# DONALDSON-THOMAS INVARIANTS OF LOCAL ELLIPTIC SURFACES VIA THE TOPOLOGICAL VERTEX

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex (some previously known, some new), we derive product formulas for the partition functions. In the special case where the elliptic surface is a K3 surface, we get a new proof of the Katz-Klemm-Vafa formula.

## 1. INTRODUCTION

**Theorem 1.**

$$\widehat{\mathrm{DT}}(X) = \frac{1}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})^{e(B)}} \left(\sum_{\lambda} \frac{\mathsf{V}_{\lambda,\square,\varnothing}(p)}{\mathsf{V}_{\lambda,\varnothing,\varnothing}(p)} q^{|\lambda|}\right)^{e(B)-N} \left(\sum_{\mu} \frac{\mathsf{V}_{\mu,\square,\varnothing}(p)\mathsf{V}_{\mu,\mu',\varnothing}(p)p^{\binom{\mu}{2}+|\mu|}}{\mathsf{V}_{\mu,\varnothing,\varnothing}(p)} q^{|\mu|}\right)^{N}.$$

$$\widehat{\mathrm{DT}}^{\mathrm{conn}}(X) = \frac{1}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})^{e(B)}} \left(\frac{\sum_{\lambda} \frac{\mathsf{V}_{\lambda,\square,\varnothing}(p)}{\mathsf{V}_{\lambda,\varnothing,\varnothing}(p)} q^{|\lambda|}}{\sum_{\lambda} q^{|\lambda|}}\right)^{e(B)-N} \left(\frac{\sum_{\mu} \frac{\mathsf{V}_{\mu,\square,\varnothing}(p)\mathsf{V}_{\mu,\mu',\varnothing}(p)p^{\binom{\mu}{2}+|\mu|}}{\mathsf{V}_{\mu,\varnothing,\varnothing}(p)} q^{|\mu|}}{\sum_{\mu} \mathsf{V}_{\mu,\mu',\varnothing}(p)p^{\binom{\mu}{2}+|\mu|}q^{|\mu|}}\right)^{N}.$$

**Corollary 2** (Bryan-Kool-Young)**.**

$$\widehat{\mathrm{DT}}^{\mathrm{conn}}(X) = \frac{1}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})^{2}} \prod_{k=1}^{\infty} \frac{1}{(1-pq^k)^2(1-p^{-1}q^k)^2(1-q)^{12N-4}}.$$

## 2. DEFINITIONS

Let $p : S \to B$ be an elliptic surface over a smooth projective curve $B$ of genus $g$. We make two assumptions:

- $p$ has a *unique* section $B \hookrightarrow S$,
- all singular fibres of $\pi$ are of Kodaira type $I_1$, i.e. rational nodal fibres.

Drop uniqueness so generating function becomes product with contribution from multiple sections?

We write $F_x$ for the fibre $p^{-1}(\{x\})$, for all closed points $x \in B$. We denote the classes of the fibre and section by $B, F \in H^2(S,\mathbb{Z})$. Interesting examples are the elliptic surfaces $E(n)$, where $B = \mathbb{P}^1$ and $S$ has $12n$ nodal fibres. Then $E(1)$ is the rational elliptic surface and $E(2)$ is the elliptic K3 surface.

Let $\beta \in H_2(S)$ be Poincaré dual to $B + dF$, where $d \geq 0$. Now let $X = \mathrm{Tot}(K_S)$ be the total space of the canonical bundle over $S$. Then $X$ is a non-compact Calabi-Yau 3-fold. Let

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X \ : \ [Z] = \beta,\ \chi(\mathcal{O}_Z) = n\}$$

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denote the Hilbert scheme of proper subschemes $Z \subset X$ with fixed homology class and holomorphic Euler characteristics. K. Behrend associates to any $\mathbb{C}$-scheme of finite type $Y$ a constructible function $\nu : Y \to \mathbb{Z}$ [?]. Applied to $\mathrm{Hilb}^{\beta,n}(X)$, the Donaldson-Thomas invariants of $X$ can be defined as[1]

$$\mathrm{DT}_{\beta,n}(X) := \int_{\mathrm{Hilb}^{\beta,n}(X)} \nu \; de := \sum_{k\in\mathbb{Z}} k \; e(\nu^{-1}(\{k\})),$$

where $e(\cdot)$ denotes topological Euler characteristic. Many of the key properties of DT invariants are already captured by the more classical Euler characteristic version[2]

$$\widehat{\mathrm{DT}}_{\beta,n}(X) := \int_{\mathrm{Hilb}^{\beta,n}(X)} 1 \; de = e(\mathrm{Hilb}^{\beta,n}(X)).$$

For brevity, we define

$$\begin{aligned}
\mathrm{Hilb}^{d,n}(X) &:= \mathrm{Hilb}^{B+dF,n}(X),\\
\mathrm{DT}_{\beta,n}(X) &:= \mathrm{DT}_{B+dF,n}(X),\\
\widehat{\mathrm{DT}}_{d,n}(X) &:= \mathrm{DT}_{B+dF,n}(X).
\end{aligned}$$

The generating functions of interest are

$$\begin{aligned}
\mathrm{DT}(X) &:= \sum_{d\geq 0} \mathrm{DT}_d(X) q^d := \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \mathrm{DT}_{d,n}(X) p^n q^d,\\
\widehat{\mathrm{DT}}(X) &:= \sum_{d\geq 0} \widehat{\mathrm{DT}}_d(X) q^d := \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \widehat{\mathrm{DT}}_{d,n}(X) p^n q^d.
\end{aligned}$$

Since we are dealing with generating functions and our calculations involve cut-paste methods on the moduli space, it is useful to introduce the following notation. We define

$$[\mathrm{Hilb}^{d,\bullet}(X)] := \sum_{n\in\mathbb{Z}} [\mathrm{Hilb}^{d,n}(X)] p^n,$$

which is an element of $K_0(\mathrm{Var}_{\mathbb{C}})(\!(p)\!)$, i.e. a Laurent series with coefficients in the Grothendieck group of varieties. We also write $\mathrm{Hilb}^{d,\bullet}(X)$ to denote the union of all $\mathrm{Hilb}^{d,n}(X)$. Therefore $\mathrm{Hilb}^{d,\bullet}(X)$ is a $\mathbb{C}$-scheme which is locally of finite type.

