G-fixed Hilbert schemes on $K3$ surfaces and modular forms

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Abstract

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Let

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X)^G) q^{n-1}$$

be the generating function for the Euler characteristics of Hilbert scheme of G-invariant length n subschemes. We show that its reciprocal, $Z_{X,G}(q)^{-1}$ is the Fourier expansion of a modular cusp form of weight $\frac{1}{2}e(X/G)$ and index $|G|$. We give an explicit formula for $Z_{X,G}$ in terms of the Dedekind eta function for all 82 possible (X, G). The key intermediate result we prove is of independent interest: it establishes an eta product identity for a certain shifted theta function associated to the root lattice of a simply laced root system.

1 Introduction

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Mukai showed that such G are precisely the subgroups of the Mathieu group $M_{23} \subset M_{24}$ such that the induced action on the set $\{1, \dots, 24\}$ has at least five orbits [14]. Xiao classified all possible actions into 82 possible topological types of the quotient X/G [18].

The *G-fixed Hilbert scheme* of X parameterizes G-invariant length n subschemes $Z \subset X$. It can be identified with the G-fixed point locus in the Hilbert scheme of points:

$$\mathrm{Hilb}^n(X)^G \subset \mathrm{Hilb}^n(X)$$

We define the corresponding *G-fixed partition function* of X by

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^{n-1}$$

where $e(-)$ is topological Euler characteristic.

Throughout this paper we set

$$q = \exp(2\pi i \tau)$$

so that we may regard $Z_{X,G}$ as a function of $\tau \in \mathbb{H}$ where $\mathbb{H}$ is the upper half-plane.

Our main result is the following:

Theorem 1.1. *The function $Z_{X,G}(q)^{-1}$ is a modular cusp form*[1] *of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$.*

Our theorem specializes in the case where G is the trivial group to a famous result of Göttsche [9]. The case where G is a cyclic group was proved in [2]. One can interpret our result as an instance of the Vafa-Witten S-duality conjecture for the orbifold $[X/G]$.

We give an explicit formula for $Z_{X,G}(q)$ in terms of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

as follows. Let $p_1, \ldots, p_r$ be the singular points of X/G and let $G_1, \ldots, G_r$ be the corresponding stabilizer subgroups of G. The singular points are necessarily of ADE type: they are locally given by $\mathbb{C}^2/G_i$ where $G_i \subset SU(2)$. Finite subgroups of $SU(2)$ have an ADE classification and we let $\Delta_1, \ldots, \Delta_r$ denote the corresponding ADE root systems.

For any finite subgroup $G_\Delta \subset SU(2)$ with associated root system Δ we define the *local G_Δ-fixed partition function* by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n - \frac{1}{24}}.$$

Theorem 1.2. *The local partition function for Δ of type A_n is given by*

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

and for type D_n and E_n by

$$Z_\Delta(q) = \frac{\eta^2(2\tau)\eta(4E\tau)}{\eta(\tau)\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}$$

where (E, F, V) are given as follows[2]*:*

$$(E, F, V) = \begin{cases} (n-2, 2, n-2), & \Delta = D_n \\ (6, 4, 4), & \Delta = E_6 \\ (12, 8, 6), & \Delta = E_7 \\ (30, 20, 12), & \Delta = E_8 \end{cases}$$

[1] See Section § **??** for notation and definitions regarding modular forms.

[2] The group $H = G_\Delta/\{\pm 1\} \subset SO(3)$ is the symmetry group of a polyhedral decomposition of S^2 into isomorphic regular polygons. Then E, F, and V are the number of edges, faces, and vertices of the polyhedron.

The 82 possible collections of ADE root systems $\Delta_1, \ldots, \Delta_r$ associated to (X, G) a $K3$ surface with a symplectic G action, are given in Appendix A, Table 1. We let $k = |G|$, $k_i = |G_i|$, and

$$a = e(X/G) - r = \frac{24}{k} - \sum_{i=1}^{r} \frac{1}{k_i}.$$

Our method to prove Theorem 1.1 is based on the next result, which expresses the global series $Z_{X,G}(q)$ as an eta product.

Theorem 1.3. *With the above notation we have*

$$Z_{X,G}(q) = \eta^{-a}(k\tau) \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

In Appendix A, Table 1 we have listed explictly the eta product of the modular form $(Z_{X,G})^{-1}$ for all 82 possible cases of (X, G).

We will prove in Lemma 2.2 that

$$Z_{\Delta}(q) = \frac{\theta_{\Delta}(\tau)}{\eta(k\tau)^{N+1}}$$

where

$$\theta_{\Delta}(\tau) = \sum_{\boldsymbol{m} \in M_{\Delta}} q^{\frac{k}{2}\left(\boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta} | \boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta}\right)}$$

is a shifted theta function for M_{Δ} the root lattice of Δ, N is the rank of the root system, and $k = |G_{\Delta}|$ (see section 2 for the definition of $\boldsymbol{\zeta}$).

