TRACE IDENTITIES FOR THE TOPOLOGICAL VERTEX

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ABSTRACT. The topological vertex is a universal series which can be regarded as an object in combinatorics, representation theory, geometry, or physics. It encodes the combinatorics of 3D partitions, the action of vertex operators on Fock space, the Donaldson-Thomas theory of toric Calabi-Yau threefolds, or the open string partition function of $\mathbb{C}^3$.

We prove several identities in which a sum over terms involving the topological vertex is expressed as a closed formula, often a product of simple terms, closely related to Fourier expansions of Jacobi forms. We use purely combinatorial and representation theoretic methods to prove our formulas, but we discuss applications to the Donaldson-Thomas invariants of elliptically fibered Calabi-Yau threefolds at the end of the paper.

1. INTRODUCTION

The topological vertex $\mathsf{V}_{\lambda\mu\nu} = \mathsf{V}_{\lambda\mu\nu}(p)$ is a universal formal Laurent series in p depending on a triple of partitions (λ, μ, ν). It can be considered as an object in combinatorics, representation theory, geometry, or physics. In combinatorics, $\mathsf{V}_{\lambda\mu\nu}$ is the generating function for the number of 3D partitions with asymptotic legs of type (λ, μ, ν) (see Definition 2). In representation theory, $\mathsf{V}_{\lambda\mu\nu}$ is given as the matrix coefficients of a certain vertex operator on Fock space [14]. In geometry, $\mathsf{V}_{\lambda\mu\nu}$ is the basic building block for computing the Donaldson-Thomas/Gromov-Witten invariants of toric Calabi-Yau threefolds [11]. The topological vertex was first discovered in physics as an open string partition function in type IIA string theory on $\mathbb{C}^3$ [1]. An explicit expression for $\mathsf{V}_{\lambda\mu\nu}$ in terms of Schur functions was given by [14] (see § 3).

In this paper we prove several "trace identities" in which a sum over certain combinations of the vertex is expressed as a closed formula, often a product of simple terms. The products are closely related to the Fourier expansions of Jacobi forms. Applications of these identities are used to compute the Donaldson-Thomas partition functions of certain Calabi-Yau threefolds in terms of Jacobi forms [5, 6, 7].

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2. DEFINITIONS AND THE MAIN RESULT.

In this section we give the combinatorial definition of the vertex and we state our main identities.

Definition 1. Let (λ, μ, ν) be a triple of ordinary partitions. A *3D partition* π *asymptotic to* (λ, μ, ν) is a subset

$$\pi \subset (\mathbb{Z}_{\geq 0})^3$$

satisfying

(1) if any of $(i+1, j, k)$, $(i, j+1, k)$, and $(i, j, k+1)$ is in π, then (i, j, k) is also in π, and
(2) (a) $(j, k) \in \lambda$ if and only if $(i, j, k) \in \pi$ for all $i \gg 0$,
  (b) $(k, i) \in \mu$ if and only if $(i, j, k) \in \pi$ for all $j \gg 0$,
  (c) $(i, j) \in \nu$ if and only if $(i, j, k) \in \pi$ for all $k \gg 0$.

where we regard ordinary partitions as finite subsets of $(\mathbb{Z}_{\geq 0})^2$ via their diagram.

Intuitively, π is a pile of boxes in the positive octant of 3-space. Condition (1) means that the boxes are stacked stably with gravity pulling them in the $(-1, -1, -1)$ direction; condition (2) means that the pile of boxes is infinite along the coordinate axes with cross-sections asymptotically given by λ, μ, and ν.

The subset $\{(i, j, k) : (j, k) \in \lambda\} \subset \pi$ will be called the *leg* of π in the i direction, and the legs in the j and k directions are defined analogously. Let

$$\xi_\pi(i, j, k) = 1 - \# \text{ of legs of } \pi \text{ containing } (i, j, k).$$

We define the renormalized volume of π by

$$|\pi| = \sum_{(i,j,k) \in \pi} \xi_\pi(i, j, k).$$

Note that $|\pi|$ can be negative.

Definition 2. The topological vertex $\mathsf{V}_{\lambda\mu\nu}$ is defined to be

$$\mathsf{V}_{\lambda\mu\nu} = \sum_{\pi} p^{|\pi|}$$

where the sum is taken over all 3D partitions π asymptotic to (λ, μ, ν). We regard $\mathsf{V}_{\lambda\mu\nu}$ as a formal Laurent series in p. Note that $\mathsf{V}_{\lambda\mu\nu}$ is clearly cyclically symmetric in the indices, and reflection about the $i = j$ plane yields

$$\mathsf{V}_{\lambda\mu\nu} = \mathsf{V}_{\mu'\lambda'\nu'}$$

where $'$ denotes conjugate partition:

$$\lambda' = \{(i, j) : (j, i) \in \lambda\}.$$

This definition of topological vertex differs from the vertex $C(\lambda, \mu, \nu)$ of the physics literature by a normalization factor (and we use the variable p instead of q). Our $\mathsf{V}_{\lambda\mu\nu}$ is equal to $P(\lambda, \mu, \nu)$ defined by Okounkov, Reshetikhin, and Vafa [14, eqn 3.16]. They derive an explicit formula for $\mathsf{V}_{\lambda\mu\nu} = P(\lambda, \mu, \nu)$ in terms of Schur functions [14, eqns 3.20 and 3.21].

The *rows* or *parts* of λ are the integers $\lambda_j = \min\{i \mid (i, j) \notin \lambda\}$, for $j \geq 0$. We use the notation

$$|\lambda| = \sum_j \lambda_j, \quad \|\lambda\|^2 = \sum_j \lambda_j^2.$$

Let $\square$ denote the partition with a single part of size 1.

We also use the notation

$$M(p, q) = \prod_{m=1}^{\infty} (1 - p^m q)^{-m}$$

and the shorthand $M(p) = M(p, 1)$. Here M stands for MacMahon, who proved [10] that

$$\mathsf{V}_{\varnothing\varnothing\varnothing} = M(p).$$

We can now state our main result.

