DONALDSON-THOMAS INVARIANTS OF LOCAL ELLIPTIC SURFACES VIA THE TOPOLOGICAL VERTEX

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex proved in [4], we derive product formulas for the partition functions. The connected version of the partition function is written in terms of Jacobi forms. In the special case where the elliptic surface is a K3 surface, we get a new derivation of the Katz-Klemm-Vafa formula.

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Date: August 1, 2016.

1. Introduction

Let $p : S \to B$ be a non-trivial elliptic surface over a complex smooth projective curve B. We assume p has a section and all singular fibres are irreducible rational nodal curves.

We are interested in the Donaldson-Thomas (DT) invariants of $X = \mathrm{Tot}(K_S)$, i.e. the total space of the canonical bundle K_S. This is a non-compact Calabi-Yau threefold. Let β be an effective curve class on S. Consider the Hilbert scheme

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X \; : \; [Z] = \beta, \; \chi(\mathcal{O}_Z) = n\}$$

of proper subschemes $Z \subset X$ with homology class β and holomorphic Euler characteristics n. The DT invariants of X can be defined as

$$\mathsf{DT}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X), \nu) := \sum_{k \in \mathbb{Z}} k \, e(\nu^{-1}(k)),$$

where $e(\cdot)$ denotes topological Euler characteristic and $\nu : \mathrm{Hilb}^{\beta,n}(X) \to \mathbb{Z}$ is Behrend's constructible function [2]. We also consider an unweighted Euler characteristic version of these invariants

$$\widehat{\mathsf{DT}}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X)).$$

We choose a section $B \subset S$ and focus on the primitive classes $\beta = B + dF$, where B is the class of the chosen section and F the class of the fiber. We define the partition functions by

$$\widehat{\mathsf{DT}}(X) = \sum_{d=0}^{\infty} \sum_{n \in \mathbb{Z}} \widehat{\mathsf{DT}}_{B+dF,n}(X) p^n q^d,$$

$$\mathsf{DT}(X) = \sum_{d=0}^{\infty} \sum_{n \in \mathbb{Z}} \mathsf{DT}_{B+dF,n}(X) y^n q^d.$$

We also consider the partition functions for the invariants for multiples of the fiber class:

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \widehat{\mathsf{DT}}_{dF,n}(X) p^n q^d,$$

$$\mathsf{DT}_{\mathsf{fib}}(X) = \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \mathsf{DT}_{dF,n}(X) y^n q^d.$$

The main result of this paper are closed product formulas for the partition functions $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$. Assuming a general conjecture about the Behrend function, we also determine $\mathsf{DT}(X)$ and $\mathsf{DT}_{\mathsf{fib}}(X)$.

We use the notation

$$M(p,q) = \prod_{m=1}^{\infty} (1 - p^m q)^{-m}$$

and the shorthand $M(p) = M(p,1)$.

Theorem 1. *Let $e(S)$ and $e(B)$ denote the topological Euler characteristic of the elliptic surface and the base. Then*

$$\widehat{\mathsf{DT}}(X) = \left\{ M(p) \prod_{d=1}^{\infty} \frac{M(p,q^d)}{(1-q^d)} \right\}^{e(S)} \left\{ \frac{1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})} \prod_{d=1}^{\infty} \frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)} \right\}^{e(B)}$$

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = \left\{ M(p) \prod_{d=1}^{\infty} M(p,q^d) \right\}^{e(S)} \left\{ \prod_{d=1}^{\infty} \frac{1}{(1-q^d)} \right\}^{e(B)}$$

The ratio $\widehat{\mathsf{DT}}(X)/\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ can be considered as the generating function for the connected invariants in the classes $B + dF$. This series has a particularly nice form and can be written in terms of classical Jacobi forms. Consider the Dedekind eta function and the Jacobi theta function

$$\eta = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k),$$

$$\Theta = (p^{\frac{1}{2}} - p^{-\frac{1}{2}}) \prod_{k=1}^{\infty} \frac{(1 - pq^k)(1 - p^{-1}q^k)}{(1 - q^k)^2}.$$

Corollary 2. *The partition function of the connected invariants is given as follows*

$$\frac{\widehat{\mathsf{DT}}(X)}{\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)} = \left(q^{-\frac{1}{24}}\eta\right)^{-e(S)} \Theta^{-e(B)}.$$

In the case where $S \to \mathbb{P}^1$ is an elliptically fibred $K3$ surface, the above series specializes to the well-known Katz-Klemm-Vafa formula. Because X is non-compact, the connected series is required to obtain the KKV formula. Our result provides a new derivation of the KKV formula for primitive classes. The KKV formula was recently proved in all curve classes in [14]. Ours is the first derivation of the KKV formula, which does not depend on the Kawai-Yoshioka formula [9].

Our results can be extended to apply to the usual (Behrend function weighted) Donaldson-Thomas invariants if we assume a general conjecture which we formulate in Section 7. Our conjecture relates the Behrend function at subschemes with embedded points to the value of the Behrend function at the underlying Cohen-Macaulay subscheme and may be of independent interest.

Theorem 3. *Assume that Conecture 18 holds, then*

$$\mathsf{DT}(X) = (-1)^{\chi(\mathcal{O}_S)} \widehat{\mathsf{DT}}(X)$$

and

$$\mathsf{DT}_{\mathsf{fib}}(X) = \widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$$

under the change of variables

$$y = -p.$$

A similar phenomenon to the above is known to hold when X is a toric Calabi-Yau threefold.

We expect that the method of computation that we introduce in this paper should apply to other elliptically fibered geometries. Indeed, it has already found applications to the calculation of DT generating functions on $K3 \times E$, where E is an elliptic curve [3], and abelian 3-folds [5]. Even though the geometry under consideration is not toric, we combine $\mathbb{C}^*$-localization, motivic methods, formal methods, and $(\mathbb{C}^*)^3$-localization to end up with expressions that only depend on the topological vertex $\mathsf{V}_{\lambda\mu\nu}$, and the topological Euler characteristics $e(B)$, and $e(S)$. Here is a rough sketch: rewrite sketch.

(A) The action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space[1] $\mathrm{Hilb}^{B+dF,\bullet}(X)$. Therefore, we only have to understand the fixed locus $\mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*}$. Push-forward along $X \to S \to B$ induces a morphism

$$\rho_d : \mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*} \to \mathrm{Sym}^d(B).$$

[1]The bullet indicates that we take the union of $\mathrm{Hilb}^{B+dF,n}(X)$ over all n, see Convention 3.1.

This map is constructed in Section **??** . The fibres of ρ_d decompose into components according to the shape of the underlying Cohen-Macaulay curve. This leads to a decomposition over 2D partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$.

(B) The Euler characteristics of the fibres of ρ_d define a constructible function f_d on $\mathrm{Sym}^d(B)$. In Section 4, we show that if f_d satisfies a certain product formula, then $\widehat{\mathsf{DT}}(X)$ satisfies a corresponding product formula. This follows from general power structure arguments reviewed in Appendix A.2.

(C) A component Σ of a fibre of ρ_d indexed by λ can be further broken down by taking a certain fpqc cover of the underlying (now fixed) Cohen-Macaulay curve Z_{CM} determined by λ. This cover consists of formal neighbourhoods $\widehat{X}_x$ around the singular points x of the reduced support of Z_{CM} and "tubular neighbourhoods" along the reduced support of Z_{CM} after removing the singularities. Since Z_{CM} is already fixed, gluing is automatic. Hence restriction to the elements of the cover gives a bijection morphism of Σ to local Hilbert schemes on the elements of the cover. In Section 5, we show this leads to the product formula for f_d in (B).

(D) On the formal neighbourhoods $\widehat{X}_x$, we have an action of $\mathbb{C}^{*3}$. This allows us to express their contributions to the generating function in terms of the topological vertex. The contributions of the tubular neighbourhoods along the *punctured* section and fibres can also be expressed in terms of the topological vertex (utilizing a map to $\mathrm{Sym}^n(F)$ which records the location and multiplicity of the embedded points). This is worked out in Section 6.

2. Definitions, notation, and conventions

Let $p : S \to B$ be an elliptic surface over a smooth projective curve B. We assume:

(1) S is a non-trivial fibration,
(2) p has a section $B \subset S$,
(3) all singular fibres of p are irreducible rational nodal curves.

We note that the number of singular fibers is equal to $e(S)$.

