G-FIXED HILBERT SCHEMES ON $K3$ SURFACES, MODULAR FORMS, AND ETA PRODUCTS

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ABSTRACT. Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Let

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^{n-1}$$

be the generating function for the Euler characteristics of the Hilbert schemes of G-invariant length n subschemes. We show that its reciprocal, $Z_{X,G}(q)^{-1}$ is the Fourier expansion of a modular cusp form of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$. We give an explicit formula for $Z_{X,G}$ in terms of the Dedekind eta function for all 82 possible (X, G). We extend our results to various refinements of the Euler characteristic, namely the Elliptic genus, the χ_y genus, and the motivic class. As a byproduct of our method, we prove a result which is of independent interest: it establishes an eta product identity for a certain shifted theta function of the root lattice of a simply laced root system.

1. INTRODUCTION

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Mukai showed that such G are precisely the subgroups of the Mathieu group $M_{23} \subset M_{24}$ such that the induced action on the set $\{1, \ldots, 24\}$ has at least five orbits [17]. Xiao classified all possible actions into 82 possible topological types of the quotient X/G [24].

The *G-fixed Hilbert scheme*[1] of X parameterizes G-invariant length n subschemes $Z \subset X$. It can be identified with the G-fixed point locus in the Hilbert scheme of points:

$$\mathrm{Hilb}^n(X)^G \subset \mathrm{Hilb}^n(X)$$

We define the corresponding *G-fixed partition function* of X by

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^{n-1}$$

where $e(-)$ is topological Euler characteristic.

Throughout this paper we set

$$q = \exp(2\pi i \tau)$$

so that we may regard $Z_{X,G}$ as a function of $\tau \in \mathbb{H}$ where $\mathbb{H}$ is the upper half-plane.

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[1] Some authors call this the G-equivariant Hilbert scheme or the G-invariant Hilbert scheme.

1.1. **The Main Results.** Our main result is the following:

Theorem 1.1. *The function $Z_{X,G}(q)^{-1}$ is a modular cusp form*[2] *of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$.*

Our theorem specializes in the case where G is the trivial group to a famous result of Göttsche [8]. The case where G is a cyclic group was proved in [3]. An analogous result for the case where X is an Abelian surface acted on symplectically by a finite group G has been recently proven by Pietromonaco [23].

We give an explicit formula for $Z_{X,G}(q)$ in terms of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

as follows. Let $p_1, \ldots, p_r$ be the singular points of X/G and let $G_1, \ldots, G_r$ be the corresponding stabilizer subgroups of G. The singular points are necessarily of ADE type: they are locally given by $\mathbb{C}^2/G_i$ where $G_i \subset SU(2)$. Finite subgroups of $SU(2)$ have an ADE classification and we let $\Delta_1, \ldots, \Delta_r$ denote the corresponding ADE root systems.

For any finite subgroup $G_\Delta \subset SU(2)$ with associated root system Δ we define the *local G_Δ-fixed partition function* by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}.$$

The main geometric result we prove is the following.

Theorem 1.2. *The local partition function for Δ of type A_n is given by*

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

and for type D_n and E_n by

$$Z_\Delta(q) = \frac{\eta^2(2\tau)\eta(4E\tau)}{\eta(\tau)\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}$$

where (E, F, V) are given by[3]*:*

$$(E,F,V) = \begin{cases} (n-2, 2, n-2), & \Delta = D_n \\ (6,4,4), & \Delta = E_6 \\ (12,8,6), & \Delta = E_7 \\ (30,20,12), & \Delta = E_8 \end{cases}$$

Our proof of the above Theorem uses a trick exploiting the derived McKay correspondence between $\mathfrak{X} = [\mathbb{C}^2/\{\pm 1\}]$ and $Y = \mathrm{Tot}(K_{\mathbb{P}^1})$, (see section 3).

Using the work of Nakajima, we will also prove in Lemma 4.2 that

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{n+1}}$$

[2]By cusp form, we mean that the order of vanishing at $q = 0$ is at least 1. Modular forms of half integral weight transform with respect to a multiplier system. We refer to [15] for definitions.

[3]For Δ of type D_n or E_n, the group $H = G_\Delta/\{\pm 1\} \subset SO(3)$ is the symmetry group of a polyhedral decomposition of S^2 into isomorphic regular spherical polygons. Then E, F, and V are the number of edges, faces, and vertices of the polyhedron.

where

$$\theta_\Delta(\tau) = \sum_{\boldsymbol{m}\in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)}$$

is a shifted theta function for M_Δ, the root lattice of Δ. Here n is the rank of the root system, $k = |G_\Delta|$, and $\boldsymbol{\zeta}$ is dual to the longest root (see section 4, eqn (2) for details).

Theorem 1.2 then yields an eta product identity for the theta function $\theta_\Delta(\tau)$ reminiscent of the MacDonald identities:

Theorem 1.3. *The shifted theta function $\theta_\Delta(\tau)$ defined above (c.f. § 4, eqn* (2)*) is given by an eta product as follows:*

For Δ of type A_n

$$\theta_{A_n}(\tau) = \frac{\eta^{n+1}((n+1)\tau)}{\eta(\tau)}$$

and for Δ of type D_n or E_n

$$\theta_\Delta(\tau) = \frac{\eta^2(2\tau)\,\eta^{n+2}(4E\tau)}{\eta(\tau)\,\eta(2E\tau)\,\eta(2F\tau)\,\eta(2V\tau)}$$

where E, F, V are as in Theorem 1.2.

Remark 1.4. Kac found that the Macdonald identities could be interpreted in terms of the character formula for highest weight representations of Kac-Moody algebras (c.f. [12, § 10]). It would be very interesting to find such an interpretation of the new identities in Theorem 1.3.

The 82 possible collections of ADE root systems $\Delta_1, \dots, \Delta_r$ associated to (X, G) a $K3$ surface with a symplectic G action, are given in Appendix B, Table 1. We let $k = |G|$, $k_i = |G_i|$, and

$$a = e(X/G) - r = \frac{24}{k} - \sum_{i=1}^{r} \frac{1}{k_i}.$$

The global series $Z_{X,G}(q)$ can be expressed as a product of local contributions (and thus via Theorem 1.2 as an explicit eta product) by our next result:

Theorem 1.5. *With the above notation we have*

$$Z_{X,G}(q) = \eta^{-a}(k\tau) \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

Theorem 1.1 then immediately follows from Theorem 4.1 and Theorem 1.5 using the formulas for the weight and level of an eta product given in [15, § 2.1].

In Appendix B, Table 1 we have listed explicitly the eta product of the modular form $Z_{X,G}(q)^{-1}$ for all 82 possible cases of (X, G).

1.2. **Consequences of the Main Results.** Having obtained explicit eta product expressions for $Z_{X,G}(q)$ allows us to make several observational corollaries:

Corollary 1.6. *If G is a finite subgroup of an elliptic curve E, i.e. G is isomorphic to a product of one or two cyclic groups, then $Z_{X,G}(q)^{-1}$ is a Hecke eigenform. In Table 1 these are the 13 cases having Xiao number in the set $\{0, 1, 2, 3, 4, 5, 7, 8, 11, 14, 15, 19, 25\}$. Moreover, in each of these cases, the dimension of the Hecke eigenspace is one.*

We remark that in these cases, we may form a Calabi-Yau threefold called a CHL model by taking the free group quotient

$$(X \times E)/G$$

Then the partition function $Z_{X,G}(q)$ gives the Donaldson-Thomas invariants of $(X \times E)/G$ in curve classes which are degree zero over X/G (c.f. [3]).

Remark 1.7. Hecke eigenforms of weight 3 arise in the arithmetic of $K3$ surfaces: if X is a $K3$ surface defined over $\mathbb{Q}$ and has $\rho(X) = \mathrm{rk}\,\mathrm{NS(X)} = 20$, then there is a weight 3 Hecke eigenform

$$f_X(q) = \sum_{n=1}^{\infty} a_n q^n$$

such that for almost all primes p, a_p is the trace of the p-th Frobenius morphism acting on $H^2(X)/NS(X)$. There are four cases where $Z_{X,G}(q)^{-1}$ is a weight three Hecke eigenform and they correspond to the cases where G is $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, or $\mathbb{Z}_4 \times \mathbb{Z}_4$ (numbered 8, 14, 19, 25 on Table 1). If X admits a symplectic G action for one of these four groups, then we may take X to be defined over $\mathbb{Q}$, have $\rho(X) = 20$, and then remarkably

$$Z_{X,G}(q)^{-1} = f_X(q).$$

Indeed, in each of these cases, we may take X to be elliptically fibered over $\mathbb{P}^1$ and have G as its group of sections (thus giving rise to the symplectic G action). Moreover, X is then the universal curve over the modular curve parameterizing (E, G), an elliptic curve E with a subgroup $G \subset E$. We thank Shuai Wang and Noam Elkies for noticing and elucidating this phenomenon.

For any eta product expression of a modular form, one may easily compute the order of vanishing (or pole) at any of the cusps [15, Cor 2.2]. Performing this computation on the 82 cases yields the following

Corollary 1.8. *The modular form $Z_{X,G}(q)^{-1}$ always vanishes with order 1 at the cusps $i\infty$ and 0. Moreover, $Z_{X,G}(q)^{-1}$ is holomorphic at all cusps except for the two cases with Xiao number 38 or 69, which have poles at the cusps $1/2$ and $1/8$ respectively. These are precisely the cases where X/G has two singularities of type E_6.*

Remark 1.9. The integers $e\left(\mathrm{Hilb}^n(X)^G\right)$ should have enumerative significance: they can be interpreted as virtual counts of G-invariant curves, whose quotient is rational, in a complete linear series of dimension n on X. This generalizes the famous Yau-Zaslow formula [25] in the case where G is the trivial group. The precise nature between the virtual count and the actual count is expected to be subtle for the case of general G. This has been recently explored in [26] and also in the case of G acting on an Abelian surface in [23].

