

# TRACE IDENTITIES FOR THE TOPOLOGICAL VERTEX

JIM BRYAN, MARTIJN KOOL, BENJAMIN YOUNG

ABSTRACT. The topological vertex is a universal series which can be regarded as an object in combinatorics, representation theory, geometry, or physics. It encodes the combinatorics of 3D partitions, the action of vertex operators on Fock space, the Donaldson-Thomas theory of toric Calabi-Yau threefolds, or the open string partition function of  $\mathbb{C}^3$ .

We prove several identities in which a sum over terms involving the topological vertex is expressed as a closed formula, often a product of simple terms, closely related to Fourier expansions of Jacobi forms. We use purely combinatorial and representation theoretic methods to prove our formulas, but we discuss applications to the Donaldson-Thomas invariants of elliptically fibered Calabi-Yau threefolds at the end of the paper.

## 1. INTRODUCTION

The topological vertex  $V_{\lambda\mu\nu} = V_{\lambda\mu\nu}(p)$  is a universal formal Laurent series in  $p$  depending on a triple of partitions  $(\lambda, \mu, \nu)$ . It can be considered as an object in combinatorics, representation theory, geometry, or physics. In combinatorics,  $V_{\lambda\mu\nu}$  is the generating function for the number of 3D partitions with asymptotic legs of type  $(\lambda, \mu, \nu)$  (see Definition 2). In representation theory,  $V_{\lambda\mu\nu}$  is given as the matrix coefficients of a certain vertex operator on Fock space [12]. In geometry,  $V_{\lambda\mu\nu}$  is the basic building block for computing the Donaldson-Thomas/Gromov-Witten invariants of toric Calabi-Yau threefolds [9]; in this incarnation, it can be realized as the generating function for the Euler characteristics of certain Hilbert schemes of curves in  $\mathbb{C}^3$  (see § 6). The topological vertex was first discovered in physics as an open string partition function in type IIA string theory on  $\mathbb{C}^3$  [1]. An explicit expression for  $V_{\lambda\mu\nu}$  in terms of Schur functions was given by [12] (see section 3).

In this paper we prove several “trace identities” in which a sum over certain combinations of the vertex is expressed as a closed formula, often a product of simple terms. The products are closely related to the Fourier expansions of Jacobi forms. Applications of these identities are used to

compute the Donaldson-Thomas partition functions of certain Calabi-Yau threefolds in terms of Jacobi forms [5, 6, 7].

**1.1. Acknowledgments.** We thank Paul Johnson for showing us (on Math-Overflow) how to use the Bloch-Okounkov result [3] to prove Equation (2) in our Theorem 3. We also thank Guillaume Chapuy and Sylvie Corteel for showing us their trick of doing an infinite number of cyclic permutations of vertex operators (see § 5). We thank Georg Oberdieck for helpful conversations.

## 2. DEFINITIONS AND THE MAIN RESULT.

In this section we give the combinatorial definition of the vertex and we state our main identities.

**Definition 1.** Let  $(\lambda, \mu, \nu)$  be a triple of ordinary partitions. A *3D partition*  $\pi$  *asymptotic to*  $(\lambda, \mu, \nu)$  is a subset

$$\pi \subset (\mathbb{Z}_{\geq 0})^3$$

satisfying

- (1) if any of  $(i+1, j, k)$ ,  $(i, j+1, k)$ , and  $(i, j, k+1)$  is in  $\pi$ , then  $(i, j, k)$  is also in  $\pi$ , and
- (2) (a)  $(j, k) \in \lambda$  if and only if  $(i, j, k) \in \pi$  for all  $i \gg 0$ ,  
 (b)  $(k, i) \in \mu$  if and only if  $(i, j, k) \in \pi$  for all  $j \gg 0$ ,  
 (c)  $(i, j) \in \nu$  if and only if  $(i, j, k) \in \pi$  for all  $k \gg 0$ .

where we regard ordinary partitions as finite subsets of  $(\mathbb{Z}_{\geq 0})^2$  via their diagram.

Intuitively,  $\pi$  is a pile of boxes in the positive octant of 3-space. Condition (1) means that the boxes are stacked stably with gravity pulling them in the  $(-1, -1, -1)$  direction; condition (2) means that the pile of boxes is infinite along the coordinate axes with cross-sections asymptotically given by  $\lambda$ ,  $\mu$ , and  $\nu$ .

The subset  $\{(i, j, k) : (j, k) \in \lambda\} \subset \pi$  will be called the *leg* of  $\pi$  in the  $i$  direction, and the legs in the  $j$  and  $k$  directions are defined analogously. Let

$$\xi_\pi(i, j, k) = 1 - \# \text{ of legs of } \pi \text{ containing } (i, j, k).$$

We define the renormalized volume of  $\pi$  by

$$|\pi| = \sum_{(i,j,k) \in \pi} \xi_\pi(i, j, k).$$

Note that  $|\pi|$  can be negative.

