BPS STATE COUNTS OF LOCAL ELLIPTIC SURFACES VIA FORMAL GEOMETRY AND THE TOPOLOGICAL VERTEX

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Abstract. We compute the (connected) stable pair invariants of $X = \mathrm{Tot}(K_S)$ where S is an elliptic surface with section and at worst 1-nodal singular fibres. The calculation includes thickenings to all orders in both the surface and fibre direction. We use a new method combining motivic arguments and torus localization.

We stratify the moduli space according to underlying reduced support C_{red} of the stable pair and compute the contribution of each C_{red} individually. The contribution of C_{red} can be split up into a part coming from the nodes of C_{red} and the complement of the nodes $C^{\circ}_{\mathrm{red}} \subset C_{\mathrm{red}}$. The formal neighbourhood of C°_{red} in X is isomorphic to a formal neighbourhood of C°_{red} inside its normal bundle. This gives us lots of $\mathbb{C}^*$-actions.

Localization with respect to the torus actions leads to a vertex calculation which can be performed explicitly. As special cases we find a new proof of the Katz–Klemm–Vafa formula in the primitive case (independent of Kawai–Yoshioka's formula) and the BPS spectrum of the local rational elliptic surface.

Contents

1. Introduction

Let X be a smooth projective 3-fold, $\chi \in \mathbb{Z}$, and $\beta \in H_2(X)$ a curve class. Denote by $P_\chi(X,\beta)$ the moduli space of stable pairs $I^\bullet = [\mathcal{O}_X \to \mathcal{F}]$ on X for which $\chi(\mathcal{F}) = \chi$, and the scheme theoretic support of F has curve class β. The moduli space $P_\chi(X,\beta)$ is an instance of Le Potier's more general moduli spaces of stable pairs [**?**]. The deformation-obstruction theory of stable pairs *does not* provide a perfect obstruction theory for $P_\chi(X,\beta)$. R. P. Thomas and R. Pandharipande realize $P_\chi(X,\beta)$ as a component of the moduli space of complexes in $D^b(X)$ with trivial determinant. Viewed as a moduli space of complexes $P_\chi(X,\beta)$ *does* have a perfect obstruction theory [4]. When X is in addition Calabi-Yau, this perfect obstruction theory is symmetric and the stable pair invariants of X are defined as the degree of the virtual cycle

$$P_{\chi,\beta}(X) := \int_{[P_\chi(X,\beta)]^{\mathrm{vir}}} 1.$$

By a theorem of K. Behrend [**?**]

$$\int_{[P_\chi(X,\beta)]^{\mathrm{vir}}} 1 = \int_{P_\chi(X,\beta)} \nu_B \ \mathrm{d}e,$$

where $\nu_B : P_\chi(X,\beta) \to \mathbb{Z}$ is Behrend's constructible function and $e(\cdot)$ denotes topological Euler characteristic.

In this paper $\pi : S \to B$ denotes an elliptic surface. This means S is a smooth surface, B a smooth curve of genus $g(B)$, and π a holomorphic map with general fibre a connected smooth genus 1 curve [**?**]. We make two assumptions:

Can more be said about such surfaces? I don't thin we need $B \cong \mathbb{P}^1$?

- π has a section $B \hookrightarrow S$,
- all singular fibres of π are of Kodaira type I_1, i.e. rational 1-nodal curves.

We are interested in the case $X = \mathrm{Tot}(K_S)$ and $\beta = B + dF$, where B is the class of the section and F is the class of the fibre. Since X is a non-compact Calabi-Yau 3-fold we require curves of $P_\chi(X,\beta)$ to have proper support. Non-compactness of $P_\chi(X,\beta)$ also means we do not have a virtual cycle, so one should *define* stable pair invariants in this setting eitherby Graber-Pandharipande's localization formula [**?**] or by integration of ν_B over $P_\chi(X,\beta)$. We choose the latter approach.

How are both approaches related?

EULER CHAR FOR THIS VERSION FOR NOW!

Consider the (disconnected) generating function

$$Z^{\bullet P}(q,y) := \sum_{d \geq 0} \sum_{\chi} P_{\chi,B+dF}(X) q^\chi y^d, \tag{1}$$
$$P_{\chi,B+dF}(X) := e(P_\chi(X,B+dF)).$$

The connected generating function is defined as [4]

$$Z^P(q,y) := \frac{\sum_{d\geq 0}\sum_\chi P_{\chi,B+dF}(X) q^\chi y^d}{\sum_{d\geq 0}\sum_\chi P_{\chi,dF}(X) q^\chi y^d}, \tag{2}$$
$$P_{\chi,dF}(X) := e(P_\chi(X,dF)),$$

where $P_{\chi,0}(X) = 1$ for all χ. Our main result is the following.

Theorem 1.1. *Let $X = \mathrm{Tot}(K_S)$ where $S \to B$ is an elliptic surface with section B of genus g and let $e = e(S)$ be the topological Euler characteristic of S. Then*

$$Z^P(q,y) = \left(\frac{q}{(1-q)^2}\right)^{1-g} \prod_{i=1}^{\infty} \frac{1}{(1-y^i)^{e-4+4g}(1-qy^i)^{2-2g}(1-q^{-1}y^i)^{2-2g}}.$$

The proof is divided into five movements:

Stratifation, Restriction, Formalization, Localization, Finale (Schur).

Applications for this paper: stable pair version of KKV in the primitive case independent of KY, gen fun for rational elliptic surface. Future applications: elliptically fibres CY3's, refinement and comparison to refined KKV, ...

2. Stratification

2.1. **Some geometry of curves on elliptic surfaces.** In this subsection we prove the following lemma and corollary, which will tell us what is the reduced support of all curves in the class $\beta = F + dF$.

Lemma 2.1. *For any line bundle ϵ on B, multiplication by the canonical section of $\mathcal{O}(B)$ induces an isomorphism*

$$H^0(S, \pi^*(\epsilon)(B)) \cong H^0(S, \pi^*(\epsilon)).$$

Corollary 2.2. *Let $\beta = B + dF \in H_2(S)$. Then the Chow variety of curves in the class β is isomorphic to $\mathrm{Sym}^d(B)$ where a point $\sum_i d_i x_i \in \mathrm{Sym}^d(B)$ corresponds to the curve $B + \sum_i d_i \pi^{-1}(x_i)$.*

Proof. The corollary follows immediately from the lemma since the Chow variety is the space of effective divisors and the lemma implies that any effective divisor in the class β is a union of the section B with an effective divisor pulled by from the base.

