G-fixed Hilbert schemes on $K3$ surfaces and modular forms.

Jim Bryan and Ádám Gyenge

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Abstract

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Let

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X)^G) q^{n-1}$$

be the generating function for the Euler characteristics of Hilbert scheme of G-invariant length n subschemes. We show that its reciprocal, $Z_{X,G}(q)^{-1}$ is the Fourier expansion of a modular cusp form of weight $\frac{1}{2}e(X/G)$ and index $|G|$. We give an explicit formula for $Z_{X,G}$ in terms of the Dedekind eta function for all 82 possible (X, G).

1 Introduction

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Mukai showed that such G are precisely the subgroups of the Mathieu group $M_{23} \subset M_{24}$ such that the induced action on the set $\{1, \ldots, 24\}$ has at least five orbits [5]. Xiao classified all possible actions into 82 possible topological types of the quotient X/G [7].

The *G-fixed Hilbert scheme* of X parameterizes G-invariant length n subschemes $Z \subset X$. It can be identified with the G-fixed point locus in the Hilbert scheme of points:

$$\mathrm{Hilb}^n(X)^G \subset \mathrm{Hilb}^n(X)$$

We define the corresponding *G-fixed partition function* of X by

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^{n-1}$$

where $e(-)$ is topological Euler characteristic.

Throughout this paper we set

$$q = \exp\left(2\pi i \tau\right)$$

so that we may regard $Z_{X,G}$ as a function of $\tau \in \mathbb{H}$ where $\mathbb{H}$ is the upper half-plane.

Our main result is the following:

Theorem 1. *The function $Z_{X,G}(q)^{-1}$ is a modular cusp form*[1] *of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$.*

Our theorem specializes in the case where G is the trivial group to a famous result of Göttsche [2]. The case where G is a cyclic group was proved in [1]. One can interpret our result as an instance of the Vafa-Witten S-duality conjecture for the orbifold $[X/G]$. The partition function $Z_{X,G}(q)$ also has an interpretation in enumerative geometry generalizing: its coefficients count G-invariant rational curves on X (see § 1.1), generalizing the famous Yau-Zaslow formula.

We also give an explicit formula for $Z_{X,G}(q)$ in terms of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

as follows. Let $p_1, \ldots, p_r$ be the singular points of X/G and let $G_1, \ldots, G_r$ be the corresponding stabilizer subgroups of G. The singular points are necessarily of ADE type: they are locally given by $\mathbb{C}^2/G_i$ where $G_i \subset SU(2)$. Finite subgroups of $SU(2)$ have an ADE classification and we let $\Delta_1, \ldots, \Delta_r$ denote the corresponding ADE root systems.

For any finite subgroup $G_\Delta \subset SU(2)$ with associated root system Δ we define the *local G_Δ-fixed partition function* by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}.$$

We will prove in Lemma 6 that

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{N+1}}$$

where $\theta_\Delta(\tau)$ is a shifted theta function for the root lattice of Δ, N is the rank of the root system, and $k = |G_\Delta|$.

The 82 possible collections of ADE root systems $\Delta_1, \ldots, \Delta_r$ associated to (X, G) a $K3$ surface with a symplectic G action, are given in table 1 and we note that $\Delta_i \in \{A_1, \ldots, A_7, D_4, D_5, D_6, E_6\}$. We let $k = |G|$, $k_i = |G_i|$, and

$$a = e(X/G) - r = \frac{24}{k} - \sum_{i=1}^{r} \frac{1}{k_i}.$$

Theorem 2. *With the above notation we have*

$$Z_{X,G}(q) = \eta^{-a}(k\tau) \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right)$$

[1] See section § 4 for notation and definitions regarding modular forms.

where

$$
\begin{aligned}
Z_{A_n}(\tau) &= \frac{1}{\eta(\tau)}, \quad n \geq 1 \\
Z_{D_n}(\tau) &= \frac{\eta^2(2\tau)\eta((4n-8)\tau)}{\eta(\tau)\eta(4\tau)\eta^2((2n-4)\tau)}, \quad 4 \leq n \leq 6 \\
Z_{E_6}(\tau) &= \frac{\eta^2(2\tau)\eta(24\tau)}{\eta(\tau)\eta^2(8\tau)\eta(12\tau)}
\end{aligned}
$$

We conjecture that the formula for Z_{D_n} holds for all $n \geq 4$ and we provide explicit conjectural formulas for Z_{E_7} and Z_{E_8} (see Conjecture 7). In table 1 we have listed explictly the eta product of the modular form $(Z_{X,G})^{-1}$ for all 82 possible cases of (X, G).