## 3. Push-forward to the symmetric product

The scaling action of $\mathbb{C}^*$ on the fibres of $X$ lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1 \; de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1 \; de.$$

Recall that

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1 \; de = \sum_{n\in\mathbb{Z}} p^n \int_{\mathrm{Hilb}^{d,n}(X)} 1 \; de \in \mathbb{Z}(\!(p)\!).$$

We revisit the $\mathbb{C}^*$-fixed point locus in detail in the next section.

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[1] If $X$ is a compact Calabi-Yau 3-fold, R.P. Thomas's original definition of DT invariants is by the degree of the virtual cycle of $\mathrm{Hilb}^{\beta,n}(X)$ [?]. Behrend showed that this is the same as $e(\mathrm{Hilb}^{\beta,n}(X), \nu)$ [?]. The advantage of the definition by means of virtual cycles is that the construction works relative to a base. This implies deformation invariance of the invariants.

[2] From the point of view of [?, ?], there are two natural integration maps on the semi-classical Hall-algebra. One corresponds to weighing by the Behrend function and the other to weighing by the "trivial" constructible function which is constant equal to 1.

Denote by $\mathrm{Sym}^d(B)$ the $d$th symmetric product of $B$. Recall that we have projections

$$X \xrightarrow{\pi} S \xrightarrow{p} B.$$

A subscheme $Z$ of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ always contains the zero section $B \subset S \subset X$. We can remove it and consider the scheme $\overline{Z \setminus B}$. There exists a morphism

$$\begin{aligned} \rho_d : \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*} &\longrightarrow \mathrm{Sym}^d(B), \\ Z &\mapsto \mathrm{supp}(p_*\pi_*\mathcal{O}_{\overline{Z\setminus B}}), \end{aligned}$$

where $\mathrm{supp}(\cdot)$ denotes the scheme theoretic support, which is a divisor on $B$. We obtain

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1 \; de = \int_{\mathrm{Sym}^d(B)} \rho_{d*}(1) \; de,$$

where $f_d := \rho_{d*}(1)$ is a constructible function on $\mathrm{Sym}^d(B)$. Its value at a closed point $\sum_i a_i x_i$ is

$$\int_{\rho_d^{-1}\left(\sum_i a_i x_i\right)} 1 \; de.$$

We are interested in the calculation of

$$\widehat{\mathrm{DT}}(X) = \sum_{d \geq 0} \widehat{\mathrm{DT}}_d(X) q^d = \sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d \; de.$$

We prove that the constructible function $f_d : \mathrm{Sym}^d(B) \to \mathbb{Z}(\!(p)\!)$ has two multiplicative properties. The first one is described as follows. Denote by $B^{\mathrm{sm}} \subset B$ the open subset over which the fibres $F_x$ are smooth and by $B^{\mathrm{sing}}$ the $N$ points over which the fibres $F_x$ are singular. We can consider the restrictions of $f_d$ to $\mathrm{Sym}^d(B^{\mathrm{sm}}) \subset \mathrm{Sym}^d(B)$ and $\mathrm{Sym}^d(B^{\mathrm{sing}}) \subset \mathrm{Sym}^d(B)$. Denote by $M(q)$ the MacMahon function.

**Proposition 3.** *Let $d_1, d_2 \geq 0$ be such that $d_1 + d_2 = d$. Then*

$$f_d(\mathfrak{a} + \mathfrak{b}) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \cdot f_{d_1}(\mathfrak{a}) \cdot f_{d_2}(\mathfrak{b}),$$

*for any $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$.*

We prove this proposition in Section 5.3. The following product formula is an immediate consequence of this result

$$\begin{aligned} &\sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d \; de \\ &= \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \Big( \sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sm}})} f_d \; de \Big) \cdot \Big( \sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sing}})} f_d \; de \Big). \end{aligned} \tag{1}$$

The restricted constructible functions $f_d : \mathrm{Sym}^d(B^{\mathrm{sm}}) \to \mathbb{Z}$ and $f_d : \mathrm{Sym}^d(B^{\mathrm{sing}}) \to \mathbb{Z}$ satisfy further multiplicative properties:

**Proposition 4.** *There exist functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$ and $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$, such that $g(0) = 1$, $h(0) = 1$, and*

$$f_d(\mathfrak{a}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_i g(a_i),$$
$$f_d(\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_j h(b_j),$$

*for all $\mathfrak{a} = \sum_i a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$ and $\mathfrak{b} = \sum_j b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$.*

We prove this proposition in 5.3. Together with Lemma 15 from the appendix, Proposition 4 and (1) gives

$$\sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d \, de = \frac{M(p)^N}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \Big( \sum_{a=0}^{\infty} g(a) q^a \Big)^{e(B)-N} \cdot \Big( \sum_{b=0}^{\infty} h(b) q^b \Big)^N.$$

We want to prove Lemmas 3, 4 and find formulae for $g(a)$, $h(b)$. This requires a better understanding of the strata

$$\rho_d^{-1}(\mathfrak{a} + \mathfrak{b}) \subset \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*},$$

for all $\mathfrak{a} = \sum_i a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} = \sum_j b_j y_j \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$ with $d_1 + d_2 = d$. We start with a more careful study of the $\mathbb{C}^*$-fixed locus.

## 4. The $\mathbb{C}^*$-fixed locus

As we already noted, the scaling action of $\mathbb{C}^*$ on the fibres of $X$ lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore, we only need to restrict attention to $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$.

Using the map $\pi : X \to S$, a quasi-coherent sheaf on $X$ can be viewed as a quasi-coherent sheaf $\mathcal{F}$ on $S$ together with a morphism $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$. A $\mathbb{C}^*$-equivariant structure on $\mathcal{F}$ translates into a $\mathbb{Z}$-grading

$$\pi_* \mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k,$$

such that $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$ is graded, i.e.

$$\mathcal{F}_k \otimes K_S^{-1} \longrightarrow \mathcal{F}_{k-1}.$$

The structure sheaf $\mathcal{O}_X$ corresponds to

$$\pi_* \mathcal{O}_X = \bigoplus_{k=0}^{\infty} K_S^{-k}.$$

Therefore a $\mathbb{C}^*$-fixed morphism $\mathcal{F} \to \mathcal{O}_X$ corresponds to a graded sheaf $\mathcal{F}$ as above together with maps

$$\begin{array}{ccccccccccc}
\cdots & \mathcal{F}_1 & \oplus & \mathcal{F}_0 & \oplus & \mathcal{F}_{-1} & \oplus & \mathcal{F}_{-2} & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & \mathcal{O}_S & \oplus & K_S^{-1} & \oplus & K_S^{-2} & \cdots
\end{array}$$

