

DONALDSON-THOMAS INVARIANTS OF LOCAL ELLIPTIC SURFACES VIA THE TOPOLOGICAL VERTEX

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex (some previously known, some new), we derive product formulas for the partition functions in a companion paper with B. Young. In the special case where the elliptic surface is a K3 surface, we get a new proof of the Katz-Klemm-Vafa formula.

1. INTRODUCTION

Let $p : S \rightarrow B$ be an elliptic surface over a complex smooth projective curve B . We assume p has a section and all singular fibres are rational nodal curves. Let N be the number of singular fibres. Important examples are the rational elliptic surface or elliptic K3 surface for which $B = \mathbb{P}^1$ and $N = 12$, $N = 24$ respectively.

We are interested in the Donaldson-Thomas invariants of $X = \text{Tot}(K_S)$, i.e. the total space of the canonical bundle K_S . This is a non-compact Calabi-Yau 3-fold. Let β be an effective curve class on S . Consider the Hilbert scheme

$$\text{Hilb}^{\beta,n}(X) = \{Z \subset X : [Z] = \beta, \chi(\mathcal{O}_Z) = n\}$$

of proper subschemes $Z \subset X$ with homology class β and holomorphic Euler characteristics n . The DT invariants of X can be defined as

$$\text{DT}_{\beta,n}(X) := e(\text{Hilb}^{\beta,n}(X), \nu) := \sum_{k \in \mathbb{Z}} k e(\nu^{-1}(\{k\})),$$

where $e(\cdot)$ denotes topological Euler characteristic and $\nu : \text{Hilb}^{\beta,n}(X) \rightarrow \mathbb{Z}$ is Behrend's constructible function [Beh]. We consider an Euler characteristic version of these invariants

$$\widehat{\text{DT}}_{\beta,n}(X) := e(\text{Hilb}^{\beta,n}(X)).$$

We choose a section $B \subset S$ and focus on the primitive classes $\beta = B + dF$, where B is the class of the chosen section and F the class of the fibre. Let

$$\widehat{\text{DT}}(X) := \sum_{d \geq 0} \widehat{\text{DT}}_d(X) q^d := \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \widehat{\text{DT}}_{B+dF,n}(X) p^n q^d.$$

The main result of this paper is a formula for this generating function, and its connected analog $\widehat{\text{DT}}^{\text{conn}}(X)$, in terms of the topological vertex $V_{\lambda,\mu,\nu}(p)$, $e(B)$, and N . This formula can be found in Theorem 16 of Section 6.

It turns out the formula of Theorem 16 can be expressed in terms of traces of certain natural operators on Fock space. Some of these traces are known and others are new. The way to compute these traces is of independent interest and forms the subject of a companion

paper with B. Young [BKY]. The outcome is the following. Consider the Dedekind eta function and the following Jacobi eta function [Cha]

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{k=1}^{\infty} (1 - q^k)$$

$$K(z, \tau) = \frac{i\vartheta_1(z, \tau)}{\eta(\tau)^3} = (p^{\frac{1}{2}} - p^{-\frac{1}{2}}) \prod_{k=1}^{\infty} \frac{(1 - pq^k)(1 - p^{-1}q^k)}{(1 - q^k)^2}$$

expanded in the region $|p|, |q| < 1$ where

$$p = e^{2\pi i z}, \quad q = e^{2\pi i \tau}.$$

Theorem 1 (Bryan-Kool-Young). *Let g be the genus of B . Then*

$$\widehat{\text{DT}}^{\text{conn}}(X) = (q^{-\frac{1}{24}} \eta(\tau))^{-e(S)} K(z, \tau)^{-e(B)}.$$

In the case $S \rightarrow \mathbb{P}^1$ is the elliptically fibred K3 surface, this provides a new derivation of the famous Katz-Klemm-Vafa formula. The KKV formula was recently proved in *all* curve classes in [PT]. This is the first derivation of the KKV formula, which does *not* depend on the Kawai-Yoshioka formula [KY].

The most important result of this paper is perhaps not the formula, but rather the method of calculation. This method has found further applications to the calculation of DT generating functions on $K3 \times E$, where E is an elliptic curve [Bry], and abelian 3-folds [BOPY]. Even though the geometry under consideration is not toric, we combine \mathbb{C}^* -localization, motivic methods, formal methods, and \mathbb{C}^{*3} -localization to end up with expressions that only depend on $V_{\lambda, \mu, \nu}(p)$, $e(B)$, and N . Here is a rough sketch of our method:

- (A) The action of \mathbb{C}^* on the fibres of X lifts to the moduli space¹ $\text{Hilb}^{B+dF, \bullet}(X)$. Therefore, we only have to understand $e(\text{Hilb}^{B+dF, \bullet}(X))^{\mathbb{C}^*}$. Push-forward along $X \rightarrow S \rightarrow B$ induces a morphism

$$(1) \quad \text{Hilb}^{B+dF, \bullet}(X)^{\mathbb{C}^*} \rightarrow \text{Sym}^d(B).$$

This map is constructed in Section 3. The fibres of (1) decompose into components according to the shape of the underlying Cohen-Macaulay curve. This leads to a decomposition over 2D partitions $\lambda = (\lambda_0 \geq \lambda_1 \geq \dots)$.

- (B) The Euler characteristics of the fibres of (1) give a constructible function f_d on $\text{Sym}^d(B)$. In Section 4, we show that if f_d satisfies a certain product formula, then $\widehat{\text{DT}}(X)$ satisfies a corresponding product formula. This follows from general power structure arguments reviewed in Appendix A.2.
- (C) A component Σ of a fibre of (1) indexed by λ can be further broken down by taking a certain fpqc cover of the underlying (now fixed) Cohen-Macaulay curve Z_{CM} determined by λ . This cover consists of formal neighbourhoods \widehat{X}_x around the singular points x of the reduced support of Z_{CM} and “tubular neighbourhoods” along the reduced support of Z_{CM} after removing the singularities. Since Z_{CM} is already fixed, gluing is automatic. Hence restriction to the elements of the cover gives a set theoretic bijection of Σ with Hilbert schemes on the elements of the cover. In Section 5, we show these moves lead to the product formula for f_d mentioned in (A).

¹The bullet indicates that we take the union of $\text{Hilb}^{B+dF, n}(X)$ over all n .

- (D) On the formal neighbourhoods \widehat{X}_x , we have an action of \mathbb{C}^* ³. This allows us to express their contributions to the generating function in terms of the topological vertex. The contributions of the tubular neighbourhoods along the *punctured* section and fibres can also be expressed in terms of the topological vertex (roughly speaking, by utilizing actions of the elliptic curve F or \mathbb{C}^* on itself). This is worked out in Section 6.

Many of the methods of this paper work well with the Behrend function. In particular, steps (A), (B), and (D) do not provide any problems. In Section 7...

TBC: section on Behrend function.

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Add people and funding.

2. DEFINITIONS

Let $p : S \rightarrow B$ be an elliptic surface over a smooth projective curve B . Besides assuming S is not a product, we require:

- (1) p has a section $B \subset S$,
- (2) all singular fibres of p are of Kodaira type I_1 , i.e. rational nodal curves.

We write F_x for the fibre $p^{-1}(\{x\})$, for all closed points $x \in B$. We choose a section $B \subset S$ and denote its class in $H_2(S)$ by B as well. We denote the class of the fibre by $F \in H_2(S)$. Interesting examples are the elliptic surfaces $E(n)$. Then $B = \mathbb{P}^1$ and the number of nodal fibres is $12n$. For example, $E(1)$ is the rational elliptic surface and $E(2)$ is the elliptic K3 surface.

Let $\beta = B + dF \in H_2(S)$ for arbitrary $d \geq 0$. Denote by $X = \text{Tot}(K_S)$ the total space of the canonical bundle over S . Then X is a non-compact Calabi-Yau 3-fold. Consider the Hilbert scheme

$$\text{Hilb}^{\beta,n}(X) = \{Z \subset X : [Z] = \beta, \chi(\mathcal{O}_Z) = n\}$$

of proper subschemes $Z \subset X$ with fixed homology class and holomorphic Euler characteristic. K. Behrend associates to any \mathbb{C} -scheme of finite type Y a constructible function $\nu : Y \rightarrow \mathbb{Z}$ [Beh]. Applied to $\text{Hilb}^{\beta,n}(X)$, the Donaldson-Thomas invariants of X can be defined as²

$$\text{DT}_{\beta,n}(X) := \int_{\text{Hilb}^{\beta,n}(X)} \nu \, de := \sum_{k \in \mathbb{Z}} k \, e(\nu^{-1}(\{k\})),$$

where $e(\cdot)$ denotes topological Euler characteristic. Many of the key properties of DT invariants are already captured by the more classical Euler characteristic version³

$$\widehat{\text{DT}}_{\beta,n}(X) := \int_{\text{Hilb}^{\beta,n}(X)} 1 \, de = e(\text{Hilb}^{\beta,n}(X)).$$

²If X is a compact Calabi-Yau 3-fold, R.P. Thomas's original definition of the DT invariant is the degree of the virtual cycle of $\text{Hilb}^{\beta,n}(X)$ [Tho]. Behrend showed this is the same as $e(\text{Hilb}^{\beta,n}(X), \nu)$ [Beh]. The advantage of the definition by means of virtual cycles is that the construction works relative to a base. This implies deformation invariance of the invariants.

