A TOPOLOGICAL VERTEX IDENTITY

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex (some previously known, some new), we derive product formulas for the partition functions. In the special case where the elliptic surface is a K3 surface, we get a new proof of the Katz-Klemm-Vafa formula.

1. INTRODUCTION

APPENDIX A. ODDS AND ENDS

A.1. **Weighted Euler characteristics of symmetric products.** In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

Lemma 1. *Let S be a scheme of finite type over $\mathbb{C}$ and let $e(S)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((Q))$ be any function with $g(0) = 1$. Let $f_d : \operatorname{Sym}^d(S) \to \mathbb{Z}((Q))$ be the constructible function defined by $f_d(\sum_i a_i x_i) = \prod_i g(a_i)$. Then*

$$\sum_{d=0}^{\infty} u^d \int_{\operatorname{Sym}^d(S)} f_d de = \left(\sum_{a=0}^{\infty} g(a) u^a \right)^{e(S)}.$$

Remark 2. In the special case where $g = f_d \equiv 1$, the lemma recovers Macdonald's formula: $\sum_{d=1}^{\infty} e(\operatorname{Sym}^d(S)) u^d = (1-u)^{-e(S)}$.

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties definited by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure []. For convenience's sake, we provide a direct proof here.

Proof. The dth symmetric product admits a stratification with strata labelled by partitions of d. Associated to any partition of d is a unique tuple $(m_1, m_2, \dots)$ of non-negative integers with $\sum_{j=1}^{\infty} j m_j = d$. The stratum labelled by $(m_1, m_2, \dots)$ parameterizes collections of points where there are m_j points of multiplicity j. The full stratification is given by:

$$\operatorname{Sym}^d(S) = \bigsqcup_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left\{ \left(\prod_{j=1}^{\infty} S^{m_j} \right) - \Delta \right\} / \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, S^0 is a point, Δ is the large diagonal, and σ_m is the mth symmetric group. Note that the function f_d is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

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For schemes over $\mathbb{C}$, topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\mathrm{Sym}^d(S)} f_d\, de = \sum_{\substack{(m_1,m_2,\dots)\\ \sum_{j=1}^\infty jm_j=d}} \left(\prod_{j=1}^\infty g(j)^{m_j}\right) \frac{e(S^{\sum_j m_j}-\Delta)}{m_1!\,m_2!\,m_3!\dots}.$$

For any natural number N, the projection $S^N-\Delta\to S^{N-1}-\Delta$ has fibers of the form $S-\{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(S)-(N-1)$ and consequently, $e(S^N-\Delta)=(e(S)-(N-1))\cdot e(S^{N-1}-\Delta)$. Thus by induction, we find $e(S^N-\Delta)=e(S)\cdot(e(S)-1)\cdots(e(S)-(N-1))$ and so

$$\frac{e(S^{\sum_j m_j}-\Delta)}{m_1!\,m_2!\,m_3!\cdots} = \binom{e(S)}{m_1,m_2,m_3,\cdots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\begin{aligned}\sum_{d=0}^\infty \int_{\mathrm{Sym}^d(S)} f_d\, de &= \sum_{(m_1,m_2,\dots)} \prod_{j=1}^\infty \left(g(j)u^j\right)^{m_j}\binom{e(S)}{m_1,m_2,m_3,\dots}\\ &= \left(1+\sum_{j=1}^\infty g(j)u^j\right)^{e(S)}\end{aligned}$$

which proves the lemma. □

A.2. **Some geometry of curves on elliptic surfaces.** In this subsection we prove the following lemma and corollary, which will tell us what is the reduced support of all curves in the class $\beta=B+dF$.

Lemma 3. *For any line bundle ϵ on B, multiplication by the canonical section of $\mathcal{O}(B)$ induces an isomorphism*

$$H^0(S,\pi^*(\epsilon)(B))\cong H^0(S,\pi^*(\epsilon)).$$

Corollary 4. *Let $\beta=B+dF\in H_2(S)$. Then the Chow variety of curves in the class β is isomorphic to $\mathrm{Sym}^d(B)$ where a point $\sum_i d_i x_i\in\mathrm{Sym}^d(B)$ corresponds to the curve $B+\sum_i d_i\pi^{-1}(x_i)$.*

Proof. The corollary follows immediately from the lemma since the Chow variety is the space of effective divisors and the lemma implies that any effective divisor in the class β is a union of the section B with an effective divisor pulled by from the base.

To prove lemma 3 we proceed as follows. For any line bundle δ on B, the Leray spectral sequence yields the short exact sequence:

$$0\to H^0(B,\delta\otimes R^1\pi_*\mathcal{O})\to H^1(S,\pi^*\delta)\xrightarrow{\alpha} H^1(B,\delta)\to 0,$$

in particular, α is a surjection.

Then the long exact cohomology sequence associated to

$$0\to\pi^*\delta\otimes\mathcal{O}(-B)\to\pi^*\delta\to\mathcal{O}_B\otimes\pi^*\delta\to 0$$

is

$$\cdots\to H^1(S,\pi^*(\delta))\xrightarrow{\alpha} H^1(B,\delta)\to H^2(S,\pi^*\delta\otimes\mathcal{O}(-B))\to H^2(S,\pi^*\delta)\to 0,$$

and since α is a surjection, we get an isomorphism of the last two terms. We apply Serre duality to that isomorphism and we use the fact that $K_S = \pi^*(K_B \otimes L)$ where $L = (R\pi_*\mathcal{O}_S)^\vee$ [**?**, prop?] to obtain

$$H^0(S, \pi^*(\delta^{-1} \otimes K_B \otimes L)(B)) \cong H^0(S, \pi^*(\delta^{-1} \otimes K_B \otimes L)).$$

Letting $\delta = K_B \otimes L \otimes \epsilon^{-1}$, the lemma is proved. $\square$

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