Optimal Measurement of Field Properties with Quantum Sensor Networks

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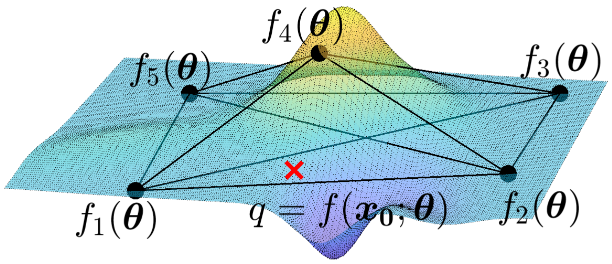

Based on arXiv:2011.01259
(Accepted as PRA Letter)

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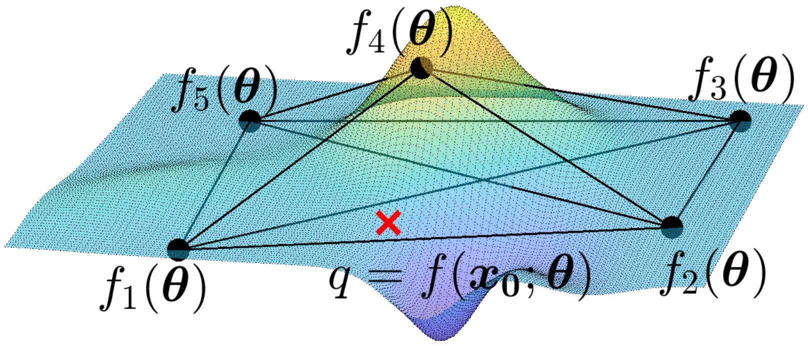
Problem Statement

Given a quantum sensor network coupled to a field $f(\vec{x};\vec{\theta})$ parameterized by $\vec{\theta}$ what is the minimal attainable variance of some analytic function $q(\vec{\theta})$?



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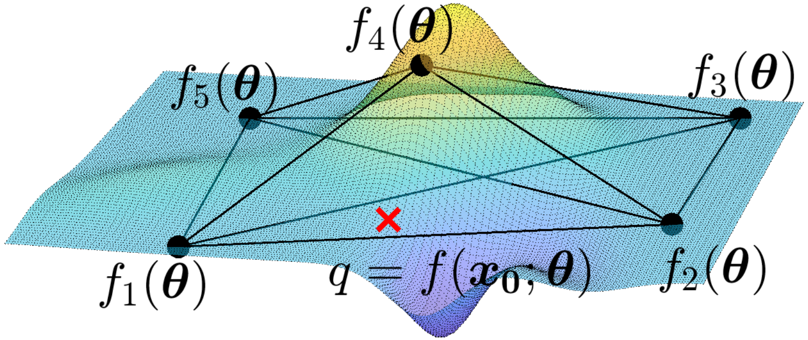

Key point: We include correlations between measured field amplitudes!

A **quantum sensor network** is a collection of localized sensors each coupled to a local field amplitude.

E.g. qubit sensors coupled via: $H = H_c(t) + \frac{1}{2}\sum_{i=1}^{d} f_i(\vec{\theta})\sigma_i^{(z)}$

Task: Optimally measure an analytic function $q(\vec{\theta})$.

Information Theory ⟷ Match ⟷ **Protocol Design**

Information Theory

- Bounds on variance of estimate for $q(\vec{\theta})$
- Cramer-Rao bounds/Fisher Information

Protocol Design

- Choose an initial state, control Hamiltonian, measurements

Previous research: Measure a linear combination of field amplitudes $q = \vec{\alpha} \cdot \vec{f}$

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Without entanglement:

$$\mathrm{Var}[f_i] = \frac{1}{t^2}$$

$$\mathrm{Var}[q] = \sum_i \alpha_i^2 \mathrm{Var}[f_i]^2 = \frac{||\vec{\alpha}||^2}{t^2}$$

Eldredge et. al., PRA, 042337 (2018)

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With entanglement:

$$\mathrm{Var}[q] = \frac{||\vec{\alpha}||_\infty^2}{t^2}$$

If f_i independent, this is provably optimal!

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Example: Calculate an average.

$$\alpha_i = \frac{1}{\sqrt{d}} \quad \forall i$$

$$||\vec{\alpha}||^2 = 1$$

$$||\vec{\alpha}||_\infty^2 = \frac{1}{d}$$

Entanglement gives advantage!