It is useful to re-define $\mathcal{G}_k := \mathcal{F}_{-k} \otimes K_S^k$. Then the data of a $\mathbb{C}^*$-fixed morphism $\mathcal{F} \to \mathcal{O}_X$ is equivalent to the following data:

- coherent sheaves $\{\mathcal{G}_k\}_{k \in \mathbb{Z}}$ on $S$,

- morphisms $\{\mathcal{G}_k \to \mathcal{G}_{k+1}\}_{k\in\mathbb{Z}}$,
- morphisms $\mathcal{G}_k \to \mathcal{O}_S$ such that the following diagram commutes:

$$\begin{tikzcd} \cdots & \mathcal{G}_{-1} \arrow[r] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[r] \arrow[d] & \cdots \\ & & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \cdots \end{tikzcd}$$

In the case of interest to us $\mathcal{G} \to \mathcal{O}_X$ is an ideal sheaf $I_Z \hookrightarrow \mathcal{O}_X$ cutting out $Z \subset X$. In the above language, this means $\mathcal{G}_k = 0$ for $k < 0$, the morphisms $\mathcal{G}_k \to \mathcal{O}_S$ are injective (hence $\mathcal{G}_k = I_{Z_k}$ is an ideal sheaf cutting out $Z_k \subset S$), and the morphisms $\mathcal{G}_k \to \mathcal{G}_{k+1}$ are injective (hence $I_{Z_k} \subset I_{Z_{k+1}}$, i.e. $Z_k \supset Z_{k+1}$). We conclude:

**Proposition 5.** *A closed point $Z$ of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ corresponds to a finite nesting of closed subschemes of $S$*

$$Z_0 \supset Z_1 \supset \cdots \supset Z_l,$$

*for some $l \geq 0$, such such that*

$$\sum_{k=0}^{l} [Z_k] = B + dF \in H_2(S).$$

In the above proposition, each $Z_k$ contains a maximal Cohen-Macaulay subcurve $D_k$ such that $Z_k \setminus D_k$ is 0-dimensional. For $k = 0$, $D_0$ is the scheme-theoretic union of the section $B$ and thickenings of certain distinct fibres $F_{x_1}, \ldots, F_{x_n}$. Denoting the orders of thickenings by $\lambda_1^{(1)}, \ldots, \lambda_1^{(n)} > 0$, we obtain[3]

$$D_0 = B \cup \lambda_0^{(1)} F_{x_1} \cup \cdots \cup \lambda_0^{(n)} F_{x_n}.$$

This statement follows from Corollary 18 of the appendix. Next, for all $i = 1, \ldots, n$ and $k \geq 1$, there are $\lambda_k^{(i)} \leq \lambda_{k-1}^{(i)}$ such that

$$D_k = \lambda_k^{(1)} F_{x_1} \cup \cdots \cup \lambda_k^{(n)} F_{x_n}.$$

We conclude:

**Proposition 6.** *To each closed point $Z$ of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ correspond distinct closed points $x_1, \ldots, x_n \in B$ for some $n$ and (finite) 2D partitions $\lambda^{(1)}, \ldots, \lambda^{(n)}$ such that*

$$\sum_{i=1}^{n} |\lambda^{(i)}| = d.$$

*The maximal Cohen-Macaulay subcurve of $Z$ is given by the scheme-theoretic union of the zero section $B$ and the schemes with ideal sheaves*

$$\bigoplus_{k=0}^{\infty} \mathcal{O}_S(-\lambda_k^{(i)} F_{x_i}) \otimes K_S^{-k}, \tag{2}$$

*for all $i = 1, \ldots, n$.*

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[3]For any reduced curve $C$ on a surface $S$ with ideal sheaf $I_C \subset \mathcal{O}_S$ and $d > 0$, we denote by $dC$ the scheme defined by the ideal sheaf $I_C^d \subset \mathcal{O}_S$.

In the previous section, we considered the stratum

$$\rho_d^{-1}(\mathfrak{a}+\mathfrak{b}) \subset \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*},$$

for any $\mathfrak{a} = \sum_{i=1}^m a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} = \sum_{j=1}^n b_j y_j \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$ satisfying $d_1 + d_2 = d$. By the previous lemma, it has a further stratification into locally closed subsets

$$\coprod_{\substack{\lambda^{(1)} \vdash a_1 \\ \cdots \\ \lambda^{(m)} \vdash a_m}} \coprod_{\substack{\mu^{(1)} \vdash b_1 \\ \cdots \\ \mu^{(m)} \vdash b_m}} \Sigma(x_1, \ldots, x_m, y_1, \ldots, y_n, \lambda^{(1)}, \ldots, \lambda^{(m)}, \mu^{(1)}, \ldots, \mu^{(n)}). \tag{3}$$

We abbreviate a stratum on the RHS by $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$. It should be viewed as the stratum of $Z \in \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ for which the maximal CM subcurve $C \subset Z$ has been fixed by the data $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$. We are after the Euler characteristics of these strata. We will see this Euler characteristic does *not* depend on the exact location of the points $x_i$ and $y_j$, but only on their number $m$ and $n$ and the partitions $\lambda^{(i)}$ and $\mu^{(j)}$.

## 5. Restriction to formal neighbourhoods

In the previous two sections, we reduced our consideration to the stratum $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$ containing $Z \in \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ for which the maximal CM subcurve $C \subset Z$ is fixed by the data $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$. In this section, we further break down this stratum by cutting it up in pieces covered by formal neighbourhoods. For notational simplicity, we first consider the case where the base point is

$$ax + by \in \mathrm{Sym}^2(B),$$

with $x \in B^{\mathrm{sm}}$ and $y \in B^{\mathrm{sing}}$ and compute the Euler characteristic of $\Sigma(x, y, \lambda, \mu)$. Once this is eastablished, it is not hard to calculate the Euler characteristic of any stratum $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$. This leads to a proof of Propositions 3, 4, and a geometric characterization of the functions $g(a)$, $h(b)$ of Section 3.