We write F_x for the fibre $p^{-1}(\{x\})$ for all closed points $x \in B$. We choose a section $B \subset S$ and denote its class in $H_2(S)$ by B as well. We denote the class of the fibre by $F \in H_2(S)$.

For brevity, we define

$$\mathrm{Hilb}^{d,n}(X) := \mathrm{Hilb}^{B+dF,n}(X),$$
$$\widehat{\mathsf{DT}}_{d,n}(X) := \widehat{\mathsf{DT}}_{B+dF,n}(X).$$

Since we are dealing with generating functions and our calculations involve cut-paste methods on the Hilbert schemes, it is useful to introduce the following notation. We define

$$\mathrm{Hilb}^{d,\bullet}(X) := \sum_{n \in \mathbb{Z}} \mathrm{Hilb}^{d,n}(X)\, p^n,$$

where we view the right hand side as a formal Laurent series whose coefficients elements in the Grothendieck ring of varieties, i.e. $K_0(\mathrm{Var}_{\mathbb{C}})(\!(p)\!)$.

Convention 3.1. *When an index is replaced by a bullet, we will sum over the index, multiplying by the appropriate variable. We regard the result as a formal power (or Laurent) series whose coefficients lie in $K_0(\mathrm{Var}_{\mathbb{C}})$ and we extend operations of the Grothendieck group (addition, multiplication, Euler characteristic) to the series in the obvious way.*

For example

$$\mathrm{Hilb}^{\bullet,\bullet}(X) = \sum_{d=0}^{\infty}\sum_{n\in\mathbb{Z}} \mathrm{Hilb}^{d,n}(X) q^d p^n,$$

so that we can write

$$\widehat{\mathsf{DT}}(X) = e(\mathrm{Hilb}^{\bullet,\bullet}(X)).$$

It is notationally convenient to treat an Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, the measurable functions are constructible functions, and the measure of a set is given by its Euler characteristic. In this language we have

$$\widehat{\mathsf{DT}}_{\beta,n}(X) = \int_{\mathrm{Hilb}^{\beta,n}(X)} 1\, de, \qquad \mathsf{DT}_{\beta,n}(X) = \int_{\mathrm{Hilb}^{\beta,n}(X)} \nu\, de,$$

and following the bullet convention we have

$$\widehat{\mathsf{DT}}(X) = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)} 1\, de, \qquad \mathsf{DT}(X) = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)} \nu\, de.$$

We will also need notation for subsets of the Hilbert scheme which parameterize those subschemes obtained by adding embedded points to some fixed Cohen-Macaulay curve.

Definition 4. Let $U \subset X$ be an open set (possibly in the fpcq topology) and let $C \subset U$ be a (not necessarily reduced) Cohen-Macaulay subscheme of dimension 1 which we assume is the restriction of some $\overline{C} \subset X$ to U. We define

$$\mathrm{Hilb}^n(U, C) = \{Z \subset U \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n\}.$$

Via the inclusion $U \subset X$, $\mathrm{Hilb}^n(Y, C)$ can be viewed as a constructible subscheme of $\mathrm{Hilb}(X)$. It parameterizes subschemes which roughly speaking are obtained from C by adding n embedded points.

3. The reduction to partition thickened comb curves

The action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1\, de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\, de.$$

The main result of this section is a classification of the subschemes parameterized by $\mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*}$, namely the $\mathbb{C}^*$-invariant subschemes. We find that the maximal Cohen-Macaulay subscheme of a $\mathbb{C}^*$-invariant subscheme is determined by a point in $\mathrm{Sym}^d(B)$ along with some discrete data (a collection of integer partitions). We begin with some notation.

Definition 5. Let $T = \mathrm{Tot}(K_S|_B)$ and let $p : X \to T$ be the map induced by the elliptic fibration. We say that a subscheme $C \subset X$ is a **comb curve** if $C = B \cup p^{-1}(Z)$ where $Z \subset T$ is a 0-dimensional subscheme which is set-theoretically supported on B.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ be an integer partition. Then λ determines a 0-dimensional subscheme $Z_\lambda \subset \mathrm{Spec}\,\mathbb{C}[r, s]$ given by the monomial ideal $I_\lambda = (r^{\lambda_1}, r^{\lambda_2}s, \ldots, r^{\lambda_l}s^{l-1}, r^l)$. In terms of λ as a Young diagram, we note $(\rho, \sigma) \in \lambda$ if and only if $r^\rho s^\sigma \notin I_\lambda$.

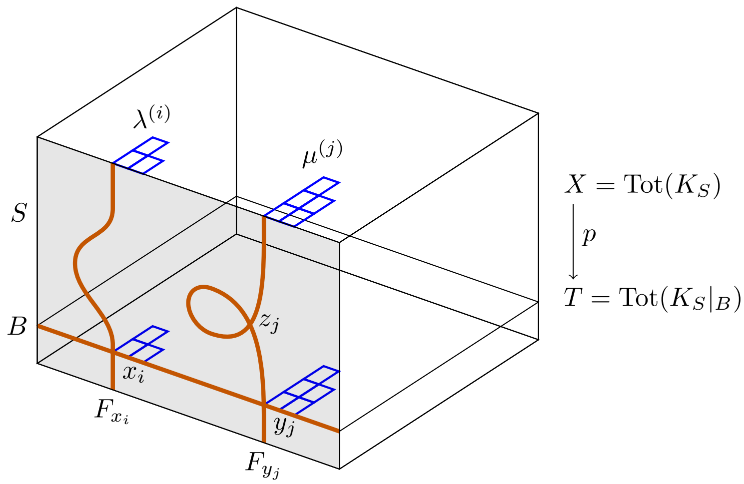

FIGURE 1. A partition thickened comb curve $C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right) \cup_j \left(\mu^{(j)} F_{y_j}\right)$.

Definition 6. Let $C = B \cup p^{-1}(Z)$ be a comb curve, let $x_1, \ldots, x_n \in B \subset T$ be the points where Z is supported, an let (r_i, s_i) be formal local coordinates on T about each point x_i so that s_i vanishes on $S \cap T$ and r_i vanishes on $R_i \cap T$ where $R_i = \mathrm{Tot}(K_S|_{F_{x_i}})$. We say that C is a **partition thickened comb curve** if there exists partitions $\lambda^{(1)}, \ldots, \lambda^{(n)}$ such that Z is given by $Z_{\lambda^{(i)}}$ in the local coordinates (r_i, s_i) about x_i. We denote such a curve by $B \cup_i \left(\lambda^{(i)} F_{x_i}\right)$. We say that a subscheme $Z \subset X$ is a **partition thickened comb curve with points (PCP)** if the maximal Cohen-Macaulay subscheme $Z_{CM} \subset Z$ is a partition thickened comb curve, in other words, Z is obtained from a partition thickened comb curve by adding embedded points and/or zero dimensional components. We denote by

$$\mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X) \subset \mathrm{Hilb}^{d,n}(X)$$

the locus in the Hilbert scheme parameterizing partition thickened comb curves with points.

In the next section it will be important to notationally distinguish between singular and smooth fibers. See figure 1 for an illustration of a partition thickened comb curve with smooth fibers $\{F_{x_i}\}$ thickened by partitions $\{\lambda^{(i}\}$ and nodal fibers $\{F_{y_j}\}$ thickened by partitions $\{\mu^{(j}\}$.

The main result of this section is the following

Theorem 7. *If a subscheme $Z \subset X$ is $\mathbb{C}^*$-invariant, then it is a partition thickened comb curve with points. That is*

$$\mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*} \subset \mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X) \subset \mathrm{Hilb}^{d,n}(X).$$

Moreover, $\mathbb{C}^*$ acts on $\mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X)$ and there exists a morphism

$$\rho_d : \mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X) \to \mathrm{Sym}^d(B) \tag{1}$$

such that for $[Z] \in \mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X)$ where the maximal Cohen-Macaulay subscheme of Z is $B \cup_i \left(\lambda^{(i)} F_{x_i}\right)$, then

$$\rho_d([Z]) = \sum_i |\lambda^{(i)}|\, x_i.$$

Insert Martijn's proof but adapted to the theorem as stated and maybe streamlined a bit.

