G-FIXED HILBERT SCHEMES ON $K3$ SURFACES, MODULAR FORMS, AND ETA PRODUCTS

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ABSTRACT. Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Let

$$Z_{X,G}(q)=\sum_{n=0}^{\infty} e\left(\operatorname{Hilb}^n(X)^G\right)\, q^{n-1}$$

be the generating function for the Euler characteristics of the Hilbert schemes of G-invariant length n subschemes. We show that its reciprocal, $Z_{X,G}(q)^{-1}$ is the Fourier expansion of a modular cusp form of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$. We give an explicit formula for $Z_{X,G}$ in terms of the Dedekind eta function for all 82 possible (X,G). The key intermediate result we prove is of independent interest: it establishes an eta product identity for a certain shifted theta function of the root lattice of a simply laced root system. We extend our results to various refinements of the Euler characteristic, namely the Elliptic genus, the Chi-y genus, and the motivic class.

1. INTRODUCTION

Let X be a complex $K3$ surface with an effective action of a group G which preserves the holomorphic symplectic form. Mukai showed that such G are precisely the subgroups of the Mathieu group $M_{23} \subset M_{24}$ such that the induced action on the set $\{1,\dots,24\}$ has at least five orbits [13]. Xiao classified all possible actions into 82 possible topological types of the quotient X/G [18].

The *G-fixed Hilbert scheme* of X parameterizes G-invariant length n subschemes $Z \subset X$. It can be identified with the G-fixed point locus in the Hilbert scheme of points:

$$\operatorname{Hilb}^n(X)^G \subset \operatorname{Hilb}^n(X)$$

We define the corresponding *G-fixed partition function* of X by

$$Z_{X,G}(q)=\sum_{n=0}^{\infty} e\left(\operatorname{Hilb}^n(X)^G\right) q^{n-1}$$

where $e(-)$ is topological Euler characteristic.

Throughout this paper we set

$$q=\exp(2\pi i\tau)$$

so that we may regard $Z_{X,G}$ as a function of $\tau \in \mathbb{H}$ where $\mathbb{H}$ is the upper half-plane.

Our main result is the following:

Theorem 1.1. *The function $Z_{X,G}(q)^{-1}$ is a modular cusp form[1] of weight $\frac{1}{2}e(X/G)$ for the congruence subgroup $\Gamma_0(|G|)$.*

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[1]By cusp form, we mean that the order of vanishing at $q=0$ is at least 1. Modular forms of half integral weight transform with respect to a multiplier system. We refer to [11] for definitions.

Our theorem specializes in the case where G is the trivial group to a famous result of Göttsche [5]. The case where G is a cyclic group was proved in [2].

We give an explicit formula for $Z_{X,G}(q)$ in terms of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$$

as follows. Let $p_1, \ldots, p_r$ be the singular points of X/G and let $G_1, \ldots, G_r$ be the corresponding stabilizer subgroups of G. The singular points are necessarily of ADE type: they are locally given by $\mathbb{C}^2/G_i$ where $G_i \subset SU(2)$. Finite subgroups of $SU(2)$ have an ADE classification and we let $\Delta_1, \ldots, \Delta_r$ denote the corresponding ADE root systems.

For any finite subgroup $G_\Delta \subset SU(2)$ with associated root system Δ we define the *local G_Δ-fixed partition function* by

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right) q^{n-\frac{1}{24}}.$$

The main geometric result we prove is the following.

Theorem 1.2. *The local partition function for Δ of type A_n is given by*

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

and for type D_n and E_n by

$$Z_\Delta(q) = \frac{\eta^2(2\tau)\eta(4E\tau)}{\eta(\tau)\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}$$

where (E, F, V) are given by[2]*:*

$$(E,F,V) = \begin{cases} (n-2, 2, n-2), & \Delta = D_n \\ (6,4,4), & \Delta = E_6 \\ (12,8,6), & \Delta = E_7 \\ (30,20,12), & \Delta = E_8 \end{cases}$$

Our proof of the above Theorem uses a trick exploiting the derived McKay correspondence between $\mathfrak{X} = [\mathbb{C}^2/\{\pm 1\}]$ and $Y = \mathrm{Tot}(K_{\mathbb{P}^1})$, (see sections 4 and 5).

We will also prove in Lemma 2.2 that

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{n+1}}$$

where

$$\theta_\Delta(\tau) = \sum_{\boldsymbol{m} \in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta} | \boldsymbol{m} + \frac{1}{k}\boldsymbol{\zeta}\right)}$$

is a shifted theta function for M_Δ, the root lattice of Δ. Here n is the rank of the root system, $k = |G_\Delta|$, and $\boldsymbol{\zeta}$ is dual to the longest root (see section 2 for details).

Theorem 1.2 then yields an eta product identity for the theta function $\theta_\Delta(\tau)$ reminiscent of the MacDonald identities:

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[2]For Δ of type D_n or E_n, the group $H = G_\Delta/\{\pm 1\} \subset SO(3)$ is the symmetry group of a polyhedral decomposition of S^2 into isomorphic regular spherical polygons. Then E, F, and V are the number of edges, faces, and vertices of the polyhedron.

Theorem 1.3. *The shifted theta function $\theta_\Delta(\tau)$ defined above (c.f. § 2) is given by an eta product as follows:*

For Δ of type A_n

$$\theta_{A_n}(\tau) = \frac{\eta^{n+1}((n+1)\tau)}{\eta(\tau)}$$

and for Δ of type D_n or E_n

$$\theta_\Delta(\tau) = \frac{\eta^2(2\tau)\,\eta^{n+2}(4E\tau)}{\eta(\tau)\,\eta(2E\tau)\,\eta(2F\tau)\,\eta(2V\tau)}$$

where E, F, V are as in Theorem 1.2.

The 82 possible collections of ADE root systems $\Delta_1, \ldots, \Delta_r$ associated to (X, G) a $K3$ surface with a symplectic G action, are given in Appendix B, Table 1. We let $k = |G|$, $k_i = |G_i|$, and

$$a = e(X/G) - r = \frac{24}{k} - \sum_{i=1}^{r} \frac{1}{k_i}.$$

The global series $Z_{X,G}(q)$ can be expressed as a product of local contributions (and thus via Theorem 1.2 as an explicit eta product) by our next result:

Theorem 1.4. *With the above notation we have*

$$Z_{X,G}(q) = \eta^{-a}(k\tau) \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

Theorem 1.1 then immediately follows from Theorem 1.4 and Theorem 2.1 using the formulas for the weight and level of an eta product given in [11, § 2.1].

In Appendix B, Table 1 we have listed explictly the eta product of the modular form $Z_{X,G}(q)^{-1}$ for all 82 possible cases of (X, G). Having obtained such expressions allows us to make several observational corollaries:

Corollary 1.5. *If G is a finite subgroup of an elliptic curve E, i.e. G is isomorphic to a product of one or two cyclic groups, then $Z_{X,G}(q)^{-1}$ is a Hecke eigenform. In Table 1 these are the 13 cases having Xiao number in the set $\{0, 1, 2, 3, 4, 5, 7, 8, 11, 14, 15, 19, 25\}$. Moreover, in each of these cases, the dimension of the Hecke eigenspace is one.*

We remark that in these cases, we may form a Calabi-Yau threefold called a CHL model by taking the free group quotient

$$(X \times E)/G$$

Then the partition function $Z_{X,G}(q)$ gives the Donaldson-Thomas invariants of $(X\times E)/G$ in curve classes which are degree zero over X/G (c.f. [2]).

For any eta product expression of a modular form, one may easily compute the order of vanishing (or pole) at any of the cusps [11, Cor 2.2]. Performing this computation on the 82 cases yields the following

Corollary 1.6. *The modular form $Z_{X,G}(q)^{-1}$ always vanishes with order 1 at the cusps $i\infty$ and 0. Moreover, $Z_{X,G}(q)^{-1}$ is holomorphic at all cusps except for the two cases with Xiao number 38 or 69, which have poles at the cusps $1/2$ and $1/8$ respectively. These are precisely the cases where X/G has two singularities of type E_6.*

Remark 1.7. The integers $e\left(\mathrm{Hilb}^n(X)^G\right)$ should have enumerative significance: they can be interpreted as a virtual counts of G-invariant rational curves in a complete linear series of dimension n on X. This generalizes the famous Yau-Zaslow formula in the case where

G is the trivial group. The precise nature between the virtual count and the actual count is expected to be subtle for the case of general G. This has been recently explored in [19].

We can extend our results to various refinements of the Euler characteristic, namely the elliptic genus, the χ_y genus, and the motivic class. These refinements all stem from the following theorem which we prove in Section 6.

Theorem 1.8. *Let*

$$Z^{\mathsf{bir}}_{X,G}(q) = \sum_{n=0}^{\infty} [\mathrm{Hilb}^n(X)^G]_{\mathsf{bir}}\; q^{n-1}$$

be a formal series whose coefficients we regard as birational equivalence classes of compact hyperkahler manifolds. Such equivalence classes form a semi-ring under disjoint union and Cartesian product. Let Y be the minimal resolution of X/G, then

$$Z^{\mathsf{bir}}_{X,G}(q) = Z^{\mathsf{bir}}_{Y}(q^k) \cdot Z_{X,G}(q) \cdot \Delta(k\tau)$$

where $k = |G|$, $\Delta(\tau) = \eta(\tau)^{24}$, and we have suppressed the trivial group from the notation in the series $Z^{\mathsf{bir}}_{Y}(q^k)$.

