DONALDSON-THOMAS INVARIANTS OF LOCAL ELLIPTIC SURFACES VIA THE TOPOLOGICAL VERTEX

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex (some previously known, some new), we derive product formulas for the partition functions. In the special case where the elliptic surface is a K3 surface, we get a new proof of the Katz-Klemm-Vafa formula.

1. INTRODUCTION

2. DEFINITIONS

Let $p : S \to B$ be an elliptic surface over a smooth projective curve B of genus g. We make two assumptions:

- p has a section $B \hookrightarrow S$,
- all singular fibres of π are of Kodaira type I_1, i.e. rational nodal fibres.

We write F_x for the fibre $p^{-1}(\{x\})$, for all closed points $x \in B$. We denote the classes of the fibre and section by $B, F \in H^2(S, \mathbb{Z})$. Interesting examples are the elliptic surfaces $E(n)$, where $B = \mathbb{P}^1$ and S has $12n$ nodal fibres. Then $E(1)$ is the rational elliptic surface and $E(2)$ is the elliptic K3 surface.

Let $\beta \in H_2(S)$ be Poincaré dual to $B + dF$, where $d \geq 0$. Now let $X = \mathrm{Tot}(K_S)$ be the total space of the canonical bundle over S. Then X is a non-compact Calabi-Yau 3-fold. Let

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X \ : \ [Z] = \beta, \ \chi(\mathcal{O}_Z) = n\}$$

denote the Hilbert scheme of proper subschemes $Z \subset X$ with fixed homology class and holomorphic Euler characteristics. K. Behrend associates to any $\mathbb{C}$-scheme of finite type Y a constructible function $\nu : Y \to \mathbb{Z}$ [**?**]. Applied to $\mathrm{Hilb}^{\beta,n}(X)$, the Donaldson-Thomas invariants of X can be defined as[1]

$$\mathrm{DT}_{\beta,n}(X) := \int_{\mathrm{Hilb}^{\beta,n}(X)} \nu \, de := \sum_{k \in \mathbb{Z}} k \, e(\nu^{-1}(\{k\})),$$

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[1] If X is a compact Calabi-Yau 3-fold, R.P. Thomas's original definition of DT invariants is by the degree of the virtual cycle of $\mathrm{Hilb}^{\beta,n}(X)$ [**?**]. Behrend showed that this is the same as $e(\mathrm{Hilb}^{\beta,n}(X), \nu)$ [**?**]. The advantage of the definition by means of virtual cycles is that the construction works relative to a base. This implies deformation invariance of the invariants.

where $e(\cdot)$ denotes topological Euler characteristic. Many of the key properties of DT invariants are already captured by the more classical Euler characteristic version[2]

$$\widehat{\mathrm{DT}}_{\beta,n}(X) := \int_{\mathrm{Hilb}^{\beta,n}(X)} 1 \, de = e(\mathrm{Hilb}^{\beta,n}(X)).$$

For brevity, we define

$$\begin{aligned} \mathrm{Hilb}^{d,n}(X) &:= \mathrm{Hilb}^{B+dF,n}(X), \\ \mathrm{DT}_{\beta,n}(X) &:= \mathrm{DT}_{B+dF,n}(X), \\ \widehat{\mathrm{DT}}_{d,n}(X) &:= \mathrm{DT}_{B+dF,n}(X). \end{aligned}$$

The generating functions of interest are

$$\begin{aligned} \mathrm{DT}(X) &:= \sum_{d\geq 0} \mathrm{DT}_d(X) q^d := \sum_{d\geq 0} \sum_{n\in\mathbb{Z}} \mathrm{DT}_{d,n}(X) p^n q^d, \\ \widehat{\mathrm{DT}}(X) &:= \sum_{d\geq 0} \widehat{\mathrm{DT}}_d(X) q^d := \sum_{d\geq 0} \sum_{n\in\mathbb{Z}} \widehat{\mathrm{DT}}_{d,n}(X) p^n q^d. \end{aligned}$$