1.3. **Refinements of the Euler Characteristic.** We can extend our results to various refinements of the Euler characteristic, namely the elliptic genus, the χ_y genus, and the motivic class. These refinements all stem from the next result which we prove in Section 5. Let

$$Z^{\mathsf{bir}}_{X,G}(q) = \sum_{n=0}^{\infty} [\mathrm{Hilb}^n(X)^G]_{\mathsf{bir}}\; q^{n-1}$$

be a formal series whose coefficients we regard as birational equivalence classes of compact hyperkahler manifolds. Such equivalence classes form a semi-ring under disjoint union and Cartesian product.

Theorem 1.10. *Let Y be the minimal resolution of X/G, then*

$$Z^{\mathsf{bir}}_{X,G}(q) = Z^{\mathsf{bir}}_{Y}(q^k) \cdot Z_{X,G}(q) \cdot \Delta(k\tau)$$

where $k = |G|$, $\Delta(\tau) = \eta(\tau)^{24}$, and we have suppressed the trivial group from the notation in the series $Z^{\mathsf{bir}}_{Y}(q^k)$.

A famous theorem of Huybrechts [11, Thm 4.6] asserts that birational compact hyperkahler manifolds are deformation equivalent. Moreover, combining Huybrechts' theorem with [20, Prop 3.21] it follows that birational compact hyperkahler manifolds are equal in $K_0(\mathrm{Var}_{\mathbb{C}})$, the Grothendieck group of varieties.

Thus we may specialize the series $Z^{\mathsf{bir}}_{X,G}(q)$ to Elliptic genus, motivic class, and χ_y genus since these are all well defined on birational equivalence classes of compact hyperkahler manifolds. These specializations are all well known for the series Z^{bir}_{Y} and hence we easily get the following corollaries.

Corollary 1.11. *Let $Q = \exp(2\pi i\sigma)$, $q = \exp(2\pi i\tau)$, $y = \exp(2\pi i z)$, and let*

$$Z^{\mathsf{Ell}}_{X,G}(Q,q,y) = \sum_{n=0}^{\infty} \mathrm{Ell}_{q,y}\left(\mathrm{Hilb}^n(X)^G\right) Q^{n-1}$$

where $\mathrm{Ell}_{q,y}(-)$ is elliptic genus. Then

$$Z^{\mathsf{Ell}}_{X,G}(Q,q,y) = \frac{\phi_{10,1}(\tau,z)}{\chi_{10}(k\sigma,\tau,z)} \cdot Z_{X,G}(q) \cdot \Delta(k\tau)$$

where $\phi_{10,1}(q,y)$ is the unique Jacobi cusp form of weight 10 and index 1 and $\chi_{10}(\sigma,\tau,z)$ is Igusa's genus 2 Siegel cusp form of weight 10.

We refer the reader to [22] (§5, §6, and eqn 6.9.8) for definitions of $\mathrm{Ell}_{q,y}$, $\phi_{10,1}$, χ_{10}, and the formula for the elliptic genera of $\mathrm{Hilb}^n(Y)$.

A further specialization of particular interest is the (normalized) χ_y genus. Let

$$\begin{aligned}\overline{\chi}_{-y}(M) &= y^{-\frac{1}{2}\dim M}\ \chi_{-y}(M) \\ &= y^{-\frac{1}{2}\dim M} \sum_{p,q} (-1)^{p+q} y^q \ \dim H^{p,q}(M)\end{aligned}$$

and we note that $\overline{\chi}_{-y}(M) = \mathrm{Ell}_{q,y}(M)|_{q=0}$.

Corollary 1.12. *Let*

$$Z^{\overline{\chi}}_{X,G}(q,y) = \sum_{n=0}^{\infty} \overline{\chi}_{-y}\left(\mathrm{Hilb}^n(X)^G\right) q^{n-1}.$$

Then

$$Z^{\overline{\chi}}_{X,G}(q,y) = y^{-1}(1-y)^2 \frac{Z_{X,G}(q)}{\phi_{-2,1}(q^k,y)}$$

where $\phi_{-2,1}$ is the unique weak Jacobi form of weight -2 and index 1. In particular,

$$y^{-1}(1-y)^2 Z^{\overline{\chi}}_{X,G}(q,y)^{-1} = \frac{\phi_{-2,1}(q^k,y)}{Z_{X,G}(q)}$$

is a Jacobi form of index 1 and weight

$$\frac{1}{2}e(X/G) - 2 = 10 - \frac{1}{2}\sum_{i=1}^{r} \mathrm{rank}\,\Delta_i$$

for the congruence subgroup $\Gamma_1(k)$.

We note that for G cyclic, the series $Z_{X,G}(q)/\phi_{-2,1}(q^k,y)$ is the leading coefficient in the expansion of the Donaldson-Thomas partition function of $(X\times E)/G$ in the variable tracking the curve class in X (see [3, Thm 0.1]).

We also get a formula for the motivic classes of the G-fixed Hilbert schemes:

Corollary 1.13. *Let*

$$Z^{K_0}_{X,G}(q)=\sum_{n=0}^{\infty}[\mathrm{Hilb}^n(X)^G]_{K_0}\;q^{n-1}$$

where $[\mathrm{Hilb}^n(X)^G]_{K_0}\in K_0(\mathrm{Var}_{\mathbb{C}})$ *denotes the motivic class of the G-fixed Hilbert scheme. Then*

$$Z^{K_0}_{X,G}(q)=q^{-1}\cdot\prod_{m=1}^{\infty}\left(1-\mathbb{L}^{m-1}q^{km}\right)^{-[Y]}\cdot Z_{X,G}(q)\cdot\Delta(k\tau)$$

where $\mathbb{L}=[\mathbb{A}^1_{\mathbb{C}}]\in K_0(\mathrm{Var}_{\mathbb{C}})$.

We refer the reader to [9] for the meaning of $[Y]$ in the exponent and the formula for the motivic class of $\mathrm{Hilb}^n(Y)$. The above series has further specializations giving formulas for the Hodge polynomials and Poincare polynomials of the G-fixed Hilbert schemes.

1.4. **Structure of paper.** In Section 2 we express the global partition function in terms of the local partition functions and deduce Theorem 1.5. In Section 3 we prove our main geometric result Theorem 1.2 which gives the eta product expression of the local partition functions. In Section 4 we express the local partition functions in terms of certain theta functions and thus prove our Theorem 1.3 which gives us the new theta function identities. In Section 5 we obtain the enhanced result of Theorem 1.10 on the partition function birational equivalence class of the G-fixed Hilbert schemes. Appendix A contains a proof of a root theoretic identity we need and Appendix B contains a table listing the modular form $Z^{-1}_{X,G}$ in all 82 topological types of symplectic actions on a $K3$ surface.

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2. THE GLOBAL PARTITION FUNCTION

As in the introduction, let X be a $K3$ surface with a symplectic action of a finite group G. Recall that $p_1,\ldots,p_r\in X/G$ are the singular points of X/G with corresponding stabilizer subgroups $G_i\subset G$ of order k_i and ADE type Δ_i. Let $\{x_i^1,\ldots,x_i^{k/k_i}\}$ be the orbit of G in X corresponding to the point p_i (recall that $k=|G|$). We may stratify $\mathrm{Hilb}(X)^G$ according to the orbit types of subscheme as follows.

Let $Z\subset X$ be a G-invariant subscheme of length nk whose support lies on free orbits. Then Z determines and is determined by a length n subscheme of

$$(X/G)^o=X/G\setminus\{p_1,\ldots,p_r\},$$

i.e. a point in $\mathrm{Hilb}^n((X/G)^o)$.

On the other hand, suppose $Z\subset X$ is a G-invariant subscheme of length $\frac{nk}{k_i}$ supported on the orbit $\{x_i^1,\ldots,x_i^{k/k_i}\}$. Then Z determines and is determined by the length n component of Z supported on a formal neighborhood of one of the points, say x_i^1. Choosing a G_i-equivariant isomorphism of the formal neighborhood of x_i^1 in X with the formal neighborhood of the origin in $\mathbb{C}^2$, we see that Z determines and is determined by a point in $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$, where $\mathrm{Hilb}_0^n(\mathbb{C}^2) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ is the punctual Hilbert scheme parameterizing subschemes supported on a formal neighborhood of the origin in $\mathbb{C}^2$.

By decomposing an arbitrary G-invariant subscheme into components of the above types, we obtain a stratification of $\mathrm{Hilb}(X)^G$ into strata which are given by products of $\mathrm{Hilb}((X/G)^o)$ and $\mathrm{Hilb}_0(\mathbb{C}^2)^{G_1}, \ldots, \mathrm{Hilb}_0(\mathbb{C}^2)^{G_r}$. Then using the fact that Euler characteristic is additive under stratifications and multiplicative under products, we arrive at the following equation of generating functions:

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^n &= \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^o)\right) q^{kn}\right) \\ &\quad \cdot \prod_{i=1}^{r} \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}}\right). \end{aligned} \tag{1}$$

As in the introduction, let $a = e(X/G) - r = e\left((X/G)^o\right)$. Then by Göttsche's formula [8],

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^0\right) q^{kn} &= \prod_{m=1}^{\infty} (1-q^{km})^{-a} \\ &= q^{\frac{ak}{24}} \cdot \eta(k\tau)^{-a}.\end{aligned}$$

We also note that $e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) = e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right)$ since the natural $\mathbb{C}^*$ action on both $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$ and $\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}$ have the same fixed points. Thus we may write

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}} &= \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}} \\ &= q^{\frac{k}{24k_i}} \cdot Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).\end{aligned}$$

Multiplying equation (1) by q^{-1} and substituting the above formulas, we find that

$$Z_{X,G}(q) = q^{-1+\frac{ak}{24}+\sum \frac{k}{24k_i}} \cdot \eta(k\tau)^{-a} \cdot \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

From the following Euler characteristic calculation, we see that the exponent of q in the above equation is zero:

$$\begin{aligned}24 = e(X) &= e\left(X - \cup_{i=1}^{r}\{x_i^1, \ldots, x_i^{k/k_i}\}\right) + \sum_{i=1}^{r} \frac{k}{k_i} \\ &= k \cdot e\left((X/G)^o\right) + \sum_{i=1}^{r} \frac{k}{k_i} \\ &= k \cdot a + \sum_{i=1}^{r} \frac{k}{k_i}\end{aligned}$$

This completes the proof of Theorem 1.5. $\square$

3. The local partition function

Recall that the local partition function is defined by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}$$

where $G_\Delta \subset SU(2)$ is the finite subgroup corresponding to the ADE root system Δ. In this section, we prove Theorem 1.2 which provides an explicit formula for $Z_\Delta(q)$ in terms of the Dedekind eta function. We regard this as the main geometric result of this paper.