To prove lemma 2.1 we proceed as follows. For any line bundle δ on B, the Leray spectral sequence yields the short exact sequence:

$$0 \to H^0(B, \delta \otimes R^1\pi_*\mathcal{O}) \to H^1(S, \pi^*\delta) \xrightarrow{\alpha} H^1(B,\delta) \to 0,$$

in particular, α is a surjection.

Then the long exact cohomology sequence associated to

$$0 \to \pi^*\delta \otimes \mathcal{O}(-B) \to \pi^*\delta \to \mathcal{O}_B \otimes \pi^*\delta \to 0$$

is

$$\cdots \to H^1(S, \pi^*(\delta)) \xrightarrow{\alpha} H^1(B, \delta) \to H^2(S, \pi^*\delta \otimes \mathcal{O}(-B)) \to H^2(S, \pi^*\delta) \to 0,$$

and since α is a surjection, we get an isomorphism of the last two terms. We apply Serre duality to that isomorphism and we use the fact that $K_S = \pi^*(K_B \otimes L)$ where $L = (R\pi_*\mathcal{O}_S)^\vee$ [1, prop?] to obtain

$$H^0(S, \pi^*(\delta^{-1} \otimes K_B \otimes L)(B)) \cong H^0(S, \pi^*(\delta^{-1} \otimes K_B \otimes L)).$$

Letting $\delta = K_B \otimes L \otimes \epsilon^{-1}$, the lemma is proved. □

Let β be Poincaré dual to $B + dF$. The projections

$$\varpi : X \longrightarrow S \longrightarrow B$$

induce a push-forward map

$$P_\chi(X, \beta) \longrightarrow \mathrm{Sym}^d(B), \; I^\bullet = [\mathcal{O}_X \to \mathcal{F}] \mapsto \varpi_* \mathcal{F}.$$

We denote the fibre of $S \to B$ over $p \in B$ by F_p. Let

$$\mathbf{p} := \sum_{i=1}^m d_i p_i \subset B$$

be an effective divisor with all $d_i > 0$ and $\sum_{i=1}^m d_i = d$. Consider the reduced curve

$$C_{\mathbf{p}} := \bigcup_{i=1}^m F_{p_i} \subset S \subset X,$$

where $S \subset X$ is the zero-section. The fibre of ϖ_* over $\mathbf{p}$ is

$$P_\chi(X, \mathbf{p}) := \big\{ I^\bullet = [\mathcal{O}_X \to \mathcal{F}] \in P_\chi(X, \beta) \;:\; \varpi_*\mathcal{F} = \mathbf{p} \big\},$$

i.e. the locally closed subset of stable pairs $I^\bullet = [\mathcal{O}_X \to \mathcal{F}] \in P_\chi(X, \beta)$ for which $\mathcal{F}$ has set theoretic support $C_{\mathbf{p}}$ and multiplicity d_i along F_{p_i} for all i. We are interested in the stratification

$$P_\chi(X, \beta) = \coprod_{\mathbf{p} \in \mathrm{Sym}^d(B)} P_\chi(X, \mathbf{p}).$$

Lemma 2.3.

$$e(P_\chi(X, \beta)) = \int_{\mathbf{p} \in \mathrm{Sym}^d(B)} e(P_\chi(X, \mathbf{p})) \; \mathrm{d}e.$$

Proof. [**?**]. □

3. Restriction

By Lemma 2.3 we are reduced to computing $e(P_\chi(X, \mathbf{p}))$ for any

$$\mathbf{p} = \sum_{i=1}^m d_i p_i \in \mathrm{Sym}^d(B).$$

Let $q \in C_\mathbf{p}$ be one of the *nodal* singularities (either a node in a singular fibre or an intersection point of a fibre with the section). We denote by $\widehat{X}_q$ the formal neighbourhood of $\{q\} \subset X$ and by $X \setminus q$ the complement of $\{q\} \subset X$. Let

A little more care in the def of this moduli space is needed since $X \setminus q$ is non-compact AND the supports of the stable pairs are non-compact.

$$P_\chi(X \setminus q, \mathbf{p})$$

be the moduli space of stable pairs $I^\bullet = [\mathcal{O}_{X\setminus q} \to \mathcal{F}]$ such that $\chi(\mathcal{F}) = \chi$, $\mathcal{F}$ has set theoretic support $C_\mathbf{p} \setminus q$, $\mathcal{F}$ has multiplicity 1 along $B \setminus q$, and $\mathcal{F}$ has multiplicity d_i along $F_{p_i} \setminus q$ for all i. Moreover let

This might need a little more care too since stable pair theory is not yet defined for formal schemes.

$$P_\chi(\widehat{X}_q, \mathbf{p})$$

be the moduli spaces of stable pairs $I^\bullet = [\mathcal{O}_{\widehat{X}_q} \to \mathcal{F}]$ such that $\chi(\mathcal{F}) = \chi$, $\mathcal{F}$ has set theoretic support $\widehat{C_\mathbf{p}}$, $\mathcal{F}$ has multiplicity 1 along $\widehat{B}$, and $\mathcal{F}$ has multiplicity d_i along $\widehat{F_{p_i}}$ for all i. Here $\widehat{C_\mathbf{p}}$, $\widehat{B}$, $\widehat{F_{p_i}}$ denote the lifts[1] of $C_\mathbf{p}$, B, F_{p_i} to $\widehat{X}_q$. We are interested in the injective morphism induced by restriction

$$P_\chi(X, \mathbf{p}) \hookrightarrow \coprod_{\chi=\chi_1+\chi_2} P_{\chi_1}(X \setminus q, \mathbf{p}) \times P_{\chi_2}(\widehat{X}_q, \mathbf{p}). \tag{3}$$

The image of this morphism can be characterized as follows. Let

May not exist. In general work in stalk? See Jim's e-mail on 25.6.2014.