Having obtained explicit eta product expressions for $Z_{X,G}(q)$ in all 82 possible cases allows us to make several observational corollaries:

Corollary 3. *If G is a finite subgroup of an elliptic curve E, i.e. G is isomorphic to a product of one or two cyclic groups, then $Z_{X,G}(q)^{-1}$ is a Hecke eigenform. On table 1 these are the 13 cases having Xiao number in the set $\{0, 1, 2, 3, 4, 5, 7, 8, 11, 14, 15, 19, 25\}$. Moreover, in each of these cases, the dimension of the Hecke eigenspace is one.*

We remark that in these cases, we may form a Calabi-Yau threefold called a CHL model by taking the free group quotient

$$
(X \times E)/G
$$

Then the partition function $Z_{X,G}(q)$ gives the (modified) Donaldson-Thomas invariants of $(X \times E)/G$ in curve classes which are degree zero over X/G (see [1]). For any eta product expression of a modular form, one may easily compute the order of vanishing (or pole) at any of the cusps [4, Cor 2.2]. Performing this computation on the 82 cases yields the following

Corollary 4. *The modular form $Z_{X,G}(q)^{-1}$ always vanishes at the cusps $i\infty$ and 0. Moreover,*

- *$Z_{X,G}(q)^{-1}$ vanishes at all cusps except for the eleven cases with Xiao number in the set $\{13, 20, 27, 29, 37, 38, 45, 53, 54, 60, 69\}$.*
- *$Z_{X,G}(q)^{-1}$ is holomorphic except for the two cases with Xiao number 38 or 69, which have poles at the cusps $1/2$ and $1/8$ respectively. These are precisely the cases where X/G has two singularities of type E_6.*

1.1 Enumerative applications

We have already mentioned above the enumerative application to the CHL Calabi-Yau threefold $(X \times E)/G$ in the case where $G \subset E$ is a finite subgroup of an elliptic curve. Another application is the following generalization of the Yau-Zaslow formula counting rational curves on X.

Let $X \subset \mathbb{P}^g$ be an embedding obtained from a G-equivariant ample line bundle L with $c_1(L)$ a primitive class of square $2g-2$. Then the coefficient of q^{g-1} in $Z_{X,G}(q)$ is the number of hyperplane sections which are G-invariant rational curves, counted with multiplicity.

...add discussion of the above. Formulate as proposition?

1.2 Structure of the paper

I'm not sure we really need to outline the paper here, but we could.

2 The local partition functions

The classical McKay correspondence associates an ADE root system Δ to any finite subgroup $G_\Delta \subset SU(2)$. Using the work of Nakajima [6], the partition function of the Euler characteristics of the Hilbert scheme of points on the stack quotient $[\mathbb{C}^2/G_\Delta]$ was computed explicitly in [3] in terms of the root data of Δ.

The local partition functions $Z_\Delta(q)$ considered in this paper are obtained from a specialization of the partition functions of the stack $[\mathbb{C}^2/G_\Delta]$ and in this section, we use this to express $Z_\Delta(q)$ in terms of a shifted theta function for the root lattice of Δ.

A zero-dimensional substack $Z \subset [\mathbb{C}^2/G_\Delta]$ may be regarded as a G_Δ invariant, zero-dimensional subscheme of $\mathbb{C}^2$. Consequently, we may identify the Hilbert scheme of points on the stack $[\mathbb{C}^2/G_\Delta]$ with the G_Δ fixed locus of the Hilbert scheme of points on $\mathbb{C}^2$:

$$\mathrm{Hilb}\left([\mathbb{C}^2/G_\Delta]\right) = \mathrm{Hilb}(\mathbb{C}^2)^{G_\Delta}.$$

This Hilbert scheme has components indexed by representations ρ of G_Δ as follows

$$\mathrm{Hilb}^\rho\left([\mathbb{C}^2/G_\Delta]\right) = \left\{Z \subset \mathbb{C}^2,\ Z \text{ is } G_\Delta \text{ invariant and } H^0(\mathcal{O}_Z) \cong \rho\right\}.$$