³From the point of view of [JS, Bri], there are two natural integration maps on the semi-classical Hall-algebra. One corresponds to weighing by the Behrend function. The other corresponds to weighing by the “trivial” constant constructible function 1. The former gives rise to $\text{DT}(X)$ and the latter to $\widehat{\text{DT}}(X)$.

For brevity, we define

$$\begin{aligned}\mathrm{Hilb}^{d,n}(X) &:= \mathrm{Hilb}^{B+dF,n}(X), \\ \mathrm{DT}_{d,n}(X) &:= \mathrm{DT}_{B+dF,n}(X), \\ \widehat{\mathrm{DT}}_{d,n}(X) &:= \mathrm{DT}_{B+dF,n}(X).\end{aligned}$$

The generating functions of interest are

$$\begin{aligned}\mathrm{DT}(X) &:= \sum_{d \geq 0} \mathrm{DT}_d(X) q^d := \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \mathrm{DT}_{d,n}(X) p^n q^d, \\ \widehat{\mathrm{DT}}(X) &:= \sum_{d \geq 0} \widehat{\mathrm{DT}}_d(X) q^d := \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \widehat{\mathrm{DT}}_{d,n}(X) p^n q^d.\end{aligned}$$

Note that the corresponding connected series $\mathrm{DT}^{\mathrm{conn}}(X)$, $\widehat{\mathrm{DT}}^{\mathrm{conn}}(X)$ are obtained after dividing by

$$\begin{aligned}\sum_{d \geq 0} \sum_{n \in \mathbb{Z}} e(\mathrm{Hilb}^{dF,n}(X), \nu) p^n q^d \\ \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} e(\mathrm{Hilb}^{dF,n}(X)) p^n q^d\end{aligned}$$

respectively.

Since we are dealing with generating functions and our calculations involve cut-paste methods on the moduli space, it is useful to introduce the following notation. We define

$$[\mathrm{Hilb}^{d,\bullet}(X)] := \sum_{n \in \mathbb{Z}} [\mathrm{Hilb}^{d,n}(X)] p^n,$$

which is an element of $K_0(\mathrm{Var}_{\mathbb{C}})((p))$, i.e. a Laurent series with coefficients in the Grothendieck group of varieties. We also write $\mathrm{Hilb}^{d,\bullet}(X)$ to denote the union of all $\mathrm{Hilb}^{d,n}(X)$. Therefore $\mathrm{Hilb}^{d,\bullet}(X)$ is a \mathbb{C} -scheme locally of finite type.

3. THE \mathbb{C}^* -FIXED LOCUS

The action of \mathbb{C}^* on the fibres of X lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1 \, de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1 \, de.$$

Recall that

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1 \, de = \sum_{n \in \mathbb{Z}} p^n \int_{\mathrm{Hilb}^{d,n}(X)} 1 \, de \in \mathbb{Z}((p)).$$

Using the map $\pi : X \rightarrow S$, a quasi-coherent sheaf on X can be viewed as a quasi-coherent sheaf \mathcal{F} on S together with a morphism $\mathcal{F} \otimes K_S^{-1} \rightarrow \mathcal{F}$. A \mathbb{C}^* -equivariant structure on \mathcal{F} translates into a \mathbb{Z} -grading

$$\pi_* \mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k,$$

such that $\mathcal{F} \otimes K_S^{-1} \rightarrow \mathcal{F}$ is graded, i.e.

$$\mathcal{F}_k \otimes K_S^{-1} \longrightarrow \mathcal{F}_{k-1}.$$

Here \mathcal{F}_k has weight k and K_S weight -1 under the \mathbb{C}^* -action. The structure sheaf \mathcal{O}_X corresponds to

$$\pi_* \mathcal{O}_X = \bigoplus_{k=0}^{\infty} K_S^{-k}.$$

Therefore a \mathbb{C}^* -equivariant morphism $\mathcal{F} \rightarrow \mathcal{O}_X$ corresponds to a graded sheaf \mathcal{F} as above together with maps

$$\begin{array}{ccccccc} \cdots & \mathcal{F}_1 & \oplus & \mathcal{F}_0 & \oplus & \mathcal{F}_{-1} & \oplus & \mathcal{F}_{-2} & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & 0 & \oplus & \mathcal{O}_S & \oplus & K_S^{-1} & \oplus & K_S^{-2} & \cdots, \end{array}$$

where

$$\begin{array}{ccc} \mathcal{F}_k \otimes K_S^{-1} & \longrightarrow & \mathcal{F}_{k-1} \\ \downarrow & & \downarrow \\ K_S^k \otimes K_S^{-1} & \xlongequal{\quad} & K_S^{k-1} \end{array}$$

commute for all $k \leq 0$ and the composition $\mathcal{F}_1 \otimes K_S^{-1} \rightarrow \mathcal{F}_0 \rightarrow \mathcal{O}_S$ is to zero.

It is useful to re-define $\mathcal{G}_k := \mathcal{F}_{-k} \otimes K_S^k$. Then a \mathbb{C}^* -equivariant morphism $\mathcal{F} \rightarrow \mathcal{O}_X$ is equivalent to the following data:

- quasi-coherent sheaves $\{\mathcal{G}_k\}_{k \in \mathbb{Z}}$ on S ,
- morphisms $\{\mathcal{G}_k \rightarrow \mathcal{G}_{k+1}\}_{k \in \mathbb{Z}}$,
- morphisms $\mathcal{G}_k \rightarrow \mathcal{O}_S$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \mathcal{G}_{-1} & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & 0 & \longrightarrow & \mathcal{O}_S & \xlongequal{\quad} & \mathcal{O}_S & \xlongequal{\quad} & \mathcal{O}_S & \xlongequal{\quad} & \cdots \end{array}$$

In the case of interest to us $\mathcal{G} \rightarrow \mathcal{O}_X$ is an ideal sheaf $I_Z \hookrightarrow \mathcal{O}_X$ cutting out $Z \subset X$. In the above language, this means $\mathcal{G}_k = 0$ for $k < 0$, the morphisms $\mathcal{G}_k \rightarrow \mathcal{O}_S$ are injective (hence $\mathcal{G}_k = I_{Z_k \subset S}$ is an ideal sheaf cutting out $Z_k \subset S$), and the morphisms $\mathcal{G}_k \rightarrow \mathcal{G}_{k+1}$ are injective (hence $I_{Z_k \subset S} \subset I_{Z_{k+1} \subset S}$, i.e. $Z_k \supset Z_{k+1}$). We conclude:

Proposition 2. *A closed point Z of $\text{Hilb}^{d, \bullet}(X)^{\mathbb{C}^*}$ corresponds to a finite nesting of closed subschemes of S*

$$Z_0 \supset Z_1 \supset \cdots \supset Z_l,$$

for some $l \geq 0$, such such that

$$\sum_{k=0}^l [Z_k] = B + dF \in H_2(S).$$

Let $\text{Hilb}^{B+dF}(S)$ be the Hilbert scheme of effective divisors on S with class

$$B + dF \in H_2(S).$$

Let g be the genus of B . In Lemma 17 of the Appendix A.1, we prove that pull-back along p and adding the section B induces an isomorphism

$$\text{Sym}^d(B) \cong \text{Hilb}^{B+dF}(S),$$

for all $d > 2g - 2$. This allows us to see the curves on S .