$$U = \mathrm{Spec}\ \mathbb{C}[x, y] \subset S$$

be an open affine neighbourhood of q over which $X = \mathrm{Tot}(K_S)$ trivializes with fibre coordinate z. Then $\widehat{X}_q$ is the reduced point q with sheaf of rings

$$\mathcal{O}_{\widehat{X}_q} \cong \widehat{\mathcal{O}}_{X,q} \cong \mathbb{C}[\![x, y, z]\!].$$

Suppose the coordinates are chosen such that $C_\mathbf{p}$ is defined by $xy = z = 0$. Define open subsets

$$V = \{x \neq 0\} \subset U, \ W = \{y \neq 0\} \subset U.$$

[1] Let $\widehat{X}_Z$ be the formal completion of any scheme along a closed subset Z. If $\mathcal{E}$ is a coherent sheaf on X then one can define a lift $\mathcal{E}^\Delta$ to $\widehat{X}_Z$ [**?**]. In the case $\mathcal{E} = \mathscr{I} \subset \mathcal{O}_X$ is an ideal sheaf, this provides an ideal sheaf $\mathscr{I}^\Delta \subset \mathcal{O}_{\widehat{X}_Z}$ [**?**].

Lemma 3.1. *An element*

$$([s_1 : \mathcal{O}_{X \setminus q} \to \mathcal{F}_1], [s_2 : \mathcal{O}_{\widehat{X}_q} \to \mathcal{F}_2])$$

lies in the image of the embedding (3) *if and only if the Cohen-Macaulay support curves* $C_{\mathcal{F}_1}$, $C_{\mathcal{F}_2}$ *underlying both stable pairs glue i.e.*

$$\Gamma(\widehat{X}_q, \mathscr{I}_{C_{\mathcal{F}_2}}) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x^{\pm}, y, z]\!] \cong \widehat{\Gamma}(V \times \mathbb{C}, \mathscr{I}_{C_{\mathcal{F}_1}}|_{V \times \mathbb{C}}),$$
$$\Gamma(\widehat{X}_q, \mathscr{I}_{C_{\mathcal{F}_2}})) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x, y^{\pm}, z]\!] \cong \widehat{\Gamma}(W \times \mathbb{C}, \mathscr{I}_{C_{\mathcal{F}_1}}|_{W \times \mathbb{C}}),$$

where $\Gamma(\cdot)$ *denotes the global section functor,* $\widehat{(\cdot)}$ *is the formal completion of the module* $(\cdot)$*, and* $\mathscr{I}_{C_{\mathcal{F}_1}}$, $\mathscr{I}_{C_{\mathcal{F}_2}}$ *are ideal sheaves.*

See Jim's fpqc e-mail on 25.6.2014.

Proof. Perhaps Ben-Bassat–Temkin's [**?**] abstract setup (or a stable pairs version) reduces to this when Z (in their notation) is just a point. Note: life is not too bad because only the support curve has to glue. This is because the section of a stable pair is an isomorphism outside a 0-dim subscheme. □

We want to apply the above construction not just for one point q. Let $q_1, \ldots, q_n \in C_{\mathbf{p}}$ be all nodes. For notational simplicity we write

$$X^\circ := X \setminus \{q_1, \ldots, q_n\}.$$

We embed

$$P_\chi(X, \mathbf{p}) \hookrightarrow \coprod_{\chi = \chi' + \chi_1 + \cdots + \chi_n} P_{\chi'}(X^\circ, \mathbf{p}) \times \prod_{j=1}^{n} P_{\chi_j}(\widehat{X}_{q_j}, \mathbf{p}).$$

The image is characterized by gluing conditions as in Lemma 3.1 at each of the nodes q_j.

4. Formalization

In the previous section we characterized the image of $P_\chi(X, \mathbf{p})$ under restriction to special points and their complements

$$P_\chi(X, \mathbf{p}) \hookrightarrow \coprod_{\chi = \chi' + \chi_1 + \cdots + \chi_n} P_{\chi'}(X^\circ, \mathbf{p}) \times \prod_{i=1}^{n} P_{\chi_j}(\widehat{X}_{q_j}, \mathbf{p})$$

In this section we relate $P_\chi(X^\circ, \mathbf{p})$ to moduli spaces of stable pairs on the (punctured) fibres/section inside their normal bundle.

Recall that

$$\mathbf{p} := \sum_{i=1}^{m} d_i p_i \in \mathrm{Sym}^d(B),\ C_{\mathbf{p}} := \bigcup_{i=1}^{m} F_{p_i},$$

and $q_1, \ldots, q_n$ are all nodes of $C_{\mathbf{p}}$. We have an inclusion

$$P_\chi(X^\circ, \mathbf{p}) \subset P_\chi(X^\circ, \beta),$$

where $P_\chi(X^\circ, \beta)$ denotes the moduli space of stable pairs $I^\bullet = [\mathcal{O}_{X^\circ} \to \mathcal{F}]$ on X° such that $\chi(\mathcal{F}) = \chi$ and the closure of the scheme theoretic support of $\mathcal{F}$ in X is proper with class β. We can make a formal completion of the former space along the latter

$$\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ, \mathbf{p})}.$$

Obviously the underlying topological space is unchanged so

$$e(\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ, \mathbf{p})}) = e(P_\chi(X^\circ, \mathbf{p})).$$

Passing to the formal completion allows us to consider stable pairs on the formal completion of X° along $C_{\mathbf{p}}^\circ := C_{\mathbf{p}} \setminus \{q_1, \ldots, q_n\}$. This formal completion is denoted by

$$\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}.$$

Lemma 4.1. *There exists a canonical isomorphism*

$$\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ, \mathbf{p})} \cong P_\chi(\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}, \mathbf{p}),$$

where $P_\chi(\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}, \mathbf{p})$ is the moduli space of stable pairs $I^\bullet = [\mathcal{O} \to \mathcal{F}]$ on $\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}$ such that $\chi(\mathcal{F}) = \chi$, $\mathcal{F}$ has multiplicity 1 along $\widehat{B^\circ}$, and $\mathcal{F}$ has multiplicity d_i along $\widehat{F_{p_i}^\circ}$ for all i. Here $\widehat{B^\circ}$, $\widehat{F_{p_i}^\circ}$ denote the lifts of B°, $F_{p_i}^\circ := F_{p_i} \setminus \{q_1, \ldots, q_n\}$ to $\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}$.