**Definition 2.** The topological vertex  $V_{\lambda\mu\nu}$  is defined to be

$$V_{\lambda\mu\nu} = \sum_{\pi} p^{|\pi|}$$

where the sum is taken over all 3D partitions  $\pi$  asymptotic to  $(\lambda, \mu, \nu)$ . We regard  $V_{\lambda\mu\nu}$  as a formal Laurent series in  $p$ . Note that  $V_{\lambda\mu\nu}$  is clearly cyclically symmetric in the indices, and reflection about the  $i = j$  plane yields

$$V_{\lambda\mu\nu} = V_{\mu'\lambda'\nu'}$$

where  $'$  denotes conjugate partition:

$$\lambda' = \{(i, j) : (j, i) \in \lambda\}.$$

This definition of topological vertex differs from the vertex  $C(\lambda, \mu, \nu)$  of the physics literature by a normalization factor (and we use the variable  $p$  instead of  $q$ ). Our  $V_{\lambda\mu\nu}$  is equal to  $P(\lambda, \mu, \nu)$  defined by Okounkov, Reshetikhin, and Vafa [12, eqn 3.16]. They derive an explicit formula for  $V_{\lambda\mu\nu} = P(\lambda, \mu, \nu)$  in terms of Schur functions [12, eqns 3.20 and 3.21].

The *rows* or *parts* of  $\lambda$  are the integers  $\lambda_j = \min\{i \mid (i, j) \notin \lambda\}$ , for  $j \geq 0$ . We use the notation

$$|\lambda| = \sum_j \lambda_j, \quad \|\lambda\|^2 = \sum_j \lambda_j^2.$$

Let  $\square$  denote the partition with a single part of size 1.

We also use the notation

$$M(p, q) = \prod_{m=1}^{\infty} (1 - p^m q)^{-m}$$

and the shorthand  $M(p) = M(p, 1)$ .

We can now state our main result.

**Theorem 3.** *The following identities hold as formal power series in  $q$  whose coefficients are formal Laurent series in  $p$ :*

(1)

$$\sum_{\lambda} q^{|\lambda|} p^{\|\lambda'\|^2} V_{\lambda'\lambda\emptyset} = M(p) \prod_{d=1}^{\infty} (1 - q^d)^{-1} M(p, q^d)$$

(2)

$$\sum_{\lambda} q^{|\lambda|} \frac{V_{\lambda\Box\emptyset}}{V_{\lambda\emptyset\emptyset}} = (1 - p)^{-1} \prod_{d=1}^{\infty} \frac{(1 - q^d)}{(1 - pq^d)(1 - p^{-1}q^d)}$$

(3)

$$\sum_{\lambda} q^{|\lambda|} p \frac{V_{\Box\Box\lambda}}{V_{\emptyset\emptyset\lambda}} = \prod_{m=1}^{\infty} (1 - q^m)^{-1} \cdot \left\{ 1 + \frac{p}{(1 - p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k})q^d \right\}$$

(4)

$$\sum_{\lambda} q^{|\lambda|} p^{\|\lambda\|^2} V_{\lambda\lambda'\emptyset} \frac{V_{\lambda\Box\emptyset}}{V_{\lambda\emptyset\emptyset}} = (1 - p)^{-1} M(p) \prod_{d=1}^{\infty} \frac{M(p, q^d)}{(1 - pq^d)(1 - p^{-1}q^d)}$$

*The sums in the left hand sides of the above formulas run over all partitions.*

We call these formulas “trace formulas” since the left hand side can be expressed as the traces of certain operators on Fock space. This will be made explicit in section 5.

We prove Formula (1) in section 3 using the orthogonality properties of skew Schur functions. Formulas (2) and (3) are proved in section 4 using a theorem of Bloch-Okounkov [3]. The most difficult identity to prove is equation (4) which we do in section 5. There we prove that the left hand side of equation (4) is given as the trace of a certain product of operators on Fock space. To compute the trace, we use a trick which involves an “infinite number” of permutations of the operators.

### 3. THE TOPOLOGICAL VERTEX AND SCHUR FUNCTIONS

Okounkov-Reshetikhin-Vafa derived a formula for the topological vertex in terms of skew Schur functions. Translating their formulas [12, 3.20& 3.21] into our notation, we get:

(5)

$$V_{\lambda\mu\nu}(p) = M(p) p^{-\frac{1}{2}(\|\lambda\|^2 + \|\mu'\|^2 + \|\nu\|^2)} s_{\nu'}(p^{-\rho}) \sum_{\eta} s_{\lambda'/\eta}(p^{-\nu-\rho}) s_{\mu/\eta}(p^{-\nu'-\rho}).$$

Here,  $s_{\alpha/\beta}(x_1, x_2, \dots)$  is the skew Schur function (see for example [8, § 5]) and

$$\rho = \left( -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots \right)$$

so that  $p^{-\nu-\rho}$  is notation for the variable list

$$p^{-\nu-\rho} = \left( p^{-\nu_1+\frac{1}{2}}, p^{-\nu_2+\frac{3}{2}}, \dots \right).$$

We prove equation (1) as follows. Using equation (5) we see

$$V_{\lambda'\lambda\emptyset} = M(p)p^{-\|\lambda'\|^2} \sum_{\eta} s_{\lambda/\eta}(p^{-\rho})^2$$

and so (using orthogonality of skew Schur functions [8, 28(a) pg 94] in the second line below) we see

$$\begin{aligned} \sum_{\lambda} q^{|\lambda|} p^{\|\lambda'\|^2} V_{\lambda'\lambda\emptyset} &= M(p) \sum_{\lambda, \eta} q^{|\lambda|} (s_{\lambda/\eta}(p^{\frac{1}{2}}, p^{\frac{3}{2}}, \dots))^2 \\ &= M(p) \prod_{d=1}^{\infty} \left( (1 - q^d)^{-1} \prod_{j,k=1}^{\infty} (1 - q^d p^{i-\frac{1}{2}+j-\frac{1}{2}})^{-1} \right) \\ &= M(p) \prod_{d=1}^{\infty} (1 - q^d)^{-1} \prod_{m=1}^{\infty} (1 - q^d p^m)^{-m} \\ &= M(p) \prod_{d=1}^{\infty} (1 - q^d)^{-1} M(p, q^d). \end{aligned}$$