For any reduced curve $C \subset S$ defined by ideal sheaf $I_{C \subset S}$ and $d > 0$, we denote by dC the Gorenstein curve defined by the ideal sheaf $I_{C \subset S}^d$ (the d th power of $I_{C \subset S}$). We combine Lemma 17 with a (family version of) Proposition 2 to prove the following:

Proposition 3. *Let $n \in \mathbb{Z}$ and $d > 2g - 2$. Then there exists a morphism*

$$\rho_d : \text{Hilb}^{d,n}(X)^{\mathbb{C}^*} \longrightarrow \text{Sym}^d(B).$$

At the level of closed points this morphism has the following description. Let $Z \in \text{Hilb}^{d,n}(X)^{\mathbb{C}^}$ and let $Z_{\text{CM}} \subset Z$ be the maximal Cohen-Macaulay subcurve of Z . Since Z_{CM} is \mathbb{C}^* -fixed, its ideal sheaf decomposes*

$$I_{Z_{\text{CM}}} = \bigoplus_{k=0}^{\infty} I_{Y_k \subset S} \otimes K_S^{-k},$$

where

$$Y_0 = B \cup \lambda_0^{(1)} F_{x_1} \cup \dots \cup \lambda_0^{(n)} F_{x_n}$$

for some distinct closed points $x_i \in B$ and $\lambda_0^{(i)} > 0$ and

$$Y_k = \lambda_k^{(1)} F_{x_1} \cup \dots \cup \lambda_k^{(n)} F_{x_n}.$$

for some $\lambda_{k-1}^{(i)} \geq \lambda_k^{(i)}$. Here $\lambda^{(i)} = (\lambda_0^{(i)} \geq \lambda_1^{(i)} \geq \dots)$ define 2D partitions satisfying

$$\sum_{i=1}^n |\lambda^{(i)}| = d.$$

The map ρ_d sends Z to

$$\sum_{i=1}^n |\lambda^{(i)}| x_i \in \text{Sym}^d(B).$$

Remark 4. The morphism of this proposition is perhaps somewhat surprising. Since we are on a 3-fold, the map which sends a closed subscheme of $Z \in \text{Hilb}^{d,n}(X)$ to its underlying Cohen-Macaulay curve Z_{CM} is *not* a morphism. Nevertheless, the map ρ_d which records the location of the fibres in Z_{CM} and their multiplicities, is a morphism.

Proof. The description of ρ_d at the level of closed points is clear. We must construct ρ_d as a morphism from Proposition 2 and Lemma 17 of Appendix A.1.

Let T be an arbitrary base scheme of finite type and let

$$\mathcal{Y} \subset X \times T$$

be a \mathbb{C}^* -fixed and T -flat closed subscheme. Assume for each $t \in T$ the fibre \mathcal{Y}_t has class $B + dF \in H_2(S)$ and $\chi(\mathcal{O}_{\mathcal{Y}_t}) = n$. Since \mathcal{Y} is \mathbb{C}^* -fixed, Proposition 2 implies that its ideal sheaf decomposes⁴

$$I_{\mathcal{Y}} = \bigoplus_{k=0}^{\infty} I_{\mathcal{Y}_k \subset S \times T} \otimes K_S^{-k},$$

where K_S is pulled-back along $S \times T \rightarrow S$ and

$$\mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \dots.$$

Then each $\mathcal{Y}_k \subset S \times T$ is T -flat as well. The maximal CM subschemes $\mathcal{Y}_{k,\text{CM}} \subset \mathcal{Y}_k \subset S \times T$ are T -flat as well and induces morphisms

⁴The arguments leading to Proposition 2 hold equally well for T -flat families over a base T .

I can provide explanation why $\mathcal{Y}_{\text{CM},k}$ also T -flat. Needed?

$$\begin{aligned} T &\longrightarrow \mathrm{Hilb}^{B+d_0F}(S), \\ T &\longrightarrow \mathrm{Hilb}^{d_kF}(S), \text{ for } k > 0 \end{aligned}$$

where $\sum_k d_k = d$. Adding divisors gives a morphism $T \longrightarrow \mathrm{Hilb}^{B+dF}(S)$. By Lemma 17, we obtain a morphism $T \rightarrow \mathrm{Sym}^d(B)$. This morphism corresponds to a T -flat family for $\mathrm{Sym}^d(B)$. We have defined ρ_d as a morphism. \square

4. PUSH-FORWARD TO THE SYMMETRIC PRODUCT

In the previous section we constructed a morphism (Proposition 3)

$$(2) \quad \rho_d : \mathrm{Hilb}^{d, \bullet}(X)^{\mathbb{C}^*} \longrightarrow \mathrm{Sym}^d(B).$$

We obtain

$$\int_{\mathrm{Hilb}^{d, \bullet}(X)^{\mathbb{C}^*}} 1 \, de = \int_{\mathrm{Sym}^d(B)} \rho_{d*}(1) \, de,$$

where $f_d := \rho_{d*}(1)$ is a constructible function on $\mathrm{Sym}^d(B)$. Its value at a closed point $\mathfrak{a} \in \mathrm{Sym}^d(B)$ is

$$f_d(\mathfrak{a}) = \int_{\rho_d^{-1}(\mathfrak{a})} 1 \, de.$$

We are interested in the calculation of

$$\widehat{\mathrm{DT}}(X) = \sum_{d \geq 0} \widehat{\mathrm{DT}}_d(X) q^d = \sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d \, de.$$

It turns out that the constructible function $f_d : \mathrm{Sym}^d(B) \rightarrow \mathbb{Z}((p))$ has two multiplicative properties. The first one is described as follows. Denote by $B^{\mathrm{sm}} \subset B$ the open subset over which the fibres are smooth and by B^{sing} the N points over which the fibres are singular. We can consider the restrictions of f_d to $\mathrm{Sym}^d(B^{\mathrm{sm}}) \subset \mathrm{Sym}^d(B)$ and $\mathrm{Sym}^d(B^{\mathrm{sing}}) \subset \mathrm{Sym}^d(B)$. Denote by $M(p)$ the MacMahon function.

Proposition 5. *Let $d_1, d_2 \geq 0$ be such that $d_1 + d_2 = d$. Then*

$$f_d(\mathfrak{a} + \mathfrak{b}) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \cdot f_{d_1}(\mathfrak{a}) \cdot f_{d_2}(\mathfrak{b}),$$

for any $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$.

We prove this proposition in Section 5.3. The following product formula is an immediate consequence of this result

$$(3) \quad \begin{aligned} &\sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d \, de = \\ &\frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \left(\sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sm}})} f_d \, de \right) \cdot \left(\sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sing}})} f_d \, de \right). \end{aligned}$$

The restricted constructible functions $f_d : \mathrm{Sym}^d(B^{\mathrm{sm}}) \rightarrow \mathbb{Z}((p))$ and $f_d : \mathrm{Sym}^d(B^{\mathrm{sing}}) \rightarrow \mathbb{Z}((p))$ satisfy further multiplicative properties:

Proposition 6. *There exist functions $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}((p))$ and $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}((p))$ taking values in formal Laurent series $\mathbb{Z}((p))$, such that $g(0) = 1$, $h(0) = 1$, and*

$$f_d(\mathbf{a}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{i=1}^m g(a_i),$$

$$f_d(\mathbf{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{j=1}^n h(b_j),$$

for all $d \geq 0$, $\mathbf{a} = \sum_{i=1}^m a_i x_i \in \text{Sym}^d(B^{\text{sm}})$, and $\mathbf{b} = \sum_{j=1}^n b_j y_j \in \text{Sym}^d(B^{\text{sing}})$, where $x_i \in B^{\text{sm}}$ and $y_j \in B^{\text{sing}}$ are collections of distinct closed points.