A famous theorem of Huybrechts [8, Thm 4.6] asserts that birational compact hyperkahler manifolds are deformation equivalent. Moreover, combining Huybrecht's theorem with [16, Prop 3.21] it follows that birational compact hyperkahler manifolds are equal in $K_0(\mathrm{Var}_{\mathbb{C}})$, the Grothendieck group of varieties.

Thus we may specialize the series $Z^{\mathsf{bir}}_{X,G}(q)$ to Elliptic genus, motivic class, and Chi-y genus since these are all well defined on birational equivalence classes of compact hyperkahler manifolds. These specializations are all well known for the series Z^{bir}_{Y} and hence we easily get the following corollaries.

Corollary 1.9. *Let $Q = \exp(2\pi i\sigma)$ and let*

$$Z^{\mathsf{Ell}}_{X,G}(Q,q,y) = \sum_{n=0}^{\infty} \mathrm{Ell}_{q,y}\left(\mathrm{Hilb}^n(X)^G\right) Q^{n-1}$$

where $\mathrm{Ell}_{q,y}(-)$ is elliptic genus. Then

$$Z^{\mathsf{Ell}}_{X,G}(Q,q,y) = \frac{\phi_{10,1}(q,y)}{\chi_{10}(k\sigma,\tau,z)} \cdot Z_{X,G}(q) \cdot \Delta(k\tau)$$

where $\phi_{10,1}(q,y)$ is the Fourier expansion of the unique Jacobi cusp form of weight 10 and index 1 and $\chi_{10}(\sigma,\tau,z)$ is Igusa's genus 2 Siegel cusp form of weight 10.

We refer the reader to [17] (§5, §6, and eqn 6.9.8) for definitions of $\mathrm{Ell}_{q,y}$, $\phi_{10,1}$, χ_{10}, and the formula for the elliptic genera of $\mathrm{Hilb}^n(Y)$.

A further specialization of particular interest is the (normalized) χ_y genus. Let

$$\overline{\chi}_{-y}(M) = y^{-\frac{1}{2}\dim M}\chi_{-y}(M)$$

and we note that $\overline{\chi}_{-y}(M) = \mathrm{Ell}_{q,y}(M)|_{q=0}$.

Corollary 1.10. *Let*

$$Z^{\overline{\chi}}_{X,G}(q,y) = \sum_{n=0}^{\infty} \overline{\chi}_{y}\left(\mathrm{Hilb}^n(X)^G\right) q^{n-1}.$$

Then

$$Z^{\overline{\chi}}_{X,G}(q,y) = y^{-1}(1-y)^2 \frac{Z_{X,G}(q)}{\phi_{-2,1}(q^k,y)}$$

where $\phi_{-2,1}$ is the unique weak Jacobi form of weight -2 and index 1. In particular,

$$y^{-1}(1-y)^2 Z_{X,G}^{\overline{X}}(q,y)^{-1} = \frac{\phi_{-2,1}(q^k,y)}{Z_{X,G}(q)}$$

is a Jacobi form of index 1 and weight

$$\frac{1}{2}e(X/G) - 2 = 10 - \frac{1}{2}\sum_{i=1}^{r} \operatorname{rank} \Delta_i$$

for the congruence subgroup $\Gamma_1(k)$.

We note that for G cyclic, the series $Z_{X,G}(q)/\phi_{-2,1}(q^k,y)$ is the leading coefficient in the expansion of the Donaldson-Thomas partition function of $(X \times E)/G$ in the variable tracking the curve class in X (see [2, Thm 0.1]).

We also get a formula for the motivic classes of the G-fixed Hilbert schemes:

Corollary 1.11. *Let*

$$Z_{X,G}^{K_0}(q) = \sum_{n=0}^{\infty} [\operatorname{Hilb}^n(X)^G]_{K_0}\; q^{n-1}$$

where $[\operatorname{Hilb}^n(X)^G]_{K_0} \in K_0(\operatorname{Var}_{\mathbb{C}})$ denotes the motivic class of the G-fixed Hilbert scheme. Then

$$Z_{X,G}^{K_0}(q) = q^{-1} \cdot \prod_{m=1}^{\infty} \left(1 - \mathbb{L}^{m-1} q^{km}\right)^{-[Y]} \cdot Z_{X,G}(q) \cdot \Delta(k\tau)$$

where $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}] \in K_0(\operatorname{Var}_{\mathbb{C}})$.

We refer the reader to [6] for the meaning of $[Y]$ in the exponent and the formula for the motivic class of $\operatorname{Hilb}^n(Y)$.

The above series has further specializations giving formulas for the Hodge polynomials and Poincare polynomials of the G-fixed Hilbert schemes.

2. THE LOCAL PARTITION FUNCTIONS

The classical McKay correspondence associates an ADE root system Δ to any finite subgroup $G_\Delta \subset SU(2)$. Using the work of Nakajima [14], the partition function of the Euler characteristics of the Hilbert scheme of points on the stack quotient $[\mathbb{C}^2/G_\Delta]$ was computed explicitly in [7] in terms of the root data of Δ.

The local partition functions $Z_\Delta(q)$ considered in this paper are obtained from a specialization of the partition functions of the stack $[\mathbb{C}^2/G_\Delta]$ and in this section, we use this to express $Z_\Delta(q)$ in terms of a shifted theta function for the root lattice of Δ.

A zero-dimensional substack $Z \subset [\mathbb{C}^2/G_\Delta]$ may be regarded as a G_Δ invariant, zero-dimensional subscheme of $\mathbb{C}^2$. Consequently, we may identify the Hilbert scheme of points on the stack $[\mathbb{C}^2/G_\Delta]$ with the G_Δ fixed locus of the Hilbert scheme of points on $\mathbb{C}^2$:

$$\operatorname{Hilb}\left([\mathbb{C}^2/G_\Delta]\right) = \operatorname{Hilb}(\mathbb{C}^2)^{G_\Delta}.$$

This Hilbert scheme has components indexed by representations ρ of G_Δ as follows

$$\operatorname{Hilb}^\rho\left([\mathbb{C}^2/G_\Delta]\right) = \left\{Z \subset \mathbb{C}^2,\ Z \text{ is } G_\Delta \text{ invariant and } H^0(\mathcal{O}_Z) \cong \rho\right\}.$$

Let $\{\rho_0, \ldots, \rho_n\}$ be the irreducible representations of G_Δ where ρ_0 is the trivial representation. We note that n is also the rank of Δ. We define

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n) = \sum_{m_0,\ldots,m_n=0}^{\infty} e\left(\operatorname{Hilb}^{m_0\rho_0+\cdots+m_n\rho_n}([\mathbb{C}^2/G_\Delta])\right) q_0^{m_0} \cdots q_n^{m_n}.$$

Recall that our local partition function $Z_\Delta(q)$ is defined by

$$Z_\Delta(q)=\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}\right)q^{n-\frac{1}{24}}.$$

We then readily see that

$$Z_\Delta(q)=q^{\frac{-1}{24}}\cdot Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n)\big|_{q_i=q^{d_i}}$$

where

$$d_i=\dim\rho_i.$$

The following formula is given explicitly in [7, Thm 1.3], but its content is already present in the work of Nakajima [14]:

Theorem 2.1. *Let C_Δ be the Cartan matrix of the root system Δ, then*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n)=\prod_{m=1}^{\infty}(1-Q^m)^{-n-1}\cdot\sum_{\boldsymbol{m}\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n}\cdot Q^{\frac{1}{2}\boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{m}}$$

where $Q=q_0^{d_0}q_1^{d_1}\cdots q_n^{d_n}$.

We note that under the specialization $q_i=q^{d_i}$,

$$Q=q^{d_0^2+\cdots+d_n^2}=q^k$$

where $k=|G|$ is the order of the group G.

We then obtain

$$Z_\Delta(q)=q^{\frac{-1}{24}}\cdot\prod_{m=1}^{\infty}(1-q^{km})^{-n-1}\cdot\sum_{\boldsymbol{m}\in\mathbb{Z}^n} q^{\boldsymbol{m}^{\mathsf{t}}\cdot\boldsymbol{d}}\cdot q^{\frac{k}{2}\boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{m}}$$

where $\boldsymbol{d}=(d_1,\dots,d_n)$.

