BPS STATE COUNTS OF LOCAL ELLIPTIC SURFACES VIA FORMAL GEOMETRY AND THE TOPOLOGICAL VERTEX

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Abstract. We compute the (connected) stable pair invariants of $X = \mathrm{Tot}(K_S)$ where S is an elliptic surface with section and at worst 1-nodal singular fibres. The calculation includes thickenings to all orders in both the surface and fibre direction. We use a new method combining motivic arguments and torus localization.

We stratify the moduli space according to underlying reduced support C_{red} of the stable pair and compute the contribution of each C_{red} individually. The contribution of C_{red} can be split up into a part coming from the nodes of C_{red} and the complement of the nodes $C_{\mathrm{red}}^{\circ} \subset C_{\mathrm{red}}$. The formal neighbourhood of C_{red}° in X is isomorphic to a formal neighbourhood of C_{red}° inside its normal bundle. This gives us lots of $\mathbb{C}^*$-actions.

Localization with respect to the torus actions leads to a vertex calculation which can be performed explicitly. As special cases we find a new proof of the Katz–Klemm–Vafa formula in the primitive case (independent of Kawai–Yoshioka's formula) and the BPS spectrum of the local rational elliptic surface.

Contents

1. Introduction

Let X be a smooth projective 3-fold, $\chi \in \mathbb{Z}$, and $\beta \in H_2(X)$ a curve class. Denote by $P_\chi(X, \beta)$ the moduli space of stable pairs $I^\bullet = [\mathcal{O}_X \to \mathcal{F}]$ on X for which $\chi(\mathcal{F}) = \chi$, and the scheme theoretic support of F has curve class β. The moduli space $P_\chi(X, \beta)$ is an instance of Le Potier's more general moduli spaces of stable pairs [LeP]. The deformation-obstruction theory of stable pairs *does not* provide a perfect obstruction theory for $P_\chi(X, \beta)$. R. P. Thomas and R. Pandharipande realize $P_\chi(X, \beta)$ as a component of the moduli space of complexes in $D^b(X)$ with trivial determinant. Viewed as a moduli space of complexes $P_\chi(X, \beta)$ *does* have a perfect obstruction theory [PT1]. When X is in addition Calabi-Yau, this perfect obstruction theory is symmetric and the stable pair invariants of X are defined as the degree of the virtual cycle

$$P_{\chi,\beta}(X) := \int_{[P_\chi(X,\beta)]^{\mathrm{vir}}} 1.$$

By a theorem of K. Behrend [Beh]

$$\int_{[P_\chi(X,\beta)]^{\mathrm{vir}}} 1 = \int_{P_\chi(X,\beta)} \nu_B \ \mathrm{d}e,$$

where $\nu_B : P_\chi(X, \beta) \to \mathbb{Z}$ is Behrend's constructible function and $e(\cdot)$ denotes topological Euler characteristic.

In this paper $\pi : S \to B$ denotes an elliptic surface. This means S is a smooth surface, B a smooth curve of genus $g(B)$, and π a holomorphic map with general fibre a connected smooth genus 1 curve [Mir]. We make two assumptions:Can more be said about such surfaces? I don't thin we need $B \cong \mathbb{P}^1$?

- π has a section $B \hookrightarrow S$,
- all singular fibres of π are of Kodaira type I_1, i.e. rational 1-nodal curves.

We are interested in the case $X = \mathrm{Tot}(K_S)$ and $\beta = B + dF$, where B is the class of the section and F is the class of the fibre. Since X is a non-compact Calabi-Yau 3-fold we require curves of $P_\chi(X, \beta)$ to have proper support. Non-compactness of $P_\chi(X, \beta)$ also means we do not have a virtual cycle, so one should *define* stable pair invariants in this setting eitherHow are both approaches related? by Graber-Pandharipande's localization formula [GP] or by integration of ν_B over $P_\chi(X, \beta)$. We choose the latter approach.EULER CHAR FOR THIS VERSION FOR NOW!

Consider the (disconnected) generating function

$$
\begin{aligned}
(1) \qquad Z^{\bullet P}(q,y) &:= \sum_{d\geq 0}\sum_{\chi} P_{\chi,B+dF}(X) q^{\chi} y^d, \\
P_{\chi,B+dF}(X) &:= e(P_{\chi}(X,B+dF)).
\end{aligned}
$$

The connected generating function is defined as [PT1]

$$
\begin{aligned}
(2) \qquad Z^{P}(q,y) &:= \frac{\sum_{d\geq 0}\sum_{\chi} P_{\chi,B+dF}(X) q^{\chi} y^d}{\sum_{d\geq 0}\sum_{\chi} P_{\chi,dF}(X) q^{\chi} y^d}, \\
P_{\chi,dF}(X) &:= e(P_{\chi}(X,dF)),
\end{aligned}
$$

where $P_{\chi,0}(X)=1$ for all χ. Our main result is the following.

Theorem 1.1. *Let $X=\mathrm{Tot}(K_S)$ where $S\to B$ is an elliptic surface with section B of genus $g(B)$ and N 1-nodal fibres. Then*

$$
Z^P(q,y) = \left(\frac{q}{(1-q)^2}\right)^{1-g(B)} \prod_{i=1}^{\infty} \frac{1}{(1-y^i)^{N-2e(B)}(1-qy^i)^{e(B)}(1-q^{-1}y^i)^{e(B)}}.
$$

The proof is divided into five movements:

Stratifation, *Restriction*, *Formalization*, *Localization*, *Finale (Schur).*

Applications for this paper: stable pair version of KKV in the primitive case independent of KY, gen fun for rational elliptic surface. Future applications: elliptically fibres CY3's, refinement and comparison to refined KKV, ...

