G-fixed Hilbert schemes on $K3$ surfaces and modular forms

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Abstract

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Let

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(X)^G) q^{n-1}$$

be the generating function for the Euler characteristics of Hilbert scheme of G-invariant length n subschemes. We show that its reciprocal, $Z_{X,G}(q)^{-1}$ is the Fourier expansion of a modular cusp form of weight $\frac{1}{2}e(X/G)$ and index $|G|$. We give an explicit formula for $Z_{X,G}$ in terms of the Dedekind eta function for all 82 possible (X, G).

1 Introduction

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Mukai showed that such G are precisely the subgroups of the Mathieu group $M_{23} \subset M_{24}$ such that the induced action on the set $\{1, \ldots, 24\}$ has at least five orbits [14]. Xiao classified all possible actions into 82 possible topological types of the quotient X/G [18].

The *G-fixed Hilbert scheme* of X parameterizes G-invariant length n subschemes $Z \subset X$. It can be identified with the G-fixed point locus in the Hilbert scheme of points:

$$\mathrm{Hilb}^n(X)^G \subset \mathrm{Hilb}^n(X)$$

We define the corresponding *G-fixed partition function* of X by

$$Z_{X,G}(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^{n-1}$$

where $e(-)$ is topological Euler characteristic.

Throughout this paper we set

$$q = \exp\left(2\pi i \tau\right)$$

so that we may regard $Z_{X,G}$ as a function of $\tau \in \mathbb{H}$ where $\mathbb{H}$ is the upper half-plane.

Our main result is the following:

Theorem 1.1. *The function $Z_{X,G}(q)^{-1}$ is a modular cusp form*[1] *of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$.*

Our theorem specializes in the case where G is the trivial group to a famous result of Göttsche [9]. The case where G is a cyclic group was proved in [2]. One can interpret our result as an instance of the Vafa-Witten S-duality conjecture for the orbifold $[X/G]$. The partition function $Z_{X,G}(q)$ also has an interpretation in enumerative geometry: its coefficients count G-invariant rational curves on X (see § 1.1), generalizing the famous Yau-Zaslow formula.

We also give an explicit formula for $Z_{X,G}(q)$ in terms of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

[1] See Section § 4 for notation and definitions regarding modular forms.

as follows. Let $p_1, \ldots, p_r$ be the singular points of X/G and let $G_1, \ldots, G_r$ be the corresponding stabilizer subgroups of G. The singular points are necessarily of ADE type: they are locally given by $\mathbb{C}^2/G_i$ where $G_i \subset SU(2)$. Finite subgroups of $SU(2)$ have an ADE classification and we let $\Delta_1, \ldots, \Delta_r$ denote the corresponding ADE root systems.

For any finite subgroup $G_\Delta \subset SU(2)$ with associated root system Δ we define the *local G_Δ-fixed partition function* by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right)\, q^{n-\frac{1}{24}}.$$

We will prove in Lemma 2.2 that

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{N+1}}$$

where $\theta_\Delta(\tau)$ is a shifted theta function for the root lattice of Δ, N is the rank of the root system, and $k = |G_\Delta|$.

The 82 possible collections of ADE root systems $\Delta_1, \ldots, \Delta_r$ associated to (X, G) a $K3$ surface with a symplectic G action, are given in Appendix C, Table 3 and we note that $\Delta_i \in \{A_1, \ldots, A_7, D_4, D_5, D_6, E_6\}$. We let $k = |G|$, $k_i = |G_i|$, and

$$a = e(X/G) - r = \frac{24}{k} - \sum_{i=1}^{r} \frac{1}{k_i}.$$

Our method to prove Theorem 1.1 is based on the next result, which expresses the global series $Z_{X,G}(q)$ as an eta product[2].

Theorem 1.2. *With the above notation we have*

$$Z_{X,G}(q) = \eta^{-a}(k\tau) \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right)$$

where

$$\begin{aligned}
Z_{A_n}(\tau) &= \frac{1}{\eta(\tau)}, \quad n \geq 1 \\
Z_{D_n}(\tau) &= \frac{\eta^2(2\tau)\eta((4n-8)\tau)}{\eta(\tau)\eta(4\tau)\eta^2((2n-4)\tau)}, \quad 4 \leq n \leq 6 \\
Z_{E_6}(\tau) &= \frac{\eta^2(2\tau)\eta(24\tau)}{\eta(\tau)\eta^2(8\tau)\eta(12\tau)}
\end{aligned}$$

We conjecture that the formula for Z_{D_n} holds for all $n \geq 4$ and we provide explicit conjectural formulas for Z_{E_7} and Z_{E_8} (see Conjecture 2.3). In Appendix C, Table 3 we have listed explictly the eta product of the modular form $(Z_{X,G})^{-1}$ for all 82 possible cases of (X, G).

Having obtained explicit eta product expressions for $Z_{X,G}(q)$ in all 82 possible cases allows us to make several observational corollaries:

[2]See Section § 5 for notation and definitions regarding eta products.

Corollary 1.3. *If G is a finite subgroup of an elliptic curve E, i.e. G is isomorphic to a product of one or two cyclic groups, then $Z_{X,G}(q)^{-1}$ is a Hecke eigenform. Appendix C, Table 3 these are the 13 cases having Xiao number in the set* $\{0, 1, 2, 3, 4, 5, 7, 8, 11, 14, 15, 19, 25\}$. *Moreover, in each of these cases, the dimension of the Hecke eigenspace is one.*

We remark that in these cases, we may form a Calabi-Yau threefold called a CHL model by taking the free group quotient

$$(X \times E)/G$$

Then the partition function $Z_{X,G}(q)$ gives the (modified) Donaldson-Thomas invariants of $(X \times E)/G$ in curve classes which are degree zero over X/G (see [2]). For any eta product expression of a modular form, one may easily compute the order of vanishing (or pole) at any of the cusps [13, Cor 2.2]. Performing this computation on the 82 cases yields the following

Corollary 1.4. *The modular form $Z_{X,G}(q)^{-1}$ always vanishes with order 1 at the cusps $i\infty$ and 0. Moreover,*

- $Z_{X,G}(q)^{-1}$ *vanishes at all cusps except for the eleven cases with Xiao number in the set* $\{13, 20, 27, 29, 37, 38, 45, 53, 54, 60, 69\}$.
- $Z_{X,G}(q)^{-1}$ *is holomorphic except for the two cases with Xiao number 38 or 69, which have poles at the cusps $1/2$ and $1/8$ respectively. These are precisely the cases where X/G has two singularities of type E_6.*

1.1 Enumerative applications

We have already mentioned above the enumerative application to the CHL Calabi-Yau threefold $(X \times E)/G$ in the case where $G \subset E$ is a finite subgroup of an elliptic curve. Another application is the following generalization of the Yau-Zaslow formula counting rational curves on X.

Let $X \subset \mathbb{P}^g$ be an embedding obtained from a G-equivariant ample line bundle L with $c_1(L)$ a primitive class of square $2g-2$. Then the coefficient of q^{g-1} in $Z_{X,G}(q)$ is the number of hyperplane sections which are G-invariant rational curves, counted with multiplicity.

…add discussion of the above. Formulate as proposition?

1.2 Structure of the paper

I'm not sure we really need to outline the paper here, but we could.

2 The local partition functions

The classical McKay correspondence associates an ADE root system Δ to any finite subgroup $G_\Delta \subset SU(2)$. Using the work of Nakajima [15], the partition function of the Euler characteristics of the Hilbert scheme of points on the stack quotient $[\mathbb{C}^2/G_\Delta]$ was computed explicitly in [11] in terms of the root data of Δ.

The local partition functions $Z_\Delta(q)$ considered in this paper are obtained from a specialization of the partition functions of the stack $[\mathbb{C}^2/G_\Delta]$ and in this section, we use this to express $Z_\Delta(q)$ in terms of a shifted theta function for the root lattice of Δ.

A zero-dimensional substack $Z \subset [\mathbb{C}^2/G_\Delta]$ may be regarded as a G_Δ invariant, zero-dimensional subscheme of $\mathbb{C}^2$. Consequently, we may identify the Hilbert scheme of points on the stack $[\mathbb{C}^2/G_\Delta]$ with the G_Δ fixed locus of the Hilbert scheme of points on $\mathbb{C}^2$:

$$\mathrm{Hilb}\left([\mathbb{C}^2/G_\Delta]\right) = \mathrm{Hilb}(\mathbb{C}^2)^{G_\Delta}.$$

This Hilbert scheme has components indexed by representations ρ of G_Δ as follows

$$\mathrm{Hilb}^\rho\left([\mathbb{C}^2/G_\Delta]\right) = \left\{Z \subset \mathbb{C}^2,\ Z \text{ is } G_\Delta \text{ invariant and } H^0(\mathcal{O}_Z) \cong \rho\right\}.$$

Let $\{\rho_0, \dots, \rho_N\}$ be the irreducible representations of G_Δ where ρ_0 is the trivial representation. We note that N is also the rank of Δ. We define

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \dots, q_N) = \sum_{m_0,\dots,m_N=0}^{\infty} e\left(\mathrm{Hilb}^{m_0\rho_0+\dots+m_N\rho_M}([\mathbb{C}^2/G_\Delta])\right) q_0^{m_0}\cdots q_N^{m_N}.$$

Recall that our local partition function $Z_\Delta(q)$ is defined by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}.$$

We then readily see that

$$Z_\Delta(q) = q^{\frac{-1}{24}} \cdot Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \dots, q_N)|_{q_i=q^{d_i}}$$

where

$$d_i = \dim \rho_i.$$

The following formula is given explicitely in [11, Thm 1.3], but its content is already present in the work of Nakajima [15]:

Theorem 2.1. *Let C_Δ be the Cartan matrix of the root system Δ, then*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \dots, q_N) = \prod_{m=1}^{\infty}(1-Q^m)^{-N-1} \cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^N} q_1^{m_1}\cdots q_N^{m_N} \cdot Q^{\frac{1}{2}\boldsymbol{m}^{\mathrm{t}}\cdot C_\Delta\cdot\boldsymbol{m}}$$

where $Q = q_0^{d_0} q_1^{d_1} \cdots d_N^{d_N}$.