Theorem 3. *The following identities hold as formal power series in q whose coefficients are formal Laurent series in p:*

(1)
$$\sum_{\lambda} q^{|\lambda|} p^{\|\lambda'\|^2} \mathsf{V}_{\lambda'\lambda\varnothing} = M(p) \prod_{d=1}^{\infty} (1-q^d)^{-1} M(p, q^d)$$

(2)
$$\sum_{\lambda} q^{|\lambda|} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = (1-p)^{-1} \prod_{d=1}^{\infty} \frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)}$$

(3)
$$\sum_{\lambda} q^{|\lambda|} p \frac{\mathsf{V}_{\square\square\lambda}}{\mathsf{V}_{\varnothing\varnothing\lambda}} = \prod_{m=1}^{\infty} (1-q^m)^{-1} \cdot \left\{ 1 + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k}) q^d \right\}$$

(4)
$$\sum_{\lambda} q^{|\lambda|} p^{\|\lambda\|^2} \mathsf{V}_{\lambda\lambda'\varnothing} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = (1-p)^{-1} M(p) \prod_{d=1}^{\infty} \frac{M(p, q^d)}{(1-pq^d)(1-p^{-1}q^d)}.$$

The sums in the left hand sides of the above formulas run over all partitions.

We prove Formula (1) in section 3 using the orthogonality properties of skew Schur functions. Formulas (2) and (3) are proved in section 4 using a theorem of Bloch-Okounkov [3]. The most difficult identity to prove is equation (4) which we do in section 5. There we prove that the left hand side of equation (4) is given as the trace of a certain product of operators on Fock space (hence the term "trace identities" in the title). To compute the trace, we use a trick which involves an "infinite number" of permutations of the operators.

3. The topological vertex and Schur functions

Okounkov-Reshetikhin-Vafa derived a formula for the topological vertex in terms of skew Schur functions. Translating their formulas [14, 3.20& 3.21] into our notation, we get:

(5)
$$\mathsf{V}_{\lambda\mu\nu}(p) = M(p) p^{-\frac{1}{2}(\|\lambda\|^2 + \|\mu'\|^2 + \|\nu\|^2)} s_{\nu'}(p^{-\rho}) \sum_{\eta} s_{\lambda'/\eta}(p^{-\nu-\rho}) s_{\mu/\eta}(p^{-\nu'-\rho}).$$

Here, $s_{\alpha/\beta}(x_1, x_2, \dots)$ is the skew Schur function (see for example [9, § 5]) and

$$\rho = \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\right)$$

so that $p^{-\nu-\rho}$ is notation for the variable list

$$p^{-\nu-\rho} = \left(p^{-\nu_1+\frac{1}{2}}, p^{-\nu_2+\frac{3}{2}}, \dots\right).$$

We prove equation (1) as follows. Using equation (5) we see

$$\mathsf{V}_{\lambda'\lambda\varnothing} = M(p) p^{-\|\lambda'\|^2} \sum_{\eta} s_{\lambda/\eta}(p^{-\rho})^2$$

and so (using orthogonality of skew Schur functions [9, 28(a) pg 94] in the second line below) we see

$$\begin{aligned}
\sum_{\lambda} q^{|\lambda|} p^{\|\lambda'\|^2} \mathsf{V}_{\lambda'\lambda\varnothing} &= M(p) \sum_{\lambda,\eta} q^{|\lambda|} (s_{\lambda/\eta}(p^{\frac{1}{2}}, p^{\frac{3}{2}}, \dots))^2 \\
&= M(p) \prod_{d=1}^{\infty} \left((1-q^d)^{-1} \prod_{j,k=1}^{\infty} (1 - q^d p^{i-\frac{1}{2}+j-\frac{1}{2}})^{-1} \right) \\
&= M(p) \prod_{d=1}^{\infty} (1-q^d)^{-1} \prod_{m=1}^{\infty} (1-q^d p^m)^{-m} \\
&= M(p) \prod_{d=1}^{\infty} (1-q^d)^{-1} M(p, q^d).
\end{aligned}$$

We also use equation (5) to derive the following key formulas:

Lemma 4. *The following hold:*

$$\begin{aligned}
p^{\frac{1}{2}} \, \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}} \\
p \, \frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= 1 - \left(\sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}}\right) \left(\sum_{j=1}^{\infty} p^{\lambda_j-j+\frac{1}{2}}\right).
\end{aligned}$$

Proof. Applying equation (5) to $\mathsf{V}_{\lambda\square\varnothing}/\mathsf{V}_{\lambda\varnothing\varnothing} = \mathsf{V}_{\square\varnothing\lambda}/\mathsf{V}_{\varnothing\varnothing\lambda}$ we see that

$$p^{\frac{1}{2}} \frac{\mathsf{V}_{\lambda\square\varnothing}}{V_{\lambda\varnothing\varnothing}} = s_{\square}(p^{-\lambda-\rho}) = s_{\square}(p^{-\lambda_1+\frac{1}{2}}, p^{-\lambda_2+\frac{3}{2}}, \dots) = \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}}.$$

Similarly,

$$p\frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = p\frac{\mathsf{V}_{\square\square\lambda}}{\mathsf{V}_{\varnothing\varnothing\lambda}} = \sum_{\eta} s_{\square/\eta}(p^{-\lambda-\rho})s_{\square/\eta}(p^{-\lambda'-\rho})$$
$$= 1 + s_{\square}(p^{-\lambda-\rho})s_{\square}(p^{-\lambda'-\rho}).$$

In general we have the following relation (see [14, Eqn (3.10)])[1]

$$s_{\lambda/\mu}(p^{\nu+\rho}) = (-1)^{|\lambda|-|\mu|}s_{\lambda'/\mu'}(p^{-\nu'-\rho})$$

so in particular

$$s_{\square}(p^{\nu+\rho}) = -s_{\square}(p^{-\nu'-\rho}) \tag{6}$$

and thus

$$p\frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = 1 - s_{\square}(p^{-\lambda-\rho})s_{\square}(p^{\lambda+\rho})$$
$$= 1 - \left(\sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}}\right)\left(\sum_{j=1}^{\infty} p^{\lambda_j-j+\frac{1}{2}}\right)$$

which proves the lemma.

4. Applications of a theorem of Bloch-Okounkov

We summarize a result of Bloch-Okounkov [3] and use it to prove equations (2) and (3).