We prove this proposition in Section 5.3. Together with Lemma 18 from the appendix, Proposition 6 and equation (3) imply

$$(4) \quad \sum_{d \geq 0} q^d \int_{\text{Sym}^d(B)} f_d de = \frac{M(p)^N}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)-N} \cdot \left(\sum_{b=0}^{\infty} h(b) q^b \right)^N.$$

Our goal is to prove Propositions 5, 6, and find formulae for $g(a)$, $h(b)$. This requires a better understanding of the strata

$$\rho_d^{-1}(\mathbf{a} + \mathbf{b}) \subset \text{Hilb}^{d, \bullet}(X)^{\mathbb{C}^*},$$

for all $\mathbf{a} \in \text{Sym}^{d_1}(B^{\text{sm}})$ and $\mathbf{b} \in \text{Sym}^{d_2}(B^{\text{sing}})$ with $d_1 + d_2 = d$. Now suppose

$$\mathbf{a} = \sum_{i=1}^m a_i x_i \in \text{Sym}^{d_1}(B^{\text{sm}}),$$

$$\mathbf{b} = \sum_{j=1}^n b_j y_j \in \text{Sym}^{d_2}(B^{\text{sing}}),$$

where $x_i \in B^{\text{sm}}$ and $y_j \in B^{\text{sing}}$ are collections of distinct closed points and $d_1 + d_2 = d$. Proposition 3 gives a decomposition of $\rho_d^{-1}(\mathbf{a} + \mathbf{b})$ into components⁵

$$(5) \quad \bigsqcup_{\substack{\lambda^{(1)} \vdash a_1 \\ \vdots \\ \lambda^{(m)} \vdash a_m}} \bigsqcup_{\substack{\mu^{(1)} \vdash b_1 \\ \vdots \\ \mu^{(n)} \vdash b_n}} \Sigma(x_1, \dots, x_m, y_1, \dots, y_n, \lambda^{(1)}, \dots, \lambda^{(m)}, \mu^{(1)}, \dots, \mu^{(n)}).$$

We abbreviate these components by $\Sigma(\mathbf{x}; \mathbf{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$. We see that $\Sigma(\mathbf{x}; \mathbf{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$ can be characterized as the stratum of points $Z \in \text{Hilb}^{d, \bullet}(X)^{\mathbb{C}^*}$, for which the maximal Cohen-Macaulay subcurve $Z_{\text{CM}} \subset Z$ is determined by the data $\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$ as in Proposition 3. Note that these strata have a natural scheme structure: the fibres of ρ_d are closed subschemes of $\text{Hilb}^{d, \bullet}(X)^{\mathbb{C}^*}$ and these strata are components of them. We are interested in the Euler characteristics of these strata. In the next section, we will see the Euler characteristic of $\Sigma(\mathbf{x}; \mathbf{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$ does *not* depend on the exact location of the points $x_i \in B^{\text{sm}}$ and $y_j \in B^{\text{sing}}$, but only on their number m and n and the partitions $\lambda^{(i)}$ and $\mu^{(j)}$.

⁵We use the term component somewhat loose: it means a subset which is both open and closed. We do not care whether it is connected.

5. RESTRICTION TO FORMAL NEIGHBOURHOODS

In the previous two sections we reduced our consideration to the strata $\Sigma(\mathbf{x}; \mathbf{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$ of $Z \in \text{Hilb}^{d, \bullet}(X)^{\mathbb{C}^*}$ for which the maximal Cohen-Macaulay subcurve $Z_{\text{CM}} \subset Z$ is determined by the data $\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$. In this section we further break down this stratum by cutting it up in pieces covered by formal neighbourhoods. For notational simplicity, we first consider the case where the base point is

$$ax + by \in \text{Sym}^d(B),$$

with $x \in B^{\text{sm}}, y \in B^{\text{sing}}$, and $d = a + b$. We show how to compute $e(\Sigma(x, y, \lambda, \mu))$. Once this case is established, it is not hard to generalize to arbitrary $e(\Sigma(\mathbf{x}; \mathbf{y}; \boldsymbol{\lambda}; \boldsymbol{\mu}))$. This leads to a proof of Propositions 5, 6, and a geometric characterization of the functions $g(a), h(b)$ of Section 4.

5.1. fpqc cover. The idea is to use an appropriate cover of X and calculate on pieces of the cover. We first give a complex analytic definition of the cover to aid the intuition and then give the actual “algebraic-geometric cover”:

- (1) The reduced support $B \cup F_x \cup F_y$ has three singular points⁶: $x, y \in B$ and $z \in F_y^{\text{sing}}$. We take small open balls around these points.
- (2) Consider the punctured curve $B^\circ := B \setminus \{x, y\}$ and let $X^\circ := X \setminus (F_x \cup F_y)$. We take a tubular neighbourhood of $B^\circ \subset X^\circ$.
- (3) Consider the punctured curve $F_x^\circ := F_x \setminus \{x\}$ and let $X^\circ := X \setminus B$. We take a tubular neighbourhood of $F_x^\circ \subset X^\circ$.
- (4) Consider the punctured curve $F_y^\circ := F_y \setminus \{y, z\}$ and let $X^\circ := X \setminus (B \cup \{z\})$. We take a tubular neighbourhood of $F_y^\circ \subset X^\circ$.
- (5) Finally, we take $W = X \setminus (B \cup F_x \cup F_y)$.

In order to work in algebraic geometry, in (1) we take the formal neighbourhood \widehat{X}_x of $\{x\}$ in X . Denote the local ring at x by (R, \mathfrak{m}) . Then by \widehat{X}_x we mean the (non-noetherian) scheme

$$\text{Spec } \varprojlim R/\mathfrak{m}^n$$

and *not* the formal scheme

$$\text{Spf } \varprojlim R/\mathfrak{m}^n.$$

Similarly in (2) and (3), let \widehat{X}_y be the formal neighbourhood of $\{y\}$ in X and \widehat{X}_z the formal neighbourhood of $\{z\}$ in X . Note that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \text{Spec } \mathbb{C}[[x_1, x_2, x_3]].$$

Even though \widehat{X}_x is non-noetherian, the morphism $\widehat{X}_x \rightarrow X$ has a good property: it is fpqc so can be used as part of a cover [Vis, Sect. 2.3.2]. Flatness of this map follows from the fact that formal completion is an exact operation [AM, Prop. 10.14].

In (2) we consider $B^\circ := B \setminus \{x, y\}$, $X^\circ := X \setminus (F_x \cup F_y)$ and let $\widehat{X}_{B^\circ}^\circ$ be the formal neighbourhood of F_x° in X° . For (3) and (4) the formal neighbourhoods $\widehat{X}_{F_x^\circ}^\circ$ and $\widehat{X}_{F_y^\circ}^\circ$ are defined analogously. Note that the definition of X° in (2)–(4) varies slightly. Finally, in (5) we take $W = X \setminus (B \cup F_x \cup F_y)$. Then

$$\mathcal{U} = \{\widehat{X}_x \rightarrow X, \widehat{X}_y \rightarrow X, \widehat{X}_z \rightarrow X, \widehat{X}_{B^\circ}^\circ \rightarrow X, \widehat{X}_{F_x^\circ}^\circ \rightarrow X, \widehat{X}_{F_y^\circ}^\circ \rightarrow X, W \subset X\}$$

⁶Recall that $x, y \in B$ in the base can be viewed as points on S and X via the sections $B \subset S \subset X$.

Completion=faithfully flat: stacks project Lem. 10.96.3.

forms an fpqc cover of X . Consequently the data of a quasi-coherent sheaf on X is equivalent to the data of quasi-coherent sheaves on each of the opens of \mathcal{U} and gluing isomorphisms between the restrictions on the overlaps. Technically: quasi-coherent sheaves on X form a stack with respect to the fpqc topology [Vis, Thm. 4.23].

5.2. Local moduli spaces. We now introduce moduli spaces of closed subschemes of dimension ≤ 1 on each of the pieces of the cover \mathcal{U} . Assume the coordinates on

$$\widehat{X}_x \cong \operatorname{Spec} \mathbb{C}[[x_1, x_2, x_3]]$$

are chosen such that $x_2 = x_3 = 0$ corresponds to the intersection $\widehat{X}_x \times_X B$ and $x_1 = x_3 = 0$ corresponds to $\widehat{X}_x \times_X F_x$. Define

$$\begin{aligned} \operatorname{Hilb}^{(1,d),n}(\widehat{X}_x) &:= \\ \{I_Z \subset \mathcal{O}_{\widehat{X}_x} : [Z] &= [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x] \text{ and } h^0(I_{Z_{\text{CM}}}/I_Z) = n\}. \end{aligned}$$

Here the equation

$$[Z] = [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x]$$

means Z is supported along

$$(\widehat{X}_x \times_X B) \cup (\widehat{X}_x \times_X F_x)$$

with multiplicity 1 along $(\widehat{X}_x \times_X B)$ and multiplicity d along $(\widehat{X}_x \times_X F_x)$. Furthermore, Z_{CM} denotes the maximal Cohen-Macaulay subcurve of Z . The ideal sheaves fit into a short exact sequence

$$0 \longrightarrow I_Z \longrightarrow I_{Z_{\text{CM}}} \longrightarrow Q \longrightarrow 0,$$

where Q is a 0-dimensional. The Hilbert scheme $\operatorname{Hilb}^{(1,d),n}(\widehat{X}_y)$ is defined likewise replacing the point x by y . For \widehat{X}_z , we define

$$\operatorname{Hilb}^{d,n}(\widehat{X}_z) := \{I_Z \subset \mathcal{O}_{\widehat{X}_z} : [Z] = d[\widehat{X}_z \times_X F_y] \text{ and } h^0(I_{Z_{\text{CM}}}/I_Z) = n\}.$$

Each of $\widehat{X}_x, \widehat{X}_y, \widehat{X}_z$ has an action of \mathbb{C}^* compatible with the fibre scaling on X . This action lifts to the moduli space. Moreover, since each of these formal neighbourhoods is isomorphic to $\operatorname{Spec} \mathbb{C}[[x_1, x_2, x_3]]$, the bigger torus \mathbb{C}^{*3} acts on it and this action lifts to the moduli space. The existence of these “extra actions” will be used in Section 6.