Let $\{\rho_0, \ldots, \rho_N\}$ be the irreducible representations of G_Δ where ρ_0 is the trivial representation. We note that N is also the rank of Δ. We define

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_N) = \sum_{m_0,\ldots,m_N=0}^{\infty} e\left(\mathrm{Hilb}^{m_0\rho_0+\cdots+m_N\rho_M}([\mathbb{C}^2/G_\Delta])\right) q_0^{m_0} \cdots q_N^{m_N}.$$

Recall that our local partition function $Z_\Delta(q)$ is defined by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}.$$

We then readily see that

$$Z_\Delta(q) = q^{\frac{-1}{24}} \cdot Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_N)|_{q_i=q^{d_i}}$$

where

$$d_i = \dim \rho_i.$$

The following theorem is given in [3, Thm 1.3] where it is attributed to Nakajima [6]:

Theorem 5. *Let C_Δ be the Cartan matrix of the root system Δ, then*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_N) = \prod_{m=1}^{\infty}(1-Q^m)^{-N-1}\cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^N} q_1^{m_1}\cdots q_N^{m_N}\cdot Q^{\frac{1}{2}\boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot \boldsymbol{m}}$$

where $Q = q_0^{d_0}q_1^{d_1}\cdots d_N^{d_N}$.

We note that under the specialization $q_i = q^{d_i}$,

$$Q = q^{d_0^2+\dots+d_N^2} = q^k$$

where $k = |G|$ is the order of the group G.

We then obtain

$$Z_\Delta(q) = q^{\frac{-1}{24}}\cdot \prod_{m=1}^{\infty}(1-q^{km})^{-N-1}\cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^N} q^{\boldsymbol{m}^{\mathsf{t}}\cdot\boldsymbol{d}}\cdot q^{\frac{k}{2}\boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{m}}$$

where $\boldsymbol{d} = (d_1,\dots,d_N)$.

Let M_Δ be the root lattice of Δ which we identify with $\mathbb{Z}^N$ via the basis given by $\alpha_1,\dots,\alpha_N$, the simple positive roots of Δ. Under this identification, the standard Weyl invariant bilinear form is given by

$$(\boldsymbol{u}|\boldsymbol{v}) = \boldsymbol{u}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{v}.$$

We define

$$\boldsymbol{\zeta} = C_\Delta^{-1}\cdot\boldsymbol{d}$$

so that

$$\boldsymbol{m}^{\mathsf{t}}\cdot\boldsymbol{d} = \boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{\zeta} = (\boldsymbol{m}|\boldsymbol{\zeta}).$$

We may then write

$$\begin{aligned}
Z_\Delta(q) &= q^{\frac{-1}{24}}\cdot\prod_{m=1}^{\infty}(1-q^{km})^{-N-1}\cdot\sum_{\boldsymbol{m}\in M_\Delta} q^{(\boldsymbol{m}|\boldsymbol{\zeta})+\frac{k}{2}(\boldsymbol{m}|\boldsymbol{m})}\\
&= q^A\cdot\left(q^{\frac{k}{24}}\prod_{m=1}^{\infty}(1-q^{km})\right)^{-N-1}\cdot\sum_{\boldsymbol{m}\in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)}\\
&= q^A\cdot\eta(k\tau)^{-N-1}\cdot\theta_\Delta(\tau)
\end{aligned}$$

where

$$A = \frac{-1}{24}+\frac{k(N+1)}{24}-\frac{1}{2k}(\boldsymbol{\zeta}|\boldsymbol{\zeta}) = \frac{k(N+1)-1}{24}-\frac{1}{2k}\boldsymbol{d}^{\mathsf{t}}\cdot C_\Delta^{-1}\cdot\boldsymbol{d}$$

and $\theta_\Delta(\tau)$ is the shifted theta function:

$$\theta_\Delta(\tau) = \sum_{\boldsymbol{m}\in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)}$$

where as throughout this paper we have identified $q = \exp(2\pi i\tau)$.

In section ???, lemma ??? we will prove that the identity $A = 0$ holds for all Δ and hence we obtain the following:

Lemma 6. *The local series $Z_\Delta(q)$ is given by*

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{N+1}}.$$

We make the following conjecture which provides explicit eta product expressions for the theta function $\theta_\Delta(\tau)$.