2. Stratification

Let β be Poincaré dual to $B+dF$. The projections

$$
\varpi : X \longrightarrow S \longrightarrow B
$$

induce a push-forward map

$$
P_{\chi}(X,\beta) \longrightarrow \mathrm{Sym}^d(B),\ I^{\bullet} = [\mathcal{O}_X \to \mathcal{F}] \mapsto \varpi_* \mathcal{F}.
$$

We denote the fibre of $S\to B$ over $p\in B$ by F_p. Let

$$
\mathbf{p} := \sum_{i=1}^{m} d_i p_i \subset B
$$

be an effective divisor with all $d_i>0$ and $\sum_{i=1}^m d_i = d$. Consider the reduced curve

$$
C_{\mathbf{p}} := \bigcup_{i=1}^{m} F_{p_i} \subset S \subset X,
$$

where $S \subset X$ is the zero-section. The fibre of ϖ_* over $\mathbf{p}$ is

$$P_\chi(X,\mathbf{p}) := \left\{ I^\bullet = [\mathcal{O}_X \to \mathcal{F}] \in P_\chi(X,\beta) \ : \ \varpi_*\mathcal{F} = \mathbf{p} \right\},$$

i.e. the locally closed subset of stable pairs $I^\bullet = [\mathcal{O}_X \to \mathcal{F}] \in P_\chi(X,\beta)$ for which $\mathcal{F}$ has set theoretic support $C_{\mathbf{p}}$ and multiplicity d_i along F_{p_i} for all i. We are interested in the stratification

$$P_\chi(X,\beta) = \coprod_{\mathbf{p} \in \mathrm{Sym}^d(B)} P_\chi(X,\mathbf{p}).$$

Lemma 2.1.

$$e(P_\chi(X,\beta)) = \int_{\mathbf{p} \in \mathrm{Sym}^d(B)} e(P_\chi(X,\mathbf{p})) \ \mathrm{d}e.$$

Proof. [MacP]. □

3. Restriction

By Lemma 2.1 we are reduced to computing $e(P_\chi(X,\mathbf{p}))$ for any

$$\mathbf{p} = \sum_{i=1}^m d_i p_i \in \mathrm{Sym}^d(B).$$

Let $q \in C_{\mathbf{p}}$ be one of the *nodal* singularities (either a node in a singular fibre or an intersection point of a fibre with the section). We denote by $\widehat{X}_q$ the formal neighbourhood of $\{q\} \subset X$ and by $X \setminus q$ the complement of $\{q\} \subset X$. LetA little more care in the def of this moduli space is needed since $X \setminus q$ is non-compact AND the supports of the stable pairs are non-compact.

$$P_\chi(X \setminus q, \mathbf{p})$$

be the moduli space of stable pairs $I^\bullet = [\mathcal{O}_{X\setminus q} \to \mathcal{F}]$ such that $\chi(\mathcal{F}) = \chi$, $\mathcal{F}$ has set theoretic support $C_{\mathbf{p}} \setminus q$, $\mathcal{F}$ has multiplicity 1 along $B \setminus q$, and $\mathcal{F}$ has multiplicity d_i along $F_{p_i} \setminus q$ for all i. Moreover letThis might need a little more care too since stable pair theory is not yet defined for formal schemes.

$$P_\chi(\widehat{X}_q, \mathbf{p})$$

be the moduli spaces of stable pairs $I^\bullet = [\mathcal{O}_{\widehat{X}_q} \to \mathcal{F}]$ such that $\chi(\mathcal{F}) = \chi$, $\mathcal{F}$ has set theoretic support $\widehat{C_{\mathbf{p}}}$, $\mathcal{F}$ has multiplicity 1 along $\widehat{B}$, and $\mathcal{F}$ has multiplicity d_i along $\widehat{F_{p_i}}$ for all i. Here $\widehat{C_{\mathbf{p}}}$, $\widehat{B}$, $\widehat{F_{p_i}}$ denote the lifts[1] of $C_{\mathbf{p}}$, B,

[1]Let $\widehat{X}_Z$ be the formal completion of any scheme along a closed subset Z. If $\mathcal{E}$ is a coherent sheaf on X then one can define a lift $\mathcal{E}^\Delta$ to $\widehat{X}_Z$ [Har]. In the case $\mathcal{E} = \mathscr{I} \subset \mathcal{O}_X$ is an ideal sheaf, this provides an ideal sheaf $\mathscr{I}^\Delta \subset \mathcal{O}_{\widehat{X}_Z}$ [Har].

F_{p_i} to $\widehat{X}_q$. We are interested in the injective morphism induced by restriction

$$P_\chi(X, \mathbf{p}) \hookrightarrow \coprod_{\chi=\chi_1+\chi_2} P_{\chi_1}(X \setminus q, \mathbf{p}) \times P_{\chi_2}(\widehat{X}_q, \mathbf{p}). \tag{3}$$

The image of this morphism can be characterized as follows. LetMay not exist. In general work in stalk? See Jim's e-mail on 25.6.2014.

$$U = \operatorname{Spec} \mathbb{C}[x, y] \subset S$$

be an open affine neighbourhood of q over which $X = \operatorname{Tot}(K_S)$ trivializes with fibre coordinate z. Then $\widehat{X}_q$ is the reduced point q with sheaf of rings

$$\mathcal{O}_{\widehat{X}_q} \cong \widehat{\mathcal{O}}_{X,q} \cong \mathbb{C}[\![x, y, z]\!].$$

Suppose the coordinates are chosen such that $C_{\mathbf{p}}$ is defined by $xy = z = 0$. Define open subsets

$$V = \{x \neq 0\} \subset U, \ W = \{y \neq 0\} \subset U.$$

Lemma 3.1. *An element*

$$([s_1 : \mathcal{O}_{X\setminus q} \to \mathcal{F}_1], [s_2 : \mathcal{O}_{\widehat{X}_q} \to \mathcal{F}_2])$$

lies in the image of the embedding (3) *if and only if the Cohen-Macaulay support curves $C_{\mathcal{F}_1}$, $C_{\mathcal{F}_2}$ underlying both stable pairs glue i.e.*

$$\Gamma(\widehat{X}_q, \mathscr{I}_{C_{\mathcal{F}_2}}) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x^{\pm}, y, z]\!] \cong \widehat{\Gamma}(V \times \mathbb{C}, \mathscr{I}_{C_{\mathcal{F}_1}}|_{V\times\mathbb{C}}),$$
$$\Gamma(\widehat{X}_q, \mathscr{I}_{C_{\mathcal{F}_2}})) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x, y^{\pm}, z]\!] \cong \widehat{\Gamma}(W \times \mathbb{C}, \mathscr{I}_{C_{\mathcal{F}_1}}|_{W\times\mathbb{C}}),$$

where $\Gamma(\cdot)$ denotes the global section functor, $\widehat{(\cdot)}$ is the formal completion of the module $(\cdot)$, and $\mathscr{I}_{C_{\mathcal{F}_1}}$, $\mathscr{I}_{C_{\mathcal{F}_2}}$ are ideal sheaves.