We note that under the specialization $q_i = q^{d_i}$,

$$Q = q^{d_0^2+\dots+d_N^2} = q^k$$

where $k = |G|$ is the order of the group G.

We then obtain

$$Z_\Delta(q) = q^{\frac{-1}{24}} \cdot \prod_{m=1}^{\infty}(1-q^{km})^{-N-1} \cdot \sum_{\boldsymbol{m}\in\mathbb{Z}^N} q^{\boldsymbol{m}^{\mathrm{t}}\cdot \boldsymbol{d}} \cdot q^{\frac{k}{2}\boldsymbol{m}^{\mathrm{t}}\cdot C_\Delta \cdot \boldsymbol{m}}$$

where $\boldsymbol{d} = (d_1, \ldots, d_N)$.

Let M_Δ be the root lattice of Δ which we identify with $\mathbb{Z}^N$ via the basis given by $\alpha_1, \ldots, \alpha_N$, the simple positive roots of Δ. Under this identification, the standard Weyl invariant bilinear form is given by

$$(\boldsymbol{u}|\boldsymbol{v}) = \boldsymbol{u}^{\mathrm{t}} \cdot C_\Delta \cdot \boldsymbol{v}.$$

We define

$$\boldsymbol{\zeta} = C_\Delta^{-1} \cdot \boldsymbol{d}$$

so that

$$\boldsymbol{m}^{\mathrm{t}} \cdot \boldsymbol{d} = \boldsymbol{m}^{\mathrm{t}} \cdot C_\Delta \cdot \boldsymbol{\zeta} = (\boldsymbol{m}|\boldsymbol{\zeta}).$$

We may then write

$$\begin{aligned} Z_\Delta(q) &= q^{\frac{-1}{24}} \cdot \prod_{m=1}^{\infty}(1-q^{km})^{-N-1} \cdot \sum_{\boldsymbol{m}\in M_\Delta} q^{(\boldsymbol{m}|\boldsymbol{\zeta})+\frac{k}{2}(\boldsymbol{m}|\boldsymbol{m})} \\ &= q^A \cdot \left(q^{\frac{k}{24}} \prod_{m=1}^{\infty}(1-q^{km})\right)^{-N-1} \cdot \sum_{\boldsymbol{m}\in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)} \\ &= q^A \cdot \eta(k\tau)^{-N-1} \cdot \theta_\Delta(\tau) \end{aligned}$$

where

$$A = \frac{-1}{24} + \frac{k(N+1)}{24} - \frac{1}{2k}(\boldsymbol{\zeta}|\boldsymbol{\zeta}) = \frac{k(N+1)-1}{24} - \frac{1}{2k}\boldsymbol{d}^{\mathrm{t}} \cdot C_\Delta^{-1} \cdot \boldsymbol{d}$$

and $\theta_\Delta(\tau)$ is the shifted theta function:

$$\theta_\Delta(\tau) = \sum_{\boldsymbol{m}\in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)}$$

where as throughout this paper we have identified $q = \exp(2\pi i\tau)$.

In Section 4, Proposition 4.11 we will prove that the identity $A = 0$ holds for all Δ. Hence we obtain the following:

Lemma 2.2. *The local series $Z_\Delta(q)$ is given by*

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{N+1}}.$$

Moreover, $Z_\Delta(q)$ is a (possibly meromorphic) modular form of weight $-1/2$.

We make the following conjecture which provides explicit eta product expressions for the theta function $\theta_\Delta(\tau)$.

Conjecture 2.3. *$\theta_\Delta(\tau)$ is given by*

$$\theta_{A_n}(\tau) = \frac{\eta^{n+1}((n+1)\tau)}{\eta(\tau)}, \quad n \geq 1 \tag{1}$$

$$\theta_{D_n}(\tau) = \frac{\eta^2(2\tau)\,\eta^{n+2}((4n-8)\tau)}{\eta(\tau)\,\eta(4\tau)\,\eta^2((2n-4)\tau)}, \quad n \geq 4 \tag{2}$$

$$\theta_{E_6}(\tau) = \frac{\eta^2(2\tau)\,\eta^8(24\tau)}{\eta(\tau)\,\eta^2(8\tau)\,\eta(12\tau)}, \tag{3}$$

$$\theta_{E_7}(\tau) = \frac{\eta^2(2\tau)\,\eta^9(48\tau)}{\eta(\tau)\,\eta(12\tau)\,\eta(16\tau)\,\eta(24\tau)}, \tag{4}$$

$$\theta_{E_8}(\tau) = \frac{\eta^2(2\tau)\,\eta^{10}(120\tau)}{\eta(\tau)\,\eta(24\tau)\,\eta(40\tau)\,\eta(60\tau)}. \tag{5}$$

¡¡¡¡¡¡¡ HEAD Since both sides of the above equations are explicit modular forms of known weight and index, any given formula can be proved with a finite number of computations. We will give a uniform geometric proof in the A_n case for all n below, and we will give computational proofs for the cases of D_4, D_5, D_6, and E_6 (Theorem **??**). These are the only cases needed for our application to K3 surfaces. It would be desirable to have a purely root theoretic way of writing the eta products and a pure root theoretic proof of the conjecture. [3]

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Theorem 2.4. *Conjecture 2.3 holds for the case of A_n.*

Proof. By Lemma 2.2 , the conjecture is equivalent to the statment that

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

which is in turn equivalent to the statement

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}(\mathbb{C}^2)^{\mathbb{Z}/(n+1)}\right) q^n = \prod_{m=1}^{\infty} (1-q^m)^{-1}.$$

[3]One uniform way to write the conjecture is as follows. Consider $E, F, V, M \in \mathbb{N}$ where E, F, and V each divide M and satisfy $E+F+V = M+2$. Such tuples (E, F, V, M) can be seen to have an ADE classification so that to each ADE root system Δ, we have an associated (E, F, V, M) Then our conjecture is equivalent to

$$Z_\Delta(q) = \frac{\eta^2(2\tau)\eta(2M\tau)}{\eta(\tau)\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}.$$

The action of $\mathbb{Z}/(n+1)$ on $\mathbb{C}^2$ commutes with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ and consequently, the Euler characteristics on the left hand side may be computed by counting the $\mathbb{C}^* \times \mathbb{C}^*$-fixed subschemes, namely those given by monomial ideals. Such subschemes of length n have a well known bijection with integer partitions of n, whose generating function is given by the right hand side. $\square$

3 The global series

Recall that $p_1, \ldots, p_r \in X/G$ are the singular points of X/G with corresponding stabilizer subgroups $G_i \subset G$ of order k_i and ADE type Δ_i. Let $\{x_i^1, \ldots, x_i^{k/k_i}\}$ be the orbit of G in X corresponding to the point p_i (recall that $k = |G|$). We may stratify $\mathrm{Hilb}(X)^G$ according to the orbit types of subscheme as follows:

Let $Z \subset X$ be a G-invariant subscheme of length nk whose support lies on free orbits. Then Z determines and is determined by a length n subscheme of

$$(X/G)^o = X/G \setminus \{p_1, \ldots, p_r\},$$

i.e. a point in $\mathrm{Hilb}^n((X/G)^o)$.

On the other hand, suppose $Z \subset X$ is a G-invariant subscheme of length $\frac{nk}{k_i}$ supported on the orbit $\{x_i^1, \ldots, x_i^{k/k_i}\}$. Then Z determines and is determined by the length n component of Z supported on a formal neighborhood of one of the points, say x_i^1. Choosing a G_i-equivariant isomorphism of the formal neighborhood of x_i^1 in X with the formal neighborhood of the origin in $\mathbb{C}^2$, we see that Z determines and is determined by a point in $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$, where $\mathrm{Hilb}_0^n(\mathbb{C}^2) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ is the punctual Hilbert scheme parameterizing subschemes supported on a formal neighborhood of the origin in $\mathbb{C}^2$.

By decomposing an arbitrary G-invariant subscheme into components of the above types, we obtain a stratification of $\mathrm{Hilb}(X)^G$ into strata which are given by products of $\mathrm{Hilb}((X/G)^o)$ and $\mathrm{Hilb}_0(\mathbb{C}^2)^{G_1}, \ldots, \mathrm{Hilb}_0(\mathbb{C}^2)^{G_r}$. Then using the fact that Euler characteristic is additive under stratifications and multiplicative under products, we arrive at the following equation of generating functions:

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^n = & \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^o)\right) q^{kn}\right) \\ & \cdot \prod_{i=1}^{r}\left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}}\right). \qquad (6)\end{aligned}$$

As in the introduction, let $a = e(X/G) - r = e\left((X/G)^o\right)$. Then by Göttsche's formula [9],

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^0\right) q^{kn} &= \prod_{m=1}^{\infty} (1 - q^{km})^{-a} \\ &= q^{\frac{ak}{24}} \cdot \eta(k\tau)^{-a}.\end{aligned}$$

We also note that $e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) = e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right)$ since the natural $\mathbb{C}^*$ action on both $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$ and $\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}$ have the same fixed points. Thus we may write

$$\begin{aligned}\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right)\, q^{\frac{nk}{k_i}} &= \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right)\, q^{\frac{nk}{k_i}} \\ &= q^{\frac{k}{24k_i}} \cdot Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).\end{aligned}$$

Multiplying equation (6) by q^{-1} and substituting the above formulas, we find that

$$Z_{X,G}(q) = q^{-1+\frac{ak}{24}+\sum\frac{k}{24k_i}} \cdot \eta(k\tau)^{-a} \cdot \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

The exponent of q in the above equation is zero as is readily seen from the following Euler characteristic calculation:

$$\begin{aligned}24 = e(X) &= e\left(X - \cup_{i=1}^{r}\{x_i^1,\ldots,x_i^{k/k_i}\}\right) + \sum_{i=1}^{r}\frac{k}{k_i} \\ &= k\cdot e\left((X/G)^o\right) + \sum_{i=1}^{r}\frac{k}{k_i} \\ &= k\cdot a + \sum_{i=1}^{r}\frac{k}{k_i}\end{aligned}$$

We have thus proved that the first equation in Theorem 1.2 always holds. Then since the only root systems which can occur as singularities of X/G are of type A_n or D_4, D_5, D_6, or E_6, we may now use Theorem 2.4 and Proposition 5.10 to complete the proof of Theorem 1.2. $\square$

4 Modular forms I: Theta functions

4.1 Modular forms with multiplier systems and congruence subgroups

Fix a subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$, a function $\vartheta\colon \Gamma \to \mathbb{C}^*$ with $|\vartheta(A)| = 1$ for $A \in \Gamma$, and a half-integer k. Then a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is said to transform as a modular form of weight k with the multiplier system ϑ for Γ if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \vartheta(A)(c\tau+d)^k f(\tau) \quad \text{for all } A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma.$$

When k is not an integer, $(c\tau+d)^k$ is understood to be a principal value. If moreover f is holomorphic at all the cusps of Γ on $\mathbb{Q}\cup\{i\infty\}$, then f is said to be a modular form. We will denote the space of modular forms of weight k and multiplier systems χ for Γ by $M_k(\Gamma,\vartheta)$.