We also use equation (5) to derive the following key formulas:

**Lemma 4.** *The following hold:*

$$\begin{aligned} p^{\frac{1}{2}} \frac{V_{\lambda\Box\emptyset}}{V_{\lambda\emptyset\emptyset}} &= \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}} \\ p \frac{V_{\lambda\Box\Box}}{V_{\lambda\emptyset\emptyset}} &= 1 - \left( \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}} \right) \left( \sum_{j=1}^{\infty} p^{\lambda_j-j+\frac{1}{2}} \right) \end{aligned}$$

*Proof.* Applying equation (5) to  $V_{\lambda\Box\emptyset}/V_{\lambda\emptyset\emptyset} = V_{\Box\emptyset\lambda}/V_{\emptyset\emptyset\lambda}$  we see that

$$p^{\frac{1}{2}} \frac{V_{\lambda\Box\emptyset}}{V_{\lambda\emptyset\emptyset}} = s_{\Box}(p^{-\lambda-\rho}) = s_{\Box}(p^{-\lambda_1+\frac{1}{2}}, p^{-\lambda_2+\frac{3}{2}}, \dots) = \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}}.$$

Similarly,

$$\begin{aligned} p \frac{V_{\lambda \square \square}}{V_{\lambda \emptyset \emptyset}} &= p \frac{V_{\square \square \lambda}}{V_{\emptyset \emptyset \lambda}} = \sum_{\eta} s_{\square/\eta}(p^{-\lambda-\rho}) s_{\square/\eta}(p^{-\lambda'-\rho}) \\ &= 1 + s_{\square}(p^{-\lambda-\rho}) s_{\square}(p^{-\lambda'-\rho}). \end{aligned}$$

In general we have the following relation (see [12, Eqn (3.10)])<sup>1</sup>

$$s_{\lambda/\mu}(p^{\nu+\rho}) = (-1)^{|\lambda|-|\mu|} s_{\lambda'/\mu'}(p^{-\nu'-\rho})$$

so in particular

$$(6) \quad s_{\square}(p^{\nu+\rho}) = -s_{\square}(p^{-\nu'-\rho})$$

and thus

$$\begin{aligned} p \frac{V_{\lambda \square \square}}{V_{\lambda \emptyset \emptyset}} &= 1 - s_{\square}(p^{-\lambda-\rho}) s_{\square}(p^{\lambda+\rho}) \\ &= 1 - \left( \sum_{i=1}^{\infty} p^{-\lambda_i + i - \frac{1}{2}} \right) \left( \sum_{j=1}^{\infty} p^{\lambda_j - j + \frac{1}{2}} \right) \end{aligned}$$

which proves the lemma.

#### 4. APPLICATIONS OF A THEOREM OF BLOCH-OKOUNKOV

We summarize a result of Bloch-Okounkov [3] and use it to prove equations (2) and (3).

We define the following theta function

$$\Theta(p, q) = \eta(q)^{-3} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} p^{n+\frac{1}{2}}$$

which, by the Jacobi triple product formula is given by

$$\Theta(p, q) = (p^{\frac{1}{2}} - p^{-\frac{1}{2}}) \prod_{m=1}^{\infty} \frac{(1 - pq^m)(1 - p^{-1}q^m)}{(1 - q^m)^2}.$$

We suppress the  $q$  from the notation:  $\Theta(p) = \Theta(p, q)$ , and we note that

$$\Theta(p) = -\Theta(p^{-1}).$$

**Theorem 5** (Bloch-Okounkov [3]). *Define the  $n$  point correlation function by the formula*

$$F(p_1, \dots, p_n) = \prod_{m=1}^{\infty} (1 - q^m) \sum_{\lambda} q^{|\lambda|} \prod_{k=1}^n \left( \sum_{i=1}^{\infty} p_k^{\lambda_i - i + \frac{1}{2}} \right).$$

---

<sup>1</sup>There is a typo in equation 3.10 in [12] — the exponent on the right hand side should be  $-\nu' - \rho$ .

Then

$$F(p) = \frac{1}{\Theta(p)}$$

and

$$F(p_1, p_2) = \frac{1}{\Theta(p_1 p_2)} \left( p_1 \frac{d}{dp_1} \log(\Theta(p_1)) + p_2 \frac{d}{dp_2} \log(\Theta(p_2)) \right).$$

In [3], formulas for the general  $n$  variable function are given, but we will only need the cases of  $n = 1$  and  $n = 2$ .

Using this theorem, we will prove equations (2) and (3) of the main theorem.

#### 4.1. Proofs of equations (2) and (3).

We apply Lemma 4 and Theorem 5:

$$\begin{aligned} \sum_{\lambda} (1-p) q^{|\lambda|} \frac{V_{\lambda \square \emptyset}}{V_{\lambda \emptyset \emptyset}} &= (p^{-\frac{1}{2}} - p^{\frac{1}{2}}) \sum_{\lambda} q^{|\lambda|} \sum_{i=1}^{\infty} p^{-\lambda_i + i - \frac{1}{2}} \\ &= (p^{-\frac{1}{2}} - p^{\frac{1}{2}}) \prod_{m=1}^{\infty} (1 - q^m)^{-1} F(p^{-1}) \\ &= (p^{-\frac{1}{2}} - p^{\frac{1}{2}}) \prod_{m=1}^{\infty} (1 - q^m)^{-1} \frac{1}{-\Theta(p)} \\ &= \prod_{m=1}^{\infty} \frac{(1 - q^m)}{(1 - pq^m)(1 - p^{-1}q^m)} \end{aligned}$$

which proves equation (2).