5.1. **Fpqc cover.** The idea is to use an appropriate cover of $X$ and calculate on pieces of the cover. We first give a complex analytic definition of the cover to aid the intuition and then give the actual "algebro-geometric cover":

- The reduced support $B \cup F_x \cup F_y$ has three singular points: $x, y, F_y^{\mathrm{sing}}$. We take small open balls $U_1, U_2, U_3$ around these points.
- Consider the punctured space $X \setminus \{x, y, F_y^{\mathrm{sing}}\}$ and the following three disjoint closed curves in it $B \setminus \{x,y\}$, $F_x \setminus \{x\}$, $F_y \setminus \{y, F_y^{\mathrm{sing}}\}$. We take small tubular neighbourhoods $V_1, V_2, V_3$ of each of these curves. Note how the union of the $U_i$ and $V_i$ covers the reduced support $B \cup F_x \cup F_y$.
- Finally we take $W = X \setminus (B \cup F_x \cup F_y)$.

In order to work in algebraic geometry, we let $U_1$ be the formal neighbourhood of $\{x\}$ in $X$. If $R$ is the local ring at $x$, then we mean by $U_1$ the (non-noetherian) scheme

$$\mathrm{Spec} \varprojlim R/\mathfrak{m}^n$$

and *not* the formal scheme $\mathrm{Spf} \varprojlim R/\mathfrak{m}^n$. Similarly, we let $U_2$ be the formal neighbourhood of $\{y\}$ on $X$ and $U_3$ the formal neighbourhood of $F_y^{\mathrm{sing}}$ in $X$. Note that

$$U_i \cong \mathrm{Spec}\, \mathbb{C}[\![x_1, x_2, x_3]\!],$$

for all $i = 1, 2, 3$. Even though the $U_i$ are non-noetherian, the maps $U_i \to X$ are fpqc morphisms [**?**], so can be used as part of a cover. Flatness follows from the fact that localization and formal completion are exact operations [**?**, Cor. 3.6, Prop. 10.12, Prop. 10.13].

Next, in the punctured space $X \setminus \{x, y, F_y^{\mathrm{sing}}\}$, we let $V_1$ be the formal neighbourhood of $B\setminus\{x, y\}$. Similarly, let $V_2$ be the formal neighbourhood of $F_x\setminus\{x\}$ in $X\setminus\{x, y, F_y^{\mathrm{sing}}\}$ and $V_3$ the formal neighbourhood of $F_y \setminus \{x, F_y^{\mathrm{sing}}\}$ in $X \setminus \{x, y, F_y^{\mathrm{sing}}\}$. Again, formal neighbourhoods are meant in the sense of taking Spec of of inverse limits as above. Finally, $W = X \setminus (B \cup F_x \cup F_y)$. Then

$$\mathfrak{U} = \{U_1 \to X, U_2 \to X, U_3 \to X, V_1 \to X, V_2 \to X, V_3 \to X, W \subset X\}$$

forms an fpqc cover of $X$. This means the data of a quasicoherent sheaf on $X$ is equivalent to the data of quasicoherent sheaves on each of the opens of $\mathfrak{U}$ together gluing isomorphisms on overlaps. Technically: quasi-coherent sheaves on $X$ form a stack with respect to the fpqc topology [**?**, Thm. 4.23].

5.2. **Local moduli spaces.** We now introduce moduli spaces of closed subschemes of dimension $\leq 1$ on each of the pieces of the cover $\mathfrak{U}$. Consider $U_1 \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$ and assume the coordinates are chosen such that $x_2 = x_3 = 0$ corresponds to the intersection of $U_1 \times_X B$ and $x_1 = x_3 = 0$ corresponds to $U_1 \times_X F_x$. Define

$$\mathrm{Hilb}^{(1,d),n}(U_1) = \big\{I_Z \subset \mathcal{O}_{U_1} \;:\; [Z] = [U_1 \times_X B] + d[U_1 \times_X F_x] \text{ and } h^0(I_{Z_{CM}}/I_Z) = n\big\}.$$

Here the equation

$$[Z] = [U_1 \times_X B] + d[U_1 \times_X F_x]$$

means $Z$ is supported along $(U_1 \times_X B) \cup (U_1 \times_X F_x)$ with multiplicity 1 along $(U_1 \times_X B)$ and multiplicity $d$ along $(U_1 \times_X F_x)$. Furthermore, $Z_{CM}$ denotes the maximal Cohen-Macaulay subcurve of $Z$ which fits into a short exact sequence

$$0 \longrightarrow I_Z \longrightarrow I_{Z_{CM}} \longrightarrow Q \longrightarrow 0,$$

where $Q$ is a 0-dimensional. The Hilbert scheme $\mathrm{Hilb}^{(1,d),n}(U_2)$ is defined likewise replacing the point $x$ by $y$. (Recall that $x, y$ are intersections of $B$ with the fibres $F_x$, $F_y$.) For the point $F_y^{\mathrm{sing}}$ and its formal neighbourhood $U_3$, we define

$$\mathrm{Hilb}^{d,n}(U_3) = \big\{I_Z \subset \mathcal{O}_{U_3} \;:\; [Z] = d[U_3 \times_X F_y] \text{ and } h^0(I_{Z_{CM}}/I_Z) = n\big\}.$$

Each $U_i$ has an action of $\mathbb{C}^*$ compatible with the fibre scaling on $X$. This action lifts to the moduli space. Moreover, since $U_i \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$ the bigger torus $\mathbb{C}^{*3}$ acts on $U_i$ and lifts to the moduli space. The existence of these "extra actions" are one of our main computational tools and will be used in Section 6.

Next consider $V_1$, the formal neighbourhood of the punctured zero section $B \subset X$. Define

$$\mathrm{Hilb}^{1,n}(V_1) = \big\{I_Z \subset \mathcal{O}_{V_1} \;:\; [Z] = [V_1 \times_X B] \text{ and } h^0(I_{Z_{CM}}/I_Z) = n\big\}.$$

For $V_2$, $V_3$ we define

$$\mathrm{Hilb}^{d,n}(V_i) = \big\{I_Z \subset \mathcal{O}_{V_i} \;:\; [Z] = d[V_i \times_X B] \text{ and } h^0(I_{Z_{CM}}/I_Z) = n\big\}.$$

Finally, for $W$ we define

$$\mathrm{Hilb}^{0,n}(W) = \big\{I_Z \subset \mathcal{O}_{V_1} \;:\; \dim(Z) = 0 \text{ and } h^0(\mathcal{O}_Z) = n\big\}.$$

Each $V_i$ and $W$ also have an action of $\mathbb{C}^*$ compatible with the fibre scaling on $X$. These actions lift to the moduli space. However, *unlike* the $U_i$, no additional tori act on $V_i$, $W$.