Theorem 1.2 then yields an eta product identity for the theta function $\theta_{\Delta}(\tau)$ reminiscent of the MacDonald identities (???), namely

Corollary 1.4. *For Δ of type A_n*

$$\theta_{A_n}(\tau) = \frac{\eta^{n+1}((n+1)\tau)}{\eta(\tau)}$$

and for Δ of type D_n or E_n

$$\theta_{\Delta}(\tau) = \theta_{A_1}(\tau) \frac{\eta^{N+2}(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}.$$

Having obtained explicit eta product expressions for $Z_{X,G}(q)$ in all 82 possible cases allows us to make several observational corollaries:

Corollary 1.5. *If G is a finite subgroup of an elliptic curve E, i.e. G is isomorphic to a product of one or two cyclic groups, then $Z_{X,G}(q)^{-1}$ is a Hecke eigenform. Appendix A, Table 1 these are the 13 cases having Xiao number in the set* $\{0, 1, 2, 3, 4, 5, 7, 8, 11, 14, 15, 19, 25\}$. *Moreover, in each of these cases, the dimension of the Hecke eigenspace is one.*

We remark that in these cases, we may form a Calabi-Yau threefold called a CHL model by taking the free group quotient

$$(X \times E)/G$$

Then the partition function $Z_{X,G}(q)$ gives the (modified) Donaldson-Thomas invariants of $(X \times E)/G$ in curve classes which are degree zero over X/G (see [2]).

For any eta product expression of a modular form, one may easily compute the order of vanishing (or pole) at any of the cusps [13, Cor 2.2]. Performing this computation on the 82 cases yields the following

Corollary 1.6. *The modular form $Z_{X,G}(q)^{-1}$ always vanishes with order 1 at the cusps $i\infty$ and 0. Moreover, $Z_{X,G}(q)^{-1}$ is holomorphic at all cusps except for the two cases with Xiao number 38 or 69, which have poles at the cusps $1/2$ and $1/8$ respectively. These are precisely the cases where X/G has two singularities of type E_6.*

2 The local partition functions

The classical McKay correspondence associates an ADE root system Δ to any finite subgroup $G_\Delta \subset SU(2)$. Using the work of Nakajima [15], the partition function of the Euler characteristics of the Hilbert scheme of points on the stack quotient $[\mathbb{C}^2/G_\Delta]$ was computed explicitly in [11] in terms of the root data of Δ.

The local partition functions $Z_\Delta(q)$ considered in this paper are obtained from a specialization of the partition functions of the stack $[\mathbb{C}^2/G_\Delta]$ and in this section, we use this to express $Z_\Delta(q)$ in terms of a shifted theta function for the root lattice of Δ.

A zero-dimensional substack $Z \subset [\mathbb{C}^2/G_\Delta]$ may be regarded as a G_Δ invariant, zero-dimensional subscheme of $\mathbb{C}^2$. Consequently, we may identify the Hilbert scheme of points on the stack $[\mathbb{C}^2/G_\Delta]$ with the G_Δ fixed locus of the Hilbert scheme of points on $\mathbb{C}^2$:

$$\mathrm{Hilb}\left([\mathbb{C}^2/G_\Delta]\right) = \mathrm{Hilb}(\mathbb{C}^2)^{G_\Delta}.$$

This Hilbert scheme has components indexed by representations ρ of G_Δ as follows

$$\mathrm{Hilb}^\rho\left([\mathbb{C}^2/G_\Delta]\right) = \left\{Z \subset \mathbb{C}^2,\ Z \text{ is } G_\Delta \text{ invariant and } H^0(\mathcal{O}_Z) \cong \rho\right\}.$$

Let $\{\rho_0, \dots, \rho_N\}$ be the irreducible representations of G_Δ where ρ_0 is the trivial representation. We note that N is also the rank of Δ. We define

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \dots, q_N) = \sum_{m_0,\dots,m_N=0}^{\infty} e\left(\mathrm{Hilb}^{m_0\rho_0+\dots+m_N\rho_M}([\mathbb{C}^2/G_\Delta])\right) q_0^{m_0} \cdots q_N^{m_N}.$$

Recall that our local partition function $Z_\Delta(q)$ is defined by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}.$$

We then readily see that

$$Z_\Delta(q) = q^{\frac{-1}{24}} \cdot Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\ldots,q_N)|_{q_i=q^{d_i}}$$

where

$$d_i = \dim \rho_i.$$

The following formula is given explicitely in [11, Thm 1.3], but its content is already present in the work of Nakajima [15]:

Theorem 2.1. *Let C_Δ be the Cartan matrix of the root system Δ, then*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\ldots,q_N) = \prod_{m=1}^{\infty}(1-Q^m)^{-N-1} \cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^N} q_1^{m_1}\cdots q_N^{m_N} \cdot Q^{\frac{1}{2}\boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta \cdot \boldsymbol{m}}$$

where $Q = q_0^{d_0} q_1^{d_1}\cdots d_N^{d_N}$.

