

G -fixed Hilbert schemes on $K3$ surfaces and modular forms.

Jim Bryan and Ádám Gyenge

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Abstract

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Let

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X)^G) q^{n-1}$$

be the generating function for the Euler characteristics of Hilbert scheme of G -invariant length n subschemes. We show that its reciprocal, $Z_{X,G}(q)^{-1}$ is the Fourier expansion of a modular cusp form of weight $\frac{1}{2}e(X/G)$ and index $|G|$. We give an explicit formula for $Z_{X,G}$ in terms of the Dedekind eta function for all 82 possible (X, G) .

1 Introduction

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Mukai showed that such G are precisely the subgroups of the Mathieu group $M_{23} \subset M_{24}$ such that the induced action on the set $\{1, \dots, 24\}$ has at least five orbits [?]. Xiao classified all possible actions into 82 possible topological types of the quotient X/G [?].

The G -fixed Hilbert scheme of X parameterizes G -invariant length n subschemes $Z \subset X$. It can be identified with the G -fixed point locus in the Hilbert scheme of points:

$$\mathrm{Hilb}^n(X)^G \subset \mathrm{Hilb}^n(X)$$

We define the corresponding G -fixed partition function of X by

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X)^G) q^{n-1}$$

where $e(-)$ is topological Euler characteristic.

Throughout this paper we set

$$q = \exp(2\pi i\tau)$$

so that we may regard $Z_{X,G}$ as a function of $\tau \in \mathbb{H}$ where \mathbb{H} is the upper half-plane.

Our main result is the following:

Theorem 1. *The function $Z_{X,G}(q)^{-1}$ is a modular cusp form¹ of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$.*

Our theorem specializes in the case where G is the trivial group to a famous result of Göttsche [?]. The case where G is a cyclic group was proved in [?]. One can interpret our result as an instance of the Vafa-Witten S-duality conjecture for the orbifold $[X/G]$ (see Remark ???). The partition function $Z_{X,G}(q)$ also has an interpretation in enumerative geometry: its coefficients count G -invariant rational curves on X (see Remark ???).

We also give an explicit formula for $Z_{X,G}(q)$ in terms of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

as follows. Let p_1, \dots, p_r be the singular points of X/G and let G_1, \dots, G_r be the corresponding stabilizer subgroups of G . The singular points are necessarily of ADE type: they are locally given by \mathbb{C}^2/G_i where $G_i \subset SU(2)$. Finite subgroups of $SU(2)$ have an ADE classification and we let $\Delta_1, \dots, \Delta_r$ denote the corresponding ADE root systems.

For any finite subgroup $G_\Delta \subset SU(2)$ with associated root system Δ we define the *local G_Δ -fixed partition function* by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e(\text{Hilb}^n(\mathbb{C}^2)^{G_\Delta}) q^{n - \frac{1}{24}}.$$

We will prove in Theorem ?? that

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{N+1}}$$

where $\theta_\Delta(\tau)$ is a certain theta function associated to the root system of Δ , N is the rank of the root system, and $k = |G_\Delta|$.

The 82 possible collections of ADE root systems $\Delta_1, \dots, \Delta_r$ associated to (X, G) a $K3$ surface with a symplectic G action, are given in table ??? and we note that $\Delta_i \in \{A_1, \dots, A_7, D_4, D_5, D_6, E_6\}$. We let $k = |G|$, $k_i = |G_i|$, and

$$a = \frac{24}{k} - \sum_{i=1}^r \frac{1}{k_i} = e(X/G) - r.$$

Theorem 2. *With the above notation we have*

$$Z_{X,G}(q) = \eta^{-a}(k\tau) \prod_{i=1}^r Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right)$$

¹See section § ?? for notation and definitions regarding modular forms.

where

$$\begin{aligned} Z_{A_n}(\tau) &= \frac{1}{\eta(\tau)}, \quad n \geq 1 \\ Z_{D_n}(\tau) &= \frac{\eta^2(2\tau)\eta((4n-8)\tau)}{\eta(\tau)\eta(4\tau)\eta^2((2n-4)\tau)}, \quad 4 \leq n \leq 6 \\ Z_{E_6}(\tau) &= \frac{\eta^2(2\tau)\eta(24\tau)}{\eta(\tau)\eta^2(8\tau)\eta(12\tau)} \end{aligned}$$

We conjecture in ?? that the formula for Z_{D_n} holds for all $n \geq 4$ and we provide explicit conjectural formulas for Z_{E_7} and Z_{E_8} . In table 1 we have listed explicitly the eta product of the modular form $(Z_{X,G})^{-1}$ for all 82 possible cases of (X, G) .

