
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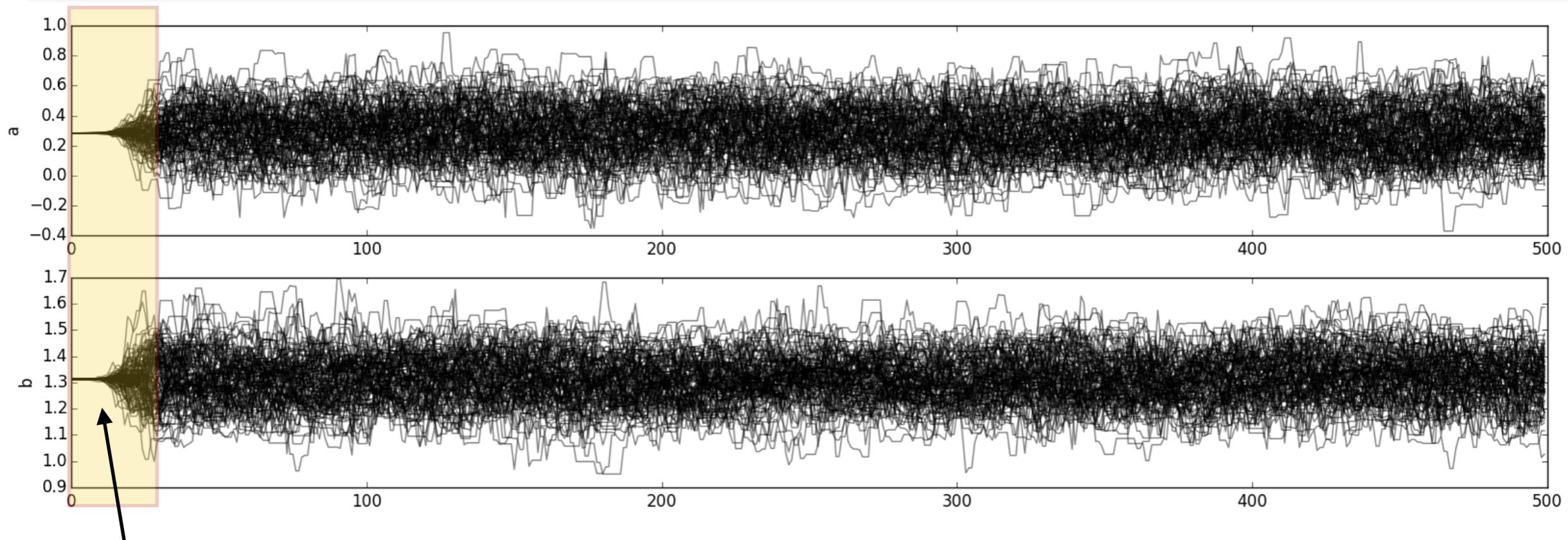
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    plt.ylabel(labels[i_dim])

    for i in range(100):
        plt.plot(chain[i,:,:i_dim],color='black', alpha=0.5)
```



Burn-in - we need to wait until this has happened before we start to use the samples.

# A practical example - showing the result

Let us extract the samples from 50 onwards and collapse the different walkers