We note that under the specialization $q_i = q^{d_i}$,

$$Q = q^{d_0^2+\cdots+d_N^2} = q^k$$

where $k = |G|$ is the order of the group G.

We then obtain

$$Z_\Delta(q) = q^{\frac{-1}{24}} \cdot \prod_{m=1}^{\infty}(1-q^{km})^{-N-1} \cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^N} q^{\boldsymbol{m}^{\mathsf{t}}\cdot\boldsymbol{d}} \cdot q^{\frac{k}{2}\boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{m}}$$

where $\boldsymbol{d} = (d_1,\ldots,d_N)$.

Let M_Δ be the root lattice of Δ which we identify with $\mathbb{Z}^N$ via the basis given by $\alpha_1,\ldots,\alpha_N$, the simple positive roots of Δ. Under this identification, the standard Weyl invariant bilinear form is given by

$$(\boldsymbol{u}|\boldsymbol{v}) = \boldsymbol{u}^{\mathsf{t}} \cdot C_\Delta \cdot \boldsymbol{v}.$$

We define

$$\boldsymbol{\zeta} = C_\Delta^{-1}\cdot\boldsymbol{d}$$

so that

$$\boldsymbol{m}^{\mathsf{t}}\cdot\boldsymbol{d} = \boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{\zeta} = (\boldsymbol{m}|\boldsymbol{\zeta}).$$

We may then write

$$\begin{aligned}
Z_\Delta(q) &= q^{\frac{-1}{24}} \cdot \prod_{m=1}^{\infty}(1-q^{km})^{-N-1} \cdot \sum_{\boldsymbol{m}\in M_\Delta} q^{(\boldsymbol{m}|\boldsymbol{\zeta})+\frac{k}{2}(\boldsymbol{m}|\boldsymbol{m})} \\
&= q^A \cdot \left(q^{\frac{k}{24}}\prod_{m=1}^{\infty}(1-q^{km})\right)^{-N-1} \cdot \sum_{\boldsymbol{m}\in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)} \\
&= q^A\cdot\eta(k\tau)^{-N-1}\cdot\theta_\Delta(\tau)
\end{aligned}$$

where

$$A = \frac{-1}{24} + \frac{k(N+1)}{24} - \frac{1}{2k}(\boldsymbol{\zeta}|\boldsymbol{\zeta}) = \frac{k(N+1)-1}{24} - \frac{1}{2k}\boldsymbol{d}^{\mathrm{t}} \cdot C_{\Delta}^{-1} \cdot \boldsymbol{d}$$

and $\theta_\Delta(\tau)$ is the shifted theta function:

$$\theta_\Delta(\tau) = \sum_{\boldsymbol{m} \in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)}$$

where as throughout this paper we have identified $q = \exp(2\pi i\tau)$.

In Section **??**, Proposition **??** we will prove that the identity $A = 0$ holds for all Δ. Hence we obtain the following:

Lemma 2.2. *The local series $Z_\Delta(q)$ is given by*

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{N+1}}.$$

3 The global series

Recall that $p_1, \ldots, p_r \in X/G$ are the singular points of X/G with corresponding stabilizer subgroups $G_i \subset G$ of order k_i and ADE type Δ_i. Let $\{x_i^1, \ldots, x_i^{k/k_i}\}$ be the orbit of G in X corresponding to the point p_i (recall that $k = |G|$). We may stratify $\mathrm{Hilb}(X)^G$ according to the orbit types of subscheme as follows:

Let $Z \subset X$ be a G-invariant subscheme of length nk whose support lies on free orbits. Then Z determines and is determined by a length n subscheme of

$$(X/G)^o = X/G \setminus \{p_1, \ldots, p_r\},$$

i.e. a point in $\mathrm{Hilb}^n((X/G)^o)$.

On the other hand, suppose $Z \subset X$ is a G-invariant subscheme of length $\frac{nk}{k_i}$ supported on the orbit $\{x_i^1, \ldots, x_i^{k/k_i}\}$. Then Z determines and is determined by the length n component of Z supported on a formal neighborhood of one of the points, say x_i^1. Choosing a G_i-equivariant isomorphism of the formal neighborhood of x_i^1 in X with the formal neighborhood of the origin in $\mathbb{C}^2$, we see that Z determines and is determined by a point in $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$, where $\mathrm{Hilb}_0^n(\mathbb{C}^2) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ is the punctual Hilbert scheme parameterizing subschemes supported on a formal neighborhood of the origin in $\mathbb{C}^2$.