Proof. Jim's idea of categorical limits. This should be formal. □

Let us take a closer look at the formal scheme $\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}$. Removing the nodes points $q_1, \ldots, q_n$ we obtain smooth curves B°, $F_{p_i}^\circ$ and

$$C_{\mathbf{p}}^\circ \cong B^\circ \sqcup F_{p_1}^\circ \sqcup \cdots \sqcup F_{p_m}^\circ.$$

This isomorphism also holds at the level of formal schemes.

Lemma 4.2. *There exists a canonical isomorphism*

$$\widehat{X^\circ}_{C_{\mathbf{p}}^\circ} \cong \widehat{X^\circ}_{B^\circ} \sqcup \widehat{X^\circ}_{F_{p_1}^\circ} \sqcup \cdots \sqcup \widehat{X^\circ}_{F_{p_m}^\circ},$$

where $\widehat{X^\circ}_{B^\circ}$, $\widehat{X^\circ}_{F_{p_i}^\circ}$ are the formal completions of X along B°, $F_{p_i}^\circ$.

Proof. Disjoint union commutes with formal completion. □

IS THIS REALLY TRUE? Sounds plausible.

This lemma allows us to pass to the normal bundles of $B^\circ \subset X^\circ$, $F_{p_i}^\circ \subset X^\circ$.

Lemma 4.3. *There exists natural isomorphisms*

$$\widehat{X^\circ}_{B^\circ} \cong \widehat{N_{B^\circ/X^\circ}}_{B^\circ}, \ \widehat{X^\circ}_{F^\circ_{p_i}} \cong \widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}},$$

where $\widehat{N_{B^\circ/X^\circ}}_{B^\circ}$, $\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}$ *are the formal completions of the normal bundles* N_{B°/X°, $N_{F^\circ_{p_i}/X^\circ}$ *along their zero sections* B°, $F^\circ_{p_i}$.

We have to be careful here because of the discussions in the first two weeks of July/the discussions with Davesh. Something like this presumably only works on open affine pieces. That should be enough though since we first remove the nodes.

Proof. For the fibres we proved this rigorously using sections of $\mathcal{O}/\mathscr{I}^{r+1} \to \mathcal{O}/\mathscr{I}^r$ pulled back from the base B. This requires flatness of π. For the section we use Davesh's argument. □

Lemmas 4.1, 4.2, 4.3 allow us write

$$\widehat{P}_\chi(X^\circ,\beta)_{P_\chi(X^\circ,\mathbf{p})} \cong \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_m} P_{\chi'}(\widehat{N_{B^\circ/X^\circ}}_{B^\circ},\mathbf{p}) \times \prod_{i=1}^m P_{\chi_i}(\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}},\mathbf{p}),$$

where $P_\chi(\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}},\mathbf{p})$ is the moduli space of stable pairs $I^\bullet = [\mathcal{O} \to \mathcal{F}]$ on $\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}$ with $\chi(\mathcal{F}) = \chi$ and $\mathcal{F}$ has set theoretic support $\widehat{F^\circ_{p_i}}$ with multiplicity d_i. Here $\widehat{F^\circ_{p_i}}$ denotes the lift of $F^\circ_{p_i}$ to $\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}$. Similar for $P_\chi(\widehat{N_{B^\circ/X^\circ}}_{B^\circ},\mathbf{p})$ where the multiplicity along $\widehat{B^\circ}$ is required to be one.

Finally we want to "undo" the formal completion on the normal bundles by using categorical limits as in Lemma 4.1. We denote by

$$P_\chi(N_{F^\circ_{p_i}/X^\circ},\mathbf{p}) \subset P_\chi(N_{F^\circ_{p_i}/X^\circ}, d_i F^\circ_{p_i})$$

moduli spaces of stable pairs $I^\bullet = [\mathcal{O} \to \mathcal{F}]$ on N_{B°/X° with $\chi(\mathcal{F}) = \chi$. The first has $\mathcal{F}$ with set theoretic support $F^\circ_{p_i}$ and multiplicity d_i. The second has $\mathcal{F}$ such that the closure of its set theoretic support in $N_{F_{p_i}/X}$ is proper with class $d_i F_{p_i}$. Similarly we consider

$$P_\chi(N_{B^\circ/X^\circ},\mathbf{p}) \subset P_\chi(N_{B^\circ/X^\circ}, B^\circ).$$

The argument presented in the proof of Lemma 4.1 gives

$$P_\chi(\widehat{N_{B^\circ/X^\circ}}_{B^\circ},\mathbf{p}) \cong \widehat{P}_\chi(N_{B^\circ/X^\circ}, B^\circ)_{P_\chi(N_{B^\circ/X^\circ},\mathbf{p})}$$
$$P_\chi(\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}},\mathbf{p}) \cong \widehat{P}_\chi(N_{F^\circ_{p_i}/X^\circ}, d_i F^\circ_{p_i})_{P_\chi(N_{F^\circ_{p_i}/X^\circ},\mathbf{p})}.$$

Combining all arguments of this section gives the following result.

Proposition 4.4. *We have natural isomorphisms*

$$\widehat{P}_\chi(X^\circ,\beta)_{P_\chi(X^\circ,\mathbf{p})} \cong \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_m} \widehat{P}_{\chi'}(N_{B^\circ/X^\circ},B^\circ)_{P_\chi(N_{B^\circ/X^\circ},\mathbf{p})} \times \prod_{i=1}^m \widehat{P}_{\chi_i}(N_{F^\circ_{p_i}/X^\circ}, d_i F^\circ_{p_i})_{P_\chi(N_{F^\circ_{p_i}/X^\circ},\mathbf{p})}.$$

In particular on the underlying topological space we have a homeomorphism

$$P_\chi(X^\circ,\mathbf{p}) \approx \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_m} P_\chi(N_{B^\circ/X^\circ},\mathbf{p}) \times \prod_{i=1}^m P_\chi(N_{F^\circ_{p_i}/X^\circ},\mathbf{p}).$$