3.1. **Proof of Theorem 1.2 in the A_n case.** We wish to prove

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

which is equivalent to the statement

$$\sum_{m=0}^{\infty} e\left(\mathrm{Hilb}^m(\mathbb{C}^2)^{\mathbb{Z}/(n+1)}\right) q^m = \prod_{m=1}^{\infty}(1-q^m)^{-1}.$$

The action of $\mathbb{Z}/(n+1)$ on $\mathbb{C}^2$ commutes with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ and consequently, the Euler characteristics on the left hand side may be computed by counting the $\mathbb{C}^* \times \mathbb{C}^*$-fixed subschemes, namely those given by monomial ideals. Such subschemes of length m have a well known bijection with integer partitions of m, whose generating function is given by the right hand side. □

3.2. **Proof of Theorem 1.2 in the D_n and E_n cases.** Our proof of Theorem 1.2 in the D_n and E_n cases uses a trick exploiting the derived McKay correspondence between $\mathfrak{X} = [\mathbb{C}^2/\{\pm 1\}]$ and $Y = \mathrm{Tot}(K_{\mathbb{P}^1})$.

Let $G \subset SU(2)$ be a subgroup where the corresponding root system Δ is of D or E type. Then $\{\pm 1\} \subset G$ and let $H \subset SO(3)$ be the quotient

$$H = G/\{\pm 1\}.$$

The induced action of H on $\mathbb{P}^1 \cong S^2$ is by rotations. Indeed, H is the symmetry group of a regular polyhedral decomposition of S^2 which is given by the platonic solids in the E_n case and the decomposition into two hemispherical $(n-2)$-gons in the D_n case. H is generated by rotations of order p, q, r, obtained by rotating about the center of an edge, a face, or a vertex respectively. H has a group presentation:

$$H = \{\langle a, b, c\rangle : \quad a^{\mathsf{p}} = b^{\mathsf{q}} = c^{\mathsf{r}} = abc = 1\}.$$

Let $M = |H|$ be the order of H and let E, F, V be the number of edges, faces, and vertices respectively. Then

$$M = \mathsf{p}E = \mathsf{q}F = \mathsf{r}V$$

and since the stabilizer of an edge is always order 2 we have $\mathsf{p} = 2$ and so $M = 2E$. Then since $F + V - E = 2$ we find

$$E + F + V = 2 + M$$

We summarize this information below:

Type	H	M	$(\mathsf{p},\mathsf{q},\mathsf{r})$	(E,F,V)
D_n	dihedral	$2n-2$	$(2,n-2,2)$	$(n-1,2,n-1)$
E_6	tetrahedral	12	$(2,3,3)$	$(6,4,4)$
E_7	octahedral	24	$(2,3,4)$	$(12,8,6)$
E_8	icosohedral	60	$(2,3,5)$	$(30,20,12)$

Now let $\mathfrak{X}$ be the stack quotient

$$\mathfrak{X}=[\mathbb{C}^2/\{\pm 1\}]$$

and let

$$Y\cong \operatorname{Tot}(K_{\mathbb{P}^1})$$

be the minimal resolution of the singular space $X=\mathbb{C}^2/\{\pm 1\}$.

The stack quotient $[\mathbb{P}^1/H]$ has three stacky points with stabilizers of order $\mathsf{p},\mathsf{q},\mathsf{r}$, and consequently the stack quotient $[Y/H]$ has three orbifold points locally of the form $[\mathbb{C}^2/\mathbb{Z}_{\mathsf{a}}]$ for $\mathsf{a}\in\{\mathsf{p},\mathsf{q},\mathsf{r}\}$.

We observe that

$$[\mathbb{C}^2/G]\cong[\mathfrak{X}/H]$$

and consequently

$$\operatorname{Hilb}^n(\mathbb{C}^2)^G\cong \operatorname{Hilb}^n(\mathfrak{X})^H.$$

The scheme $\operatorname{Hilb}^n(\mathfrak{X})$ decomposes into components $\operatorname{Hilb}^{m_0,m_1}(\mathfrak{X})$ with $n=m_0+m_1$ where the corresponding $\{\pm 1\}$ invariant subschemes $Z\subset\mathbb{C}^2$ have the property that as a $\{\pm 1\}$-representation, $H^0(\mathcal{O}_Z)$ has m_0 copies of the trivial representation and m_1 copies of the non-trivial representation.

We will prove in section 3.3 that as a consequence of the derived McKay correspondence between $\mathfrak{X}$ and Y, we have the following:

Proposition 3.1. $\operatorname{Hilb}^{m_0,m_1}(\mathfrak{X})^H$ *is deformation equivalent to and hence diffeomorphic to* $\operatorname{Hilb}^{m_0-(m_0-m_1)^2}(Y)^H$. *In particular*

$$e\left(\operatorname{Hilb}^{m_0,m_1}(\mathfrak{X})^H\right)=e\left(\operatorname{Hilb}^{m_0-(m_0-m_1)^2}(Y)^H\right).$$

Let

$$j=m_1-m_0,\quad n=m_0-(m_0-m_1)^2$$

so that

$$m_0+m_1=2n+j+2j^2.$$

We then can compute:

$$\begin{aligned} q^{\frac{1}{24}}Z_\Delta(q)&=\sum_{m_0,m_1=0}^{\infty} e\left(\operatorname{Hilb}^{m_0,m_1}(\mathfrak{X})^H\right)\, q^{m_0+m_1}\\ &=\sum_{j\in\mathbb{Z}}\sum_{n=0}^{\infty} e\left(\operatorname{Hilb}^{n}(Y)^H\right)\, q^{2n+j+2j^2}. \end{aligned}$$

The following identity follows easily from the Jacobi triple product formula:

$$\sum_{j\in\mathbb{Z}} q^{2j^2+j+\frac{1}{8}}=\frac{\eta^2(2\tau)}{\eta(\tau)}.$$

Substituting this into the previous equation multiplied by $q^{\frac{1}{8}}$ we find

$$q^{\frac{1}{6}} Z_{\Delta}(q) = \frac{\eta^2(2\tau)}{\eta(\tau)} \cdot \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(Y)^H\right) q^{2n}.$$

We can now compute the summation factor in the above equation by the same method we used to compute the global series in section 2. Here we use the fact that the singularities of Y/H are all of type A and we have already proven our formula for the local series in the A_n case. Indeed, the quotient $[Y/H]$ has three stacky points with stabilizers $\mathbb{Z}_{\mathsf{p}}$, $\mathbb{Z}_{\mathsf{q}}$, and $\mathbb{Z}_{\mathsf{r}}$ and the complement of those points $(Y/H)^o$ has Euler characteristic -1. Proceeding then by the same argument we used in section 2 to get equation (1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(Y)^H\right) q^{2n} &= \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n\left((Y/H)^o\right)\right) q^{2Mn}\right) \\
&\quad \cdot \prod_{\mathsf{a}\in\{\mathsf{p},\mathsf{q},\mathsf{r}\}} \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n_0(\mathbb{C}^2)^{\mathbb{Z}_{\mathsf{a}}}\right) q^{\frac{2Mn}{\mathsf{a}}}\right) \\
&= \prod_{m=1}^{\infty} \frac{\left(1-q^{2Mn}\right)}{\left(1-q^{\frac{2Mn}{\mathsf{p}}}\right)\left(1-q^{\frac{2Mn}{\mathsf{q}}}\right)\left(1-q^{\frac{2Mn}{\mathsf{r}}}\right)} \\
&= \prod_{m=1}^{\infty} \frac{\left(1-q^{4En}\right)}{\left(1-q^{2En}\right)\left(1-q^{2Fn}\right)\left(1-q^{2Vn}\right)} \\
&= q^{\frac{1}{24}(-2E+2F+2V)} \cdot \frac{\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)} \\
&= \frac{q^{\frac{1}{6}}\,\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}.
\end{aligned}$$

Substituting into the previous equation and cancelling the factors of $q^{\frac{1}{6}}$, we have thus proved

$$Z_{\Delta}(q) = \frac{\eta^2(2\tau)}{\eta(\tau)} \cdot \frac{\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)},$$

which completes the proof of Theorem 1.2 in the general case. □

3.3. **The derived McKay correspondence and the proof of Proposition 3.1.** The derived McKay correspondence uses a Fourier-Mukai transform to give an equivalence of derived categories [1, 13]:

$$\mathsf{FM} : D^b(\mathfrak{X}) \to D^b(Y).$$

We will want to track discrete parameters through the above equivalence and to do so it will be convenient to compactify $\mathfrak{X}$ and Y. Let $\overline{Y}$ be the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus K_{\mathbb{P}^1})$ which then contains Y as a Zariski open set. Let $\overline{\mathfrak{X}}$ be the corresponding compactification of $\mathfrak{X}$, namely $\mathfrak{X}$ is given by the weighted projective space $\mathbb{P}(2,1,1)$. The derived Fourier-Mukai correspondence extends to the compactifications [4]:

$$\mathsf{FM} : D^b(\overline{\mathfrak{X}}) \to D^b(\overline{Y}).$$

The induced map on the numerical K-groups

$$K_0(\overline{\mathfrak{X}}) \to K_0(\overline{Y})$$

is well known to take

$$[\mathcal{O}_{\overline{\mathfrak{X}}}] \longmapsto \quad [\mathcal{O}_{\overline{Y}}]$$

$$\begin{aligned}[][\mathcal{O}_0 \otimes \rho_0] &\longmapsto \quad [\mathcal{O}_C] \\ [\mathcal{O}_0 \otimes \rho_1] &\longmapsto -[\mathcal{O}_C(-1)]\end{aligned}$$

where $\mathcal{O}_0$ is the skyscraper sheaf of the origin in $[\mathbb{C}^2/\{\pm 1\}] \subset \overline{\mathfrak{X}}$, ρ_0 and ρ_1 are the trivial and non-trivial irreducible representations of $\{\pm 1\}$, and $C \subset Y \subset \overline{Y}$ is the exceptional curve (see for example [7] or [13]).