By decomposing an arbitrary G-invariant subscheme into components of the above types, we obtain a stratification of $\mathrm{Hilb}(X)^G$ into strata which are given by products of $\mathrm{Hilb}((X/G)^o)$ and $\mathrm{Hilb}_0(\mathbb{C}^2)^{G_1}, \ldots, \mathrm{Hilb}_0(\mathbb{C}^2)^{G_r}$. Then using the fact that Euler characteristic is additive under stratifications and multiplicative under products, we arrive at the following equation of generating functions:

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^n = \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^o)\right) q^{kn}\right) \cdot \prod_{i=1}^{r} \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}}\right). \tag{1}$$

As in the introduction, let $a = e(X/G) - r = e\left((X/G)^o\right)$. Then by Göttsche's formula [9],

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^0\right) q^{kn} = \prod_{m=1}^{\infty}(1-q^{km})^{-a}$$
$$= q^{\frac{ak}{24}} \cdot \eta(k\tau)^{-a}.$$

We also note that $e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) = e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right)$ since the natural $\mathbb{C}^*$ action on both $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$ and $\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}$ have the same fixed points. Thus we may write

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right)\, q^{\frac{nk}{k_i}} = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right)\, q^{\frac{nk}{k_i}}$$
$$= q^{\frac{k}{24k_i}} \cdot Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

Multiplying equation (1) by q^{-1} and substituting the above formulas, we find that

$$Z_{X,G}(q) = q^{-1+\frac{ak}{24}+\sum\frac{k}{24k_i}} \cdot \eta(k\tau)^{-a} \cdot \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

The exponent of q in the above equation is zero as is readily seen from the following Euler characteristic calculation:

$$24 = e(X) = e\left(X - \cup_{i=1}^{r}\{x_i^1, \ldots, x_i^{k/k_i}\}\right) + \sum_{i=1}^{r}\frac{k}{k_i}$$
$$= k \cdot e\left((X/G)^o\right) + \sum_{i=1}^{r}\frac{k}{k_i}$$
$$= k \cdot a + \sum_{i=1}^{r}\frac{k}{k_i}$$

We have thus proved that the first equation in Theorem 1.3 always holds. Then since the only root systems which can occur as singularities of X/G are of type A_n or D_4, D_5, D_6, or E_6, we may now use Theorem **??** and Proposition **??** to complete the proof of Theorem 1.3. □

4 Proof of Theorem 1.2

4.1 Proof of Theorem 1.2 in the A_n case.

We wish to prove

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

which is equivalent to the statement

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}(\mathbb{C}^2)^{\mathbb{Z}/(n+1)}\right) q^n = \prod_{m=1}^{\infty} (1-q^m)^{-1}.$$

The action of $\mathbb{Z}/(n+1)$ on $\mathbb{C}^2$ commutes with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ and consequently, the Euler characteristics on the left hand side may be computed by counting the $\mathbb{C}^* \times \mathbb{C}^*$-fixed subschemes, namely those given by monomial ideals. Such subschemes of length n have a well known bijection with integer partitions of n, whose generating function is given by the right hand side. $\square$

4.2 Proof of Theorem 1.2 in the D_n and E_n cases.

Let $G \subset SU(2)$ be a subgroup where the corresponding Δ is of D or E type. Then $\{\pm 1\} \subset G$ and let $H \subset SO(3)$ be the quotient

$$H = G/\{\pm 1\}.$$

The action of H on $\mathbb{P}^1 \cong S^2$ is by rotations. Indeed, H is the symmetry group of a regular polyhedral decomposition of S^2 which is given by the platonic solids in the E_n case and the decomposition into two hemispherical $(n-2)$-gons in the D_n case. H is generated by rotations of order p, q, r, obtained by rotating about the center of an edge, a face, or a vertex respectively. H has a group presentation:

$$H = \{\langle a, b, c\rangle : \quad a^{\mathsf{p}} = b^{\mathsf{q}} = c^{\mathsf{r}} = abc = 1\}.$$

Let $M = |H|$ be the order of H and let E, F, V be the number of edges, faces, and vertices respectively. Then

$$M = \mathsf{p}E = \mathsf{q}F = \mathsf{r}V$$

and since the stabilizer of an edge is always order 2 we have $\mathsf{p} = 2$ and so $M = 2E$. Then since $F + V - E = 2$ we find

$$E + F + V = 2 + M$$

We summarize this information below:

Type	H	M	$(\mathsf{p}, \mathsf{q}, \mathsf{r})$	(E, F, V)
D_n	dihedral	$2n-2$	$(2, n-2, 2)$	$(n-1, 2, n-1)$
E_6	tetrahedral	12	$(2,3,3)$	$(6,4,4)$
E_7	octahedral	24	$(2,3,4)$	$(12,8,6)$
E_8	icosohedral	60	$(2,3,5)$	$(30,20,12)$

Now let $\mathfrak{X}$ be the stack quotient

$$\mathfrak{X} = [\mathbb{C}^2/\{\pm 1\}]$$

and let

$$Y \cong \mathrm{Tot}(K_{\mathbb{P}^1})$$

be the minimal resolution of the singular space $X = \mathbb{C}^2/\{\pm 1\}$. Let $C \subset Y$ be the exceptional curve.

The stack quotient $[\mathbb{P}^1/H]$ has three stacky points with stabilizers of order $\mathsf{p}, \mathsf{q}, \mathsf{r}$, and consequently the stack quotient $[Y/H]$ has three orbifold points locally of the form $[\mathbb{C}^2/\mathbb{Z}_a]$ for $a \in \{\mathsf{p}, \mathsf{q}, \mathsf{r}\}$.