We define the following theta function

$$\Theta(p,q) = \eta(q)^{-3}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}p^{n+\frac{1}{2}}$$

where

$$\eta(q) = q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m).$$

By the Jacobi triple product formula, Θ is given by

$$\Theta(p,q) = (p^{\frac{1}{2}} - p^{-\frac{1}{2}})\prod_{m=1}^{\infty}\frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}.$$

We suppress the q from the notation: $\Theta(p) = \Theta(p,q)$, and we note that

$$\Theta(p) = -\Theta(p^{-1}).$$

[1] There is a typo in equation 3.10 in [14] — the exponent on the right hand side should be $-\nu' - \rho$.

Theorem 5 (Bloch-Okounkov [3])**.** *Define the n point correlation function by the formula*

$$F(p_1,\ldots,p_n)=\prod_{m=1}^{\infty}(1-q^m)\sum_{\lambda}q^{|\lambda|}\prod_{k=1}^{n}\left(\sum_{i=1}^{\infty}p_k^{\lambda_i-i+\frac{1}{2}}\right).$$

Then

$$F(p)=\frac{1}{\Theta(p)}$$

and

$$F(p_1,p_2)=\frac{1}{\Theta(p_1p_2)}\left(p_1\frac{d}{dp_1}\log(\Theta(p_1))+p_2\frac{d}{dp_2}\log(\Theta(p_2))\right).$$

In [3], formulas for the general n variable function are given, but we will only need the cases of $n=1$ and $n=2$.

Using this theorem, we will prove equations (2) and (3) of the main theorem.

4.1. **Proofs of equations** (2) **and** (3)**.**

We apply Lemma 4 and Theorem 5:

$$\begin{aligned}\sum_{\lambda}(1-p)q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}&=(p^{-\frac{1}{2}}-p^{\frac{1}{2}})\sum_{\lambda}q^{|\lambda|}\sum_{i=1}^{\infty}p^{-\lambda_i+i-\frac{1}{2}}\\&=(p^{-\frac{1}{2}}-p^{\frac{1}{2}})\prod_{m=1}^{\infty}(1-q^m)^{-1}F(p^{-1})\\&=(p^{-\frac{1}{2}}-p^{\frac{1}{2}})\prod_{m=1}^{\infty}(1-q^m)^{-1}\frac{1}{-\Theta(p)}\\&=\prod_{m=1}^{\infty}\frac{(1-q^m)}{(1-pq^m)(1-p^{-1}q^m)}\end{aligned}$$

which proves equation (2).

Again we apply Lemma 4 and Theorem 5:

$$\begin{aligned}\sum_{\lambda}q^{|\lambda|}p\frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}}&=\sum_{\lambda}q^{|\lambda|}\left\{1-\left(\sum_{i=1}^{\infty}p^{-\lambda_i+i-\frac{1}{2}}\right)\left(\sum_{j=1}^{\infty}p^{\lambda_j-j+\frac{1}{2}}\right)\right\}\\&=\prod_{m=1}^{\infty}(1-q^m)^{-1}\left(1-F(p,p^{-1})\right).\end{aligned}$$

From Theorem 5, we see that

$$F(p,p^{-1})=\lim_{(p_1,p_2)\to(p,p^{-1})}\frac{1}{\Theta(p_1p_2)}\left(p_1\frac{d}{dp_1}\log(\Theta(p_1))+p_2\frac{d}{dp_2}\log(\Theta(p_2))\right).$$

To evaluate this limit, we simplify the above expression. A short computation shows that

$$p\frac{d}{dp}\log(\Theta(p)) = \frac{1}{2}\frac{p+1}{p-1} + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(-p^k+p^{-k}\right)q^{mk}.$$

Thus

$$\begin{aligned}
F(p,p^{-1}) = &\lim_{\substack{(p_1,p_2)\to\\(p,p^{-1})}} \left((p_1p_2)^{\frac{1}{2}}-(p_1p_2)^{-\frac{1}{2}}\right)^{-1}\prod_{m=1}^{\infty}\frac{(1-q^m)^2}{(1-(p_1p_2)q^m)(1-(p_1p_2)^{-1}q^m)}\\
&\cdot\left\{\frac{1}{2}\cdot\frac{p_1+1}{p_1-1}+\frac{1}{2}\cdot\frac{p_2+1}{p_2-1}+\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(-p_1^k-p_2^k+p_1^{-k}+p_2^{-k}\right)q^{mk}\right\}\\
= &\lim_{\substack{(p_1,p_2)\to\\(p,p^{-1})}} \frac{-(p_1p_2)^{\frac{1}{2}}}{1-p_1p_2}\cdot\left\{\frac{p_1p_2-1}{(p_1-1)(p_2-1)}+\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(1-p_1^kp_2^k)(p_1^{-k}+p_2^{-k})q^{mk}\right\}\\
= &\lim_{\substack{(p_1,p_2)\to\\(p,p^{-1})}} (p_1p_2)^{\frac{1}{2}}\left\{\frac{1}{(1-p_1)(1-p_2)}-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1-(p_1p_2)^k}{1-p_1p_2}(p_1^{-k}+p_2^{-k})q^{mk}\right\}\\
= &\frac{1}{(1-p)(1-p^{-1})}-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}k(p^k+p^{-k})q^{mk}.
\end{aligned}$$

Therefore

$$1-F(p,p^{-1}) = 1+\frac{p}{(1-p)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(p^k+p^{-k})q^d$$

which finishes the proof of equation (3).

5. VERTEX OPERATORS AND THE PROOF OF EQUATION (4)

There are several sources for vertex operators and the infinite wedge formalism. For consistency, we will follow the notation and conventions of [13, Appendix A].

Let V be the vector space with basis $\{\underline{k}\}$, $k\in\mathbb{Z}+\frac{1}{2}$. We define *Fock space* $\Lambda^{\frac{\infty}{2}}V$ to be the vector space spanned by vectors

$$v_S = \underline{s_1}\wedge\underline{s_2}\wedge\dots$$

where $S=\{s_1>s_2>\dots\}\subset\mathbb{Z}+\frac{1}{2}$ is any subset such that the sets

$$S_+ = S\cap\left(\mathbb{Z}+\frac{1}{2}\right)_{>0} \quad\text{and}\quad S_- = S^c\cap\left(\mathbb{Z}+\frac{1}{2}\right)_{<0}$$

are both finite. Let $(\cdot,\cdot)$ be the inner product on $\Lambda^{\frac{\infty}{2}}V$ such that the basis $\{v_S\}$ is orthonormal.