Proof. Adapt and streamline Martijn's proof for the theorem as stated. □

4. PUSH-FORWARD TO THE SYMMETRIC PRODUCT

From the inclusions in Theorem 7, we have

$$\widehat{\mathsf{DT}}(X) = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)} 1\, de = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)^{\mathbb{C}^*}} 1\, de = \int_{\mathrm{Hilb}^{\bullet,\bullet}_{\mathsf{PCP}}(X)} 1\, de.$$

We compute these Euler characteristics by pushing forward along the map ρ_d constructed in Theorem 7. That is we use

$$\int_{\mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)} 1\, de = \int_{\mathrm{Sym}^d(B)} \rho_{d*}(1)\, de,$$

where $f_d := \rho_{d*}(1)$ is the $\mathbb{Z}((p))$-valued constructible function on $\mathrm{Sym}^d(B)$ given by pushing forward the Euler characteristic measure[10]. Its value at a closed point $\mathfrak{a} \in \mathrm{Sym}^d(B)$ is

$$f_d(\mathfrak{a}) = \int_{\rho_d^{-1}(\mathfrak{a})} 1\, de.$$

It turns out that the constructible function $f_d : \mathrm{Sym}^d(B) \to \mathbb{Z}((p))$ satisfies two multiplicative properties. The first one is described as follows. Denote by $B^{\mathrm{sm}} \subset B$ the open subset over which the fibres are smooth and by B^{sing} the $e(S)$ points over which the fibres are singular. We can consider the restrictions of f_d to $\mathrm{Sym}^d(B^{\mathrm{sm}}) \subset \mathrm{Sym}^d(B)$ and $\mathrm{Sym}^d(B^{\mathrm{sing}}) \subset \mathrm{Sym}^d(B)$.

Proposition 8. *Let $d_1, d_2 \geq 0$ be such that $d_1 + d_2 = d$. Then*

$$f_d(\mathfrak{a}+\mathfrak{b}) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \cdot f_{d_1}(\mathfrak{a}) \cdot f_{d_2}(\mathfrak{b}),$$

for any $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$.

We prove this proposition in Section **??**. The following product formula is an immediate consequence of this result

(2)
$$\widehat{\mathsf{DT}}(X) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \left(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sm}})} f_d\, de \right) \cdot \left(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sing}})} f_d\, de \right).$$

The restricted constructible functions $f_d : \mathrm{Sym}^d(B^{\mathrm{sm}}) \to \mathbb{Z}((p))$ and $f_d : \mathrm{Sym}^d(B^{\mathrm{sing}}) \to \mathbb{Z}((p))$ satisfy further multiplicative properties:

Proposition 9. *There exist functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$ and $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$ taking values in formal Laurent series $\mathbb{Z}((p))$, such that $g(0) = 1$, $h(0) = 1$, and*

$$f_d(\mathfrak{a}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{i=1}^{l} g(a_i),$$

$$f_d(\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{j=1}^{m} h(b_j),$$

for all $\mathfrak{a} = \sum_{i=1}^n a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$, and $\mathfrak{b} = \sum_{j=1}^m b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$, where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.

We prove this proposition in Section **??**. Together with Lemma 26 of Appendix A.2, Proposition 9 and equation (2) imply

$$\widehat{\mathsf{DT}}(X) = \frac{M(p)^{e(S)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)-e(S)} \cdot \left(\sum_{b=0}^{\infty} h(b) q^b \right)^{e(S)}. \tag{3}$$

Our goal is to prove Propositions 8 and 9, and find formulae for $g(a)$, $h(b)$. This requires a better understanding of the strata

$$\rho_d^{-1}(\mathfrak{a} + \mathfrak{b}) \subset \mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X),$$

for all $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$ with $d_1 + d_2 = d$. Suppose

$$\mathfrak{a} = \sum_{i=1}^{n} a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}}),$$
$$\mathfrak{b} = \sum_{j=1}^{m} b_j y_j \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}}),$$

where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points. Theorem 7 gives the following decomposition of the fibers of ρ_d into components[2]

$$\rho_d^{-1}(\mathfrak{a} + \mathfrak{b}) = \bigsqcup_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \bigsqcup_{\boldsymbol{\mu} \vdash \boldsymbol{b}} \Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{4}$$

where

$$\boldsymbol{a} = (a_1, \ldots, a_n), \quad \boldsymbol{b} = (b_1, \ldots, b_m),$$
$$\boldsymbol{x} = (x_1, \ldots, x_n), \quad \boldsymbol{y} = (y_1, \ldots, y_m),$$
$$\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)}), \quad \boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(n)}),$$

and the meaning of $\boldsymbol{\lambda} \vdash \boldsymbol{a}$ and $\boldsymbol{\mu} \vdash \boldsymbol{b}$ is that $\lambda^{(i)} \vdash a_i$ and $\mu^{(j)} \vdash b_j$ for all i and j. Therefore $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is the stratum of points $Z \in \mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)$, for which the maximal Cohen-Macaulay subcurve $Z_{\mathrm{CM}} \subset Z$ is given by

$$C = B \bigcup_{i=1}^{n} \left(\lambda^{(i)} F_{x_i} \right) \bigcup_{j=1}^{m} \left(\mu^{(j)} F_{y_j} \right)$$

The bullet reminds us that we are multiplying by $p^{\chi(\mathcal{O}_Z)}$ and summing over all possible holomorphic Euler characteristics. We regard $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ as an element in $K_0(Var_{\mathbb{C}}((p))$. In the next section, we will see that the Euler characteristic of $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ does *not* depend on the exact location of the points $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$, but only on their number m and n and the partitions $\lambda^{(i)}$ and $\mu^{(j)}$.

5. RESTRICTION TO FORMAL NEIGHBOURHOODS

In the previous two sections we reduced our consideration to the strata $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ of $\mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)$ which parameterize subschemes Z whose maximal Cohen-Macaulay subscheme $Z_{\mathrm{CM}} \subset Z$ is the partition thickened comb curve

$$C = B \cup_i (\lambda^{(i)} F_{x_i}) \cup_j (\mu^{(j)} F_{x_j}).$$

[2]We use the term component somewhat loose: it means a subset which is both open and closed. We do not care whether it is connected.

5.1. **The fpqc cover.** The reduced support of C is $B \cup_i F_{x_i} \cup_j F_{y_j}$ which is a nodal curve with nodes at $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ and $(z_1, \dots, z_m)$ where z_j is the node of the nodal fiber F_{y_j} (see figure 1).

We wish to make an open cover of X which is compatible with the support of C. Intuitively, one can think of the complex analytic open cover consisting of small balls around the points $\{x_i, y_j, z_j\}$, a small tubular neighborhood of $B - \{x_i, y_j\}$, small tubular neighborhoods of $F_{x_i} - \{x_i\}$ and $F_{y_j} - \{y_j, z_j\}$, and finally the complement of C.

To work algebraically, we use formal neighborhoods instead of tubular neighborhoods and balls. This also has an addition advantage over an analytic cover: the intersection of these open sets is very small in the sense that no embedded points or zero-dimensional components can occur inside the intersections of distinct open sets in the cover.

Definition 10. We define $\mathfrak{U}$, a cover of X, to be the collection of the follow sets, open in the fpqc topology:

(1) W, the complement of the support of C.
(2) $\widehat{X}_{x_i}$, for $i = 1, \dots, n$, the formal neighborhoods of the points x_i.
(3) $\widehat{X}_{y_j}$, for $i = 1, \dots, m$, the formal neighborhoods of the points y_j.
(4) $\widehat{X}_{z_j}$, for $i = 1, \dots, m$, the formal neighborhoods of the points z_j.
(5) $\widehat{X}_{B^\circ}$, the formal neighborhood of $B^\circ \subset X$ where $B^\circ = B - \{x_1, \dots, x_n, y_1 \cdots, y_m\}$.
(6) $\widehat{X}_{F^\circ_{x_i}}$, for $i = 1, \dots, n$ the formal neighborhood of $F^\circ_{x_i} \subset X$ where $F^\circ_{x_i} = F_{x_i} - \{x_i\}$.
(7) $\widehat{X}_{F^\circ_{y_j}}$, for $j = 1, \dots, m$ the formal neighborhood of $F^\circ_{y_j} \subset X$ where $F^\circ_{y_j} = F_{y_j} - \{y_j, z_j\}$.

The subschemes B°, $F^\circ_{x_i}$, and $F^\circ_{y_j}$ are not closed in X but they are locally closed. To define $\widehat{X}_{V^\circ}$, the formal neighborhood of a locally closed subscheme V°, we take any open $X^\circ \subset X$ such that $V^\circ \subset X^\circ$ is closed and then define $\widehat{X}_{V^\circ}$ to be $\widehat{X}^\circ_{V^\circ}$.