Let $\mathfrak{M}^{m_0,m_1}(\overline{\mathfrak{X}})$ be the moduli stack of objects $F^\bullet$ in $D^b(\overline{\mathfrak{X}})$ having numerical K-theory class given by

$$[F^\bullet] = [\mathcal{O}_{\overline{\mathfrak{X}}}] - m_0[\mathcal{O}_0 \otimes \rho_0] - m_1[\mathcal{O}_0 \otimes \rho_1].$$

Then $\operatorname{Hilb}^{m_0,m_1}(\mathfrak{X})$ may be regarded as the open substack of $\mathfrak{M}^{m_0,m_1}(\overline{\mathfrak{X}})$ parameterizing ideal sheaves I_Z, $Z \subset \mathfrak{X} \subset \overline{\mathfrak{X}}$, viewed as objects in $D^b(\overline{\mathfrak{X}})$ supported in degree 0.

The derived McKay equivalence then induces an equivalence of stacks

$$\mathfrak{M}^{m_0,m_1}(\overline{\mathfrak{X}}) \cong \mathfrak{M}^{m_0,m_1}(\overline{Y})$$

where $\mathfrak{M}^{m_0,m_1}(\overline{Y})$ is the moduli space of objects $F^\bullet$ in $D^b(\overline{Y})$ having numerical K-theory class given by

$$[F^\bullet] = [\mathcal{O}_{\overline{Y}}] - m_0[\mathcal{O}_C] + m_1[\mathcal{O}_C(-1)].$$

Numerical K-theory on $\overline{Y}$ is isomorphic to $H^*(\overline{Y}) \cong H^0(\overline{Y}) \oplus H^2(\overline{Y}) \oplus H^4(\overline{Y})$ via the Chern character. Then since

$$\begin{aligned} ch([\mathcal{O}_{\overline{Y}}]) &= (1,0,0) \\ ch([\mathcal{O}_C]) &= (0,C,1) \\ ch([\mathcal{O}_C(-1)]) &= (0,C,0)\end{aligned}$$

we have

$$ch([\mathcal{O}_{\overline{Y}}] - m_0[\mathcal{O}_C] + m_1[\mathcal{O}_C(-1)]) = (1,(m_1-m_0)C,-m_0).$$

We switch notation so that $\mathfrak{M}_{(a,b,c)}(\overline{Y})$ denotes the moduli stack of objects in $D^b(\overline{Y})$ having Chern character $(a,b,c) \in H^*(\overline{Y})$. Then we can rewrite the derived McKay equivalence as

$$\mathfrak{M}^{m_0,m_1}(\overline{\mathfrak{X}}) \cong \mathfrak{M}_{(1,(m_1-m_0)C,-m_0)}(\overline{Y}).$$

Tensoring by the line bundle

$$L = \mathcal{O}_{\overline{Y}}((m_0-m_1)C)$$

induces an equivalence

$$\mathfrak{M}_{(1,(m_1-m_0)C,-m_0)}(\overline{Y}) \cong \mathfrak{M}_{(1,0,-n)}(\overline{Y})$$

where

$$n = m_0 - (m_1-m_0)^2.$$

Indeed, this follows from

$$ch(L) = \big(1,(m_0-m_1)C,\tfrac{1}{2}(m_0-m_1)^2C^2\big)$$

and

$$\begin{aligned}(1,(m_1-m_0)C,-m_0)&\cdot(1,(m_0-m_1)C,\tfrac{1}{2}(m_0-m_1)^2C^2) \\ &= (1,0,-(m_0-m_1)^2C^2 + \tfrac{1}{2}(m_0-m_1)^2C^2 - m_0) \\ &= (1,0,(m_0-m_1)^2 - m_0)\end{aligned}$$

where we have used $C^2 = -2$.

Thus we have an equivalence of stacks

$$\mathfrak{M}^{m_0,m_1}(\overline{\mathfrak{X}}) \cong \mathfrak{M}_{(1,0,-n)}(\overline{Y})$$

which takes the open substack $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ isomorphically onto an open substack of $\mathfrak{M}_{(1,0,-n)}(\overline{Y})$. Noting that $\mathrm{Hilb}^n(Y)$ is also an open substack of $\mathfrak{M}_{(1,0,-n)}(\overline{Y})$ which intersects the image of $\mathrm{Hilb}^{m_1,m_1}(\mathfrak{X})$ non-trivially, we find that the equivalence induces a birational map

$$\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X}) \dashrightarrow \mathrm{Hilb}^n(Y).$$

Both sides of the above birational equivalence are smooth, holomorphic symplectic varieties, and they are both symplectic resolutions of the singular affine symplectic variety $\mathrm{Sym}^n(\mathbb{C}^2/\{\pm 1\})$. [I think the correct line of argument for the birational equivalence is as follows:] When $n \geq 1$, both open substacks are symplectic resolutions of the singular affine symplectic variety $\mathrm{Sym}^n(\mathbb{C}^2/\{\pm 1\})$ [19, Sections 2 and 3]. Hence, they intersect non-trivially, and we find that the equivalence induces a birational map

$$\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X}) \dashrightarrow \mathrm{Hilb}^n(Y).$$

Consequently, $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ is deformation equivalent to, and hence diffeomorphic to $\mathrm{Hilb}^n(Y)$. This assertion follows from Nakajima [18, Cor 4.2] by viewing both Hilbert schemes as moduli spaces of quiver representations of the A_1 Nakajima quiver variety having the same dimension vector but using different stability conditions. Writing these Hilbert schemes as Nakajima quiver varieties is discussed in detail in [16], specifically the identification of $\mathrm{Hilb}(\mathfrak{X})$ and $\mathrm{Hilb}(Y)$ with Nakajima quiver varieties for specific stability conditions is given in [16] at the bottom of page 2. [I think the correct line of argument for the deformation equivalence is as follows:] Both sides of the above birational equivalence are smooth, holomorphic symplectic varieties, and as we already mentioned they are both symplectic resolutions of the singular affine symplectic variety $\mathrm{Sym}^n(\mathbb{C}^2/\{\pm 1\})$. By [5, Proposition 4.8], there is a birational equivalence

$$\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X}) \dashrightarrow \mathrm{Hilb}^{n,n}(\mathfrak{X}),$$

and therefore both Hilbert schemes can be realised as moduli spaces of quiver representations of the A_1 Nakajima quiver variety of the same dimension but using different stability conditions. Note that our n coincides with the length $v_0 - v^T Cv/2$ of deHority.

The above constructions are compatible with the H action. Indeed, the derived McKay equivalence and tensoring with the line bundle L both commute with the H action and so we have an equivalence of stacks:

$$\mathfrak{M}^{m_0,m_1}(\overline{\mathfrak{X}})^H \cong \mathfrak{M}_{(1,0,-n)}(\overline{Y})^H$$

and a birational equivalence of holomorphic symplectic varieties:

$$\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})^H \dashrightarrow \mathrm{Hilb}^n(Y)^H.$$

As before, both sides of the above birational equivalence can be viewed as Nakajima quiver varieties (in this case for the Nakajima quiver associated to the Dynkin diagram corresponding to G), and so by [18, Cor 4.2] are deformation equivalent. This then completes the proof of Proposition 3.1. $\square$

4. The local partition function as a theta function via Nakajima

The local partition functions $Z_\Delta(q)$ considered in this paper are obtained from a specialization of the partition functions of the stack $[\mathbb{C}^2/G_\Delta]$. Using the work of Nakajima [19], the partition function of the Euler characteristics of the Hilbert scheme of points on the stack quotient $[\mathbb{C}^2/G_\Delta]$ was computed explicitly in [10] in terms of the root data of Δ. We use this to express $Z_\Delta(q)$ in terms of $\theta_\Delta(\tau)$, a shifted theta function for the root lattice of Δ. As a byproduct we obtain an eta product formula for the associated shifted theta function (Theorem 1.3).

A zero-dimensional substack $Z \subset [\mathbb{C}^2/G_\Delta]$ may be regarded as a G_Δ invariant, zero-dimensional subscheme of $\mathbb{C}^2$. Consequently, we may identify the Hilbert scheme of points on the stack $[\mathbb{C}^2/G_\Delta]$ with the G_Δ fixed locus of the Hilbert scheme of points on $\mathbb{C}^2$:

$$\mathrm{Hilb}\left([\mathbb{C}^2/G_\Delta]\right) = \mathrm{Hilb}(\mathbb{C}^2)^{G_\Delta}.$$

This Hilbert scheme has components indexed by representations ρ of G_Δ as follows

$$\mathrm{Hilb}^\rho\left([\mathbb{C}^2/G_\Delta]\right) = \left\{Z \subset \mathbb{C}^2,\ Z \text{ is } G_\Delta \text{ invariant and } H^0(\mathcal{O}_Z) \cong \rho\right\}.$$

Let $\{\rho_0, \ldots, \rho_n\}$ be the irreducible representations of G_Δ where ρ_0 is the trivial representation. We note that n is also the rank of Δ. We define

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n) = \sum_{m_0,\ldots,m_n=0}^{\infty} e\left(\mathrm{Hilb}^{m_0\rho_0+\cdots+m_n\rho_n}([\mathbb{C}^2/G_\Delta])\right) q_0^{m_0}\cdots q_n^{m_n}.$$

Recall that our local partition function $Z_\Delta(q)$ is defined by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}.$$

We then readily see that

$$Z_\Delta(q) = q^{\frac{-1}{24}} \cdot Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n)|_{q_i = q^{d_i}}$$

where

$$d_i = \dim \rho_i.$$

The following formula is given explicitly in [10, Thm 1.3], but its content is already present in the work of Nakajima [19]:

Theorem 4.1. *Let C_Δ be the Cartan matrix of the root system Δ, then*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n) = \prod_{m=1}^{\infty}(1-Q^m)^{-n-1} \cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n} \cdot Q^{\frac{1}{2}\boldsymbol{m}^{\mathrm{t}}\cdot C_\Delta \cdot \boldsymbol{m}}$$

where $Q = q_0^{d_0} q_1^{d_1}\cdots q_n^{d_n}$.

