EQUIVARIANT HILBERT SCHEME OF POINTS ON K3 SURFACES AND MODULAR FORMS

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Contents

1. Introduction

Let X be a smooth projective surface equipped with the action of a finite group G. Let $\mathrm{Hilb}^m(X)$ denote the Hilbert scheme of m points on X, the projective scheme parametrizing 0-dimensional subschemes of X of length m. Consider also the moduli space of G-invariant length m subschemes of X. In other words this is the invariant part of $\mathrm{Hilb}^m(X)$ under the lifted action of G. This Hilbert scheme is variously called the *orbifold Hilbert scheme* [23] or *equivariant Hilbert scheme* [11]. We will denote it as $\mathrm{Hilb}^m(X)^G$. We collect the topological Euler characteristics of these moduli spaces into generating functions. The *G-fixed generating series* is defined as:

$$Z_{[X/G]}(q) = 1 + \sum_{m=1}^{\infty} \chi(\mathrm{Hilb}^m(X)^G) q^m.$$

When the group action is trivial, we get back the much investigated generating series of the Euler characteristics of the usual Hilbert scheme of points on X:

$$Z_X(q) = 1 + \sum_{m=1}^{\infty} \chi(\mathrm{Hilb}^m(X)) q^m.$$

In the case when X is K3 surface it is known from [10] that the function

$$f_X(\tau) = \frac{q}{Z_X(q)}\Big|_{q=e^{2\pi i \tau}}$$

is equal to

$$f_X(\tau) = \eta^{24}(\tau),$$

where $\eta(\tau) = q^{1/24}\prod_{m\geq 1}(1-q^m)$ is the Dedekind eta function. The function $f_X(\tau)$ in turn the discriminant form $\Delta(\tau)$, which is a cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. In this paper we consider the G-fixed generating series when the finite group G acts on X symplectically. More precisely, our protagonist will be

$$f_{[X/G]}(\tau) = \frac{q}{Z_{[X/G]}}(q)\Big|_{q=\mathrm{e}^{2\pi i\tau}}.$$

The main result of the paper is the following.

Theorem 1.1. *If a finite group G acts on a K3 surface X symplectically, then $f_{[X/G]}(\tau)$ is a modular form of weight $\frac{\chi(X)}{2}$ for $\Gamma_0(|G|)$ whose order of vanishing at the cusps at 0 and $i\infty$ is 1. The order of vanishing of $f_{[X/G]}(\tau)$ at the other cusps is always nonnegative if and only if the group action has at least two points whose stabilizer is the finite group of type E_6. If there are two points with E_6 stabilizer, then there is exactly one cusp where the order of vanishing is negative.*

In the special case when G is cyclic Theorem 1.1 was already deduced in [2]. As we recall in Section 5 below, a symplectic action of a finite group G on a K3 surface has at worst ADE singularities. That is, the stabilizers are always finite groups of type ADE. It is also known that an action as in Theorem 1.1 cannot have more than two points with E_6 stabilizer.

Our method to prove Theorem 1.1 is to show that at each such singularity a local analog of the function $f_X(\tau)$ can be written as a product of scaled Dedekind eta functions with (possibly negative) integer powers. Functions of this form are called eta products; see Section 4 below for a precise definition. We conjecture the explicit formulas of the local generating series for all ADE cases in Conjecture 4.2 below. An equivalent statement for the global generating series is as follows.

Conjecture 1.2. *Suppose that a finite group G acts on a surface X with singularities $\{P_1,\dots,P_r\}$ which are at worst of ADE type. Let $\{\Delta_1,\dots,\Delta_r\}$ be the root systems associated with the corresponding stabilizer groups $\{G_1,\dots,G_r\}$. Let $k_i = |G_i|$ and $k = |G|$. Then the function*

$$\frac{q^\alpha}{Z_{[X/G]}}(q)\Big|_{q=\mathrm{e}^{2\pi i\tau}},$$

for some suitable $\alpha\in\mathbb{Q}$, can be written as an eta product:

$$\frac{q^\alpha}{Z_{[X/G]}}(q)\Big|_{q=\mathrm{e}^{2\pi i\tau}} = \eta^{\chi(X^0/G)}(k\tau)\cdot\prod_{i=1}^{r} Z_{\Delta_i}^{-1}\left(\frac{k}{k_i}\tau\right).$$

Here $X^0\subset X$ is the part where G acts freely, and the local terms $Z_\Delta(\tau)$ are as in Conjecture 4.2 (2).

Our result for K3 surfaces is the following.

Theorem 1.3. *For X a K3 surface, Conjecture 1.2 is true with $\alpha = 1$. In particular, $f_{[X/G]}(\tau)$ can be written as an eta product with terms as in Conjecture 4.2 (2).*

The structure of the paper is the following. In Section 2 we summarize the necessary notions related to generating series of Hilbert schemes of points. In particular, we show how the global generating series decomposes to the product of local generating series. We also recall the most relevant notions of the theory of modular forms. In Section 3 we prove explicit results about the modularity of the local generating series. In Section 4 we express the local generating series as eta products. These results imply immediately Theorem 1.3. In Section 5 we investigate explicitely the modularity properties of the global generating series of K3 surfaces, and finish the proof of Theorem 1.1.

Acknowledgement. The authors would like to thank to Federico Amadio Guidi, Ken Ono and Balázs Szendrői for helpful comments and discussions.

2. Generating series and modular forms

2.1. **Generating series of Hilbert schemes of points.** Throughout the paper we work over the complex field $\mathbb{C}$. For a projective variety X let $\mathrm{Hilb}^m(X)$ denote the Hilbert scheme of m points on X, the projective scheme parametrizing 0-dimensional subschemes of X of length m. When X is equipped with the action of a finite group G, let $\mathrm{Hilb}^m(X)^G \subset \mathrm{Hilb}^m(X)$ be the orbifold Hilbert scheme consisting of the subschemes fixed by G. Being components of fixed point sets of a finite group acting on smooth projective varieties, the orbifold Hilbert schemes themselves are smooth and projective [3].

We assume that G has generically trivial stabilizers and that the points where the stabilizer is not trivial are isolated. By compactness there can only be a finite number of such points. Let $X^0 \subset X$ be the locus of points with trivial stabilizer, and let $\{P_1, \dots, P_k\}$ be the set of points with nontrivial stabilizers. Due to the by-now standard techniques of [12] the generating function $Z_{[X/G]}(q)$ has a product decomposition

$$Z_{[X/G]}(q) = Z_{[X^0/G]}(q) \cdot \prod_{j=1}^{k} Z^{(P_i,X)}(q). \tag{1}$$

Here the local term at P_i is

$$Z^{(P_i,X)}(q) = 1 + \sum_{m=1}^{\infty} \chi(\mathrm{Hilb}^m(X, P_i)^G) q^m,$$

where $\mathrm{Hilb}^m(X, P_i)^G \subset \mathrm{Hilb}^m(X)^G$ is the subscheme of (G-equivariant) ideals supported set-theoretically at P_i. We are interested only in motivic invariants of these moduli spaces, and these are well defined even if X is quasi-projective. Therefore, although the free locus X^0 is in general only quasi-projective, this does not cause any problem.

Let $k = |G|$. Since the degree k covering map $X^0 \to X^0/G$ is étale, the contribution of $[X^0/G]$ to $Z_{[X/G]}(q)$ can be written in the following product form using the main result of [10]:

$$Z_{[X^0/G]}(q) = Z_{X^0/G}(q^k) = \left(\prod_{m=1}^{\infty} (1 - q^{km})^{-1} \right)^{\chi(X^0/G)}. \tag{2}$$

Suppose that the quotient X/G is a surface with at worst simple (Kleinian, rational double point) singularities, and the projection $X \to X/G$ is unramified outside these singular points. It is known that locally analytically a simple surface singularity is isomorphic to a quotient $S = \mathbb{C}^2/G_\Delta$. Here $G_\Delta < \mathrm{SL}(2, \mathbb{C})$ is a finite subgroup corresponding to an irreducible simply-laced Dynkin diagram Δ, the dual graph of the exceptional components in the minimal resolution of the singularity. There are three possible types: Δ can be of type A_n for $n \geq 1$, type D_n for $n \geq 4$ and type E_n for $n = 6, 7, 8$. Let $\rho_0, \dots, \rho_n \in \mathrm{Rep}(G_\Delta)$ denote the (isomorphism classes of) irreducible representations of G_Δ, with ρ_0 the trivial representation. These irreducible representations correspond to vertices of the affine Dynkin diagram associated with

Δ. We will call Δ simply laced if it is any of types A_n for $n \geq 1$, D_n for $n \geq 4$, E_6, E_7 or E_8. We will refer to the corresponding orbifold $[\mathbb{C}^2/G_\Delta]$ as the simple singularity orbifold.