1.1 Examples.

1.2 Interpretation of results.

1.3 Structure of the paper

A Table of eta products

The following table provides the list of modular forms, expressed as eta products, attached to the 82 possible symplectic actions on a $K3$ surface. We follow the numbering in Xiao's list [?].

Xiao #	$ G $	Singularities of X/G	The modular form $Z_{X,G}^{-1}$	Weight
0	1		$\eta(\tau)^{24}$	12
1	2	$8A_1$	$\eta(2\tau)^8 \eta(\tau)^8$	8
2	3	$6A_2$	$\eta(3\tau)^6 \eta(\tau)^6$	6
3	4	$12A_1$	$\eta(2\tau)^{12}$	6
4	4	$2A_1 + 4A_3$	$\eta(4\tau)^4 \eta(2\tau)^2 \eta(\tau)^4$	5
5	5	$4A_4$	$\eta(5\tau)^4 \eta(\tau)^4$	4
6	6	$8A_1 + 3A_2$	$\frac{\eta(3\tau)^8 \eta(2\tau)^3}{\eta(6\tau)}$	5
7	6	$2A_1 + 2A_2 + 2A_5$	$\eta(6\tau)^2 \eta(3\tau)^2 \eta(2\tau)^2 \eta(\tau)^2$	4
8	7	$3A_6$	$\eta(7\tau)^3 \eta(\tau)^3$	3
9	8	$14A_1$	$\frac{\eta(4\tau)^{14}}{\eta(8\tau)^4}$	5
10	8	$9A_1 + 2A_3$	$\frac{\eta(4\tau)^9 \eta(2\tau)^2}{\eta(8\tau)^2}$	9/2
11	8	$4A_1 + 4A_3$	$\eta(4\tau)^4 \eta(2\tau)^4$	4
12	8	$3A_3 + 2D_4$	$\frac{\eta(\tau)^2 \eta(4\tau)^6}{\eta(2\tau)}$	7/2
13	8	$A_1 + 4D_4$	$\frac{\eta(4\tau)^{13} \eta(\tau)^4}{\eta(8\tau)^2 \eta(2\tau)^8}$	7/2
14	8	$A_1 + A_3 + 2A_7$	$\eta(8\tau)^2 \eta(4\tau) \eta(2\tau) \eta(\tau)^2$	3
15	9	$8A_2$	$\eta(3\tau)^8$	4
16	10	$8A_1 + 2A_4$	$\frac{\eta(5\tau)^8 \eta(2\tau)^2}{\eta(10\tau)^2}$	4
17	12	$4A_1 + 6A_2$	$\frac{\eta(6\tau)^4 \eta(4\tau)^6}{\eta(12\tau)^2}$	4
18	12	$9A_1 + A_2 + A_5$	$\frac{\eta(6\tau)^9 \eta(4\tau) \eta(2\tau)}{\eta(12\tau)^3}$	4
19	12	$3A_1 + 3A_5$	$\eta(6\tau)^3 \eta(2\tau)^3$	3
20	12	$A_2 + 2A_3 + 2D_5$	$\frac{\eta(4\tau)^3 \eta(3\tau)^2 \eta(\tau)^2 \eta(6\tau)^4}{\eta(12\tau) \eta(2\tau)^4}$	3
21	16	$15A_1$	$\frac{\eta(8\tau)^{15}}{\eta(16\tau)^6}$	9/2
22	16	$10A_1 + 2A_3$	$\frac{\eta(8\tau)^{10} \eta(4\tau)^2}{\eta(16\tau)^4}$	4
23	16	$5A_1 + 4A_3$	$\frac{\eta(8\tau)^5 \eta(4\tau)^4}{\eta(16\tau)^2}$	7/2
24	16	$6A_1 + A_3 + 2D_4$	$\frac{\eta(8\tau)^{12} \eta(2\tau)^2}{\eta(16\tau)^4 \eta(4\tau)^3}$	7/2