Proof. Perhaps Ben-Bassat–Temkin's [BT] abstract setup (or a stable pairs version) reduces to this when Z (in their notation) is just a point.See Jim's fpqc e-mail on 25.6.2014. Note: life is not too bad because only the support curve has to glue. This is because the section of a stable pair is an isomorphism outside a 0-dim subscheme. □

We want to apply the above construction not just for one point q. Let $q_1, \ldots, q_n \in C_{\mathbf{p}}$ be all nodes. For notational simplicity we write

$$X^\circ := X \setminus \{q_1, \ldots, q_n\}.$$

We embed

$$P_\chi(X, \mathbf{p}) \hookrightarrow \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_n} P_{\chi'}(X^\circ, \mathbf{p}) \times \prod_{j=1}^{n} P_{\chi_j}(\widehat{X}_{q_j}, \mathbf{p}).$$

The image is characterized by gluing conditions as in Lemma 3.1 at each of the nodes q_j.

4. Formalization

In the previous section we characterized the image of $P_\chi(X, \mathbf{p})$ under restriction to special points and their complements

$$P_\chi(X, \mathbf{p}) \hookrightarrow \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_n} P_{\chi'}(X^\circ, \mathbf{p}) \times \prod_{i=1}^{n} P_{\chi_j}(\widehat{X}_{q_j}, \mathbf{p})$$

In this section we relate $P_\chi(X^\circ, \mathbf{p})$ to moduli spaces of stable pairs on the (punctured) fibres/section inside their normal bundle.

Recall that

$$\mathbf{p} := \sum_{i=1}^{m} d_i p_i \in \mathrm{Sym}^d(B),\ C_{\mathbf{p}} := \bigcup_{i=1}^{m} F_{p_i},$$

and $q_1, \ldots, q_n$ are all nodes of $C_{\mathbf{p}}$. We have an inclusion

$$P_\chi(X^\circ, \mathbf{p}) \subset P_\chi(X^\circ, \beta),$$

where $P_\chi(X^\circ, \beta)$ denotes the moduli space of stable pairs $I^\bullet = [\mathcal{O}_{X^\circ} \to \mathcal{F}]$ on X° such that $\chi(\mathcal{F}) = \chi$ and the closure of the scheme theoretic support of $\mathcal{F}$ in X is proper with class β. We can make a formal completion of the former space along the latter

$$\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ, \mathbf{p})}.$$

Obviously the underlying topological space is unchanged so

$$e(\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ, \mathbf{p})}) = e(P_\chi(X^\circ, \mathbf{p})).$$

Passing to the formal completion allows us to consider stable pairs on the formal completion of X° along $C_{\mathbf{p}}^\circ := C_{\mathbf{p}} \setminus \{q_1, \ldots, q_n\}$. This formal completion is denoted by

$$\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}.$$

Lemma 4.1. *There exists a canonical isomorphism*

$$\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ, \mathbf{p})} \cong P_\chi(\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}, \mathbf{p}),$$

where $P_\chi(\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}, \mathbf{p})$ is the moduli space of stable pairs $I^\bullet = [\mathcal{O} \to \mathcal{F}]$ on $\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}$ such that $\chi(\mathcal{F}) = \chi$, $\mathcal{F}$ has multiplicity 1 along $\widehat{B^\circ}$, and $\mathcal{F}$ has multiplicity d_i along $\widehat{F_{p_i}^\circ}$ for all i. Here $\widehat{B^\circ}$, $\widehat{F_{p_i}^\circ}$ denote the lifts of B°, $F_{p_i}^\circ := F_{p_i} \setminus \{q_1, \ldots, q_n\}$ to $\widehat{X^\circ}_{C_{\mathbf{p}}^\circ}$.

Proof. Jim's idea of categorical limits. This should be formal. □

Let us take a closer look at the formal scheme $\widehat{X^\circ}_{C^\circ_{\mathbf{p}}}$. Removing the nodes points $q_1, \ldots, q_n$ we obtain smooth curves B°, $F^\circ_{p_i}$ and

$$C^\circ_{\mathbf{p}} \cong B^\circ \sqcup F^\circ_{p_1} \sqcup \cdots \sqcup F^\circ_{p_m}.$$

This isomorphism also holds at the level of formal schemes.

Lemma 4.2. *There exists a canonical isomorphism*

$$\widehat{X^\circ}_{C^\circ_{\mathbf{p}}} \cong \widehat{X^\circ}_{B^\circ} \sqcup \widehat{X^\circ}_{F^\circ_{p_1}} \sqcup \cdots \sqcup \widehat{X^\circ}_{F^\circ_{p_m}},$$

where $\widehat{X^\circ}_{B^\circ}$, $\widehat{X^\circ}_{F^\circ_{p_i}}$ *are the formal completions of* X *along* B°, $F^\circ_{p_i}$.

Proof. Disjoint union commutes with formal completionIS THIS REALLY TRUE? Sounds plausible.. $\square$

This lemma allows us to pass to the normal bundles of $B^\circ \subset X^\circ$, $F^\circ_{p_i} \subset X^\circ$.