Conjecture 7. *$\theta_\Delta(\tau)$ is given by*

$$\theta_{A_n}(\tau) = \frac{\eta^{n+1}((n+1)\tau)}{\eta(\tau)}, \quad n \geq 1 \tag{1}$$

$$\theta_{D_n}(\tau) = \frac{\eta^2(2\tau)\,\eta^{n+2}((4n-8)\tau)}{\eta(\tau)\,\eta(4\tau)\,\eta^2((2n-4)\tau)}, \quad n \geq 4 \tag{2}$$

$$\theta_{E_6}(\tau) = \frac{\eta^2(2\tau)\,\eta^8(24\tau)}{\eta(\tau)\,\eta^2(8\tau)\,\eta(12\tau)}, \tag{3}$$

$$\theta_{E_7}(\tau) = \frac{\eta^2(2\tau)\,\eta^9(48\tau)}{\eta(\tau)\,\eta(12\tau)\,\eta(16\tau)\,\eta(24\tau)}, \tag{4}$$

$$\theta_{E_8}(\tau) = \frac{\eta^2(2\tau)\,\eta^{10}(120\tau)}{\eta(\tau)\,\eta(24\tau)\,\eta(40\tau)\,\eta(60\tau)}. \tag{5}$$

Since both sides of the above equations are explicit modular forms of known weight and index, any given formula can be proved with a finite number of computations. We will give a uniform geometric proof in the A_n case for all n below, and we will give computational proofs for the cases of D_4, D_5, D_6, and E_6 (Theorem **??**). These are the only cases needed for our application to K3 surfaces. It would be desirable to have a purely root theoretic way of writing the eta products and a pure root theoretic proof of the conjecture.

Theorem 8. *Conjecture 7 holds for the case of A_n.*

Proof. By Lemma 6 , the conjecture is equivalent to the statment that

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

which is in turn equivalent to the statement

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}(\mathbb{C}^2)^{\mathbb{Z}/(n+1)}\right) q^n = \prod_{m=1}^{\infty} (1-q^m)^{-1}.$$

The action of $\mathbb{Z}/(n+1)$ on $\mathbb{C}^2$ commutes with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ and consequently, the Euler characteristics on the left hand side may be computed by counting the $\mathbb{C}^* \times \mathbb{C}^*$-fixed subschemes, namely those given by monomial ideals. Such subschemes of length n have a well known bijection with integer partitions of n, whose generating function is given by the right hand side. $\square$

3 The Global series

Recall that $p_1, \ldots, p_r \in X/G$ are the singular points of X/G with corresponding stabilizer subgroups $G_i \subset G$ of order k_i and ADE type Δ_i. Let $\{x_i^1, \ldots, x_i^{k/k_i}\}$ be the orbit of G in X corresponding to the point p_i (recall that $k = |G|$). We may stratify $\mathrm{Hilb}(X)^G$ according to the orbit types of subscheme as follows:

Let $Z \subset X$ be a G-invariant subscheme of length nk whose support lies on free orbits. Then Z determines and is determined by a length n subscheme of

$$(X/G)^o = X/G \setminus \{p_1, \ldots, p_r\},$$

i.e. a point in $\mathrm{Hilb}^n((X/G)^o)$.

On the other hand, suppose $Z \subset X$ is a G-invariant subscheme of length $\frac{nk}{k_i}$ supported on the orbit $\{x_i^1, \ldots, x_i^{k/k_i}\}$. Then Z determines and is determined by the length n component of Z supported on a formal neighborhood of one of the points, say x_i^1. Choosing a G_i-equivariant isomorphism of the formal neighborhood of x_i^1 in X with the formal neighborhood of the origin in $\mathbb{C}^2$, we see that Z determines and is determined by a point in $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$, where $\mathrm{Hilb}_0^n(\mathbb{C}^2) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ is the punctual Hilbert scheme parameterizing subschemes supported on a formal neighborhood of the origin in $\mathbb{C}^2$.