Next consider $\widehat{X}_{B^\circ}^\circ$, the formal neighbourhood of the punctured zero section $B^\circ \subset X^\circ$. Define

$$\operatorname{Hilb}^{1,n}(\widehat{X}_{B^\circ}^\circ) := \{I_Z \subset \mathcal{O}_{\widehat{X}_{B^\circ}^\circ} : [Z] = [\widehat{X}_{B^\circ}^\circ \times_X B] \text{ and } h^0(I_{Z_{\text{CM}}}/I_Z) = n\}.$$

For $\widehat{X}_{F_x^\circ}^\circ, \widehat{X}_{F_y^\circ}^\circ$ we define

$$\operatorname{Hilb}^{d,n}(\widehat{X}_{F_x^\circ}^\circ) := \{I_Z \subset \mathcal{O}_{\widehat{X}_{F_x^\circ}^\circ} : [Z] = d[\widehat{X}_{F_x^\circ}^\circ \times_X B] \text{ and } h^0(I_{Z_{\text{CM}}}/I_Z) = n\},$$

$$\operatorname{Hilb}^{d,n}(\widehat{X}_{F_y^\circ}^\circ) := \{I_Z \subset \mathcal{O}_{\widehat{X}_{F_y^\circ}^\circ} : [Z] = d[\widehat{X}_{F_y^\circ}^\circ \times_X B] \text{ and } h^0(I_{Z_{\text{CM}}}/I_Z) = n\}.$$

Finally, for W we define

$$\operatorname{Hilb}^{0,n}(W) := \{I_Z \subset \mathcal{O}_W : \dim(Z) = 0 \text{ and } h^0(\mathcal{O}_Z) = n\}.$$

On $\widehat{X}_{B^\circ}^\circ, \widehat{X}_{F_x^\circ}^\circ, \widehat{X}_{F_y^\circ}^\circ$, and W we have an action of \mathbb{C}^* compatible with the fibre scaling on X . These actions lift to the moduli space. However, *unlike* for $\widehat{X}_x, \widehat{X}_y, \widehat{X}_z$, no additional tori act.

As before, we use the notation $\text{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ for the union of all $\text{Hilb}^{(1,d),n}(\widehat{X}_x)$ (and similarly for all other moduli spaces of this section). Like in Section 3, the components of the \mathbb{C}^* -fixed locus of $\text{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ are indexed by 2D partitions

$$\text{Hilb}^{(1,d),\bullet}(\widehat{X}_x)^{\mathbb{C}^*} = \bigsqcup_{\lambda \vdash d} \text{Hilb}^{(1,d),\bullet}(\widehat{X}_x)_{\lambda}^{\mathbb{C}^*}.$$

Proposition 7. *Consider the stratum $\Sigma(x, y, a, b)$, where $a = |\lambda|$ and $b = |\mu|$. Restriction from X to the elements of the cover \mathfrak{U} induces a morphism*

$$(6) \quad \begin{aligned} \Sigma(x, y, \lambda, \mu) \longrightarrow & \text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_{\lambda}^{\mathbb{C}^*} \times \text{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_{\mu}^{\mathbb{C}^*} \times \text{Hilb}^{b,\bullet}(\widehat{X}_z)_{\mu}^{\mathbb{C}^*} \times \\ & \text{Hilb}^{1,\bullet}(\widehat{X}_{B^\circ})^{\mathbb{C}^*} \times \text{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ})_{\lambda}^{\mathbb{C}^*} \times \text{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ})_{\mu}^{\mathbb{C}^*} \times \\ & \text{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}, \end{aligned}$$

which is a bijection on closed points.

Proof. Since pull-back works in families, restriction defines a morphism between LHS and RHS. For the rest of the proof, we work on closed points only.

Since \mathfrak{U} is an fpqc cover, fpqc descent implies that any ideal sheaf $I_Z \subset \mathcal{O}_X$ is entirely determined by its restriction along the morphisms of the elements of \mathfrak{U} . This proves injectivity.

Conversely, given local ideal sheaves in the image of (6), their restrictions to overlaps only depend on the underlying Cohen-Macaulay curve (and not on the embedded points). Since we chose the strata such that the underlying Cohen-Macaulay curve automatically glues, there are no further gluing conditions and fpqc descent implies surjectivity. \square

Remark 8. Note that the argument of Proposition 7 is purely set-theoretic in nature. We do *not* claim (6) is an isomorphism of schemes.

Remark 9. It is important to relate holomorphic Euler characteristic of domain and target in (6). For any subscheme Z in the domain $\Sigma(x, y, \lambda, \mu)$, denote the corresponding maximal Cohen-Macaulay curve of its elements by Z_{CM} (Proposition 3). Then

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{Z_{\text{CM}}}) + \chi(I_{Z_{\text{CM}}}/I_Z).$$

Recall that Z_{CM} is entirely determined by the data x, y, λ, μ , where $\lambda = (\lambda_0 \geq \lambda_1 \geq \dots)$ and $\mu = (\mu_0 \geq \mu_1 \geq \dots)$ are 2D partitions (equation (5)). An easy calculation shows

$$\chi(\mathcal{O}_{Z_{\text{CM}}}) = \chi(\mathcal{O}_B) - \lambda_0 - \mu_0.$$

We conclude

$$(7) \quad \chi(\mathcal{O}_Z) = \frac{e(B)}{2} - \lambda_0 - \mu_0 + \chi(I_{Z_{\text{CM}}}/I_Z).$$

Proposition 7 allows us to calculate

$$f_d(ax + by) = e(\rho_d^{-1}(ax + by)) = \sum_{\lambda \vdash a} \sum_{\mu \vdash b} e(\Sigma(x, y, a, b)).$$

By Proposition 7 and (7) this equals

$$(8) \quad \begin{aligned} f_d(ax + by) = & p^{\frac{e(B)}{2}} e(\text{Hilb}^{1,\bullet}(\widehat{X}_{B^\circ})^{\mathbb{C}^*}) e(\text{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) \times \\ & \sum_{\lambda \vdash a} \sum_{\mu \vdash b} p^{-\lambda_0 - \mu_0} e(\text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_{\lambda}^{\mathbb{C}^*}) e(\text{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_{\mu}^{\mathbb{C}^*}) \times \\ & e(\text{Hilb}^{b,\bullet}(\widehat{X}_z)_{\mu}^{\mathbb{C}^*}) e(\text{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ})_{\lambda}^{\mathbb{C}^*}) e(\text{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ})_{\mu}^{\mathbb{C}^*}). \end{aligned}$$

Before we proceed, we calculate $e(\text{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*})$ and $e(\text{Hilb}^{1,\bullet}(\widehat{X}_{B^\circ}^\circ)^{\mathbb{C}^*})$. The first follows from a formula of J. Cheah [Che]

$$(9) \quad e(\text{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) = M(p)^{e(W)}.$$

For the second we use the following proposition:

Proposition 10. *Let $x_1, \dots, x_n \in B$ be any number of closed points. Define*

$$B^\circ := B \setminus \{x_1, \dots, x_n\},$$

$$X^\circ := X \setminus \bigcup_{i=1}^n F_{x_i}.$$

Let $\widehat{X}_{B^\circ}^\circ$ be the formal neighbourhood of B° in X° . Define $\text{Hilb}^{1,n}(\widehat{X}_{B^\circ}^\circ)$ as the Hilbert scheme of subschemes $Z \subset \widehat{X}_{B^\circ}^\circ$, such that $Z_{\text{CM}} = B^\circ$ and $\chi(I_{Z_{\text{CM}}}/I_Z) = n$. Then

$$e(\text{Hilb}^{1,\bullet}(\widehat{X}_{B^\circ}^\circ)) = \left(\frac{M(p)}{(1-p)} \right)^{e(B^\circ)}.$$

Proof. Let $x \in B^\circ$ and let $\widehat{X}_x \cong \text{Spec } \mathbb{C}[[x_1, x_2, x_3]]$ be the formal neighbourhood of x in X° . Denote by

$$\text{Hilb}^{1,n}(\widehat{X}_x^\circ)$$

the Hilbert scheme of subschemes $Z \subset \widehat{X}_x^\circ$, such that $Z_{\text{CM}} = \{x_2 = x_3 = 0\}$ and $\chi(I_{Z_{\text{CM}}}/I_Z) = n$.

