Sample Semantic Proof Method to Determine Logical Consequence

The Semantic Proof Method can be used to both prove true claims of logical consequence (by reducing to an absurdity the assumption of a countermodel when there is none to be found) and to prove false claims of logical consequence (by writing a recipe for the construction of a countermodel by describing the truth values of the atomic parts). The first two phases of both kinds of semantic proofs (the *reductio* assumption stage and the regressing of the truth complexity of the sentences involved) match. When the claim of logical consequence is true, the truth assignments to atomics unpacked in the second stage are contradictory and so you can move to the third phase of explicitly drawing that contradiction out. When the claim of logical consequence is false, by contrast, the truth assignments to atomics unpacked in the second stage are perfectly consistent and describe a countermodel, which you can then go on to make explicit as a row of the truth table. The following two cases illustrate. (These are examples used in lecture when discussing logical equivalence.)

Determine whether or not the following claims of logical consequence are true using the method of Semantic Proof. If true, show that the assumption that there is a countermodel is contradictory. If false, use the semantic proof method to describe a countermodel.

1) $(p\to p)\to q \vDash p\to(p\to q)$ 2) $p\to(p\to q) \vDash (p\to p)\to q$

Semantic Proof for 1.
Stage 1: Suppose that there is an interpretation I such that
1. $(p\to p)\to q$ is true and
2. $p\to(p\to q)$ is false in I.
Stage 2: Unpack the truth conditions
By 1 and $\to$T, EITHER
3. $p\to p$ is false in I, in which case, by 3 and $\to$F, 4. p is true in I and 5. p is false in I.
OR
6. q is true in I.
By 2 and $\to$F, 7. p is true in I and 8. $p\to q$ is false in I, in which case, by 8 and $\to$F, 9. p is true in I and 10. q is false in I.
Stage 3: Make explicit the contradiction.
By 1, either 3 or 6. Suppose first 3. Then, 4. p is true. But by 5, p is false. Contradiction. Suppose then 6. Then q is true. But, by 10, q is false. Contradiction. So, either way, a contradiction follows. So, 1 and 2 are contradictory. So, there is no such countermodel I and so $(p\to p)\to q \vDash p\to(p\to q)$.

Semantic Proof for 2.

Stage 1: Suppose that there is an interpretation I such that

1. $p \to (p \to q)$ is true and
2. $(p \to p) \to q$ is false in I.

Stage 2: Unpack the truth conditions

By 1 and $\to$T, EITHER

3. p is false in I

OR

4. $p \to q$ is true in I, in which case, by 4 and $\to$T, EITHER

    5. p is false in I

    OR

    6. q is true in I.

By 2 and $\to$F, 7. $p \to p$ is true in I and <u>8. q is false</u> in I. By 7 and $\to$T, EITHER

9. p is true in I

OR

10. p is false in I.

Let both p and q be false. Then all of the above conditions are satisfied and so $p \to (p \to q)$ is true and $(p \to p) \to q$ is false (NOTE: This is not easy to see because of all of the ‘either/or’s in the regression. However, remember that a disjunction is true when at least one disjunct is true and look back through stage 2 focusing on the highlighted disjunct; you will see that each has being false and so all of those disjunctions are true, ignoring the innermost second, as we don’t need to make it true (although we do, as we have also satisfied the first of its disjuncts!)– and we make q false because of step 8, which occurs outside and ‘either/or’), as follows:

p q	$p \to (p \to q)$	$(p \to p) \to q$
F F	F T	F T F F

So, the last row of their joint truth table is a countermodel. So, $p \to (p \to q) \not\models (p \to p) \to q$.