Proof. Combination of the above. □

5. Localization

5.1. **Localization I.** In the previous two sections we constructed an embedding

$$P_\chi(X,\mathbf{p}) \hookrightarrow \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_n} P_{\chi'}(X^\circ,\mathbf{p}) \times \prod_{j=1}^n P_{\chi_j}(\widehat{X}_{q_i},\mathbf{p}) \tag{4}$$

and homeomorphisms

$$P_\chi(X^\circ,\mathbf{p}) \approx \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_m} P_\chi(N_{B^\circ/X^\circ},\mathbf{p}) \times \prod_{i=1}^m P_\chi(N_{F^\circ_{p_i}/X^\circ},\mathbf{p}). \tag{5}$$

Each normal bundle has a natural $\mathbb{C}^{*2}$-action given by scaling the fibres. The action of $\mathbb{C}^{*2}$ on $P_\chi(N_{B^\circ/X^\circ},\mathbf{p})$ is trivial[2] so we ignore it. Therefore $\mathbb{C}^{*2m}$ acts naturally on $P_\chi(X^\circ,\mathbf{p})$ by (5).

Since each $\widehat{X}_{q_j}$ is just the reduced point q_j with structure sheaf

$$\mathcal{O}_{\widehat{X}_{q_j}} \cong \widehat{\mathcal{O}}_{X,q_j} \cong \mathbb{C}[\![x,y,z]\!],$$

we have $\mathbb{C}^{*3}$ acting on this space by $(s_1,s_2,s_3)\cdot(x,y,z) = (s_1x,s_2y,s_3z)$. In total we get an action of $\mathbb{C}^{*(2m+3n)}$ on the RHS of (4). However $P_\chi(X,\mathbf{p})$ is not invariant under this full torus.

Lemma 5.1. *Define the a $2m$-dimensional subtorus $T\subset\mathbb{C}^{*(2m+3n)}$ by the following equations. For any nodal fibre F_{p_i} with node q_j let $(t_1^{(i)},t_2^{(i)})$ be the*

[2]This action is transverse to the section and our stable pairs have multiplicity 1 along B.

coordinates of $\mathbb{C}^{*2}$ *acting on* $N_{F^\circ_{p_i}/X^\circ}$ *and let* $(s_1^{(j)}, s_2^{(j)}, s_3^{(j)})$ *be the coordinates of* $\mathbb{C}^{*3}$ *acting on* $\widehat{X}_{q_j}$*, then*

$$s_1^{(j)} = s_2^{(j)} = t_1^{(i)}, \;\; s_3^{(j)} = t_2^{(i)}.$$

For any (not necessarily nodal) fibre F_{p_i} *and* $\{q_j\} = F_{p_i} \cap B$ *let* $(t_1^{(i)}, t_2^{(i)})$ *be the coordinates of* $\mathbb{C}^{*2}$ *acting on* $N_{F^\circ_{p_i}/X^\circ}$ *and let* $(s_1^{(j)}, s_2^{(j)}, s_3^{(j)})$ *be the coordinates of* $\mathbb{C}^{*3}$ *acting on* $\widehat{X}_{q_j}$*, then*

$$s_1^{(j)} = 1, \;\; s_2^{(j)} = t_1^{(i)}, \;\; s_3^{(j)} = t_2^{(i)}.$$

Then T *leaves* $P_\chi(X, \mathbf{p})$ *invariant.*

How tedious will this be...?

Proof. Use the gluing conditions of Lemma 3.1. This does require passing through several isomorphisms which could be tricky. □

Since $e(P_\chi(X,\mathbf{p})) = e(P_\chi(X,\mathbf{p})^T)$ we are reduced to understanding the fixed point locus $P_\chi(X,\mathbf{p})^T$. Let

$$([s : \mathcal{O}_{X^\circ} \to \mathcal{E}], \{[s_j : \mathcal{O}_{\widehat{X}_{q_j}} \to \mathcal{F}_j]\}_{j=1}^n) \in \coprod_{\chi = \chi' + \chi_1 + \cdots + \chi_n} P_{\chi'}(X^\circ, \mathbf{p}) \times \prod_{j=1}^n P_{\chi_j}(\widehat{X}_{q_j}, \mathbf{p}).$$

This element lies in $P_\chi(X,\mathbf{p})$ if and only if the underlying Cohen-Macaulay curves $C_{\mathcal{E}}$, $C_{\mathcal{F}_j}$ glue as described in Lemma 3.1. This element is in addition T-fixed if and only if each of the restrictions

$$\Gamma(\widehat{X}_{q_j}, \mathscr{I}_{C_{\mathcal{F}_j}}) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x^{\pm}, y, z]\!] \cong \widehat{\Gamma}(V \times \mathbb{C}, \mathscr{I}_{C_{\mathcal{E}}}|_{V \times \mathbb{C}})$$
$$\Gamma(\widehat{X}_{q_j}, \mathscr{I}_{C_{\mathcal{F}_j}})) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x, y^{\pm}, z]\!] \cong \widehat{\Gamma}(W \times \mathbb{C}, \mathscr{I}_{C_{\mathcal{E}}}|_{W \times \mathbb{C}})$$

is given by a monomial ideal in two variables, i.e. a (2-dimensional) partition. For each node which is the intersection point of a (not necessarily nodal) fibre F_{p_i} with the zero section, this amounts to specifying a partition λ_i of d_i in the fibre direction. The partition in the section direction is (1), because the multiplicity of $C_{\mathcal{E}}$ along B is 1. For each node of a nodal fibre F_{p_i} the cross-section of the Cohen-Macaulay support curve has to be given by the same partitions λ_i. Altogether we have fixed partitions $\boldsymbol{\lambda} = \{\lambda_i \vdash d_i\}_{i=1}^m$. Denote by

$$P_\chi(X^\circ, \mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(X^\circ, \mathbf{p}), \;\; P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(\widehat{X}_{q_j}, \mathbf{p})$$

the locally closed subsets for which the underlying Cohen-Macaulay curves have restrictions described by partitions $\boldsymbol{\lambda}$ as above. We arrive at the following conclusion.

Lemma 5.2. *The embedding* (4) *induces a bijective morphism*

$$P_\chi(X,\mathbf{p})^T \cong \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_n} \coprod_{\boldsymbol{\lambda}=\{\lambda_i\vdash d_i\}_{i=1}^m} P_{\chi'}(X^\circ,\mathbf{p})_{\boldsymbol{\lambda}} \times \prod_{j=1}^n P_{\chi_j}(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}},$$

where T is the torus of Lemma 4.3.