We observe that

$$[\mathbb{C}^2/G] \cong [\mathfrak{X}/H]$$

and consequently

$$\mathrm{Hilb}^n(\mathbb{C}^2)^G \cong \mathrm{Hilb}^n(\mathfrak{X})^H.$$

Recall from section 2 that $\mathrm{Hilb}^n(\mathfrak{X})$ decomposes into components $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ with $n = m_0 + m_1$ where the corresponding $\{\pm 1\}$ invariant subschemes $Z \subset \mathbb{C}^2$ have the property that as a $\{\pm 1\}$-representation, $H^0(\mathcal{O}_Z)$ has m_0 copies of the trivial representation and m_1 copies of the non-trivial representation.

We will prove in section **??** that as a consequence of the derived McKay correspondence we have the following:

Proposition 4.1. $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ *is deformation equivalent to and hence diffeomorphic to* $\mathrm{Hilb}^{m_0-(m_0-m_1)^2}(Y)$*. Moreover, this deformation and diffeomorphism are* H*-equivariant and so in particular*

$$e\left(\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})^H\right) = e\left(\mathrm{Hilb}^{m_0-(m_0-m_1)^2}(Y)^H\right).$$

Let

$$k = m_1 - m_0, \quad n = m_0 - (m_0 - m_1)^2$$

so that

$$m_0 + m_1 = 2n + k + 2k^2.$$

We then can compute:

$$\begin{aligned} q^{\frac{1}{24}} Z_\Delta(q) &= \sum_{m_0,m_1=0}^{\infty} e\left(\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})^H\right) q^{m_0+m_1} \\ &= \sum_{k\in\mathbb{Z}} \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(Y)^H\right) q^{2n+k+2k^2}. \end{aligned}$$

For the root lattice of A_1, we have $M_{A_1} \cong \mathbb{Z}$, $C_{A_1} = (2)$, and $k = 2$, $\boldsymbol{\zeta} = \frac{1}{2}$ so by definition

$$\begin{aligned} \theta_{A_1}(\tau) &= \sum_{m\in\mathbb{Z}} q^{\frac{2}{2}\left(m+\frac{1}{4}|m+\frac{1}{4}\right)} \\ &= \sum_{m\in\mathbb{Z}} q^{2(m+\frac{1}{4})^2} \\ &= q^{\frac{1}{8}} \sum_{k\in\mathbb{Z}} q^{2k^2+k} \end{aligned}$$

Substituting into the previous equation multiplied by $q^{\frac{1}{8}}$ we find

$$q^{\frac{1}{6}} Z_{\Delta}(q) = \theta_{A_1}(\tau) \cdot \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(Y)^H\right) q^{2n}.$$

We can now compute the summation factor in the above equation by the same method we used to compute the global series in section 3. Here we utilize the fact that the singularities of Y/H are all of type A and we have already proven our formula for the local series in the A_n case. Indeed, the quotient $[Y/H]$ has three stacky points with stabilizers $\mathbb{Z}_{\mathsf{p}}$, $\mathbb{Z}_{\mathsf{q}}$, and $\mathbb{Z}_{\mathsf{r}}$ and the complement of those points $(Y/H)^o$ has Euler characteristic -1. Proceeding then by the same argument we used in section 3 to get equation (1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(Y)^H\right) q^{2n} &= \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n\left((Y/H)^o\right)\right) q^{2Mn}\right) \\
&\quad \cdot \prod_{\mathsf{a}\in\{\mathsf{p},\mathsf{q},\mathsf{r}\}} \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n_0(\mathbb{C}^2)^{\mathbb{Z}_{\mathsf{a}}}\right) q^{\frac{2Mn}{\mathsf{a}}}\right) \\
&= \prod_{m=1}^{\infty} \frac{\left(1-q^{2Mn}\right)}{\left(1-q^{\frac{2Mn}{\mathsf{p}}}\right)\left(1-q^{\frac{2Mn}{\mathsf{q}}}\right)\left(1-q^{\frac{2Mn}{\mathsf{r}}}\right)} \\
&= \prod_{m=1}^{\infty} \frac{\left(1-q^{4En}\right)}{\left(1-q^{2En}\right)\left(1-q^{2Fn}\right)\left(1-q^{2Vn}\right)} \\
&= q^{\frac{1}{24}(-2E+2F+2V)} \cdot \frac{\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)} \\
&= \frac{q^{\frac{1}{6}}\, \eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}.
\end{aligned}$$

Substituting into the previous equation and cancelling the factors of $q^{\frac{1}{6}}$, we have thus proved

$$Z_{\Delta}(q) = \theta_{A_1}(\tau) \cdot \frac{\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}.$$

Finally, by Lemma 2.2 and the A_1 case of Theorem 1.2 (which we've already proved), we have that

$$\theta_{A_1}(\tau) = \frac{\eta^2(2\tau)}{\eta(\tau)}$$

which, when substituted into the above, completes the proof of Theorem 1.2 in the general case. □

5 The derived McKay correspondence and the proof of Proposition 4.1

A Table of eta products

The following table provides the list of the modular forms $Z_{X,G}^{-1}$, expressed as eta products, for each of the 82 possible symplectic actions of a group G on a $K3$ surface X. Our numbering matches Xiao's [18] whose table we refer to for a description of each group.