We note that under the specialization $q_i = q^{d_i}$,

$$Q = q^{d_0^2+\cdots+d_n^2} = q^k$$

where $k = |G|$ is the order of the group G.

We then obtain

$$Z_\Delta(q) = q^{\frac{-1}{24}} \cdot \prod_{m=1}^{\infty}(1-q^{km})^{-n-1} \cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^n} q^{\boldsymbol{m}^{\mathrm{t}}\cdot \boldsymbol{d}} \cdot q^{\frac{k}{2}\boldsymbol{m}^{\mathrm{t}}\cdot C_\Delta\cdot\boldsymbol{m}}$$

where $\boldsymbol{d} = (d_1, \ldots, d_n)$.

Let M_Δ be the root lattice of Δ which we identify with $\mathbb{Z}^n$ via the basis given by $\alpha_1, \ldots, \alpha_n$, the simple positive roots of Δ. Under this identification, the standard Weyl invariant bilinear form is given by

$$(\boldsymbol{u}|\boldsymbol{v}) = \boldsymbol{u}^{\mathsf{t}} \cdot C_\Delta \cdot \boldsymbol{v}$$

and $\boldsymbol{d}$ is identified with the longest root. We define

$$\boldsymbol{\zeta} = C_\Delta^{-1} \cdot \boldsymbol{d}$$

so that

$$\boldsymbol{m}^{\mathsf{t}} \cdot \boldsymbol{d} = \boldsymbol{m}^{\mathsf{t}} \cdot C_\Delta \cdot \boldsymbol{\zeta} = (\boldsymbol{m}|\boldsymbol{\zeta}).$$

We may then write

$$\begin{aligned} Z_\Delta(q) &= q^{\frac{-1}{24}} \cdot \prod_{m=1}^{\infty} (1-q^{km})^{-n-1} \cdot \sum_{\boldsymbol{m} \in M_\Delta} q^{(\boldsymbol{m}|\boldsymbol{\zeta}) + \frac{k}{2}(\boldsymbol{m}|\boldsymbol{m})} \\ &= q^A \cdot \left(q^{\frac{k}{24}} \prod_{m=1}^{\infty} (1-q^{km}) \right)^{-n-1} \cdot \sum_{\boldsymbol{m} \in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta} | \boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta}\right)} \\ &= q^A \cdot \eta(k\tau)^{-n-1} \cdot \theta_\Delta(\tau) \end{aligned}$$

where

$$A \quad = \quad \frac{-1}{24} + \frac{k(n+1)}{24} - \frac{1}{2k}(\boldsymbol{\zeta}|\boldsymbol{\zeta})$$

and $\theta_\Delta(\tau)$ is the shifted theta function:

$$\theta_\Delta(\tau) = \sum_{\boldsymbol{m} \in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta} | \boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta}\right)} \tag{2}$$

where as throughout this paper we have identified $q = \exp(2\pi i \tau)$.

In Appendix A, we will prove the following formula which for $\Delta = A_n$ coincides with the "strange formula" of Freudenthal and de Vries [6]:

$$\frac{k(n+1)-1}{24} = \frac{(\boldsymbol{\zeta}|\boldsymbol{\zeta})}{2k}.$$

It follows that $A = 0$ and we obtain the following:

Lemma 4.2. *The local series $Z_\Delta(q)$ is given by*

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{n+1}}.$$

5. Proof of Theorem 1.10

Let $\mathcal{Z} = [X/G]$ be the stack quotient of X by G and let $Y \to X/G$ be the minimal resolution. The Hilbert scheme of zero dimensional substacks of $\mathcal{Z}$ is naturally identified with the G-fixed Hilbert scheme of X:

$$\operatorname{Hilb}(\mathcal{Z}) \cong \operatorname{Hilb}(X)^G.$$

We emphasize that $\operatorname{Hilb}(\mathcal{Z})$ is itself a scheme, not just a stack, as the objects it parameterizes (substacks $V \subset \mathcal{Z}$) do not have automorphisms (see [21] or [2, § 2.3]). Components of $\operatorname{Hilb}(\mathcal{Z})$ are indexed by the numerical K-theory class of $\mathcal{O}_Z$ for $Z \subset \mathcal{Z}$. The K-theory class of $\mathcal{O}_Z$ can be written in a basis for K-theory as follows:

$$[\mathcal{O}_Z] = n[\mathcal{O}_p] + \sum_{i=1}^{r} \sum_{j=1}^{n(i)} m_j(i) [\mathcal{O}_{p_i} \otimes \rho_j(i)]$$

where $p \in \mathcal{Z}$ is a generic point and $p_1, \ldots, p_r \in \mathcal{Z}$ are the orbifold points. The local group of $\mathcal{Z}$ at p_i is $G_{\Delta(i)} \subset SU(2)$ and has corresponding root system $\Delta(i)$ of rank $n(i)$, and has irreducible representations $\rho_0(i), \rho_1(i), \ldots, \rho_{n(i)}(i)$ where $\rho_0(i)$ is the trivial representation. We note that we do not need to include $[\mathcal{O}_{p_i} \otimes \rho_0(i)]$ in our basis for K-theory because of the following relation in K-theory which holds for all i:

$$[\mathcal{O}_p] = [\mathcal{O}_{p_i} \otimes \rho_{\mathsf{reg}}(i)] \tag{3}$$

where $\rho_{\mathsf{reg}}(i)$ is the regular representation of $G_{\Delta(i)}$.

We abbreviate the data $\{m_j(i)\}$ appearing in the K-theory class above by the symbol $\mathfrak{m}$ and we denote by

$$\operatorname{Hilb}^{n,\mathfrak{m}}(\mathcal{Z}) \subset \operatorname{Hilb}(\mathcal{Z})$$

the corresponding component. Let

$$D_{\mathfrak{m}} = \sum_{i=1}^{r} \sum_{j=1}^{n(i)} m_j(i)\, E_j(i)$$

where $E_1(i), \ldots, E_{n(i)}(i)$ are the exceptional curves over p_i. We can organize the data $\mathfrak{m} = \{m_j(i)\}$ into $\boldsymbol{m}(i) \in M_{\Delta(i)}$, i.e. the vectors in the root lattice of $\Delta(i)$ having components $m_1(i), \ldots, m_{n(i)}(i)$. Under this identification

$$D_{\mathfrak{m}}^2 = -\sum_{i=1}^{r} (\boldsymbol{m}(i)|\boldsymbol{m}(i))_{\Delta(i)}$$

since the intersection form of the exceptional curves over p_i is the negative of the corresponding Cartan matrix $C_{\Delta(i)}$.

Proposition 5.1. $\operatorname{Hilb}^{n,\mathfrak{m}}(\mathcal{Z})$ *is birational to* $\operatorname{Hilb}^{n+\frac{1}{2}D_{\mathfrak{m}}^2}(Y)$.

To prove this we will first need the following lemma:

Lemma 5.2.

(1) *Let Δ be a rank n ADE root system, let M_Δ be the corresponding root lattice, and let $G_\Delta \subset SU(2)$ the corresponding finite subgroup. To any element $\boldsymbol{m} \in M_\Delta$ there is a unique rigid*[4] *substack $Z_{\boldsymbol{m}} \subset [\mathbb{C}^2/G_\Delta]$ with K-theory class*

$$\frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})[\mathcal{O}_p] + \sum_{j=1}^{n} m_j [\mathcal{O}_0 \otimes \rho_j]$$

where $p \in [\mathbb{C}^2/G]$ is a generic point.

(2) *For every datum $\mathfrak{m}$ there is a unique rigid substack $Z_{\mathfrak{m}} \subset \mathcal{Z}$ with K-theory class*

$$\begin{aligned} \sum_{i=1}^{r} &\left(\frac{1}{2}(\boldsymbol{m}(i)|\boldsymbol{m}(i))_{\Delta(i)}[\mathcal{O}_p] + \sum_{j=1}^{n(i)} m_j(i)[\mathcal{O}_{p_i} \otimes \rho_j(i)] \right) \\ &= -\frac{D_{\mathfrak{m}}^2}{2}[\mathcal{O}_p] + \sum_{j=1}^{r} \sum_{j=1}^{n(i)} m_j(i)[\mathcal{O}_{p_i} \otimes \rho_j(i)] \end{aligned}$$

where $p \in \mathcal{Z}$ is a generic point.

---

[4] By definition, a substack Z is *rigid* if it corresponds to an isolated point in the Hilbert scheme.

Proof. Part (2) is implied by Part (1) since we can take the union of the rigid subschemes supported at the orbifold points $p_1, \ldots, p_r \in \mathcal{Z}$. So we need only prove the local case.

To prove Part (1) we need to show that component of $\operatorname{Hilb}([\mathbb{C}^2/G_\Delta])$ corresponding to substacks with K-theory class

$$\frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})[\mathcal{O}_p] + \sum_{j=1}^{n} m_j[\mathcal{O}_0 \otimes \rho_j]$$

is a single isolated point. This component corresponds to the coefficient of

$$Q^{\frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})} \cdot q_1^{m_1} \cdots q_n^{m_n}$$

in Theorem 4.1. It follows immediately from the formula in Theorem 4.1 that this coefficient is 1, and thus to prove this component is a single point, we need only prove that it has dimension 0.