Let M_Δ be the root lattice of Δ which we identify with $\mathbb{Z}^n$ via the basis given by $\alpha_1,\dots,\alpha_n$, the simple positive roots of Δ. Under this identification, the standard Weyl invariant bilinear form is given by

$$(\boldsymbol{u}|\boldsymbol{v})=\boldsymbol{u}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{v}$$

and $\boldsymbol{d}$ is identified with the longest root. We define

$$\boldsymbol{\zeta}=C_\Delta^{-1}\cdot\boldsymbol{d}$$

so that

$$\boldsymbol{m}^{\mathsf{t}}\cdot\boldsymbol{d}=\boldsymbol{m}^{\mathsf{t}}\cdot C_\Delta\cdot\boldsymbol{\zeta}=(\boldsymbol{m}|\boldsymbol{\zeta}).$$

We may then write

$$\begin{aligned}Z_\Delta(q)&=q^{\frac{-1}{24}}\cdot\prod_{m=1}^{\infty}(1-q^{km})^{-n-1}\cdot\sum_{\boldsymbol{m}\in M_\Delta} q^{(\boldsymbol{m}|\boldsymbol{\zeta})+\frac{k}{2}(\boldsymbol{m}|\boldsymbol{m})}\\&=q^A\cdot\left(q^{\frac{k}{24}}\prod_{m=1}^{\infty}(1-q^{km})\right)^{-n-1}\cdot\sum_{\boldsymbol{m}\in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)}\\&=q^A\cdot\eta(k\tau)^{-n-1}\cdot\theta_\Delta(\tau)\end{aligned}$$

where

$$A\quad=\quad\frac{-1}{24}+\frac{k(n+1)}{24}-\frac{1}{2k}(\boldsymbol{\zeta}|\boldsymbol{\zeta})$$

and $\theta_\Delta(\tau)$ is the shifted theta function:

$$\theta_\Delta(\tau) = \sum_{\boldsymbol{m} \in M_\Delta} q^{\frac{k}{2}\left(\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}|\boldsymbol{m}+\frac{1}{k}\boldsymbol{\zeta}\right)}$$

where as throughout this paper we have identified $q = \exp(2\pi i \tau)$.

In Appendix A, we will prove the following formula which for $\Delta = A_n$ coincides with the "strange formula" of Freudenthal and de Vries [3]:

$$\frac{k(n+1)-1}{24} = \frac{(\boldsymbol{\zeta}|\boldsymbol{\zeta})}{2k}.$$

It follows that $A = 0$ and we obtain the following:

Lemma 2.2. *The local series $Z_\Delta(q)$ is given by*

$$Z_\Delta(q) = \frac{\theta_\Delta(\tau)}{\eta(k\tau)^{n+1}}.$$

3. The global series

Recall that $p_1, \ldots, p_r \in X/G$ are the singular points of X/G with corresponding stabilizer subgroups $G_i \subset G$ of order k_i and ADE type Δ_i. Let $\{x_i^1, \ldots, x_i^{k/k_i}\}$ be the orbit of G in X corresponding to the point p_i (recall that $k = |G|$). We may stratify $\mathrm{Hilb}(X)^G$ according to the orbit types of subscheme as follows:

Let $Z \subset X$ be a G-invariant subscheme of length nk whose support lies on free orbits. Then Z determines and is determined by a length n subscheme of

$$(X/G)^o = X/G \setminus \{p_1, \ldots, p_r\},$$

i.e. a point in $\mathrm{Hilb}^n((X/G)^o)$.

On the other hand, suppose $Z \subset X$ is a G-invariant subscheme of length $\frac{nk}{k_i}$ supported on the orbit $\{x_i^1, \ldots, x_i^{k/k_i}\}$. Then Z determines and is determined by the length n component of Z supported on a formal neighborhood of one of the points, say x_i^1. Choosing a G_i-equivariant isomorphism of the formal neighborhood of x_i^1 in X with the formal neighborhood of the origin in $\mathbb{C}^2$, we see that Z determines and is determined by a point in $\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}$, where $\mathrm{Hilb}_0^n(\mathbb{C}^2) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ is the punctual Hilbert scheme parameterizing subschemes supported on a formal neighborhood of the origin in $\mathbb{C}^2$.

By decomposing an arbitrary G-invariant subscheme into components of the above types, we obtain a stratification of $\mathrm{Hilb}(X)^G$ into strata which are given by products of $\mathrm{Hilb}((X/G)^o)$ and $\mathrm{Hilb}_0(\mathbb{C}^2)^{G_1}, \ldots, \mathrm{Hilb}_0(\mathbb{C}^2)^{G_r}$. Then using the fact that Euler characteristic is additive under stratifications and multiplicative under products, we arrive at the following equation of generating functions:

$$\begin{aligned} \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(X)^G\right) q^n = & \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^o)\right) q^{kn}\right) \\ & \cdot \prod_{i=1}^{r} \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}}\right). \end{aligned} \tag{1}$$

As in the introduction, let $a = e(X/G) - r = e\left((X/G)^o\right)$. Then by Göttsche's formula [5],

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n((X/G)^0\right) q^{kn} = \prod_{m=1}^{\infty} (1-q^{km})^{-a}$$
$$= q^{\frac{ak}{24}} \cdot \eta(k\tau)^{-a}.$$

We also note that $e\left(\mathrm{Hilb}^n_0(\mathbb{C}^2)^{G_i}\right) = e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right)$ since the natural $\mathbb{C}^*$ action on both $\mathrm{Hilb}^n_0(\mathbb{C}^2)^{G_i}$ and $\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}$ have the same fixed points. Thus we may write

$$\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n_0(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}} = \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_i}\right) q^{\frac{nk}{k_i}}$$
$$= q^{\frac{k}{24k_i}} \cdot Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

Multiplying equation (1) by q^{-1} and substituting the above formulas, we find that

$$Z_{X,G}(q) = q^{-1+\frac{ak}{24}+\sum\frac{k}{24k_i}} \cdot \eta(k\tau)^{-a} \cdot \prod_{i=1}^{r} Z_{\Delta_i}\left(\frac{k\tau}{k_i}\right).$$

The exponent of q in the above equation is zero as is readily seen from the following Euler characteristic calculation:

$$24 = e(X) = e\left(X - \cup_{i=1}^{r}\{x_i^1, \dots, x_i^{k/k_i}\}\right) + \sum_{i=1}^{r} \frac{k}{k_i}$$
$$= k \cdot e\left((X/G)^o\right) + \sum_{i=1}^{r} \frac{k}{k_i}$$
$$= k \cdot a + \sum_{i=1}^{r} \frac{k}{k_i}$$

This completes the proof of Theorem 1.4. □

4. Proof of Theorem 1.2

4.1. **Proof of Theorem 1.2 in the A_n case.** We wish to prove

$$Z_{A_n}(q) = \frac{1}{\eta(\tau)}$$

which is equivalent to the statement

$$\sum_{m=0}^{\infty} e\left(\mathrm{Hilb}^m(\mathbb{C}^2)^{\mathbb{Z}/(n+1)}\right) q^m = \prod_{m=1}^{\infty} (1-q^m)^{-1}.$$

The action of $\mathbb{Z}/(n+1)$ on $\mathbb{C}^2$ commutes with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ and consequently, the Euler characteristics on the left hand side may be computed by counting the $\mathbb{C}^* \times \mathbb{C}^*$-fixed subschemes, namely those given by monomial ideals. Such subschemes of length m have a well known bijection with integer partitions of m, whose generating function is given by the right hand side. □

4.2. **Proof of Theorem 1.2 in the D_n and E_n cases.** Our proof of Theorem 1.2 in the D_n and E_n cases uses a trick exploiting the derived McKay correspondence between $\mathfrak{X} = [\mathbb{C}^2/\{\pm 1\}]$ and $Y = \mathrm{Tot}(K_{\mathbb{P}^1})$.

Let $G \subset SU(2)$ be a subgroup where the corresponding root system Δ is of D or E type. Then $\{\pm 1\} \subset G$ and let $H \subset SO(3)$ be the quotient

$$H = G/\{\pm 1\}.$$

The induced action of H on $\mathbb{P}^1 \cong S^2$ is by rotations. Indeed, H is the symmetry group of a regular polyhedral decomposition of S^2 which is given by the platonic solids in the E_n case and the decomposition into two hemispherical $(n-2)$-gons in the D_n case. H is generated by rotations of order p, q, r, obtained by rotating about the center of an edge, a face, or a vertex respectively. H has a group presentation:

$$H = \{\langle a,b,c\rangle : \quad a^{\mathsf{p}} = b^{\mathsf{q}} = c^{\mathsf{r}} = abc = 1\}.$$

Let $M = |H|$ be the order of H and let E, F, V be the number of edges, faces, and vertices respectively. Then

$$M = \mathsf{p}E = \mathsf{q}F = \mathsf{r}V$$

and since the stabilizer of an edge is always order 2 we have $\mathsf{p} = 2$ and so $M = 2E$. Then since $F + V - E = 2$ we find

$$E + F + V = 2 + M$$

We summarize this information below:

Type	H	M	$(\mathsf{p},\mathsf{q},\mathsf{r})$	(E,F,V)
D_n	dihedral	$2n-2$	$(2,n-2,2)$	$(n-1,2,n-1)$
E_6	tetrahedral	12	$(2,3,3)$	$(6,4,4)$
E_7	octahedral	24	$(2,3,4)$	$(12,8,6)$
E_8	icosohedral	60	$(2,3,5)$	$(30,20,12)$

Now let $\mathfrak{X}$ be the stack quotient

$$\mathfrak{X} = [\mathbb{C}^2/\{\pm 1\}]$$

and let

$$Y \cong \mathrm{Tot}(K_{\mathbb{P}^1})$$

be the minimal resolution of the singular space $X = \mathbb{C}^2/\{\pm 1\}$.