Again we apply Lemma 4 and Theorem 5:

$$\begin{aligned} \sum_{\lambda} q^{|\lambda|} p \frac{V_{\lambda \square \square}}{V_{\lambda \emptyset \emptyset}} &= \sum_{\lambda} q^{|\lambda|} \left\{ 1 - \left( \sum_{i=1}^{\infty} p^{-\lambda_i + i - \frac{1}{2}} \right) \left( \sum_{j=1}^{\infty} p^{\lambda_j - j + \frac{1}{2}} \right) \right\} \\ &= \prod_{m=1}^{\infty} (1 - q^m)^{-1} (1 - F(p, p^{-1})). \end{aligned}$$

From Theorem 5, we see that

$$F(p, p^{-1}) = \lim_{(p_1, p_2) \rightarrow (p, p^{-1})} \frac{1}{\Theta(p_1 p_2)} \left( p_1 \frac{d}{dp_1} \log(\Theta(p_1)) + p_2 \frac{d}{dp_2} \log(\Theta(p_2)) \right).$$

To evaluate this limit, we simplify the above expression. A short computation shows that

$$p \frac{d}{dp} \log(\Theta(p)) = \frac{1}{2} \frac{p+1}{p-1} + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} (-p^k + p^{-k}) q^{mk}.$$

Thus

$$\begin{aligned}
F(p, p^{-1}) &= \lim_{\substack{(p_1, p_2) \rightarrow \\ (p, p^{-1})}} \left( (p_1 p_2)^{\frac{1}{2}} - (p_1 p_2)^{-\frac{1}{2}} \right)^{-1} \prod_{m=1}^{\infty} \frac{(1 - q^m)^2}{(1 - (p_1 p_2) q^m)(1 - (p_1 p_2)^{-1} q^m)} \\
&\quad \cdot \left\{ \frac{1}{2} \cdot \frac{p_1 + 1}{p_1 - 1} + \frac{1}{2} \cdot \frac{p_2 + 1}{p_2 - 1} + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} (-p_1^k - p_2^k + p_1^{-k} + p_2^{-k}) q^{mk} \right\} \\
&= \lim_{\substack{(p_1, p_2) \rightarrow \\ (p, p^{-1})}} \frac{-(p_1 p_2)^{\frac{1}{2}}}{1 - p_1 p_2} \cdot \left\{ \frac{p_1 p_2 - 1}{(p_1 - 1)(p_2 - 1)} + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} (1 - p_1^k p_2^k)(p_1^{-k} + p_2^{-k}) q^{mk} \right\} \\
&= \lim_{\substack{(p_1, p_2) \rightarrow \\ (p, p^{-1})}} (p_1 p_2)^{\frac{1}{2}} \left\{ \frac{1}{(1 - p_1)(1 - p_2)} - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1 - (p_1 p_2)^k}{1 - p_1 p_2} (p_1^{-k} + p_2^{-k}) q^{mk} \right\} \\
&= \frac{1}{(1 - p)(1 - p^{-1})} - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} k(p^k + p^{-k}) q^{mk}.
\end{aligned}$$

Therefore

$$1 - F(p, p^{-1}) = 1 + \frac{p}{(1 - p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k}) q^d$$

which finishes the proof of equation (3).

## 5. VERTEX OPERATORS AND THE PROOF OF EQUATION (4)

There are several sources for vertex operators and the infinite wedge formalism. For consistency, we will follow the notation and conventions of [11, Appendix A].

Let  $V$  be the vector space with basis  $\{\underline{k}\}$ ,  $k \in \mathbb{Z} + \frac{1}{2}$ . We define *Fock space*  $\Lambda^{\frac{\infty}{2}} V$  to be the vector space spanned by vectors

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \dots$$

where  $S = \{s_1 > s_2 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$  is any subset such that the sets

$$S_+ = S \cap \left( \mathbb{Z} + \frac{1}{2} \right)_{>0} \quad \text{and} \quad S_- = S^c \cap \left( \mathbb{Z} + \frac{1}{2} \right)_{<0}$$

are both finite. Let  $(\cdot, \cdot)$  be the inner product on  $\Lambda^{\frac{\infty}{2}} V$  such that the basis  $\{v_S\}$  is orthonormal.

For any  $k \in \mathbb{Z} + \frac{1}{2}$  let  $\psi_k$  be the operator

$$\psi_k(f) = \underline{k} \wedge f$$

and let  $\psi_k^*$  be its adjoint.



For any partition  $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ , we define the vector

$$v_\lambda = \underbrace{(\lambda_1 - \frac{1}{2})}_{\text{}} \wedge \underbrace{(\lambda_2 - \frac{3}{2})}_{\text{}} \wedge \dots$$

Let  $\Lambda_0^{\frac{\infty}{2}} V \subset \Lambda^{\frac{\infty}{2}} V$  be the subspace spanned by the vectors  $\{v_\lambda\}$  where  $\lambda$  runs over all partitions. We call this *charge zero Fock space*.

The *energy operator*

$$H = \sum_{k>0} k (\psi_k \psi_k^* + \psi_{-k}^* \psi_{-k})$$

acts on the basis  $v_\lambda$  by

$$H v_\lambda = |\lambda| v_\lambda$$

and so the operator  $q^H$  acts by

$$q^H v_\lambda = q^{|\lambda|} v_\lambda$$

where  $q$  is a formal parameter.