For any $k \in \mathbb{Z} + \frac{1}{2}$ let ψ_k be the operator

$$\psi_k(f) = \underline{k} \wedge f$$

and let ψ_k^* be its adjoint.

For any partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$, we define the vector

$$v_\lambda = \underline{(\lambda_1 - \frac{1}{2})} \wedge \underline{(\lambda_2 - \frac{3}{2})} \wedge \dots$$

Let $\Lambda_0^{\frac{\infty}{2}} V \subset \Lambda^{\frac{\infty}{2}} V$ be the subspace spanned by the vectors $\{v_\lambda\}$ where λ runs over all partitions. We call this *charge zero Fock space.*

The *energy operator*

$$H = \sum_{k>0} k \left(\psi_k \psi_k^* + \psi_{-k}^* \psi_{-k} \right)$$

acts on the basis v_λ by

$$H v_\lambda = |\lambda| v_\lambda$$

and so the operator q^H acts by

$$q^H v_\lambda = q^{|\lambda|} v_\lambda$$

where q is a formal parameter.

For $n \in \mathbb{Z}$, $n \neq 0$ define

$$\alpha_n = \sum_k \psi_{k-n} \psi_k^*$$

and observe that $\alpha_n^* = \alpha_{-n}$.

Following [13], we define the *vertex operators* $\Gamma_\pm(\mathbf{x})$ which are operators on $\Lambda_0^{\frac{\infty}{2}} V$ over the coefficient ring given by symmetric functions in an infinite set of variables $\mathbf{x} = (x_1, x_2, x_3, \dots)$. Let $\mathbf{s} = (s_1, s_2, \dots)$

$$s_k = \frac{1}{k} \sum_{i=1}^{\infty} x_i^k$$

be the power sum basis for the ring of symmetric functions and let [2]

$$\Gamma_\pm(\mathbf{x}) = \exp\left(\sum_{n=1}^{\infty} s_n \alpha_{\pm n} \right).$$

Observe that $\Gamma_\pm^* = \Gamma_\mp$.

The matrix coefficients of the vertex operators in the $\{v_\lambda\}$ basis are given by skew Schur functions:

$$(\Gamma_-(\mathbf{x}) v_\mu, v_\lambda) = (v_\mu, \Gamma_+(\mathbf{x}) v_\lambda) = s_{\lambda/\mu}(\mathbf{x}). \tag{7}$$

[2]In [13], the argument of $\Gamma_\pm$ is $\mathbf{s}$, and the dependence on the underlying set of variables $\mathbf{x}$ is left implicit. We prefer to make $\mathbf{x}$ the explicit argument.

Orthogonality of the skew Schur functions then gives rise to the following commutation equation:

$$\Gamma_+(\mathbf{x})\Gamma_-(\mathbf{y}) = \prod_{i,j}(1-x_iy_j)^{-1}\Gamma_-(\mathbf{y})\Gamma_+(\mathbf{x}),$$

in particular

$$\Gamma_+(up^{-\rho})\Gamma_-(vp^{-\rho}) = M(p,uv)\Gamma_-(vp^{-\rho})\Gamma_+(up^{-\rho}) \tag{8}$$

where recall that $up^{-\rho} = (up^{\frac{1}{2}}, up^{\frac{3}{2}}, up^{\frac{5}{2}}, \dots)$.

We let

$$\psi(z) = \sum_i z^i\psi_i \quad \text{and} \quad \psi^*(w) = \sum_j w^{-j}\psi^*_j.$$

The commutation relations of these operators with the vertex operators are given by

$$\begin{aligned} \Gamma_\pm(\mathbf{x})\psi(z) &= \prod_{i=1}^{\infty}(1-x_iz^{\pm1})^{-1}\ \psi(z)\Gamma_\pm(\mathbf{x}) \\ \Gamma_\pm(\mathbf{x})\psi^*(w) &= \prod_{i=1}^{\infty}(1-x_iw^{\pm1})\ \psi^*(w)\Gamma_\pm(\mathbf{x}). \end{aligned} \tag{9}$$

We use operators $\mathcal{E}_r$ introduced by Okounkov-Pandharipande in [12, § 2.2.4]. For $r\in\mathbb{Z}$, let[3]

$$\mathcal{E}_r(p) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} p^{k-\frac{r}{2}}\ \psi_{k-r}\psi^*_k.$$

Our variable p is related to the variable z in [12] by $p = e^z$.

From [12, Eqns 2.9 and 0.18], we see that $\mathcal{E}_0$ is the diagonal operator given by

$$\mathcal{E}_0(p)v_\lambda = \left(\sum_{i=1}^{\infty} p^{\lambda_i - i + \frac{1}{2}}\right) v_\lambda.$$

By equation (6) we have

$$\mathcal{E}_0(p)v_\lambda = -\left(\sum_{i=1}^{\infty} p^{-\lambda'_i + i - \frac{1}{2}}\right) v_\lambda. \tag{10}$$

We define

$$\mathcal{E}(a,p) = \sum_{r\in\mathbb{Z}} a^r\mathcal{E}_r(p)$$

where a is a formal parameter. A short computation shows that

$$\mathcal{E}(a,p) = \psi(a^{-1}p^{\frac{1}{2}})\psi^*(a^{-1}p^{-\frac{1}{2}}).$$

[3]We avoid the use of normal ordering by allowing coefficients which are Laurent series in $p^{\frac{1}{2}}$.

From equation (9) we get

$$\Gamma_\pm(\mathbf{x})\mathcal{E}(a,p) = \prod_{i=1}^{\infty} \frac{(1-a^{\mp 1}p^{\mp\frac{1}{2}}x_i)}{(1-a^{\mp 1}p^{\pm\frac{1}{2}}x_i)}\ \mathcal{E}(a,p)\Gamma_\pm(\mathbf{x}).$$

For $\mathbf{x} = up^{-\rho} = (up^{\frac{1}{2}}, up^{\frac{3}{2}}, up^{\frac{5}{2}}, \dots)$ the above simplifies to

$$\begin{aligned} \Gamma_+(up^{-\rho})\mathcal{E}(a,p) &= (1-a^{-1}u)\mathcal{E}(a,p)\Gamma_+(up^{-\rho}) \\ \mathcal{E}(a,p)\Gamma_-(up^{-\rho}) &= (1-au)\Gamma_-(up^{-\rho})\mathcal{E}(a,p). \end{aligned} \tag{11}$$

Finally, it follows from equation (7) that

$$\Gamma_\pm(\mathbf{x})q^H = q^H\Gamma_\pm(q^{\pm 1}\mathbf{x}). \tag{12}$$

We now write the left hand side of equation (4) in the main theorem as a trace of operators on charge zero Fock space.