Suppose that in (1) the singular point P_i is of type Δ_i, $1 \leq i \leq n$. Then the local terms in (1) can be expressed as

$$Z^{(P_i,X)}(q) = Z_{[\mathbb{C}^2/G_{\Delta_i}]}(q). \tag{3}$$

2.2. **Some notions of modular forms.** In our discussion we will need to work with modular forms of half-integer weight. Fix a subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$, a function $\vartheta\colon \Gamma \to \mathbb{C}^*$ with $|\vartheta(A)| = 1$ for $A \in \Gamma$, and a half-integer k. Then a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is said to transform as a modular form of weight k with the multiplier system ϑ for Γ if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \vartheta(A)(c\tau+d)^k f(\tau) \quad \text{for all } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

When k is not an integer, $(c\tau+d)^k$ is understood to be a principal value. If moreover f is holomorphic at all the cusps of Γ on $\mathbb{Q} \cup \{i\infty\}$, then f is said to be a modular form. We will denote the space of modular forms of weight k and multiplier systems χ for Γ by $M_k(\Gamma, \vartheta)$.

We will need the following congruence subgroups:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\};$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\};$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod N \right\}.$$

Moreover, for a divisor $m|N$ let us introduce the following subset of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma(N,m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv \pm 1 \pmod{N/m}, b \equiv 0 \pmod{N/m}, c \equiv 0 \pmod N \right\}.$$

Lemma 2.1. *$\Gamma(N,m)$ is a congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. The two calculations are very similar; we just do it for $\Gamma(N,m)$. Since $ad - bc = 1$ and $c \equiv 0 \pmod{N/m}$, we have that $ad \equiv 1 \pmod{N/m}$. This and $a \equiv \pm 1 \pmod{N/m}$ implies that $a \equiv d \pmod{N/m}$. Hence the inverse

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ of } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N,m)$$

belongs to $\Gamma(N,m)$ as well. Similarly, the product

$$\begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ a_2c_1 + c_2d_1 & b_2c_1 + d_1d_2 \end{pmatrix}$$

of two matrices

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma(N,m)$$

is contained in $\Gamma(N,m)$. Finally, it follows from the definition of $\Gamma(N,m)$ that it contains $\Gamma(N)$. $\square$

Δ	n	m
A_n	odd	$2(n+1)$
	even	$n+1$
D_n	odd	8
	$n \equiv 2 \pmod 4$	4
	$n \equiv 0 \pmod 4$	2
E_6		3
E_7		4
E_8		1

TABLE 1. The numbers m.

Lemma 2.2. *The index of* $\Gamma(N,m)$ *inside* $\mathrm{SL}_2(\mathbb{Z})$ *is*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N,m)] = \begin{cases} 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), & \text{if } N/m = 2 \\ 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \prod_{p|N/m} \left(\frac{p}{p-1}\right), & \text{if } N/m > 2. \end{cases}$$

The proof of this statement will be given in Appendix **??**.

For a congruence subgroup Γ we will denote by $M_k(\Gamma, \vartheta)$ the space of modular forms with respect to the multiplier system ϑ.

3. LOCAL CONTRIBUTIONS OF SIMPLE SINGULARITIES

Proposition 3.1. *Let* $[\mathbb{C}^2/G_\Delta]$ *be a simple singularity orbifold and let*

$$N = \left(\sum_{i=0}^{n} (\mathrm{dim}\rho_i)^2 \right) m = |G_\Delta| m,$$

where m is as in Table 1. Then $q^{\frac{1}{24}}(Z_{[\mathbb{C}^2/G_\Delta]}(q))^{-1}$ *with the substitution* $q = \mathrm{e}^{2\pi i \tau}$ *is a (possibly meromorphic) modular form of weight* $1/2$ *in the variable* τ *for the congruence subgroup* $\Gamma(N,m)$.

3.1. **Orbifold Hilbert schemes.** To prove Proposition 3.1 we will make a digression to the orbifold viewpoint of $\mathrm{Hilb}(X)^G$. In the context of orbifolds or stacks this moduli space is also denoted as $\mathrm{Hilb}([X/G])$, because it is associated with the quotient stack $[X/G]$. Moreover, it decomposes also as

$$\mathrm{Hilb}([X/G]) = \bigsqcup_{\rho \in \mathrm{Rep}(G)} \mathrm{Hilb}^\rho([X/G]),$$

where

$$\mathrm{Hilb}^\rho([X/G]) = \{ I \in \mathrm{Hilb}(X)^G \colon H^0(\mathcal{O}_{\mathbb{C}^2}/I) \simeq_G \rho \}$$

for any finite-dimensional representation $\rho \in \mathrm{Rep}(G)$ of G; here $\mathrm{Hilb}(\mathbb{C}^2)^G$ is the set of G-invariant ideals of $\mathbb{C}[x,y]$, and $\simeq_G$ means G-equivariant isomorphism. Another generating series is associated with this decomposition. Recall that $\rho_0, \ldots, \rho_n \in \mathrm{Rep}(G)$ denotes the (isomorphism classes of) irreducible representations of G, with ρ_0 the trivial representation. Then, the *orbifold generating series* of the global quotient orbifold $[X/G]$ is defined as

$$Z_{[X/G]}(q_0, \ldots, q_n) = \sum_{m_0, \ldots, m_n = 0}^{\infty} \chi\left(\mathrm{Hilb}^{m_0\rho_0 + \ldots + m_n\rho_n}([X/G])\right) q_0^{m_0} \cdot \ldots \cdot q_n^{m_n}.$$

Lemma 3.2. *The G-fixed generating series of $[X/G]$ is obtained as the following specialization of the orbifold generating series:*

$$Z_{[X/G]}(q) = Z_{[X/G]}(q_0, \ldots, q_n)\Big|_{q_i = q^{\dim \rho_i}}.$$

Proof. Let I be an equivariant ideal such that $H^0(\mathcal{O}_{\mathbb{C}^2}/I) \simeq_G \rho$, where $\rho \simeq m_0\rho_0 + \ldots + m_n\rho_n$. This implies that

$$\dim H^0(\mathcal{O}_{\mathbb{C}^2}/I) = \sum_{i=0}^{n} m_i \dim \rho_i,$$

when I is considered as a non-equivariant ideal of $\mathbb{C}^2$ □

The orbifold generating series of a simple singularity orbifold is given explicitely by the following result.

Theorem 3.3 ([18])**.** *Let $[\mathbb{C}^2/G_\Delta]$ be a simple singularity orbifold. Then its orbifold generating series can be expressed as*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n) = \left(\prod_{m=1}^{\infty} (1 - \mathbf{q}^m)^{-1} \right)^{n+1} \cdot \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n} q_1^{m_1} \ldots q_n^{m_n} (\mathbf{q}^{1/2})^{\mathbf{m}^\top \cdot C_\Delta \cdot \mathbf{m}},$$

where $\mathbf{q} = \prod_{i=0}^n q_i^{d_i}$ with $d_i = \dim \rho_i$, and C_Δ is the finite type Cartan matrix corresponding to Δ.

3.2. **Theta functions of lattices.** It turns out that the function appearing in the numerator of Theorem 3.3 is a theta function associated with a shift of the root lattice of the corresponding finite Lie algebra. For theta functions and their modularity properties we will refer to [4, Chapter 14].

Let $L \cong \mathbb{Z}^n$ be an n dimensional lattice equipped with a real quadratic form Q, which we suppose to be integral and positive definite. That is $Q(L) \in \mathbb{Z}^+$. The associated symmetric bilinear form is obtained as

$$B(\mathbf{a}, \mathbf{b}) = Q(\mathbf{a} + \mathbf{b}) - Q(\mathbf{a}) - Q(\mathbf{b}).$$

Then

$$Q(\mathbf{a}) = \frac{1}{2} B(\mathbf{a}, \mathbf{a}).$$

Let moreover

$$L^* = \{\mathbf{b} \in \mathbb{R}^n : B(\mathbf{a}, \mathbf{b}) \in \mathbb{Z} \text{ for all } \mathbf{a} \in L\}$$

be the dual lattice to L in $L \otimes \mathbb{R} \cong \mathbb{R}^n$ with respect to Q.

Let $\mathbf{a} \in L^*$. The theta function associated with the lattice L and shifted by $\mathbf{a}$ is defined as

$$\Theta_{\mathbf{a}}(\tau) = \sum_{m \in L + \mathbf{a}} q^{Q(\mathbf{m})}\Big|_{q = e^{2\pi i \tau}}. \tag{4}$$

Remark 3.4. In [4, Definition 14.3.3] a much more general class of theta functions is introduced. We will use the conventions of [4, Example 14.2.5]. In our case the spherical polynomial $P(X)$ which appears in [4, 14.2.5] is equal to the constant function 1. Moreover, the number k appearing in [4, Example 14.2.5 and Definition 14.3.3] is equal to n in our case. In particular, $k \equiv n \pmod 2$.