We will need the following congruence subgroups:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\},$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod N \right\}.$$

Moreover, for a divisor $m|N$ let us introduce the following subset of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma(N,m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv \pm 1 \pmod{N/m}, b \equiv 0 \pmod{N/m}, c \equiv 0 \pmod N \right\}.$$

Lemma 4.1. *$\Gamma(N,m)$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. Since $ad - bc = 1$ and $c \equiv 0 \pmod{N/m}$, we have that $ad \equiv 1 \pmod{N/m}$. This and $a \equiv \pm 1 \pmod{N/m}$ implies that $a \equiv d \pmod{N/m}$. Hence the inverse

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ of } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N,m)$$

belongs to $\Gamma(N,m)$ as well. Similarly, the product

$$\begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ a_2c_1 + c_2d_1 & b_2c_1 + d_1d_2 \end{pmatrix}$$

of two matrices

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma(N,m)$$

is contained in $\Gamma(N,m)$. Finally, it follows from the definition of $\Gamma(N,m)$ that it contains $\Gamma(N)$. □

Lemma 4.2. *The index of $\Gamma(N,m)$ inside $\mathrm{SL}_2(\mathbb{Z})$ is*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N,m)] = \begin{cases} 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), & \textit{if } N/m = 2 \\ 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \prod_{p|N/m} \left(\frac{p}{p-1}\right), & \textit{if } N/m > 2. \end{cases}$$

The proof of this statement will be given in Appendix A.1.

4.2 Theta functions

As mentioned in Section 2 the numerator of the partition function in Theorem 2.1 is (up to a rational exponent factor of q) a shifted theta function of the root lattice of the corresponding finite type Lie algebra. We now investigate the modularity properties of such shifted theta functions. For the notations and many results we will refer to [3, Chapter 14].

Let $L \cong \mathbb{Z}^n$ be an n dimensional lattice equipped with a real quadratic form Q, which we suppose to be integral and positive definite. That is $Q(L) \in \mathbb{Z}^+$. The associated symmetric bilinear form is obtained as

$$B(\mathbf{a}, \mathbf{b}) = Q(\mathbf{a} + \mathbf{b}) - Q(\mathbf{a}) - Q(\mathbf{b}).$$

Then

$$Q(\mathbf{a}) = \frac{1}{2} B(\mathbf{a}, \mathbf{a}).$$

Let moreover

$$L^* = \{\mathbf{b} \in \mathbb{R}^n : B(\mathbf{a}, \mathbf{b}) \in \mathbb{Z} \text{ for all } \mathbf{a} \in L\}$$

be the dual lattice to L in $L \otimes \mathbb{R} \cong \mathbb{R}^n$ with respect to Q.

Let $\mathbf{a} \in L^*$. The theta function associated with the lattice L and shifted by $\mathbf{a}$ is defined as

$$\Theta_{\mathbf{a}}(\tau) = \sum_{m \in L + \mathbf{a}} q^{Q(\mathbf{m})}. \tag{7}$$

Remark 4.3. In [3, Definition 14.3.3] a much more general class of theta functions is introduced. We will use the conventions of [3, Example 14.2.5]. In our case the spherical polynomial $P(X)$ which appears in [3, 14.2.5] is equal to the constant function 1. Moreover, the number k appearing in [3, Example 14.2.5 and Definition 14.3.3] is equal to n in our case. In particular, $k \equiv n \pmod 2$.

Recall that L^* is the dual lattice to L with respect to B. Then $L \subset L^*$ always, and there is a smallest positive integer N for which

$$NL^* \subset L \text{ and } NQ(\mathbf{a}) \in \mathbb{Z} \text{ for all } \mathbf{a} \in L^*. \tag{8}$$

This number N is called the level in [3, Definition 14.3.15].

Elements of a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ will be denoted from now on as follows:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{9}$$

Then $A \in \Gamma_0(N)$ if and only if

$$cL^* \subset L \text{ and } cQ(\mathbf{a}) \in \mathbb{Z} \text{ for all } \mathbf{a} \in L^*.$$

We also introduce the symbol

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv -1 \pmod 4. \end{cases} \tag{10}$$

Theorem 4.4. *Let $A \in \Gamma_0(N)$ and $D = \det(B)$. Then*

$$\Theta_{\mathbf{a}}(A\tau) = \vartheta(A)(c\tau + d)^{n/2} \Theta_{a\mathbf{a}}(\tau)$$

for a multiplier system ϑ such that

$$\vartheta(A) = e^{2\pi i ab Q(\mathbf{a})} \left(\frac{D}{d}\right) \left(\epsilon_d^{-1} \left(\frac{2c}{d}\right)\right)^n$$

when $c \neq 0$ and d is odd.

Proof. Corollary 14.3.8 and Theorem 14.3.11 from [3] implies that

$$\Theta_{\mathbf{a}}(A\tau) = (d, -1^q D)_\infty \vartheta(A)(c\tau + d)^{n/2}\Theta_{a\mathbf{a}}(\tau),$$

where q is the number of negative eigenvalues of Q, $(d, (-1)^q D)_\infty = -1$ if $d < 0$ and $(-1)^q D < 0$, and $(d, (-1)^q D)_\infty = 1$ otherwise. The form Q is positive definite. Hence $q = 0$ and $D > 0$. In turn $(d, (-1)^q D)_\infty = 1$ always. $\square$

Let s be the smallest integer, such $s\mathbf{a} \in L$. This is, in general, not the same as N, but $s|N$ always.

Corollary 4.5. *Suppose that in* (9) *the element* $a \equiv \pm 1$ $(mod\ s)$*. Then*

$$\Theta_{\mathbf{a}}(A\tau) = \vartheta(A)(c\tau + d)^{n/2}\Theta_{\mathbf{a}}(\tau)$$

where ϑ *is as in Theorem 4.4.*

Proof. Since $a \equiv \pm 1$ $(mod\ s)$, $a\mathbf{a} \equiv \pm\mathbf{a}$ $(mod\ L)$. It follows from the definition (11) that $\Theta_{\mathbf{a}}(\tau)$ depends only on the class of $\mathbf{a}$ modulo L. Furthermore, since

$$Q(\mathbf{m} - \mathbf{a}) = Q(-\mathbf{m} + \mathbf{a}),$$

$\Theta_{-\mathbf{a}}(\tau) = \Theta_{\mathbf{a}}(\tau)$. $\square$

Lemma 4.6. *Let* $\Gamma \subset \Gamma_0(N)$ *be a subgroup such that* $a \equiv \pm 1$ $(mod\ s)$ *for all* $A \in \Gamma$*. Then* $\Theta_{\mathbf{a}}(\tau) \in M_{n/2}(\Gamma, \vartheta)$*, where* ϑ *is as in Theorem 4.4.*

Proof. Due to Corollary 4.5, $\Theta_{\mathbf{a}}(\tau)$ transforms as a modular form with the multiplier system ϑ for the elements of Γ.

Showing that it is holomorphic at the cusps is analogous to the proof of [3, Corollary 14.3.16]. By [3, Theorem 14.3.7] when an element $A \in \mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane, $\Theta_{\mathbf{a}}(A\tau)$ decomposes into a finite linear combination:

$$\Theta_{\mathbf{a}}(A\tau) = \sum_{\mathbf{b} \in L^*/L} c_{\mathbf{b}}(c\tau + d)^{n/2}\Theta_{\mathbf{b},k}(\tau).$$

It is known that the group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the cusps of Γ. Hence, to prove that $\Theta_{\mathbf{a}}(\tau)$ is holomorphic at all the cusps of Γ it is enough to show that $\Theta_{\mathbf{b}}(\tau)$ is holomorphic as $\tau \to i\infty$ for any $\mathbf{b} \in L^*/L$. Since the bilinear form is positive definite, $Q(\mathbf{b}) > 0$ for any $\mathbf{b} \neq 0$. Therefore the only term in

$$\Theta_{\mathbf{b}}(\tau) = \sum_{\mathbf{m} \in L + \mathbf{b}} e^{\pi i \tau k Q(\mathbf{m})}$$

which could not tend to 0 as $\tau \to i\infty$ is the one with $\mathbf{m} = -\mathbf{b}$. This term exists only if $\mathbf{b} \in L$, and in this case the limit is 1. The theorem follows. $\square$

Corollary 4.7. $\Theta_{\mathbf{a}}(\tau) \in M_{n/2}(\Gamma(N, s), \vartheta)$ *for a multiplier system* ϑ *such that*

$$\vartheta(A) = \left(\frac{D}{d}\right)\left(\epsilon_d^{-1}\left(\frac{2c}{d}\right)\right)^n$$

when $c \neq 0$ *and* d *is odd.*

Proof. For the elements of $\Gamma(N,s)$, $a \equiv \pm 1 \ (mod\ s)$. Thus the conditions of Lemma 4.6 are satisfied. Moreover, since $b \equiv 0 \ (mod\ s)$, $abQ(\mathbf{a})$ is an integer. This implies that the term

$$e^{2\pi i abQ(\mathbf{a})}$$

in Theorem 4.4 is equal to 1. □

Remark 4.8. Suppose that the rank n of the lattice L is even. Then the multiplier system in Corollary 4.7 simplifies as

$$\vartheta(A) = \left(\frac{(-1)^{n/2}D}{d}\right),$$

because $\epsilon_d^{-2} = -1$ and $\left(\frac{2c}{d}\right)^2 = 1$.