#	$\lvert G\rvert$	Singularities of X/G	The modular form $Z_{X,G}^{-1}$	Weight
0	1		$\eta(\tau)^{24}$	12
1	2	$8A_1$	$\eta(2\tau)^8\,\eta(\tau)^8$	8
2	3	$6A_2$	$\eta(3\tau)^6\,\eta(\tau)^6$	6
3	4	$12A_1$	$\eta(2\tau)^{12}$	6
4	4	$2A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^2\,\eta(\tau)^4$	5
5	5	$4A_4$	$\eta(5\tau)^4\,\eta(\tau)^4$	4
6	6	$8A_1+3A_2$	$\frac{\eta(3\tau)^8\eta(2\tau)^3}{\eta(6\tau)}$	5
7	6	$2A_1+2A_2+2A_5$	$\eta(6\tau)^2\,\eta(3\tau)^2\,\eta(2\tau)^2\,\eta(\tau)^2$	4
8	7	$3A_6$	$\eta(7\tau)^3\,\eta(\tau)^3$	3
9	8	$14A_1$	$\frac{\eta(4\tau)^{14}}{\eta(8\tau)^4}$	5
10	8	$9A_1+2A_3$	$\frac{\eta(4\tau)^9\eta(2\tau)^2}{\eta(8\tau)^2}$	9/2
11	8	$4A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^4$	4
12	8	$3A_3+2D_4$	$\frac{\eta(\tau)^2\eta(4\tau)^6}{\eta(2\tau)}$	7/2
13	8	A_1+4D_4	$\frac{\eta(4\tau)^{13}\eta(\tau)^4}{\eta(8\tau)^2\eta(2\tau)^8}$	7/2
14	8	$A_1+A_3+2A_7$	$\eta(8\tau)^2\,\eta(4\tau)\,\eta(2\tau)\,\eta(\tau)^2$	3
15	9	$8A_2$	$\eta(3\tau)^8$	4
16	10	$8A_1+2A_4$	$\frac{\eta(5\tau)^8\eta(2\tau)^2}{\eta(10\tau)^2}$	4
17	12	$4A_1+6A_2$	$\frac{\eta(6\tau)^4\eta(4\tau)^6}{\eta(12\tau)^2}$	4
18	12	$9A_1+A_2+A_5$	$\frac{\eta(6\tau)^9\eta(4\tau)\eta(2\tau)}{\eta(12\tau)^3}$	4
19	12	$3A_1+3A_5$	$\eta(6\tau)^3\,\eta(2\tau)^3$	3
20	12	$A_2+2A_3+2D_5$	$\frac{\eta(4\tau)^3\eta(3\tau)^2\eta(\tau)^2\eta(6\tau)^4}{\eta(12\tau)\eta(2\tau)^4}$	3
21	16	$15A_1$	$\frac{\eta(8\tau)^{15}}{\eta(16\tau)^6}$	9/2
22	16	$10A_1+2A_3$	$\frac{\eta(8\tau)^{10}\eta(4\tau)^2}{\eta(16\tau)^4}$	4
23	16	$5A_1+4A_3$	$\frac{\eta(8\tau)^5\eta(4\tau)^4}{\eta(16\tau)^2}$	7/2
24	16	$6A_1+A_3+2D_4$	$\frac{\eta(8\tau)^{12}\eta(2\tau)^2}{\eta(16\tau)^4\eta(4\tau)^3}$	7/2