By decomposing an arbitrary G-invariant subscheme into components of the above types, we obtain a stratification of $\mathrm{Hilb}(X)^G$ into strata which are given by products of $\mathrm{Hilb}((X/G)^o)$ and $\mathrm{Hilb}_0(\mathbb{C}^2)^{G_1}, \ldots, \mathrm{Hilb}_0(\mathbb{C}^2)^{G_r}$. Then using the fact that Euler characteristic is additive under stratifications and multiplicative under products, we arrive at the following equation of generating functions:

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^n = &\left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^o)\right) q^{kn}\right) \\ &\cdot \prod_{i=1}^{r} \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}}\right). \end{aligned} \tag{6}$$

As in the introduction, let $a = e(X/G) - r = e\left((X/G)^o\right)$. Then by Göttsche's formula [2],

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^0\right) q^{kn} &= \prod_{m=1}^{\infty} (1 - q^{km})^{-a} \\ &= q^{\frac{ak}{24}} \cdot \eta(k\tau)^{-a}.\end{aligned}$$

We also note that $e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) = e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right)$ since the natural $\mathbb{C}^*$ action on both $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$ and $\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}$ have the same fixed points. Thus we may write

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}} &= \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}} \\ &= q^{\frac{k}{24k_i}} \cdot Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).\end{aligned}$$

Multiplying equation (6) by q^{-1} and substituting the above formulas, we find that

$$Z_{X,G}(q) = q^{-1+\frac{ak}{24}+\sum \frac{k}{24k_i}} \cdot \eta(k\tau)^{-a} \cdot \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

The exponent of q in the above equation is zero as is readily seen from the following Euler characteristic calculation:

$$\begin{aligned} 24 = e(X) &= e\left(X - \cup_{i=1}^{r}\{x_i^1, \ldots, x_i^{k/k_i}\}\right) + \sum_{i=1}^{r} \frac{k}{k_i} \\ &= k \cdot e\left((X/G)^o\right) + \sum_{i=1}^{r} \frac{k}{k_i} \\ &= k \cdot a + \sum_{i=1}^{r} \frac{k}{k_i} \end{aligned}$$

We have thus proved that the first equation in Theorem 2 always holds. Then since the only root systems which can occur as singularities of X/G are of type A_n or D_4, D_5, D_6, or E_6, we may now use Theorem 8 and Theorem **??** to complete the proof of Theorem 2. □

4 Modular forms

4.1 Modular forms with multiplier systems and congruence subgroups

4.2 Multiplier systems and congruence subgroups of shifted theta functions

4.3 Multiplier systems and congruence subgroups of eta products

4.4 Sturm bounds and the proof of Theorem ???

4.5 Miscellany

A Table of eta products

The following table provides the list of the modular forms $Z_{X,G}^{-1}$, expressed as eta products, for each of the 82 possible symplectic actions of a group G on a $K3$ surface X. Our numbering matches Xiao's [7] whose table we refer to for a description of each group.