Lemma 4.9. *Let* $\mathbf{a} \in L^*$. *Then*

$$\Theta_{\mathbf{a}}(\tau) = (q^{k/2})^{\mathbf{a}^t \cdot C_\Delta \cdot \mathbf{a}} \cdot \sum_{\mathbf{m} \in \mathbb{Z}^n} q^{k(\mathbf{m}^{\mathrm{t}} \cdot \mathbf{a})} (q^{k/2})^{\mathbf{m}^{\mathrm{t}} \cdot C_\Delta \cdot \mathbf{m}}.$$

Proof. Since $L \cong \mathbb{Z}^n$, one can rewrite (11) as

$$\sum_{\mathbf{m} \in \mathbb{Z}^n} (q^{k/2})^{(\mathbf{m}+\mathbf{a}|\mathbf{m}+\mathbf{a})} = \sum_{\mathbf{m} \in \mathbb{Z}^n} (q^{k/2})^{(\mathbf{m}|\mathbf{m})+2(\mathbf{m}|\mathbf{a})+(\mathbf{a}|\mathbf{a})}.$$

The pairing between $\mathbf{a} \in L^*$ and $\mathbf{m} \in L$ is just

$$(\mathbf{m}|\mathbf{a}) = \sum_{i=1}^{n} a_i m_i = \mathbf{m}^{\mathrm{t}} \cdot \mathbf{a}.$$

□

4.3 Theta functions of root systems

Let Δ be an irreducible root system of finite type, and let L be its root lattice. If $B = (|)$, the standard Weyl invariant bilinear form on L, then the level N of the lattice, which was defined in (8), is equal to the number m listed in Table 1 [12, page 261]. In this case in the standard basis given by the roots $\alpha_1, \ldots, \alpha_n$, $L \cong \mathbb{Z}^n$,

$$B(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\mathrm{t}} \cdot C_\Delta \cdot \mathbf{b}, \quad Q(\mathbf{a}) = \frac{1}{2}\mathbf{a}^{\mathrm{t}} \cdot C_\Delta \cdot \mathbf{a}$$

and $D = \det(B) = |C_\Delta|$.

We now set instead $B = k(|)$, where $k = |G_\Delta|$. Then the level N of L becomes km,

$$B(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\mathrm{t}} \cdot kC_\Delta \cdot \mathbf{b}, \quad Q(\mathbf{a}) = \frac{1}{2}\mathbf{a}^{\mathrm{t}} \cdot kC_\Delta \cdot \mathbf{a}$$

and $D = \det(B) = k^n |C_\Delta|$.

Δ	n	m
A_n	odd	$2(n+1)$
	even	$n+1$
D_n	odd	8
	$n \equiv 2 \pmod 4$	4
	$n \equiv 0 \pmod 4$	2
E_6		3
E_7		4
E_8		1

Table 1: The numbers m.

Furthermore, we fix a particular shift vector $\mathbf{a}$. Let us denote the standard basis of L by $\{\alpha_1, \ldots, \alpha_n\}$ and the corresponding dual basis of L^* by $\{\omega_1, \ldots, \omega_n\}$. Let

$$\theta = (\mathrm{dim}\rho_1, \ldots, \mathrm{dim}\rho_n) = \sum_{i=1}^{n} (\mathrm{dim}\rho_i)\alpha_i \in L. \tag{11}$$

Our $\mathbf{a} \in L^*$ will be the dual of θ with respect to $k(|)$. Explicitely, this means that

$$\mathbf{a} = \frac{1}{k} \sum_{i=1}^{n} (\mathrm{dim}\rho_i)\omega_i = \sum_{i=1}^{n} a_i \alpha_i \tag{12}$$

where $(a_1, \ldots, a_n) = (kC_\Delta)^{-1} \cdot \theta$. Finally, we introduce the notation

$$\Theta_\Delta(\tau) = \Theta_{\mathbf{a}}(\tau).$$

The next statement follows immediately from Corollary 4.7.

Corollary 4.10. *The function $\Theta_\Delta(\tau)$ is a modular form of weight $n/2$ for $\Gamma(km, m)$ with a multiplier system ϑ such that*

$$\vartheta(A) = \left(\frac{k^n |C_\Delta|}{d} \right) \left(\epsilon_d^{-1} \left(\frac{2c}{d} \right) \right)^n$$

when $c \neq 0$ and d is odd.

Proposition 4.11. *Let $[\mathbb{C}^2/G_\Delta]$ be a simple singularity orbifold with Δ of rank n, $k = |G_\Delta|$, and let m be is as in Table 1. Then*

$$q^{-\frac{1}{24}} Z_{[\mathbb{C}^2/G_\Delta]}(q) = \frac{\Theta_\Delta(\tau)}{(\eta(k\tau))^{n+1}}.$$

In particular, $q^{\frac{1}{24}} (Z_{[\mathbb{C}^2/G_\Delta]}(q))^{-1}$ is a (possibly meromorphic) modular form of weight $1/2$ in the variable τ for the congruence subgroup $\Gamma(km, m)$.

Proof. Let

$$\zeta = k\mathbf{a} = \sum_{i=1}^{n} (\mathrm{dim}\rho_i)\omega_i = \sum_{i=1}^{n} b_i \alpha_i \tag{13}$$

where $(b_1, \ldots, b_n) = (C_\Delta)^{-1} \cdot \theta$. Then

$$Q(\mathbf{a}) = \frac{k}{2}\left(\mathbf{a}^{\mathsf{t}} \cdot C_\Delta \cdot \mathbf{a}\right) = \frac{\zeta^{\mathsf{t}} \cdot C_\Delta \cdot \zeta}{2k}.$$

Substituting this into Lemma 4.9 yields

$$\Theta_\Delta(\tau) = q^{\frac{\zeta^{\mathsf{t}} \cdot C_\Delta \cdot \zeta}{2k}} \sum_{\mathbf{m} \in \mathbb{Z}^n} q^{\mathbf{m}^{\mathsf{t}} \cdot (\mathrm{dim}\rho_1, \ldots, \mathrm{dim}\rho_n)} (q^{k/2})^{\mathbf{m}^{\mathsf{t}} \cdot C_\Delta \cdot \mathbf{m}}.$$

Up to the factor $q^{\frac{\zeta^{\mathsf{t}} \cdot C_\Delta \cdot \zeta}{2k}}$ this is exactly the numerator of $Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n)$ appearing in Theorem 2.1 when we substitute $q_i = q^{\mathrm{dim}\rho_i}$, $0 \leq i \leq n$. In the denominator of $Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n)$, after the same substitution, a product of $n+1$ terms of

$$\prod_{m=1}^{\infty} (1 - q^{km})$$

appears. As a consequence,

$$q^{\frac{\zeta^{\mathsf{t}} \cdot C_\Delta \cdot \zeta}{2k} - \frac{k(n+1)}{24}} Z_{[\mathbb{C}^2/G_\Delta]}(q) = \frac{\Theta_\Delta(\tau)}{(\eta(k\tau))^{n+1}}.$$

The Dedeking eta function $\eta(\tau)$ is a modular form of weight $1/2$ for $\Gamma(1)$. Hence, $\eta(k\tau)$ is a modular form of weight $1/2$ for $\Gamma(k)$. By Lemma 4.12 below

$$\frac{k(n+1)}{24} - \frac{\zeta^{\mathsf{t}} \cdot C_\Delta \cdot \zeta}{2k} = \frac{1}{24}.$$

Hence,

$$Z_\Delta(\tau) = q^{-\frac{1}{24}} Z_{[\mathbb{C}^2/G_\Delta]}(q)$$

is the quotient of two holomorphic modular forms. It transforms as a modular form for $\Gamma(km, m) \cap \Gamma(k) = \Gamma(km, m)$ with weight

$$\frac{n}{2} - \frac{n+1}{2} = -\frac{1}{2}.$$

□

Lemma 4.12. *Let Δ be a simply laced root system, and let ζ be defined as in* (13)*. Then*

$$\frac{(\zeta|\zeta)}{2k} = \frac{\zeta^{\mathsf{t}} \cdot C_\Delta \cdot \zeta}{2k} = \frac{(n+1)k - 1}{24}.$$

Remark 4.13. Lemma 4.12 expresses the *modular anomaly* of the numerator of $Z_{[\mathbb{C}^2/G_\Delta]}(q)$ (see [12, 12.7.5]). It is proved in Appendix A.2 below. We have not found it in this generality in the literature, but in type A it turns out to be another form of the "strange formula" of Freudenthal–de Vries [5]:

$$\frac{(\rho|\rho)}{2h} = \frac{\rho^{\mathsf{t}} \cdot C_\Delta \cdot \rho}{2h} = \frac{\dim \mathfrak{g}_\Delta}{24}$$

where ρ is the sum of the positive roots of Δ, h is the (dual) Coxeter number, and $\mathfrak{g}_\Delta$ is the corresponding Lie algebra. See Appendix A.2 for the details. We expect that the identity of Lemma 4.12 holds in the non-simply laced cases as well.