25	16	$6A_3$	$\eta(4\tau)^6$	3
26	16	$4A_1 + A_3 + A_7 + D_4$	$\frac{\eta(8\tau)^7 \eta(2\tau)^2}{\eta(16\tau)^2 \eta(4\tau)}$	3
27	16	$2A_1 + 4D_4$	$\frac{\eta(8\tau)^{14} \eta(2\tau)^4}{\eta(4\tau)^8 \eta(16\tau)^4}$	3
28	16	$2A_1 + A_3 + 2A_7$	$\eta(8\tau)^2 \eta(4\tau) \eta(2\tau)^2$	5/2
29	16	$A_3 + D_4 + 2D_6$	$\frac{\eta(4\tau) \eta(8\tau)^7 \eta(\tau)^2}{\eta(16\tau)^2 \eta(2\tau)^3}$	5/2
30	18	$8A_1 + 4A_2$	$\frac{\eta(9\tau)^8 \eta(6\tau)^4}{\eta(18\tau)^4}$	4
31	18	$2A_1 + 3A_2 + 2A_5$	$\frac{\eta(9\tau)^2 \eta(6\tau)^3 \eta(3\tau)^2}{\eta(18\tau)}$	3
32	20	$2A_1 + 4A_3 + A_4$	$\frac{\eta(10\tau)^2 \eta(5\tau)^4 \eta(4\tau)}{\eta(20\tau)}$	3
33	21	$6A_2 + A_6$	$\frac{\eta(7\tau)^6 \eta(3\tau)}{\eta(21\tau)}$	3
34	24	$5A_1 + 3A_2 + 2A_3$	$\frac{\eta(12\tau)^5 \eta(8\tau)^3 \eta(6\tau)^2}{\eta(24\tau)^3}$	7/2
35	24	$4A_1 + 2A_2 + 2A_5$	$\frac{\eta(12\tau)^4 \eta(8\tau)^2 \eta(4\tau)^2}{\eta(24\tau)^2}$	3
36	24	$5A_1 + A_3 + A_5 + D_5$	$\frac{\eta(12\tau)^7 \eta(6\tau) \eta(2\tau) \eta(8\tau)}{\eta(24\tau)^3 \eta(4\tau)}$	3
37	24	$2A_2 + A_5 + D_4 + E_6$	$\frac{\eta(8\tau)^4 \eta(4\tau) \eta(3\tau) \eta(12\tau)^4 \eta(\tau)}{\eta(6\tau)^2 \eta(24\tau)^2 \eta(2\tau)^2}$	5/2
38	24	$2A_2 + A_3 + 2E_6$	$\frac{\eta(8\tau)^6 \eta(6\tau) \eta(\tau)^2 \eta(12\tau)^2}{\eta(2\tau)^4 \eta(24\tau)^2}$	5/2
39	32	$8A_1 + 3A_3$	$\frac{\eta(16\tau)^8 \eta(8\tau)^3}{\eta(32\tau)^4}$	7/2
40	32	$9A_1 + 2D_4$	$\frac{\eta(16\tau)^{15} \eta(4\tau)^2}{\eta(32\tau)^6 \eta(8\tau)^4}$	7/2
41	32	$3A_1 + 5A_3$	$\frac{\eta(16\tau)^3 \eta(8\tau)^5}{\eta(32\tau)^2}$	3
42	32	$4A_1 + 2A_3 + 2D_4$	$\frac{\eta(16\tau)^{10} \eta(4\tau)^2}{\eta(32\tau)^4 \eta(8\tau)^2}$	3
43	32	$5A_1 + 2A_7$	$\frac{\eta(16\tau)^5 \eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
44	32	$2A_1 + 2A_3 + A_7 + D_4$	$\frac{\eta(16\tau)^5 \eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
45	32	$3A_1 + D_4 + 2D_6$	$\frac{\eta(16\tau)^{10} \eta(2\tau)^2}{\eta(32\tau)^4 \eta(4\tau)^3}$	5/2
46	36	$2A_1 + 2A_2 + 4A_3$	$\frac{\eta(18\tau)^2 \eta(12\tau)^2 \eta(9\tau)^4}{\eta(36\tau)^2}$	3
47	36	$A_1 + 6A_2 + A_5$	$\frac{\eta(18\tau) \eta(12\tau)^6 \eta(6\tau)}{\eta(36\tau)^2}$	3
48	36	$6A_1 + A_2 + 2A_5$	$\frac{\eta(18\tau)^6 \eta(12\tau) \eta(6\tau)^2}{\eta(36\tau)^3}$	3
49	48	$5A_1 + 6A_2$	$\frac{\eta(24\tau)^5 \eta(16\tau)^6}{\eta(48\tau)^4}$	7/2
50	48	$6A_2 + 2A_3$	$\frac{\eta(16\tau)^6 \eta(12\tau)^2}{\eta(48\tau)^2}$	3
51	48	$5A_1 + A_2 + 2A_3 + A_5$	$\frac{\eta(24\tau)^5 \eta(16\tau) \eta(12\tau)^2 \eta(8\tau)}{\eta(48\tau)^3}$	3
52	48	$4A_1 + 3A_5$	$\frac{\eta(24\tau)^4 \eta(8\tau)^3}{\eta(48\tau)^2}$	5/2
53	48	$A_1 + A_2 + 2A_3 + 2D_5$	$\frac{\eta(24\tau)^5 \eta(16\tau)^3 \eta(12\tau)^2 \eta(4\tau)^2}{\eta(48\tau)^3 \eta(8\tau)^4}$	5/2
54	48	$4A_1 + A_2 + A_7 + E_6$	$\frac{\eta(24\tau)^5 \eta(16\tau)^3 \eta(6\tau) \eta(2\tau)}{\eta(48\tau)^3 \eta(4\tau)^2}$	5/2