Since we are dealing with generating functions and our calculations involve cut-paste methods on the moduli space, it is useful to introduce the following notation. We define

$$[\mathrm{Hilb}^{d,\bullet}(X)] := \sum_{n\in\mathbb{Z}} [\mathrm{Hilb}^{d,n}(X)] p^n,$$

which is an element of $K_0(\mathrm{Var}_{\mathbb{C}})((p))$, i.e. a Laurent series with coefficients in the Grothendieck group of varieties. We also write $\mathrm{Hilb}^{d,\bullet}(X)$ to denote the union of all $\mathrm{Hilb}^{d,n}(X)$. Therefore $\mathrm{Hilb}^{d,\bullet}(X)$ is a $\mathbb{C}$-scheme which is locally of finite type.

3. Reduction to strata with fixed curve class

The scaling action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1 \, de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1 \, de.$$

Recall that

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1 \, de = \sum_{n\in\mathbb{Z}} p^n \int_{\mathrm{Hilb}^{d,n}(X)} 1 \, de \in \mathbb{Z}((p)).$$

We revisit the $\mathbb{C}^*$-fixed point locus in detail in the next section.

Denote by $\mathrm{Sym}^d(B)$ the dth symmetric product of B. Recall that we have projections

$$X \xrightarrow{\pi} S \xrightarrow{p} B.$$

A subscheme Z of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ always contains the zero section $B \subset S \subset X$. We can remove it and consider the scheme $\overline{Z \setminus B}$. There exists a morphism

Can/do we want to write this easier?

$$\begin{aligned} \rho_d : \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*} &\longrightarrow \mathrm{Sym}^d(B), \\ Z &\mapsto \mathrm{supp}(p_* \pi_* \mathcal{O}_{\overline{Z\setminus B}}), \end{aligned}$$

[2]From the point of view of [?, ?]: there are two natural integration maps on the semi-classical Hall-algebra. One corresponds to weighing by the Behrend function and the other to weighing by the "trivial" constructible function which is constant equal to 1.

where $\mathrm{supp}(\cdot)$ denotes the scheme theoretic support, which is a divisor on B. We obtain

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\,de = \int_{\mathrm{Sym}^d(B)} \rho_{d*}(1)\,de,$$

where $f_d := \rho_{d*}(1)$ is a constructible function on $\mathrm{Sym}^d(B)$. Its value at a closed point $\sum_i a_i x_i$ is

$$\int_{\rho_d^{-1}\left(\sum_i a_i x_i\right)} 1\,de.$$

We are interested in the calculation of

$$\widehat{\mathrm{DT}}(X) = \sum_{d\geq 0} \widehat{\mathrm{DT}}_d(X) q^d = \sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d\,de.$$

We prove that the constructible function $f_d : \mathrm{Sym}^d(B) \to \mathbb{Z}(\!(p)\!)$ has two multiplicative properties. The first is one is described as follows. Denote by $B^{\mathrm{sm}} \subset B$ the open subset over which the fibres F_x are smooth and by B^{sing} the N points over which the fibres F_x are singular.

Lemma 1. *Let $d_1, d_2 \geq 0$ be such that $d_1 + d_2 = d$. Then there are constructible functions*

$$g_{d_1} : \mathrm{Sym}^{d_1}(B^{\mathrm{sm}}) \longrightarrow \mathbb{Z}(\!(p)\!)$$
$$h_{d_2} : \mathrm{Sym}^{d_2}(B^{\mathrm{sing}}) \longrightarrow \mathbb{Z}(\!(p)\!),$$

such that for any $\sum_i a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\sum_j b_j y_j \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$

Forgot: points which can are off the curve.

$$f_d\Big(\sum_i a_i x_i + \sum_j b_j y_j\Big) = g_{d_1}\Big(\sum_i a_i x_i\Big) \cdot h_{d_2}\Big(\sum_j b_j y_j\Big).$$

We prove this lemma in **??**. The following product formula is an immediate consequence of this result

$$\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d\,de = \Big(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sm}})} g_d\,de\Big) \cdot \Big(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sing}})} h_d\,de\Big). \tag{1}$$

The constructible functions $g_d, h_d : \mathrm{Sym}^d(B) \to \mathbb{Z}$ satisfy further multiplicative properties:

Lemma 2. *There exist functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ and $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$, such that $g(0) = 1$, $h(0) = 1$, and*

$$g_d\Big(\sum_i a_i x_i\Big) = \prod_i g(a_i),$$
$$h_d\Big(\sum_j b_j y_j\Big) = \prod_j h(b_j),$$

for all $\sum_i a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$ and $\sum_j b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$.