As before, we use the notation $\mathrm{Hilb}^{(1,d),\bullet}(U_1)$ for the union of all $\mathrm{Hilb}^{(1,d),n}(U_1)$ (and similarly for all other moduli spaces of this section). Like in Section 4, the components of the $\mathbb{C}^*$-fixed locus of $\mathrm{Hilb}^{(1,d),\bullet}(U_1)$ are indexed by partitions $\lambda \vdash d$

$$\mathrm{Hilb}^{(1,d),\bullet}(U_1)^{\mathbb{C}^*} = \coprod_{\lambda \vdash d} \mathrm{Hilb}^{(1,d),\bullet}(U_1)^{\mathbb{C}^*}_{\lambda}.$$

**Proposition 7.** *Consider the stratum $\Sigma(x,y,a,b)$, where $a = |\lambda|$ and $b = |\mu|$. Restriction from $X$ to the open subsets of the cover $\mathfrak{U}$ induces a morphism*

$$\begin{aligned} \Sigma(x,y,\lambda,\mu) \longrightarrow\, & \mathrm{Hilb}^{(1,a),\bullet}(U_1)^{\mathbb{C}^*}_{\lambda} \times \mathrm{Hilb}^{(1,b),\bullet}(U_2)^{\mathbb{C}^*}_{\mu} \times \mathrm{Hilb}^{b,\bullet}(U_3)^{\mathbb{C}^*}_{\mu} \times \\ & \mathrm{Hilb}^{1,\bullet}(V_1)^{\mathbb{C}^*} \times \mathrm{Hilb}^{a,\bullet}(V_2)^{\mathbb{C}^*}_{\lambda} \times \mathrm{Hilb}^{b,\bullet}(V_3)^{\mathbb{C}^*}_{\mu} \times \\ & \mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}, \end{aligned} \tag{4}$$

*which is a bijection on closed points.*

*Proof.* Throughout this proof, we work on closed points only.

Restriction along $U_i \to X$, $V_i \longrightarrow X$, $W \subset X$ gives the map set-theoretically. For any reduced base $B$, restriction along $U_i \times B \to X \times B$, $V_i \times B \longrightarrow X \times B$, $W \times B \subset X \times B$ gives a map between moduli functors and hence induces the above morphism.

Since $\mathfrak{U}$ is an fpqc cover, fpqc descent implies that any ideal sheaf $I_Z \subset \mathcal{O}_X$ is entirely determined by its restriction along $U_i \to X$, $V_i \longrightarrow X$, $W \subset X$ proving injectivity.

Conversely, given local ideal sheaves in the image of (4), their restrictions to overlaps only depend on the underlying Cohen-Macaulay curve (and not on the embedded points). Since we chose the strata such that the underlying Cohen-Macauly curve automatically glues, there are no further gluing conditions and fpqc descent implies surjectivity. □

**Remark 8.** Note that the argument of Proposition 7 is purely set-theoretic in nature. We do *not* claim (4) is an isomorphism of schemes.

**Remark 9.** It is important to relate holomorphic Euler characteristic of domain and target in (4). For any subscheme $Z$ in the domain $\Sigma(x,y,\lambda,\mu)$, denote the corresponding maximal Cohen-Macaulay curve by $Z_{CM}$ (Proposition 6). Then

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{Z_{CM}}) + \chi(I_{Z_{CM}}/I_Z).$$

Recall that $Z_{CM}$ is entirely determined by the data $x, y, \lambda, \mu$, where $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$ and $\mu = (\mu_0 \geq \mu_1 \geq \cdots)$ are 2D partitions (3). An easy calculation shows

$$\chi(\mathcal{O}_{Z_{CM}}) = \chi(\mathcal{O}_B) - \lambda_0 - \mu_0.$$

We conclude

$$\chi(\mathcal{O}_Z) = -\frac{e(B)}{2} - \lambda_0 - \mu_0 + \chi(I_{Z_{CM}}/I_Z). \tag{5}$$

Proposition 7 allows us to calculate the Euler characteristic of the stratum $\Sigma(x,y,\lambda,\mu)$. Recall that $a = |\lambda|$, $b = |\mu|$ so $d = a+b$. Then by (3), Proposition 7, and (5)

$$\begin{aligned} f_d(ax+by) &= e(\rho_d^{-1}(ax+by)) \\ &= p^{-\frac{e(B)}{2}} e(\mathrm{Hilb}^{1,\bullet}(V_1)^{\mathbb{C}^*}) e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) \times \\ &\quad \sum_{\lambda \vdash a} \sum_{\mu \vdash b} p^{-\frac{e(B)}{2} - \lambda_0 - \mu_0} e(\mathrm{Hilb}^{(1,a),\bullet}(U_1)^{\mathbb{C}^*}_{\lambda}) e(\mathrm{Hilb}^{(1,b),\bullet}(U_2)^{\mathbb{C}^*}_{\mu}) \times \\ &\qquad e(\mathrm{Hilb}^{b,\bullet}(U_3)^{\mathbb{C}^*}_{\mu}) e(\mathrm{Hilb}^{a,\bullet}(V_2)^{\mathbb{C}^*}_{\lambda}) e(\mathrm{Hilb}^{b,\bullet}(V_3)^{\mathbb{C}^*}_{\mu}). \end{aligned} \tag{6}$$

Before we proceed, we need to calculate $e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*})$ and $e(\mathrm{Hilb}^{1,\bullet}(V_1)^{\mathbb{C}^*})$. The first follows from a formula of J. Cheah []

$$e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) = M(p)^{e(W)}. \tag{7}$$

For the second we use the following proposition:

**Proposition 10.** *Let $B^\circ$ be the section $B \subset S \subset X$ with any number of punctures and let $X^\circ$ be obtained from $X$ by removing the same punctures. Let $V$ be the formal neighbourhoods $V$ of $B^\circ$ in $X^\circ$. Define $\mathrm{Hilb}^{1,n}(V)$ as the Hilbert scheme of subschemes $Z \subset V$, such that $Z_{CM} = C$ and $\chi(I_{Z_{CM}}/I_Z) = n$, where $Z_{CM}$ denotes the maximal Cohen-Macaulay subscheme contained in $Z$ (Proposition 6). Then*

$$e(\mathrm{Hilb}^{1,\bullet}(V)) = \left(\frac{M(p)}{(1-p)}\right)^{e(B^\circ)}.$$