For  $n \in \mathbb{Z}$ ,  $n \neq 0$  define

$$\alpha_n = \sum_k \psi_{k-n} \psi_k^*$$

and observe that  $\alpha_n^* = \alpha_{-n}$ .

Following [11], we define the *vertex operators*  $\Gamma_\pm(\mathbf{x})$  which are operators on  $\Lambda_0^{\frac{\infty}{2}} V$  over the coefficient ring given by symmetric functions in an infinite set of variables  $\mathbf{x} = (x_1, x_2, x_3, \dots)$ . Let  $\mathbf{s} = (s_1, s_2, \dots)$

$$s_k = \frac{1}{k} \sum_{i=1}^{\infty} x_i^k$$

be the power sum basis for the ring of symmetric functions and let <sup>2</sup>

$$\Gamma_\pm(\mathbf{x}) = \exp \left( \sum_{n=1}^{\infty} s_n \alpha_{\pm n} \right).$$

Observe that  $\Gamma_\pm^* = \Gamma_\mp$ .

The matrix coefficients of the vertex operators in the  $\{v_\lambda\}$  basis are given by skew Schur functions:

$$(7) \quad (\Gamma_-(\mathbf{x}) v_\mu, v_\lambda) = (v_\mu, \Gamma_+(\mathbf{x}) v_\lambda) = s_{\lambda/\mu}(\mathbf{x}).$$

---

<sup>2</sup>In [11], the argument of  $\Gamma_\pm$  is  $\mathbf{s}$ , and the dependence on the underlying set of variables  $\mathbf{x}$  is left implicit. We prefer to make  $\mathbf{x}$  the explicit argument.

Orthogonality of the skew Schur functions then gives rise to the following commutation equation:

$$\Gamma_+(\mathbf{x})\Gamma_-(\mathbf{y}) = \prod_{i,j} (1 - x_j y_i)^{-1} \Gamma_-(\mathbf{y})\Gamma_+(\mathbf{x}),$$

in particular

$$(8) \quad \Gamma_+(up^{-\rho})\Gamma_-(vp^{-\rho}) = M(p, uv)\Gamma_-(vp^{-\rho})\Gamma_+(up^{-\rho})$$

where recall that  $up^{-\rho} = (up^{\frac{1}{2}}, up^{\frac{3}{2}}, up^{\frac{5}{2}}, \dots)$ .

We let

$$\psi(z) = \sum_i z^i \psi_i \quad \text{and} \quad \psi^*(w) = \sum_j w^{-j} \psi_j^*.$$

The commutation relations of these operators with the vertex operators is given by

$$(9) \quad \begin{aligned} \Gamma_{\pm}(\mathbf{x})\psi(z) &= \prod_{i=1}^{\infty} (1 - x_i z^{\pm 1})^{-1} \psi(z)\Gamma_{\pm}(\mathbf{x}) \\ \Gamma_{\pm}(\mathbf{x})\psi^*(w) &= \prod_{i=1}^{\infty} (1 - x_i w^{\pm 1}) \psi^*(w)\Gamma_{\pm}(\mathbf{x}). \end{aligned}$$

We use operators  $\mathcal{E}_r$  introduced by Okounkov-Pandharipande in [10, § 2.2.4]. For  $r \in \mathbb{Z}$ , let<sup>3</sup>

$$\mathcal{E}_r(p) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} p^{k - \frac{r}{2}} \psi_{k-r} \psi_k^*.$$

Our variable  $p$  is related to the variable  $z$  in [10] by  $p = e^z$ .

From [10, Eqns 2.9 and 0.18], we see that  $\mathcal{E}_0$  is the diagonal operator given by

$$\mathcal{E}_0(p)v_{\lambda} = \left( \sum_{i=1}^{\infty} p^{\lambda_i - i + \frac{1}{2}} \right) v_{\lambda}.$$

By equation (6) we have

$$(10) \quad \mathcal{E}_0(p)v_{\lambda} = - \left( \sum_{i=1}^{\infty} p^{-\lambda'_i + i - \frac{1}{2}} \right) v_{\lambda}.$$

We define

$$\mathcal{E}(a, p) = \sum_{r \in \mathbb{Z}} a^r \mathcal{E}_r(p)$$

where  $a$  is a formal parameter. A short computation shows that

$$\mathcal{E}(a, p) = \psi(a^{-1}p^{\frac{1}{2}})\psi^*(a^{-1}p^{-\frac{1}{2}}).$$

---

<sup>3</sup>We avoid the use of normal ordering by allowing coefficients which are Laurent series in  $p^{\frac{1}{2}}$ .

From equation (9) we get

$$\Gamma_{\pm}(\mathbf{x})\mathcal{E}(a, p) = \prod_{i=1}^{\infty} \frac{(1 - a^{\mp 1} p^{\mp \frac{1}{2}} x_i)}{(1 - a^{\mp 1} p^{\pm \frac{1}{2}} x_i)} \mathcal{E}(a, p) \Gamma_{\pm}(\mathbf{x}).$$

For  $\mathbf{x} = up^{-\rho} = (up^{\frac{1}{2}}, up^{\frac{3}{2}}, up^{\frac{5}{2}}, \dots)$  the above simplifies to

$$(11) \quad \begin{aligned} \Gamma_+(up^{-\rho})\mathcal{E}(a, p) &= (1 - a^{-1}u)\mathcal{E}(a, p)\Gamma_+(up^{-\rho}) \\ \mathcal{E}(a, p)\Gamma_-(up^{-\rho}) &= (1 - au)\Gamma_-(up^{-\rho})\mathcal{E}(a, p). \end{aligned}$$

Finally, it follows from equation (7) that

$$(12) \quad \Gamma_{\pm}(\mathbf{x})q^H = q^H \Gamma_{\pm}(q^{\pm 1}\mathbf{x}).$$

We now write the left hand side of equation (4) in the main theorem as a trace of operators on charge zero Fock space.