Let Δ_1 (resp. Δ_2) be a root system of rank n_1 (resp. n_2). Denote by G_{Δ_1} $(resp. G_{\Delta_2})$ the corresponding finite group, whose order is $k_1 = |G_{\Delta_1}|$ (resp. $k_2 = |G_{\Delta_2}|$). Let θ_1 (resp. θ_2) be as in (11). Let $\mathbf{a}_1$ (resp. $\mathbf{a}_2$) be the vector dual to θ_1 (resp. θ_2) with respect to the form $k_1(|)_1$ (resp. $k_2(|)_2$). We define

$$\Theta_{\Delta_1 \oplus \Delta_2}(\tau) = \Theta_{\mathbf{a}_1 \oplus \mathbf{a}_2}(\tau)$$

where the right side is the theta function of the lattice $L_1 \oplus L_2$ equipped with the form $k_1(|)_1 \oplus k_2(|)_2$. The next statement is a straightforward calculation.

Lemma 4.14.

$$\Theta_{\Delta_1 \oplus \Delta_2}(\tau) = \Theta_{\Delta_1}(\tau) \cdot \Theta_{\Delta_2}(\tau).$$

Corollary 4.15. *Let Δ be an irreducible, finite type root system.*

1. *If the rank of Δ is even, then $\Theta_\Delta(\tau) \in M_{n/2}(\Gamma(N,k),\vartheta)$, where $N = km$, and*

$$\vartheta(A) = \left(\frac{(-1)^{n/2} k^n |C_\Delta|}{d} \right).$$

2. *If the rank of Δ is odd and Δ is not of type A, then $\Theta_{\Delta \oplus A_1}(\tau) \in M_{(n+1)/2}(\Gamma(N,k),\vartheta)$, where $N = km$, and the multiplier system is*

$$\vartheta(A) = \left(\frac{(-1)^{(n+1)/2} 4k^n |C_\Delta|}{d} \right).$$

Proof. Part (1) follows from Remark 4.8 and Corollary 4.10.

If Δ is not of type A, then $2|k$ and $8|km$. By Corollary 4.10, $\Theta_\Delta(\tau)$ is a modular form for $\Gamma(km,k)$, and $\Theta_{A_1}(\tau)$ is a modular form for $\Gamma(8,2)$. Hence, their product is a modular form for $\Gamma(km,k)$. The formula of the multiplier system follows from Part (1) and from that $2|C_{A_1}| = 4$. □

Remark 4.16. In Section **??** below we perform computer calculations. In the odd rank cases we found it better to work with $\Theta_\Delta(\tau) \cdot \Theta_{A_1}(\frac{k}{2}\tau)$. With the same reasoning as above, $\Theta_{A_1}(\frac{k}{2}\tau)$ is a modular form for $\Gamma(4k,k)$. Since in all non-type A, odd rank cases $4|m$, we have that $\Theta_\Delta(\tau) \cdot \Theta_{A_1}(\frac{k}{2}\tau)$ is a modular form for $\Gamma(km,k)$. Moreover, as the determinant of $kC_\Delta \oplus kC_{A_1}$ is $2k^{n+1}|C_\Delta|$, the multiplier system is

$$\vartheta(A) = \left(\frac{(-1)^{(n+1)/2} 2k^{n+1} |C_\Delta|}{d} \right).$$

5 Modular forms II: Eta products

5.1 Eta products

An eta products is a finite product

$$f(\tau) = \prod_m \eta(m\tau)^{a_m} \tag{14}$$

where m runs through a finite set of positive integers and the exponents a_m may take values from $\mathbb{Z}$. The least common multiple of all m such that $a_m \neq 0$ will be denoted by N; it is called the minimum level of $f(\tau)$.

For a general eta quotient $f(\tau)$ as in (14), let $k = \sum_m a_m$. The expression $f(\tau)(\mathrm{d}\tau)^{k/2}$ transforms as a $k/2$-differential due to the transformation law of the Dedekind eta function. Since $\eta(\tau)$ is nonzero on $\mathbb{H}$, (quotients of) eta products never has finite poles. The only issue for an eta product to be a (possibly half-integral weight) modular form is whether the numerator vanishes to at least the same order as the denominator at each cusp.

Theorem 5.1 ([7, Theorem 3])**.** *Let f be an eta product as in* (14) *such that $n = \sum_{m|N} a_m$ is even. Let $s = \prod_{m|N} m^{a_m}$, $\frac{1}{24}\sum_{m|N} ma_m = c/e$ and $\frac{1}{24}\sum_{m|N} \frac{N}{m} a_m = c_0/e_0$, both in lowest terms. Then $f(\tau)$ is a modular form of weight $n/2$ for $\Gamma_0(Ne_0) \cap \Gamma^0(e)$ with the multiplier system defined by the Dirichlet character (mod Ne_0)*

$$\gamma(A) = \left(\frac{(-1)^{n/2}s}{a}\right)$$

for $a > 0$, $gcd(a, 6) = 1$.

Remark 5.2.

1. The fact that the $\gamma(A)$ values for $a > 0$, $gcd(a, 6) = 1$ are enough to define a multiplier system follows from [16, Lemma 3], and the multiplier system was calculated originally in [16, Theorem 1].

2. Since $N|c$ and $ad - bc = 1$, we have that $ad \equiv 1 \pmod{m}$ for all $m|N$. This means that
$$\left(\frac{a}{m}\right) = \left(\frac{d}{m}\right),$$
or equivalently, that
$$\left(\frac{m}{a}\right) = \left(\frac{m}{d}\right).$$
Hence, the multiplier system in Theorem 5.1 can also be written as
$$\gamma(A) = \left(\frac{(-1)^{n/2}s}{d}\right).$$
In this case $d > 0$, $gcd(d, 6) = 1$ is required.

The content of Theorem 5.1 is explained in [8, Section 1]. In the case when $\sum_{m|N} \frac{N}{m} a_m \equiv 0 \pmod{24}$, $f(\tau)$ has an integral order at 0. If this condition is not satisfied for N, it can be guaranteed by replacing N with Ne_0. In effect this *widens* the cusp of $\Gamma_0(N)$ at 0 by a factor of e_0. Similarly, $\sum_{m|N} ma_m \equiv 0 \pmod{24}$ if and only if $f(\tau)$ has an integral order at the cusp at $i\infty$. If this is not the case, widening the cusp $\Gamma_0(Ne_0)$ at $i\infty$ can be achieved by passing to the subgroup $\Gamma_0^0(Ne_0, e) = \Gamma_0(Ne_0) \cap \Gamma^0(e)$. The numbers e_0 and e are called the ramification numbers of $f(\tau)$ at 0 and $i\infty$ respectively. We will say that $f(\tau)$ is unramified if $e = e_0 = 1$.

5.2 Sturm bounds and the proof of Theorem 1.2

Let Δ be a simply laced root system. We introduce the notations

$$Z_\Delta(\tau) = q^{-\frac{1}{24}} Z_{[\mathbb{C}^2/G_\Delta]}(q) = \frac{\Theta_\Delta(\tau)}{\eta(k\tau)^n}.$$

We will show that in the cases when $\Delta = A_n$, $n \geq 1$, D_4, D_6, D_7 or E_6 the functions $\Theta_\Delta(\tau)$, and hence $Z_\Delta(\tau)$, can be expressed as eta products. Conjecturally the same statement holds for all ADE types.

We will denote the eta products on the right hand sides of Conjecture 2.3 by $\eta_\Delta(\tau)$. Then Theorem 1.2 boils down to showing that $\Theta_\Delta(\tau) = \eta_\Delta(\tau)$. To do this we want to show that both of them are holomorphic modular forms for the same congruence subgroup and with the same multiplier system.

Theorem 5.1 provides the multiplier system of $\eta_\Delta(\tau)$ only for root systems of even rank. It is possible to obtain an analogue of Theorem 5.1 for root systems of odd rank. Since these calculations would be too circuitous, we reduce instead to the case of root systems of even rank by taking a direct sum with A_1. We can do this because the identity $\Theta_{A_1}(\tau) = \eta_{A_1}(\tau)$ holds by Theorem 2.4. More precisely, for computational reasons in the odd rank cases we will show that

$$\Theta_\Delta(\tau) \cdot \Theta_{A_1}\left(\frac{k}{2}\tau\right) = \eta_\Delta(\tau) \cdot \eta_{A_1}\left(\frac{k}{2}\tau\right).$$

Lemma 5.3. *If Δ is an irreducible simply laced root system of rank n with $|G_\Delta| = k$, then for $\eta_\Delta(\tau)$ the numbers appearing in Theorem 5.1 are as follows:*

$$e = \frac{24}{gcd(24, k(n+1)-1)}; \quad e_0 = 1; \quad s = k^n |C_\Delta|; \quad N = k.$$

Proof. Direct calculation shows that in each case $\sum_{m|N} ma_m = k(n+1) - 1$, and $\sum_{m|N} \frac{N}{m} a_m = 0$. The third identity is also straightforward. $\square$

Lemma 5.4. *If Δ is an irreducible simply laced root system of odd rank n with $|G_\Delta| = k$, then for $\eta_\Delta(\tau) \cdot \eta_{A_1}(\frac{k}{2}\tau)$ the numbers appearing in Theorem 5.1 are as follows:*

$$e = \frac{24}{gcd(24, k(n+\frac{5}{2})-1)}; \quad e_0 = 1; \quad s = 2k^{n+1}|C_\Delta|; \quad N = k.$$

Proof. For $\eta_{A_1}(\frac{k}{2}\tau) = \eta^2(k\tau)\eta^{-1}(\frac{k}{2}\tau)$, $\sum_{m|N} ma_m = \frac{3k}{2}$ and $\prod_{m|N} m^{a_m} = 2k$. $\square$

Corollary 5.5. *Let Δ be a simply laced root system.*

1. *If n is even, let e be as Lemma 5.3. Then the function $\eta_\Delta(\tau)$ is a modular form of weight $\frac{n}{2}$ for $\Gamma_0(k) \cap \Gamma^0(e)$ with the multiplier system defined by*
$$\gamma(A) = \left(\frac{(-1)^{n/2}k^n|C_\Delta|}{d}\right).$$
2. *If n is odd, let e be as Lemma 5.4. Then the function $\eta_\Delta(\tau)\cdot\eta_{A_1}(\frac{k}{2}\tau)$ if a modular form of weight $\frac{n+1}{2}$ for $\Gamma_0(k) \cap \Gamma^0(e)$ with the multiplier system defined by*
$$\gamma(A) = \left(\frac{(-1)^{(n+1)/2}2k^{n+1}|C_\Delta|}{d}\right).$$

Proof. Follows from Theorem 5.1. $\square$

Lemma 5.6. *Let Δ be an irreducible simply laced root system. Let $k = |G_\Delta|$, and let e be as in Corollary 5.5.*

1. *If Δ is either of type D and even rank or of type E, then $e|k$. As a consequence, $\Gamma(km, m) \cap \Gamma_0(k) \cap \Gamma^0(e) = \Gamma(km, m)$.*
2. *If Δ is of type D and odd rank, then $e|2k$. As a consequence, $\Gamma(km, m) \cap \Gamma_0(k) \cap \Gamma^0(e) = \Gamma(km, m/2)$.*

Proof. We will show that $\Gamma(km, m)$ (resp. $\Gamma(km, m/2)$) is contained in $\Gamma_0(k) \cap \Gamma^0(e)$. For this we only need that $e|k$ (resp. $e|2k$). In the type E case this is automatic, since $e|24$ always and $24|k$ in all three cases.