By equation (3), we have

$$\frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})[\mathcal{O}_p]+\sum_{j=1}^{n} m_j[\mathcal{O}_0\otimes\rho_j] = \frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})[\mathcal{O}_0\otimes\rho_0]+\sum_{j=1}^{n}\left(\frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})d_j + m_j\right)[\mathcal{O}_0\otimes\rho_j]$$

and so the component in question is

$$\operatorname{Hilb}^{v_0\rho_0+\cdots+v_n\rho_n}([\mathbb{C}^2/G_\Delta])$$

where

$$\boldsymbol{v} = (v_0, v_1, \ldots, v_n) = \left(\frac{1}{2}(\boldsymbol{m}|\boldsymbol{m}), \frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})d_1 + m_1, \ldots, \frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})d_n + m_n\right).$$

We define

$$\begin{aligned} \boldsymbol{\delta} &= (1, d_1, \ldots, d_n), \text{ and} \\ \boldsymbol{\mu} &= (0, m_1, \ldots, m_n) \end{aligned}$$

so that our $\boldsymbol{v}$ of interest may be written

$$\boldsymbol{v} = \frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})\boldsymbol{\delta} + \boldsymbol{\mu}.$$

Nakajima has shown [19, § 2] that

$$\operatorname{Hilb}^{v_0\rho_0+\cdots+v_n\rho_n}([\mathbb{C}^2/G_\Delta]) = M(\boldsymbol{v}, \boldsymbol{w})$$

where $\boldsymbol{w} = (1, 0, \ldots, 0)$ and $M(\boldsymbol{v}, \boldsymbol{w})$ is the Nakajima quiver variety associated to the affine Dynkin diagram of Δ with framing vector $\boldsymbol{w}$ and dimension vector $\boldsymbol{v}$. By [18, (2.6)] we have

$$\begin{aligned} \dim M(\boldsymbol{v}, \boldsymbol{w}) &= 2\boldsymbol{v} \cdot \boldsymbol{w} - \langle \boldsymbol{v}, \boldsymbol{v} \rangle \\ &= 2v_0 - \langle \boldsymbol{v}, \boldsymbol{v} \rangle \\ &= (\boldsymbol{m}|\boldsymbol{m}) - \langle \boldsymbol{v}, \boldsymbol{v} \rangle \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product given by the Cartan matrix associated to the affine Dynkin diagram.

We also have

$$\langle \boldsymbol{\mu}, \boldsymbol{\mu} \rangle = (\boldsymbol{m}|\boldsymbol{m}), \qquad \langle \boldsymbol{\delta}, \boldsymbol{\delta} \rangle = 0, \qquad \langle \boldsymbol{\mu}, \boldsymbol{\delta} \rangle = 0.$$

The first follows directly from our definitions, and the later two are well known properties of the vector $\boldsymbol{\delta}$. Using the above we compute the dimension of the Hilbert scheme of interest:

$$\begin{aligned}\dim M(\boldsymbol{v},\boldsymbol{w}) &= (\boldsymbol{m}|\boldsymbol{m}) - \left\langle \frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})\boldsymbol{\delta}+\boldsymbol{\mu}, \frac{1}{2}(\boldsymbol{m}|\boldsymbol{m})\boldsymbol{\delta}+\boldsymbol{\mu}\right\rangle \\ &= (\boldsymbol{m}|\boldsymbol{m}) - \frac{1}{4}(\boldsymbol{m}|\boldsymbol{m})\langle\boldsymbol{\delta},\boldsymbol{\delta}\rangle - (\boldsymbol{m}|\boldsymbol{m})\langle\boldsymbol{\mu},\boldsymbol{\delta}\rangle - \langle\boldsymbol{\mu},\boldsymbol{\mu}\rangle \\ &= 0.\end{aligned}$$

We thus can conclude that $\mathrm{Hilb}^{v_0\rho_0+\cdots+v_n\rho_n}([\mathbb{C}^2/G_\Delta]) = M(\boldsymbol{v},\boldsymbol{w})$ is a single point which finishes the proof of the lemma. $\square$

Proof of Proposition 5.1. Let $U = \mathcal{Z}\setminus\{p_1,\ldots,p_r\}$ be the Zariski open part with trivial stabilizers. to the scheme W. Let $V\subset Y$ be the complement of the exceptional divisors. Let furthermore $Z_{\mathfrak{m}}\subset\mathcal{Z}$ be the rigid substack corresponding to the K-theory datum $\mathfrak{m}$ provided by Lemma 5.2. The Zariski open substack of $\mathrm{Hilb}^{n,\mathfrak{m}}(\mathcal{Z})$ parameterizing substacks of $\mathcal{Z}$ of the form $P\cup Z_{\mathfrak{m}}$ where P is a colength $n+\frac{1}{2}D^2_{\mathfrak{m}}$ subscheme of U is isomorphic to $\mathrm{Hilb}^{n+\frac{1}{2}D^2_{\mathfrak{m}}}(U)$. This is because $Z_{\mathfrak{m}}$ was rigid and it had the K-theory class $-\frac{1}{2}D^2_{\mathfrak{m}}[\mathcal{O}_p]+\sum_{j=1}^n m_j(i)[\mathcal{O}_{p_i}\otimes\rho_j(i)]$. On the other hand, the Zariski open subset of $\mathrm{Hilb}^{n+\frac{1}{2}D^2_{\mathfrak{m}}}(Y)$ parameterizing subschemes supported on $V\subset Y$ is isomorphic to $\mathrm{Hilb}^{n+\frac{1}{2}D^2_{\mathfrak{m}}}(V)$. Finally, $\mathrm{Hilb}^{n+\frac{1}{2}D^2_{\mathfrak{m}}}(U)\cong\mathrm{Hilb}^{n+\frac{1}{2}D^2_{\mathfrak{m}}}(V)$ since U and V are canonically isomorphic. $\square$

With Proposition 5.1, we can now prove Theorem 1.10. Using the identification

$$\mathrm{Hilb}(X)^G\cong\mathrm{Hilb}(\mathcal{Z})$$

and identifying discrete parameters we get

$$\begin{aligned}Z^{\mathsf{bir}}_{X,G}(q) &= \sum_{a=0}^{\infty}\left[\mathrm{Hilb}^a(X)^G\right]_{\mathsf{bir}}\ q^{a-1} \\ &= \sum_{n,\mathfrak{m}}\left[\mathrm{Hilb}^{n,\mathfrak{m}}(\mathcal{Z})\right]_{\mathsf{bir}}\ q^{D(n,\mathfrak{m})-1}\end{aligned}$$

where (recalling that $d_j(i)=\dim\rho_j(i)$),

$$D(n,\mathfrak{m}) = kn+\sum_{i=1}^r\frac{k}{k_i}\sum_{j=1}^{n(i)}m_j(i)d_j(i).$$

Let

$$d = n+\frac{1}{2}D^2_{\mathfrak{m}} = n-\frac{1}{2}\sum_{i=1}^r(\boldsymbol{m}(i)|\boldsymbol{m}(i))_{\Delta(i)}.$$

Then

$$Z^{\mathsf{bir}}_{X,G}(q) = \sum_{d=0}\left[\mathrm{Hilb}^d(Y)\right]_{\mathsf{bir}}\prod_{i=1}^r\sum_{\boldsymbol{m}(i)\in M_{\Delta(i)}} q^{D(n,\mathfrak{m})-1}$$

with

$$D(n,\mathfrak{m})-1 = -1+k\left(d+\frac{1}{2}\sum_{i=1}^r(\boldsymbol{m}(i)|\boldsymbol{m}(i))_{\Delta(i)}\right)+\sum_{i=1}^r\frac{k}{k_i}\sum_{j=1}^{n(i)}m_j(i)d_j(i)$$

$$= kd - 1 + \frac{k}{2}\sum_{i=1}^{r}\left\{(\boldsymbol{m}(i)|\boldsymbol{m}(i))_{\Delta(i)} + \frac{2}{k_i}\,(\boldsymbol{m}(i)|\boldsymbol{\zeta}(i))_{\Delta(i)}\right\}$$

where $\boldsymbol{\zeta}(i) \in M_{\Delta(i)} \otimes \mathbb{Q}$ is as in Section 4.

Completing the square and using the formula

$$\frac{1}{k_i^2}\,(\boldsymbol{\zeta}(i)|\boldsymbol{\zeta}(i))_{\Delta(i)} = \frac{2}{k_i}\left(\frac{k_i(n(i)+1)-1}{24}\right),$$

which follows from Lemma A.1, we get

$$D(n,\mathfrak{m}) - 1 = kd - 1$$
$$-\sum_{i=1}^{r}\frac{k}{k_i}\left(\frac{k_i(n(i)+1)-1}{24}\right) + \frac{k}{2}\sum_{i=1}^{r}\left(\boldsymbol{m}(i) + \frac{1}{k_i}\boldsymbol{\zeta}(i)\;\Big|\;\boldsymbol{m}(i) + \frac{1}{k_i}\boldsymbol{\zeta}(i)\right)_{\Delta(i)}.$$

It then follows that

$$Z^{\mathsf{bir}}_{X,G}(q) =$$
$$q^A\sum_{d=0}^{\infty}\left[\mathrm{Hilb}^d(Y)\right]_{\mathsf{bir}}\; q^{kd-k}\;\prod_{i=1}^{r}\;\sum_{\boldsymbol{m}(i)\in M_{\Delta(i)}} q^{\frac{k}{2}\left(\boldsymbol{m}(i)+\frac{1}{k_i}\boldsymbol{\zeta}(i)\;\big|\;\boldsymbol{m}(i)+\frac{1}{k_i}\boldsymbol{\zeta}(i)\right)_{\Delta(i)}}$$

where

$$A = k - 1 - \frac{k}{24}\sum_{i=1}^{r}\left(n(i) + 1 - \frac{1}{k_i}\right).$$

Since

$$\begin{aligned}
24 = e(Y) &= e\left(X/G - \{p_1,\dots,p_r\}\right) + \sum_{i=1}^{r}(n(i)+1)\\
&= \frac{1}{k}\left(24 - \sum_{i=1}^{r}\frac{k}{k_i}\right) + \sum_{i=1}^{r}(n(i)+1)\\
&= \frac{24}{k} + \sum_{i=1}^{r}\left(n(i) + 1 - \frac{1}{k_i}\right)
\end{aligned}$$

we see that $A = 0$.