We consider the formal neighborhoods which are members of the cover $\mathfrak{U}$ not as formal schemes, but as the associated non-Noetherian schemes. The scheme maps

$$\widehat{X}_{x_i} \to X, \quad \widehat{X}_{B^\circ} \to X, \dots$$

are open in the fpqc topology [,] so $\mathfrak{U}$ forms an fpqc cover. For $U \in \mathfrak{U}$, we suppress the associated map $U \to X$ from the notation and use the usual restriction notation to denote the pullback, for example:

$$U|_C := U \times_X C.$$

5.2. **Writing $\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ as a product of Local Hilbert schemes.** Let

$$C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right) \cup_j \left(\mu^{(j)} F_{y_j}\right).$$

Recall that the component

$$\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \subset \operatorname{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)$$

parameterizes subschemes $Z \subset X$ whose minimal Cohen-Macaulay subscheme is C, i.e. Z is obtained from C by adding embedded points and/or 0-dimensional components. We apply the definition of $\operatorname{Hilb}^n(U, C)$ (Definition 4) to the open sets in our cover $\mathfrak{U}$. The following lemma writes the stratum $\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ as a product of those schemes.

Lemma 11. *The following equation holds in $K_0(Var_{\mathbb{C}})((p))$:*

$$\begin{aligned}\Sigma^\bullet(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\mu}) &= p^{\chi(\mathcal{O}_C)}\cdot\prod_{U\in\mathfrak{U}}\mathrm{Hilb}^\bullet(U,C|_U)\\ &= p^{\chi(\mathcal{O}_C)}\cdot\mathrm{Hilb}^\bullet(W)\cdot\mathrm{Hilb}^\bullet(\widehat{X}^\circ_B,B^\circ)\\ &\quad\cdot\prod_{i=1}^n\mathrm{Hilb}^\bullet(\widehat{X}_{x_i},\widehat{C}_{x_i})\cdot\mathrm{Hilb}^\bullet(\widehat{X}^\circ_{F_{x_i}},\lambda^{(i)}F^\circ_{x_i})\\ &\quad\cdot\prod_{j=1}^m\mathrm{Hilb}^\bullet(\widehat{X}_{y_j},\widehat{C}_{y_j})\cdot\mathrm{Hilb}^\bullet(\widehat{X}_{z_j},\widehat{C}_{z_j})\cdot\mathrm{Hilb}^\bullet(\widehat{X}^\circ_{F_{y_j}},\mu^{(j)}F^\circ_{y_j})\end{aligned}$$

Note that we have implicitly introduced notation for our curve restricted to the various open sets: $C|_{\widehat{X}_{x_i}}=\widehat{C}_{x_i}$, $C|_{\widehat{X}^\circ_{F_{y_j}}}=\mu^{(j)}F^\circ_{y_j}$, etc.

Proof. Let Z be a subscheme corresponding to a point in $\Sigma^\bullet(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\mu})$. Our fpqc cover $\mathfrak{U}$ has the property that every 0-dimensional component or embedded point of Z is contained in a unique open set in the cover. In other words, the restriction of Z to any intersection of sets in the cover $\mathfrak{U}$ has no embedded points or 0-dimensional components. fpqc decent then tells us that Z is uniquely determined by its restriction to the open sets of the cover. This yields a constructible bijective morphism from the product of Hilbert schemes on the open sets to the Hilbert scheme on X which induces the isomorphism in the Grothendieck group. Finally, we verify that the powers of p correctly match up each side. For a subscheme with Z maximal Cohen-Macaulay subscheme C, we have

$$\begin{aligned}\chi(\mathcal{O}_Z) &= \chi(\mathcal{O}_C)+\mathrm{length}(\mathcal{O}_Z/\mathcal{O}_C)\\ &= \chi(\mathcal{O}_C)+\sum_{U\in\mathfrak{U}}\mathrm{length}(\mathcal{O}_Z/\mathcal{O}_C|_U)\end{aligned}$$

therefore the $p^{\chi(\mathcal{O}_Z)}$ term on the left hand side of the equation correctly matches the

$$p^{\chi(\mathcal{O}_C)}\cdot\prod_{U\in\mathfrak{U}}p^{\mathrm{length}(\mathcal{O}_Z/\mathcal{O}_C|_U)}$$

term on the right hand side. □

Lemma 12. *Let*

$$C=B\cup_i\left(\lambda^{(i)}F_{x_i}\right)\cup_j\left(\mu^{(j)}F_{y_j}\right)$$

then

$$\chi(\mathcal{O}_C)=\chi(\mathcal{O}_B)-\sum_{i=1}^n\lambda_1^{(i)}-\sum_{j=1}^m\mu_1^{(j)}.$$

Proof. Since $\lambda^{(i)}F_{x_i}=p^{-1}(Z_{\lambda^{(i)}}$ and p is an elliptic fibration, $\chi(\mathcal{O}_{F_{x_i}})=0$ and similarly we have $\chi(\mathcal{O}_{F_{y_j}})=0$. Note that $B\cap\lambda^{(i)}F_{x_i}$ and $B\cap\mu^{(j)}F_{y_j}$ are zero dimensional subschemes of length $\lambda_1^{(i)}$ and $\mu_1^{(j)}$ respectively. The lemma then follows from the exact sequence

$$0\to\mathcal{O}_C\to\mathcal{O}_B\oplus_i\mathcal{O}_{\lambda^{(i)}F_{x_i}}\oplus_j\mathcal{O}_{\mu^{(j)}F_{y_j}}\to\oplus_i\mathcal{O}_{B\cap\lambda^{(i)}F_{x_i}}\oplus_j\mathcal{O}_{B\cap\mu^{(j)}F_{y_j}}\to 0.$$

□

Combining. . .

5.3. **Formal coordinates and reduction to Hilbert schemes of** $\operatorname{Spec}\mathbb{C}[[r,s,t]]$**.**

Let λ,μ,ν be integer partitions which we also regard as subsets in $(\mathbb{Z}_{\geq 0})^2$ by their diagram. Consider the subscheme

$$Z_{\lambda\mu\nu} = Z_{\lambda\emptyset\emptyset} \cup Z_{\emptyset\mu\emptyset} \cup Z_{\emptyset\emptyset\nu} \subset \mathbb{C}^3 = \operatorname{Spec}\mathbb{C}[r,s,t]$$

defined by the monomial ideal

$$I_{\lambda\mu\nu} = I_{\lambda\emptyset\emptyset} \cap I_{\emptyset\mu\emptyset} \cap I_{\emptyset\emptyset\nu}$$

where

$$\begin{aligned} r^\rho s^\sigma t^\tau \in I_{\lambda\emptyset\emptyset} &\iff (\sigma,\tau) \notin \lambda, \\ r^\rho s^\sigma t^\tau \in I_{\emptyset\mu\emptyset} &\iff (\tau,\rho) \notin \mu, \\ r^\rho s^\sigma t^\tau \in I_{\emptyset\emptyset\nu} &\iff (\rho,\sigma) \notin \nu. \end{aligned}$$

Let $\widehat{Z}_{\lambda\mu\nu}$ be the restriction of $Z_{\lambda\mu\nu}$ to $\widehat{\mathbb{C}}^3_0 = \operatorname{Spec}\mathbb{C}[[r,s,t]]$. Let

$$\operatorname{Hilb}^n(\lambda,\mu,\nu) := \operatorname{Hilb}^n(\widehat{\mathbb{C}}^3_0, \widehat{Z}_{\lambda\mu\nu})$$

be the Hilbert scheme parameterizing the subschemes obtained by adding a length n embedded point to $Z_{\lambda\mu\nu}$ at the origin (see Definition 4.

We now choose formal local coordinates at x_i, y_j, and z_j so that we can identify $\operatorname{Hilb}^n(\widehat{X}_{x_i}, \widehat{C}_{x_i})$, $\operatorname{Hilb}^n(\widehat{X}_{y_j}, \widehat{C}_{y_j})$, and $\operatorname{Hilb}^n(\widehat{X}_{z_j}, \widehat{C}_{z_j})$ in terms of $\operatorname{Hilb}^n(\lambda,\mu,\nu)$ for appropriate choices of λ,μ,ν.