```
samples = sampler.chain[:, 50:, :].reshape((-1, 2))
```

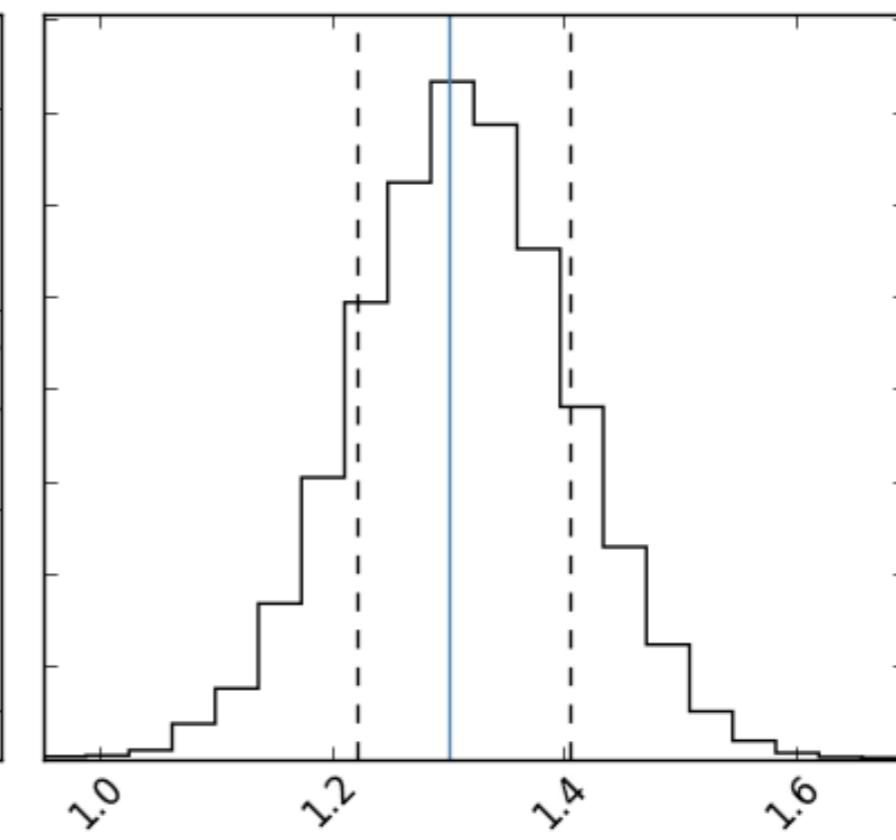
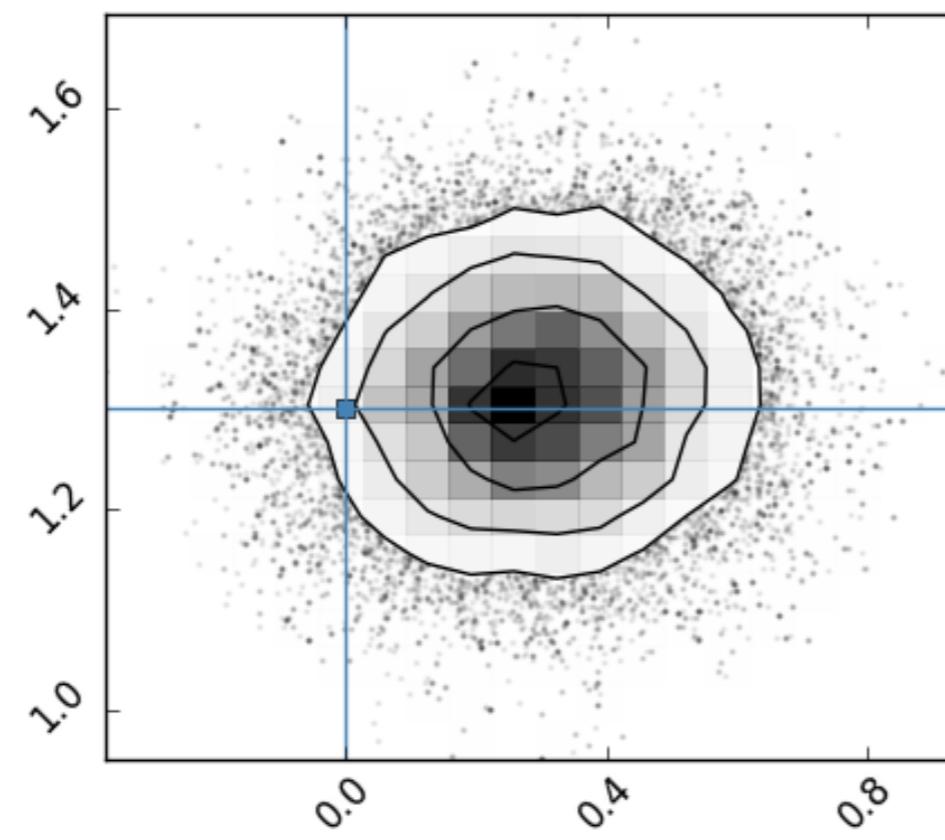
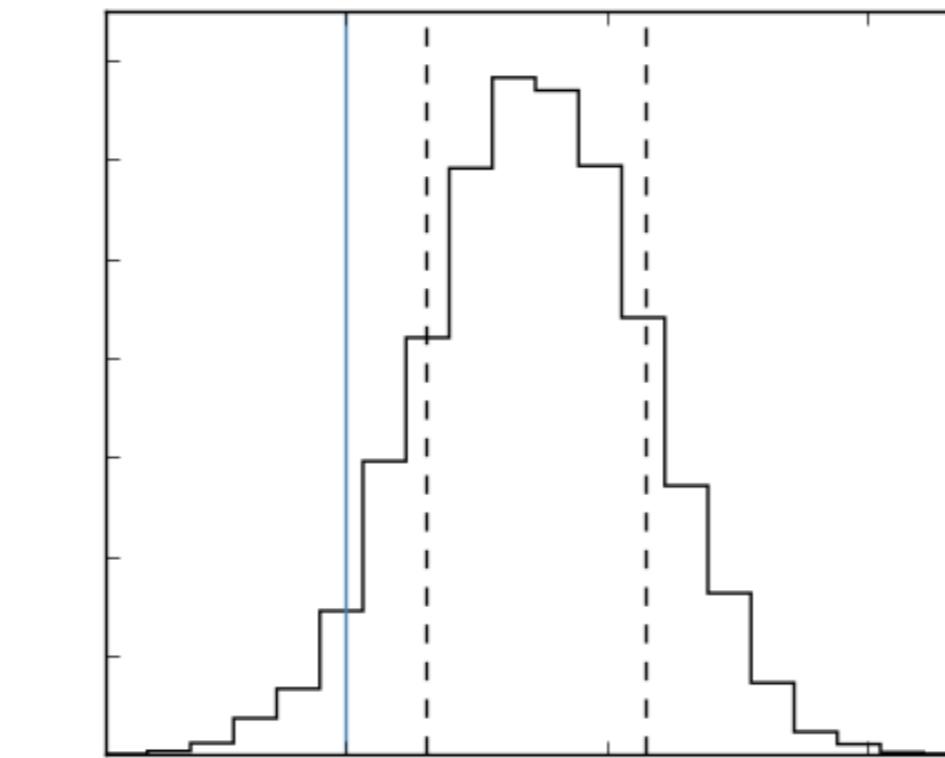
We then show the result using the corner package:

```
import corner

fig = corner.corner(samples, labels=["$a$", "$b$"],
                     truths=[0.0, 1.3], quantiles=[0.16, 0.84])
fig.show()
```

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