Proof. Easy from the above. □

5.2. **Localization II.** In this subsection we focus attention on $e(P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}})$ for any $\boldsymbol{\lambda}=\{\lambda_i\vdash d_i\}_{i=1}^m$. On each moduli space $P_\chi(\widehat{X}_{q_j},\mathbf{p})$ we have a $\mathbb{C}^{*3}$-action as described in the previous subsection. This action leaves

$$P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(\widehat{X}_{q_j},\mathbf{p})$$

invariant. The fixed point locus $P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}$ consists of *isolated* fixed points which can be counted using the vertex/edge formalism for stable pairs developed by R. Pandharipande and R. P. Thomas [6]. Note that the fixed loci indeed consist of isolated reduced points since one leg is always empty [6]. There are two cases:

Case 1: q_j is a node of a nodal fibre F_{p_i}. In this case the legs of the elements of $P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}$ are fixed by the partitions $(\lambda_i,\lambda_i^t,\varnothing)$ where $(\cdot)^t$ denotes the dual partition and we use the ordering convention of [3]. The generating function is given by the stable pairs vertex

$$W_{\lambda_i,\lambda_i^t,\varnothing}(q)=\sum_\chi e(P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}})q^\chi = \sum_{\mathcal{Q}\in P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}} w(\mathcal{Q})q^{l(\mathcal{Q})+2|\lambda_i|}, \tag{6}$$

where we use the notation of [6].

[6, 3] are signed Euler chars, whereas for the moment we are doing ordinary Euler chars. $W_{\lambda,\mu,\nu}(q)$ are understood in this way for now.

Case 2: q_j is a node arising from B intersecting a fibre F_{p_i}. In this case the legs of the elements of $P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}$ are fixed by the partitions $(\lambda_i,(1),\varnothing)$. The generating function is given by the stable pairs vertex

$$W_{\lambda_i,(1),\varnothing}(q)=\sum_\chi e(P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}})q^\chi = \sum_{\mathcal{Q}\in P_\chi(\widehat{X}_{q_j},\mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}} w(\mathcal{Q})q^{l(\mathcal{Q})+|\lambda_i|+1}. \tag{7}$$

5.3. **Punctured curves.** In this subsection we consider $e(P_\chi(X^\circ,\mathbf{p})_{\boldsymbol{\lambda}})$ for any $\boldsymbol{\lambda}=\{\lambda_i\vdash d_i\}_{i=1}^m$. Recall the homeomorphism (5) and define locally closed subsets

$$P_\chi(N_{B^\circ/X^\circ},\mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(N_{B^\circ/X^\circ},\mathbf{p}),\ P_\chi(N_{F_{p_i}^\circ/X^\circ},\mathbf{p})_\lambda \subset P_\chi(N_{F_{p_i}^\circ/X^\circ},\mathbf{p})$$

with specified "cross-sections" $\boldsymbol{\lambda}$ of the underlying Cohen-Macaulay curves. Since the Cohen-Macaulay curves underlying the stable pairs in $P_\chi(N_{B^\circ/X^\circ},\mathbf{p})$ have multiplicity 1, this space is just a Hilbert scheme of points on B° [5]

$$P_\chi(N_{B^\circ/X^\circ},\mathbf{p}) \cong \mathrm{Hilb}^n(B^\circ)$$

where

$$\chi = 1-g(B)+n.$$

Therefore

$$\sum_\chi e(P_\chi(N_{B^\circ/X^\circ},\mathbf{p}))q^\chi = q^{1-g(B)}\sum_{n=0}^\infty e(\mathrm{Hilb}^n(B^\circ))q^n = \frac{q^{1-g(B)}}{(1-q)^{e(B^\circ)}}. \tag{8}$$

The curves $F_{p_i}^\circ$ coming from a nodal fibre are punctured $\mathbb{P}^1$'s

$$F_{p_i}^\circ \cong \mathbb{P}^1 \setminus \{3\ pts\} \cong \mathbb{C}^* \setminus pt.$$

The curves $F_{p_i}^\circ$ coming from a smooth fibre are smooth elliptic curves E with one puncture. Moreover all normal bundles are in fact *trivial*. Indeed for any fibre F of the elliptic surface $\pi : S \to B$ we have

$$N_{F/X} \cong N_{F/S} \oplus N_{S/X}|_F \cong \mathcal{O}_F(F) \oplus K_S|_F \cong \mathcal{O}_F \oplus \mathcal{O}_F.$$

The last isomorphism follows from $F^2 = 0$ and the formula for the canonical divisor of an elliptic fibration [**?**]

$$K_S = \pi^* D,$$

where D is a divisor of degree $\chi(\mathcal{O}_S) - \chi(\mathcal{O}_B)$ on B. Therefore

$$N_{F_{p_i}^\circ/X^\circ} \cong F_{p_i}^\circ \times \mathbb{C}^2 \cong \begin{cases} (\mathbb{C}^*\setminus pt)\times\mathbb{C}^2 & \text{if } F_{p_i} \text{ is nodal} \\ (E\setminus pt)\times\mathbb{C}^2 & \text{if } F_{p_i} \text{ is smooth.} \end{cases}$$

The generating functions of the trivial rank 2 bundles over $F_{p_i}\setminus q_j \cong \mathbb{C}^*$ (when F_{p_i} is nodal with node q_j) and $F_{p_i} \cong E$ (when F_{p_i} is smooth) are easy. Indeed in the former case case $\mathbb{C}^*$ acts (freely) on itself by multiplication and in the latter case E acts (freely) on itself by addition. These actions lift to free actions on the moduli spaces. We obtain the following result.