Recall that $S \subset X$ is the elliptic surface and $T = \operatorname{Tot}(K_S|_B)$. For any point $p \in B$, let $R_p = \operatorname{Tot}(K_S|F_p)$. We choose isomorphisms

$$\widehat{X}_{x_i} \cong \widehat{X}_{y_j} \cong \widehat{X}_{z_j} \cong \operatorname{Spec}\mathbb{C}[[r,s,t]]$$

such that on $\widehat{X}_{x_i}$

$$R_{x_i} = \{r=0\}, \quad S = \{s=0\}, \quad T = \{t=0\},$$

on $\widehat{X}_{y_j}$

$$R_{y_j} = \{r=0\}, \quad S = \{s=0\}, \quad T = \{t=0\},$$

and on $\widehat{X}_{z_j}$

$$R_{z_j} = \{r=0\}, \quad S = \{s=0\}, \quad T = \{t=0\}.$$

6. Reduction to the topological vertex

In this section, we obtain (Theorem 16) expressions for $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ in terms of the topological vertex $\mathsf{V}_{\lambda\mu\nu}(p)$, $e(B)$, and N (the number of nodal fibres). The theorem follows by expressing $g(a)$ and $h(b)$ of Proposition **??** in terms of the topological vertex.

6.1. **Point contributions.** Following the conventions of [4], we denote by

$$\mathsf{V}_{\lambda\mu\nu} = \sum_\pi p^{|\pi|},$$

the topological vertex. Here the sum is over all 3D partitions π with outgoing legs λ,μ,ν and $|\pi|$ denotes renormalized volume (see Definitions (1) and (2) in [4]). For a 2D partition

$\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$, we write λ' for the corresponding transposed partition and

$$|\lambda| := \sum_{k=0}^{\infty} \lambda_k,$$
$$\|\lambda\|^2 := \sum_{k=0}^{\infty} \lambda_k^2.$$

Proposition 13. *Let F_x be a smooth fibre and F_y a singular fibre with singularity z. Then for any $\lambda \vdash a$, $\mu \vdash b$*

$$p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda) = \mathsf{V}_{\lambda\square\varnothing},$$
$$p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu) = \mathsf{V}_{\mu\square\varnothing},$$
$$p^{-\|\mu\|^2} e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu) = \mathsf{V}_{\mu\mu'\varnothing}.$$

Proof. Recall that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!].$$

Therefore, $\mathbb{C}^{*3}$ acts on each of these schemes and their moduli spaces

$$\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda, \ \mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu, \ \mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu.$$

The coordinates can be chosen such that the action of the last factor of $\mathbb{C}^{*3}$ corresponds to $x_3 \mapsto t_3 x_3$. This component acts trivially since we are already on the $\mathbb{C}^*$-fixed locus. The $\mathbb{C}^{*3}$-fixed locus consists of isolated reduced points corresponding to monomial ideals with asymptotics $(\lambda, \varnothing, \varnothing)$, $(\mu, \varnothing, \varnothing)$, $(\mu, \mu', \varnothing)$ respectively[3]. These monomial ideals are exactly what the topological vertex counts.

Finally, note that the generating functions $e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)$, $e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)$, $e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)$ all start with 1. On the other hand, from the definition

$$\mathsf{V}_{\lambda\square\varnothing} = p^{-\lambda_0} + \cdots,$$
$$\mathsf{V}_{\mu\square\varnothing} = p^{-\mu_0} + \cdots,$$
$$\mathsf{V}_{\mu\mu'\varnothing} = p^{-\sum_{k=0}^{\infty} \mu_k^2} + \cdots,$$

where $\cdots$ stands for higher order terms in p. The proposition follows. □

6.2. **Fibre contribution.** Let F_x be a smooth fibre and F_y a singular fibre. Recall the formal neighbourhoods $\widehat{X}_{F_x^\circ}$, $\widehat{X}_{F_y^\circ}$ of Section 5.

Proposition 14. *For any $\lambda \vdash a$ and $\mu \vdash b$, we have*

$$e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ})^{\mathbb{C}^*}_\lambda) = \frac{1}{\mathsf{V}_{\lambda\varnothing\varnothing}},$$
$$e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ})^{\mathbb{C}^*}_\mu) = \frac{1}{\mathsf{V}_{\mu\varnothing\varnothing}}.$$

Proof. ******** □

[3]The transpose in μ' occurs, because we follow the orientation convention of [4].

6.3. **Putting it together.** Combining Proposition **??** with Propositions 13, 14 gives:

Proposition 15. *For any* $a, b > 0$

$$g(a) = (1-p)\sum_{\lambda\vdash a}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}},$$
(5)
$$h(b) = \frac{1-p}{M(p)}\sum_{\mu\vdash b}p^{\|\mu\|^2}\frac{\mathsf{V}_{\mu\square\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}}\mathsf{V}_{\mu\mu'\varnothing}.$$

Putting all our results together, we obtain formulas for the partition functions in terms of the vertex:

Theorem 16.

$$\widehat{\mathsf{DT}}(X) = \frac{1}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})^{e(B)}}\left((1-p)\sum_{\lambda}q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}\right)^{e(B)-N}\left((1-p)\sum_{\mu}q^{|\mu|}p^{\|\mu\|^2}\frac{\mathsf{V}_{\mu\square\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}}\mathsf{V}_{\mu\mu'\varnothing}\right)^{N}$$

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = \left(\sum_{\lambda}q^{|\lambda|}\right)^{e(B)-N}\left(\sum_{\mu}q^{|\mu|}p^{\|\mu\|^2}\mathsf{V}_{\mu\mu'\varnothing}\right)^{N}.$$

Proof. Inserting the equations for $g(a)$, $h(b)$ of Proposition 15 into (3) gives the formula for $\widehat{\mathsf{DT}}(X)$. Similar reasoning. . . □

Go back and put in the necessary stuff so that the $\widehat{\mathsf{DT}}_{\mathsf{fib}}$ computation isn't just hand waving.

Corollary 17. *Theorem 1 is true.*

Proof. We apply the main theorem of [4]. In particular, we substitute [4, Eqns (2)&(4)] into the formula for $\widehat{\mathsf{DT}}(X)$ and we substitute [4, Eqn (1)], as well as the well-known formula

$$\sum_{\lambda}q^{|\lambda|} = \prod_{d=1}^{\infty}\frac{1}{(1-q^d)}$$

into the formula for $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$. □

7. Putting in the Behrend function

The aim of this section is to show that the partition functions $\widehat{\mathsf{DT}}(X)$ and $\mathsf{DT}(X)$ are equal after the simple change of variables $y = -p$. In order to do this we will need to assume a conjecture about the Behrend function which we formulate below for general Calabi-Yau threefolds.

Let Y be any quasi-projective Calabi-Yau threefold. Let $C \subset Y$ be a (not necessarily reduced) Cohen-Macaulay curve with proper support. Assume that the singularities of C_{red} are locally toric[4]. Define

$$\mathrm{Hilb}^{C,n}(Y) = \{Z \subset Y \textit{ such that } C \subset Z \textit{ and } I_C/I_Z \textit{ has finite length } n\}.$$

Note that $\mathrm{Hilb}^{C,n}(Y) \subset \mathrm{Hilb}(Y)$ and let ν denote the Behrend function on $\mathrm{Hilb}(Y)$. Our conjecture is the following:

[4]This means that formally locally C_{red} is either smooth, nodal, or the union of the three coordinate axes. That is at $p \in C_{\mathrm{red}} \subset Y$ the ideal $\widehat{I}_{C_{\mathrm{red}}} \subset \widehat{\mathcal{O}}_{Y,p}$ is given by (x_1, x_2), (x_1, x_2x_3), or (x_1x_2, x_2x_3, x_1x_3) for some isomorphism $\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[[x_1, x_2, x_3]]$.

Conjecture 18.
$$\int_{\mathrm{Hilb}^{C,n}(Y)} \nu \, de = (-1)^n \nu([C]) \int_{\mathrm{Hilb}^{C,n}(Y)} de$$
where $\nu([C])$ is the value of the Behrend function at the point $[C] \in \mathrm{Hilb}(Y)$.

Remark 19. Conceivably, the condition that C_{red} has locally toric singularities could be weakened, although we do not have any evidence for this case. Our conjecture is true for Y a (globally) toric Calabi-Yau. This follows from the computations in [11].

One could also make the much stronger conjecture that
$$\nu([Z]) = (-1)^n \nu([C])$$
for all $[Z] \in \mathrm{Hilb}^{C,n}(Y)$. This would of course imply our conjecture as stated. However, we do not know if this stronger version holds, even in the case where Y is $\mathbb{C}^3$ and C is empty. In this case, this stronger conjecture says that the Behrend function on $\mathrm{Hilb}^n(\mathbb{C}^3)$ is the constant function $(-1)^n$.