55	60	$4A_1 + 3A_2 + 2A_4$	$\frac{\eta(30\tau)^4 \eta(20\tau)^3 \eta(12\tau)^2}{\eta(60\tau)^3}$	3
56	64	$5A_1 + 3A_3 + D_4$	$\frac{\eta(32\tau)^8 \eta(16\tau) \eta(8\tau)}{\eta(64\tau)^4}$	3
57	64	$6A_1 + 3D_4$	$\frac{\eta(32\tau)^{15} \eta(8\tau)^3}{\eta(64\tau)^6 \eta(16\tau)^6}$	3
58	64	$3A_1 + 3A_3 + A_7$	$\frac{\eta(32\tau)^3 \eta(16\tau)^3 \eta(8\tau)}{\eta(64\tau)^2}$	5/2
59	64	$5A_3 + D_4$	$\frac{\eta(32\tau)^3 \eta(16\tau)^3 \eta(8\tau)}{\eta(64\tau)^2}$	5/2
60	64	$4A_1 + A_3 + 2D_6$	$\frac{\eta(32\tau)^8 \eta(16\tau)^3 \eta(4\tau)^2}{\eta(64\tau)^4 \eta(8\tau)^4}$	5/2
61	72	$4A_1 + 3A_2 + A_3 + D_5$	$\frac{\eta(36\tau)^6 \eta(24\tau)^4 \eta(18\tau) \eta(6\tau)}{\eta(72\tau)^4 \eta(12\tau)^2}$	3
62	72	$3A_1 + 2A_3 + 2A_5$	$\frac{\eta(36\tau)^3 \eta(18\tau)^2 \eta(12\tau)^2}{\eta(72\tau)^2}$	5/2
63	72	$A_2 + 3A_3 + 2D_4$	$\frac{\eta(24\tau) \eta(9\tau)^2 \eta(36\tau)^6}{\eta(72\tau)^3 \eta(18\tau)}$	5/2
64	80	$3A_1 + 4A_4$	$\frac{\eta(40\tau)^3 \eta(16\tau)^4}{\eta(80\tau)^2}$	5/2
65	96	$3A_1 + 3A_2 + 3A_3$	$\frac{\eta(48\tau)^3 \eta(32\tau)^3 \eta(24\tau)^3}{\eta(96\tau)^3}$	3
66	96	$2A_1 + 2A_2 + A_3 + 2A_5$	$\frac{\eta(48\tau)^2 \eta(32\tau)^2 \eta(24\tau) \eta(16\tau)^2}{\eta(96\tau)^2}$	5/2
67	96	$2A_1 + 3A_2 + A_7 + D_4$	$\frac{\eta(48\tau)^5 \eta(32\tau)^3 \eta(12\tau)^2}{\eta(96\tau)^3 \eta(24\tau)^2}$	5/2
68	96	$3A_1 + 2A_3 + A_5 + D_5$	$\frac{\eta(48\tau)^5 \eta(24\tau)^2 \eta(8\tau) \eta(32\tau)}{\eta(96\tau)^3 \eta(16\tau)}$	5/2
69	96	$3A_1 + 2A_2 + 2E_6$	$\frac{\eta(48\tau)^5 \eta(32\tau)^6 \eta(4\tau)^2}{\eta(96\tau)^4 \eta(8\tau)^4}$	5/2