We introduce the symbol

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv -1 \pmod 4). \end{cases} \tag{5}$$

Recall that L^* is the dual lattice to L with respect to B. Then $L \subset L^*$ always, and there is a smallest positive integer N for which

$$NL^* \subset L \text{ and } NQ(\mathbf{a}) \in \mathbb{Z} \text{ for all } \mathbf{a} \in L^*. \tag{6}$$

This number N is called the level in [4, Definition 14.3.15].

Elements of a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ will be denoted as follows:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7}$$

Then $A \in \Gamma_0(N)$ if and only if

$$cL^* \subset L \text{ and } cQ(\mathbf{a}) \in \mathbb{Z} \text{ for all } \mathbf{a} \in L^*.$$

Theorem 3.5. *Let $A \in \Gamma_0(N)$ and $D = \det(Q)$. Then*

$$\Theta_{\mathbf{a}}(A\tau) = \vartheta(A)(c\tau + d)^{n/2}\Theta_{a\mathbf{a}}(\tau)$$

for a multiplier system ϑ such that

$$\vartheta(A) = e^{2\pi i ab Q(\mathbf{a})} \left(\frac{D}{d}\right)\left(\epsilon_d^{-1}\left(\frac{2c}{d}\right)\right)^n,$$

when $c \neq 0$ and d is odd.

Proof. Corollary 14.3.8 and Theorem 14.3.11 from [4] implies that

$$\Theta_{\mathbf{a}}(A\tau) = (d, -1^q D)_\infty \vartheta(A)(c\tau + d)^{n/2}\Theta_{a\mathbf{a}}(\tau),$$

where q is the number of negative eigenvalues of Q, $(d, (-1)^q D)_\infty = -1$ if $d < 0$ and $(-1)^q D < 0$, and $(d, (-1)^q D)_\infty = 1$ otherwise. The form Q is positive definite. Hence $q = 0$ and $D > 0$. In turn $(d, (-1)^q D)_\infty = 1$ always.

□

Let s be the smallest integer, such $s\mathbf{a} \in L$. This is, in general, not the same as N, but $s|N$ always.

Corollary 3.6. *Suppose that in (6) the element $a \equiv \pm 1$ (mod s). Then*

$$\Theta_{\mathbf{a}}(A\tau) = \vartheta(A)(c\tau + d)^{n/2}\Theta_{\mathbf{a}}(\tau),$$

where ϑ is as in Theorem **??**.

Proof. Since $a \equiv \pm 1$ (*mod* s), $a\mathbf{a} \equiv \pm\mathbf{a}$ (*mod* L). It follows from the definition (4) that $\Theta_{\mathbf{a}}(\tau)$ depends only on the class of $\mathbf{a}$ modulo L. Furthermore, since

$$Q(\mathbf{m} - \mathbf{a}) = Q(-\mathbf{m} + \mathbf{a}),$$

$\Theta_{-\mathbf{a}}(\tau) = \Theta_{\mathbf{a}}(\tau)$. □

Lemma 3.7. *Let $\Gamma \subset \Gamma_0(N)$ be a subgroup such that $a \equiv \pm 1$ (mod s) for all $A \in \Gamma$. Then $\Theta_{\mathbf{a}}(\tau) \in M_{n/2}(\Gamma, \vartheta)$, where ϑ is as in Theorem* **??**.

Proof. Due to Corollary 3.8, $\Theta_{\mathbf{a}}(\tau)$ transforms as a modular form with the multiplier system ϑ for the elements of Γ.

Showing that it is holomorphic at the cusps is analogous to the proof of [4, Corollary 14.3.16]. By [4, Theorem 14.3.7] when an element $A \in \mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane, $\Theta_{\mathbf{a}}(A\tau)$ decomposes into a finite linear combination:

$$\Theta_{\mathbf{a}}(A\tau) = \sum_{\mathbf{b} \in L^*/L} c_{\mathbf{b}}(c\tau + d)^{n/2} \Theta_{\mathbf{b},k}(\tau).$$

It is known that the group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the cusps of Γ. Hence, to prove that $\Theta_{\mathbf{a}}(\tau)$ is holomorphic at all the cusps of Γ it is enough to show that $\Theta_{\mathbf{b}}(\tau)$ is holomorphic as $\tau \to i\infty$ for any $\mathbf{b} \in L^*/L$. Since the bilinear form is positive definite, $Q(\mathbf{b}) > 0$ for any $\mathbf{b} \neq 0$. Therefore the only term in

$$\Theta_{\mathbf{b}}(\tau) = \sum_{\mathbf{m} \in L+\mathbf{b}} e^{\pi i \tau k Q(\mathbf{m})}$$

which could not tend to 0 as $\tau \to i\infty$ is the one with $\mathbf{m} = -\mathbf{b}$. This term exists only if $\mathbf{b} \in L$, and in this case the limit is 1. The theorem follows. $\square$

Corollary 3.8. *$\Theta_{\mathbf{a}}(\tau) \in M_{n/2}(\Gamma(N,s), \vartheta)$ for a multiplier system ϑ such that*

$$\vartheta(A) = \left(\frac{D}{d}\right)\left(\epsilon_d^{-1}\left(\frac{2c}{d}\right)\right)^n,$$

when $c \neq 0$ and d is odd.

Proof. For the elements of $\Gamma(N,s)$, $a \equiv \pm 1$ $(mod\ s)$. Thus the conditions of Lemma 3.9 are satisfied. Moreover, since $b \equiv 0$ $(mod\ s)$, $abQ(\mathbf{a})$ is an integer. This implies that the term

$$e^{2\pi i ab Q(\mathbf{a})}$$

in Theorem **??** is equal to 1. $\square$

Remark 3.9. Suppose that the rank n of the lattice L is even. Then then the multiplier system in Corollary **??** simplifies as

$$\vartheta(A) = \left(\frac{(-1)^{n/2} D}{d}\right),$$

because $\epsilon_d^{-2} = -1$ and $\left(\frac{2c}{d}\right)^2 = 1$.

3.3. **Theta functions of root systems.** Let Δ be a root system of finite type, and let L be its root lattice.

If Δ is an irreducible root system of finite type and $B = (|)$, the standard invariant form, then the level N of the root lattice L defined in in (**??**) is equal to the number m listed in Table 1 [14, page 261]. In the standard basis $L \cong \mathbb{Z}^n$

$$B(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot C_\Delta \cdot \mathbf{b}^\top, \quad Q(\mathbf{a}) = \frac{1}{2}\mathbf{a} \cdot C_\Delta \cdot \mathbf{a}^\top,$$

and $D = \det(Q) = |C_\Delta|$.

We now set instead $B = k(|)$, where $k = |G_\Delta|$. In this case the level N of L is km,

$$B(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot kC_\Delta \cdot \mathbf{b}^\top, \quad Q(\mathbf{a}) = \frac{1}{2}\mathbf{a} \cdot kC_\Delta \cdot \mathbf{a}^\top,$$

and $D = \det(Q) = k^n |C_\Delta|$.

Lemma 3.10. *Let* $\mathbf{a} \in L^*$. *Then*

$$\Theta_{\mathbf{a}}(\tau) = (q^{k/2})^{\mathbf{a}^\top \cdot C_\Delta \cdot \mathbf{a}} \cdot \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n} q^{k(\mathbf{m}^\top \cdot \mathbf{a})}(q^{k/2})^{\mathbf{m}^\top \cdot C_\Delta \cdot \mathbf{m}}\Big|_{q=e^{2\pi i\tau}}.$$

Proof. Since $L \cong \mathbb{Z}^n$, one can rewrite (4) as

$$\sum_{\mathbf{m}\in\mathbb{Z}^n} (q^{k/2})^{(\mathbf{m}+\mathbf{a}|\mathbf{m}+\mathbf{a})}\Big|_{q=e^{2\pi i\tau}} = \sum_{\mathbf{m}\in\mathbb{Z}^n} (q^{k/2})^{(\mathbf{m}|\mathbf{m})+2(\mathbf{m}|\mathbf{a})+(\mathbf{a}|\mathbf{a})}\Big|_{q=e^{2\pi i\tau}}.$$

The pairing between $\mathbf{a} \in L^*$ and $\mathbf{m} \in L$ is just

$$(\mathbf{m}|\mathbf{a}) = \sum_{i=1}^n a_i m_i = \mathbf{m}^\top \cdot \mathbf{a}.$$

□

We now fix a particular shift vector $\mathbf{a}$. Let us denote the standard basis of L by $\{\alpha_1, \ldots, \alpha_n\}$ and the corresponding dual basis of L^* by $\{\omega_1, \ldots, \omega_n\}$. Let

$$\theta = (\text{dim}\rho_1, \ldots, \text{dim}\rho_n) = \sum_{i=1}^n (\text{dim}\rho_i)\alpha_i \in L. \tag{8}$$

Our $\mathbf{a} \in L^*$ will be the dual of θ with respect to $k(|)$. Explicitely, this means that

$$\mathbf{a} = \frac{1}{k}\sum_{i=1}^n (\text{dim}\rho_i)\omega_i = \sum_{i=1}^n a_i\alpha_i, \tag{9}$$

where $(a_1, \ldots, a_n) = (kC_\Delta)^{-1} \cdot \theta$. Finally, we introduce the notation

$$\Theta_\Delta(\tau) = \Theta_{\mathbf{a}}(\tau).$$

The next statement follows immediately from Corollary **??**.