Thus we have

$$\begin{aligned}
Z^{\mathsf{bir}}_{X,G}(q) &= Z^{\mathsf{bir}}_{Y}(q^k)\prod_{i=1}^{r}\theta_{\Delta(i)}\left(\frac{k\tau}{k_i}\right)\\
&= Z^{\mathsf{bir}}_{Y}(q^k)\prod_{i=1}^{r}Z_{\Delta(i)}\left(\frac{k\tau}{k_i}\right)\eta(k\tau)^{n(i)+1}\\
&= Z^{\mathsf{bir}}_{Y}(q^k)\cdot\eta(k\tau)^B\cdot Z_{X,G}(q)
\end{aligned}$$

where we've used Theorem 1.3, Theorem 1.5, and we've set

$$B = \frac{24}{k} + \sum_{i=1}^{r}\left(n(i) + 1 - \frac{1}{k_i}\right).$$

The previous equation which showed that $A = 0$ also shows that $B = 24$. Then since $\Delta(\tau) = \eta(\tau)^{24}$, we see that Theorem 1.10 follows. $\square$

APPENDIX A. ANOTHER STRANGE FORMULA

We recall the notation from Section 4. Let Δ be an ADE root system of rank n. Let $\alpha_1, \ldots \alpha_n$ be a system of positive simple roots and let

$$\boldsymbol{d} = \sum_{i=1}^{n} d_i \alpha_i$$

be the largest root. Let $(\cdot|\cdot)$ be the Weyl invariant bilinear form with $(\alpha_i|\alpha_i) = 2$ and let $\boldsymbol{\zeta}$ be the dual vector to $\boldsymbol{d}$ in the sense that

$$\sum_{i=1}^{n} (\boldsymbol{\zeta}|\alpha_i)\alpha_i = \boldsymbol{d}. \tag{4}$$

Let

$$k = 1 + \sum_{i=1}^{n} d_i^2 = 1 + (\boldsymbol{\zeta}|\boldsymbol{d}). \tag{5}$$

The identity of the following lemma coincides with Freudenthal and de Vries's "strange formula" when Δ is A_n.

Lemma A.1. *Let k, n, and $\boldsymbol{\zeta}$ be as above. Then,*

$$\frac{k(n+1)-1}{24} = \frac{(\boldsymbol{\zeta}|\boldsymbol{\zeta})}{2k}.$$

Proof. **The case of $\Delta = A_n$:** For any ADE root system we have $(\boldsymbol{\rho}|\alpha) = 1$ for all positive roots where $\boldsymbol{\rho} = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ is half the sum of the positive roots. Since for A_n, $d_i = 1$, it follows from equation (4) that $\boldsymbol{\zeta} = \boldsymbol{\rho}$, and it follows from equation (5) that $k = n+1 = h$ is the Coxeter number. The lemma is then

$$\frac{(n+1)^2-1}{24} = \frac{(\boldsymbol{\rho}|\boldsymbol{\rho})}{2h}.$$

Since the Lie algebra associated to A_n, namely $\mathfrak{sl}_{n+1}$, has dimension $(n+1)^2 - 1$ and the Killing form satisfies $\kappa(\cdot,\cdot) = \frac{1}{2h}(\cdot|\cdot)$, the lemma may be rewritten as

$$\frac{\dim \mathfrak{sl}_{n+1}}{24} = \kappa(\boldsymbol{\rho}, \boldsymbol{\rho})$$

which is Freudenthal and de Vries's "strange formula" [6, 47.11].

The case of $\Delta = D_n$: Let $e_1, \ldots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$. Then the collection $\{\pm e_i \pm e_j, i < j\}$ is a D_n root system and we may take

$$\alpha_i = \begin{cases} e_i - e_{i+1}, & i = 1, \ldots, n-1 \\ e_{n-1} + e_n, & i = n \end{cases}$$

as a system of simple positive roots. Then the fundamental weights ω_i, which are defined by the condition $(\omega_i|\alpha_j) = \delta_{ij}$, are given by [14, Appendix C]

$$\omega_i = \begin{cases} e_1 + \cdots + e_i, & i \leq n-2 \\ \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n), & i = n-1 \\ \frac{1}{2}(e_1 + \cdots + e_{n-1} + e_n), & i = n. \end{cases}$$

Then since

$$d_i = \begin{cases} 1 & i = 1, n-1, n, \\ 2 & i = 2, \ldots, n-2, \end{cases}$$

we have

$$\begin{aligned}\boldsymbol{\zeta} &= \omega_1 + 2\omega_2 + 2\omega_3 + \cdots + 2\omega_{n-2} + \omega_{n-1} + \omega_n \\ &= 2(n-2)e_1 + \sum_{i=2}^{n-2} \left(2(n-1-i)+1\right)\, e_i + e_{n-1}\end{aligned}$$

and so

$$\begin{aligned}(\boldsymbol{\zeta}|\boldsymbol{\zeta}) &= 4(n-2)^2 + 1 + \sum_{i=2}^{n-2} \left(2(n-1-i)+1\right)^2 \\ &= \frac{4}{3}n^3 - 4n^2 - \frac{1}{3}n + 6.\end{aligned}$$

Finally since $k = 1 + \sum_{i=1}^{n} d_i = 4(n-2)$ the lemma becomes

$$\frac{4(n-2)(n+1)-1}{24} = \frac{\frac{4}{3}n^3 - 4n^2 - \frac{1}{3}n + 6}{8(n-2)}$$

which is readily verified.

The case of $\Delta = E_6, E_7, E_8$: These three individual cases are easily checked one by one.

$\square$

APPENDIX B. TABLE OF ETA PRODUCTS

The following table provides the list of the modular forms $Z_{X,G}^{-1}$, expressed as eta products, for each of the 82 possible symplectic actions of a group G on a $K3$ surface X. Our numbering matches Xiao's [24] whose table we refer to for a description of each group.

#	$\lvert G\rvert$	Singularities of X/G	The modular form $Z_{X,G}^{-1}$	Weight
0	1		$\eta(\tau)^{24}$	12
1	2	$8A_1$	$\eta(2\tau)^8\,\eta(\tau)^8$	8
2	3	$6A_2$	$\eta(3\tau)^6\,\eta(\tau)^6$	6
3	4	$12A_1$	$\eta(2\tau)^{12}$	6
4	4	$2A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^2\,\eta(\tau)^4$	5
5	5	$4A_4$	$\eta(5\tau)^4\,\eta(\tau)^4$	4
6	6	$8A_1+3A_2$	$\frac{\eta(3\tau)^8\eta(2\tau)^3}{\eta(6\tau)}$	5
7	6	$2A_1+2A_2+2A_5$	$\eta(6\tau)^2\,\eta(3\tau)^2\,\eta(2\tau)^2\,\eta(\tau)^2$	4
8	7	$3A_6$	$\eta(7\tau)^3\,\eta(\tau)^3$	3
9	8	$14A_1$	$\frac{\eta(4\tau)^{14}}{\eta(8\tau)^4}$	5
10	8	$9A_1+2A_3$	$\frac{\eta(4\tau)^9\eta(2\tau)^2}{\eta(8\tau)^2}$	9/2
11	8	$4A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^4$	4
12	8	$3A_3+2D_4$	$\frac{\eta(\tau)^2\eta(4\tau)^6}{\eta(2\tau)}$	7/2
13	8	A_1+4D_4	$\frac{\eta(4\tau)^{13}\eta(\tau)^4}{\eta(8\tau)^2\eta(2\tau)^8}$	7/2
14	8	$A_1+A_3+2A_7$	$\eta(8\tau)^2\,\eta(4\tau)\,\eta(2\tau)\,\eta(\tau)^2$	3
15	9	$8A_2$	$\eta(3\tau)^8$	4
16	10	$8A_1+2A_4$	$\frac{\eta(5\tau)^8\eta(2\tau)^2}{\eta(10\tau)^2}$	4
17	12	$4A_1+6A_2$	$\frac{\eta(6\tau)^4\eta(4\tau)^6}{\eta(12\tau)^2}$	4
18	12	$9A_1+A_2+A_5$	$\frac{\eta(6\tau)^9\eta(4\tau)\eta(2\tau)}{\eta(12\tau)^3}$	4
19	12	$3A_1+3A_5$	$\eta(6\tau)^3\,\eta(2\tau)^3$	3
20	12	$A_2+2A_3+2D_5$	$\frac{\eta(4\tau)^3\eta(3\tau)^2\eta(\tau)^2\eta(6\tau)^4}{\eta(12\tau)\eta(2\tau)^4}$	3
21	16	$15A_1$	$\frac{\eta(8\tau)^{15}}{\eta(16\tau)^6}$	9/2
22	16	$10A_1+2A_3$	$\frac{\eta(8\tau)^{10}\eta(4\tau)^2}{\eta(16\tau)^4}$	4
23	16	$5A_1+4A_3$	$\frac{\eta(8\tau)^5\eta(4\tau)^4}{\eta(16\tau)^2}$	7/2
24	16	$6A_1+A_3+2D_4$	$\frac{\eta(8\tau)^{12}\eta(2\tau)^2}{\eta(16\tau)^4\eta(4\tau)^3}$	7/2
25	16	$6A_3$	$\eta(4\tau)^6$	3
26	16	$4A_1+A_3+A_7+D_4$	$\frac{\eta(8\tau)^7\eta(2\tau)^2}{\eta(16\tau)^2\eta(4\tau)}$	3