*Proof.* Let $p \in B^\circ$ and let $U \cong \operatorname{Spec}\mathbb{C}[x_1, x_2, x_3]$ the formal neighbourhood of $p$ in $X^\circ$. Denote by $\mathrm{Hilb}^{1,n}(U)$ the Hilbert scheme of subschemes $Z \subset U$, such that $Z_{CM} = \{x_2 = x_3 = 0\}$ and $\chi(I_{Z_{CM}}/I_Z) = n$. We have projections

$$X^\circ \longrightarrow S^\circ \longrightarrow B^\circ.$$

Similar to Proposition 7, this map induces a morphism

$$\mathrm{Hilb}^{1,d}(V) \longrightarrow \mathrm{Sym}^d(B^\circ).$$

The fibre over a point $\mathfrak{a} = \sum_i a_i x_i$ equals

$$\prod_i e(\mathrm{Hilb}^{1,a_i}(U)).$$

This follows by using an appropriate fpqc cover of $B^\circ$ similar to Proposition 7. Therefore, Lemma 15 of the appendix implies

$$e(\mathrm{Hilb}^{1,\bullet}(V)) = \left(\sum_{a=0}^{\infty} e(\mathrm{Hilb}^{1,a}(U))p^a\right)^{e(B^\circ)}.$$

Now $U$ has an action of $\mathbb{C}^{*3}$ which lifts to $\mathrm{Hilb}^{1,a}(U)$. The fixed locus consists of a finite number of points counted by the topological vertex (discussed in general in Section 6)

$$\mathsf{V}_{\square,\varnothing,\varnothing}(p) = \frac{M(p)}{(1-p)}.$$

□

Using (7) and Proposition 10, equation (6) becomes

$$\begin{aligned} f_d(ax+by) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \Big( (1-p) \sum_{\lambda \vdash a} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(U_1)_\lambda^{\mathbb{C}^*}) e(\mathrm{Hilb}^{a,\bullet}(V_2)_\lambda^{\mathbb{C}^*}) \Big) \times \\ \Big( (1-p) M(p)^{-1} \sum_{\mu \vdash b} p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(U_2)_\mu^{\mathbb{C}^*}) e(\mathrm{Hilb}^{b,\bullet}(U_3)_\mu^{\mathbb{C}^*}) e(\mathrm{Hilb}^{b,\bullet}(V_3)_\mu^{\mathbb{C}^*}) \Big). \end{aligned}$$

5.3. **Geometric characterization of $g(a)$ and $h(b)$.** The arguments of the preceding two sections are straightforwardly modified to any stratum $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$. Let $U$ be the formal neighbourhood of any point on $B \subset S \subset X$ and define $\mathrm{Hilb}^{(1,a),\bullet}(U)$ as in Section 5.2. Let $U'$ be the formal neighbourhood of the singular point of any singular fibre $F \subset S \subset X$ and define $\mathrm{Hilb}^{b,\bullet}(U')$ as in Section 5.2. Let $V$ be the formal neighbourhood of any smooth fibre $F \setminus B$ in $X \setminus B$ and define $\mathrm{Hilb}^{a,\bullet}(V)$ as in Section 5.2. Let $V'$ be the formal neighbourhood of any singular fibre $F \setminus (B \cup F^{\mathrm{sing}})$ in $X \setminus (B \cup F^{\mathrm{sing}})$ and define $\mathrm{Hilb}^{b,\bullet}(V')$ as in Section 5.2.

**Proposition 11.** *For any $a, b > 0$ define*

(8)
$$g(a) := (1-p)\sum_{\lambda \vdash a} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(U)^{\mathbb{C}^*}_{\lambda}) e(\mathrm{Hilb}^{a,\bullet}(V)^{\mathbb{C}^*}_{\lambda}),$$
$$h(b) := (1-p)M(p)^{-1}\sum_{\mu \vdash b} p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(U)^{\mathbb{C}^*}_{\mu}) e(\mathrm{Hilb}^{b,\bullet}(U')^{\mathbb{C}^*}_{\mu}) e(\mathrm{Hilb}^{b,\bullet}(V')^{\mathbb{C}^*}_{\mu}),$$

*and let $g(0) := 1$, $h(0) := 1$. Then*

$$f_d(\mathfrak{a}+\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_i g(a_i) \cdot \prod_j h(b_j),$$

*for any $\mathfrak{a} = \sum_i a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$ and $\mathfrak{b} = \sum_j b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$.*

We immediately deduce:

**Corollary 12.** *Propositions 3 and 4 are true for $g(a)$ and $h(b)$ defined by* (10).

## 6. Reduction to the topological vertex

In this section, we prove Theorem 1 of the introduction by expressing $g(a)$ and $h(b)$ in terms of the topological vertex.

Finish this subsection.

6.1. **The topological vertex and point contributions.**

$$p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(U)^{\mathbb{C}^*}_{\lambda}) = \mathsf{V}_{\lambda,\varnothing,\varnothing}(p),$$
$$p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(U)^{\mathbb{C}^*}_{\mu}) = \mathsf{V}_{\mu,\varnothing,\varnothing}(p),$$
$$e(\mathrm{Hilb}^{b,\bullet}(U')^{\mathbb{C}^*}_{\mu}) = p^{\binom{\mu}{2}+|\mu|}\mathsf{V}_{\mu,\mu',\varnothing}(p),$$

where $\mu'$ denotes the transposed partition [4].

6.2. **Fibre contribution.** Recall that we denote by $V$ the formal neighbourhood of any smooth fibre $F \setminus B$ in $X \setminus B$ and by $V'$ be the formal neighbourhood of any singular fibre $F \setminus (B \cup F^{\mathrm{sing}})$ in $X \setminus (B \cup F^{\mathrm{sing}})$.