Lemma 4.3. *There exists natural isomorphisms*

$$\widehat{X^\circ}_{B^\circ} \cong \widehat{N_{B^\circ/X^\circ}}_{B^\circ}, \ \widehat{X^\circ}_{F^\circ_{p_i}} \cong \widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}},$$

where $\widehat{N_{B^\circ/X^\circ}}_{B^\circ}$, $\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}$ *are the formal completions of the normal bundles* N_{B°/X°, $N_{F^\circ_{p_i}/X^\circ}$ *along their zero sections* B°, $F^\circ_{p_i}$.

Proof. For the fibres we proved this rigorously using sections of $\mathcal{O}/\mathscr{I}^{r+1} \to \mathcal{O}/\mathscr{I}^r$ pulled back from the base B. This requires flatness of π. For the section we use Davesh's argument. $\square$

Lemmas 4.1, 4.2, 4.3 allow us write

$$\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ, \mathbf{p})} \cong \coprod_{\chi = \chi' + \chi_1 + \cdots + \chi_m} P_{\chi'}(\widehat{N_{B^\circ/X^\circ}}_{B^\circ}, \mathbf{p}) \times \prod_{i=1}^m P_{\chi_i}(\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}, \mathbf{p}),$$

where $P_\chi(\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}, \mathbf{p})$ is the moduli space of stable pairs $I^\bullet = [\mathcal{O} \to \mathcal{F}]$ on $\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}$ with $\chi(\mathcal{F}) = \chi$ and $\mathcal{F}$ has set theoretic support $\widehat{F^\circ_{p_i}}$ with multiplicity d_i. Here $\widehat{F^\circ_{p_i}}$ denotes the lift of $F^\circ_{p_i}$ to $\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}$. Similar for $P_\chi(\widehat{N_{B^\circ/X^\circ}}_{B^\circ}, \mathbf{p})$ where the multiplicity along $\widehat{B^\circ}$ is required to be one.

Finally we want to "undo" the formal completion on the normal bundles by using categorical limits as in Lemma 4.1. We denote by

$$P_\chi(N_{F^\circ_{p_i}/X^\circ}, \mathbf{p}) \subset P_\chi(N_{F^\circ_{p_i}/X^\circ}, d_i F^\circ_{p_i})$$

moduli spaces of stable pairs $I^\bullet = [\mathcal{O} \to \mathcal{F}]$ on N_{B°/X° with $\chi(\mathcal{F}) = \chi$. The first has $\mathcal{F}$ with set theoretic support $F^\circ_{p_i}$ and multiplicity d_i. The second has $\mathcal{F}$ such that the closure of its set theoretic support in $N_{F_{p_i}/X}$ is proper with class $d_iF_{p_i}$. Similarly we consider

$$P_\chi(N_{B^\circ/X^\circ}, \mathbf{p}) \subset P_\chi(N_{B^\circ/X^\circ}, B^\circ).$$

The argument presented in the proof of Lemma 4.1 gives

$$P_\chi(\widehat{N_{B^\circ/X^\circ}}_{B^\circ}, \mathbf{p}) \cong \widehat{P}_\chi(N_{B^\circ/X^\circ}, B^\circ)_{P_\chi(N_{B^\circ/X^\circ},\mathbf{p})}$$
$$P_\chi(\widehat{N_{F^\circ_{p_i}/X^\circ}}_{F^\circ_{p_i}}, \mathbf{p}) \cong \widehat{P}_\chi(N_{F^\circ_{p_i}/X^\circ}, d_iF^\circ_{p_i})_{P_\chi(N_{F^\circ_{p_i}/X^\circ},\mathbf{p})}.$$

Combining all arguments of this section gives the following result.

Proposition 4.4. *We have natural isomorphisms*

$$\widehat{P}_\chi(X^\circ, \beta)_{P_\chi(X^\circ,\mathbf{p})} \cong \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_m} \widehat{P}_{\chi'}(N_{B^\circ/X^\circ}, B^\circ)_{P_\chi(N_{B^\circ/X^\circ},\mathbf{p})} \times \prod_{i=1}^m \widehat{P}_{\chi_i}(N_{F^\circ_{p_i}/X^\circ}, d_iF^\circ_{p_i})_{P_\chi(N_{F^\circ_{p_i}/X^\circ},\mathbf{p})}.$$

In particular on the underlying topological space we have a homeomorphism

$$P_\chi(X^\circ, \mathbf{p}) \approx \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_m} P_\chi(N_{B^\circ/X^\circ}, \mathbf{p}) \times \prod_{i=1}^m P_\chi(N_{F^\circ_{p_i}/X^\circ}, \mathbf{p}).$$

Proof. Combination of the above. □

5. Localization

5.1. **Localization I.** In the previous two sections we constructed an embedding

$$P_\chi(X, \mathbf{p}) \hookrightarrow \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_n} P_{\chi'}(X^\circ, \mathbf{p}) \times \prod_{j=1}^n P_{\chi_j}(\widehat{X}_{q_i}, \mathbf{p}) \tag{4}$$

and homeomorphisms

$$P_\chi(X^\circ, \mathbf{p}) \approx \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_m} P_\chi(N_{B^\circ/X^\circ}, \mathbf{p}) \times \prod_{i=1}^m P_\chi(N_{F^\circ_{p_i}/X^\circ}, \mathbf{p}). \tag{5}$$

Each normal bundle has a natural $\mathbb{C}^{*2}$-action given by scaling the fibres. The action of $\mathbb{C}^{*2}$ on $P_\chi(N_{B^\circ/X^\circ}, \mathbf{p})$ is trivial[2] so we ignore it. Therefore $\mathbb{C}^{*2m}$ acts naturally on $P_\chi(X^\circ, \mathbf{p})$ by (5).

[2]This action is transverse to the section and our stable pairs have multiplicity 1 along B.

Since each $\widehat{X}_{q_j}$ is just the reduced point q_j with structure sheaf

$$\mathcal{O}_{\widehat{X}_{q_j}} \cong \widehat{\mathcal{O}}_{X,q_j} \cong \mathbb{C}[\![x,y,z]\!],$$

we have $\mathbb{C}^{*3}$ acting on this space by $(s_1,s_2,s_3)\cdot(x,y,z) = (s_1x, s_2y, s_3z)$. In total we get an action of $\mathbb{C}^{*(2m+3n)}$ on the RHS of (4). However $P_\chi(X,\mathbf{p})$ is not invariant under this full torus.