The stack quotient $[\mathbb{P}^1/H]$ has three stacky points with stabilizers of order $\mathsf{p},\mathsf{q},\mathsf{r}$, and consequently the stack quotient $[Y/H]$ has three orbifold points locally of the form $[\mathbb{C}^2/\mathbb{Z}_{\mathsf{a}}]$ for $\mathsf{a} \in \{\mathsf{p},\mathsf{q},\mathsf{r}\}$.

We observe that

$$[\mathbb{C}^2/G] \cong [\mathfrak{X}/H]$$

and consequently

$$\mathrm{Hilb}^n(\mathbb{C}^2)^G \cong \mathrm{Hilb}^n(\mathfrak{X})^H.$$

Recall from section 2 that $\mathrm{Hilb}^n(\mathfrak{X})$ decomposes into components $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ with $n = m_0 + m_1$ where the corresponding $\{\pm 1\}$ invariant subschemes $Z \subset \mathbb{C}^2$ have the property that as a $\{\pm 1\}$-representation, $H^0(\mathcal{O}_Z)$ has m_0 copies of the trivial representation and m_1 copies of the non-trivial representation.

We will prove in section 5 that as a consequence of the derived McKay correspondence between $\mathfrak{X}$ and Y, we have the following:

Proposition 4.1. $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})^H$ *is deformation equivalent to and hence diffeomorphic to* $\mathrm{Hilb}^{m_0-(m_0-m_1)^2}(Y)^H$*. In particular*

$$e\left(\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})^H\right) = e\left(\mathrm{Hilb}^{m_0-(m_0-m_1)^2}(Y)^H\right).$$

Let

$$j = m_1 - m_0, \quad n = m_0 - (m_0 - m_1)^2$$

so that

$$m_0 + m_1 = 2n + j + 2j^2.$$

We then can compute:

$$\begin{aligned} q^{\frac{1}{24}} Z_\Delta(q) &= \sum_{m_0,m_1=0}^{\infty} e\left(\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})^H\right) q^{m_0+m_1} \\ &= \sum_{j\in\mathbb{Z}} \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^{n}(Y)^H\right) q^{2n+j+2j^2}. \end{aligned}$$

For the root lattice of A_1, we have $M_{A_1} \cong \mathbb{Z}$, $C_{A_1} = (2)$, and $k = 2$, $\boldsymbol{\zeta} = \frac{1}{2}$ so by definition

$$\begin{aligned} \theta_{A_1}(\tau) &= \sum_{m\in\mathbb{Z}} q^{\frac{2}{2}\left(m+\frac{1}{4}|m+\frac{1}{4}\right)} \\ &= \sum_{m\in\mathbb{Z}} q^{2(m+\frac{1}{4})^2} \\ &= q^{\frac{1}{8}} \sum_{j\in\mathbb{Z}} q^{2j^2+j} \end{aligned}$$

Substituting into the previous equation multiplied by $q^{\frac{1}{8}}$ we find

$$q^{\frac{1}{6}} Z_\Delta(q) = \theta_{A_1}(\tau) \cdot \sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^{n}(Y)^H\right) q^{2n}.$$

We can now compute the summation factor in the above equation by the same method we used to compute the global series in section 3. Here we utilize the fact that the singularities of Y/H are all of type A and we have already proven our formula for the local series in the A_n case. Indeed, the quotient $[Y/H]$ has three stacky points with stabilizers $\mathbb{Z}_\mathsf{p}$, $\mathbb{Z}_\mathsf{q}$, and $\mathbb{Z}_\mathsf{r}$ and the complement of those points $(Y/H)^o$ has Euler characteristic -1.

Proceeding then by the same argument we used in section 3 to get equation (1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n(Y)^H\right) q^{2n} &= \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}^n\left((Y/H)^o\right)\right) q^{2Mn}\right) \\
&\quad \cdot \prod_{\mathsf{a}\in\{\mathsf{p},\mathsf{q},\mathsf{r}\}} \left(\sum_{n=0}^{\infty} e\left(\mathrm{Hilb}_0^n(\mathbb{C}^2)^{\mathbb{Z}_{\mathsf{a}}}\right) q^{\frac{2Mn}{\mathsf{a}}}\right) \\
&= \prod_{m=1}^{\infty} \frac{\left(1-q^{2Mn}\right)}{\left(1-q^{\frac{2Mn}{\mathsf{p}}}\right)\left(1-q^{\frac{2Mn}{\mathsf{q}}}\right)\left(1-q^{\frac{2Mn}{\mathsf{r}}}\right)} \\
&= \prod_{m=1}^{\infty} \frac{\left(1-q^{4En}\right)}{\left(1-q^{2En}\right)\left(1-q^{2Fn}\right)\left(1-q^{2Vn}\right)} \\
&= q^{\frac{1}{24}(-2E+2F+2V)} \cdot \frac{\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)} \\
&= \frac{q^{\frac{1}{6}}\,\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}.
\end{aligned}$$

Substituting into the previous equation and cancelling the factors of $q^{\frac{1}{6}}$, we have thus proved

$$Z_\Delta(q) = \theta_{A_1}(\tau) \cdot \frac{\eta(4E\tau)}{\eta(2E\tau)\eta(2F\tau)\eta(2V\tau)}.$$

Finally, by Lemma 2.2 and the A_1 case of Theorem 1.2 (which we've already proved), we have that

$$\theta_{A_1}(\tau) = \frac{\eta^2(2\tau)}{\eta(\tau)}$$

which, when substituted into the above, completes the proof of Theorem 1.2 in the general case. □

5. THE DERIVED MCKAY CORRESPONDENCE AND THE PROOF OF PROPOSITION 4.1

The derived McKay correspondence uses a Fourier-Mukai transform to give an equivalence of derived categories [1, 9]:

$$\mathsf{FM} : D^b(\mathfrak{X}) \to D^b(Y).$$

The induced map on the numerical K-groups

$$K_0(\mathfrak{X}) \to K_0(Y)$$

is well known to take

$$\begin{aligned}
[\mathcal{O}_{\mathfrak{X}}] &\longmapsto \quad [\mathcal{O}_Y] \\
[\mathcal{O}_0 \otimes \rho_0] &\longmapsto \quad [\mathcal{O}_C] \\
[\mathcal{O}_0 \otimes \rho_1] &\longmapsto -[\mathcal{O}_C(-1)]
\end{aligned}$$

where $\mathcal{O}_0$ is the skyscraper sheaf of the origin in $\mathbb{C}^2$, ρ_0 and ρ_1 are the trivial and non-trivial irreducible representations of $\{\pm 1\}$, and $C \subset Y$ is the exceptional curve (see for example [4] or [9]).

Let $\mathfrak{M}^{m_0,m_1}(\mathfrak{X})$ be the moduli stack of objects $F_\bullet$ in $D^b(\mathfrak{X})$ having numerical K-theory class given by

$$[F_\bullet] = [\mathcal{O}_{\mathfrak{X}}] - m_0[\mathcal{O}_0 \otimes \rho_0] - m_1[\mathcal{O}_0 \otimes \rho_1].$$

Then $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ may be regarded as the open substack of $\mathfrak{M}^{m_0,m_1}(\mathfrak{X})$ parameterizing ideal sheaves I_Z viewed as objects in $D^b(\mathfrak{X})$ supported in degree 0.

The Fourier-Mukai equivalence then induces an equivalence of stacks

$$\mathfrak{M}^{m_0,m_1}(\mathfrak{X}) \cong \mathfrak{M}^{m_0,m_1}(Y)$$

where $\mathfrak{M}^{m_0,m_1}(Y)$ is the moduli space of objects $F_\bullet$ in $D^b(Y)$ having numerical K-theory class given by

$$[F_\bullet] = [\mathcal{O}_Y] - m_0[\mathcal{O}_C] + m_1[\mathcal{O}_C(-1)].$$

Numerical K-theory on Y is isomorphic to $H^*(Y)$ via the Chern character. Then since

$$\begin{aligned} ch([\mathcal{O}_Y]) &= (1,0,0) \\ ch([\mathcal{O}_C]) &= (0,C,1) \\ ch([\mathcal{O}_C(-1)]) &= (0,C,0) \end{aligned}$$

we have

$$ch([\mathcal{O}_Y] - m_0[\mathcal{O}_C] + m_1[\mathcal{O}_C(-1)]) = (1,(m_1-m_0)C,-m_0).$$

We switch notation so that $\mathfrak{M}_{(a,b,c)}(Y)$ denotes the moduli stack of objects in $D^b(Y)$ having Chern character $(a,b,c) \in H^*(Y)$. Then we can rewrite the Fourier-Mukai equivalence as

$$\mathfrak{M}^{m_0,m_1}(\mathfrak{X}) \cong \mathfrak{M}_{(1,(m_1-m_0)C,-m_0)}(Y).$$

Tensoring by the line bundle

$$L = \mathcal{O}_Y((m_0-m_1)C)$$

induces an equivalence

$$\mathfrak{M}_{(1,(m_1-m_0)C,-m_0)}(Y) \cong \mathfrak{M}_{(1,0,-n)}(Y)$$

where

$$n = m_0 - (m_1-m_0)^2.$$

Indeed, this follows from

$$ch(L) = \left(1,(m_0-m_1)C,\tfrac{1}{2}(m_0-m_1)^2C^2\right)$$

and

$$\begin{aligned} (1,(m_1-m_0)C,-m_0)&\cdot(1,(m_0-m_1)C,\tfrac{1}{2}(m_0-m_1)^2C^2) \\ &= (1,0,-(m_0-m_1)^2C^2 + \tfrac{1}{2}(m_0-m_1)^2C^2 - m_0) \\ &= (1,0,(m_0-m_1)^2 - m_0) \end{aligned}$$

where we have used $C^2=-2$.