**Lemma 6.**

$$\begin{aligned} \sum_{\lambda} q^{|\lambda|} p^{\|\lambda\|^2} V_{\lambda\lambda'\emptyset} \frac{V_{\lambda\Box\emptyset}}{V_{\lambda\emptyset\emptyset}} &= -p^{-\frac{1}{2}} \text{tr} (\mathcal{E}_0(p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) \\ &= -p^{-\frac{1}{2}} \text{Coeff}_{a^0} \{ \text{tr} (\mathcal{E}(a, p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) \}. \end{aligned}$$

*Proof.* The following computation uses, in order, the commutation relation for  $\Gamma_+$  and  $\Gamma_-$  (equation (8)), the definition of trace, the formula for  $\mathcal{E}_0(p)$  (equation (10)), Lemma 4, equation (7), equation (5), and finally switching  $\lambda$  to  $\lambda'$  in the sum:

$$\begin{aligned} &-p^{-\frac{1}{2}} \text{tr} (\mathcal{E}_0(p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) \\ &= -p^{-\frac{1}{2}} M(p) \text{tr} (\mathcal{E}_0(p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^H) \\ &= -p^{-\frac{1}{2}} M(p) \sum_{\lambda} (v_{\lambda}, \mathcal{E}_0(p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^H v_{\lambda}) \\ &= p^{-\frac{1}{2}} M(p) \sum_{\lambda} q^{|\lambda|} \left( \sum_{i=1}^{\infty} p^{-\lambda'_i + i - \frac{1}{2}} \right) (v_{\lambda}, \Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})v_{\lambda}) \\ &= M(p) \sum_{\lambda} q^{|\lambda|} \frac{V_{\lambda'\Box\emptyset}}{V_{\lambda'\emptyset\emptyset}} (\Gamma_+(p^{-\rho})v_{\lambda}, \Gamma_+(p^{-\rho})v_{\lambda}) \\ &= \sum_{\lambda} q^{|\lambda|} \frac{V_{\lambda'\Box\emptyset}}{V_{\lambda'\emptyset\emptyset}} M(p) \sum_{\eta} (s_{\lambda/\eta}(p^{-\rho}))^2 \\ &= \sum_{\lambda} q^{|\lambda|} \frac{V_{\lambda'\Box\emptyset}}{V_{\lambda'\emptyset\emptyset}} p^{\|\lambda'\|^2} V_{\lambda'\lambda\emptyset} \\ &= \sum_{\lambda} q^{|\lambda|} p^{\|\lambda\|^2} V_{\lambda\lambda'\emptyset} \frac{V_{\lambda\Box\emptyset}}{V_{\lambda\emptyset\emptyset}}. \end{aligned}$$

□

While the operator  $\mathcal{E}_0(p)$  does not have good commutation relations with the vertex operators, the operator  $\mathcal{E}(a, p)$  does. Hence we first replace  $\mathcal{E}_0(p)$  with the more general  $\mathcal{E}(a, p)$ , compute the trace, and then specialize to the  $a^0$  coefficient.

**Lemma 7.**

$$\begin{aligned} \text{tr} \left( \mathcal{E}(a, p) \Gamma_+(p^{-\rho}) \Gamma_-(p^{-\rho}) q^H \right) = \\ \frac{1}{p^{-\frac{1}{2}} - p^{\frac{1}{2}}} M(p) \prod_{m=1}^{\infty} \frac{(1 - q^m a^{-1})(1 - q^{m-1} a)(1 - q^m) M(p, q^m)}{(1 - pq^m)(1 - p^{-1} q^m)}. \end{aligned}$$

*Proof.* Our strategy is the following. We use the cyclic invariance of trace along with the commutation relations for  $\Gamma_+$  to move the operator  $\Gamma_+$  past the other operators cyclically to the right until the operators are back to their original positions, but with new arguments. We perform this operation a countable number of times, eventually making the  $\Gamma_+$  operator disappear<sup>4</sup>. We then employ the same strategy moving  $\Gamma_-$  cyclically to the left a countable number of times until it disappears and we are left with a term which we can evaluate with the Bloch-Okounkov theorem (Theorem 5).

We first cyclically commute the operator  $\Gamma_+$  to the right using equations (8), (11), and (12):

$$\begin{aligned} & \text{tr}(\mathcal{E}(a, p) \Gamma_+(p^{-\rho}) \Gamma_-(p^{-\rho}) q^H) \\ &= M(p) \text{tr}(\mathcal{E}(a, p) \Gamma_-(p^{-\rho}) \Gamma_+(p^{-\rho}) q^H) \\ &= M(p) \text{tr}(\mathcal{E}(a, p) \Gamma_-(p^{-\rho}) q^H \Gamma_+(qp^{-\rho})) \\ &= M(p) \text{tr}(\Gamma_+(qp^{-\rho}) \mathcal{E}(a, p) \Gamma_-(p^{-\rho}) q^H) \\ &= M(p)(1 - qa^{-1}) \text{tr}(\mathcal{E}(a, p) \Gamma_+(qp^{-\rho}) \Gamma_-(p^{-\rho}) q^H). \end{aligned}$$

Cyclically commuting  $\Gamma_+$  to the right a second time we get:

$$\begin{aligned} & \text{tr}(\mathcal{E}(a, p) \Gamma_+(p^{-\rho}) \Gamma_-(p^{-\rho}) q^H) = \\ & M(p)(1 - qa^{-1}) M(p, q)(1 - q^2 a^{-1}) \text{tr}(\mathcal{E}(a, p) \Gamma_+(q^2 p^{-\rho}) \Gamma_-(p^{-\rho}) q^H). \end{aligned}$$

---

<sup>4</sup>The third author thanks Guillaume Chapuy and Sylvie Corteel for teaching him this trick at a conference lunch in 2014. Bouttier, Chapuy, and Corteel used the trick in the paper [4] in the proof of theorem 12 therein.