Lemma 6.

$$\begin{aligned} \sum_\lambda q^{|\lambda|}p^{\|\lambda\|^2}\mathsf{V}_{\lambda\lambda'\varnothing}\ \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= -p^{-\frac{1}{2}}\operatorname{tr}\left(\mathcal{E}_0(p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right) \\ &= -p^{-\frac{1}{2}}\operatorname{Coeff}_{a^0}\left\{\operatorname{tr}\left(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right)\right\}. \end{aligned}$$

Proof. The following computation uses, in order, the commutation relation for Γ_+ and Γ_- (equation (8)), the definition of trace, the formula for $\mathcal{E}_0(p)$ (equation (10)), Lemma 4, equation (7), equation (5), and finally switching λ to λ' in the sum:

$$\begin{aligned} &-p^{-\frac{1}{2}}\operatorname{tr}\left(\mathcal{E}_0(p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right) \\ &= -p^{-\frac{1}{2}}M(p)\operatorname{tr}\left(\mathcal{E}_0(p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^H\right) \\ &= -p^{-\frac{1}{2}}M(p)\sum_\lambda\left(v_\lambda, \mathcal{E}_0(p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^H v_\lambda\right) \\ &= p^{-\frac{1}{2}}M(p)\sum_\lambda q^{|\lambda|}\left(\sum_{i=1}^\infty p^{-\lambda_i'+i-\frac{1}{2}}\right)\left(v_\lambda, \Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})v_\lambda\right) \\ &= M(p)\sum_\lambda q^{|\lambda|}\frac{\mathsf{V}_{\lambda'\square\varnothing}}{\mathsf{V}_{\lambda'\varnothing\varnothing}}\left(\Gamma_+(p^{-\rho})v_\lambda, \Gamma_+(p^{-\rho})v_\lambda\right) \\ &= \sum_\lambda q^{|\lambda|}\frac{\mathsf{V}_{\lambda'\square\varnothing}}{\mathsf{V}_{\lambda'\varnothing\varnothing}}M(p)\sum_\eta\left(s_{\lambda/\eta}(p^{-\rho})\right)^2 \\ &= \sum_\lambda q^{|\lambda|}\frac{\mathsf{V}_{\lambda'\square\varnothing}}{\mathsf{V}_{\lambda'\varnothing\varnothing}}\,p^{\|\lambda'\|^2}V_{\lambda'\lambda\varnothing} \\ &= \sum_\lambda q^{|\lambda|}p^{\|\lambda\|^2}V_{\lambda\lambda'\varnothing}\,\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \end{aligned}$$

□

While the operator $\mathcal{E}_0(p)$ does not have good commutation relations with the vertex operators, the operator $\mathcal{E}(a,p)$ does. Hence we first replace $\mathcal{E}_0(p)$ with the more general $\mathcal{E}(a,p)$, compute the trace, and then specialize to the a^0 coefficient.

Lemma 7.

$$\operatorname{tr}\left(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right) = \frac{M(p)}{p^{\frac{1}{2}}-p^{-\frac{1}{2}}}\prod_{m=1}^{\infty}\frac{(1-q^m a^{-1})(1-q^{m-1}a)(1-q^m)M(p,q^m)}{(1-pq^m)(1-p^{-1}q^m)}.$$

Proof. Our strategy is the following. We use the cyclic invariance of trace along with the commutation relations for Γ_+ to move the operator Γ_+ past the other operators cyclically to the right until the operators are back to their original positions, but with new arguments. We perform this operation a countable number of times, eventually making the Γ_+ operator disappear[4]. We then employ the same strategy moving Γ_- cyclically to the left a countable number of times until it disappears and we are left with a term which we can evaluate with the Bloch-Okounkov theorem (Theorem 5).

We first cyclically commute the operator Γ_+ to the right using equations (8), (11), and (12):

$$\begin{aligned}
&\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H)\\
&= M(p)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^H)\\
&= M(p)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H\Gamma_+(qp^{-\rho}))\\
&= M(p)\operatorname{tr}(\Gamma_+(qp^{-\rho})\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H)\\
&= M(p)(1-qa^{-1})\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(qp^{-\rho})\Gamma_-(p^{-\rho})q^H).
\end{aligned}$$

Cyclically commuting Γ_+ to the right a second time we get:

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = M(p)(1-qa^{-1})M(p,q)(1-q^2a^{-1})\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(q^2p^{-\rho})\Gamma_-(p^{-\rho})q^H).$$

After performing N iterations of this strategy, we arrive at

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \prod_{d=1}^{N} M(p,q^{d-1})(1-q^d a^{-1})\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(q^N p^{-\rho})\Gamma_-(p^{-\rho})q^H).$$

[4]The third author thanks Guillaume Chapuy and Sylvie Corteel for teaching him this trick at a conference lunch in 2014. Bouttier, Chapuy, and Corteel used the trick in the paper [4] in the proof of Theorem 12 therein.

It follows from equation (7) that

$$\Gamma_\pm(q^N p^{-\rho}) \equiv \mathrm{Id} \mod q^N.$$

So the above two equations imply that the equation

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \prod_{d=1}^{\infty} M(p,q^{d-1})(1-q^d a^{-1}) \operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H)$$

holds to all orders in q and is hence true as a formal power series in q.

