K3 surfaces with group actions, enumerative geometry, & modular forms

Let X be a K3 surface: a smooth, projective surface with $K_X = \mathcal{O}_X$ and $h^{0,1}(X) = 0$. Example quartic in $\mathbb{P}^3$ or a double cover $X \to \mathbb{P}^2$ branched over a smooth sextic.

Let $\mathrm{Hilb}^n(X)$ be the Hilbert scheme of n points on X.

$\mathrm{Hilb}^n(X)$ is a Hyperkähler manifold, holo. symplectic variety, resolution of $\mathrm{Sym}^n(X) = X^n/S_n$.

Gottsche 1990 $\quad e(\mathrm{Hilb}(X)) \rightsquigarrow$ Number theory

$$Z_X(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X))\, q^{n-1}$$

then $Z_X(q)^{-1} = \Delta(q) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$

Fourier expansion of the unique modular cusp form of weight 12

$q = \exp(2\pi i \tau) \quad \tau \in \mathbb{H}$.

Yau-Zaslow 1995 $\quad e(\mathrm{Hilb}(X)) \rightsquigarrow$ enumerative geometry

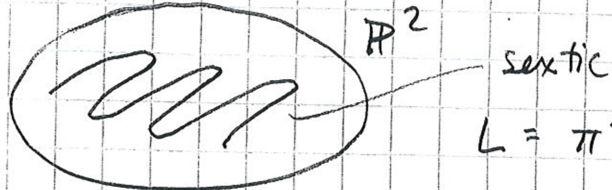
$|L|$ complete integral linear system of dim n on X

then $e(\mathrm{Hilb}^n(X)) = \#$ of rational curves C in $|L|$

(counted with multiplicity $e(\mathrm{Jac}\,C)$ if only nodal singularities $e(\mathrm{Jac}\,C) = e(\mathrm{Jac}\,C_{norm}) = 1$)

Solution to infinite # of enumerative problems.

e.g. $X \longrightarrow \mathbb{P}^2$



$L = \pi^* \mathcal{O}(1)$

$C \in |L|$ $\quad C$ double cover of lines in $\mathbb{P}^2$ $\quad e(\mathrm{Hilb}^2(X)) = 324$

$= \#$ bitangents

Above story has been vastly generalized over the last two decades, although not before in the following direction.

Let G be a finite group acting effectively on X preserving K_X.

<u>Mukai 1988</u> $\quad$ 82 possibilities $\quad G \subset M_{23} \subset M_{24}$ $\quad$ Mathieu gp

such that the action of G on $\{1, \cdots, 24\}$ has at least 5 orbits

e.g. $\mathbb{Z}/n$ $\quad n = 1, \cdots, 8$. biggest gp $\mathbb{Z}/2^4 \rtimes A_5$ $\quad |G| = 960$.

If $X \to \mathbb{P}^2$ branched over sextic invariant under $\mathbb{P}^2 \circlearrowleft \mathbb{Z}/2$

then combining branched cover involution with involution on base gives symplectic involution on X.

G-fixed Hilbert scheme $\mathrm{Hilb}^n(X)^G$ parameterizes G-invariant collections of n points.

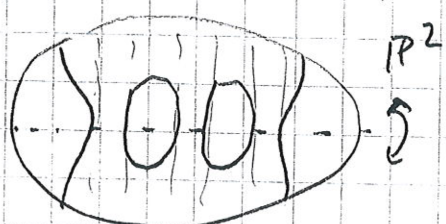
$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X)^G)\, q^{n-1}$$

Theorem (B. – Gyenge) $Z_{X,G}(q)^{-1}$ is a modular cusp form for $\Gamma_0(|G|) < SL_2\mathbb{Z}$ of weight $\frac{1}{2}e(X/G)$

Enumerative geometry: Suppose that $|L|$ is a complete, integral, linear system of dim n on X with an action of G

$e(\mathrm{Hilb}^n(X)^G)$ = # of curves $C \in |L|$ C/G rational counted by $e((\overline{\mathrm{Jac}\, C})^G) \underset{\substack{\text{if}\\ \text{nodes}}}{=} e((\mathrm{Jac}\, C_{norm})^G)$

example (Sailun Zhan) $\mathbb{Z}/2$ acting on $X \xrightarrow{\pi} \mathbb{P}^2$



$L = \pi^* \mathcal{O}(1)$

$$Z_{X,\mathbb{Z}/2}(q)^{-1} = q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8$$

unique cusp form weight 8 for $\Gamma_1(2)$

$$e(\mathrm{Hilb}^2(X)^{\mathbb{Z}/2}) = 52$$

C/G rational:
1. C/G preimage of $\mathbb{Z}/2$ fixed line $g(C) = 2$
2. C/G " " one of 6 invariant tangent lines $g(C) = 1$
3. C/G " " 12 $\mathbb{Z}/2$ invariant bitangents $g(C) = 0$

count with multiplicity

$$2^{2g(C)} = e((\mathrm{Jac}(C_{norm})^{\mathbb{Z}/2}))$$

$$52 = 16 \cdot 1 + 4 \cdot 6 + 1 \cdot 12$$

We can make the theorem explicit: $Z_{X,G}$ is a product of local contributions (depending on singularities of X/G)

$p \in X/G$ singular point locally $\mathbb{C}^2/G$ $G \subset SU(2)$

such subgroups classified by ADE root systems

$$G_\Delta \subset SU(2) \iff \Delta \quad \text{ADE root system.}$$

Define $$Z_\Delta(q) = \sum_{n=0}^{\infty}{}' e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}$$

Let $$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$ Dedekind eta fnc.