70	120	$2A_1 + A_2 + 2A_3 + A_4 + A_5$	$\frac{\eta(60\tau)^2 \eta(40\tau) \eta(30\tau)^2 \eta(24\tau) \eta(20\tau)}{\eta(120\tau)^2}$	5/2
71	128	$3A_1 + 2A_3 + D_4 + D_6$	$\frac{\eta(64\tau)^8 \eta(32\tau) \eta(8\tau)}{\eta(128\tau)^4 \eta(16\tau)}$	5/2
72	144	$A_1 + 4A_2 + 2A_5$	$\frac{\eta(72\tau) \eta(48\tau)^4 \eta(24\tau)^2}{\eta(144\tau)^2}$	5/2
73	160	$2A_1 + 3A_3 + 2A_4$	$\frac{\eta(80\tau)^2 \eta(40\tau)^3 \eta(32\tau)^2}{\eta(160\tau)^2}$	5/2
74	168	$A_1 + 3A_2 + 2A_3 + A_6$	$\frac{\eta(84\tau) \eta(56\tau)^3 \eta(42\tau)^2 \eta(24\tau)}{\eta(168\tau)^2}$	5/2
75	192	$2A_1 + 6A_2 + D_4$	$\frac{\eta(96\tau)^5 \eta(64\tau)^6 \eta(24\tau)}{\eta(192\tau)^4 \eta(48\tau)^2}$	3
76	192	$2A_1 + A_2 + 2A_3 + A_5 + D_4$	$\frac{\eta(96\tau)^5 \eta(64\tau) \eta(32\tau) \eta(24\tau)}{\eta(192\tau)^3}$	5/2
77	192	$2A_1 + A_2 + 3A_3 + E_6$	$\frac{\eta(96\tau)^3 \eta(64\tau)^3 \eta(48\tau)^3 \eta(8\tau)}{\eta(192\tau)^3 \eta(16\tau)^2}$	5/2
78	288	$2A_1 + 2A_2 + A_3 + 2D_5$	$\frac{\eta(144\tau)^6 \eta(96\tau)^4 \eta(72\tau) \eta(24\tau)^2}{\eta(288\tau)^4 \eta(48\tau)^4}$	5/2
79	360	$A_1 + 2A_2 + 2A_3 + 2A_4$	$\frac{\eta(180\tau) \eta(120\tau)^2 \eta(90\tau)^2 \eta(72\tau)^2}{\eta(360\tau)^2}$	5/2
80	384	$A_1 + 3A_2 + 2A_3 + D_6$	$\frac{\eta(192\tau)^3 \eta(128\tau)^3 \eta(96\tau)^3 \eta(24\tau)}{\eta(384\tau)^3 \eta(48\tau)^2}$	5/2
81	960	$A_1 + 3A_2 + 2A_4 + D_4$	$\frac{\eta(480\tau)^4 \eta(320\tau)^3 \eta(192\tau)^2 \eta(120\tau)}{\eta(960\tau)^3 \eta(240\tau)^2}$	5/2

Table 1: Table of the modular forms $Z_{X,G}^{-1}$ for all symplectic G actions.

References