Let Δ be of type D whose rank n is even. Then $k(n+1)-1 = (4n-8)(n+1)-1 = 4n^2 - 4n - 9$, which is always an odd number. The divisors of 24 are 2 and 3. So the only possibilities for e are 8 and 24 depending on whether $4n^2 - 4n - 9$ is divisible by 3 or not. Now $4n^2 - 4n - 9 = 4n(n-1) - 9$, so it is not divisible by 3 if and only if $n \equiv -1(\mod 3)$. Hence $e = 24$ if and only if $n = 6l + 2$ for some integer l. But this means that $k = 4n - 8 = 4(6l + 2) - 8 = 24l$, so $24|k$. In the cases when $n = 6l$ (resp. $n = 6l + 4$) the order $k = 24l - 8$ (resp. $k = 24l + 8$). So in both cases $8|k$.

Suppose now that Δ is of type D whose rank n is odd. Then $k(n + \frac{5}{2}) - 1) = (4n-8)(n+\frac{5}{2}) - 1 = 4n^2 + 2n - 21$, which is again always an odd number. Similarly as above, it is not divisible by 3 if and only if $n \equiv -1(\mod 3)$. If this is the case, then $e = 24$ and $n = 6l+5$ for some integer l. Then, $k = 4n-8 = 4(6l+5)-8 = 24l-12$, so $e|2k$. The other case is when $e = 8$. Then either $n = 6l + 1$ for some integer l and hence $k = 4n - 8 = 4(6l + 1) - 8 = 24l - 4$, or $n = 6l + 3$ and hence $k = 4n - 8 = 4(6l + 3) - 8 = 24l - 12$. In both cases $e|2k$. $\square$

Δ	k	m	Γ	$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$	Sturm bound
D_4	8	2	$\Gamma(16,2)$	768	128
$D_5 \oplus A_1$	12	8	$\Gamma(96,4)$	36864	9216
D_6	16	4	$\Gamma(64,4)$	12288	3072
E_6	24	3	$\Gamma(72,3)$	20736	5184

Table 2: Sturm bounds

Corollary 5.7. *Let Δ be an irreducible simply laced root system of rank n. Let $k = |G_\Delta|$, and let e be as in Corollary 5.5. Let*

$$\Gamma = \begin{cases} \Gamma(km, m), & \text{if } \Delta \text{ is of type D and } n \text{ is even, or } \Delta \text{ is of type E} \\ \Gamma(km, m/2), & \text{if } \Delta \text{ is of type D and } n \text{ is odd.} \end{cases}$$

Then both

$$\begin{cases} \eta_\Delta(\tau) \text{ and } \Theta_\Delta(\tau), & \text{if } n \text{ is even,} \\ \eta_\Delta(\tau) \cdot \eta_{A_1}(\frac{k}{2}\tau) \text{ and } \Theta_\Delta(h\tau) \cdot \Theta_{A_1}(\frac{k}{2}\tau), & \text{if } n \text{ is odd,} \end{cases}$$

are modular forms for Γ of the same weight and they have the same multiplier system.

Proof. Follows from Corollary 4.10, Corollary 5.5 and Lemma 5.6. □

The next result gives a limit up to which the vanishing of the Fourier coefficients of a modular form guarantees the vanishing of the modular form. It is generally known as the Sturm bound.

Theorem 5.8 ([17, Theorem 1])**.** *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, n be a positive even integer, and ϑ be a multiplier system for Γ. Let $f = \sum_{m=0}^{\infty} a(m)q^m \in M_{n/2}(\Gamma, \vartheta)$. If $a(m) = 0$ for all $m \leq \frac{n}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$, then $f = 0$. As a consequence, if the Fourier coefficients of two modular forms in $M_{n/2}(\Gamma, \vartheta)$ agree at least up to degree $\frac{n}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$, then the two modular forms are equal.*

Corollary 5.9. *For every fixed Δ Conjecture 2.3 can be checked numerically.*

Proof. Because of Theorem 5.8 one only has to check whether the q-expansions of the two functions from Corollay 5.7 agree at least up to order $\frac{n}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N, m)]$. □

Proposition 5.10. *Conjecture 2.3, and hence the local expressions in Theorem 1.2 are true in the cases when $\Delta = D_4, D_5, D_6$ or E_6.*

Proof. Taking into account the index formula from Lemma 4.2, the groups $\Gamma(km, m)$ (resp. $\Gamma(km, m/2)$) and the corresponding Sturm bounds are calculated in Table 2. We performed a computer check in each case and found that the Fourier coefficients agree at least up to the appropriate bound.

□

A Some proofs

A.1 Proof of Lemma 4.2

The following indices are known [4, Section 1.2]:

$$[\mathrm{Sl}_2(\mathbb{Z}) \,:\, \Gamma(N)] = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), \tag{15}$$

$$[\Gamma_0(N) \,:\, \Gamma_1(N)] = \phi(N), \tag{16}$$

$$[\Gamma_1(N) \,:\, \Gamma(N)] = N. \tag{17}$$

The index (16) is because of the following. Due to $c \equiv 0 \pmod N$ we must have $ad \equiv 1 \pmod N$. This equation has $\phi(N)$ solutions, but we only allow one of these in $\Gamma_1(N)$: the one with $a \equiv d \equiv 1 \pmod N$. The $\phi(N)$ residue classes modulo N are distributed uniformly into the $\phi(N/m)$ relative prime residue classes modulo N/m. Hence, the congruence $a \equiv 1 \pmod{N/m}$ has $\phi(N)/\phi(N/m)$ residue classes as solutions. If $N/m = 2$, this is all the solutions of $a \equiv \pm 1 \pmod{N/m}$. If $N/m > 2$, then the number of solutions of $a \equiv \pm 1 \pmod{N/m}$ is $2\phi(N)/\phi(N/m)$. As a consequence, the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv \pm 1 \pmod{N/m}, c \equiv 0 \pmod N \right\} \tag{18}$$

in $\Gamma_0(N)$ has index $\phi(N)/\phi(N/m) = \phi(N)$ if $N/m = 2$, and $2\phi(N)/\phi(N/m)$ if $N/m > 2$.

Second, the index (17) comes from requireing $b \equiv 1 \pmod N$. Similarly the index of $\Gamma(N, m)$ inside the group defined in (18) is N/m. Combining all these we obtain that

$$[\Gamma_0(N) \,:\, \Gamma(N,m)] = \begin{cases} 2N\phi(N), & \text{if } N/m = 2 \\ \frac{2N\phi(N)}{m\phi(N/m)}, & \text{if } N/m > 2. \end{cases}$$

The index of $\Gamma_0(N)$ inside $\mathrm{SL}_2(\mathbb{Z})$ turns out to be

$$[\mathrm{SL}_2(\mathbb{Z}) \,:\, \Gamma_0(N)] = \frac{N^2}{\phi(N)} \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Hence,

$$\begin{aligned} [\mathrm{SL}_2(\mathbb{Z}) \,:\, \Gamma(N,m)] &= [\mathrm{SL}_2(\mathbb{Z}) \,:\, \Gamma_0(N)] \cdot [\Gamma_0(N) \,:\, \Gamma(N,m)] \\ &= \begin{cases} 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), & \text{if } N/m = 2 \\ 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \prod_{p|N/m} \left(\frac{p}{p-1}\right), & \text{if } N/m > 2 \end{cases} \end{aligned}$$

where we have used the following expression for the ϕ function:

$$\phi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right).$$

A.2 Proof of Lemma 4.12

We follow the notations of [1].

Type A_n, $n \geq 1$. In this case $\dim\rho_i = 1$, $0 \leq i \leq n$. This implies that

- $k = n + 1 = h$, the (dual) Coxeter number of the root system,
- and $\zeta = C_\Delta^{-1} \cdot (1, \ldots, 1) = \rho$, the sum of the positive roots.

The "strange formula" of Freudenthal-de Vries [5] says that for any simple Lie algebra:

$$\frac{(\rho|\rho)}{2h} = \frac{\rho^{\mathsf{t}} \cdot C_\Delta \cdot \rho}{2h} = \frac{\dim\mathfrak{g}_\Delta}{24}.$$

It is known that $\dim\mathfrak{g}_{A_n} = n(n+2)$. Hence,

$$\frac{(\zeta|\zeta)}{2k} = \frac{\rho^{\mathsf{t}} \cdot C_\Delta \cdot \rho}{2h} = \frac{n(n+2)}{24} = \frac{(n+1)^2 - 1}{24}$$

as claimed. □

Type D_n, $n \geq 4$. In this and the remaining cases we will do direct calculation.