27	16	$2A_1+4D_4$	$\frac{\eta(8\tau)^{14}\eta(2\tau)^4}{\eta(4\tau)^8\eta(16\tau)^4}$	3
28	16	$2A_1+A_3+2A_7$	$\eta(8\tau)^2\,\eta(4\tau)\,\eta(2\tau)^2$	5/2
29	16	$A_3+D_4+2D_6$	$\frac{\eta(4\tau)\eta(8\tau)^7\eta(\tau)^2}{\eta(16\tau)^2\eta(2\tau)^3}$	5/2
30	18	$8A_1+4A_2$	$\frac{\eta(9\tau)^8\eta(6\tau)^4}{\eta(18\tau)^4}$	4
31	18	$2A_1+3A_2+2A_5$	$\frac{\eta(9\tau)^2\eta(6\tau)^3\eta(3\tau)^2}{\eta(18\tau)}$	3
32	20	$2A_1+4A_3+A_4$	$\frac{\eta(10\tau)^2\eta(5\tau)^4\eta(4\tau)}{\eta(20\tau)}$	3
33	21	$6A_2+A_6$	$\frac{\eta(7\tau)^6\eta(3\tau)}{\eta(21\tau)}$	3
34	24	$5A_1+3A_2+2A_3$	$\frac{\eta(12\tau)^5\eta(8\tau)^3\eta(6\tau)^2}{\eta(24\tau)^3}$	7/2
35	24	$4A_1+2A_2+2A_5$	$\frac{\eta(12\tau)^4\eta(8\tau)^2\eta(4\tau)^2}{\eta(24\tau)^2}$	3
36	24	$5A_1+A_3+A_5+D_5$	$\frac{\eta(12\tau)^7\eta(6\tau)\eta(2\tau)\eta(8\tau)}{\eta(24\tau)^3\eta(4\tau)}$	3
37	24	$2A_2+A_5+D_4+E_6$	$\frac{\eta(8\tau)^4\eta(4\tau)\eta(3\tau)\eta(12\tau)^4\eta(\tau)}{\eta(6\tau)^2\eta(24\tau)^2\eta(2\tau)^2}$	5/2
38	24	$2A_2+A_3+2E_6$	$\frac{\eta(8\tau)^6\eta(6\tau)\eta(\tau)^2\eta(12\tau)^2}{\eta(2\tau)^4\eta(24\tau)^2}$	5/2
39	32	$8A_1+3A_3$	$\frac{\eta(16\tau)^8\eta(8\tau)^3}{\eta(32\tau)^4}$	7/2
40	32	$9A_1+2D_4$	$\frac{\eta(16\tau)^{15}\eta(4\tau)^2}{\eta(32\tau)^6\eta(8\tau)^4}$	7/2
41	32	$3A_1+5A_3$	$\frac{\eta(16\tau)^3\eta(8\tau)^5}{\eta(32\tau)^2}$	3
42	32	$4A_1+2A_3+2D_4$	$\frac{\eta(16\tau)^{10}\eta(4\tau)^2}{\eta(32\tau)^4\eta(8\tau)^2}$	3
43	32	$5A_1+2A_7$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
44	32	$2A_1+2A_3+A_7+D_4$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
45	32	$3A_1+D_4+2D_6$	$\frac{\eta(16\tau)^{10}\eta(2\tau)^2}{\eta(32\tau)^4\eta(4\tau)^3}$	5/2
46	36	$2A_1+2A_2+4A_3$	$\frac{\eta(18\tau)^2\eta(12\tau)^2\eta(9\tau)^4}{\eta(36\tau)^2}$	3
47	36	$A_1+6A_2+A_5$	$\frac{\eta(18\tau)\eta(12\tau)^6\eta(6\tau)}{\eta(36\tau)^2}$	3
48	36	$6A_1+A_2+2A_5$	$\frac{\eta(18\tau)^6\eta(12\tau)\eta(6\tau)^2}{\eta(36\tau)^3}$	3
49	48	$5A_1+6A_2$	$\frac{\eta(24\tau)^5\eta(16\tau)^6}{\eta(48\tau)^4}$	7/2
50	48	$6A_2+2A_3$	$\frac{\eta(16\tau)^6\eta(12\tau)^2}{\eta(48\tau)^2}$	3
51	48	$5A_1+A_2+2A_3+A_5$	$\frac{\eta(24\tau)^5\eta(16\tau)\eta(12\tau)^2\eta(8\tau)}{\eta(48\tau)^3}$	3
52	48	$4A_1+3A_5$	$\frac{\eta(24\tau)^4\eta(8\tau)^3}{\eta(48\tau)^2}$	5/2
53	48	$A_1+A_2+2A_3+2D_5$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(12\tau)^2\eta(4\tau)^2}{\eta(48\tau)^3\eta(8\tau)^4}$	5/2
54	48	$4A_1+A_2+A_7+E_6$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(6\tau)\eta(2\tau)}{\eta(48\tau)^3\eta(4\tau)^2}$	5/2
55	60	$4A_1+3A_2+2A_4$	$\frac{\eta(30\tau)^4\eta(20\tau)^3\eta(12\tau)^2}{\eta(60\tau)^3}$	3
56	64	$5A_1+3A_3+D_4$	$\frac{\eta(32\tau)^8\eta(16\tau)\eta(8\tau)}{\eta(64\tau)^4}$	3
57	64	$6A_1+3D_4$	$\frac{\eta(32\tau)^{15}\eta(8\tau)^3}{\eta(64\tau)^6\eta(16\tau)^6}$	3

58	64	$3A_1+3A_3+A_7$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
59	64	$5A_3+D_4$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
60	64	$4A_1+A_3+2D_6$	$\frac{\eta(32\tau)^8\eta(16\tau)^3\eta(4\tau)^2}{\eta(64\tau)^4\eta(8\tau)^4}$	5/2
61	72	$4A_1+3A_2+A_3+D_5$	$\frac{\eta(36\tau)^6\eta(24\tau)^4\eta(18\tau)\eta(6\tau)}{\eta(72\tau)^4\eta(12\tau)^2}$	3
62	72	$3A_1+2A_3+2A_5$	$\frac{\eta(36\tau)^3\eta(18\tau)^2\eta(12\tau)^2}{\eta(72\tau)^2}$	5/2
63	72	$A_2+3A_3+2D_4$	$\frac{\eta(24\tau)\eta(9\tau)^2\eta(36\tau)^6}{\eta(72\tau)^3\eta(18\tau)}$	5/2
64	80	$3A_1+4A_4$	$\frac{\eta(40\tau)^3\eta(16\tau)^4}{\eta(80\tau)^2}$	5/2
65	96	$3A_1+3A_2+3A_3$	$\frac{\eta(48\tau)^3\eta(32\tau)^3\eta(24\tau)^3}{\eta(96\tau)^3}$	3
66	96	$2A_1+2A_2+A_3+2A_5$	$\frac{\eta(48\tau)^2\eta(32\tau)^2\eta(24\tau)\eta(16\tau)^2}{\eta(96\tau)^2}$	5/2
67	96	$2A_1+3A_2+A_7+D_4$	$\frac{\eta(48\tau)^5\eta(32\tau)^3\eta(12\tau)^2}{\eta(96\tau)^3\eta(24\tau)^2}$	5/2
68	96	$3A_1+2A_3+A_5+D_5$	$\frac{\eta(48\tau)^5\eta(24\tau)^2\eta(8\tau)\eta(32\tau)}{\eta(96\tau)^3\eta(16\tau)}$	5/2
69	96	$3A_1+2A_2+2E_6$	$\frac{\eta(48\tau)^5\eta(32\tau)^6\eta(4\tau)^2}{\eta(96\tau)^4\eta(8\tau)^4}$	5/2
70	120	$2A_1+A_2+2A_3+A_4+A_5$	$\frac{\eta(60\tau)^2\eta(40\tau)\eta(30\tau)^2\eta(24\tau)\eta(20\tau)}{\eta(120\tau)^2}$	5/2
71	128	$3A_1+2A_3+D_4+D_6$	$\frac{\eta(64\tau)^8\eta(32\tau)\eta(8\tau)}{\eta(128\tau)^4\eta(16\tau)}$	5/2
72	144	$A_1+4A_2+2A_5$	$\frac{\eta(72\tau)\eta(48\tau)^4\eta(24\tau)^2}{\eta(144\tau)^2}$	5/2
73	160	$2A_1+3A_3+2A_4$	$\frac{\eta(80\tau)^2\eta(40\tau)^3\eta(32\tau)^2}{\eta(160\tau)^2}$	5/2
74	168	$A_1+3A_2+2A_3+A_6$	$\frac{\eta(84\tau)\eta(56\tau)^3\eta(42\tau)^2\eta(24\tau)}{\eta(168\tau)^2}$	5/2
75	192	$2A_1+6A_2+D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)^6\eta(24\tau)}{\eta(192\tau)^4\eta(48\tau)^2}$	3
76	192	$2A_1+A_2+2A_3+A_5+D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)\eta(32\tau)\eta(24\tau)}{\eta(192\tau)^3}$	5/2
77	192	$2A_1+A_2+3A_3+E_6$	$\frac{\eta(96\tau)^3\eta(64\tau)^3\eta(48\tau)^3\eta(8\tau)}{\eta(192\tau)^3\eta(16\tau)^2}$	5/2
78	288	$2A_1+2A_2+A_3+2D_5$	$\frac{\eta(144\tau)^6\eta(96\tau)^4\eta(72\tau)\eta(24\tau)^2}{\eta(288\tau)^4\eta(48\tau)^4}$	5/2
79	360	$A_1+2A_2+2A_3+2A_4$	$\frac{\eta(180\tau)\eta(120\tau)^2\eta(90\tau)^2\eta(72\tau)^2}{\eta(360\tau)^2}$	5/2
80	384	$A_1+3A_2+2A_3+D_6$	$\frac{\eta(192\tau)^3\eta(128\tau)^3\eta(96\tau)^3\eta(24\tau)}{\eta(384\tau)^3\eta(48\tau)^2}$	5/2
81	960	$A_1+3A_2+2A_4+D_4$	$\frac{\eta(480\tau)^4\eta(320\tau)^3\eta(192\tau)^2\eta(120\tau)}{\eta(960\tau)^3\eta(240\tau)^2}$	5/2

Table 1: Table of the modular forms $Z_{X,G}^{-1}$ for all symplectic G actions.

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