After performing  $N$  iterations of this strategy, we arrive at

$$\begin{aligned} \text{tr}(\mathcal{E}(a, p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \\ \prod_{d=1}^N M(p, q^{d-1})(1 - q^d a^{-1}) \text{tr}(\mathcal{E}(a, p)\Gamma_+(q^N p^{-\rho})\Gamma_-(p^{-\rho})q^H). \end{aligned}$$

It follows from Equation (7) that

$$\Gamma_{\pm}(q^N p^{-\rho}) \equiv \text{Id} \pmod{q^N}.$$

So the above two equations imply that the equation

$$\begin{aligned} \text{tr}(\mathcal{E}(a, p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \\ \prod_{d=1}^{\infty} M(p, q^{d-1})(1 - q^d a^{-1}) \text{tr}(\mathcal{E}(a, p)\Gamma_-(p^{-\rho})q^H) \end{aligned}$$

holds to all orders in  $q$  and is hence true as a formal power series in  $q$ .

We now apply the same strategy commuting  $\Gamma_-$  to the left:

$$\begin{aligned} \text{tr}(\mathcal{E}(a, p)\Gamma_-(p^{-\rho})q^H) &= (1 - a) \text{tr}(\mathcal{E}(a, p)\Gamma_-(qp^{-\rho})q^H) \\ &= (1 - a)(1 - aq) \text{tr}(\mathcal{E}(a, p)\Gamma_-(q^2 p^{-\rho})q^H) \\ &= \dots \\ &= \prod_{d=1}^{\infty} (1 - aq^{d-1}) \text{tr}(\mathcal{E}(a, p)q^H) \end{aligned}$$

and so we have proved

$$\begin{aligned} \text{tr}(\mathcal{E}(a, p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \\ \prod_{d=1}^{\infty} M(p, q^{d-1})(1 - q^d a^{-1})(1 - aq^{d-1}) \text{tr}(\mathcal{E}(a, p)q^H). \end{aligned}$$

From the definition of  $\mathcal{E}_r(p)$  we see that its matrix entries are all off-diagonal if  $r \neq 0$ . Therefore

$$\begin{aligned} \text{tr}(\mathcal{E}(a, p)q^H) &= \text{tr}(\mathcal{E}_0(p)q^H) \\ &= \sum_{\lambda} q^{|\lambda|} \sum_{i=1}^{\infty} p^{\lambda - i + \frac{1}{2}} \\ &= (p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{-1} \prod_{m=1}^{\infty} \frac{(1 - q^m)}{(1 - pq^m)(1 - p^{-1}q^m)} \end{aligned}$$

where the last equality follows from the computation in the proof of equation (2) in § 4.1. Combining this with the previous computations finishes the proof of the lemma.  $\square$

Combining Lemmas 6 and 7, we get

$$\sum_{\lambda} q^{|\lambda|} p^{\|\lambda'\|^2} \frac{V_{\lambda \square \emptyset}}{V_{\lambda \emptyset \emptyset}} V_{\lambda' \lambda \emptyset} = \frac{1}{1-p} M(p) \prod_{m=1}^{\infty} \frac{M(p, q^m)}{(1-pq^m)(1-p^{-1}q^m)} \\ \cdot \text{Coeff}_{a^0} \left\{ \prod_{m=1}^{\infty} (1-q^m a^{-1})(1-q^{m-1}a)(1-q^m) \right\}.$$

By the Jacobi triple product identity, we have

$$\prod_{m=1}^{\infty} (1-q^m a^{-1})(1-q^{m-1}a)(1-q^m) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} (-a)^n$$

whose  $a^0$  coefficient is 1. Plugging into the previous equation we finish the proof of equation (4).

## 6. GEOMETRY AND APPLICATIONS

**6.1. Overview:** In this section we briefly outline the applications of our trace formulas to the Donaldson-Thomas theory of elliptically fibered Calabi-Yau threefolds [6, 5, 7]. As we will explain below, each formula in our main theorem gives a certain contribution to the Donaldson-Thomas partition function of such threefolds. Namely, we have the following.

- The contribution of multiples of a nodal elliptic fiber is given by Equation (1).
- The contribution of multiples of a smooth elliptic fiber attached to a smooth section curve is given by Equation (2).
- The contribution of multiples of a smooth elliptic fiber attached to a nodal section curve is given by Equation (3).
- The contribution of multiples of a nodal elliptic fiber attached to a smooth section curve is given by Equation (4).

**6.2. An absurdly short recap of Donaldson-Thomas theory.** Donaldson-Thomas theory is a curve counting theory for Calabi-Yau threefolds which is conjecturally equivalent to Gromov-Witten theory [9]. Let  $X$  be a Calabi-Yau threefold, let  $\beta \in H_2(X, \mathbb{Z})$  be a curve class, and let  $n \in \mathbb{Z}$ .