25	16	$6A_3$	$\eta\left(4\tau\right)^6$	3
26	16	$4A_1+A_3+A_7+D_4$	$\frac{\eta(8\tau)^7\eta(2\tau)^2}{\eta(16\tau)^2\eta(4\tau)}$	3
27	16	$2A_1+4D_4$	$\frac{\eta(8\tau)^{14}\eta(2\tau)^4}{\eta(4\tau)^8\eta(16\tau)^4}$	3
28	16	$2A_1+A_3+2A_7$	$\eta\left(8\tau\right)^2\eta\left(4\tau\right)\eta\left(2\tau\right)^2$	5/2
29	16	$A_3+D_4+2D_6$	$\frac{\eta(4\tau)\eta(8\tau)^7\eta(\tau)^2}{\eta(16\tau)^2\eta(2\tau)^3}$	5/2
30	18	$8A_1+4A_2$	$\frac{\eta(9\tau)^8\eta(6\tau)^4}{\eta(18\tau)^4}$	4
31	18	$2A_1+3A_2+2A_5$	$\frac{\eta(9\tau)^2\eta(6\tau)^3\eta(3\tau)^2}{\eta(18\tau)}$	3
32	20	$2A_1+4A_3+A_4$	$\frac{\eta(10\tau)^2\eta(5\tau)^4\eta(4\tau)}{\eta(20\tau)}$	3
33	21	$6A_2+A_6$	$\frac{\eta(7\tau)^6\eta(3\tau)}{\eta(21\tau)}$	3
34	24	$5A_1+3A_2+2A_3$	$\frac{\eta(12\tau)^5\eta(8\tau)^3\eta(6\tau)^2}{\eta(24\tau)^3}$	7/2
35	24	$4A_1+2A_2+2A_5$	$\frac{\eta(12\tau)^4\eta(8\tau)^2\eta(4\tau)^2}{\eta(24\tau)^2}$	3
36	24	$5A_1+A_3+A_5+D_5$	$\frac{\eta(12\tau)^7\eta(6\tau)\eta(2\tau)\eta(8\tau)}{\eta(24\tau)^3\eta(4\tau)}$	3
37	24	$2A_2+A_5+D_4+E_6$	$\frac{\eta(8\tau)^4\eta(4\tau)\eta(3\tau)\eta(12\tau)^4\eta(\tau)}{\eta(6\tau)^2\eta(24\tau)^2\eta(2\tau)^2}$	5/2
38	24	$2A_2+A_3+2E_6$	$\frac{\eta(8\tau)^6\eta(6\tau)\eta(\tau)^2\eta(12\tau)^2}{\eta(2\tau)^4\eta(24\tau)^2}$	5/2
39	32	$8A_1+3A_3$	$\frac{\eta(16\tau)^8\eta(8\tau)^3}{\eta(32\tau)^4}$	7/2
40	32	$9A_1+2D_4$	$\frac{\eta(16\tau)^{15}\eta(4\tau)^2}{\eta(32\tau)^6\eta(8\tau)^4}$	7/2
41	32	$3A_1+5A_3$	$\frac{\eta(16\tau)^3\eta(8\tau)^5}{\eta(32\tau)^2}$	3
42	32	$4A_1+2A_3+2D_4$	$\frac{\eta(16\tau)^{10}\eta(4\tau)^2}{\eta(32\tau)^4\eta(8\tau)^2}$	3
43	32	$5A_1+2A_7$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
44	32	$2A_1+2A_3+A_7+D_4$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
45	32	$3A_1+D_4+2D_6$	$\frac{\eta(16\tau)^{10}\eta(2\tau)^2}{\eta(32\tau)^4\eta(4\tau)^3}$	5/2
46	36	$2A_1+2A_2+4A_3$	$\frac{\eta(18\tau)^2\eta(12\tau)^2\eta(9\tau)^4}{\eta(36\tau)^2}$	3
47	36	$A_1+6A_2+A_5$	$\frac{\eta(18\tau)\eta(12\tau)^6\eta(6\tau)}{\eta(36\tau)^2}$	3
48	36	$6A_1+A_2+2A_5$	$\frac{\eta(18\tau)^6\eta(12\tau)\eta(6\tau)^2}{\eta(36\tau)^3}$	3
49	48	$5A_1+6A_2$	$\frac{\eta(24\tau)^5\eta(16\tau)^6}{\eta(48\tau)^4}$	7/2
50	48	$6A_2+2A_3$	$\frac{\eta(16\tau)^6\eta(12\tau)^2}{\eta(48\tau)^2}$	3
51	48	$5A_1+A_2+2A_3+A_5$	$\frac{\eta(24\tau)^5\eta(16\tau)\eta(12\tau)^2\eta(8\tau)}{\eta(48\tau)^3}$	3
52	48	$4A_1+3A_5$	$\frac{\eta(24\tau)^4\eta(8\tau)^3}{\eta(48\tau)^2}$	5/2
53	48	$A_1+A_2+2A_3+2D_5$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(12\tau)^2\eta(4\tau)^2}{\eta(48\tau)^3\eta(8\tau)^4}$	5/2
54	48	$4A_1+A_2+A_7+E_6$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(6\tau)\eta(2\tau)}{\eta(48\tau)^3\eta(4\tau)^2}$	5/2