**Proposition 13.** *For any $\lambda \vdash a$ and $\mu \vdash b$, we have*

$$e(\mathrm{Hilb}^{a,\bullet}(V)^{\mathbb{C}^*}_{\lambda}) = \frac{1}{\mathsf{V}_{\lambda,\varnothing,\varnothing}(p)},$$
$$e(\mathrm{Hilb}^{b,\bullet}(V')^{\mathbb{C}^*}_{\mu}) = \frac{1}{\mathsf{V}_{\mu,\varnothing,\varnothing}(p)}.$$

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*Proof.* We start with the first equation. Let $F = F_x \subset S$ be a smooth fibre over a closed point $x \in B$. Consider the auxiliary surface $\tilde{S} = B \times F$ ($F$ a smooth elliptic curve) and $\tilde{X} = \mathrm{Tot}(K_{\tilde{S}})$. let $e \in F$ be any point and consider the embeddings

$$B \hookrightarrow \tilde{S},\ b \mapsto (b, e)$$
$$\tilde{S} \hookrightarrow \tilde{X},\ p \mapsto (p, 0).$$

Consider the embeddings $B \subset \tilde{S} \subset \tilde{X}$ (where $\tilde{S} \subset \tilde{X}$ is the zero section) and $B \subset S \subset X$. Denote by $T$ a formal neighbourhood of $F \subset S \subset X$ and $\tilde{T}$ a formal neighbourhood of $F \cong \{x\} \times F \subset \tilde{S}\tilde{X}$ ($T$ stands for "tubular"). Certainly $T$ and $\tilde{T}$ are *not* isomorphic. Next, denote by $V$ the formal neighbourhood of $F \setminus B$ on $X \setminus B$ and by $\tilde{V}$ the formal neighbourhood of $F \setminus B$ on $\tilde{X} \setminus B$ (using the same embeddings as above). Then

$$V \cong \tilde{V}. \tag{9}$$

We are interested in the moduli space $\mathrm{Hilb}^{a,\bullet}(V)$ and the correspondingly defined moduli space $\mathrm{Hilb}^{a,\bullet}(\tilde{V})$. Since $V$, $\tilde{V}$ have (compatible) $\mathbb{C}^*$-actions coming from scaling the fibres of $X, \tilde{X}$, we can consider their fixed loci and stratify them according to 2D partitions as in 3. By (9), we have

$$\mathrm{Hilb}^{a,\bullet}(V)^{\mathbb{C}^*}_{\lambda} \cong \mathrm{Hilb}^{a,\bullet}(\tilde{V})^{\mathbb{C}^*}_{\lambda}.$$

This observation allows us to work in the much simpler geometry of $\tilde{X}$.

Let $F \cong \{x\} \times F \subset \tilde{S}$ as above. Denote the zero section by $\tilde{S} \subset \tilde{X}$. We will use the following formal neighbourhoods (in the sense of Section 5):

- Let $\tilde{U}$ be the formal neighbourhood of $(x, e, 0) \in \tilde{X}$.
- Let $\tilde{V}$ be the formal neighbourhood of $F \setminus \{(x, e, 0)\}$ inside $\tilde{X} \setminus \{(b, e, 0)\}$ (introduced above).
- Let $\tilde{T}$ be the formula neighbourhood of $F$ inside $\tilde{X}$ (introduced above).

Then $\tilde{U} \to \tilde{T}$, $\tilde{V} \to \tilde{T}$ forms an fpqc cover of $\tilde{T}$. Note that $\tilde{U} \cong \mathrm{Spec}\, \mathbb{C}[x_1, x_2, x_3]$. On these pieces, we introduce moduli spaces

$$\mathrm{Hilb}^{a,\bullet}(\tilde{U}),\ \mathrm{Hilb}^{a,\bullet}(\tilde{V}),\ \mathrm{Hilb}^{a,\bullet}(\tilde{T})$$

exactly as in Section 5. As in Proposition 7, restriction gives a bijective morphism on closed points

$$\mathrm{Hilb}^{a,\bullet}(\tilde{T})^{\mathbb{C}^*}_{\lambda} \to \mathrm{Hilb}^{a,\bullet}(\tilde{U})^{\mathbb{C}^*}_{\lambda} \times \mathrm{Hilb}^{a,\bullet}(\tilde{V})^{\mathbb{C}^*}_{\lambda}.$$

Recall that $\tilde{S} = B \times F$. Therefore $F$ does not only act on $F \subset \tilde{S}$, but for any thickening $dF \subset \tilde{S}$ it acts on $dF$. This is because

$$\mathcal{O}_{dF} = \mathcal{O}_{db} \otimes \mathcal{O}_F,$$

where $db \subset B$ denotes the $d$ times thickening of $b \in B$. Moreover, $F$ acts on the thickened curve $\lambda F \subset \tilde{X}$ defined by

$$\bigoplus_{k=0}^{\infty} \mathcal{O}_{\tilde{S}}(-\lambda_k F) \otimes K_{\tilde{S}}^{-k}.$$

Since the action of $F$ on itself is fixed-point-free, it lifts to a free action on acts freely on $\mathrm{Hilb}^{a,\bullet}(\tilde{T})^{\mathbb{C}^*}_{\lambda}$. Since $e(F) = 0$, we deduce

$$e(\mathrm{Hilb}^{a,\bullet}(\tilde{T})^{\mathbb{C}^*}_{\lambda}) = 1,$$

where 1 comes $\bullet = 0$.

Finally, since $U \cong \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3]$, we have an action of $\mathbb{C}^{*3}$ on it and

$$e(\mathrm{Hilb}^{a,\bullet}(\tilde{U})^{\mathbb{C}^*}_{\lambda}) = e(\mathrm{Hilb}^{a,\bullet}(\tilde{U})^{\mathbb{C}^{*3}}_{\lambda}) = \mathsf{V}_{\lambda,\varnothing,\varnothing}(p).$$

The equation for $e(\mathrm{Hilb}^{b,\bullet}(V')^{\mathbb{C}^*}_{\mu})$ follows similarly. This time, the smooth fibre $F \subset S \subset X$ is replaced by the punctured singular fibre $F' \setminus F'^{\,\mathrm{sing}} \subset S \setminus F'^{\,\mathrm{sing}} \subset X \setminus F'^{\,\mathrm{sing}}$. Note that

$$F' \setminus F'^{\,\mathrm{sing}} \cong \mathbb{P}^1 \setminus \{2 \text{ points}\} \cong \mathbb{C}^*.$$

Therefore, we again have a free action of $F' \setminus F'^{\,\mathrm{sing}}$ on itself and $e(F' \setminus F'^{\,\mathrm{sing}}) = 0$. The proof follows the same steps. □