We have projections

$$X^\circ \longrightarrow S^\circ \longrightarrow B^\circ.$$

Similar to Proposition 7, these map induces a morphism

$$\text{Hilb}^{1,n}(\widehat{X}_{B^\circ}^\circ) \longrightarrow \text{Sym}^n(B^\circ).$$

The fibre over a point $\mathfrak{a} = \sum_i a_i x_i$ equals

$$\prod_i \text{Hilb}^{1,a_i}(\widehat{X}_{x_i}).$$

This follows by using an appropriate fpqc cover of B° similar to Proposition 7. Therefore, Lemma 18 of the appendix implies

$$e(\text{Hilb}^{1,\bullet}(\widehat{X}_{B^\circ}^\circ)) = \left(\sum_{a=0}^{\infty} e(\text{Hilb}^{1,a}(\widehat{X}_x)) p^a \right)^{e(B^\circ)}.$$

The formal neighbourhood \widehat{X}_x has an action of \mathbb{C}^{*3} and this action lifts to $\text{Hilb}^{1,a}(\widehat{X}_x)$. The fixed locus consists of a finite number of points counted by the topological vertex⁷

$$V_{\square, \emptyset, \emptyset}(p) = \frac{M(p)}{1-p}.$$

The proof follows. □

⁷Discussed in general in Section 6.

Using (9) and Proposition 10, equation (8) becomes

$$\begin{aligned} f_d(ax + by) &= \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \times \\ &\quad (1-p) \sum_{\lambda \vdash a} p^{-\lambda_0} e(\text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_\lambda^{\mathbb{C}^*}) e(\text{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ})_\lambda^{\mathbb{C}^*}) \times \\ &\quad \frac{1-p}{M(p)} \sum_{\mu \vdash b} p^{-\mu_0} e(\text{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_\mu^{\mathbb{C}^*}) e(\text{Hilb}^{b,\bullet}(\widehat{X}_z)_\mu^{\mathbb{C}^*}) e(\text{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ})_\mu^{\mathbb{C}^*}). \end{aligned}$$

5.3. Geometric characterization of $g(a)$ and $h(b)$. The arguments of the preceding two sections are straightforwardly modified to any stratum $\Sigma(\mathbf{x}; \mathbf{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$. Fix a smooth fibre F_x and a singular fibre F_y . Denote the singular point of F_y by z . Let $\widehat{X}_x, \widehat{X}_z$ be the formal neighbourhoods of x, z in X . Define $\text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)$, $\text{Hilb}^{b,\bullet}(\widehat{X}_z)$ as in Section 5.2. As in Section 5.2, we also consider the “tubular” formal neighbourhoods $\widehat{X}_{F_x^\circ}^\circ, \widehat{X}_{F_y^\circ}^\circ$ and corresponding Hilbert schemes $\text{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ}^\circ)$, $\text{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ}^\circ)$. The arguments of this section yield:

Proposition 11. *For any $a, b > 0$ define*

$$\begin{aligned} g(a) &:= (1-p) \sum_{\lambda \vdash a} p^{-\lambda_0} e(\text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_\lambda^{\mathbb{C}^*}) e(\text{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ}^\circ)_\lambda^{\mathbb{C}^*}), \\ h(b) &:= \frac{1-p}{M(p)} \sum_{\mu \vdash b} p^{-\mu_0} e(\text{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_\mu^{\mathbb{C}^*}) e(\text{Hilb}^{b,\bullet}(\widehat{X}_z)_\mu^{\mathbb{C}^*}) e(\text{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ}^\circ)_\mu^{\mathbb{C}^*}), \end{aligned}$$

and let $g(0) := 1, h(0) := 1$. Then

$$f_d(\mathbf{a} + \mathbf{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_i g(a_i) \cdot \prod_j h(b_j),$$

for any $\mathbf{a} = \sum_i a_i x_i \in \text{Sym}^d(B^{\text{sm}})$ and $\mathbf{b} = \sum_j b_j y_j \in \text{Sym}^d(B^{\text{sing}})$, where $x_i \in B^{\text{sm}}$ and $y_j \in B^{\text{sing}}$ are collections of distinct closed points.

We immediately deduce:

Corollary 12. *Propositions 5 and 6 are true for $g(a)$ and $h(b)$ defined in Proposition 11.*

6. REDUCTION TO THE TOPOLOGICAL VERTEX

In this section, we prove the main result of this paper: expressions for $\widehat{\text{DT}}(X)$ and $\widehat{\text{DT}}^{\text{conn}}(X)$ in terms of the topological vertex $V_{\lambda,\mu,\nu}(p)$, $e(B)$, and N (the number of nodal fibres). This is Theorem 16 below. The theorem follows by expressing $g(a)$ and $h(b)$ of Proposition 11 in terms of the topological vertex.

6.1. Point contributions. We denote by

$$V_{\lambda,\mu,\nu}(p) = \sum_{\pi} p^{|\pi|},$$

the topological vertex of DT theory⁸ [MNOP1, MNOP2]. Here the sum is over all 3D partitions π with outgoing legs λ, μ, ν and $|\pi|$ denotes renormalized volume. For a 2D

⁸In general this depends on an equivariant measure $w(\pi)$ depending on equivariant parameters s_1, s_2, s_3 . If $s_1 + s_2 + s_3 = 0$, then $w(\pi) = \pm 1$. Since we are working with an Euler characteristic version of DT invariants, we take all signs to be $+1$.

partition $\lambda = (\lambda_0 \geq \lambda_1 \geq \dots)$, we write λ' for the corresponding transposed partition and

$$|\lambda| := \sum_{k=0}^{\infty} \lambda_k,$$

$$\|\lambda\| := \sum_{k=0}^{\infty} \lambda_k^2.$$

Proposition 13. *Let F_x be a smooth fibre and F_y a singular fibre with singularity z . Then for any $\lambda \vdash a$, $\mu \vdash b$*

$$p^{-\lambda_0} e(\text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_\lambda^{\mathbb{C}^*}) = V_{\lambda,\square,\emptyset}(p),$$

$$p^{-\mu_0} e(\text{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_\mu^{\mathbb{C}^*}) = V_{\mu,\square,\emptyset}(p),$$

$$p^{-\|\mu\|} e(\text{Hilb}^{b,\bullet}(\widehat{X}_z)_\mu^{\mathbb{C}^*}) = V_{\mu,\mu',\emptyset}(p).$$

Proof. Recall that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \text{Spec } \mathbb{C}[[x_1, x_2, x_3]].$$

Therefore, \mathbb{C}^{*3} acts on each of these schemes and their moduli spaces

$$\text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_\lambda^{\mathbb{C}^*}, \text{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_\mu^{\mathbb{C}^*}, \text{Hilb}^{b,\bullet}(\widehat{X}_z)_\mu^{\mathbb{C}^*}.$$

The coordinates can be chosen such that the action of the last factor of \mathbb{C}^{*3} corresponds to $x_3 \mapsto t_3 x_3$. This component acts trivially since we are already on the \mathbb{C}^* -fixed locus. The \mathbb{C}^{*3} -fixed locus consists of isolated reduced points corresponding to monomial ideals with asymptotics $(\lambda, \emptyset, \emptyset)$, $(\mu, \emptyset, \emptyset)$, (μ, μ', \emptyset) respectively⁹. These monomial ideals are exactly what the topological vertex counts.

Finally, note that the generating functions $e(\text{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_\lambda^{\mathbb{C}^*})$, $e(\text{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_\mu^{\mathbb{C}^*})$, $e(\text{Hilb}^{b,\bullet}(\widehat{X}_z)_\mu^{\mathbb{C}^*})$ all start with 1. On the other hand, from the definition

$$V_{\lambda,\square,\emptyset}(p) = p^{-\lambda_0} + \dots,$$

$$V_{\mu,\square,\emptyset}(p) = p^{-\mu_0} + \dots,$$

$$V_{\mu,\mu',\emptyset}(p) = p^{-\sum_{k=0}^{\infty} \mu_k^2} + \dots,$$

where \dots stands for higher order terms in p . The proposition follows. \square

6.2. Fibre contribution. Let F_x be a smooth fibre and F_y a singular fibre. Recall the formal neighbourhoods $\widehat{X}_{F_x}^\circ$, $\widehat{X}_{F_y}^\circ$ of Section 5.