Xiao #	$\lvert G\rvert$	Singularities of X/G	The modular form $Z_{X,G}^{-1}$	Weight
0	1		$\eta(\tau)^{24}$	12
1	2	$8A_1$	$\eta(2\tau)^8\,\eta(\tau)^8$	8
2	3	$6A_2$	$\eta(3\tau)^6\,\eta(\tau)^6$	6
3	4	$12A_1$	$\eta(2\tau)^{12}$	6
4	4	$2A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^2\,\eta(\tau)^4$	5
5	5	$4A_4$	$\eta(5\tau)^4\,\eta(\tau)^4$	4
6	6	$8A_1+3A_2$	$\frac{\eta(3\tau)^8\eta(2\tau)^3}{\eta(6\tau)}$	5
7	6	$2A_1+2A_2+2A_5$	$\eta(6\tau)^2\,\eta(3\tau)^2\,\eta(2\tau)^2\,\eta(\tau)^2$	4
8	7	$3A_6$	$\eta(7\tau)^3\,\eta(\tau)^3$	3
9	8	$14A_1$	$\frac{\eta(4\tau)^{14}}{\eta(8\tau)^4}$	5
10	8	$9A_1+2A_3$	$\frac{\eta(4\tau)^9\eta(2\tau)^2}{\eta(8\tau)^2}$	9/2
11	8	$4A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^4$	4
12	8	$3A_3+2D_4$	$\frac{\eta(\tau)^2\eta(4\tau)^6}{\eta(2\tau)}$	7/2
13	8	A_1+4D_4	$\frac{\eta(4\tau)^{13}\eta(\tau)^4}{\eta(8\tau)^2\eta(2\tau)^8}$	7/2
14	8	$A_1+A_3+2A_7$	$\eta(8\tau)^2\,\eta(4\tau)\,\eta(2\tau)\,\eta(\tau)^2$	3
15	9	$8A_2$	$\eta(3\tau)^8$	4
16	10	$8A_1+2A_4$	$\frac{\eta(5\tau)^8\eta(2\tau)^2}{\eta(10\tau)^2}$	4
17	12	$4A_1+6A_2$	$\frac{\eta(6\tau)^4\eta(4\tau)^6}{\eta(12\tau)^2}$	4
18	12	$9A_1+A_2+A_5$	$\frac{\eta(6\tau)^9\eta(4\tau)\eta(2\tau)}{\eta(12\tau)^3}$	4
19	12	$3A_1+3A_5$	$\eta(6\tau)^3\,\eta(2\tau)^3$	3
20	12	$A_2+2A_3+2D_5$	$\frac{\eta(4\tau)^3\eta(3\tau)^2\eta(\tau)^2\eta(6\tau)^4}{\eta(12\tau)\eta(2\tau)^4}$	3
21	16	$15A_1$	$\frac{\eta(8\tau)^{15}}{\eta(16\tau)^6}$	9/2
22	16	$10A_1+2A_3$	$\frac{\eta(8\tau)^{10}\eta(4\tau)^2}{\eta(16\tau)^4}$	4
23	16	$5A_1+4A_3$	$\frac{\eta(8\tau)^5\eta(4\tau)^4}{\eta(16\tau)^2}$	7/2
24	16	$6A_1+A_3+2D_4$	$\frac{\eta(8\tau)^{12}\eta(2\tau)^2}{\eta(16\tau)^4\eta(4\tau)^3}$	7/2