Lemma 5.1. *Define the a $2m$-dimensional subtorus $T \subset \mathbb{C}^{*(2m+3n)}$ by the following equations. For any nodal fibre F_{p_i} with node q_j let $(t_1^{(i)}, t_2^{(i)})$ be the coordinates of $\mathbb{C}^{*2}$ acting on $N_{F_{p_i}^\circ/X^\circ}$ and let $(s_1^{(j)}, s_2^{(j)}, s_3^{(j)})$ be the coordinates of $\mathbb{C}^{*3}$ acting on $\widehat{X}_{q_j}$, then*

$$s_1^{(j)} = s_2^{(j)} = t_1^{(i)},\ s_3^{(j)} = t_2^{(i)}.$$

For any (not necessarily nodal) fibre F_{p_i} and $\{q_j\} = F_{p_i}\cap B$ let $(t_1^{(i)}, t_2^{(i)})$ be the coordinates of $\mathbb{C}^{*2}$ acting on $N_{F_{p_i}^\circ/X^\circ}$ and let $(s_1^{(j)}, s_2^{(j)}, s_3^{(j)})$ be the coordinates of $\mathbb{C}^{*3}$ acting on $\widehat{X}_{q_j}$, then

$$s_1^{(j)} = 1,\ s_2^{(j)} = t_1^{(i)},\ s_3^{(j)} = t_2^{(i)}.$$

Then T leaves $P_\chi(X,\mathbf{p})$ invariant.

Proof. Use the gluing conditions of Lemma 3.1. This does require passing through several isomorphisms which could be tricky.How tedious will this be...? □

Since $e(P_\chi(X,\mathbf{p})) = e(P_\chi(X,\mathbf{p})^T)$ we are reduced to understanding the fixed point locus $P_\chi(X,\mathbf{p})^T$. Let

$$([s:\mathcal{O}_{X^\circ}\to\mathcal{E}], \{[s_j:\mathcal{O}_{\widehat{X}_{q_j}}\to\mathcal{F}_j]\}_{j=1}^n) \in \coprod_{\chi=\chi'+\chi_1+\cdots+\chi_n} P_{\chi'}(X^\circ,\mathbf{p})\times\prod_{j=1}^n P_{\chi_j}(\widehat{X}_{q_j},\mathbf{p}).$$

This element lies in $P_\chi(X,\mathbf{p})$ if and only if the underlying Cohen-Macaulay curves $C_\mathcal{E}$, $C_{\mathcal{F}_j}$ glue as described in Lemma 3.1. This element is in addition T-fixed if and only if each of the restrictions

$$\Gamma(\widehat{X}_{q_j}, \mathscr{I}_{C_{\mathcal{F}_j}}) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x^\pm,y,z]\!] \cong \widehat{\Gamma}(V\times\mathbb{C}, \mathscr{I}_{C_\mathcal{E}}|_{V\times\mathbb{C}})$$
$$\Gamma(\widehat{X}_{q_j}, \mathscr{I}_{C_{\mathcal{F}_j}})) \otimes_{\mathbb{C}[\![x,y,z]\!]} \mathbb{C}[\![x,y^\pm,z]\!] \cong \widehat{\Gamma}(W\times\mathbb{C}, \mathscr{I}_{C_\mathcal{E}}|_{W\times\mathbb{C}})$$

is given by a monomial ideal in two variables, i.e. a (2-dimensional) partition. For each node which is the intersection point of a (not necessarily nodal) fibre F_{p_i} with the zero section, this amounts to specifying a partition λ_i of d_i in the fibre direction. The partition in the section direction is (1), because the multiplicity of $C_{\mathcal{E}}$ along B is 1. For each node of a nodal fibre F_{p_i} the cross-section of the Cohen-Macaulay support curve has to be given by the same partitions λ_i. Altogether we have fixed partitions $\boldsymbol{\lambda} = \{\lambda_i \vdash d_i\}_{i=1}^m$. Denote by

$$P_\chi(X^\circ, \mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(X^\circ, \mathbf{p}),\ P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(\widehat{X}_{q_j}, \mathbf{p})$$

the locally closed subsets for which the underlying Cohen-Macaulay curves have restrictions described by partitions $\boldsymbol{\lambda}$ as above. We arrive at the following conclusion.

Lemma 5.2. *The embedding* (4) *induces a bijective morphism*

$$P_\chi(X, \mathbf{p})^T \cong \coprod_{\chi = \chi' + \chi_1 + \cdots + \chi_n} \coprod_{\boldsymbol{\lambda} = \{\lambda_i \vdash d_i\}_{i=1}^m} P_{\chi'}(X^\circ, \mathbf{p})_{\boldsymbol{\lambda}} \times \prod_{j=1}^n P_{\chi_j}(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}},$$

where T is the torus of Lemma 4.3.

Proof. Easy from the above. □

5.2. **Localization II.** In this subsection we focus attention on $e(P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}})$ for any $\boldsymbol{\lambda} = \{\lambda_i \vdash d_i\}_{i=1}^m$. On each moduli space $P_\chi(\widehat{X}_{q_j}, \mathbf{p})$ we have a $\mathbb{C}^{*3}$-action as described in the previous subsection. This action leaves

$$P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(\widehat{X}_{q_j}, \mathbf{p})$$

invariant. The fixed point locus $P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}$ consists of *isolated* fixed points which can be counted using the vertex/edge formalism for stable pairs developed by R. Pandharipande and R. P. Thomas [PT2]. Note that the fixed loci indeed consist of isolated reduced points since one leg is always empty [PT2]. There are two cases:

Case 1: q_j is a node of a nodal fibre F_{p_i}. In this case the legs of the elements of $P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}$ are fixed by the partitions $(\lambda_i, \lambda_i^t, \varnothing)$ where $(\cdot)^t$ denotes the dual partition and we use the ordering convention of [ORV]. The generating function is given by the stable pairs vertex[PT2, ORV] are signed Euler chars, whereas for the moment we are doing ordinary Euler chars. $W_{\lambda,\mu,\nu}(q)$ are understood in this way for now.

$$W_{\lambda_i, \lambda_i^t, \varnothing}(q) = \sum_\chi e(P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}) q^\chi = \sum_{\mathcal{Q} \in P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}} w(\mathcal{Q}) q^{l(\mathcal{Q}) + 2|\lambda_i|}, \tag{6}$$

where we use the notation of [PT2].