Corollary 3.11. *The function* $\Theta_\Delta(\tau)$ *is a modular form of weight* $n/2$ *for* $\Gamma(km, k)$ *with a multiplier system* ϑ *such that*

$$\vartheta(A) = \left(\frac{k^n|C_\Delta|}{d}\right)\left(\epsilon_d^{-1}\left(\frac{2c}{d}\right)\right)^n,$$

when $c \neq 0$ *and* d *is odd.*

Proof of Proposition 3.1. Let

$$\zeta = k\mathbf{a} = \sum_{i=1}^n (\text{dim}\rho_i)\omega_i = \sum_{i=1}^n b_i\alpha_i, \tag{10}$$

where $(b_1, \ldots, b_n) = (C_\Delta)^{-1} \cdot \theta$. Then

$$Q(\mathbf{a}) = \frac{k}{2}\left(\mathbf{a}^\top \cdot C_\Delta \cdot \mathbf{a}\right) = \frac{\zeta^\top \cdot C_\Delta \cdot \zeta}{2k}.$$

Substituting this into the equation in Lemma 3.4 yields

$$\Theta_\Delta(\tau) = q^{\frac{\zeta^\top \cdot C_\Delta \cdot \zeta}{2k}} \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n} q^{\mathbf{m}^\top \cdot (\text{dim}\rho_1,\ldots,\text{dim}\rho_n)}(q^{k/2})^{\mathbf{m}^\top \cdot C_\Delta \cdot \mathbf{m}}\Big|_{q=e^{2\pi i\tau}}.$$

Up to the factor $q^{\frac{\zeta^\top \cdot C_\Delta \cdot \zeta}{2k}}$ this is exactly the numerator of $Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \dots, q_n)$ appearing in Theorem 3.3 when we substitute $q_i = q^{\dim \rho_i}$, $0 \leq i \leq n$. In the denominator of $Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \dots, q_n)$, after the same substitution, a product of $n+1$ terms of

$$\prod_{m=1}^{\infty} (1 - q^{km})$$

appears. As a consequence,

$$q^{\frac{k(n+1)}{24} - \frac{\zeta^\top \cdot C_\Delta \cdot \zeta}{2k}} (Z_{[\mathbb{C}^2/G_\Delta]}(q))^{-1}\Big|_{q=\mathrm{e}^{2\pi i \tau}} = \frac{(\eta(k\tau))^{n+1}}{\Theta_{\zeta/k,k}(\tau)}.$$

The Dedeking eta function $\eta(\tau)$ is a modular form of weight 1/2 for $\Gamma(1)$. Hence, $\eta(k\tau)$ is a modular form of weight 1/2 for $\Gamma(k)$. By Lemma 3.13 below,

$$\frac{k(n+1)}{24} - \frac{\zeta^\top \cdot C_\Delta \cdot \zeta}{2k} = \frac{1}{24}.$$

Hence,

$$Z_\Delta(\tau) = q^{-\frac{1}{24}} Z_{[\mathbb{C}^2/G_\Delta]}(q)\Big|_{q=\mathrm{e}^{2\pi i \tau}}$$

is the quotient of two holomorphic modular forms. It transforms as a modular form for $\Gamma(km, k) \cap \Gamma(k) = \Gamma(km, k) = \Gamma(N, k)$ with weight

$$\frac{n+1}{2} - \frac{n}{2} = \frac{1}{2}.$$

□

Lemma 3.12. *Let Δ be a simply laced root system, and let ζ be defined as in* (9)*. Then*

$$\frac{(\zeta|\zeta)}{2k} = \frac{\zeta^\top \cdot C_\Delta \cdot \zeta}{2k} = \frac{(n+1)k - 1}{24}.$$

Remark 3.13. Lemma 3.13 expresses the *modular anomaly* of the numerator of $Z_{[\mathbb{C}^2/G_\Delta]}(q)$ (see [14, 12.7.5]). It is proved in Appendix A.1 below. We have not found it in this generality in the literature, but in type A it turns out to be another form of the "strange formula" of Freudenthal–de Vries [6]:

$$\frac{(\rho|\rho)}{2h} = \frac{\rho^\top \cdot C_\Delta \cdot \rho}{2h} = \frac{\dim \mathfrak{g}_\Delta}{24},$$

where ρ is the sum of the positive roots of Δ, h is the (dual) Coxeter number, and $\mathfrak{g}_\Delta$ is the corresponding Lie algebra. See Appendix A.1 for the details. We expect that the identity of Lemma 3.13 holds in the non-simply laced cases as well.

Let Δ_1 (resp. Δ_2) be a root system of rank n_1 (resp. n_2). Denote by G_{Δ_1} $(resp. G_{\Delta_2})$ the corresponding finite group, whose order is $k_1 = |G_{\Delta_1}|$ (resp. $k_2 = |G_{\Delta_2}|$). Let θ_1 (resp. θ_2) be as in (4). Let $\mathbf{a}_1$ (resp. $\mathbf{a}_2$) be the vector dual to θ_1 (resp. θ_2) with respect to the form $k_1(|)_1$ (resp. $k_2(|)_2$). We define

$$\Theta_{\Delta_1 \oplus \Delta_2}(\tau) = \Theta_{\mathbf{a}_1 \oplus \mathbf{a}_2}(\tau),$$

where the right side is the theta function of the lattice $L_1 \oplus L_2$ equipped with the form $k_1(|)_1 \oplus k_2(|)_2$. The next statement is a straightforward calculation.

Lemma 3.14.

$$\Theta_{\Delta_1 \oplus \Delta_2}(\tau) = \Theta_{\Delta_1}(\tau) \cdot \Theta_{\Delta_2}(\tau).$$

Corollary 3.15. *Let Δ be an irreducible, finite type root system.*

(1) *If the rank of Δ is even, then $\Theta_\Delta(\tau) \in M_{n/2}(\Gamma(N,k),\vartheta)$, where $N = km$, and*
$$\vartheta(A) = \left(\frac{(-1)^{n/2}k^n|C_\Delta|}{d}\right).$$

(2) *If the rank of Δ is odd and Δ is not of type A, then $\Theta_{\Delta\oplus A_1}(\tau) \in M_{(n+1)/2}(\Gamma(N,k),\vartheta)$, where $N = km$, and the multiplier system is*
$$\vartheta(A) = \left(\frac{(-1)^{(n+1)/2}4k^n|C_\Delta|}{d}\right).$$

Proof. Part (1) follows from Corollary 3.10

If Δ is not of type A, then $2|k$ and $8|km$. By Corollary 3.10, $\Theta_\Delta(\tau)$ is a modular form for $\Gamma(km,k)$, and $\Theta_{A_1}(\tau)$ is a modular form for $\Gamma(8,2)$. Hence, their product is a modular form for $\Gamma(km,k)$. The formula of the multiplier system follows from Part (1) and from that $2|C_{A_1}| = 4$. □

Remark 3.16. In Section 4 below we perform computer calculations. In the odd rank cases we found it better to work with $\Theta_\Delta(\tau)\cdot\Theta_{A_1}(\frac{k}{2}\tau)$. With the same reasoning as above, $\Theta_{A_1}(\frac{k}{2}\tau)$ is a modular form for $\Gamma(4k,k)$. Since in all non-type A, odd rank cases $4|m$, we have that $\Theta_\Delta(\tau)\cdot\Theta_{A_1}(\frac{k}{2}\tau)$ is a modular form for $\Gamma(km,k)$. Moreover, as the determinant of $kC_\Delta \oplus kC_{A_1}$ is $2k^{n+1}|C_\Delta|$, the multiplier system is
$$\vartheta(A) = \left(\frac{(-1)^{(n+1)/2}2k^{n+1}|C_\Delta|}{d}\right).$$

4. The local contributions as eta products

4.1. **Eta products.** An eta products is a finite product

$$f(\tau) = \prod_m \eta(m\tau)^{a_m} \tag{11}$$

where m runs through a finite set of positive integers and the exponents a_m may take values from $\mathbb{Z}$. The least common multiple of all m such that $a_m \neq 0$ will be denoted by N; it is called the minimum level of $f(\tau)$.