Thus we have an equivalence of stacks

$$\mathfrak{M}^{m_0,m_1}(\mathfrak{X}) \cong \mathfrak{M}_{(1,0,-n)}(Y)$$

which takes the open stack $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ isomorphically onto an open substack of $\mathfrak{M}_{(1,0,-n)}(Y)$. Noting that $\mathrm{Hilb}^n(Y)$ is also an open substack of $\mathfrak{M}_{(1,0,-n)}(Y)$ which intersects the image of $\mathrm{Hilb}^{m_1,m_1}(\mathfrak{X})$ non-trivially, we find the equivalence induces a birational map

$$\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X}) \dashrightarrow \mathrm{Hilb}^n(Y).$$

Both sides of the above birational equivalence are smooth, holomorphic symplectic varieties, and they are both symplectic resolutions of the singular affine symplectic variety $\mathrm{Sym}^n(\mathbb{C}^2/\{\pm 1\})$. Consequently, $\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})$ is deformation equivalent to, and hence diffeomorphic to $\mathrm{Hilb}^n(Y)$. This assertion follows from Nakajima [15, Cor 4.2] by viewing both Hilbert schemes as moduli spaces of quiver representations of the A_1 Nakajima quiver variety having the same dimension vector but using different stability conditions. Writing these Hilbert schemes as Nakajima quiver varieties is discussed in detail in [12].

The above constructions are compatible with the H action. Indeed, the Fourier-Mukai equivalence and tensoring with the line bundle L both commute with the H action and so we have an equivalence of stacks:

$$\mathfrak{M}^{m_0,m_1}(\mathfrak{X})^H \cong \mathfrak{M}_{(1,0,-n)}(Y)^H$$

and a birational equivalence of holomorphic symplectic varieties:

$$\mathrm{Hilb}^{m_0,m_1}(\mathfrak{X})^H \dashrightarrow \mathrm{Hilb}^n(Y)^H.$$

As before, both sides of the above birational equivalence can be viewed as Nakajima quiver varieties (in this case for the Δ Nakajima quiver), and so by [15, Cor 4.2] are deformation equivalent. This then completes the proof of Proposition 4.1. □

6. Proof of Theorem 1.8

Let $\mathcal{Z} = [X/G]$ be the stack quotient of X by G and let $Y \to X/G$ be the minimal resolution. Our approach to proving theorem 1.8 follows a similar idea as our proof of theorem 1.2 in Section 4 . We exploit the derived McKay equivalence:

$$\mathsf{FM} : D^b(\mathcal{Z}) \to D^b(Y).$$

The Hilbert scheme of zero dimensional substacks of $\mathcal{Z}$ is naturally identified with the G-fixed Hilbert scheme of X:

$$\mathrm{Hilb}(\mathcal{Z}) \cong \mathrm{Hilb}(X)^G.$$

Components of $\mathrm{Hilb}(\mathcal{Z})$ are indexed by the numerical K-theory class of $\mathcal{O}_Z$ for $Z \subset \mathcal{Z}$. The K-theory class of $\mathcal{O}_Z$ can be written in a basis for K-theory as follows:

$$[\mathcal{O}_Z] = n[\mathcal{O}_p] + \sum_{i=1}^{r}\sum_{j=1}^{n(i)} m_j(i)[\mathcal{O}_{p_i} \otimes \rho_j(i)]$$

where $p \in \mathcal{Z}$ is a generic point and $p_1, \dots, p_r \in \mathcal{Z}$ are the orbifold points. The local group of $\mathcal{Z}$ at p_i is $G_{\Delta(i)} \subset SL_2\mathbb{C}$ and has corresponding root system $\Delta(i)$ of rank $n(i)$, and has irreducible representations $\rho_0(i), \rho_1(i), \dots, \rho_{n(i)}(i)$ where $\rho_0(i)$ is the trivial representation. We note that we have the K-theory relation

$$[\mathcal{O}_p] = [\mathcal{O}_{p_i} \otimes \rho_{\mathsf{reg}}(i)]$$

where $\rho_{\mathsf{reg}}(i)$ is the regular representation of $G_{\Delta(i)}$, which explains why we do not need to include $[\mathcal{O}_{p_i} \otimes \rho_0(i)]$ in our basis for K-theory.

We appreviate the data $\{m_j(i)\}$ by the symbol $\mathfrak{m}$ and we denote by

$$\mathrm{Hilb}^{n,\mathfrak{m}}(\mathcal{Z}) \subset \mathrm{Hilb}(\mathcal{Z})$$

the component corresponding to the K-theory class given above. We may also regard $\mathrm{Hilb}^{n,\mathfrak{m}}(\mathcal{Z})$ as parameterizing ideal sheaves and as such, it is given as an open substack

$$\mathrm{Hilb}^{n,\mathfrak{m}}(\mathcal{Z}) \subset \mathfrak{M}^{\alpha}(\mathcal{Z})$$

of the moduli stack of objects in $D^b(\mathcal{Z})$ having K-theory class

$$\alpha = [\mathcal{O}_{\mathcal{Z}}] - n[\mathcal{O}_p] - \sum_{i=1}^{r}\sum_{j=1}^{n(i)} m_j(i)[\mathcal{O}_{p_i} \otimes \rho_j(i)].$$

The derived McKay equivalence then induces a stack equivalence

$$\mathfrak{M}^{\alpha}(\mathcal{Z}) \cong \mathfrak{M}^{\mathsf{FM}(\alpha)}(Y).$$

We identify numerical K-theory on Y with cohomology via the Chern character and index the moduli stack of objects on Y by their Chern character so that our equivalence reads

$$\mathfrak{M}^{\alpha}(\mathcal{Z}) \cong \mathfrak{M}_{\beta}(Y)$$

where $\beta = ch(\mathsf{FM}(\alpha))$. The Mukai equivalence induces the following map in K-theory:

$$\begin{array}{rcl} [\mathcal{O}_{\mathcal{Z}}] & \longmapsto & [\mathcal{O}_Y] \\ [\mathcal{O}_p] & \longmapsto & [\mathcal{O}_p] \\ [\mathcal{O}_{p_i} \otimes \rho_j(i)] & \longmapsto & -[\mathcal{O}_{E_j(i)}(-1)], \quad j = 1, \ldots, n(i) \\ [\mathcal{O}_{p_i} \otimes \rho_0(i)] & \longmapsto & \displaystyle\sum_{j=1}^{\infty} d_j(i)[\mathcal{O}_{E_j(i)}] \end{array}$$

where $p \in Y$ is a generic point, $E_1(i), \ldots, E_{n(i)}(i)$ are the exceptional curves over p_i, and

$$d_j(i) = \dim \rho_j(i).$$

The Chern characters are given by

$$\begin{aligned} ch(\mathcal{O}_p) &= (0, 0, 1) \\ ch(\mathcal{O}_Y) &= (1, 0, 0) \\ ch(\mathcal{O}_{E_j(i)}(-1)) &= (0, E_j(i), 0) \end{aligned}$$

where the right hand side is in $H^0(Y) \oplus H^2(Y) \oplus H^4(Y)$. The fact that $ch_2(\mathcal{O}_{E_j(i)}(-1)) = 0$ follows from Hirzebruch-Riemann-Roch and the fact that $\chi(\mathcal{O}_{E_j(i)}(-1)) = 0$.

We then see that

$$\beta = ch(\mathsf{FM}(\alpha)) = (1, D_{\mathfrak{m}}, -n)$$

where

$$D_{\mathfrak{m}} = \sum_{i=1}^{r} \sum_{j=1}^{n(i)} m_j(i)\, E_j(i).$$

Tensoring by $L_{\mathfrak{m}} = \mathcal{O}(-D_{\mathfrak{m}})$ induces an equivalence of categories and thus the composition

$$\mathfrak{M}^{\alpha}(\mathcal{Z}) \xrightarrow{\mathsf{FM}} \mathfrak{M}_{\beta}(Y) \xrightarrow{L_{\mathfrak{m}} \otimes -} \mathfrak{M}_{\beta \cdot (1, -D_{\mathfrak{m}}, \frac{1}{2} D_{\mathfrak{m}}^2)}(Y)$$

induces an equivalence

$$\mathfrak{M}^{\alpha}(\mathcal{Z}) \cong \mathfrak{M}_{(1, 0, -n - \frac{1}{2} D_{\mathfrak{m}}^2)}(Y).$$

As a consequence of the above equivalence, we get the following

Proposition 6.1. $\operatorname{Hilb}^{n,\mathfrak{m}}(\mathcal{Z})$ *is birational to* $\operatorname{Hilb}^{n + \frac{1}{2} D_{\mathfrak{m}}^2}(Y)$.