Let $V = \mathbb{R}^n$ and let $\varepsilon_1, \ldots, \varepsilon_n$ be the canonical basis of V. Put

$$\begin{aligned} \alpha_1 &= \varepsilon_1 - \varepsilon_2 \\ \alpha_2 &= \varepsilon_2 - \varepsilon_3 \\ &\vdots \\ \alpha_{n-1} &= \varepsilon_{n-1} - \varepsilon_n \\ \alpha_n &= \varepsilon_{n-1} + \varepsilon_n. \end{aligned}$$

Then $\alpha_1, \ldots, \alpha_n$ is the set of simple positive roots for Δ of type D_n. The fundamental weights are

$$\begin{aligned} \omega_i &= \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \\ &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} \cdots + \alpha_{n-2}) + \frac{1}{2}i(\alpha_{n-1} + \alpha_n) \end{aligned}$$

for $i < n - 1$, and

$$\begin{aligned} \omega_{n-1} &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-2} + \varepsilon_{n-1} - \varepsilon_n) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n), \\ \omega_n &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n). \end{aligned}$$

Moreover,

$$\zeta = C_\Delta^{-1} \cdot (1, 2, \ldots, 2, 1, 1) = \omega_1 + 2\omega_2 + \cdots + 2\omega_{n-2} + \omega_{n-1} + \omega_{n-2}.$$

A quick computation shows that in terms of the roots $\alpha_1, \dots, \alpha_n$

$$\zeta = \sum_{i=1}^{n-2} 2\left(in - \frac{(i+1)(i+1)}{2}\right)\alpha_i + \frac{n^2-2n}{2}(\alpha_{n-1}+\alpha_n).$$

Hence,

$$\begin{aligned}(\zeta|\zeta) &= (1,2,\dots,2,1,1)^{\mathsf{t}} \cdot C_\Delta^{-1} \cdot (1,2,\dots,2,1,1)\\ &= -(2n-4) + 4\left((n-1)\sum_{i=1}^{n-2} i - \frac{1}{2}\sum_{i=1}^{n-2} i^2 - \frac{n-2}{2}\right) + n^2 - 2n\\ &= \frac{4(n-1)(n-2)(n-1)}{2} - \frac{4(n-2)(n-1)(2n-3)}{12} + n^2 - 6n + 8\\ &= \frac{4n^3 - 12n^2 - n + 8}{3} = \frac{(n-2)(4n^2-4n-9)}{3}.\end{aligned}$$

This implies that

$$\frac{(\zeta|\zeta)}{2k} = \frac{(n-2)(4n^2-4n-9)}{24(n-2)} = \frac{4n^2-4n-9}{24}.$$

This is equal to

$$\frac{(n+1)k-1}{24} = \frac{(n+1)(4n-8)-1}{24}$$

as claimed. $\square$

Type E_6. In this case $\mathbf{d} = (1,2,3,2,2,1)^{\mathsf{t}}$, and

$$\frac{\mathbf{d}^{\mathsf{t}} \cdot C_{E_6}^{-1} \cdot \mathbf{d}}{2 \cdot 24} = \frac{7 \cdot 24 - 1}{24}.$$

$\square$

Type E_7. In this case $\mathbf{d} = (1,2,3,4,2,3,2)^{\mathsf{t}}$, and

$$\frac{\mathbf{d}^{\mathsf{t}} \cdot C_{E_7}^{-1} \cdot \mathbf{d}}{2 \cdot 48} = \frac{8 \cdot 48 - 1}{24}.$$

$\square$

Type E_8. In this case $\mathbf{d} = (2,3,4,5,6,3,4,2)^{\mathsf{t}}$, and

$$\frac{\mathbf{d}^{\mathsf{t}} \cdot C_{E_8}^{-1} \cdot \mathbf{d}}{2 \cdot 120} = \frac{9 \cdot 120 - 1}{24}.$$

$\square$

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B Formulations of the conjecture

This section is temporary, just to record this somewhere.

B.1 Number theoretic formulation

The natural number solutions (E, F, V, M) to the equation

$$E + F + V = M + 2$$

where E, F, and V each divide M have an ADE classification. For a given ADE root system Δ and corresponding solution (E, F, V, M), our conjecture is then equivalent to

$$Z_\Delta(q) = \frac{\eta^2(2\tau)}{\eta(\tau)} \cdot \left\{ \frac{\eta(2M\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)} \right\}.$$

B.2 Group theoretic formulation

Solutions to the number theoretic formulation have a group theory interpretation: Let $H \subset SO(3)$ be a finite subgroup. Then H has a presentation of the form

$$H = \{\langle \alpha, \beta, \gamma \rangle : \quad \alpha\beta\gamma = \alpha^p = \beta^q = \gamma^r = 1\}$$

where

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{2}{M}, \quad p, q, r | M$$

and $M = |H|$.

The above equation is equivalent to the previous formulation with

$$pE = qF = rV = M.$$

The classification of solutions is given by the following table.

Type	H	M	(p, q, r)	(E, F, V)
A_n	cyclic	$n+1$	$(1, n+1, n+1)$	$(n+1, 1, 1)$
D_n	dihedral	$2n-2$	$(2, 2, n-2)$	$(n-1, n-1, 2)$
E_6	tetrahedral	12	$(2, 3, 3)$	$(6, 4, 4)$
E_7	octahedral	24	$(2, 3, 4)$	$(12, 8, 6)$
E_8	icosohedral	60	$(2, 3, 5)$	$(30, 20, 12)$

Note that in D_n and E_n cases, the numbers (E, F, V) can be interpreted as the number of edges, faces, and vertices of the regular polyhedron with the corresponding symmetry group. For E_n these are the platonic solids; for D_n these are the degenerate regular polyhedrons obtained from gluing two regular $(n-1)$-gons to each other.

The groups $H \subset SO(3)$ are closely related to the groups $G \subset SU(2)$ also classified by ADE root systems. For the A_n case, both are isomorphic to $\mathbb{Z}_{n+1}$. For the D_n and E_n cases, the group G is the double cover of the group H induced by the degree 2 map $SU(2) \to SO(3)$.

C Table of eta products

The following table provides the list of the modular forms $Z_{X,G}^{-1}$, expressed as eta products, for each of the 82 possible symplectic actions of a group G on a $K3$ surface X. Our numbering matches Xiao's [18] whose table we refer to for a description of each group.

#	$\lvert G\rvert$	Singularities of X/G	The modular form $Z_{X,G}^{-1}$	Weight
0	1		$\eta(\tau)^{24}$	12
1	2	$8A_1$	$\eta(2\tau)^8\,\eta(\tau)^8$	8
2	3	$6A_2$	$\eta(3\tau)^6\,\eta(\tau)^6$	6
3	4	$12A_1$	$\eta(2\tau)^{12}$	6
4	4	$2A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^2\,\eta(\tau)^4$	5
5	5	$4A_4$	$\eta(5\tau)^4\,\eta(\tau)^4$	4
6	6	$8A_1+3A_2$	$\frac{\eta(3\tau)^8\eta(2\tau)^3}{\eta(6\tau)}$	5
7	6	$2A_1+2A_2+2A_5$	$\eta(6\tau)^2\,\eta(3\tau)^2\,\eta(2\tau)^2\,\eta(\tau)^2$	4
8	7	$3A_6$	$\eta(7\tau)^3\,\eta(\tau)^3$	3
9	8	$14A_1$	$\frac{\eta(4\tau)^{14}}{\eta(8\tau)^4}$	5
10	8	$9A_1+2A_3$	$\frac{\eta(4\tau)^9\eta(2\tau)^2}{\eta(8\tau)^2}$	9/2
11	8	$4A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^4$	4
12	8	$3A_3+2D_4$	$\frac{\eta(\tau)^2\eta(4\tau)^6}{\eta(2\tau)}$	7/2
13	8	A_1+4D_4	$\frac{\eta(4\tau)^{13}\eta(\tau)^4}{\eta(8\tau)^2\eta(2\tau)^8}$	7/2
14	8	$A_1+A_3+2A_7$	$\eta(8\tau)^2\,\eta(4\tau)\,\eta(2\tau)\,\eta(\tau)^2$	3
15	9	$8A_2$	$\eta(3\tau)^8$	4
16	10	$8A_1+2A_4$	$\frac{\eta(5\tau)^8\eta(2\tau)^2}{\eta(10\tau)^2}$	4
17	12	$4A_1+6A_2$	$\frac{\eta(6\tau)^4\eta(4\tau)^6}{\eta(12\tau)^2}$	4
18	12	$9A_1+A_2+A_5$	$\frac{\eta(6\tau)^9\eta(4\tau)\eta(2\tau)}{\eta(12\tau)^3}$	4
19	12	$3A_1+3A_5$	$\eta(6\tau)^3\,\eta(2\tau)^3$	3
20	12	$A_2+2A_3+2D_5$	$\frac{\eta(4\tau)^3\eta(3\tau)^2\eta(\tau)^2\eta(6\tau)^4}{\eta(12\tau)\eta(2\tau)^4}$	3
21	16	$15A_1$	$\frac{\eta(8\tau)^{15}}{\eta(16\tau)^6}$	9/2
22	16	$10A_1+2A_3$	$\frac{\eta(8\tau)^{10}\eta(4\tau)^2}{\eta(16\tau)^4}$	4
23	16	$5A_1+4A_3$	$\frac{\eta(8\tau)^5\eta(4\tau)^4}{\eta(16\tau)^2}$	7/2
24	16	$6A_1+A_3+2D_4$	$\frac{\eta(8\tau)^{12}\eta(2\tau)^2}{\eta(16\tau)^4\eta(4\tau)^3}$	7/2