8. Smoothness and infinitesimal deformations

In this section, we show that the locus of comb curves is a smooth open set in the Hilbert scheme and we compute its dimension. This is the key result required in Section 7 to promote our computation of $\widehat{\mathsf{DT}}(X)$ to the Behrend function version $\mathsf{DT}(X)$. We begin by stating the three main results of this section.

Theorem 20. *Let $B \subset T$ be a smooth curve contained in a smooth surface T. Define $V_l \subset \mathrm{Hilb}^n(T)$ to be the locus of points parameterizing dimension 0 subschemes $Z \subset T$ of length n such that the length of the scheme theoretic intersection $Z \cap B$ is l. Then V_l is locally closed and smooth of dimension $2n - l$.*

Theorem 21. *Let $\lambda^{(1)}, \dots, \lambda^{(n)}$ be partitions and let $C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right)$ be a partition thickened comb curve. The the Zariski tangent space of $\mathrm{Hilb}(X)$ at the point $[C]$, which is given by $\mathrm{Hom}(I_C, \mathcal{O}_C) \cong \mathrm{Ext}^1_0(I_C, I_C)$, has dimension*
$$h^0(N_{B/X}) + \sum_{i=1}^{n} \left(2|\lambda^{(i)}| - \lambda^{(i)}_1\right).$$

Theorem 22. *The locus of comb curves is a non-singular subset in $\mathrm{Hilb}(X)$.*

Our method for computing the dimension of deformation spaces is an adaption of Haiman's method for computing infinitesimal deformations of 0-dimensional subschemes on a surface [8]. Indeed, the proof of Theorem 20 follows directly using Haiman's argument. For Theorem 21, we use Haiman's method to study local deformations of C in the formal neighborhoods of the points x_i, but we use the global geometry to keep track of which local deformations extend.

8.1. **Notation and setup for proof of Theorem 21.** For notational simplicity we first treat the case where there is a singleparititon thickened fiber $F = F_x$ at $x \in B$, that is
$$C = B \cup \lambda F$$
where $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ is an integer partition of length l.

Consider the divisors
$$S, \quad R = \mathrm{Tot}(K_S|_F), \quad T = \mathrm{Tot}(K_S|_B)$$

and let (r,s,t) be formal local coordinates at x such that

$$R=\{r=0\}\,,\quad S=\{s=0\}\,,\quad T=\{t=0\}\,.$$

The formal local ring $\widehat{\mathcal{O}}_{X,x}\cong\mathbb{C}[[r,s,t]]$ has a basis as a $\mathbb{C}$-vector space given by monomials $\{r^\rho s^\sigma t^\tau\}$ for $(\rho,\sigma,\tau)\in(\mathbb{Z}_{\geq 0})^3$. We visualize these basis vectors as unit cubes in the positive octant of $\mathbb{R}^3$ with the monomial $r^\rho s^\sigma t^\tau$ corresponding to the cube whose corner closest to the origin is at (ρ,σ,τ).

8.2. **Exact sequences.** The ideal sheaf I_C ha a locally free resolution of the form

$$0\to\oplus_\beta R_\beta\to\oplus_\alpha G_\alpha\to I_C\to 0 \tag{6}$$

where G_α (the "generators") and R_β (the "relations") are of the form $\mathcal{O}(-\rho R-\sigma S-\tau T)$.

Indeed, we can explicitly take the collection of (ρ,σ,τ) for G_α to be

$$\{(\lambda_1,0,1),(\lambda_2,1,0),(\lambda_3,2,0),\ldots,(\lambda_l,l-1,0),(0,l,0)\}\,.$$

We also have the sequence

$$0\to\mathcal{O}_C\to\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}\to\mathcal{O}_{\lambda_1 x}\to 0$$

where $\lambda_1 x=B\cap\lambda F$ is the length λ_1 subscheme of B supported at x.

By standard homological algebra, we have that $\operatorname{Hom}(I_C,\mathcal{O}_C)$ is H^0 of the complex

$$\operatorname{Hom}\left([\oplus_\beta R_\beta\to\oplus_\alpha G_\alpha]\,,[\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}\to\mathcal{O}_{\lambda_1 x}]\right).$$

Namely, we have that $\operatorname{Hom}(I_C,\mathcal{O}_C)$ is given by the kernel of the map

$$\operatorname{Hom}(\oplus_\alpha G_\alpha,\mathcal{O}_B\oplus\mathcal{O}_{\lambda F})\xrightarrow{\Phi_1\oplus\Phi_2}\operatorname{Hom}(\oplus_\alpha G_\alpha,\mathcal{O}_{\lambda_1 x})\oplus\operatorname{Hom}(\oplus_\beta R_\beta,\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}).$$

This identification of $\operatorname{Hom}(I_C,\mathcal{O}_C)$ has a straight forward interpretation: a homomorphism $I_C\to\mathcal{O}_C$ is determined by what it is on each of the generators of I_C, considered as maps to $\mathcal{O}_B$ and to $\mathcal{O}_{\lambda F}$. To be in the kernel of Φ_1 just means that these maps should agree on $B\cap\lambda F$ and to be in the kernel of Φ_2 means that the images must obey the module relations. We will make this combinatorially more explicit by studying the restriction of the homomorphisms $\oplus_\alpha G_\alpha\to\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}$ to $\widehat{X}_x\cong\operatorname{Spec}\mathbb{C}[[r,s,t]]$.

8.3. **Combinatorics of Haiman arrows.** Restricted to the local ring $\widehat{\mathcal{O}}_{X,x}\cong\mathbb{C}[[r,s,t]]$, $\mathcal{O}_C$ is spanned over $\mathbb{C}$ by the monomials $r^\rho s^\sigma t^\tau$ where (ρ,σ,τ) are of the form $(\rho,0,0)$ or $(\rho,\sigma,\tau)_{(\rho,\sigma)\in\lambda}$ and I_C is spanned by the complementary monomials. As previously discussed, we view these monomials as cubes in the positive octant, see figure 2.

We call the cubes corresponding to $(\rho,0,0)$ and $(\rho,\sigma,\tau)_{(\rho,\sigma)\in\lambda}$ the B-cubes and λF-cubes respectively and the cubes in the union are called C-cubes. The complement of the C-cubes are the I_C-cubes.

A **Haiman arrow**

$$(\rho,\sigma,\tau)\to(\rho',\sigma',\tau')$$

is a vector whose tail (ρ,σ,τ) is an I_C-cube and whose head (ρ',σ',τ') is a C-cube.

The Haiman arrows form a basis for the $\mathbb{C}$-linear maps from $\widehat{I}_{C,x}$ to $\widehat{\mathcal{O}}_{C,x}$. We wish to determine a basis for $\operatorname{Hom}(I_C,\mathcal{O}_C)$ in terms of Haiman arrows.

The generators of I_C correspond to the cubes in the corners of the set of I_C cubes. They are located at (ρ,σ,τ) where (ρ,σ) are the corners just outside of λ and $\tau=0$ unless $\sigma=0$ in which case $\tau=1$. A generator at (ρ,σ,τ) corresponds to the image of $G_\alpha\to\mathcal{O}$ where $g_\alpha\cong\mathcal{O}(-\rho R-\sigma S-\tau T)$. The summands of

$$\operatorname{Hom}(\oplus_\alpha G_\alpha,\mathcal{O}_B\oplus\mathcal{O}_{\lambda F})\cong\oplus_\alpha H^0(B,G_\alpha^\vee\otimes\mathcal{O}_B)\oplus H^0(F,G_\alpha^\vee\otimes\mathcal{O}_{\lambda F})$$

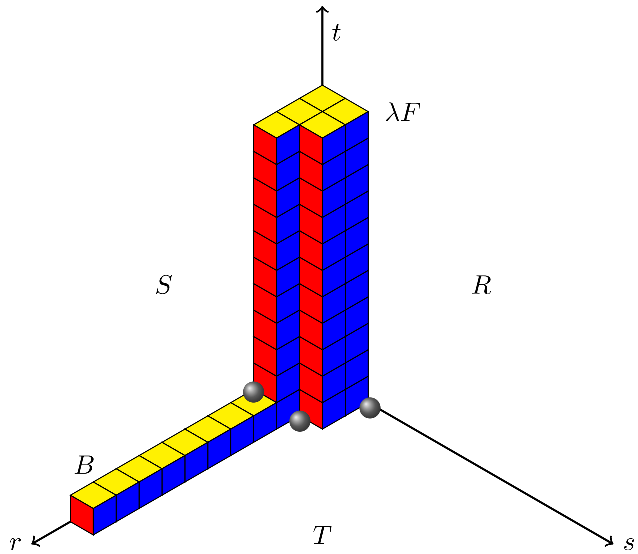

FIGURE 2. Monomials in the local ring $\widehat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[r,s,t]]$ are represented by cubes. Cubes shown are the monomials in $\widehat{\mathcal{O}}_{C,x}$. The gray balls are located at monomials which generate $\widehat{I}_{C,x}$

are interpreted as follows. For $G_\alpha \cong \mathcal{O}(-\rho R - \sigma S - \tau T)$, a homomorphism in $\operatorname{Hom}(G_\alpha, \mathcal{O}_B)$ or $\operatorname{Hom}(G_\alpha, \mathcal{O}_{\lambda F})$ is determined by a Haiman arrow from (ρ,σ,τ) to (respectively) some B-cube or λF-cube (ρ', σ', τ'). The location of the head of such a Haiman arrow is determined by the order of vanishing of the corresponding section of $H^0(B, G_\alpha^\vee \otimes \mathcal{O}_B)$ or $H^0(F, G_\alpha^\vee \otimes \mathcal{O}_{\lambda F})$ — the head will occur at (ρ', σ', τ') if the corresponding section is order $r^{\rho'} s^{\sigma'} t^{\tau'}$.