Case 2: q_j is a node arising from B intersecting a fibre F_{p_i}. In this case the legs of the elements of $P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}$ are fixed by the partitions $(\lambda_i, (1), \varnothing)$. The generating function is given by the stable pairs vertex

$$W_{\lambda_i,(1),\varnothing}(q) = \sum_\chi e(P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}})q^\chi = \sum_{\mathcal{Q} \in P_\chi(\widehat{X}_{q_j}, \mathbf{p})_{\boldsymbol{\lambda}}^{\mathbb{C}^{*3}}} w(\mathcal{Q}) q^{l(\mathcal{Q})+|\lambda_i|+1}. \tag{7}$$

5.3. **Punctured curves.** In this subsection we consider $e(P_\chi(X^\circ, \mathbf{p})_{\boldsymbol{\lambda}})$ for any $\boldsymbol{\lambda} = \{\lambda_i \vdash d_i\}_{i=1}^m$. Recall the homeomorphism (5) and define locally closed subsets

$$P_\chi(N_{B^\circ/X^\circ}, \mathbf{p})_{\boldsymbol{\lambda}} \subset P_\chi(N_{B^\circ/X^\circ}, \mathbf{p}),\ P_\chi(N_{F^\circ_{p_i}/X^\circ}, \mathbf{p})_\lambda \subset P_\chi(N_{F^\circ_{p_i}/X^\circ}, \mathbf{p})$$

with specified "cross-sections" $\boldsymbol{\lambda}$ of the underlying Cohen-Macaulay curves. Since the Cohen-Macaulay curves underlying the stable pairs in $P_\chi(N_{B^\circ/X^\circ}, \mathbf{p})$ have multiplicity 1, this space is just a Hilbert scheme of points on B° [PT3]

$$P_\chi(N_{B^\circ/X^\circ}, \mathbf{p}) \cong \mathrm{Hilb}^n(B^\circ)$$

where

$$\chi = 1 - g(B) + n.$$

Therefore

$$\sum_\chi e(P_\chi(N_{B^\circ/X^\circ}, \mathbf{p}))q^\chi = q^{1-g(B)} \sum_{n=0}^\infty e(\mathrm{Hilb}^n(B^\circ))q^n = \frac{q^{1-g(B)}}{(1-q)^{e(B^\circ)}}. \tag{8}$$

The curves $F^\circ_{p_i}$ coming from a nodal fibre are punctured $\mathbb{P}^1$'s

$$F^\circ_{p_i} \cong \mathbb{P}^1 \setminus \{3\ pts\} \cong \mathbb{C}^* \setminus pt.$$

The curves $F^\circ_{p_i}$ coming from a smooth fibre are smooth elliptic curves E with one puncture. Moreover all normal bundles are in fact *trivial*. Indeed for any fibre F of the elliptic surface $\pi : S \to B$ we have

$$N_{F/X} \cong N_{F/S} \oplus N_{S/X}|_F \cong \mathcal{O}_F(F) \oplus K_S|_F \cong \mathcal{O}_F \oplus \mathcal{O}_F.$$

The last isomorphism follows from $F^2 = 0$ and the formula for the canonical divisor of an elliptic fibration [Mir]

$$K_S = \pi^* D,$$

where D is a divisor of degree $\chi(\mathcal{O}_S) - \chi(\mathcal{O}_B)$ on B. Therefore

$$N_{F^\circ_{p_i}/X^\circ} \cong F^\circ_{p_i} \times \mathbb{C}^2 \cong \begin{cases} (\mathbb{C}^* \setminus pt) \times \mathbb{C}^2 & \text{if } F_{p_i} \text{ is nodal} \\ (E \setminus pt) \times \mathbb{C}^2 & \text{if } F_{p_i} \text{ is smooth.} \end{cases}$$

The generating functions of the trivial rank 2 bundles over $F_{p_i} \setminus q_j \cong \mathbb{C}^*$ (when F_{p_i} is nodal with node q_j) and $F_{p_i} \cong E$ (when F_{p_i} is smooth) are easy. Indeed in the former case case $\mathbb{C}^*$ acts (freely) on itself by multiplication and in the latter case E acts (freely) on itself by addition. These actions lift to free actions on the moduli spaces. We obtain the following result.

Lemma 5.3. *The following equalities hold*

$$\sum_{\chi} e(P_{\chi}(\mathbb{C}^* \times \mathbb{C}^2, \mathbf{p})_{\boldsymbol{\lambda}} q^{\chi} = 1,$$
$$\sum_{\chi} e(P_{\chi}(E \times \mathbb{C}^2, \mathbf{p})_{\boldsymbol{\lambda}} q^{\chi} = 1.$$

Proof. Easy using freeness of the action and $e(\mathbb{C}^*) = e(E) = 0$. □

The required generating functions can be computed by using the restriction argument of Section 3 once more. Let C be any smooth curve and consider the 3-fold $C \times \mathbb{C}^2$. Let $p \in C$ and consider the embedding

$$P_{\chi}(C \times \mathbb{C}^2, d) \hookrightarrow \coprod_{\chi=\chi_1+\chi_2} P_{\chi_1}((C \setminus p) \times \mathbb{C}^2, d) \times P_{\chi_2}(\widehat{C}_p \times \mathbb{C}^2, d),$$

where d denotes the degree of the curve class[3]. The torus $T = \mathbb{C}^{2*}$ is acting on both spaces by scaling of the factors of $\mathbb{C}^2$ and the fixed loci are indexed by partitions $\lambda \vdash d$ as earlier in this section. Again we use the notation $(\cdot)_\lambda$ to indicate that the "cross-section" of the underlying Cohen-Macaulay support curve has been fixed to be the monomial ideal corresponding to λ. We obtain a bijective morphism

$$P_{\chi}(C \times \mathbb{C}^2, d)_\lambda \cong \coprod_{\chi=\chi_1+\chi_2} P_{\chi_1}((C \setminus p) \times \mathbb{C}^2, d)_\lambda \times P_{\chi_2}(\widehat{C}_p \times \mathbb{C}^2, d)_\lambda.$$

Summing over all χ gives the following lemma.