Proposition 14. *For any $\lambda \vdash a$ and $\mu \vdash b$, we have*

$$e(\text{Hilb}^{a,\bullet}(\widehat{X}_{F_x}^\circ)_\lambda^{\mathbb{C}^*}) = \frac{1}{V_{\lambda,\emptyset,\emptyset}(p)},$$

$$e(\text{Hilb}^{b,\bullet}(\widehat{X}_{F_y}^\circ)_\mu^{\mathbb{C}^*}) = \frac{1}{V_{\mu,\emptyset,\emptyset}(p)}.$$

Proof. We start with the first equation. Let $F := F_x \subset S$ be a smooth fibre. Consider the auxiliary surface

$$\tilde{S} = B \times F$$

and let $Y = \text{Tot}(K_{\tilde{S}})$. Denote by \widehat{X}_F the formal neighbourhood of F in X and by \widehat{Y}_F the formal neighbourhood of

$$F \cong \{x\} \times F \subset \tilde{S} \subset Y,$$

⁹The transpose in μ' occurs, because we follow the convention of [ORV].

where $\tilde{S} \subset Y$ denotes the zero section. Certainly \hat{X}_F and \hat{Y}_F are *not* isomorphic.

Next, denote by $\hat{X}_{F^\circ}^\circ$ the formal neighbourhood of $F \setminus B$ in $X \setminus B$. Moreover, we denote by

$$\hat{Y}_{F^\circ}^\circ$$

the formal neighbourhood of $F \setminus (B \times \{x\})$ inside $Y \setminus (B \times \{x\})$. Recall that $x \in F \cap B$ and $B \times \{x\} \subset \tilde{S} \subset Y$. After removing the section B , there exists an isomorphism

$$(10) \quad \hat{X}_{F^\circ}^\circ \cong \hat{Y}_{F^\circ}^\circ.$$

We are interested in the moduli space $\text{Hilb}^{a,\bullet}(\hat{X}_{F^\circ}^\circ)$ and the correspondingly defined moduli space $\text{Hilb}^{a,\bullet}(\hat{Y}_{F^\circ}^\circ)$. Since $\hat{X}_{F^\circ}^\circ, \hat{Y}_{F^\circ}^\circ$ have (compatible) \mathbb{C}^* -actions coming from scaling the fibres of X, Y , we can consider their \mathbb{C}^* -fixed loci and stratify them according to 2D partitions as in (5). By (10), we have obtained an isomorphism

$$\text{Hilb}^{a,\bullet}(\hat{X}_{F^\circ}^\circ)_{\lambda}^{\mathbb{C}^*} \cong \text{Hilb}^{a,\bullet}(\hat{Y}_{F^\circ}^\circ)_{\lambda}^{\mathbb{C}^*}.$$

This observation allows us to work in the much simpler geometry of Y . Note that this is only possible because we removed the section B from X and Y .

Let $F \subset \tilde{S} \subset Y$ be as above. Denote by \hat{Y}_y the formal neighbourhood $y \in Y$, where y is the intersection of $F \cong \{x\} \times F$ and $B \times \{x\}$ inside the zero section $\tilde{S} \subset Y$. Let $\hat{Y}_F, \hat{Y}_{F^\circ}^\circ$ be the formal neighbourhoods introduced above. Then we have an fpqc cover

$$\{\hat{Y}_y \rightarrow \hat{Y}_F, \hat{Y}_{F^\circ}^\circ \rightarrow \hat{Y}_F\}.$$

On these pieces, we introduce moduli spaces as in Section 5

$$\text{Hilb}^{a,\bullet}(\hat{Y}_y), \text{Hilb}^{a,\bullet}(\hat{Y}_F), \text{Hilb}^{a,\bullet}(\hat{Y}_{F^\circ}^\circ).$$

Similar to Proposition 7, restriction gives a bijective morphism on closed points

$$\text{Hilb}^{a,\bullet}(\hat{Y}_F)_{\lambda}^{\mathbb{C}^*} \rightarrow \text{Hilb}^{a,\bullet}(\hat{Y}_y)_{\lambda}^{\mathbb{C}^*} \times \text{Hilb}^{a,\bullet}(\hat{Y}_{F^\circ}^\circ)_{\lambda}^{\mathbb{C}^*}.$$

Recall that $\tilde{S} = B \times F$. Therefore F does not only act on $F \subset \tilde{S}$, but on any thickening $dF \subset \tilde{S}$. This is because

$$\mathcal{O}_{dF} = \mathcal{O}_{dx} \otimes \mathcal{O}_F,$$

where $dx \subset B$ denotes the d times thickening of the point $x \in B$. Moreover, F acts on the thickened curve defined by the ideal sheaf

$$\bigoplus_{k=0}^{\infty} \mathcal{O}_{\tilde{S}}(-\lambda_k F) \otimes K_{\tilde{S}}^{-k}.$$

The action of the elliptic curve F on itself is fixed-point-free, so it lifts to a free action on $\text{Hilb}^{a,\bullet}(\hat{Y}_F)_{\lambda}^{\mathbb{C}^*}$. Since $e(F) = 0$, we deduce

$$(11) \quad e(\text{Hilb}^{a,\bullet}(\hat{Y}_F)_{\lambda}^{\mathbb{C}^*}) = 1.$$

Finally, since $\hat{Y}_y \cong \text{Spec } \mathbb{C}[[x_1, x_2, x_3]]$, we have an action of \mathbb{C}^{*3} and

$$(12) \quad e(\text{Hilb}^{a,\bullet}(\hat{Y}_y)_{\lambda}^{\mathbb{C}^*}) = e(\text{Hilb}^{a,\bullet}(\hat{Y}_y)_{\lambda}^{\mathbb{C}^{*3}}) = V_{\lambda, \emptyset, \emptyset}(p).$$

We conclude from equations (10), (11), (12) that

$$\begin{aligned} 1 &= e(\text{Hilb}^{a,\bullet}(\hat{Y}_F)_{\lambda}^{\mathbb{C}^*}) = e(\text{Hilb}^{a,\bullet}(\hat{Y}_y)_{\lambda}^{\mathbb{C}^*}) e(\text{Hilb}^{a,\bullet}(\hat{Y}_{F^\circ}^\circ)_{\lambda}^{\mathbb{C}^*}) \\ &= V_{\lambda, \emptyset, \emptyset}(p) \cdot e(\text{Hilb}^{a,\bullet}(\hat{Y}_{F^\circ}^\circ)_{\lambda}^{\mathbb{C}^*}) \\ &= V_{\lambda, \emptyset, \emptyset}(p) \cdot e(\text{Hilb}^{a,\bullet}(\hat{X}_{F^\circ}^\circ)_{\lambda}^{\mathbb{C}^*}). \end{aligned}$$

Provide more argument here?

The equation for $e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ}^\circ)_\mu^{\mathbb{C}^*})$ can be deduced similarly. This time, the smooth fibre $F = F_x \subset S \subset X$ is replaced by the *smooth locus* of the singular fibre, i.e.

$$F' := F_y^{\mathrm{sm}} = F_y \setminus \{z\},$$

where z denotes the singularity of F_y . Note that

$$F' \cong \mathbb{P}^1 \setminus \{2 \text{ points}\} \cong \mathbb{C}^*.$$

Therefore, we again have a free action of F' on itself and $e(F') = 0$. The rest of the proof follows the same steps. \square

6.3. Main theorem. Combining Proposition 11 with Propositions 13, 14 immediately gives:

Proposition 15. *For any $a, b > 0$*

$$(13) \quad \begin{aligned} g(a) &= (1-p) \sum_{\lambda \vdash a} \frac{V_{\lambda, \square, \emptyset}(p)}{V_{\lambda, \emptyset, \emptyset}(p)}, \\ h(b) &= \frac{1-p}{M(p)} \sum_{\mu \vdash b} \frac{V_{\mu, \square, \emptyset}(p) V_{\mu, \mu', \emptyset}(p) p^{\|\mu\|}}{V_{\mu, \emptyset, \emptyset}(p)}. \end{aligned}$$

Putting all our results together, we obtain our main theorem:

Theorem 16.