We wish to determine a basis for $\operatorname{Hom}(I_C, \mathcal{O}_C) \cong \operatorname{Ker}(\Phi_1 \oplus \Phi_2)$ in terms of Haiman arrows. To be in the kernel of Φ_1 just means that a Haiman arrow whose head is both a B-cube and a λF-cube must arise as sections of both $H^0(B, G_\alpha^\vee \otimes \mathcal{O}_B)$ and $H^0(F, G_\alpha^\vee \otimes \mathcal{O}_{\lambda F})$.

A key observation is the following:

Remark 23. The equations defining the kernel of Φ_2 equate two Haiman arrows which are obtained from one another by translation through other Haiman arrows. Moreover, if a Haiman arrow can be translated so that its head passes into an octant with a negative coordinate (without its tail leaving the I_C-cubes) then in must be zero.

We now analyze the possible equivalence classes of Haiman arrows.

8.4. **Haiman arrows to λF-cubes.** Let $G_\alpha \cong \mathcal{O}(-\rho R - \sigma S - \tau T)$ be a generating line bundle and consider the sections $H^0(F, G_\alpha^\vee \otimes \mathcal{O}_{\lambda F})$. These correspond to the possible Haiman arrows $(\rho, \sigma, \tau) \to (\rho', \sigma', \tau')$ to λF-cubes. Since the normal bundle of F in X is trivial, $\mathcal{O}(R)$ and $\mathcal{O}(S)$ are trivial restricted to F. Thus

$$G_\alpha^\vee \otimes \mathcal{O}_{\lambda F} \cong \mathcal{O}_{\lambda F}(\rho R + \sigma S + \tau T) \cong \mathcal{O}_{\lambda F}(\tau x).$$

Since τ is either 0 or 1 for the generators G_α, the Haiman arrows correspond to

$$H^0(F, \mathcal{O}_{\lambda F}) \text{ if } \tau = 0, \quad H^0(F, \mathcal{O}_{\lambda F}(x)) \text{ if } \tau = 1.$$

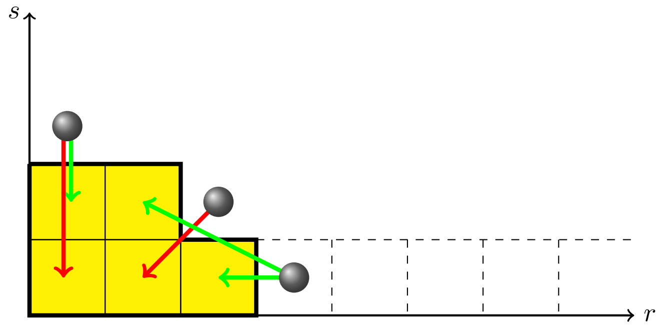

FIGURE 3. Examples of Haiman arrows to λF-cubes. The green arrows are non-zero and the red arrows are necessarily zero.

In both cases, this space has a basis of sections which in the local coordinates are given by $\{r^{\rho'}s^{\sigma'}t^{\tau}\}_{(\rho',\sigma')\in\lambda}$. Note that the section we consider above are uniquely determined by their value on the formal neighborhood $\widehat{X}_x \cong \operatorname{Spec}\mathbb{C}[[r,s,t]]$, a property which uses crucially the fact that the genus of F is 1.

We have seen that the possible Haiman arrows to λF-cubes are given by

$$(\rho,\sigma,\tau) \to (\rho',\sigma',\tau')$$

where (ρ,σ,τ) is a generating cube, $\tau' = \tau$ and $(\rho',\sigma') \in \Lambda$. In particular, the direction of the arrows is horizontal since there is no τ component in $(\rho-\rho',\sigma-\sigma',0)$.

Since all the Haiman arrows to λF-cubes are horizontal, we view them from above in the (r,s) plane (see figure 3).

If the direction of the Haiman arrow is strictly southwest (i.e. it has strictly negative ρ and σ components), then by translating (see Remark 23) along the contour of λ to the edge of the s-axis, the arrow can be equated to an arrow whose head has a negative ρ component and is thus zero. There are no strictly Northeast pointing Haiman arrows, so all non-zero Haiman arrows must be weakly northwest pointing or weakly southeast pointing. Translating a weakly northwest pointing arrow as far to the northwest as possible, we find that its head will either cross the s-axis (and hence be 0) or it will be at the top of a column of λ and its tail just outside a row. Indeed, for each square in λ, there is exactly one equivalence class of weakly northwest pointing Haiman arrows represented by the arrow going from just outside the box's row to the top of the box's column. Similarly, there is one equivalence class of weakly southeast pointing Haiman arrows for each box in λ representated by the arrow going from just outside the top of the box's column to the end of the box's row. This accounts for exactly $2|\lambda|$ different equivalence classes of Haiman arrows to λF-cubes. However, λ_1 of these arrows have their head a B-cube, namely the southeast pointing arrows whose tails are just above the top of λ and whose head is the last square in the first row of λ[5].

We thus have exactly $2|\lambda| - \lambda_1$ distinct equivalence classes of Haiman arrows to λF-cubes which are not also arrows to B-cubes.

[5]Note that the northwest pointing Haiman arrows whose head are in the first row of λ are necessarily strictly west pointing and hence originate at the generator whose τ component is 1. Therefore the head of these arrows also have τ component 1 and so they are not B-cubes.

8.5. **Haiman arrows to B-cubes.** Any non-zero Haiman arrow to a B-cube must have a tail with coordinates $(\rho, 0, 1)$ or $(\rho, 1, 0)$ since if not, it could be translated (see Remark 23) to an arrow whose head has negative τ or σ coordinates by first translating sufficiently far in the ρ and then translating the tail so that it is just outside of the B-cubes. A Haiman arrow to a B-cube whose tail is $(\rho, 0, 1)$ or $(\rho, 1, 0)$ corresponds respectively to a section in $H^0(B, \mathcal{O}_B(\rho R + T))$ or $H^0(B, \mathcal{O}_B(\rho R + S))$. Since

$$\mathcal{O}_B(R) \cong \mathcal{O}_B(x), \quad \mathcal{O}_B(T) \cong N_{B/S}, \quad \mathcal{O}_B(S) \cong N_{B/T},$$

we see that the Haiman arrows from $(\rho, 0, 1)$ or $(\rho, 1, 0)$ to B-cubes are given by

$$H^0(B, N_{B/S}(\rho x)) \quad \text{or} \quad H^0(B, N_{B/T}(\rho x))$$

respectively. The head of such a Haiman arrow is $(\rho', 0, 0)$ where ρ' is the order of vanishing at x of the corresponding section.