We prove this lemma in **??**. Together with Lemma **??** from the appendix, this lemma and (1) gives

$$\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d\,de = \Big(\sum_{a=0}^{\infty} g(a) q^a\Big)^{e(B)-N} \cdot \Big(\sum_{b=0}^{\infty} h(b) q^b\Big)^N.$$

We want to prove Lemmas **??**, **??** and find formulae for $g(a)$, $h(b)$. This requires a better understanding of the strata

$$\rho_d^{-1}\Big(\sum_i a_i x_i + \sum_j b_j y_j\Big) \subset \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*},$$

for all $\sum_i a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\sum_j b_j y_j \in \mathrm{Sym}^{d_1}(B^{\mathrm{sing}})$ with $d_1 + d_2 = d$. We start with a more careful study of the $\mathbb{C}^*$-fixed locus.

4. The $\mathbb{C}^*$-fixed locus

As we already noted, the scaling action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore, we only need to restrict attention to $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$.

Using the map $\pi : X \to S$, a quasi-coherent sheaf on X can be viewed as a quasi-coherent sheaf $\mathcal{F}$ on S together with a morphism $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$. A $\mathbb{C}^*$-equivariant structure on $\mathcal{F}$ translates into a $\mathbb{Z}$-grading

$$\pi_* \mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k,$$

such that $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$ is graded, i.e.

$$\mathcal{F}_k \otimes K_S^{-1} \longrightarrow \mathcal{F}_{k-1}.$$

The structure sheaf $\mathcal{O}_X$ corresponds to

$$\pi_* \mathcal{O}_X = \bigoplus_{k=0}^{\infty} K_S^{-k}.$$

Therefore a $\mathbb{C}^*$-fixed morphism $\mathcal{F} \to \mathcal{O}_X$ corresponds to a graded sheaf $\mathcal{F}$ as above together with maps

$$\begin{array}{ccccccccc}
\cdots & \mathcal{F}_1 & \oplus & \mathcal{F}_0 & \oplus & \mathcal{F}_{-1} & \oplus & \mathcal{F}_{-2} & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & \mathcal{O}_S & \oplus & K_S^{-1} & \oplus & K_S^{-2} & \cdots
\end{array}$$

It is useful to re-define $\mathcal{G}_k := \mathcal{F}_{-k} \otimes K_S^k$. Then the data of a $\mathbb{C}^*$-fixed morphism $\mathcal{F} \to \mathcal{O}_X$ is equivalent to the following data:

- coherent sheaves $\{\mathcal{G}_k\}_{k \in \mathbb{Z}}$ on S,
- morphisms $\{\mathcal{G}_k \to \mathcal{G}_{k+1}\}_{k \in \mathbb{Z}}$,
- morphisms $\mathcal{G}_k \to \mathcal{O}_S$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots & \mathcal{G}_{-1} \arrow[r] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G}_{-1} \arrow[r] \arrow[d] & \mathcal{G}_{-2} \arrow[r] \arrow[d] & \cdots \\
 & & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \cdots
\end{tikzcd}$$

In the case of interest to us $\mathcal{G} \to \mathcal{O}_X$ is an ideal sheaf $I_Z \hookrightarrow \mathcal{O}_X$ cutting out $Z \subset X$. In the above language, this means $\mathcal{G}_k = 0$ for $k < 0$, the morphisms $\mathcal{G}_k \to \mathcal{O}_S$ are injective (hence $\mathcal{G}_k = I_{Z_k}$ is an ideal sheaf cutting out $Z_k \subset S$), and the morphisms $\mathcal{G}_k \to \mathcal{G}_{k+1}$ are injective (hence $I_{Z_k} \subset I_{Z_{k+1}}$, i.e. $Z_k \supset Z_{k+1}$). We conclude:

Lemma 3. *A closed point Z of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ corresponds to a finite nesting of closed subschemes of S*

$$Z_0 \supset Z_1 \supset \cdots \supset Z_l,$$

for some $l \geq 0$, such such that

$$\sum_{k=0}^{l} [Z_k] = B + dF \in H_2(S).$$

In the above lemma, each Z_k contains a maximal Cohen-Macaulay subcurve D_k such that $Z_k \backslash D_k$ is 0-dimensional. For $k = 0$, D_0 is the scheme-theoretic union of the section B and thickenings of certain distinct fibres $F_{x_1}, \ldots, F_{x_n}$. Denoting the orders of thickenings by $\lambda_1^{(1)}, \ldots, \lambda_1^{(n)} > 0$, we obtain[3]

$$D_0 = B \cup \lambda_0^{(1)} F_{x_1} \cup \cdots \cup \lambda_0^{(n)} F_{x_n}.$$

This statement follows from Corollary 8 of the appendix. Next, for all $i = 1, \ldots, n$ and $k \geq 1$, there are $\lambda_k^{(i)} \leq \lambda_{k-1}^{(i)}$ such that

$$D_k = \lambda_k^{(1)} F_{x_1} \cup \cdots \cup \lambda_k^{(n)} F_{x_n}.$$

We conclude:

Lemma 4. *To each closed point Z of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ correspond distinct closed points $x_1, \ldots, x_n \in B$ and (finite) 2D partitions $\lambda^{(1)}, \ldots, \lambda^{(n)}$ such that*

$$\sum_{i=1}^{n} |\lambda^{(i)}| = d.$$

The maximal Cohen-Macaulay subcurve of Z is given by the scheme-theoretic union of the zero section B and the schemes with ideal sheaves

$$\bigoplus_{k=0}^{\infty} \mathcal{O}_S(-\lambda_k^{(i)} F_{x_i}) \otimes K_S^{-k}, \tag{2}$$

for all $i = 1, \ldots, n$.

Appendix A. Odds and Ends

A.1. **Weighted Euler characteristics of symmetric products.** In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

Lemma 5. *Let S be a scheme of finite type over $\mathbb{C}$ and let $e(S)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((Q))$ be any function with $g(0) = 1$. Let $f_d : \mathrm{Sym}^d(S) \to \mathbb{Z}((Q))$ be the constructible function defined by $f_d(\sum_i a_i x_i) = \prod_i g(a_i)$. Then*

$$\sum_{d=0}^{\infty} u^d \int_{\mathrm{Sym}^d(S)} f_d de = \left(\sum_{a=0}^{\infty} g(a) u^a \right)^{e(S)}.$$

[3] For any reduced curve C on a surface S with ideal sheaf $I_C \subset \mathcal{O}_S$ and $d > 0$, we denote by dC the scheme defined by the ideal sheaf $I_C^d \subset \mathcal{O}_S$.

Remark 6. In the special case where $g = f_d \equiv 1$, the lemma recovers Macdonald's formula: $\sum_{d=1}^{\infty} e(\mathrm{Sym}^d(S))u^d = (1-u)^{-e(S)}$.

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties definited by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure []. For convenience's sake, we provide a direct proof here.

