Question 2

Evaluating the integral of:

$$Z(A,b) = \int e^{-\frac{1}{2}x^T A x + b^T x} \, dx$$

Using completing the square method to rewrite the exponent term. Also

Since A is symmetric and positive definite, we can write it as $A = LL^T$ with L denoted as the lower triangular matrix

this simplifies as

$$x^T A x = x^T L L^T x = (L^T x)^T (L^T x)$$

Let $y = L^T x \Rightarrow x = (L^T)^{-1} y$

$$dx = |\det(L^T)| \, dy = |\det(L)| \, dy$$

Now, substituting into the integral

$$Z(A,b) = \int e^{-\frac{1}{2}y^T y + b^T (L^T)^{-1} y} \, |\det(L)| \, dy$$

Since $A = LL^T$. If L is Lower triangular

A is symmetric, then L^T is upper triangular

Completing the Square in the Exponent

$$-\frac{1}{2}y^T y + b^T (L^T)^{-1} y = -\frac{1}{2}\left(y - 2(L^T)^{-1} b\right)^T \left(y - 2(L^T)^{-1} b\right) + \frac{1}{2} b^T (L^T)^{-1} b$$

$$\Rightarrow Z(A,b) = e^{\frac{1}{2} b^T (L^T)^{-1} b} \, |\det(L)| \int e^{-\frac{1}{2}\left(y - 2(L^T)^{-1} b\right)^T \left(y - 2(L^T)^{-1} b\right)} \, dy$$

$$\Rightarrow Z(A,b) = e^{\frac{1}{2} b^T (L^T)^{-1} b} \, |\det(L)| \int e^{-\frac{1}{2}(y-c)^T (y-c)} \, dy$$

Where $c = 2(L^T)^{-1} b$

This yield a gaussian integral with its value proportional to the square root of the determinant of the matrix in the exponent.

$$\Rightarrow Z(A,b) = e^{\frac{1}{2} b^T (L^T)^{-1} b} \, |\det(L)| \, \sqrt{(2\pi)^N}$$

$$\Rightarrow Z(A,b) = e^{\frac{1}{2} b^T A^{-1} b} \sqrt{\det(A) \, 2\pi^N}$$

$$\Rightarrow Z(A,b) = e^{\frac{1}{2} b^T A^{-1} b} \sqrt{\det(A) \cdot 2\pi}^{\,N/2}$$