25	16	$6A_3$	$\eta\left(4\tau\right)^6$	3
26	16	$4A_1+A_3+A_7+D_4$	$\frac{\eta(8\tau)^7\eta(2\tau)^2}{\eta(16\tau)^2\eta(4\tau)}$	3
27	16	$2A_1+4D_4$	$\frac{\eta(8\tau)^{14}\eta(2\tau)^4}{\eta(4\tau)^8\eta(16\tau)^4}$	3
28	16	$2A_1+A_3+2A_7$	$\eta\left(8\tau\right)^2\eta\left(4\tau\right)\eta\left(2\tau\right)^2$	5/2
29	16	$A_3+D_4+2D_6$	$\frac{\eta(4\tau)\eta(8\tau)^7\eta(\tau)^2}{\eta(16\tau)^2\eta(2\tau)^3}$	5/2
30	18	$8A_1+4A_2$	$\frac{\eta(9\tau)^8\eta(6\tau)^4}{\eta(18\tau)^4}$	4
31	18	$2A_1+3A_2+2A_5$	$\frac{\eta(9\tau)^2\eta(6\tau)^3\eta(3\tau)^2}{\eta(18\tau)}$	3
32	20	$2A_1+4A_3+A_4$	$\frac{\eta(10\tau)^2\eta(5\tau)^4\eta(4\tau)}{\eta(20\tau)}$	3
33	21	$6A_2+A_6$	$\frac{\eta(7\tau)^6\eta(3\tau)}{\eta(21\tau)}$	3
34	24	$5A_1+3A_2+2A_3$	$\frac{\eta(12\tau)^5\eta(8\tau)^3\eta(6\tau)^2}{\eta(24\tau)^3}$	7/2
35	24	$4A_1+2A_2+2A_5$	$\frac{\eta(12\tau)^4\eta(8\tau)^2\eta(4\tau)^2}{\eta(24\tau)^2}$	3
36	24	$5A_1+A_3+A_5+D_5$	$\frac{\eta(12\tau)^7\eta(6\tau)\eta(2\tau)\eta(8\tau)}{\eta(24\tau)^3\eta(4\tau)}$	3
37	24	$2A_2+A_5+D_4+E_6$	$\frac{\eta(8\tau)^4\eta(4\tau)\eta(3\tau)\eta(12\tau)^4\eta(\tau)}{\eta(6\tau)^2\eta(24\tau)^2\eta(2\tau)^2}$	5/2
38	24	$2A_2+A_3+2E_6$	$\frac{\eta(8\tau)^6\eta(6\tau)\eta(\tau)^2\eta(12\tau)^2}{\eta(2\tau)^4\eta(24\tau)^2}$	5/2
39	32	$8A_1+3A_3$	$\frac{\eta(16\tau)^8\eta(8\tau)^3}{\eta(32\tau)^4}$	7/2
40	32	$9A_1+2D_4$	$\frac{\eta(16\tau)^{15}\eta(4\tau)^2}{\eta(32\tau)^6\eta(8\tau)^4}$	7/2
41	32	$3A_1+5A_3$	$\frac{\eta(16\tau)^3\eta(8\tau)^5}{\eta(32\tau)^2}$	3
42	32	$4A_1+2A_3+2D_4$	$\frac{\eta(16\tau)^{10}\eta(4\tau)^2}{\eta(32\tau)^4\eta(8\tau)^2}$	3
43	32	$5A_1+2A_7$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
44	32	$2A_1+2A_3+A_7+D_4$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
45	32	$3A_1+D_4+2D_6$	$\frac{\eta(16\tau)^{10}\eta(2\tau)^2}{\eta(32\tau)^4\eta(4\tau)^3}$	5/2
46	36	$2A_1+2A_2+4A_3$	$\frac{\eta(18\tau)^2\eta(12\tau)^2\eta(9\tau)^4}{\eta(36\tau)^2}$	3
47	36	$A_1+6A_2+A_5$	$\frac{\eta(18\tau)\eta(12\tau)^6\eta(6\tau)}{\eta(36\tau)^2}$	3
48	36	$6A_1+A_2+2A_5$	$\frac{\eta(18\tau)^6\eta(12\tau)\eta(6\tau)^2}{\eta(36\tau)^3}$	3
49	48	$5A_1+6A_2$	$\frac{\eta(24\tau)^5\eta(16\tau)^6}{\eta(48\tau)^4}$	7/2
50	48	$6A_2+2A_3$	$\frac{\eta(16\tau)^6\eta(12\tau)^2}{\eta(48\tau)^2}$	3
51	48	$5A_1+A_2+2A_3+A_5$	$\frac{\eta(24\tau)^5\eta(16\tau)\eta(12\tau)^2\eta(8\tau)}{\eta(48\tau)^3}$	3
52	48	$4A_1+3A_5$	$\frac{\eta(24\tau)^4\eta(8\tau)^3}{\eta(48\tau)^2}$	5/2
53	48	$A_1+A_2+2A_3+2D_5$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(12\tau)^2\eta(4\tau)^2}{\eta(48\tau)^3\eta(8\tau)^4}$	5/2
54	48	$4A_1+A_2+A_7+E_6$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(6\tau)\eta(2\tau)}{\eta(48\tau)^3\eta(4\tau)^2}$	5/2