Proof. The dth symmetric product admits a stratification with strata labelled by partitions of d. Associated to any partition of d is a unique tuple $(m_1, m_2, \dots)$ of non-negative integers with $\sum_{j=1}^{\infty} jm_j = d$. The stratum labelled by $(m_1, m_2, \dots)$ parameterizes collections of points where there are m_j points of multiplicity j. The full stratification is given by:

$$\mathrm{Sym}^d(S) = \bigsqcup_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} jm_j = d}} \left\{ \left(\prod_{j=1}^{\infty} S^{m_j} \right) - \Delta \right\} / \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, S^0 is a point, Δ is the large diagonal, and σ_m is the mth symmetric group. Note that the function f_d is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

For schemes over $\mathbb{C}$, topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\mathrm{Sym}^d(S)} f_d \, de = \sum_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} jm_j = d}} \left(\prod_{j=1}^{\infty} g(j)^{m_j} \right) \frac{e(S^{\sum_j m_j} - \Delta)}{m_1! \, m_2! \, m_3! \dots}.$$

For any natural number N, the projection $S^N - \Delta \to S^{N-1} - \Delta$ has fibers of the form $S - \{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(S) - (N-1)$ and consequently, $e(S^N - \Delta) = (e(S) - (N-1)) \cdot e(S^{N-1} - \Delta)$. Thus by induction, we find $e(S^N - \Delta) = e(S) \cdot (e(S) - 1) \cdots (e(S) - (N-1))$ and so

$$\frac{e(S^{\sum_j m_j} - \Delta)}{m_1! \, m_2! \, m_3! \cdots} = \binom{e(S)}{m_1, m_2, m_3, \cdots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\begin{aligned} \sum_{d=0}^{\infty} \int_{\mathrm{Sym}^d(S)} f_d \, de &= \sum_{(m_1, m_2, \dots)} \prod_{j=1}^{\infty} \left(g(j)u^j \right)^{m_j} \binom{e(S)}{m_1, m_2, m_3, \dots} \\ &= \left(1 + \sum_{j=1}^{\infty} g(j)u^j \right)^{e(S)} \end{aligned}$$

which proves the lemma. □

A.2. **Some geometry of curves on elliptic surfaces.** In this subsection we prove the following lemma and corollary, which will tell us what is the reduced support of all curves in the class $\beta = B + dF$.

Lemma 7. *For any line bundle ϵ on B, multiplication by the canonical section of $\mathcal{O}(B)$ induces an isomorphism*

$$H^0(S,\pi^*(\epsilon)(B)) \cong H^0(S,\pi^*(\epsilon)).$$

Corollary 8. *Let $\beta = B + dF \in H_2(S)$. Then the Chow variety of curves in the class β is isomorphic to $\mathrm{Sym}^d(B)$ where a point $\sum_i d_i x_i \in \mathrm{Sym}^d(B)$ corresponds to the curve $B + \sum_i d_i F_{x_i}$.*

Proof. The corollary follows immediately from the lemma since the Chow variety is the space of effective divisors and the lemma implies that any effective divisor in the class β is a union of the section B with an effective divisor pulled by from the base.

To prove lemma 7 we proceed as follows. For any line bundle δ on B, the Leray spectral sequence yields the short exact sequence:

$$0 \to H^0(B, \delta \otimes R^1\pi_*\mathcal{O}) \to H^1(S,\pi^*\delta) \xrightarrow{\alpha} H^1(B,\delta) \to 0,$$

in particular, α is a surjection.

Then the long exact cohomology sequence associated to

$$0 \to \pi^*\delta \otimes \mathcal{O}(-B) \to \pi^*\delta \to \mathcal{O}_B \otimes \pi^*\delta \to 0$$

is

$$\cdots \to H^1(S,\pi^*(\delta)) \xrightarrow{\alpha} H^1(B,\delta) \to H^2(S,\pi^*\delta\otimes\mathcal{O}(-B)) \to H^2(S,\pi^*\delta) \to 0,$$

and since α is a surjection, we get an isomorphism of the last two terms. We apply Serre duality to that isomorphism and we use the fact that $K_S = \pi^*(K_B \otimes L)$ where $L = (R\pi_*\mathcal{O}_S)^\vee$ [**?**, prop?] to obtain

$$H^0(S,\pi^*(\delta^{-1}\otimes K_B\otimes L)(B)) \cong H^0(S,\pi^*(\delta^{-1}\otimes K_B \otimes L)).$$

Letting $\delta = K_B \otimes L \otimes \epsilon^{-1}$, the lemma is proved. □

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