Proof. Since

$$\operatorname{Hilb}^{n,\mathfrak{m}}(\mathcal{Z}) \subset \mathfrak{M}^{\alpha}(\mathcal{Z}) \quad \text{and} \quad \operatorname{Hilb}^{n + \frac{1}{2} D_{\mathfrak{m}}^2}(Y) \subset \mathfrak{M}_{(1, 0, -n - \frac{1}{2} D_{\mathfrak{m}}^2)}(Y)$$

are open substacks, we need only show that the image of $\operatorname{Hilb}^{n,\mathfrak{m}}(\mathcal{Z})$ under the equivalence $\mathsf{FM} \circ (L_{\mathfrak{m}} \otimes -)$ intersects $\operatorname{Hilb}^{n + \frac{1}{2} D_{\mathfrak{m}}^2}(Y)$ non-trivially. To this end, let $l = n + \frac{1}{2} D_{\mathfrak{m}}^2$, let $q_1, \ldots, q_l \in Y$ be distinct points disjoint from the exceptional curves $E_j(i)$, and let $r_1, \ldots, r_l \in \mathcal{Z}$ denote the corresponding (non-stacky) points in $\mathcal{Z}$. We show that

$\mathsf{FM}^{-1}(I_{\{q_1,\dots,q_l\}}\otimes L_{\mathfrak{m}}^{-1})$ is the ideal sheaf of a substack $Z\subset\mathcal{Z}$ where $Z=\{r_1,\dots,r_l\}\cup Z_{\mathfrak{m}}$ where is a certain rigid substack supported at the stacky points of $\mathcal{Z}$.

PROOF NEEDS TO BE FINISHED!

□

With the proposition, we can now prove Theorem 1.8. Using the identification $\mathrm{Hilb}(X)^G=\mathrm{Hilb}(\mathcal{Z})$ and identifying discrete parameters we get

$$\begin{aligned} Z^{\mathsf{bir}}_{X,G}(q) &= \sum_{a=0}^{\infty}\left[\mathrm{Hilb}^a(X)^G\right]_{\mathsf{bir}}\; q^{a-1} \\ &= \sum_{n,\mathfrak{m}}\mathrm{Hilb}^{n,\mathfrak{m}}(\mathcal{Z})\, q^{D(n,\mathfrak{m})-1} \end{aligned}$$

where

$$D(n,\mathfrak{m})=kn+\sum_{i=1}^{r}\frac{k}{k_i}\sum_{j=1}^{n(i)}m_j(i)d_j(i).$$

We can organize the data $\mathfrak{m}=\{m_j(i)\}$ into $\boldsymbol{m}(i)\in M_{\Delta(i)}$, i.e. the vectors in the root lattice of $\Delta(i)$ having components $m_1(i),\dots,m_{n(i)}(i)$. Under this identification, we see that

$$D^2_{\mathfrak{m}}=-\sum_{i=1}^{r}\left(\boldsymbol{m}(i)|\boldsymbol{m}(i)\right)_{\Delta(i)}$$

since the intersection form of the exceptional curves over p_i is the negative of the corresponding Cartan matrix $C_{\Delta(i)}$. Let

$$d=n+\frac{1}{2}D^2_{\mathfrak{m}}=n-\frac{1}{2}\sum_{i=1}^{r}\left(\boldsymbol{m}(i)|\boldsymbol{m}(i)\right)_{\Delta(i)}.$$

Then

$$Z^{\mathsf{bir}}_{X,G}(q)=\sum_{d=0}\left[\mathrm{Hilb}^d(Y)\right]_{\mathsf{bir}}\prod_{i=1}^{r}\sum_{\boldsymbol{m}(i)\in M_{\Delta(i)}}q^{D(n,\mathfrak{m})-1}$$

with

$$\begin{aligned} D(n,\mathfrak{m})-1 &= -1+k\left(d+\frac{1}{2}\sum_{i=1}^{r}\left(\boldsymbol{m}(i)|\boldsymbol{m}(i)\right)_{\Delta(i)}\right)+\sum_{i=1}^{r}\frac{k}{k_i}\sum_{j=1}^{n(i)}m_j(i)d_j(i) \\ &= kd-1+\frac{k}{2}\sum_{i=1}^{r}\left\{\left(\boldsymbol{m}(i)|\boldsymbol{m}(i)\right)_{\Delta(i)}+\frac{2}{k_i}\left(\boldsymbol{m}(i)|\boldsymbol{\zeta}(i)\right)_{\Delta(i)}\right\} \end{aligned}$$

where $\boldsymbol{\zeta}(i)\in M_{\Delta(i)}\otimes\mathbb{Q}$ is as in Section 2.

Completing the square and using the formula

$$\frac{1}{k_i^2}\left(\boldsymbol{\zeta}(i)|\boldsymbol{\zeta}(i)\right)_{\Delta(i)}=\frac{2}{k_i}\left(\frac{k_i(n(i)+1)-1}{24}\right),$$

which follows from Lemma A.1, we get

$$D(n,\mathfrak{m})-1=kd-1-\sum_{i=1}^{r}\frac{k}{k_i}\left(\frac{k_i(n(i)+1)-1}{24}\right)+\frac{k}{2}\sum_{i=1}^{r}\left(\boldsymbol{m}(i)+\frac{1}{k_i}\boldsymbol{\zeta}(i)\;\middle|\;\boldsymbol{m}(i)+\frac{1}{k_i}\boldsymbol{\zeta}(i)\right)_{\Delta(i)}.$$

It then follows that

$$Z^{\mathsf{bir}}_{X,G}(q) = q^{A} \sum_{d=0}^{\infty} \left[\mathrm{Hilb}^{d}(Y)\right]_{\mathsf{bir}} \; q^{kd-k} \; \prod_{i=1}^{r} \sum_{\boldsymbol{m}(i)\in M_{\Delta(i)}} q^{\frac{k}{2}\left(\boldsymbol{m}(i)+\frac{1}{k_i}\boldsymbol{\zeta}(i) \,\middle|\, \boldsymbol{m}(i)+\frac{1}{k_i}\boldsymbol{\zeta}(i)\right)_{\Delta(i)}}$$

where

$$A = k-1-\frac{k}{24}\sum_{i=1}^{r}\left(n(i)+1-\frac{1}{k_i}\right).$$

Since

$$\begin{aligned} 24 = e(Y) &= e\left(X/G-\{p_1,\dots,p_r\}\right)+\sum_{i=1}^{r}(n(i)+1) \\ &= \frac{1}{k}\left(24-\sum_{i=1}^{r}\frac{k}{k_i}\right)+\sum_{i=1}^{r}(n(i)+1) \\ &= \frac{24}{k}+\sum_{i=1}^{r}\left(n(i)+1-\frac{1}{k_i}\right) \end{aligned}$$

we see that $A=0$.

Thus we have

$$\begin{aligned} Z^{\mathsf{bir}}_{X,G}(q) &= Z^{\mathsf{bir}}_{Y}(q^k)\prod_{i=1}^{r}\theta_{\Delta(i)}\left(\frac{k\tau}{k_i}\right) \\ &= Z^{\mathsf{bir}}_{Y}(q^k)\prod_{i=1}^{r}Z_{\Delta(i)}\left(\frac{k\tau}{k_i}\right)\eta(k\tau)^{n(i)+1} \\ &= Z^{\mathsf{bir}}_{Y}(q^k)\cdot\eta(k\tau)^{B}\cdot Z_{X,G}(q) \end{aligned}$$

where we've used Theorem 1.3, Theorem 1.4, and we've set

$$B = \frac{24}{k}+\sum_{i=1}^{r}\left(n(i)+1-\frac{1}{k_i}\right).$$

The previous equation which showed that $A=0$ also shows that $B=24$. Then since $\Delta(\tau)=\eta(\tau)^{24}$, we see that Theorem 1.8 follows. □

Appendix A. Another Strange Formula

We recall the notation from Section 2. Let Δ be an ADE root system of rank n. Let $\alpha_1,\dots\alpha_n$ be a system of positive simple roots and let

$$\boldsymbol{d}=\sum_{i=1}^{n}d_i\alpha_i$$

be the largest root. Let $(\cdot|\cdot)$ be the Weyl invariant bilinear form with $(\alpha_i|\alpha_i)=2$ and let $\boldsymbol{\zeta}$ be the dual vector to $\boldsymbol{d}$ in the sense that

$$\sum_{i=1}^{n}(\boldsymbol{\zeta}|\alpha_i)\alpha_i=\boldsymbol{d}. \tag{2}$$

Let

$$k = 1+\sum_{i=1}^{n}d_i^2 = 1+(\boldsymbol{\zeta}|\boldsymbol{d}). \tag{3}$$

The identity of the following lemma coincides with Freudenthal and de Vries's "strange formula" when Δ is A_n.