$$\begin{aligned} \widehat{\mathrm{DT}}(X) &= \frac{1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \left(\sum_{\lambda} \frac{V_{\lambda, \square, \emptyset}(p)}{V_{\lambda, \emptyset, \emptyset}(p)} q^{|\lambda|} \right)^{e(B)-N} \left(\sum_{\mu} \frac{V_{\mu, \square, \emptyset}(p) V_{\mu, \mu', \emptyset}(p) p^{\|\mu\|}}{V_{\mu, \emptyset, \emptyset}(p)} q^{|\mu|} \right)^N \\ \widehat{\mathrm{DT}}^{\mathrm{conn}}(X) &= \frac{1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \left(\frac{\sum_{\lambda} \frac{V_{\lambda, \square, \emptyset}(p)}{V_{\lambda, \emptyset, \emptyset}(p)} q^{|\lambda|}}{\sum_{\lambda} q^{|\lambda|}} \right)^{e(B)-N} \left(\frac{\sum_{\mu} \frac{V_{\mu, \square, \emptyset}(p) V_{\mu, \mu', \emptyset}(p) p^{\|\mu\|}}{V_{\mu, \emptyset, \emptyset}(p)} q^{|\mu|}}{\sum_{\mu} V_{\mu, \mu', \emptyset}(p) p^{\|\mu\|} q^{|\mu|}} \right)^N. \end{aligned}$$

Proof. Inserting the equations for $g(a)$, $h(b)$ of Proposition 15 into (4) gives the formula for $\widehat{\mathrm{DT}}(X)$. Following the exact same line of reasoning of Sections 4–6, it is easy to see that

$$\sum_{d \geq 0} \sum_{n \in \mathbb{Z}} e(\mathrm{Hilb}^{dF, n}(X)) p^n q^d = \left(\sum_{\lambda} q^{|\lambda|} \right)^{e(B)-N} \left(\sum_{\mu} V_{\mu, \mu', \emptyset}(p) p^{\|\mu\|} q^{|\lambda|} \right)^N.$$

The formula for $\widehat{\mathrm{DT}}^{\mathrm{conn}}(X)$ follows after dividing $\widehat{\mathrm{DT}}(X)$ by this expression. \square

7. INTRODUCING THE BEHREND FUNCTION

APPENDIX A. ODDS AND ENDS

A.1. Curves on elliptic surfaces. Let $p : S \rightarrow B$ be an elliptic surface with section $B \subset S$. In this appendix we allow any type of singular fibres. We assume S is not a product, which implies

$$p^* : \mathrm{Pic}^0(B) \xrightarrow{\cong} \mathrm{Pic}^0(S)$$

is an isomorphism [Mir, VII.1.1]. For any $\beta \in H_2(S)$, we denote by $\mathrm{Hilb}^\beta(S)$ the Hilbert scheme of effective divisors on S in class β .

Denote by $B \in H_2(S)$ the class of the section $B \subset S$ and by $F \in H_2(S)$ the class of the fibre. Then we have the following commutative diagram

$$\begin{array}{ccc}
 \mathrm{Sym}^d(B) & \longrightarrow & \mathrm{Pic}^d(B) \\
 \downarrow p^* & & \cong \downarrow p^* \\
 \mathrm{Hilb}^{dF}(S) & \longrightarrow & \mathrm{Pic}^{dF}(S) \\
 \downarrow +B & & \cong \downarrow \otimes \mathcal{O}_S(B) \\
 \mathrm{Hilb}^{B+dF}(S) & \longrightarrow & \mathrm{Pic}^{B+dF}(S).
 \end{array}$$

The horizontal arrows are Abel-Jacobi maps. The vertical arrows are induced by pull-back and adding the section $B \subset S$.

Lemma 17. *The above maps induce an isomorphism*

$$\mathrm{Sym}^d(B) \xrightarrow{\cong} \mathrm{Hilb}^{B+dF}(S).$$

Proof. Clearly p^* gives an isomorphism $\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{dF}(S)$ and $+B$ gives a closed embedding $\mathrm{Hilb}^{dF}(S) \hookrightarrow \mathrm{Hilb}^{B+dF}(S)$. Since $\mathrm{Sym}^d(B)$ is smooth and $\mathrm{Hilb}^{B+dF}(S)$ is reduced (by [Mum, Lect. 25]), it suffices to show

$$\mathrm{Sym}^d(B) \rightarrow \mathrm{Hilb}^{B+dF}(S)$$

is surjective on closed points.

For surjectivity, suppose D' is an effective divisor with class $B + dF$ which does *not* lie in the image. Firstly, we note that for any fibre F we have $D' \cdot F = 1$. Therefore D' contains a section $B' \subset S$ as an effective summand. Moreover $B \neq B'$ or else D' would lie in the image. Next, we take any D in the image and compare D and D' . Since

$$\mathcal{O}_S(D - D') \in \mathrm{Pic}^0(S) \cong \mathrm{Pic}^0(B),$$

after re-arranging we find that there are fibres F_{x_i}, F_{y_j} and $a_i \geq 0, b_j \geq 0$ such that

$$B + \sum_i a_i F_{x_i} \sim_{\mathrm{lin}} B' + \sum_j b_j F_{y_j},$$

where \sim_{lin} denotes linear equivalence. Hence there exists a pencil $\{C_t\}_{t \in \mathbb{P}^1}$ of effective divisors such that

$$C_0 = B + \sum_i a_i F_{x_i}, \quad C_\infty = B' + \sum_j b_j F_{y_j}.$$

Now fix a smooth fibre F . Then $C_t \cdot F = 1$ for any $t \in \mathbb{P}^1$, so we get a morphism

$$\mathbb{P}^1 \longrightarrow F, \quad t \mapsto C_t \cap F.$$

But F is a smooth elliptic curve so this map is constant. We conclude

$$B \cap F = C_0 \cap F = C_\infty \cap F = B' \cap F.$$

Since F was chosen arbitrary, we deduce that $B = B'$ which is a contradiction. \square

A.2. Weighted Euler characteristics of symmetric products. In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

Lemma 18. *Let B be a scheme of finite type over \mathbb{C} and let $e(B)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}((p))$ be any function with $g(0) = 1$. Let $f_d : \text{Sym}^d(B) \rightarrow \mathbb{Z}((p))$ be the constructible function defined by*

$$f_d(\mathbf{a}) = \prod_i g(a_i),$$

for all $\mathbf{a} = \sum_i a_i x_i \in \text{Sym}^d(B)$ with $x_i \in B$ distinct closed points. Then

$$\sum_{d=0}^{\infty} q^d \int_{\text{Sym}^d(B)} f_d de = \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)}.$$

Remark 19. In the special case where $g = f_d \equiv 1$, the lemma recovers MacDonald's formula:

$$\sum_{d=0}^{\infty} e(\text{Sym}^d(B)) q^d = \frac{1}{(1-q)^{e(B)}}.$$

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties defined by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure [1]. For convenience's sake, we provide a direct proof here.

Ref? Bryan-Young?

Proof. The d th symmetric product admits a stratification with strata labelled by partitions of d . Associated to any partition of d is a unique tuple (m_1, m_2, \dots) of non-negative integers with $\sum_{j=1}^{\infty} j m_j = d$. The stratum labelled by (m_1, m_2, \dots) parameterizes collections of points where there are m_j points of multiplicity j . The full stratification is given by:

$$\text{Sym}^d(B) = \bigsqcup_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left\{ \left(\prod_{j=1}^{\infty} B^{m_j} \right) - \Delta \right\} / \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, B^0 is a point, Δ is the large diagonal, and σ_m is the m th symmetric group. Note that the function f_d is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

For schemes over \mathbb{C} , topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\text{Sym}^d(B)} f_d de = \sum_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left(\prod_{j=1}^{\infty} g(j)^{m_j} \right) \frac{e(B^{\sum_j m_j} - \Delta)}{m_1! m_2! m_3! \dots}.$$

For any natural number N , the projection $B^N - \Delta \rightarrow B^{N-1} - \Delta$ has fibers of the form $B - \{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(B) - (N-1)$ and consequently, $e(B^N - \Delta) = (e(B) - (N-1)) \cdot e(B^{N-1} - \Delta)$. Thus by induction, we find $e(B^N - \Delta) = e(B) \cdot (e(B) - 1) \cdots (e(B) - (N-1))$ and so

$$\frac{e(B^{\sum_j m_j} - \Delta)}{m_1! m_2! m_3! \dots} = \binom{e(B)}{m_1, m_2, m_3, \dots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\begin{aligned} \sum_{d=0}^{\infty} q^d \int_{\mathrm{Sym}^d(B)} f_d \, de &= \sum_{(m_1, m_2, \dots)} \prod_{j=1}^{\infty} (g(j)q^j)^{m_j} \binom{e(B)}{m_1, m_2, m_3, \dots} \\ &= \left(1 + \sum_{j=1}^{\infty} g(j)q^j \right)^{e(B)} \end{aligned}$$

which proves the lemma. \square

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