Lemma 5.3. *The following equalities hold*

$$\sum_\chi e(P_\chi(\mathbb{C}^*\times\mathbb{C}^2,\mathbf{p})_{\boldsymbol{\lambda}} q^\chi = 1,$$

$$\sum_\chi e(P_\chi(E\times\mathbb{C}^2,\mathbf{p})_{\boldsymbol{\lambda}} q^\chi = 1.$$

Proof. Easy using freeness of the action and $e(\mathbb{C}^*) = e(E) = 0$. □

The required generating functions can be computed by using the restriction argument of Section 3 once more. Let C be any smooth curve and consider the 3-fold $C \times \mathbb{C}^2$. Let $p \in C$ and consider the embedding

$$P_\chi(C \times \mathbb{C}^2, d) \hookrightarrow \coprod_{\chi=\chi_1+\chi_2} P_{\chi_1}((C \setminus p) \times \mathbb{C}^2, d) \times P_{\chi_2}(\widehat{C}_p \times \mathbb{C}^2, d),$$

where d denotes the degree of the curve class[3]. The torus $T = \mathbb{C}^{2*}$ is acting on both spaces by scaling of the factors of $\mathbb{C}^2$ and the fixed loci are indexed by partitions $\lambda \vdash d$ as earlier in this section. Again we use the notation $(\cdot)_\lambda$ to indicate that the "cross-section" of the underlying Cohen-Macaulay support curve has been fixed to be the monomial ideal corresponding to λ. We obtain a bijective morphism

$$P_\chi(C \times \mathbb{C}^2, d)_\lambda \cong \coprod_{\chi=\chi_1+\chi_2} P_{\chi_1}((C \setminus p) \times \mathbb{C}^2, d)_\lambda \times P_{\chi_2}(\widehat{C}_p \times \mathbb{C}^2, d)_\lambda.$$

Summing over all χ gives the following lemma.

Lemma 5.4.

$$\sum_\chi e(P_\chi(C \times \mathbb{C}^2, d)_\lambda) q^\chi = W_{\lambda,\varnothing,\varnothing}(q) \cdot \sum_\chi e(P_\chi(\widehat{C}_p \times \mathbb{C}^2, d)) q^\chi,$$

Proof. To obtain the stable pair vertex use a $\mathbb{C}^{*3}$-action on $\widehat{C}_p \times \mathbb{C}^2$ as in the previous subsection. □

Putting everything together we obtain the desired generating function.

Proposition 5.5. *For each fibre F_{p_i} (nodal or not) we have*

$$\sum_\chi e(P_\chi(N_{F^\circ_{p_i}/X^\circ}, \mathbf{p})_\lambda) q^\chi = \frac{1}{W_{\lambda,\varnothing,\varnothing}(q)}.$$

Proof. Combine Lemmas 5.3, 5.4. □

6. Finale (Schur)

We calculate the disconnected generating function (1) first. The connected generating function (2) then follows easily. Denote by $B^\circ \subset B$ the locus of smooth fibres and by $B^{\text{sing}} \subset B$ the locus of singular fibres. Let $\text{Conf}^i(B^\circ)$

[3]The precise definition of these moduli spaces is as in Section 3: we assume the underlying reduced supports of the stable pairs in each moduli space are C, $C \setminus p$, $\widehat{C}$ respectively and d denotes the multiplicity of the underlying Cohen-Macaulay supports along these curves.

be the configuration space of i unordered points on B° and let $N := |B^{\mathrm{sing}}|$. Lemma 2.3 implies

$$\begin{aligned}
Z^{P_\bullet}(q,y) &= \sum_{\chi}\sum_{i=0}^{\infty}\sum_{i'=0}^{N}\sum_{d_1,\ldots,d_i\geq 0}\sum_{d'_1,\ldots,d'_{i'}\geq 0} y^{\sum_{a=1}^{i} d_a + \sum_{a=1}^{i'} d'_a}\cdot e(\mathrm{Conf}^i(B^\circ))\cdot \binom{N}{i'}\times \\
&\qquad e\big(P_\chi\big(X, \sum_{a=1}^{i} d_a p_a + \sum_{a=1}^{i'} d'_a p'_a\big)\big) \\
&= \sum_{\chi}\sum_{i=0}^{\infty}\sum_{i'=0}^{N}\sum_{d_1,\ldots,d_i\geq 0}\sum_{d'_1,\ldots,d'_{i'}\geq 0} y^{\sum_{a=1}^{i} d_a + \sum_{a=1}^{i'} d'_a}\cdot \binom{e(B)-N}{i}\cdot \binom{N}{i'}\times \\
&\qquad e\big(P_\chi\big(X, \sum_{a=1}^{i} d_a p_a + \sum_{a=1}^{i'} d'_a p'_a\big)\big),
\end{aligned}$$

where $p_1,\ldots,p_i$ are any choice of distinct points on B°, $p'_1,\ldots,p'_{i'}$ are any choice of distinct points among B^{sing}, and

$$\binom{n}{k} := (-1)^k \binom{k-n-1}{k}, \tag{9}$$

for $n<0$. We abbreviate $\mathbf{p} := \sum_{a=1}^{i} d_a p_a$, $\mathbf{p}' := \sum_{a=1}^{i'} d'_a p'_a$, $\mathbf{d} := \sum_{a=1}^{i} d_a$, and $\mathbf{d}' := \sum_{a=1}^{i'} d'_a$. Lemma 5.2 gives

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{i'=0}^{N}\sum_{d_1,\ldots,d_i\geq 0}\sum_{d'_1,\ldots,d'_{i'}\geq 0}\sum_{\chi}\sum_{\chi_1,\ldots,\chi_i}\sum_{\chi'_1,\ldots,\chi'_{i'}}\sum_{\boldsymbol{\lambda}=\{\lambda_a\vdash d_a\}_{a=1}^{i}}\sum_{\boldsymbol{\lambda}'=\{\lambda'_a\vdash d'_a\}_{a=1}^{i'}} y^{\mathbf{d}+\mathbf{d}'}\cdot\binom{e(B)-N}{i}\cdot\binom{N}{i'}\times \\
&e(P_\chi(X\setminus\{q_1,\ldots,q_i,q'_1,\ldots,q'_{i'},r_1,\ldots,r_{i'}\},\mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda},\boldsymbol{\lambda}'})\times \\
&\prod_{a=1}^{i} e(P_{\chi_a}(\widehat{X}_{q_a},\mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda}})\cdot\prod_{a=1}^{i'} e(P_{\chi'_a}(\widehat{X}_{q'_a},\mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda}'})\cdot\prod_{a=1}^{i'} e(P_{\chi'_a}(\widehat{X}_{r'_a},\mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda}'}),
\end{aligned}$$

where $q_1,\ldots,q_i$ denote the nodes arising from $F_{p_1},\ldots,F_{p_i}$ intersecting the zero section, $q'_1,\ldots,q'_{i'}$ denote the nodes arising from $F_{p'_1},\ldots,F_{p'_{i'}}$ intersecting the zero section, and $r_1,\ldots,r_{i'}$ are the internal nodes of $F_{p'_1},\ldots,F_{p'_{i'}}$. The sums