We now apply the same strategy commuting Γ_- to the left:

$$\begin{aligned}\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H) &= (1-a)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(qp^{-\rho})q^H) \\ &= (1-a)(1-aq)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(q^2p^{-\rho})q^H) \\ &= \dots \\ &= \prod_{d=1}^{\infty}(1-aq^{d-1})\operatorname{tr}(\mathcal{E}(a,p)q^H)\end{aligned}$$

and so we have proved

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \prod_{d=1}^{\infty} M(p,q^{d-1})(1-q^d a^{-1})(1-aq^{d-1}) \operatorname{tr}(\mathcal{E}(a,p)q^H).$$

From the definition of $\mathcal{E}_r(p)$ we see that its matrix entries are all off-diagonal if $r \neq 0$. Therefore

$$\begin{aligned}\operatorname{tr}(\mathcal{E}(a,p)q^H) &= \operatorname{tr}(\mathcal{E}_0(p)q^H) \\ &= \sum_\lambda q^{|\lambda|}\sum_{i=1}^{\infty} p^{\lambda_i - i + \frac{1}{2}} \\ &= (p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{-1}\prod_{m=1}^{\infty}\frac{(1-q^m)}{(1-pq^m)(1-p^{-1}q^m)}\end{aligned}$$

where the last equality follows from the computation in the proof of equation (2) in § 4.1. Combining this with the previous computations finishes the proof of the lemma. □

Combining Lemmas 6 and 7, we get

$$\sum_{\lambda} q^{|\lambda|} p^{\|\lambda\|^2} \mathsf{V}_{\lambda\lambda'\varnothing} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = \frac{1}{1-p} M(p) \prod_{m=1}^{\infty} \frac{M(p,q^m)}{(1-pq^m)(1-p^{-1}q^m)} \cdot \mathrm{Coeff}_{a^0} \left\{ \prod_{m=1}^{\infty} (1-q^m a^{-1})(1-q^{m-1}a)(1-q^m) \right\}.$$

By the Jacobi triple product identity, we have

$$\prod_{m=1}^{\infty} (1-q^m a^{-1})(1-q^{m-1}a)(1-q^m) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} (-a)^n$$

whose a^0 coefficient is 1. Plugging into the previous equation we finish the proof of equation (4).

The proofs of all the formulas in Theorem 3 are now complete.

6. Geometry and Applications

6.1. **Overview.** In this section we briefly outline the applications of our trace formulas to the Donaldson-Thomas theory of elliptically fibered Calabi-Yau threefolds [5, 6, 7] and the connection with Jacobi forms. This section is logically independent from the previous sections and can be safely ignored by readers not interested in the geometric applications of our theorem.

As we will explain below, each formula in our main theorem gives a certain contribution to the Donaldson-Thomas partition function of elliptically fibered threefolds. Namely, we have the following.

- The contribution of multiples of a nodal elliptic fiber is given by equation (1).
- The contribution of multiples of a smooth elliptic fiber attached to a smooth section curve is given by equation (2).
- The contribution of multiples of a smooth elliptic fiber attached to a nodal section curve is given by equation (3).
- The contribution of multiples of a nodal elliptic fiber attached to a smooth section curve is given by equation (4).

6.2. **A very short description of Donaldson-Thomas theory.** Donaldson-Thomas theory is a curve counting theory for Calabi-Yau threefolds which is conjecturally equivalent to Gromov-Witten theory [11]. Let X be a Calabi-Yau threefold, let $\beta \in H_2(X,\mathbb{Z})$ be a curve class, and let $n \in \mathbb{Z}$.

Let

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X, \quad [Z] = \beta, \quad \chi(\mathcal{O}_Z) = n\}$$

be the Hilbert scheme parameterizing subschemes of class β and whose structure sheaf has holomorphic Euler characteristic n. The Donaldson-Thomas invariant $\mathsf{DT}_{\beta,n}(X)$ is defined to be the Behrend function weighted Euler characteristic of the Hilbert scheme:

$$\mathsf{DT}_{\beta,n}(X) = \sum_{k\in\mathbb{Z}} k\, e\left(\nu^{-1}(k)\right)$$

where $e(-)$ is topological Euler characteristic and

$$\nu : \mathrm{Hilb}^{\beta,n}(X) \to \mathbb{Z}$$

is Behrend's constructible function [2]. The unweighted Euler characteristics are also often considered:

$$\widehat{\mathsf{DT}}_{\beta,n}(X) = e\left(\mathrm{Hilb}^{\beta,n}(X)\right).$$

In this discussion of our applications we will only consider the unweighted invariants $\widehat{\mathsf{DT}}_{\beta,n}(X)$. For further discourse on including the Behrend function see [5, 6, 7].

The invariants are usually assembled into a generating function called the partition function:

$$\widehat{\mathsf{DT}}(X) = \sum_{n,\beta} \widehat{\mathsf{DT}}_{\beta,n}(X) q^{\beta} p^{n}.$$

The above series can be considered as an element in the Novikov ring of X with coefficients which are formal Laurent series in p.[5]

6.3. **Donaldson-Thomas invariants of a threefold with a torus action.** The topological Euler characteristic of a $\mathbb{C}$-scheme S with the action of a complex torus $T \cong (\mathbb{C}^*)^k$ is given by the Euler characteristic of the fixed locus:

$$e(S) = e\left(S^T\right).$$

Thus if X is a Calabi-Yau threefold equipped with the action of a torus T, then

$$\widehat{\mathsf{DT}}_{\beta,n}(X) = e\left(\mathrm{Hilb}^{\beta,n}(X)^T\right).$$

In the case where X is toric with the action of a torus T, then $\mathrm{Hilb}^{\beta,n}(X)^T$ parameterizes T-invariant subschemes and is a finite set. Thus $\widehat{\mathsf{DT}}_{\beta,n}(X)$ simply counts the number of T-invariant subschemes $Z \subset X$ with $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = n$.

[5] With an appropriate choice of a basis of $H_2(X,\mathbb{Z})$, the partition function can be considered as a formal power series in a set a variables q_i whose coefficients are Laurent series in p.

On a T-invariant coordinate chart $\mathbb{C}^3 \subset X$, a T-invariant subscheme is given by a monomial ideal $I \subset \mathbb{C}[x,y,z]$. There is a bijection between 3D partitions and monomial ideals $\pi \leftrightarrow I$ given by:

$$(i,j,k) \in \pi \quad \Longleftrightarrow \quad x^i y^j z^k \notin I.$$

If the partition π is asymptotic to (λ,μ,ν) as in Definition 1, then the subscheme defined by the corresponding monomial ideal I is supported on the coordinate axes of $\mathbb{C}^3$ and has nilpotent thickenings along the axes determined by (λ,μ,ν).