Lemma A.1. *Let k, n, and $\boldsymbol{\zeta}$ be as above. Then,*

$$\frac{k(n+1)-1}{24} = \frac{(\boldsymbol{\zeta}|\boldsymbol{\zeta})}{2k}.$$

Proof. **The case of $\Delta = A_n$:** For any ADE root system we have $(\boldsymbol{\rho}|\alpha) = 1$ for all positive roots where $\boldsymbol{\rho} = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ is half the sum of the positive roots. Since for A_n, $d_i = 1$, it follows from equation (2) that $\boldsymbol{\zeta} = \boldsymbol{\rho}$, and it follows from equation (3) that $k = n+1 = h$ is the Coxeter number. The lemma is then

$$\frac{(n+1)^2-1}{24} = \frac{(\boldsymbol{\rho}|\boldsymbol{\rho})}{2h}.$$

Since the Lie algebra associated to A_n, namely $\mathfrak{sl}_{n+1}$, has dimension $(n+1)^2-1$ and the Killing form satisfies $\kappa(\cdot,\cdot) = \frac{1}{2h}(\cdot|\cdot)$, the lemma may be rewritten as

$$\frac{\dim \mathfrak{sl}_{n+1}}{24} = \kappa(\boldsymbol{\rho},\boldsymbol{\rho})$$

which is Freudenthal and de Vries's "strange formula" [3, 47.11].

The case of $\Delta = D_n$: Let $e_1,\dots,e_n$ be the standard orthonormal basis of $\mathbb{R}^n$. Then the collection $\{\pm e_i \pm e_j, i<j\}$ is a D_n root system and we may take

$$\alpha_i = \begin{cases} e_i - e_{i+1}, & i = 1,\dots,n-1 \\ e_{n-1}+e_n, & i = n \end{cases}$$

as a system of simple positive roots. Then the fundamental weights ω_i, which are defined by the condition $(\omega_i|\alpha_j) = \delta_{ij}$, are given by [10, Appendix C]

$$\omega_i = \begin{cases} e_1+\dots+e_i, & i \le n-2 \\ \frac{1}{2}(e_1+\dots+e_{n-1}-e_n), & i = n-1 \\ \frac{1}{2}(e_1+\dots+e_{n-1}+e_n), & i = n. \end{cases}$$

Then since

$$d_i = \begin{cases} 1 & i = 1, n-1, n, \\ 2 & i = 2,\dots,n-2, \end{cases}$$

we have

$$\begin{aligned} \boldsymbol{\zeta} &= \omega_1 + 2\omega_2 + 2\omega_3 + \dots + 2\omega_{n-2} + \omega_{n-1} + \omega_n \\ &= 2(n-2)e_1 + \sum_{i=2}^{n-2}\left(2(n-1-i)+1\right)\, e_i + e_{n-1} \end{aligned}$$

and so

$$\begin{aligned} (\boldsymbol{\zeta}|\boldsymbol{\zeta}) &= 4(n-2)^2 + 1 + \sum_{i=2}^{n-2}\left(2(n-1-i)+1\right)^2 \\ &= \frac{4}{3}n^3 - 4n^2 - \frac{1}{3}n + 6. \end{aligned}$$

Finally since $k = 1 + \sum_{i=1}^{n} d_i = 4(n-2)$ the lemma becomes

$$\frac{4(n-2)(n+1)-1}{24} = \frac{\frac{4}{3}n^3 - 4n^2 - \frac{1}{3}n + 6}{8(n-2)}$$

which is readily verified.

The case of $\Delta = E_6, E_7, E_8$**:** These three individual cases are easily checked one by one.

□

APPENDIX B. TABLE OF ETA PRODUCTS

The following table provides the list of the modular forms $Z^{-1}_{X,G}$, expressed as eta products, for each of the 82 possible symplectic actions of a group G on a $K3$ surface X. Our numbering matches Xiao's [18] whose table we refer to for a description of each group.

#	$\lvert G\rvert$	Singularities of X/G	The modular form $Z^{-1}_{X,G}$	Weight
0	1		$\eta(\tau)^{24}$	12
1	2	$8A_1$	$\eta(2\tau)^8\,\eta(\tau)^8$	8
2	3	$6A_2$	$\eta(3\tau)^6\,\eta(\tau)^6$	6
3	4	$12A_1$	$\eta(2\tau)^{12}$	6
4	4	$2A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^2\,\eta(\tau)^4$	5
5	5	$4A_4$	$\eta(5\tau)^4\,\eta(\tau)^4$	4
6	6	$8A_1+3A_2$	$\frac{\eta(3\tau)^8\eta(2\tau)^3}{\eta(6\tau)}$	5
7	6	$2A_1+2A_2+2A_5$	$\eta(6\tau)^2\,\eta(3\tau)^2\,\eta(2\tau)^2\,\eta(\tau)^2$	4
8	7	$3A_6$	$\eta(7\tau)^3\,\eta(\tau)^3$	3
9	8	$14A_1$	$\frac{\eta(4\tau)^{14}}{\eta(8\tau)^4}$	5
10	8	$9A_1+2A_3$	$\frac{\eta(4\tau)^9\eta(2\tau)^2}{\eta(8\tau)^2}$	9/2
11	8	$4A_1+4A_3$	$\eta(4\tau)^4\,\eta(2\tau)^4$	4
12	8	$3A_3+2D_4$	$\frac{\eta(\tau)^2\eta(4\tau)^6}{\eta(2\tau)}$	7/2
13	8	A_1+4D_4	$\frac{\eta(4\tau)^{13}\eta(\tau)^4}{\eta(8\tau)^2\eta(2\tau)^8}$	7/2
14	8	$A_1+A_3+2A_7$	$\eta(8\tau)^2\,\eta(4\tau)\,\eta(2\tau)\,\eta(\tau)^2$	3
15	9	$8A_2$	$\eta(3\tau)^8$	4
16	10	$8A_1+2A_4$	$\frac{\eta(5\tau)^8\eta(2\tau)^2}{\eta(10\tau)^2}$	4
17	12	$4A_1+6A_2$	$\frac{\eta(6\tau)^4\eta(4\tau)^6}{\eta(12\tau)^2}$	4
18	12	$9A_1+A_2+A_5$	$\frac{\eta(6\tau)^9\eta(4\tau)\eta(2\tau)}{\eta(12\tau)^3}$	4
19	12	$3A_1+3A_5$	$\eta(6\tau)^3\,\eta(2\tau)^3$	3
20	12	$A_2+2A_3+2D_5$	$\frac{\eta(4\tau)^3\eta(3\tau)^2\eta(\tau)^2\eta(6\tau)^4}{\eta(12\tau)\eta(2\tau)^4}$	3
21	16	$15A_1$	$\frac{\eta(8\tau)^{15}}{\eta(16\tau)^6}$	9/2
22	16	$10A_1+2A_3$	$\frac{\eta(8\tau)^{10}\eta(4\tau)^2}{\eta(16\tau)^4}$	4
23	16	$5A_1+4A_3$	$\frac{\eta(8\tau)^5\eta(4\tau)^4}{\eta(16\tau)^2}$	7/2
24	16	$6A_1+A_3+2D_4$	$\frac{\eta(8\tau)^{12}\eta(2\tau)^2}{\eta(16\tau)^4\eta(4\tau)^3}$	7/2
25	16	$6A_3$	$\eta(4\tau)^6$	3
26	16	$4A_1+A_3+A_7+D_4$	$\frac{\eta(8\tau)^7\eta(2\tau)^2}{\eta(16\tau)^2\eta(4\tau)}$	3