$\sum_\chi \sum_{\chi_1,\dots,\chi_i} \sum_{\chi'_1,\dots,\chi'_{i'}} \cdots$ can be done using equations (6), (7), and (8)

$$\sum_{i=0}^{\infty}\sum_{i'=0}^{N}\sum_{d_1,\dots,d_i\geq 0}\sum_{d'_1,\dots,d'_{i'}\geq 0}\sum_{\boldsymbol{\lambda}=\{\lambda_a\vdash d_a\}_{a=1}^{i}}\sum_{\boldsymbol{\lambda}'=\{\lambda'_a\vdash d'_a\}_{a=1}^{i'}} y^{\mathbf{d}+\mathbf{d}'}\cdot\binom{e(B)-N}{i}\cdot\binom{N}{i'}\times$$

$$\frac{q^{1-g(B)}}{(1-q)^{e(B)-i-i'}}\cdot\prod_{a=1}^{i}\frac{W_{\lambda_a,(1),\varnothing}(q)}{W_{\lambda_a,\varnothing,\varnothing}(q)}\cdot\prod_{a=1}^{i'}\frac{W_{\lambda'_a,\lambda'^t_a,\varnothing}(q)W_{\lambda'_a,(1),\varnothing}(q)}{W_{\lambda'_a,\varnothing,\varnothing}(q)}$$

$$=\frac{q^{1-g(B)}}{(1-q)^{e(B)}}\sum_{i=0}^{\infty}\sum_{i'=0}^{N}\binom{e(B)-N}{i}\cdot\binom{N}{i'}\cdot\left((1-q)\sum_{\lambda}\frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)}y^{|\lambda|}\right)^{i}\times$$

$$\left((1-q)\sum_{\lambda}\frac{W_{\lambda,\lambda^t,\varnothing}(q)W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)}y^{|\lambda|}\right)^{i'}.$$

With our convention for binomial coefficients (9), Newton's binomial theorem and the geometric series can be combined in one formula

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k,\ \text{for all } n\in\mathbb{Z}.$$

Performing the sums $\sum_{i=0}^{\infty}\sum_{j=0}^{N}\cdots$ yields

$$\left(\frac{q}{(1-q)^2}\right)^{1-g(B)}\left((1-q)\sum_{\lambda}\frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)}y^{|\lambda|}\right)^{e(B)-N}\cdot\left((1-q)\sum_{\lambda}\frac{W_{\lambda,\lambda^t,\varnothing}(q)W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)}y^{|\lambda|}\right)^{N}.$$

Similarly (but easier) one calculates the generating function $\sum_{d\geq 0}\sum_{\chi}P_{\chi,dF}(X)q^\chi y^d$

$$\left(\sum_{\lambda}y^{|\lambda|}\right)^{e(B)-N}\cdot\left(\sum_{\lambda}W_{\lambda,\lambda^t,\varnothing}(q)y^{|\lambda|}\right)^{N}.$$

We arrive at the following proposition.

Proposition 6.1. *The connected generating series $Z^P(q,y)$ for stable pairs of $X=\mathrm{Tot}(K_S)$ of an elliptic surface $S\to B$ with section of genus $g(B)$ and N 1-nodal fibres is given by*

$$\left(\frac{q}{(1-q)^2}\right)^{1-g(B)}\left(\frac{(1-q)\sum_{\lambda}\frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)}y^{|\lambda|}}{\sum_{\lambda}y^{|\lambda|}}\right)^{e(B)-N}\cdot\left(\frac{(1-q)\sum_{\lambda}\frac{W_{\lambda,\lambda^t,\varnothing}(q)W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)}y^{|\lambda|}}{\sum_{\lambda}W_{\lambda,\lambda^t,\varnothing}(q)y^{|\lambda|}}\right)^{N},$$

where $W_{\lambda,\mu,\nu}(q)$ is the stable pairs vertex of [6].

The various generating functions of vertices appearing in this proposition can be computed. Obviously

$$\sum_{\lambda} y^{|\lambda|} = \prod_{i=1}^{\infty} (1-y^i)^{-1}.$$

More interesting is the following lemma.

I have not checked whether the overall $q^{\cdots}$ factors work out. Is the power y^{i-1} in RHS correct?

Lemma 6.2. *The following identity holds*

$$\sum_{\lambda} W_{\lambda,\lambda^t,\varnothing}(q) y^{|\lambda|} = \prod_{i=1}^{\infty} \Big((1-y^i) \prod_{j=1}^{\infty} (1-y^{i-1}q^j)^j \Big)^{-1}.$$

Proof. [3] and [2] or exercise in [**?**]. □

Less trivial is the following lemma.

Lemma 6.3. *The following identity holds*

$$(1-q) \sum_{\lambda} \frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1-y^i}{(1-qy^i)(1-q^{-1}y^i)}.$$

Proof. First apply [3]. The remaining sum appears in [**?**] as pointed out by P. Johnson answering a MathOverflow question. □

STUCK ON THIS. Do we need help from Andrei, Paul, or Ben?

The hardest is the following lemma.

Lemma 6.4. *The following identity holds*

$$(1-q) \sum_{\lambda} \frac{W_{\lambda,\lambda^t,\varnothing}(q) W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} = \left(\prod_{i=1}^{\infty} \frac{1-y^i}{(1-qy^i)(1-q^{-1}y^i)} \right) \cdot \left(\prod_{i=1}^{\infty} \Big((1-y^i) \prod_{j=1}^{\infty} (1-y^{i-1}q^j)^j \Big)^{-1} \right)$$

Proof. ????? □

We obtain a proof of the theorem in the introduction.

Proof of Theorem 1.1. Combine Proposition 6.1 and Lemmas 6.2, 6.3, and 6.4. □

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