So we see that in order to compute $\widehat{\mathsf{DT}}_{\beta,n}(X)$ for a toric Calabi-Yau threefold X, one needs to count 3D partitions (one for each coordinate chart) such that the corresponding subschemes agree on the coordinate overlaps and the curve class and holomorphic Euler characteristic are given by β and n respectively. This leads to a general formalism developed in [11] for computing $\widehat{\mathsf{DT}}(X)$ in terms of the topological vertex.

6.4. **Using the topological vertex in non-toric geometries.** Even if X has no torus action, under some favorable circumstances, we can still use the topological vertex technology to compute Donaldson-Thomas invariants. The idea is to exploit the *motivic* nature of Euler characteristic. The Euler characteristic is a ring homomorphism from the Grothendieck group of varieties over $\mathbb{C}$ to the integers. More prosaically, this means that one can compute the Euler characteristic of a scheme by stratifying it and then summing the Euler characteristics of each individual stratum.

For example, one can stratify the Hilbert scheme by specifying the support of the corresponding subschemes. Let $C \subset X$ be a (not necessarily irreducible) curve, and let

$$\mathrm{Hilb}^{\beta,n}(X,C) \subset \mathrm{Hilb}^{\beta,n}(X)$$

be the locus parameterizing subschemes Z such that the reduced support of Z is contained in C. We can then define the *contribution of C to the Donaldson-Thomas invariants* as

$$\widehat{\mathsf{DT}}(X,C) = \sum_{n,\beta} e(\mathrm{Hilb}^{\beta,n}(X,C))p^n q^\beta.$$

A key observation is that $\mathrm{Hilb}^{\beta,n}(X,C)$ only depends on $\widehat{X}_C$, the formal neighborhood of C in X. So while X itself may not admit a non-trivial $\mathbb{C}^*$-action, it is sometimes the case that $\widehat{X}_C$ does admit an action which then induces an action on the stratum $\mathrm{Hilb}^{\beta,n}(X,C)$.

This strategy can potentially be iterated: the fixed locus $\mathrm{Hilb}^{\beta,n}(X,C)^{\mathbb{C}^*}$ may be further stratified (for example by the support of the embedded points) and these substrata may admit further $\mathbb{C}^*$-actions. The upshot is that in circumstances where this strategy is successful, it reduces the computation of the Euler characteristic of the Hilbert scheme to the substratum of the Hilbert scheme parameterizing subschemes which are *formally locally monomial (FoLoMo for short)*, i.e. given by monomial ideals in the formal neighborhoods of each point. Denoting this substratum by the subscript FoLoMo , we get

$$\begin{aligned}\widehat{\mathsf{DT}}(X,C) &= \sum_{\beta,n} e\left(\mathrm{Hilb}^{\beta,n}(X,C)\right) p^n q^\beta \\ &= \sum_{\beta,n} e\left(\mathrm{Hilb}^{\beta,n}_{FoLoMo}(X,C)\right) p^n q^\beta.\end{aligned}$$

Due to the bijection between monomial ideals and 3D partitions, FoLoMo ideals in the formal neighborhood of a point can be counted with the topological vertex.

6.5. **The case of elliptically fibered Calabi-Yau threefolds.** Let X be a Calabi-Yau threefold admitting an elliptic fibration $\pi : X \to S$. We consider the following curves in X. Let $F \subset X$ be some non-singular fiber and let $N \subset X$ be a fiber having a nodal singularity. Let B be a smooth section curve of genus g, that is a non-singular curve of genus g meeting F and N once each in a node. The strategy outlined in the previous subsection works particularly well in this setting where we consider curve classes which are arbitrary multiples of the fiber class plus possibly a single multiple of the section class. In these cases, the local contributions can be expressed in terms of the topological vertex. The results are the following:

$$\begin{aligned}\widehat{\mathsf{DT}}(X,F) &= \sum_n \sum_{d=0}^{\infty} e\left(\mathrm{Hilb}^{d[F],n}_{FoLoMo}(X,F)\right) p^n q^d \\ &= \sum_\lambda q^{|\lambda|}, \\ \widehat{\mathsf{DT}}(X,B+F) &= \sum_n \sum_{d=0}^{\infty} e\left(\mathrm{Hilb}^{[B]+d[F],n}_{FoLoMo}(X,B+F)\right) p^n q^d \\ &= \sum_\lambda q^{|\lambda|} p^{1-g} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \cdot (\mathsf{V}_{\square\varnothing\varnothing})^{1-2g}, \\ \widehat{\mathsf{DT}}(X,N) &= \sum_n \sum_{d=0}^{\infty} e\left(\mathrm{Hilb}^{d[N],n}_{FoLoMo}(X,N)\right) p^n q^d \\ &= \sum_\lambda q^{|\lambda|} p^{||\lambda||^2} \mathsf{V}_{\lambda\lambda'\varnothing},\end{aligned}$$

$$\begin{aligned}\widehat{\mathsf{DT}}(X,B+N) &= \sum_n \sum_{d=0}^{\infty} e\left(\operatorname{Hilb}^{[B]+d[N],n}_{FoLoMo}(X,B+N)\right) p^n q^d \\ &= \sum_\lambda q^{|\lambda|} p^{||\lambda||^2+1-g}\, \mathsf{V}_{\lambda\lambda'\varnothing} \cdot \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \cdot (\mathsf{V}_{\square\varnothing\varnothing})^{1-2g}.\end{aligned}$$

There is one additional case to consider. Let B' be nodal section curve, that is a curve that has a single nodal singularity at $n \in B'$ and we assume that B' meets F at the point n so that the singularity of $B' \cup F$ at n is formally locally that of the coordinate axes. Let g' be the geometric genus of B'. Then

$$\begin{aligned}\widehat{\mathsf{DT}}(X,B'+F) &= \sum_n \sum_{d=0}^{\infty} e\left(\operatorname{Hilb}^{[B']+d[F],n}_{FoLoMo}(X,B'+F)\right) p^n q^d \\ &= \sum_\lambda q^{|\lambda|}\, p^{1-g'}\, \frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \cdot (\mathsf{V}_{\square\varnothing\varnothing})^{-2g'}.\end{aligned}$$

All of the above formulas have a sum over (2D) partitions. Geometrically, this is because a FoLoMo subscheme which is a nilpotent thickening of a curve is determined, away from singular points and embedded points, by a monomial ideal in the formal coordinates transverse to the curve, which is in turn determined by a 2D partition. Thus in the above formulas, the term in the sum corresponding to a partition λ counts subschemes which have thickenings determined by λ along the fiber curve (F or N).