Lemma 5.4.

$$\sum_{\chi} e(P_{\chi}(C \times \mathbb{C}^2, d)_\lambda) q^{\chi} = W_{\lambda,\varnothing,\varnothing}(q) \cdot \sum_{\chi} e(P_{\chi}(\widehat{C}_p \times \mathbb{C}^2, d)) q^{\chi},$$

Proof. To obtain the stable pair vertex use a $\mathbb{C}^{*3}$-action on $\widehat{C}_p \times \mathbb{C}^2$ as in the previous subsection. □

Putting everything together we obtain the desired generating function.

[3]The precise definition of these moduli spaces is as in Section 3: we assume the underlying reduced supports of the stable pairs in each moduli space are C, $C \setminus p$, $\widehat{C}$ respectively and d denotes the multiplicity of the underlying Cohen-Macaulay supports along these curves.

Proposition 5.5. *For each fibre F_{p_i} (nodal or not) we have*

$$\sum_{\chi} e(P_\chi(N_{F^\circ_{p_i}/X^\circ}, \mathbf{p})_\lambda) q^\chi = \frac{1}{W_{\lambda,\varnothing,\varnothing}(q)}.$$

Proof. Combine Lemmas 5.3, 5.4. □

6. Finale (Schur)

We calculate the disconnected generating function (1) first. The connected generating function (2) then follows easily. Denote by $B^\circ \subset B$ the locus of smooth fibres and by $B^{\mathrm{sing}} \subset B$ the locus of singular fibres. Let $\mathrm{Conf}^i(B^\circ)$ be the configuration space of i unordered points on B° and let $N := |B^{\mathrm{sing}}|$. Lemma 2.1 implies

$$\begin{aligned} Z^{P_\bullet}(q,y) &= \sum_\chi \sum_{i=0}^\infty \sum_{i'=0}^N \sum_{d_1,\ldots,d_i \geq 0} \sum_{d'_1,\ldots,d'_{i'} \geq 0} y^{\sum_{a=1}^i d_a + \sum_{a=1}^{i'} d'_a} \cdot e(\mathrm{Conf}^i(B^\circ)) \cdot \binom{N}{i'} \times \\ &\qquad e\big(P_\chi\big(X, \sum_{a=1}^i d_a p_a + \sum_{a=1}^{i'} d'_a p'_a\big)\big) \\ &= \sum_\chi \sum_{i=0}^\infty \sum_{i'=0}^N \sum_{d_1,\ldots,d_i \geq 0} \sum_{d'_1,\ldots,d'_{i'} \geq 0} y^{\sum_{a=1}^i d_a + \sum_{a=1}^{i'} d'_a} \cdot \binom{e(B)-N}{i} \cdot \binom{N}{i'} \times \\ &\qquad e\big(P_\chi\big(X, \sum_{a=1}^i d_a p_a + \sum_{a=1}^{i'} d'_a p'_a\big)\big), \end{aligned}$$

where $p_1, \ldots, p_i$ are any choice of distinct points on B°, $p'_1, \ldots, p'_{i'}$ are any choice of distinct points among B^{sing}, and

$$\binom{n}{k} := (-1)^k \binom{k-n-1}{k}, \tag{9}$$

for $n < 0$. We abbreviate $\mathbf{p} := \sum_{a=1}^i d_a p_a$, $\mathbf{p}' := \sum_{a=1}^{i'} d'_a p'_a$, $\mathbf{d} := \sum_{a=1}^i d_a$, and $\mathbf{d}' := \sum_{a=1}^{i'} d'_a$. Lemma 5.2 gives

$$\begin{aligned} &\sum_{i=0}^\infty \sum_{i'=0}^N \sum_{d_1,\ldots,d_i \geq 0} \sum_{d'_1,\ldots,d'_{i'} \geq 0} \sum_\chi \sum_{\chi_1,\ldots,\chi_i} \sum_{\chi'_1,\ldots,\chi'_{i'}} \sum_{\boldsymbol{\lambda}=\{\lambda_a \vdash d_a\}_{a=1}^i} \sum_{\boldsymbol{\lambda}'=\{\lambda'_a \vdash d'_a\}_{a=1}^{i'}} y^{\mathbf{d}+\mathbf{d}'} \cdot \binom{e(B)-N}{i} \cdot \binom{N}{i'} \times \\ &e(P_\chi(X \setminus \{q_1,\ldots,q_i,q'_1,\ldots,q'_{i'},r_1,\ldots,r_{i'}\}, \mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}) \times \\ &\prod_{a=1}^i e(P_{\chi_a}(\widehat{X}_{q_a}, \mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda}}) \cdot \prod_{a=1}^{i'} e(P_{\chi'_a}(\widehat{X}_{q'_a}, \mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda}'}) \cdot \prod_{a=1}^{i'} e(P_{\chi'_a}(\widehat{X}_{r'_a}, \mathbf{p}+\mathbf{p}')_{\boldsymbol{\lambda}'}), \end{aligned}$$