6.3. **Proof of Theorem 1.**

**Proposition 14.** *For any* $a, b > 0$

$$\begin{aligned} g(a) &:= (1-p) \sum_{\lambda \vdash a} \frac{\mathsf{V}_{\lambda,\square,\varnothing}(p)}{\mathsf{V}_{\lambda,\varnothing,\varnothing}(p)}, \\ h(b) &:= (1-p) M(p)^{-1} \sum_{\mu \vdash b} \frac{\mathsf{V}_{\mu,\square,\varnothing}(p) \mathsf{V}_{\mu,\mu',\varnothing}(p) p^{\binom{\mu}{2}+|\mu|}}{\mathsf{V}_{\mu,\varnothing,\varnothing}(p)}. \end{aligned} \tag{10}$$

## Appendix A. Odds and Ends

A.1. **Weighted Euler characteristics of symmetric products.** In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

**Lemma 15.** *Let $S$ be a scheme of finite type over $\mathbb{C}$ and let $e(S)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((Q))$ be any function with $g(0) = 1$. Let $f_d : \mathrm{Sym}^d(S) \to \mathbb{Z}((Q))$ be the constructible function defined by $f_d(\sum_i a_i x_i) = \prod_i g(a_i)$. Then*

$$\sum_{d=0}^{\infty} u^d \int_{\mathrm{Sym}^d(S)} f_d de = \left( \sum_{a=0}^{\infty} g(a) u^a \right)^{e(S)}.$$

**Remark 16.** In the special case where $g = f_d \equiv 1$, the lemma recovers MacDonald's formula: $\sum_{d=1}^{\infty} e(\mathrm{Sym}^d(S)) u^d = (1-u)^{-e(S)}$.

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties definited by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure []. For convenience's sake, we provide a direct proof here.

*Proof.* The $d$th symmetric product admits a stratification with strata labelled by partitions of $d$. Associated to any partition of $d$ is a unique tuple $(m_1, m_2, \dots)$ of non-negative integers with $\sum_{j=1}^{\infty} j m_j = d$. The stratum labelled by $(m_1, m_2, \dots)$ parameterizes collections of points where there are $m_j$ points of multiplicity $j$. The full stratification is given by:

$$\mathrm{Sym}^d(S) = \coprod_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left\{ \left( \prod_{j=1}^{\infty} S^{m_j} \right) - \Delta \right\} / \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, $S^0$ is a point, $\Delta$ is the large diagonal, and $\sigma_m$ is the $m$th symmetric group. Note that the function $f_d$ is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

For schemes over $\mathbb{C}$, topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\mathrm{Sym}^d(S)} f_d \; de = \sum_{\substack{(m_1,m_2,\dots) \\ \sum_{j=1}^{\infty} jm_j = d}} \left(\prod_{j=1}^{\infty} g(j)^{m_j}\right) \frac{e(S^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \dots}.$$

For any natural number $N$, the projection $S^N - \Delta \to S^{N-1} - \Delta$ has fibers of the form $S - \{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(S) - (N-1)$ and consequently, $e(S^N - \Delta) = (e(S) - (N-1)) \cdot e(S^{N-1} - \Delta)$. Thus by induction, we find $e(S^N - \Delta) = e(S) \cdot (e(S) - 1) \cdots (e(S) - (N-1))$ and so

$$\frac{e(S^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \cdots} = \binom{e(S)}{m_1, m_2, m_3, \cdots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\begin{aligned}\sum_{d=0}^{\infty} \int_{\mathrm{Sym}^d(S)} f_d \; de &= \sum_{(m_1,m_2,\dots)} \prod_{j=1}^{\infty} \left(g(j)u^j\right)^{m_j} \binom{e(S)}{m_1, m_2, m_3, \dots} \\ &= \left(1 + \sum_{j=1}^{\infty} g(j)u^j\right)^{e(S)}\end{aligned}$$

which proves the lemma. □

A.2. **Some geometry of curves on elliptic surfaces.** In this subsection we prove the following lemma and corollary, which will tell us what is the reduced support of all curves in the class $\beta = B + dF$.

**Lemma 17.** *For any line bundle $\epsilon$ on $B$, multiplication by the canonical section of $\mathcal{O}(B)$ induces an isomorphism*

$$H^0(S, \pi^*(\epsilon)(B)) \cong H^0(S, \pi^*(\epsilon)).$$

**Corollary 18.** *Let $\beta = B + dF \in H_2(S)$. Then the Chow variety of curves in the class $\beta$ is isomorphic to $\mathrm{Sym}^d(B)$ where a point $\sum_i d_i x_i \in \mathrm{Sym}^d(B)$ corresponds to the curve $B + \sum_i d_i F_{x_i}$.*

*Proof.* The corollary follows immediately from the lemma since the Chow variety is the space of effective divisors and the lemma implies that any effective divisor in the class $\beta$ is a union of the section $B$ with an effective divisor pulled by from the base.

To prove lemma 17 we proceed as follows. For any line bundle $\delta$ on $B$, the Leray spectral sequence yields the short exact sequence:

$$0 \to H^0(B, \delta \otimes R^1\pi_*\mathcal{O}) \to H^1(S, \pi^*\delta) \xrightarrow{\alpha} H^1(B, \delta) \to 0,$$

in particular, $\alpha$ is a surjection.

Then the long exact cohomology sequence associated to

$$0 \to \pi^*\delta \otimes \mathcal{O}(-B) \to \pi^*\delta \to \mathcal{O}_B \otimes \pi^*\delta \to 0$$

is

$$\cdots \to H^1(S, \pi^*(\delta)) \xrightarrow{\alpha} H^1(B, \delta) \to H^2(S, \pi^*\delta \otimes \mathcal{O}(-B)) \to H^2(S, \pi^*\delta) \to 0,$$

and since $\alpha$ is a surjection, we get an isomorphism of the last two terms. We apply Serre duality to that isomorphism and we use the fact that $K_S = \pi^*(K_B \otimes L)$ where $L = (R\pi_* \mathcal{O}_S)^\vee$ [**?**, prop?] to obtain

$$H^0(S, \pi^*(\delta^{-1} \otimes K_B \otimes L)(B)) \cong H^0(S, \pi^*(\delta^{-1} \otimes K_B \otimes L)).$$

Letting $\delta = K_B \otimes L \otimes \epsilon^{-1}$, the lemma is proved. $\square$

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