55	60	$4A_1 + 3A_2 + 2A_4$	$\frac{\eta(30\tau)^4\eta(20\tau)^3\eta(12\tau)^2}{\eta(60\tau)^3}$	3
56	64	$5A_1 + 3A_3 + D_4$	$\frac{\eta(32\tau)^8\eta(16\tau)\eta(8\tau)}{\eta(64\tau)^4}$	3
57	64	$6A_1 + 3D_4$	$\frac{\eta(32\tau)^{15}\eta(8\tau)^3}{\eta(64\tau)^6\eta(16\tau)^6}$	3
58	64	$3A_1 + 3A_3 + A_7$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
59	64	$5A_3 + D_4$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
60	64	$4A_1 + A_3 + 2D_6$	$\frac{\eta(32\tau)^8\eta(16\tau)^3\eta(4\tau)^2}{\eta(64\tau)^4\eta(8\tau)^4}$	5/2
61	72	$4A_1 + 3A_2 + A_3 + D_5$	$\frac{\eta(36\tau)^6\eta(24\tau)^4\eta(18\tau)\eta(6\tau)}{\eta(72\tau)^4\eta(12\tau)^2}$	3
62	72	$3A_1 + 2A_3 + 2A_5$	$\frac{\eta(36\tau)^3\eta(18\tau)^2\eta(12\tau)^2}{\eta(72\tau)^2}$	5/2
63	72	$A_2 + 3A_3 + 2D_4$	$\frac{\eta(24\tau)\eta(9\tau)^2\eta(36\tau)^6}{\eta(72\tau)^3\eta(18\tau)}$	5/2
64	80	$3A_1 + 4A_4$	$\frac{\eta(40\tau)^3\eta(16\tau)^4}{\eta(80\tau)^2}$	5/2
65	96	$3A_1 + 3A_2 + 3A_3$	$\frac{\eta(48\tau)^3\eta(32\tau)^3\eta(24\tau)^3}{\eta(96\tau)^3}$	3
66	96	$2A_1 + 2A_2 + A_3 + 2A_5$	$\frac{\eta(48\tau)^2\eta(32\tau)^2\eta(24\tau)\eta(16\tau)^2}{\eta(96\tau)^2}$	5/2
67	96	$2A_1 + 3A_2 + A_7 + D_4$	$\frac{\eta(48\tau)^5\eta(32\tau)^3\eta(12\tau)^2}{\eta(96\tau)^3\eta(24\tau)^2}$	5/2
68	96	$3A_1 + 2A_3 + A_5 + D_5$	$\frac{\eta(48\tau)^5\eta(24\tau)^2\eta(8\tau)\eta(32\tau)}{\eta(96\tau)^3\eta(16\tau)}$	5/2
69	96	$3A_1 + 2A_2 + 2E_6$	$\frac{\eta(48\tau)^5\eta(32\tau)^6\eta(4\tau)^2}{\eta(96\tau)^4\eta(8\tau)^4}$	5/2
70	120	$2A_1 + A_2 + 2A_3 + A_4 + A_5$	$\frac{\eta(60\tau)^2\eta(40\tau)\eta(30\tau)^2\eta(24\tau)\eta(20\tau)}{\eta(120\tau)^2}$	5/2
71	128	$3A_1 + 2A_3 + D_4 + D_6$	$\frac{\eta(64\tau)^8\eta(32\tau)\eta(8\tau)}{\eta(128\tau)^4\eta(16\tau)}$	5/2
72	144	$A_1 + 4A_2 + 2A_5$	$\frac{\eta(72\tau)\eta(48\tau)^4\eta(24\tau)^2}{\eta(144\tau)^2}$	5/2
73	160	$2A_1 + 3A_3 + 2A_4$	$\frac{\eta(80\tau)^2\eta(40\tau)^3\eta(32\tau)^2}{\eta(160\tau)^2}$	5/2
74	168	$A_1 + 3A_2 + 2A_3 + A_6$	$\frac{\eta(84\tau)\eta(56\tau)^3\eta(42\tau)^2\eta(24\tau)}{\eta(168\tau)^2}$	5/2
75	192	$2A_1 + 6A_2 + D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)^6\eta(24\tau)}{\eta(192\tau)^4\eta(48\tau)^2}$	3
76	192	$2A_1 + A_2 + 2A_3 + A_5 + D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)\eta(32\tau)\eta(24\tau)}{\eta(192\tau)^3}$	5/2
77	192	$2A_1 + A_2 + 3A_3 + E_6$	$\frac{\eta(96\tau)^3\eta(64\tau)^3\eta(48\tau)^3\eta(8\tau)}{\eta(192\tau)^3\eta(16\tau)^2}$	5/2
78	288	$2A_1 + 2A_2 + A_3 + 2D_5$	$\frac{\eta(144\tau)^6\eta(96\tau)^4\eta(72\tau)\eta(24\tau)^2}{\eta(288\tau)^4\eta(48\tau)^4}$	5/2
79	360	$A_1 + 2A_2 + 2A_3 + 2A_4$	$\frac{\eta(180\tau)\eta(120\tau)^2\eta(90\tau)^2\eta(72\tau)^2}{\eta(360\tau)^2}$	5/2
80	384	$A_1 + 3A_2 + 2A_3 + D_6$	$\frac{\eta(192\tau)^3\eta(128\tau)^3\eta(96\tau)^3\eta(24\tau)}{\eta(384\tau)^3\eta(48\tau)^2}$	5/2
81	960	$A_1 + 3A_2 + 2A_4 + D_4$	$\frac{\eta(480\tau)^4\eta(320\tau)^3\eta(192\tau)^2\eta(120\tau)}{\eta(960\tau)^3\eta(240\tau)^2}$	5/2

Table 1: Table of the modular forms $Z_{X,G}^{-1}$ for all symplectic G actions.

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