55	60	$4A_1 + 3A_2 + 2A_4$	$\frac{\eta(30\tau)^4\eta(20\tau)^3\eta(12\tau)^2}{\eta(60\tau)^3}$	3
56	64	$5A_1 + 3A_3 + D_4$	$\frac{\eta(32\tau)^8\eta(16\tau)\eta(8\tau)}{\eta(64\tau)^4}$	3
57	64	$6A_1 + 3D_4$	$\frac{\eta(32\tau)^{15}\eta(8\tau)^3}{\eta(64\tau)^6\eta(16\tau)^6}$	3
58	64	$3A_1 + 3A_3 + A_7$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
59	64	$5A_3 + D_4$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
60	64	$4A_1 + A_3 + 2D_6$	$\frac{\eta(32\tau)^8\eta(16\tau)^3\eta(4\tau)^2}{\eta(64\tau)^4\eta(8\tau)^4}$	5/2
61	72	$4A_1 + 3A_2 + A_3 + D_5$	$\frac{\eta(36\tau)^6\eta(24\tau)^4\eta(18\tau)\eta(6\tau)}{\eta(72\tau)^4\eta(12\tau)^2}$	3
62	72	$3A_1 + 2A_3 + 2A_5$	$\frac{\eta(36\tau)^3\eta(18\tau)^2\eta(12\tau)^2}{\eta(72\tau)^2}$	5/2
63	72	$A_2 + 3A_3 + 2D_4$	$\frac{\eta(24\tau)\eta(9\tau)^2\eta(36\tau)^6}{\eta(72\tau)^3\eta(18\tau)}$	5/2
64	80	$3A_1 + 4A_4$	$\frac{\eta(40\tau)^3\eta(16\tau)^4}{\eta(80\tau)^2}$	5/2
65	96	$3A_1 + 3A_2 + 3A_3$	$\frac{\eta(48\tau)^3\eta(32\tau)^3\eta(24\tau)^3}{\eta(96\tau)^3}$	3
66	96	$2A_1 + 2A_2 + A_3 + 2A_5$	$\frac{\eta(48\tau)^2\eta(32\tau)^2\eta(24\tau)\eta(16\tau)^2}{\eta(96\tau)^2}$	5/2
67	96	$2A_1 + 3A_2 + A_7 + D_4$	$\frac{\eta(48\tau)^5\eta(32\tau)^3\eta(12\tau)^2}{\eta(96\tau)^3\eta(24\tau)^2}$	5/2
68	96	$3A_1 + 2A_3 + A_5 + D_5$	$\frac{\eta(48\tau)^5\eta(24\tau)^2\eta(8\tau)\eta(32\tau)}{\eta(96\tau)^3\eta(16\tau)}$	5/2
69	96	$3A_1 + 2A_2 + 2E_6$	$\frac{\eta(48\tau)^5\eta(32\tau)^6\eta(4\tau)^2}{\eta(96\tau)^4\eta(8\tau)^4}$	5/2
70	120	$2A_1 + A_2 + 2A_3 + A_4 + A_5$	$\frac{\eta(60\tau)^2\eta(40\tau)\eta(30\tau)^2\eta(24\tau)\eta(20\tau)}{\eta(120\tau)^2}$	5/2
71	128	$3A_1 + 2A_3 + D_4 + D_6$	$\frac{\eta(64\tau)^8\eta(32\tau)\eta(8\tau)}{\eta(128\tau)^4\eta(16\tau)}$	5/2
72	144	$A_1 + 4A_2 + 2A_5$	$\frac{\eta(72\tau)\eta(48\tau)^4\eta(24\tau)^2}{\eta(144\tau)^2}$	5/2
73	160	$2A_1 + 3A_3 + 2A_4$	$\frac{\eta(80\tau)^2\eta(40\tau)^3\eta(32\tau)^2}{\eta(160\tau)^2}$	5/2
74	168	$A_1 + 3A_2 + 2A_3 + A_6$	$\frac{\eta(84\tau)\eta(56\tau)^3\eta(42\tau)^2\eta(24\tau)}{\eta(168\tau)^2}$	5/2
75	192	$2A_1 + 6A_2 + D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)^6\eta(24\tau)}{\eta(192\tau)^4\eta(48\tau)^2}$	3
76	192	$2A_1 + A_2 + 2A_3 + A_5 + D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)\eta(32\tau)\eta(24\tau)}{\eta(192\tau)^3}$	5/2
77	192	$2A_1 + A_2 + 3A_3 + E_6$	$\frac{\eta(96\tau)^3\eta(64\tau)^3\eta(48\tau)^3\eta(8\tau)}{\eta(192\tau)^3\eta(16\tau)^2}$	5/2
78	288	$2A_1 + 2A_2 + A_3 + 2D_5$	$\frac{\eta(144\tau)^6\eta(96\tau)^4\eta(72\tau)\eta(24\tau)^2}{\eta(288\tau)^4\eta(48\tau)^4}$	5/2
79	360	$A_1 + 2A_2 + 2A_3 + 2A_4$	$\frac{\eta(180\tau)\eta(120\tau)^2\eta(90\tau)^2\eta(72\tau)^2}{\eta(360\tau)^2}$	5/2
80	384	$A_1 + 3A_2 + 2A_3 + D_6$	$\frac{\eta(192\tau)^3\eta(128\tau)^3\eta(96\tau)^3\eta(24\tau)}{\eta(384\tau)^3\eta(48\tau)^2}$	5/2
81	960	$A_1 + 3A_2 + 2A_4 + D_4$	$\frac{\eta(480\tau)^4\eta(320\tau)^3\eta(192\tau)^2\eta(120\tau)}{\eta(960\tau)^3\eta(240\tau)^2}$	5/2

Table 1: Table of the modular forms $Z_{X,G}^{-1}$ for all symplectic G actions.

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