Eldredge et. al., PRA, 042337 (2018); Qian et. al. PRA 100, 042304 (2019)

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Can generalize to analytic functions $q(\vec{f})$ in two steps:

1. Spend negligible fraction of time to estimate $\vec{f}$
2. Linearize q around estimates of f_i and use optimal linear protocol

Eldredge et. al., PRA, 042337 (2018); Qian et. al. PRA 100, 042304 (2019)

Add correlations in field amplitudes: $f_i(\vec{\theta})$

Vector of k unknown parameters

Assumptions: enough sensors to measure $\vec{\theta}$
(essentially $d \geq k$)
and functions $f_i(\vec{\theta})$ are given.

Still want to measure some analytic $q(\vec{f}(\vec{\theta})) = q(\vec{\theta})$

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Motivating example:

Suppose
$$f_1 = \theta_1$$
$$f_2 = \theta_2$$
$$f_3 = \theta_1 + \theta_2$$

Want to measure
$$q(\vec{\theta}) = f_1 = \theta_1$$

Approach: Try linear protocol

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$$q(\vec{\theta}) = f_1 = \theta_1$$

Could consider measuring many different linear combinations.

e.g. $q(\vec{\theta}) = f_1 + 0f_2 + 0f_3 \implies \mathrm{Var}(q) = \dfrac{1}{t^2}$

$$\mathrm{Var}[q] = \frac{||\vec{\alpha}||_\infty^2}{t^2}$$

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Could consider measuring many different linear combinations.

or
$$q(\vec{\theta}) = \frac{3}{4} f_1 - \frac{1}{4} f_2 + \frac{1}{4} f_3 \implies \mathrm{Var}(q) = \frac{9}{16t^2}$$

$$\mathrm{Var}[q] = \frac{||\vec{\alpha}||_\infty^2}{t^2}$$

Add correlations in field amplitudes: $f_i(\vec{\theta})$

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Motivating example:

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Want to measure
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or
$$q(\vec{\theta}) = \frac{1}{2}f_1 - \frac{1}{2}f_2 + \frac{1}{2}f_3 \implies \mathrm{Var}(q) = \frac{1}{4t^2}$$

$$\mathrm{Var}[q] = \frac{||\vec{\alpha}||_\infty^2}{t^2}$$

Add correlations in field amplitudes: $f_i(\vec{\theta})$

Vector of k unknown parameters

Assumptions: enough sensors to measure $\vec{\theta}$
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Motivating example:

Suppose
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Want to measure
$$q(\vec{\theta}) = f_1 = \theta_1$$

Could consider measuring many different linear combinations.

or
$$\boxed{q(\vec{\theta}) = \frac{1}{2} f_1 - \frac{1}{2} f_2 + \frac{1}{2} f_3 \implies \mathrm{Var}(q) = \frac{1}{4t^2}}$$

Optimal?

$$\mathrm{Var}[q] = \frac{||\vec{\alpha}||_\infty^2}{t^2}$$

That is, our guess for the optimal protocol requires solving the protocol problem.

The Protocol Problem (Informal)
What is the best way to linearily combine our field amplitudes to estimate $q(\vec{\theta})$?

Recall, to prove optimality we must derive a lower bound on the variance of $q(\vec{\theta})$ from the information theory side.

Our tool: single parameter bound

$$\mathrm{Var}(q) \geq \frac{1}{\mathcal{F}_Q} \geq \frac{1}{t^2||h_q||_s^2}$$

This bound applies when one parameter controls the evolution of the state. But we have k as our evolution depends on $\vec{\theta}$

Boixo et. al., PRL, 98 090401 (2007)

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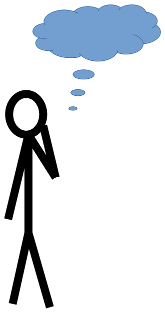

Imagine fixing $k-1$ degrees of freedom. This can only help! Gives us the bound problem.

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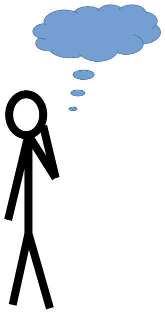

Imagine fixing $k-1$ degrees of freedom. This can only help! Gives us the bound problem.