For a general eta quotient $f(\tau)$ as in (11), let $k = \sum_m a_m$. The expression $f(\tau)(\mathrm{d}\tau)^{k/2}$ transforms as a $k/2$-differential due to the transformation law of the Dedekind eta function. Since $\eta(\tau)$ is nonzero on $\mathbb{H}$, (quotients of) eta products never has finite poles. The only issue for an eta product to be a (possibly half-integral weight) modular form is whether the numerator vanishes to at least the same order as the denominator at each cusp.

Theorem 4.1 ([8, Theorem 3]). *Let f be an eta product as in (11) such that $n = \sum_{m|N} a_m$ is even. Let $s = \prod_{m|N} m^{a_m}$, $\frac{1}{24}\sum_{m|N} ma_m = c/e$ and $\frac{1}{24}\sum_{m|N}\frac{N}{m}a_m = c_0/e_0$, both in lowest terms. Then $f(\tau)$ is a modular form of weight $n/2$ for $\Gamma_0(Ne_0)\cap\Gamma^0(e)$ with the multiplier system defined by the Dirichlet character (mod Ne_0)*
$$\gamma(A) = \left(\frac{(-1)^{n/2}s}{a}\right)$$
for $a > 0$, $gcd(a,6) = 1$.

Remark 4.2. (1) The fact that the $\gamma(A)$ values for $a > 0$, $gcd(a, 6) = 1$ are enough to define a multiplier system follows from [19, Lemma 3], and the multiplier system was calculated originally in [19, Theorem 1].

(2) Since $N|c$ and $ad - bc = 1$, we have that $ad \equiv 1 \pmod{m}$ for all $m|N$. This means that
$$\left(\frac{a}{m}\right) = \left(\frac{d}{m}\right),$$
or equivalently, that
$$\left(\frac{m}{a}\right) = \left(\frac{m}{d}\right).$$
Hence, the multiplier system in Theorem 4.4 can also be written as
$$\gamma(A) = \left(\frac{(-1)^{n/2}s}{d}\right).$$

The content of Theorem 4.4 is explained in [9, Section 1]. In the case when $\sum_{m|N} \frac{N}{m} a_m \equiv 0 \pmod{24}$, $f(\tau)$ has an integral order at 0. If this condition is not satisfied for N, it can be guaranteed by replacing N with Ne_0. In effect this *widens* the cusp of $\Gamma_0(N)$ at 0 by a factor of e_0. Similarly, $\sum_{m|N} ma_m \equiv 0 \pmod{24}$ if and only if $f(\tau)$ has an integral order at the cusp at $i\infty$. If this is not the case, widening the cusp $\Gamma_0(Ne_0)$ at $i\infty$ can be achieved by passing to the subgroup $\Gamma_0^0(Ne_0, e) = \Gamma_0(Ne_0) \cap \Gamma^0(e)$. The numbers e_0 and e are called the ramification numbers of $f(\tau)$ at 0 and $i\infty$ respectively. We will say that $f(\tau)$ is unramified if $e = e_0 = 1$.

4.2. **The local Hilbert series and their numerators as eta products.** Let Δ be a simply laced root system. We introduce the notations
$$Z_\Delta(\tau) = q^{-\frac{1}{24}} Z_{[\mathbb{C}^2/G_\Delta]}(q)\Big|_{q=e^{2\pi i\tau}}$$
and
$$\Theta_\Delta(\tau) = \Theta_{\zeta/k,k}(\tau).$$
In particular,
$$Z_\Delta(\tau) = \frac{\Theta_\Delta(\tau)}{\eta(k\tau)^n},$$
where n is the rank of Δ, and k is the order of the corresponding finite group. We will show that in the cases when $\Delta = A_n$, $n \geq 1$, D_4, D_6, D_7 or E_6 the functions $\Theta_\Delta(\tau)$, and hence $Z_\Delta(\tau)$, can be expressed as eta products. Conjecturally the same statement holds for all ADE types.

Example 4.3. For $\Delta = A_n$, $n \geq 1$
$$Z_{A_n}(\tau) = \frac{1}{\eta(\tau)}.$$
This follows from a well-investigated combinatorial argument. The series $Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n)$ in this case enumerates certain combinatorial objects called diagonally colored Young diagrams. For the precise correspondence between these and $Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n)$ see for example [7] or [13]. Here we just summarise the relevant results briefly. Let C be the set of residue classes $\mathbb{Z}/(n+1)\mathbb{Z}$. A *coloring* (or *labeling*) of a Young diagram Y with C is a function assigning an element of C to each box of Y. The *diagonal coloring* of Y is defined by associating to the (i, j)-component s of Y the residue
$$\operatorname{res}(s) = j - i + (n+1)\mathbb{Z} \in C\ .$$

The *n-core* of a diagonally colored Young diagram is the diagonally colored Young diagram obtained by successively removing border strips of length n, until this is no longer possible. Here a *border strip* is a skew Young diagram which does not contain 2×2 blocks and which contains exactly one c-labelled block for all labels $c \in C$. The removal of border strips from a diagonally colored Young diagram can be traced on another combinatorial object, the *abacus*. The abacus for n colors consists of *rulers* corresponding to the residue classes in $C = \mathbb{Z}/(n+1)\mathbb{Z}$. The i-th ruler consists of integers in the i-th residue class modulo $n+1$ in increasing order from top to bottom. Several *beads* are placed on these rulers, at most one on each integer. In particular, to a Young diagram corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ place a bead in position $\lambda_i - i + 1$ for all i, interpreting λ_i as 0 for $i > k$. The removal of a border strip from the Young diagram corresponds to moving a bead up on one of the rulers. It turns out that shifting of beads on different rulers is independent from each other. In this way, the core of a partition corresponds to the bead configuration in which all the beads are shifted up as much as possible. Let us denote by $\mathcal{C}$ the set of $(n+1)$-core partitions. It can be shown that the configuration of the beads on a ruler is described by a partition. The collection of these is called the $(n+1)$-*quotient*. Hence, we get a bijection

$$\mathcal{P} \longleftrightarrow \mathcal{C} \times \mathcal{P}^{n+1}, \tag{12}$$

compatible with the diagonal coloring. Furthermore, there is also a correspondence

$$\mathcal{C} \longleftrightarrow \mathbb{Z}^n.$$

The decomposition (12) reveals the structure of the formula of Theorem 3.3. The first term is the generating series of $(n+1)$-tuples of (uncolored) partitions. The second term is the multi variable generating series of the $(n+1)$-core Young diagrams. Applying the substitution $q_i = q$, $0 \le i \le n$ corresponds to counting the diagonally colored Young diagrams "color-blindly". That is, we just do the usual enumeration of the partitions. Hence,

$$Z_{[\mathbb{C}^2/G_\Delta]}(q) = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

Based on numerical calculations and Example 4.1 we pose the following conjecture.

Conjecture 4.4. (1) *The numerators of the local terms for simple singularities can be expressed as eta products as follows.*

(a)
$$\Theta_{A_n}(\tau) = \frac{\eta^{n+1}((n+1)\tau)}{\eta(\tau)} \quad \textit{for } n \ge 1.$$

(b)
$$\Theta_{D_n}(\tau) = \frac{\eta^2(2\tau)\eta^{n+2}((4n-8)\tau)}{\eta(\tau)\eta(4\tau)\eta^2((2n-4)\tau)} \quad \textit{for } n \ge 4.$$

(c)
$$\Theta_{E_6}(\tau) = \frac{\eta^2(2\tau)\eta^8(24\tau)}{\eta(\tau)\eta^2(8\tau)\eta(12\tau)}.$$

(d)
$$\Theta_{E_7}(\tau) = \frac{\eta^2(2\tau)\eta^9(48\tau)}{\eta(\tau)\eta(12\tau)\eta(16\tau)\eta(24\tau)}.$$

(e)
$$\Theta_{E_8}(\tau) = \frac{\eta^2(2\tau)\eta^{10}(120\tau)}{\eta(\tau)\eta(24\tau)\eta(40\tau)\eta(60\tau)}.$$

(2) *Correspondingly, the local terms for simple singularities can be expressed as eta products as follows.*

(a)
$$Z_{A_n}(\tau) = \frac{1}{\eta(\tau)} \qquad \textit{for } n \geq 1.$$

(b)
$$Z_{D_n}(\tau) = \frac{\eta^2(2\tau)\eta((4n-8)\tau)}{\eta(\tau)\eta(4\tau)\eta^2((2n-4)\tau)} \qquad \textit{for } n \geq 4.$$

(c)
$$Z_{E_6}(\tau) = \frac{\eta^2(2\tau)\eta(24\tau)}{\eta(\tau)\eta^2(8\tau)\eta(12\tau)}.$$

(d)
$$Z_{E_7}(\tau) = \frac{\eta^2(2\tau)\eta(48\tau)}{\eta(\tau)\eta(12\tau)\eta(16\tau)\eta(24\tau)}.$$

(e)
$$Z_{E_8}(\tau) = \frac{\eta^2(2\tau)\eta(120\tau)}{\eta(\tau)\eta(24\tau)\eta(40\tau)\eta(60\tau)}.$$

Example 4.5. In the case of $\Delta = A_1$, $\Theta_{A_1}(\tau) = \frac{\eta^2(2\tau)}{\eta(\tau)}$ which is a noncuspidal holomorphic modular form of weight 1/2 and level 2 [15, Example 3.12 (1)].