Lemma 24. *Let $(\rho, 0, 1) \to (\rho', 0, 0)$ or $(\rho, 1, 0) \to (\rho', 0, 0)$ be a non-zero Haiman arrow. Then $\rho' \geq \rho$.*

Proof. Consider a Haiman arrow $(\rho, 0, 1) \to (\rho', 0, 0)$ with $\rho' < \rho$. Then this arrow can be translated so that its head is a λF-cube, however we saw in the previous subsection that Haiman arrows to λF-cubes must be horizontal and so this must be zero. Consider next a Haiman arrow $(\rho, 1, 0) \to (\rho', 0, 0)$ with $\rho' < \rho$. Then this arrow maybe translated so that it is a strictly southwest pointing Haiman arrow to an λF-cube which we showed in the previous subsection must be zero.[6] □

By the lemma, we conclude that the only sections of $H^0(B, N_{B/S}(\rho x))$ or $H^0(B, N_{B/T}(\rho x))$ which correspond to non-zero Haiman arrows vanish to order at least ρ at x, and thus they are in the image of the maps

$$H^0(B, N_{B/S}) \to H^0(B, N_{B/S}(\rho x))$$
$$H^0(B, N_{B/T}) \to H^0(B, N_{B/T}(\rho x))$$

By ??? $H^0(B, N_{B/S}) = 0$. On the other hand, $H^0(B, N_{B/T})$ can be non-zero and these deformations do occur, they correspond to global deformations of B in the K_S direction.

In conclusion, we have completely classified all possible Haiman arrows upto equivalence and have thus constructed an explicit basis for $\operatorname{Hom}(I_C, \mathcal{O}_C) \cong \operatorname{Ker}(\Phi_1 \oplus \Phi_2)$. They consist of the $2|\lambda| - \lambda_1$ Haiman arrows to λF-cubes which do not go to B-cubes and the $h^0(B, N_{B/T}) = h^0(B, N_{B/X})$ dimensional space of Haiman arrows going to B-cubes. We've proved that

$$\dim \operatorname{Hom}(I_C, \mathcal{O}_C) = h^0(B, N_{B/X}) + 2|\lambda| - \lambda_1$$

for $C = B \cup \lambda F$. Our argument extends essentially word for word to the case where $C = B \cup_i (\lambda^{(i)} F_{x_i})$ has several partition thickened fibers. Whether the fiber is smooth or nodal plays no role. We have thus proved Theorem 21. □

Appendix A. Odds and Ends

A.1. **Curves on elliptic surfaces.** Let $p : S \to B$ be an elliptic surface with section $B \subset S$. In this appendix we allow any type of singular fibres. We assume S is not a product, which implies

$$p^* : \operatorname{Pic}^0(B) \xrightarrow{\cong} \operatorname{Pic}^0(S)$$

[6]Are you still with us dear reader? We are deep in the weeds now, but are almost done.

is an isomorphism [12, VII.1.1]. For any $\beta \in H_2(S)$, we denote by $\mathrm{Hilb}^{\beta}(S)$ the Hilbert scheme of effective divisors on S in class β.

Denote by $B \in H_2(S)$ the class of the section $B \subset S$ and by $F \in H_2(S)$ the class of the fibre. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sym}^d(B) \arrow[r] \arrow[d, "p^*"] & \mathrm{Pic}^d(B) \arrow[d, "p^*", "\cong"'] \\
\mathrm{Hilb}^{dF}(S) \arrow[r] \arrow[d, "+B"] & \mathrm{Pic}^{dF}(S) \arrow[d, "\otimes \mathcal{O}_S(B)", "\cong"'] \\
\mathrm{Hilb}^{B+dF}(S) \arrow[r] & \mathrm{Pic}^{B+dF}(S).
\end{tikzcd}$$

The horizontal arrows are Abel-Jacobi maps. The vertical arrows are induced by pull-back and adding the section $B \subset S$.

Lemma 25. *The above maps induce an isomorphism*

$$\mathrm{Sym}^d(B) \xrightarrow{\cong} \mathrm{Hilb}^{B+dF}(S).$$

Proof. Clearly p^* gives an isomorphism $\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{dF}(S)$ and $+B$ defines a closed embedding $\mathrm{Hilb}^{dF}(S) \hookrightarrow \mathrm{Hilb}^{B+dF}(S)$. Since $\mathrm{Sym}^d(B)$ is smooth and $\mathrm{Hilb}^{B+dF}(S)$ is reduced (by [13, Lect. 25]), it suffices to show

$$\mathrm{Sym}^d(B) \to \mathrm{Hilb}^{B+dF}(S)$$

is surjective on closed points.

For surjectivity, suppose D' is an effective divisor with class $B + dF$ which does *not* lie in the image. Firstly, we note that for any fibre F we have $D' \cdot F = 1$. Therefore D' contains a section $B' \subset S$ as an effective summand. Moreover $B \neq B'$ or else D' would lie in the image. Next, we take any D in the image and compare D and D'. Then

$$\mathcal{O}_S(D - D') \in \mathrm{Pic}^0(S) \cong \mathrm{Pic}^0(B).$$

Therefore after re-arranging we find that there are distinct fibres F_{x_i}, F_{y_j} and $a_i \geq 0$, $b_j \geq 0$ such that

$$B + \sum_i a_i F_{x_i} \sim_{\mathrm{lin}} B' + \sum_j b_j F_{y_j},$$

where $\sim_{\mathrm{lin}}$ denotes linear equivalence. Hence there exists a pencil $\{C_t\}_{t \in \mathbb{P}^1}$ of effective divisors such that

$$C_0 = B + \sum_i a_i F_{x_i},\ C_\infty = B' + \sum_j b_j F_{y_j}.$$

Now fix a smooth fibre F. Then $C_t \cdot F = 1$ for any $t \in \mathbb{P}^1$, so we get a morphism

$$\mathbb{P}^1 \longrightarrow F,\ t \mapsto C_t \cap F.$$

But F is a smooth elliptic curve so this map is constant. We conclude

$$B \cap F = C_0 \cap F = C_\infty \cap F = B' \cap F.$$

Since F was chosen arbitrary, we deduce that $B = B'$ which is a contradiction. $\square$

A.2. **Weighted Euler characteristics of symmetric products.** In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

Lemma 26. *Let B be a scheme of finite type over $\mathbb{C}$ and let $e(B)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$ be any function with $g(0) = 1$. Let $f_d : \operatorname{Sym}^d(B) \to \mathbb{Z}((p))$ be the constructible function defined by*

$$f_d(\mathfrak{a}) = \prod_i g(a_i),$$

for all $\mathfrak{a} = \sum_i a_i x_i \in \operatorname{Sym}^d(B)$ where $x_i \in B$ are distinct closed points. Then

$$\sum_{d=0}^{\infty} q^d \int_{\operatorname{Sym}^d(B)} f_d \, de = \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)}.$$

Remark 27. In the special case where $g = f_d \equiv 1$, the lemma recovers MacDonald's formula:

$$\sum_{d=0}^{\infty} e(\operatorname{Sym}^d(B))\, q^d = \frac{1}{(1-q)^{e(B)}}.$$

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties definited by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure []. For convenience's sake, we provide a direct proof here.

Ref? Bryan-Young?

Proof. The dth symmetric product admits a stratification with strata labelled by partitions of d. Associated to any partition of d is a unique tuple $(m_1, m_2, \dots)$ of non-negative integers with $\sum_{j=1}^{\infty} j m_j = d$. The stratum labelled by $(m_1, m_2, \dots)$ parameterizes collections of points where there are m_j points of multiplicity j. The full stratification is given by:

$$\operatorname{Sym}^d(B) = \bigsqcup_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left\{ \left(\prod_{j=1}^{\infty} B^{m_j} \right) - \Delta \right\} / \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, B^0 is a point, Δ is the large diagonal, and σ_m is the mth symmetric group. Note that the function f_d is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

For schemes over $\mathbb{C}$, topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\operatorname{Sym}^d(B)} f_d \; de = \sum_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left(\prod_{j=1}^{\infty} g(j)^{m_j} \right) \frac{e(B^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \dots}.$$

For any natural number N, the projection $B^N - \Delta \to B^{N-1} - \Delta$ has fibers of the form $B - \{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(B) - (N-1)$ and consequently, $e(B^N - \Delta) = (e(B) - (N-1)) \cdot e(B^{N-1} - \Delta)$. Thus by induction, we find $e(B^N - \Delta) = e(B) \cdot (e(B) - 1) \cdots (e(B) - (N-1))$ and so

$$\frac{e(B^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \cdots} = \binom{e(B)}{m_1, m_2, m_3, \cdots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\begin{aligned}\sum_{d=0}^{\infty} q^d \int_{\mathrm{Sym}^d(B)} f_d \, de &= \sum_{(m_1,m_2,\dots)} \prod_{j=1}^{\infty} \left(g(j)q^j\right)^{m_j} \binom{e(B)}{m_1, m_2, m_3, \dots} \\ &= \left(1 + \sum_{j=1}^{\infty} g(j)q^j\right)^{e(B)}\end{aligned}$$

which proves the lemma. □

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