Each term in the sum is a combination of topological vertexes which counts the ways in which embedded points can appear in the subscheme. For example, in the formula for $\widehat{\mathsf{DT}}(X,B+N)$ the $\mathsf{V}_{\lambda\lambda'\varnothing}$ counts the ways of adding embedded points at the node $n \in N$, the $\mathsf{V}_{\lambda\square\varnothing}$ counts ways of adding embedded points at $p = B \cap N$, the $\mathsf{V}_{\lambda\varnothing\varnothing}$ counts the number of ways of adding embedded points at some arbitrary point of $N - \{n,p\}$, and the $\mathsf{V}_{\square\varnothing\varnothing}$ counts the number of ways of adding embedded points at some arbitrary point of $B - p$.

The reason that the vertex terms in the above formulas appear with a (possibly negative) exponent deserves explaining. In the case of a toric Calabi-Yau threefold, the number of possible locations for embedded points is finite — they only occur at the origins of the torus invariant coordinate charts. In the case of $FoLoMo$ subschemes, embedded points can occur at an infinite number of locations. In the example at hand, the embedded points parameterized by $\mathsf{V}_{\lambda\varnothing\varnothing}$ can occur at any point of the curve N except n and p and the embedded points parameterized by $\mathsf{V}_{\square\varnothing\varnothing}$ can occur at any point of B except p. The possible locations are parameterized by symmetric products of $N - \{n,p\}$ and $B - p$ respectively. Symmetric products are very amenable to our motivic methods. Indeed, symmetric products endow the Grothendieck group of varieties with a lambda ring structure which is compatible with the Euler characteristic homomorphism. Because of the symmetric product formalism in the Grothendieck group, the $\mathsf{V}_{\lambda\varnothing\varnothing}$ and the $\mathsf{V}_{\square\varnothing\varnothing}$ terms appear with exponents given by the topological Euler characteristic of $N - \{n, p\}$ and $B - p$ respectively:

$$(\mathsf{V}_{\lambda\varnothing\varnothing})^{e(N-\{n,p\})}(\mathsf{V}_{\square\varnothing\varnothing})^{e(B-\{p\})} = \frac{1}{\mathsf{V}_{\lambda\varnothing\varnothing}} \cdot (\mathsf{V}_{\square\varnothing\varnothing})^{1-2g}.$$

6.6. **Applying our trace formulas and the appearance of Jacobi forms.** By equation (5), we see that

$$\mathsf{V}_{\square\varnothing\varnothing} = M(p)(1-p)^{-1}.$$

After substituting the above into the equations of the previous section, we may apply the formulas in our main theorem as well as the well known formula

$$\sum_{\lambda} q^{|\lambda|} = \prod_{d=1}^{\infty}(1-q^d)^{-1}$$

to the formulas in the previous subsection. We get the following results:

$$\begin{aligned}
\widehat{\mathsf{DT}}(X, F) &= \prod_{d=1}^{\infty}(1-q^d)^{-1} \\
\widehat{\mathsf{DT}}(X, B+F) &= M(p)^{1-2g}\left(p^{\frac{1}{2}} - p^{-\frac{1}{2}}\right)^{2g-2}\prod_{d=1}^{\infty}\frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)} \\
\widehat{\mathsf{DT}}(X, N) &= M(p)\prod_{d=1}^{\infty}(1-q^d)^{-1}M(p,q^d) \\
\widehat{\mathsf{DT}}(X, B+N) &= \left(p^{\frac{1}{2}} - p^{-\frac{1}{2}}\right)^{2g-2}M(p)^{2-2g}\prod_{d=1}^{\infty}\frac{M(p,q^d)}{(1-pq^d)(1-p^{-1}q^d)} \\
\widehat{\mathsf{DT}}(X, B'+F) &= \left(p^{\frac{1}{2}} - p^{-\frac{1}{2}}\right)^{2g'}M(p)^{-2g'}\prod_{d=1}^{\infty}(1-q^d)^{-1} \\
&\quad \cdot \left\{1 + \frac{p}{(1-p)^2} + \sum_{m=1}^{\infty}\sum_{k|m}k(p^k+p^{-k})q^m\right\}.
\end{aligned}$$

The above formulas have a close connection with some well known Jacobi forms and modular forms. Let $p = \exp(2\pi i z)$ and $q = \exp(2\pi i\tau)$. The Jacobi theta function Θ, the Dedekind eta function η, the Weierstrass $\wp$ function, and the Eisenstein series G_2 are given as follows:

$$\Theta = \left(p^{\frac{1}{2}} - p^{-\frac{1}{2}}\right) \prod_{m=1}^{\infty} \frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}$$
$$\eta = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1-q^m)$$
$$\wp = \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k - 2 + p^{-k})q^d$$
$$G_2 = -\frac{1}{24} + \sum_{d=1}^{\infty} \sum_{k|d} kq^d.$$

We may then observe that

$$\frac{\widehat{\mathsf{DT}}(X, B+F)}{\widehat{\mathsf{DT}}(X,F)} = \left(\frac{M(p)}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\right)^{1-2g} \frac{1}{\Theta}$$
$$\frac{\widehat{\mathsf{DT}}(X, B+N)}{\widehat{\mathsf{DT}}(X,N)} = \left(\frac{M(p)}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\right)^{1-2g} \frac{q^{\frac{1}{24}}}{\Theta\eta}$$
$$\frac{\widehat{\mathsf{DT}}(X, B'+F)}{\widehat{\mathsf{DT}}(X,F)} = \left(\frac{M(p)}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\right)^{-2g'} \left\{\wp + 2G_2 + 1\right\}.$$

The quotients on the left can be interpreted as the partition function for *connected* Donaldson-Thomas invariants. There is a general conjecture due to Huang, Katz, and Klemm [8] which predicts that the Donaldson-Thomas partition function for an elliptically fibered Calabi-Yau, in classes given by a fixed section class and multiple fiber classes, is given by a Jacobi form. Several special cases of this conjecture have been proven using the methods outlined in this section, namely when X is (1) a local elliptic surface [6], (2) a product of a $K3$ surface and an elliptic curve [5], and (3) a product of an Abelian surface and an elliptic curve [7].

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