> **The Bound Problem (Informal)**
> **What choice of fixing extra degrees of freedom gives us the least information about $q(\vec{\theta})$?**

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It so happens the solution to the protocol problem and the bound problem are the same!

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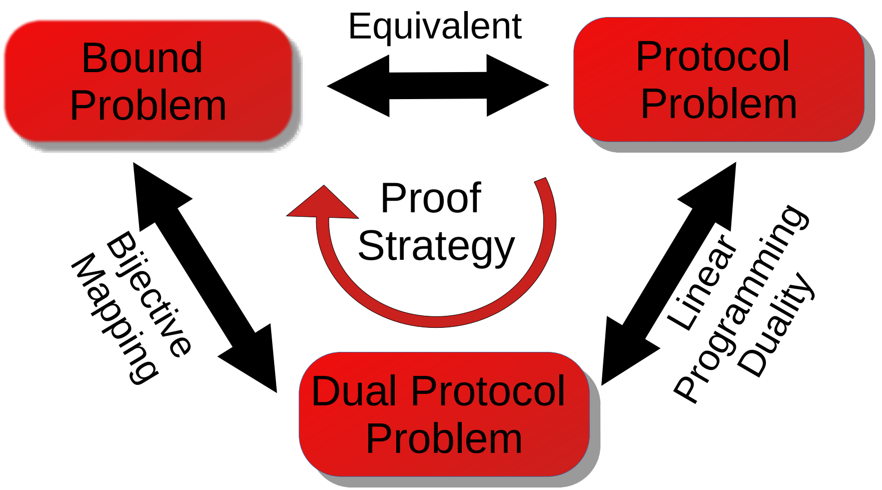

The Protocol Problem (Informal)
What is the best way to linearly combine our field amplitudes to estimate $q(\vec{\theta})$?

So the protocol is optimal and the bound is tight!

Questions?

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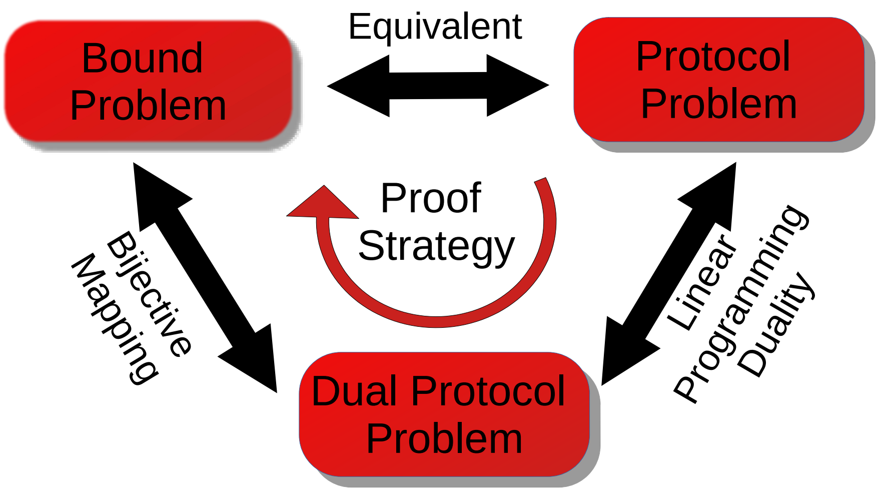
Backup Slides

Formal Statements

Bound problem: *Given a non-zero vector* $\boldsymbol{\alpha} \in \mathbb{R}^k$ *and a real* $d \times k$ *matrix* G*, compute* $u = \max_{\boldsymbol{\beta}} \frac{1}{||G\boldsymbol{\beta}||_1}$ *under the condition* $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 1$.



Protocol problem: *Given a non-zero vector* $\boldsymbol{\alpha} \in \mathbb{R}^k$ *and a real* $d \times k$ *matrix* G*, compute* $u' = \min_{\boldsymbol{w}} ||\boldsymbol{w}||_\infty$ *under the condition* $G^T\boldsymbol{w} = \boldsymbol{\alpha}$.

Dual protocol problem: *Given a non-zero vector* $\boldsymbol{\alpha} \in \mathbb{R}^k$ *and a real* $d \times k$ *matrix* G*, compute* $u'' = \max_{\boldsymbol{v}} \ \boldsymbol{\alpha} \cdot \boldsymbol{v}$ *under the condition* $||G\boldsymbol{v}||_1 \leq 1$.