We will denote the eta products on the right hand sides of Conjecture 4.2 (1) by $\eta_\Delta(\tau)$. Then Conjecture 4.2 boils down to showing that $\Theta_\Delta(\tau) = \eta_\Delta(\tau)$.

4.3. **A procedure for proving Conjecture 4.2.** To compare the eta products of Conjecture 4.2 with the theta functions of Section 3 we want to show that they are modular forms and also obtain their multiplier systems.

To prove Conjecture 4.2 we need to show that $\Theta_\Delta(\tau) = \eta_\Delta(\tau)$. Theorem 4.4 provides the multiplier system of $\eta_\Delta(\tau)$ only for root systems of even rank. It is possible to obtain an analog of Theorem 4.4 for root systems of odd rank as well. Since these calculations would be too circuitous, we instead reduce to the case of root systems of even rank by taking a direct sum with A_1, for which the identity $\Theta_{A_1}(\tau) = \eta_{A_1}(\tau)$ is known by Example 4.1. More precisely, for computational reasons in the odd rank cases we will show that
$$\Theta_\Delta(\tau) \cdot \Theta_{A_1}\left(\frac{k}{2}\tau\right) = \eta_\Delta(\tau) \cdot \eta_{A_1}\left(\frac{k}{2}\tau\right).$$

Lemma 4.6. *If Δ is an irreducible simply laced root system of rank n with $|G_\Delta| = k$, then for $\eta_\Delta(\tau)$ the numbers appearing in Theorem 4.4 are as follows:*
$$e = \frac{24}{gcd(24, k(n+1)-1)}; \quad e_0 = 1; \quad s = k^n|C_\Delta|; \quad N = k.$$

Proof. Direct calculation shows that in each case $\sum_{m|N} ma_m = k(n+1)-1$, and $\sum_{m|N} \frac{N}{m} a_m = 0$. The third identity is also straightforward. ☐

Lemma 4.7. *If Δ is an irreducible simply laced root system of odd rank n with $|G_\Delta| = k$, then for $\eta_\Delta(\tau)\cdot\eta_{A_1}(\frac{k}{2}\tau)$ the numbers appearing in Theorem 4.4 are as follows:*

$$e = \frac{24}{gcd(24, k(n+\frac{5}{2})-1)}; \quad e_0 = 1; \quad s = 2k^{n+1}|C_\Delta|; \quad N = k.$$

Proof. For $\eta_{A_1}(\frac{k}{2}\tau) = \eta^2(k\tau)\eta^{-1}(\frac{k}{2}\tau)$, $\sum_{m|N} ma_m = \frac{3k}{2}$ and $\prod_{m|N} m^{a_m} = 2k$. □

Corollary 4.8. *Let Δ be a simply laced root system.*

(1) *If n is even, let e be as Lemma 4.6. Then the function $\eta_\Delta(\tau)$ is a modular form of weight $\frac{n}{2}$ for $\Gamma_0(k)\cap\Gamma^0(e)$ with the multiplier system defined by*

$$\gamma(A) = \left(\frac{(-1)^{n/2}k^n|C_\Delta|}{d}\right).$$

(2) *If n is odd, let e be as Lemma 4.7. Then the function $\eta_\Delta(\tau)\cdot\eta_{A_1}(\frac{k}{2}\tau)$ if a modular form of weight $\frac{n+1}{2}$ for $\Gamma_0(k)\cap\Gamma^0(e)$ with the multiplier system defined by*

$$\gamma(A) = \left(\frac{(-1)^{(n+1)/2}2k^{n+1}|C_\Delta|}{d}\right).$$

Proof. Follows from Theorem 4.4. □

Lemma 4.9. *Let Δ be an irreducible simply laced root system. Let $k = |G_\Delta|$, and let e be as in Corollary 4.8.*

(1) *If Δ is either of type D and even rank or of type E, then $e|k$. As a consequence, $\Gamma(km, m)\cap\Gamma_0(k)\cap\Gamma^0(e) = \Gamma(km, m)$.*
(2) *If Δ is of type D and odd rank, then $e|2k$. As a consequence, $\Gamma(km, m)\cap\Gamma_0(k)\cap\Gamma^0(e) = \Gamma(km, m/2)$.*

Proof. We will show that $\Gamma(km, m)$ (resp. $\Gamma(km, m/2)$) is contained in $\Gamma_0(k)\cap\Gamma^0(e)$. For this we only need that $e|k$ (resp. $e|2k$). In the type E case this is automatic, since $e|24$ always and $24|k$ in all three cases.

Let Δ be of type D whose rank n is even. Then $k(n+1)-1 = (4n-8)(n+1)-1 = 4n^2-4n-9$, which is always an odd number. The divisors of 24 are 2 and 3. So the only possibilities for e are 8 and 24 depending on whether $4n^2-4n-9$ is divisible by 3 or not. Now $4n^2-4n-9 = 4n(n-1)-9$, so it is not divisible by 3 if and only if $n \equiv -1(\mod 3)$. Hence $e = 24$ if and only if $n = 6l+2$ for some integer l. But this means that $k = 4n-8 = 4(6l+2)-8 = 24l$, so $24|k$. In the cases when $n = 6l$ (resp. $n = 6l+4$) the order $k = 24l-8$ (resp. $k = 24l+8$). So in both cases $8|k$.

Suppose now that Δ is of type D whose rank n is odd. Then $k(n+\frac{5}{2})-1) = (4n-8)(n+\frac{5}{2})-1 = 4n^2+2n-21$, which is again always an odd number. Similarly as above, it is not divisible by 3 if and only if $n \equiv -1(\mod 3)$. If this is the case, then $e = 24$ and $n = 6l+5$ for some integer l. Then, $k = 4n-8 = 4(6l+5)-8 = 24l-12$, so $e|2k$. The other case is when $e = 8$. Then either $n = 6l+1$ for some integer l and hence $k = 4n-8 = 4(6l+1)-8 = 24l-4$, or $n = 6l+3$ and hence $k = 4n-8 = 4(6l+3)-8 = 24l-12$. In both cases $e|2k$. □

Corollary 4.10. *Let Δ be an irreducible simply laced root system of rank n. Let $k = |G_\Delta|$, and let e be as in Corollary 4.8. Let*

$$\Gamma = \begin{cases} \Gamma(km, m), & \text{if } \Delta \text{ is of type } D \text{ and } n \text{ is even, or } \Delta \text{ is of type } E \\ \Gamma(km, m/2), & \text{if } \Delta \text{ is of type } D \text{ and } n \text{ is odd.} \end{cases}$$

Δ	k	m	Γ	$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$	Sturm bound
D_4	8	2	$\Gamma(16,2)$	768	128
$D_5 \oplus A_1$	12	8	$\Gamma(96,4)$	36864	9216
D_6	16	4	$\Gamma(64,4)$	12288	3072
E_6	24	3	$\Gamma(72,3)$	20736	5184

TABLE 2. Sturm bounds

Then both

$$\begin{cases} \eta_\Delta(\tau) \text{ and } \Theta_\Delta(\tau), & \text{if } n \text{ is even,} \\ \eta_\Delta(\tau) \cdot \eta_{A_1}(\frac{k}{2}\tau) \text{ and } \Theta_\Delta(h\tau) \cdot \Theta_{A_1}(\frac{k}{2}\tau), & \text{if } n \text{ is odd,} \end{cases}$$

are modular forms for Γ of the same weight and they have the same multiplier system.

Proof. Follows from Corollary 3.10, Corollary 4.8 and Lemma **??**. □

The next result gives a limit up to which the vanishing of the Fourier coefficients of a modular form guarantees the vanishing of the modular form. It is generally known as the Sturm bound.