where $q_1, \ldots, q_i$ denote the nodes arising from $F_{p_1}, \ldots, F_{p_i}$ intersecting the zero section, $q'_1, \ldots, q'_{i'}$ denote the nodes arising from $F_{p'_1}, \ldots, F_{p'_{i'}}$ intersecting the zero section, and $r_1, \ldots, r_{i'}$ are the internal nodes of $F_{p'_1}, \ldots, F_{p'_{i'}}$. The sums $\sum_\chi \sum_{\chi_1,\ldots,\chi_i} \sum_{\chi'_1,\ldots,\chi'_{i'}} \cdots$ can be done using equations (6), (7), and (8)

$$\sum_{i=0}^{\infty}\sum_{i'=0}^{N} \sum_{d_1,\ldots,d_i \geq 0} \sum_{d'_1,\ldots,d'_{i'} \geq 0} \sum_{\boldsymbol{\lambda}=\{\lambda_a \vdash d_a\}_{a=1}^{i}} \sum_{\boldsymbol{\lambda}'=\{\lambda'_a \vdash d'_a\}_{a=1}^{i'}} y^{\mathbf{d}+\mathbf{d}'} \cdot \binom{e(B)-N}{i} \cdot \binom{N}{i'} \times$$

$$\frac{q^{1-g(B)}}{(1-q)^{e(B)-i-i'}} \cdot \prod_{a=1}^{i} \frac{W_{\lambda_a,(1),\varnothing}(q)}{W_{\lambda_a,\varnothing,\varnothing}(q)} \cdot \prod_{a=1}^{i'} \frac{W_{\lambda'_a,\lambda'^t_a,\varnothing}(q) W_{\lambda'_a,(1),\varnothing}(q)}{W_{\lambda'_a,\varnothing,\varnothing}(q)}$$

$$= \frac{q^{1-g(B)}}{(1-q)^{e(B)}} \sum_{i=0}^{\infty}\sum_{i'=0}^{N} \binom{e(B)-N}{i} \cdot \binom{N}{i'} \cdot \left((1-q) \sum_{\lambda} \frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} \right)^{i} \times$$

$$\left((1-q) \sum_{\lambda} \frac{W_{\lambda,\lambda^t,\varnothing}(q) W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} \right)^{i'}.$$

With our convention for binomial coefficients (9), Newton's binomial theorem and the geometric series can be combined in one formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \text{ for all } n \in \mathbb{Z}.$$

Performing the sums $\sum_{i=0}^{\infty} \sum_{j=0}^{N} \cdots$ yields

$$\left(\frac{q}{(1-q)^2}\right)^{1-g(B)} \left((1-q) \sum_{\lambda} \frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} \right)^{e(B)-N} \cdot \left((1-q) \sum_{\lambda} \frac{W_{\lambda,\lambda^t,\varnothing}(q) W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} \right)^{N}.$$

Similarly (but easier) one calculates the generating function $\sum_{d \geq 0} \sum_{\chi} P_{\chi,dF}(X) q^\chi y^d$

$$\left(\sum_{\lambda} y^{|\lambda|} \right)^{e(B)-N} \cdot \left(\sum_{\lambda} W_{\lambda,\lambda^t,\varnothing}(q) y^{|\lambda|} \right)^{N}.$$

We arrive at the following proposition.

Proposition 6.1. *The connected generating series $Z^P(q,y)$ for stable pairs of $X = \mathrm{Tot}(K_S)$ of an elliptic surface $S \to B$ with section of genus $g(B)$ and N 1-nodal fibres is given by*

$$\left(\frac{q}{(1-q)^2}\right)^{1-g(B)} \left(\frac{(1-q) \sum_{\lambda} \frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|}}{\sum_{\lambda} y^{|\lambda|}} \right)^{e(B)-N} \cdot \left(\frac{(1-q) \sum_{\lambda} \frac{W_{\lambda,\lambda^t,\varnothing}(q) W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|}}{\sum_{\lambda} W_{\lambda,\lambda^t,\varnothing}(q) y^{|\lambda|}} \right)^{N},$$

where $W_{\lambda,\mu,\nu}(q)$ is the stable pairs vertex of [PT2].

The various generating functions of vertices appearing in this proposition can be computed. Obviously

$$\sum_{\lambda} y^{|\lambda|} = \prod_{i=1}^{\infty}(1-y^i)^{-1}.$$

More interesting is the following lemma.

Lemma 6.2. *The following identity holdsI have not checked whether the overall $q^{\cdots}$ factors work out. Is the power y^{i-1} in RHS correct?*

$$\sum_{\lambda} W_{\lambda,\lambda^t,\varnothing}(q) y^{|\lambda|} = \prod_{i=1}^{\infty}\Big((1-y^i)\prod_{j=1}^{\infty}(1-y^{i-1}q^j)^j\Big)^{-1}.$$

Proof. [ORV] and [MacD] or exercise in [Sta]. □

Less trivial is the following lemma.

Lemma 6.3. *The following identity holds*

$$(1-q)\sum_{\lambda}\frac{W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} = \prod_{i=1}^{\infty}\frac{1-y^i}{(1-qy^i)(1-q^{-1}y^i)}.$$

Proof. First apply [ORV]. The remaining sum appears in [BO] as pointed out by P. Johnson answering a MathOverflow question. □

The hardest is the following lemmaSTUCK ON THIS. Do we need help from Andrei, Paul, or Ben?.

Lemma 6.4. *The following identity holds*

$$(1-q)\sum_{\lambda}\frac{W_{\lambda,\lambda^t,\varnothing}(q)W_{\lambda,(1),\varnothing}(q)}{W_{\lambda,\varnothing,\varnothing}(q)} y^{|\lambda|} = \left(\prod_{i=1}^{\infty}\frac{1-y^i}{(1-qy^i)(1-q^{-1}y^i)}\right)\cdot\left(\prod_{i=1}^{\infty}\Big((1-y^i)\prod_{j=1}^{\infty}(1-y^{i-1}q^j)^j\Big)^{-1}\right)$$

Proof. ????? □

We obtain a proof of the theorem in the introduction.

Proof of Theorem 1.1. Combine Proposition 6.1 and Lemmas 6.2, 6.3, and 6.4. □

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