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1. If U is unitary, then its rows and columns form an orthonormal basis for C^N.
2. If the rows and columns of U form an orthonormal basis for C^N, then U is unitary

Proof

1.

Let $U_1, U_2 \dots U_N$ denote the columns of U. Since U is unitary, we have $U^\dagger U = UU^\dagger = 1$, where $U^\dagger$ denotes the conjugate of transpose U.

To show that the columns of U form an orthonormal set. Let's consider the inner product of two arbitrary columns U_i and U_j:

$$\langle U_i, U_j \rangle = U_i^\dagger U_j$$

if $i = j$, then $\langle U_i, U_i \rangle = \|U_i\|^2 = 1$ since U is unitary

if $i \neq j$, then $\langle U_i, U_j \rangle = U_i^\dagger U_j = 0$ since the column of a unitary matrix are orthogonal to each other

Lets prove that columns of U span $\mathbb{C}^N$. Since U is a square matrix of size $N \times N$, the columns of U are $u_1, u_2, \dots, u_n$ which are N vectors in $\mathbb{C}^N$. Since they are linearly independent (as U is invertible) they form a basis for $\mathbb{C}^N$

$\therefore$ The columns of U form an orthonormal basis for $\mathbb{C}^N$

2. If the rows and columns of U form an orthonormal basis for $\mathbb{C}^N$, then U is unitary

Again,

Let $u_1, u_2, \dots, u_N$ denote the rows of U and let $u_{i,j}$ denote the ith component of the jth row of U.

Since the rows of U form an orthonormal basis for $\mathbb{C}^N$, we have :-

$$\langle u_i, u_j \rangle = \sum_{k=1}^{N} u_{i,k} \overline{u_{j,k}} = \delta_{ij}$$

Where δ_{ij} is the kronecker delta

Now let's consider the product $U^{+}U$. the i, j-th entry of this product is given by:

$$(U^{+}U)_{ij} = \sum_{k=1}^{N} \overline{u_{k,i}}\, u_{k,j}$$

Using the orthomality condition, we have

$$(U^{+}U)_{ij} = \sum_{k=1}^{N} \overline{u_{k,i}}\, u_{k,j} = \delta_{ij}$$

Since $(U^{+}U)_{ij} = (UU^{+})_{ij} = \delta_{ij}$ we conclude that $U^{+}U = UU^{+} = I$

$\therefore$ U is unitary