25	16	$6A_3$	$\eta\left(4\tau\right)^6$	3
26	16	$4A_1+A_3+A_7+D_4$	$\frac{\eta(8\tau)^7\eta(2\tau)^2}{\eta(16\tau)^2\eta(4\tau)}$	3
27	16	$2A_1+4D_4$	$\frac{\eta(8\tau)^{14}\eta(2\tau)^4}{\eta(4\tau)^8\eta(16\tau)^4}$	3
28	16	$2A_1+A_3+2A_7$	$\eta\left(8\tau\right)^2\eta\left(4\tau\right)\eta\left(2\tau\right)^2$	5/2
29	16	$A_3+D_4+2D_6$	$\frac{\eta(4\tau)\eta(8\tau)^7\eta(\tau)^2}{\eta(16\tau)^2\eta(2\tau)^3}$	5/2
30	18	$8A_1+4A_2$	$\frac{\eta(9\tau)^8\eta(6\tau)^4}{\eta(18\tau)^4}$	4
31	18	$2A_1+3A_2+2A_5$	$\frac{\eta(9\tau)^2\eta(6\tau)^3\eta(3\tau)^2}{\eta(18\tau)}$	3
32	20	$2A_1+4A_3+A_4$	$\frac{\eta(10\tau)^2\eta(5\tau)^4\eta(4\tau)}{\eta(20\tau)}$	3
33	21	$6A_2+A_6$	$\frac{\eta(7\tau)^6\eta(3\tau)}{\eta(21\tau)}$	3
34	24	$5A_1+3A_2+2A_3$	$\frac{\eta(12\tau)^5\eta(8\tau)^3\eta(6\tau)^2}{\eta(24\tau)^3}$	7/2
35	24	$4A_1+2A_2+2A_5$	$\frac{\eta(12\tau)^4\eta(8\tau)^2\eta(4\tau)^2}{\eta(24\tau)^2}$	3
36	24	$5A_1+A_3+A_5+D_5$	$\frac{\eta(12\tau)^7\eta(6\tau)\eta(2\tau)\eta(8\tau)}{\eta(24\tau)^3\eta(4\tau)}$	3
37	24	$2A_2+A_5+D_4+E_6$	$\frac{\eta(8\tau)^4\eta(4\tau)\eta(3\tau)\eta(12\tau)^4\eta(\tau)}{\eta(6\tau)^2\eta(24\tau)^2\eta(2\tau)^2}$	5/2
38	24	$2A_2+A_3+2E_6$	$\frac{\eta(8\tau)^6\eta(6\tau)\eta(\tau)^2\eta(12\tau)^2}{\eta(2\tau)^4\eta(24\tau)^2}$	5/2
39	32	$8A_1+3A_3$	$\frac{\eta(16\tau)^8\eta(8\tau)^3}{\eta(32\tau)^4}$	7/2
40	32	$9A_1+2D_4$	$\frac{\eta(16\tau)^{15}\eta(4\tau)^2}{\eta(32\tau)^6\eta(8\tau)^4}$	7/2
41	32	$3A_1+5A_3$	$\frac{\eta(16\tau)^3\eta(8\tau)^5}{\eta(32\tau)^2}$	3
42	32	$4A_1+2A_3+2D_4$	$\frac{\eta(16\tau)^{10}\eta(4\tau)^2}{\eta(32\tau)^4\eta(8\tau)^2}$	3
43	32	$5A_1+2A_7$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
44	32	$2A_1+2A_3+A_7+D_4$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
45	32	$3A_1+D_4+2D_6$	$\frac{\eta(16\tau)^{10}\eta(2\tau)^2}{\eta(32\tau)^4\eta(4\tau)^3}$	5/2
46	36	$2A_1+2A_2+4A_3$	$\frac{\eta(18\tau)^2\eta(12\tau)^2\eta(9\tau)^4}{\eta(36\tau)^2}$	3
47	36	$A_1+6A_2+A_5$	$\frac{\eta(18\tau)\eta(12\tau)^6\eta(6\tau)}{\eta(36\tau)^2}$	3
48	36	$6A_1+A_2+2A_5$	$\frac{\eta(18\tau)^6\eta(12\tau)\eta(6\tau)^2}{\eta(36\tau)^3}$	3
49	48	$5A_1+6A_2$	$\frac{\eta(24\tau)^5\eta(16\tau)^6}{\eta(48\tau)^4}$	7/2
50	48	$6A_2+2A_3$	$\frac{\eta(16\tau)^6\eta(12\tau)^2}{\eta(48\tau)^2}$	3
51	48	$5A_1+A_2+2A_3+A_5$	$\frac{\eta(24\tau)^5\eta(16\tau)\eta(12\tau)^2\eta(8\tau)}{\eta(48\tau)^3}$	3
52	48	$4A_1+3A_5$	$\frac{\eta(24\tau)^4\eta(8\tau)^3}{\eta(48\tau)^2}$	5/2
53	48	$A_1+A_2+2A_3+2D_5$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(12\tau)^2\eta(4\tau)^2}{\eta(48\tau)^3\eta(8\tau)^4}$	5/2
54	48	$4A_1+A_2+A_7+E_6$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(6\tau)\eta(2\tau)}{\eta(48\tau)^3\eta(4\tau)^2}$	5/2

55	60	$4A_1+3A_2+2A_4$	$\frac{\eta(30\tau)^4\eta(20\tau)^3\eta(12\tau)^2}{\eta(60\tau)^3}$	3
56	64	$5A_1+3A_3+D_4$	$\frac{\eta(32\tau)^8\eta(16\tau)\eta(8\tau)}{\eta(64\tau)^4}$	3
57	64	$6A_1+3D_4$	$\frac{\eta(32\tau)^{15}\eta(8\tau)^3}{\eta(64\tau)^6\eta(16\tau)^6}$	3
58	64	$3A_1+3A_3+A_7$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
59	64	$5A_3+D_4$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
60	64	$4A_1+A_3+2D_6$	$\frac{\eta(32\tau)^8\eta(16\tau)^3\eta(4\tau)^2}{\eta(64\tau)^4\eta(8\tau)^4}$	5/2
61	72	$4A_1+3A_2+A_3+D_5$	$\frac{\eta(36\tau)^6\eta(24\tau)^4\eta(18\tau)\eta(6\tau)}{\eta(72\tau)^4\eta(12\tau)^2}$	3
62	72	$3A_1+2A_3+2A_5$	$\frac{\eta(36\tau)^3\eta(18\tau)^2\eta(12\tau)^2}{\eta(72\tau)^2}$	5/2
63	72	$A_2+3A_3+2D_4$	$\frac{\eta(24\tau)\eta(9\tau)^2\eta(36\tau)^6}{\eta(72\tau)^3\eta(18\tau)}$	5/2
64	80	$3A_1+4A_4$	$\frac{\eta(40\tau)^3\eta(16\tau)^4}{\eta(80\tau)^2}$	5/2
65	96	$3A_1+3A_2+3A_3$	$\frac{\eta(48\tau)^3\eta(32\tau)^3\eta(24\tau)^3}{\eta(96\tau)^3}$	3
66	96	$2A_1+2A_2+A_3+2A_5$	$\frac{\eta(48\tau)^2\eta(32\tau)^2\eta(24\tau)\eta(16\tau)^2}{\eta(96\tau)^2}$	5/2
67	96	$2A_1+3A_2+A_7+D_4$	$\frac{\eta(48\tau)^5\eta(32\tau)^3\eta(12\tau)^2}{\eta(96\tau)^3\eta(24\tau)^2}$	5/2
68	96	$3A_1+2A_3+A_5+D_5$	$\frac{\eta(48\tau)^5\eta(24\tau)^2\eta(8\tau)\eta(32\tau)}{\eta(96\tau)^3\eta(16\tau)}$	5/2
69	96	$3A_1+2A_2+2E_6$	$\frac{\eta(48\tau)^5\eta(32\tau)^6\eta(4\tau)^2}{\eta(96\tau)^4\eta(8\tau)^4}$	5/2
70	120	$2A_1+A_2+2A_3+A_4+A_5$	$\frac{\eta(60\tau)^2\eta(40\tau)\eta(30\tau)^2\eta(24\tau)\eta(20\tau)}{\eta(120\tau)^2}$	5/2
71	128	$3A_1+2A_3+D_4+D_6$	$\frac{\eta(64\tau)^8\eta(32\tau)\eta(8\tau)}{\eta(128\tau)^4\eta(16\tau)}$	5/2
72	144	$A_1+4A_2+2A_5$	$\frac{\eta(72\tau)\eta(48\tau)^4\eta(24\tau)^2}{\eta(144\tau)^2}$	5/2
73	160	$2A_1+3A_3+2A_4$	$\frac{\eta(80\tau)^2\eta(40\tau)^3\eta(32\tau)^2}{\eta(160\tau)^2}$	5/2
74	168	$A_1+3A_2+2A_3+A_6$	$\frac{\eta(84\tau)\eta(56\tau)^3\eta(42\tau)^2\eta(24\tau)}{\eta(168\tau)^2}$	5/2
75	192	$2A_1+6A_2+D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)^6\eta(24\tau)}{\eta(192\tau)^4\eta(48\tau)^2}$	3
76	192	$2A_1+A_2+2A_3+A_5+D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)\eta(32\tau)\eta(24\tau)}{\eta(192\tau)^3}$	5/2
77	192	$2A_1+A_2+3A_3+E_6$	$\frac{\eta(96\tau)^3\eta(64\tau)^3\eta(48\tau)^3\eta(8\tau)}{\eta(192\tau)^3\eta(16\tau)^2}$	5/2
78	288	$2A_1+2A_2+A_3+2D_5$	$\frac{\eta(144\tau)^6\eta(96\tau)^4\eta(72\tau)\eta(24\tau)^2}{\eta(288\tau)^4\eta(48\tau)^4}$	5/2
79	360	$A_1+2A_2+2A_3+2A_4$	$\frac{\eta(180\tau)\eta(120\tau)^2\eta(90\tau)^2\eta(72\tau)^2}{\eta(360\tau)^2}$	5/2
80	384	$A_1+3A_2+2A_3+D_6$	$\frac{\eta(192\tau)^3\eta(128\tau)^3\eta(96\tau)^3\eta(24\tau)}{\eta(384\tau)^3\eta(48\tau)^2}$	5/2
81	960	$A_1+3A_2+2A_4+D_4$	$\frac{\eta(480\tau)^4\eta(320\tau)^3\eta(192\tau)^2\eta(120\tau)}{\eta(960\tau)^3\eta(240\tau)^2}$	5/2

Table 3: Table of the modular forms $Z_{X,G}^{-1}$ for all symplectic G actions.

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