27	16	$2A_1 + 4D_4$	$\frac{\eta(8\tau)^{14}\eta(2\tau)^4}{\eta(4\tau)^8\eta(16\tau)^4}$	3
28	16	$2A_1 + A_3 + 2A_7$	$\eta\left(8\tau\right)^2\eta\left(4\tau\right)\eta\left(2\tau\right)^2$	5/2
29	16	$A_3 + D_4 + 2D_6$	$\frac{\eta(4\tau)\eta(8\tau)^7\eta(\tau)^2}{\eta(16\tau)^2\eta(2\tau)^3}$	5/2
30	18	$8A_1 + 4A_2$	$\frac{\eta(9\tau)^8\eta(6\tau)^4}{\eta(18\tau)^4}$	4
31	18	$2A_1 + 3A_2 + 2A_5$	$\frac{\eta(9\tau)^2\eta(6\tau)^3\eta(3\tau)^2}{\eta(18\tau)}$	3
32	20	$2A_1 + 4A_3 + A_4$	$\frac{\eta(10\tau)^2\eta(5\tau)^4\eta(4\tau)}{\eta(20\tau)}$	3
33	21	$6A_2 + A_6$	$\frac{\eta(7\tau)^6\eta(3\tau)}{\eta(21\tau)}$	3
34	24	$5A_1 + 3A_2 + 2A_3$	$\frac{\eta(12\tau)^5\eta(8\tau)^3\eta(6\tau)^2}{\eta(24\tau)^3}$	7/2
35	24	$4A_1 + 2A_2 + 2A_5$	$\frac{\eta(12\tau)^4\eta(8\tau)^2\eta(4\tau)^2}{\eta(24\tau)^2}$	3
36	24	$5A_1 + A_3 + A_5 + D_5$	$\frac{\eta(12\tau)^7\eta(6\tau)\eta(2\tau)\eta(8\tau)}{\eta(24\tau)^3\eta(4\tau)}$	3
37	24	$2A_2 + A_5 + D_4 + E_6$	$\frac{\eta(8\tau)^4\eta(4\tau)\eta(3\tau)\eta(12\tau)^4\eta(\tau)}{\eta(6\tau)^2\eta(24\tau)^2\eta(2\tau)^2}$	5/2
38	24	$2A_2 + A_3 + 2E_6$	$\frac{\eta(8\tau)^6\eta(6\tau)\eta(\tau)^2\eta(12\tau)^2}{\eta(2\tau)^4\eta(24\tau)^2}$	5/2
39	32	$8A_1 + 3A_3$	$\frac{\eta(16\tau)^8\eta(8\tau)^3}{\eta(32\tau)^4}$	7/2
40	32	$9A_1 + 2D_4$	$\frac{\eta(16\tau)^{15}\eta(4\tau)^2}{\eta(32\tau)^6\eta(8\tau)^4}$	7/2
41	32	$3A_1 + 5A_3$	$\frac{\eta(16\tau)^3\eta(8\tau)^5}{\eta(32\tau)^2}$	3
42	32	$4A_1 + 2A_3 + 2D_4$	$\frac{\eta(16\tau)^{10}\eta(4\tau)^2}{\eta(32\tau)^4\eta(8\tau)^2}$	3
43	32	$5A_1 + 2A_7$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
44	32	$2A_1 + 2A_3 + A_7 + D_4$	$\frac{\eta(16\tau)^5\eta(4\tau)^2}{\eta(32\tau)^2}$	5/2
45	32	$3A_1 + D_4 + 2D_6$	$\frac{\eta(16\tau)^{10}\eta(2\tau)^2}{\eta(32\tau)^4\eta(4\tau)^3}$	5/2
46	36	$2A_1 + 2A_2 + 4A_3$	$\frac{\eta(18\tau)^2\eta(12\tau)^2\eta(9\tau)^4}{\eta(36\tau)^2}$	3
47	36	$A_1 + 6A_2 + A_5$	$\frac{\eta(18\tau)\eta(12\tau)^6\eta(6\tau)}{\eta(36\tau)^2}$	3
48	36	$6A_1 + A_2 + 2A_5$	$\frac{\eta(18\tau)^6\eta(12\tau)\eta(6\tau)^2}{\eta(36\tau)^3}$	3
49	48	$5A_1 + 6A_2$	$\frac{\eta(24\tau)^5\eta(16\tau)^6}{\eta(48\tau)^4}$	7/2
50	48	$6A_2 + 2A_3$	$\frac{\eta(16\tau)^6\eta(12\tau)^2}{\eta(48\tau)^2}$	3
51	48	$5A_1 + A_2 + 2A_3 + A_5$	$\frac{\eta(24\tau)^5\eta(16\tau)\eta(12\tau)^2\eta(8\tau)}{\eta(48\tau)^3}$	3
52	48	$4A_1 + 3A_5$	$\frac{\eta(24\tau)^4\eta(8\tau)^3}{\eta(48\tau)^2}$	5/2
53	48	$A_1 + A_2 + 2A_3 + 2D_5$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(12\tau)^2\eta(4\tau)^2}{\eta(48\tau)^3\eta(8\tau)^4}$	5/2
54	48	$4A_1 + A_2 + A_7 + E_6$	$\frac{\eta(24\tau)^5\eta(16\tau)^3\eta(6\tau)\eta(2\tau)}{\eta(48\tau)^3\eta(4\tau)^2}$	5/2
55	60	$4A_1 + 3A_2 + 2A_4$	$\frac{\eta(30\tau)^4\eta(20\tau)^3\eta(12\tau)^2}{\eta(60\tau)^3}$	3
56	64	$5A_1 + 3A_3 + D_4$	$\frac{\eta(32\tau)^8\eta(16\tau)\eta(8\tau)}{\eta(64\tau)^4}$	3
57	64	$6A_1 + 3D_4$	$\frac{\eta(32\tau)^{15}\eta(8\tau)^3}{\eta(64\tau)^6\eta(16\tau)^6}$	3

58	64	$3A_1 + 3A_3 + A_7$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
59	64	$5A_3 + D_4$	$\frac{\eta(32\tau)^3\eta(16\tau)^3\eta(8\tau)}{\eta(64\tau)^2}$	5/2
60	64	$4A_1 + A_3 + 2D_6$	$\frac{\eta(32\tau)^8\eta(16\tau)^3\eta(4\tau)^2}{\eta(64\tau)^4\eta(8\tau)^4}$	5/2
61	72	$4A_1 + 3A_2 + A_3 + D_5$	$\frac{\eta(36\tau)^6\eta(24\tau)^4\eta(18\tau)\eta(6\tau)}{\eta(72\tau)^4\eta(12\tau)^2}$	3
62	72	$3A_1 + 2A_3 + 2A_5$	$\frac{\eta(36\tau)^3\eta(18\tau)^2\eta(12\tau)^2}{\eta(72\tau)^2}$	5/2
63	72	$A_2 + 3A_3 + 2D_4$	$\frac{\eta(24\tau)\eta(9\tau)^2\eta(36\tau)^6}{\eta(72\tau)^3\eta(18\tau)}$	5/2
64	80	$3A_1 + 4A_4$	$\frac{\eta(40\tau)^3\eta(16\tau)^4}{\eta(80\tau)^2}$	5/2
65	96	$3A_1 + 3A_2 + 3A_3$	$\frac{\eta(48\tau)^3\eta(32\tau)^3\eta(24\tau)^3}{\eta(96\tau)^3}$	3
66	96	$2A_1 + 2A_2 + A_3 + 2A_5$	$\frac{\eta(48\tau)^2\eta(32\tau)^2\eta(24\tau)\eta(16\tau)^2}{\eta(96\tau)^2}$	5/2
67	96	$2A_1 + 3A_2 + A_7 + D_4$	$\frac{\eta(48\tau)^5\eta(32\tau)^3\eta(12\tau)^2}{\eta(96\tau)^3\eta(24\tau)^2}$	5/2
68	96	$3A_1 + 2A_3 + A_5 + D_5$	$\frac{\eta(48\tau)^5\eta(24\tau)^2\eta(8\tau)\eta(32\tau)}{\eta(96\tau)^3\eta(16\tau)}$	5/2
69	96	$3A_1 + 2A_2 + 2E_6$	$\frac{\eta(48\tau)^5\eta(32\tau)^6\eta(4\tau)^2}{\eta(96\tau)^4\eta(8\tau)^4}$	5/2
70	120	$2A_1 + A_2 + 2A_3 + A_4 + A_5$	$\frac{\eta(60\tau)^2\eta(40\tau)\eta(30\tau)^2\eta(24\tau)\eta(20\tau)}{\eta(120\tau)^2}$	5/2
71	128	$3A_1 + 2A_3 + D_4 + D_6$	$\frac{\eta(64\tau)^8\eta(32\tau)\eta(8\tau)}{\eta(128\tau)^4\eta(16\tau)}$	5/2
72	144	$A_1 + 4A_2 + 2A_5$	$\frac{\eta(72\tau)\eta(48\tau)^4\eta(24\tau)^2}{\eta(144\tau)^2}$	5/2
73	160	$2A_1 + 3A_3 + 2A_4$	$\frac{\eta(80\tau)^2\eta(40\tau)^3\eta(32\tau)^2}{\eta(160\tau)^2}$	5/2
74	168	$A_1 + 3A_2 + 2A_3 + A_6$	$\frac{\eta(84\tau)\eta(56\tau)^3\eta(42\tau)^2\eta(24\tau)}{\eta(168\tau)^2}$	5/2
75	192	$2A_1 + 6A_2 + D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)^6\eta(24\tau)}{\eta(192\tau)^4\eta(48\tau)^2}$	3
76	192	$2A_1 + A_2 + 2A_3 + A_5 + D_4$	$\frac{\eta(96\tau)^5\eta(64\tau)\eta(32\tau)\eta(24\tau)}{\eta(192\tau)^3}$	5/2
77	192	$2A_1 + A_2 + 3A_3 + E_6$	$\frac{\eta(96\tau)^3\eta(64\tau)^3\eta(48\tau)^3\eta(8\tau)}{\eta(192\tau)^3\eta(16\tau)^2}$	5/2
78	288	$2A_1 + 2A_2 + A_3 + 2D_5$	$\frac{\eta(144\tau)^6\eta(96\tau)^4\eta(72\tau)\eta(24\tau)^2}{\eta(288\tau)^4\eta(48\tau)^4}$	5/2
79	360	$A_1 + 2A_2 + 2A_3 + 2A_4$	$\frac{\eta(180\tau)\eta(120\tau)^2\eta(90\tau)^2\eta(72\tau)^2}{\eta(360\tau)^2}$	5/2
80	384	$A_1 + 3A_2 + 2A_3 + D_6$	$\frac{\eta(192\tau)^3\eta(128\tau)^3\eta(96\tau)^3\eta(24\tau)}{\eta(384\tau)^3\eta(48\tau)^2}$	5/2
81	960	$A_1 + 3A_2 + 2A_4 + D_4$	$\frac{\eta(480\tau)^4\eta(320\tau)^3\eta(192\tau)^2\eta(120\tau)}{\eta(960\tau)^3\eta(240\tau)^2}$	5/2

Table 1: Table of the modular forms $Z_{X,G}^{-1}$ for all symplectic G actions.

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