Theorem 4.11 ([21, Theorem 1])**.** *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, n be a positive even integer, and ϑ be a multiplier system for Γ. Let $f = \sum_{m=0}^{\infty} a(m)q^m \in M_{n/2}(\Gamma, \vartheta)$. If $a(m) = 0$ for all $m \leq \frac{n}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$, then $f = 0$. As a consequence, if the Fourier coefficients of two modular forms in $M_{n/2}(\Gamma, \vartheta)$ agree at least up to degree $\frac{n}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$, then the two modular forms are equal.*

Corollary 4.12. *For every fixed Δ Conjecture 4.2 can be checked numerically.*

Proof. Because of Theorem 4.10 one only has to check whether the q-expansions of the two functions from Corollay 4.9 agree at least up to order $\frac{n}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N,m)]$. □

Proposition 4.13. *Conjecture 4.2 is true in the cases when $\Delta = A_n$, $n \geq 1$, D_4, D_5, D_6 or E_6.*

Proof of Proposition 4.12. The case $\Delta = A_n$ is explained in Example 4.1 above.

In all other cases, taking into account the index formula from Lemma **??**, the groups $\Gamma(km, m)$ (resp. $\Gamma(km, m/2)$) and the corresponding Sturm bounds are calculated in Table 2. We performed a computer check in each case and found that the Fourier coefficients agree at least up to the appropriate bound.

□

5. K3 SURFACES

A smooth projective surface X is a K3 surface if the canonical line bundle is trivial $\omega_X \cong \mathcal{O}_X$ and if $h_1(X, \mathcal{O}_X) = 0$. An action of a finite group $G \subset \mathrm{Aut}(X)$ of automorphisms of X is called symplectic, if the action of G on $H^0(\omega_X)$ is trivial, or equivalently, if the resolution of the quotient X/G is also a K3 surface [22, Page 1]. In this case the stabilizers are generically trivial and the quotient X/G is a surface with at worst simple (Kleinian, rational double point) singularities. Moreover, the projection $X \to X/G$ is unramified outside these singular points [17]. Therefore the results of the previous sections apply.

Proposition 5.1. *If X is a smooth projective K3 surface and G is a finite group acting on it by symplectic automorphisms, then*

$$f_{[X/G]}(\tau) = \frac{q}{Z_{[X/G]}(q)}\Big|_{q=e^{2\pi i\tau}}$$

is an unramified modular form of level $\frac{\chi(X/G)}{2}$ for $\Gamma_0(|G|)$ (with a multiplier system which is a suitable Dirichlet character).

The complete classification of finite groups admitting a symplectic action on a K3 surface is given in [17]. It turnes out that there are 81 such finite groups. Moreover, a classification of combinatorial types of the actions was also obtained in [22]. This describes the number of fixed points of each type, or, equivalently, the singularities of the quotient. Our proof does not depend on this classification, but for examples we follow [22, Table 2] for the numbering of the possible types.

Lemma 5.2. *Let $\{P_1, \ldots, P_r\}$ denote the fixed points of a group G acting on X. Let $\{\Delta_1, \ldots, \Delta_r\}$ be the root systems associated with the corresponding stabilizers $\{G_1, \ldots, G_r\}$. Let $k_i = |G_i|$ and $k = |G|$. Then*

$$q^{\alpha} Z_{[X/G]}(q)\Big|_{q=e^{2\pi i\tau}} = \eta^{-\chi(X^0/G)}(k\tau) \cdot \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k}{k_i}\tau\right)$$

for some $\alpha \in \mathbb{Q}$.

Proof. The product structure comes directly from (1), (2) and Theorem 3.3. The coefficients of τ come from the fact that the punctual Hilbert scheme at a point in the quotient stack is expressed in terms of non-stacky punctual Hilbert scheme of the étale cover. □

Proof of Proposition 5.1. Since

$$X \to X/G$$

is a Galois cover, the identity

$$k\chi(X^0/G) + \sum_{i=1}^{r} \frac{k}{k_i} = \chi(X) = 24 \tag{13}$$

is satisfied. Hence, the exponent α in Lemma 5.2 is -1. The modularity then follows from Proposition 3.1.

The local terms $Z^{(P_i,X)}(q)$ as well as $Z_{X^0}(q)$ each count Euler characterstics and their expansion start with 1. Therefore, the q-expansion of $Z_{[X/G]}(q)$ also start with one. Hence, the same is true for $(Z_{[X/G]}(q))^{-1}$. In particulat $q(Z_{[X/G]}(q))^{-1}$ does not have a constant term, i.e. it vanishes at $q = 0$. This means that $q(Z_{[X/G]}(q))^{-1}$ is a cusp form. □

Corollary 5.3. *With the notations of Lemma 5.2 the weight of the form $q(Z_{[X/G]}(q))^{-1}$, $q = e^{2\pi i\tau}$ is*

$$\frac{\chi(X/G)}{2} = \frac{\chi(X^0/G) + r}{2} = \frac{12}{k} - \sum_{i=1}^{r} \frac{1 + k_i}{2k_i}. \tag{14}$$

Proof. Follows from Lemma 5.2 and (13). □

#	k	h
3	4	2
11	8	4
15	9	3
19	12	6
25	16	4
27	16	8
28	16	8

TABLE 3. The cases when $k \neq h$.

To compute the levels explicitly we make use of Corollary 4.12. This implies that $f_{[X/G]}(\tau)$ in all cases is expressible as an eta product. Let h be the minimal level of $f_{[X/G]}(\tau)$. The next result about h is not needed for our further calculations, but it is worth to note.

Lemma 5.4. *The identity $h = k$ is satisfied, except for the configurations in Table 3.*

Proof. When $\chi(X^0/G) \neq 0$, then by 5.2 there is a term with coefficient k. The configurations with $\chi(X^0/G) = 0$ are those appearing in Table 3 plus # 37 and #38. But in the latter two cases there are points which are fixed by the whole group, and the group is not abelian. Hence, the terms from Conjecture 4.2 (2) contribute a term with coefficient k. □

Let $\{P_1, \dots, P_r\}$ denote the fixed points of a group G acting on X. Let $\{\Delta_1, \dots, \Delta_r\}$ be the root systems associated with the corresponding stabilizers $\{G_1, \dots, G_r\}$. Let $k_i = |G_i|$ and $k = |G|$. Let

$$Z_{\Delta_i}(\tau) = \prod_m \eta(m\tau)^{a_{m_i}}$$

be the unscaled local terms. For these we define

$$l_i = \sum_{m|k_i} m a_{m_i}$$

and

$$j_i = \sum_{m|k_i} \frac{1}{m} a_{m_i}.$$

Let

$$f_{[X/G]}(\tau) = \eta^{-\chi(X^0/G)}(k\tau) \cdot \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k}{k_i}\tau\right)$$

be the global generating series with minimal level h, and write it as a single eta product:

$$f_{[X/G]}(\tau) = \prod_{m|h} \eta(m\tau)^{a_m}.$$

$s_i = \prod_{m|h} m^{a_{m_i}}$, $\frac{1}{24}\sum_{m|N} m a_m = c/e$ and $\frac{1}{24}\sum_{m|N} \frac{N}{m} a_m = c_0/e_0$, both in lowest terms.

Lemma 5.5. (1)

$$\sum_{m|h} m a_m = -k\chi(X^0/G) - \sum_i \frac{k}{k_i} l_i.$$

(2)

$$\sum_{m|h} \frac{h}{m} a_m = -\frac{h}{k}\chi(X^0/G) - \sum_i \frac{hk_i}{k} j_i.$$

Proof. Follows automatically from Lemma 5.2. □

Lemma 5.6. *For any simply laced root system Δ_i, $l_i = -1$ and $j_i = -\frac{n_i+1}{k_i}$.*

Proof. Follows by inspection on the eta product expressions in Conjecture 4.2 (2). □

Lemma 5.7. (1)

$$\sum_{m|h} m a_m = 24.$$

(2)

$$\sum_{m|h} \frac{h}{m} a_m = \frac{24h}{k}.$$

Proof. Part (1) follows by combining Lemma 5.6, Lemma 5.5 (1) and equation (13).
For Part (2) substitute $j_i = -\frac{n_i+1}{k_i}$ into Lemma 5.5 (2) to obtain

$$\sum_{m|h} \frac{h}{m} a_m = \frac{h}{k}\left(\chi(X^0/G) + \sum_i (n_i+1)\right).$$

Let $\widetilde{X/G}$ be the minimal resolution of X/G. Since $n_i + 1$ is the Euler characteristic of the exceptional locus above P_i,

$$\chi(X^0/G) + \sum_i (n_i+1) = \chi(\widetilde{X/G}) = 24.$$

Here we applied that G is symplectic, and hence $\widetilde{X/G}$ is also a K3 surface. □

Corollary 5.8. *For $f_{[X/G]}(\tau)$ the numbers appearing in Theorem 4.4 are $e = 1$ and $e_0 = \frac{k}{h}$. Hence, if the weight $\chi(X/G)/2$ is even, then $f_{[X/G]}(\tau)$ is an unramified modular form for $\Gamma_0\left(h \cdot \frac{k}{h}\right) = \Gamma_0(k)$.*