Core geometric result:

<u>Theorem</u> $Z_{A_n}(q) = \dfrac{1}{\eta(q)}$ and if Δ is D_n or E_n type

$$Z_\Delta(q) = \frac{\eta^2(q^2)\,\eta(q^{4E})}{\eta(q)\,\eta(q^{2E})\,\eta(q^{2F})\,\eta(q^{2V})}$$

$$(E,F,V) = \begin{cases} (n-2,\ 2,\ n-2) & D_n \quad n>4 \\ (6,\ 4,\ 4) & E_6 \\ (12,\ 8,\ 6) & E_7 \\ (30,\ 20,\ 12) & E_8 \end{cases}$$

$G_\Delta/\{\pm 1\} \subset SU(2)/\{\pm 1\} \cong SO(3)$ symmetries of regular polyhedral decomposition of S^2 (E, F, V = # (edges, faces, vertices)).

Main theorem follows from this result and standard Cheah–Göttsche method.

Idea of Proof $\quad G = \mathbb{Z}/_{n+1}$ (A_n case)

$$e(\mathrm{Hilb}^n(\mathbb{C}^2)^{\mathbb{Z}_{n+1}}) = e(\mathrm{Hilb}^n(\mathbb{C}^2)^{\mathbb{C}^\times \times \mathbb{C}^\times})$$

= # of monomial ideals = # integer partitions $\rightsquigarrow \eta(q)^{-1}$

$D_n E_n$ case: use Derived McKay correspondence to reduce to A_n case.

$$H = G/_{\pm 1} \subset SO(3)$$

$$\mathrm{Hilb}(\mathbb{C}^2)^G = \mathrm{Hilb}([\mathbb{C}^2/_{\pm 1}])^H$$

$$D^b([\mathbb{C}^2/_{\pm 1}]) \xrightarrow{\cong} D^b(T^*\mathbb{P}^1)$$

$$D^b([\mathbb{C}^2/G]) \xrightarrow{\cong} D^b([T^*\mathbb{P}^1/H])$$

$$[\mathbb{C}^2/_{\pm 1}] \to \mathbb{C}^2/_{\pm 1} \leftarrow T^*\mathbb{P}^1$$

3 orbifold points of order $|H|/E,\ |H|/F,\ |H|/G$

$$\mathrm{Hilb}(\mathbb{C}^2)^G \dashrightarrow \mathrm{Hilb}(T^*\mathbb{P}^1)^H$$

birational hol'c sympl. same $e(\,)$

$$\frac{\eta^2(q^2)}{\eta(q)} \quad \frac{\eta(q^{4E})}{\eta(q^{2E})\,\eta(q^{2F})\,\eta(q^{2V})}$$

contribution from non-orbifold pts.

contribution from orbifold pts

$$= \sum_{j \in \mathbb{Z}} q^{2j^2+j+\frac{1}{8}}$$

comes from matching descrete parameters, tensoring by $\mathcal{O}(j\,\mathbb{P}^1)$ leads to quadratic term.

Local formula leads to explicit eta product expression for all 82 cases.

Application to theta function identities

$\mathrm{Hilb}(\mathbb{C}^2)^{G_\Delta}$ is a Nakajima quiver variety for $\widehat{ADE}$ extended quiver.

$$\Rightarrow Z_\Delta(q) = \eta(q^K)^{-n-1} \sum_{\vec{m}\in R_\Delta}' q^{\frac{K}{2}|\vec{m}+\frac{1}{K}\vec{\xi}|^2_\Delta}$$

R_Δ: root lattice

n = rank Δ

$K = |G|$. our theorem writes this shifted theta function for the root lattice as a product (like MacDonald identities, Jacobi triple product.)

$\vec{\xi}$ = dual to longest root

Generalizations $e(\mathrm{Hilb}) \rightsquigarrow \chi_y(\mathrm{Hilb}) \rightsquigarrow \mathrm{Ell}_{q,y}(\mathrm{Hilb})$

refinement of Euler $\rightsquigarrow [\mathrm{Hilb}] \in K_0(\mathrm{Var}_{\mathbb{C}})$

We get formulas for all of these.

χ_y gives Jacobi forms $\mathrm{Ell}_{q,y}$ Siegel modular forms

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If G is a product of one or two cyclic gps (13 cases) so $G \subset E$ elliptic curve, then

$(X \times E)/G$ is a CHL CY3 DT theory

$Z_{X,G}(q)^{-1}$ are Hecke eigenforms (iff?).