Appendix A. Some proofs

A.1. **Proof of Lemma ??.** The following indices are known [5, Section 1.2]:

$$[\mathrm{Sl}_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N}\left(1 - \frac{1}{p^2}\right), \tag{15}$$

$$[\Gamma_0(N) : \Gamma_1(N)] = \phi(N), \tag{16}$$

$$[\Gamma_1(N) : \Gamma(N)] = N. \tag{17}$$

The index (**??**) is because of the following. Due to $c \equiv 0 \pmod N$ we must have $ad \equiv 1 \pmod N$. This equation has $\phi(N)$ solutions, but we only allow one of these in $\Gamma_1(N)$: the one with $a \equiv d \equiv 1 \pmod N$. The $\phi(N)$ residue classes modulo N are distributed uniformly into the $\phi(N/m)$ relative prime residue classes modulo N/m. Hence, the congruence $a \equiv 1 \pmod{N/m}$ has $\phi(N)/\phi(N/m)$ residue classes as solutions. If $N/m = 2$, this is all the solutions of $a \equiv \pm 1 \pmod{N/m}$. If $N/m > 2$, then the number of solutions of $a \equiv \pm 1 \pmod{N/m}$ is $2\phi(N)/\phi(N/m)$. As a consequence, the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv \pm 1 \ (\mathrm{mod}\ N/m), c \equiv 0 \ (\mathrm{mod}\ N) \right\} \tag{18}$$

in $\Gamma_0(N)$ has index $\phi(N)/\phi(N/m) = \phi(N)$ if $N/m = 2$, and $2\phi(N)/\phi(N/m)$ if $N/m > 2$.

Second, the index (**??**) comes from requireing $b \equiv 1 \pmod N$. Similarly the index of $\Gamma(N,m)$ inside the group defined in (**??**) is N/m. Combining all these we obtain that

$$[\Gamma_0(N) : \Gamma(N,m)] = \begin{cases} 2N\phi(N), & \text{if } N/m = 2 \\ \frac{2N\phi(N)}{m\phi(N/m)}, & \text{if } N/m > 2. \end{cases}$$

The index of $\Gamma_0(N)$ inside $\mathrm{SL}_2(\mathbb{Z})$ turns out to be

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = \frac{N^2}{\phi(N)} \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Hence,

$$\begin{aligned} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N,m)] &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \cdot [\Gamma_0(N) : \Gamma(N,m)] \\ &= \begin{cases} 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), & \text{if } N/m = 2 \\ 2N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \prod_{p|N/m} \left(\frac{p}{p-1}\right), & \text{if } N/m > 2, \end{cases} \end{aligned}$$

where we have used the following expression for the ϕ function:

$$\phi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right).$$

A.2. **Proof of Lemma 3.13.** We follow the notations of [1].

Type A_n, $n \geq 1$. In this case $\dim\rho_i = 1$, $0 \leq i \leq n$. This implies that

- $k = n + 1 = h$, the (dual) Coxeter number of the root system,
- and $\zeta = C_\Delta^{-1} \cdot (1, \ldots, 1) = \rho$, the sum of the positive roots.

The "strange formula" of Freudenthal-de Vries [6] says that for any simple Lie algebra:

$$\frac{(\rho|\rho)}{2h} = \frac{\rho^\top \cdot C_\Delta \cdot \rho}{2h} = \frac{\dim\mathfrak{g}_\Delta}{24}.$$

It is known that $\dim\mathfrak{g}_{A_n} = n(n+2)$. Hence,

$$\frac{(\zeta|\zeta)}{2k} = \frac{\rho^\top \cdot C_\Delta \cdot \rho}{2h} = \frac{n(n+2)}{24} = \frac{(n+1)^2 - 1}{24}$$

as claimed. □

Type D_n, $n \geq 4$. In this and the remaining cases we will do direct calculation.

Let $V = \mathbb{R}^n$ and let $\varepsilon_1, \ldots, \varepsilon_n$ be the canonical basis of V. Put

$$\begin{aligned} \alpha_1 &= \varepsilon_1 - \varepsilon_2 \\ \alpha_2 &= \varepsilon_2 - \varepsilon_3 \\ &\vdots \\ \alpha_{n-1} &= \varepsilon_{n-1} - \varepsilon_n \\ \alpha_n &= \varepsilon_{n-1} + \varepsilon_n. \end{aligned}$$

Then $\alpha_1, \dots, \alpha_n$ is the set of simple positive roots for Δ of type D_n. The fundamental weights are

$$\begin{aligned}\omega_i &= \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i \\ &= \alpha_1 + 2\alpha_2 + \dots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} \dots + \alpha_{n-2}) + \frac{1}{2} i(\alpha_{n-1} + \alpha_n)\end{aligned}$$

for $i < n-1$, and

$$\begin{aligned}\omega_{n-1} &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-2} + \varepsilon_{n-1} - \varepsilon_n) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2} + \frac{1}{2} n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n), \\ \omega_n &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2} n\alpha_n).\end{aligned}$$

Moreover,

$$\zeta = C_\Delta^{-1} \cdot (1, 2, \dots, 2, 1, 1) = \omega_1 + 2\omega_2 + \dots + 2\omega_{n-2} + \omega_{n-1} + \omega_{n-2}.$$

A quick computation shows that in terms of the roots $\alpha_1, \dots, \alpha_n$

$$\zeta = \sum_{i=1}^{n-2} 2\left(in - \frac{(i+1)(i+1)}{2}\right)\alpha_i + \frac{n^2 - 2n}{2}(\alpha_{n-1} + \alpha_n).$$

Hence,

$$\begin{aligned}(\zeta|\zeta) &= (1, 2, \dots, 2, 1, 1)^\top \cdot C_\Delta^{-1} \cdot (1, 2, \dots, 2, 1, 1) \\ &= -(2n-4) + 4\left((n-1)\sum_{i=1}^{n-2} i - \frac{1}{2}\sum_{i=1}^{n-2} i^2 - \frac{n-2}{2}\right) + n^2 - 2n \\ &= \frac{4(n-1)(n-2)(n-1)}{2} - \frac{4(n-2)(n-1)(2n-3)}{12} + n^2 - 6n + 8 \\ &= \frac{4n^3 - 12n^2 - n + 8}{3} = \frac{(n-2)(4n^2 - 4n - 9)}{3}.\end{aligned}$$

This implies that

$$\frac{(\zeta|\zeta)}{2k} = \frac{(n-2)(4n^2 - 4n - 9)}{24(n-2)} = \frac{4n^2 - 4n - 9}{24}.$$

This is equal to

$$\frac{(n+1)k - 1}{24} = \frac{(n+1)(4n-8) - 1}{24}$$

as claimed. $\square$

Type E_6.

$$\frac{1}{2 \cdot 24}\begin{pmatrix}1\\2\\3\\2\\2\\1\end{pmatrix}^\top \cdot \begin{pmatrix}2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & -1 & 0\\ 0 & 0 & -1 & 2 & 0 & 0\\ 0 & 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 2\end{pmatrix}^{-1} \cdot \begin{pmatrix}1\\2\\3\\2\\2\\1\end{pmatrix} = \frac{7 \cdot 24 - 1}{24}.$$

$\square$

Type E_7.

$$\frac{1}{2\cdot 48}\begin{pmatrix}1\\2\\3\\4\\2\\3\\2\end{pmatrix}^{\top}\cdot\begin{pmatrix}2&-1&0&0&0&0&0\\-1&2&-1&0&0&0&0\\0&-1&2&-1&0&0&0\\0&0&-1&2&-1&-1&0\\0&0&0&-1&2&0&0\\0&0&0&-1&0&2&-1\\0&0&0&0&0&-1&2\end{pmatrix}^{-1}\cdot\begin{pmatrix}1\\2\\3\\4\\2\\3\\2\end{pmatrix}=\frac{8\cdot 48-1}{24}.$$

□

Type E_8.

$$\frac{1}{2\cdot 120}\begin{pmatrix}2\\3\\4\\5\\6\\3\\4\\2\end{pmatrix}^{\top}\cdot\begin{pmatrix}2&-1&0&0&0&0&0&0\\-1&2&-1&0&0&0&0&0\\0&-1&2&-1&0&0&0&0\\0&0&-1&2&-1&0&0&0\\0&0&0&-1&2&-1&-1&0\\0&0&0&0&-1&2&0&0\\0&0&0&0&-1&0&2&-1\\0&0&0&0&0&0&-1&2\end{pmatrix}^{-1}\cdot\begin{pmatrix}2\\3\\4\\5\\6\\3\\4\\2\end{pmatrix}=\frac{9\cdot 120-1}{24}.$$

□

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