Let

$$\text{Hilb}^{\beta, n}(X) = \{Z \subset X, [Z] = \beta, \chi(\mathcal{O}_Z) = n\}$$

be the Hilbert scheme parameterizing subschemes of class  $\beta$  and whose structure sheaf has holomorphic Euler characteristic  $n$ . The Donaldson-Thomas invariant  $\text{DT}_{\beta, n}(X)$  is defined to be the Behrend function weighted Euler characteristic of the Hilbert scheme:

$$\text{DT}_{\beta, n}(X) = \sum_{k \in \mathbb{Z}} k e(\nu^{-1}(k))$$

where  $e(-)$  is topological Euler characteristic and

$$\nu : \text{Hilb}^{\beta,n}(X) \rightarrow \mathbb{Z}$$

is Behrend's constructible function [2]. The unweighted Euler characteristics are also often considered:

$$\widehat{\text{DT}}_{\beta,n}(X) = e(\text{Hilb}^{\beta,n}(X)).$$

In this discussion of our applications we will only consider the unweighted invariants  $\widehat{\text{DT}}_{\beta,n}(X)$ . For further discourse on including the Behrend function see [5, 6, 7].

The invariants are usually assembled into a generating function called the partition function:

$$\widehat{\text{DT}}(X) = \sum_{n,\beta} \widehat{\text{DT}}_{\beta,n}(X) q^\beta p^n.$$

The above series can be considered as an element in the Novikov ring of  $X$  with coefficients which are formal Laurent series in  $p$ .<sup>5</sup>

### 6.3. Donaldson-Thomas invariants of a threefold with a torus action.

The topological Euler characteristic of a  $\mathbb{C}$ -scheme  $S$  with the action of a complex torus  $T \cong (\mathbb{C}^*)^k$  is given by the Euler characteristic of the fixed locus:

$$e(S) = e(S^T).$$

Thus if  $X$  is a Calabi-Yau threefold equipped with the action of a torus  $T$ , then

$$\widehat{\text{DT}}_{\beta,n}(X) = e(\text{Hilb}^{\beta,n}(X)^T).$$

In the case where  $X$  is toric with action  $T$ , then  $\text{Hilb}^{\beta,n}(X)^T$  parameterizes  $T$ -invariant subschemes and is a finite set. Thus  $\widehat{\text{DT}}_{\beta,n}(X)$  simply counts the number of  $T$  invariant subschemes  $Z \subset X$  with  $[Z] = \beta$  and  $\chi(\mathcal{O}_Z) = n$ .

On a  $T$ -invariant coordinate chart  $\mathbb{C}^3 \subset X$ , a  $T$ -invariant subscheme is given by a monomial ideal  $I \subset \mathbb{C}[x, y, z]$ . There is a bijection between 3D partitions and monomial ideals given by:

$$(i, j, k) \in \pi \iff x^i y^j z^k \notin I.$$

If the partition  $\pi$  is asymptotic to  $(\lambda, \mu, \nu)$  as in Definition 1, then the subscheme defined by the corresponding monomial ideal is supported on the coordinate axes of  $\mathbb{C}^3$  and has nilpotent thickenings along the axes determined  $(\lambda, \mu, \nu)$ .

<sup>5</sup>With an appropriate choice of a basis of  $H_2(X, \mathbb{Z})$ , the partition function can be considered as a formal power series in a set of variables  $q_i$  whose coefficients are Laurent series in  $p$ .

So we see that to compute  $\widehat{\mathrm{DT}}_{\beta,n}(X)$  for a toric Calabi-Yau threefold  $X$ , we need to count 3D partitions (one for each coordinate chart) such that the corresponding subschemes agree on the coordinate overlaps and the curve class and holomorphic Euler characteristic are given by  $\beta$  and  $n$  respectively. This leads to a general formalism developed in [9] for computing  $\widehat{\mathrm{DT}}(X)$  in terms of the topological vertex.

**6.4. Using the topological vertex in non-toric geometries.** The topological vertex can also be used to compute  $\widehat{\mathrm{DT}}(X)$  for some Calabi-Yau threefolds  $X$  which do not have a global torus action, if it is known for example that the reduced support of all 1-dimensional subschemes is *formally locally monomial*. A typical setting where this can be applied is when  $X$  is elliptically fibered. Namely, there is a map  $\pi : X \rightarrow S$  to a surface such that the fibers are possibly nodal genus 1 curves. We typically require a section  $\sigma : S \rightarrow X$  and we consider curve classes of the form

$$\beta_d = \sigma_*(\beta) + d[F]$$

or simply

$$\tilde{\beta}_d = d[F].$$

Here  $\beta$  is an irreducible curve class in  $S$  and  $[F]$  is the class of the fiber.

The basic idea is to exploit the *motivic* nature of Euler characteristic. The Euler characteristic defines a ring homomorphism from the Grothendieck group of varieties to the integers. More prosaically, this means that one can compute the Euler characteristic of a scheme by stratifying the scheme and then summing the Euler characteristics of each individual stratum. If we stratify  $\mathrm{Hilb}^{\beta,n}(X)$  by the support of the corresponding subscheme, it then makes sense to talk about the contribution of schemes supported on some particular curve to the Donaldson-Thomas invariant of  $X$ . For example, given a particular fiber  $F$  and some particular curve  $C \subset S$  in the class, it makes sense to talk about the contribution of schemes supported on  $\sigma(C) \cup F$  to the invariant  $\widehat{\mathrm{DT}}_{\beta_d,n}(X)$ . Namely, it is the Euler characteristic of the substratum of the Hilbert scheme  $\mathrm{Hilb}^{\beta,n}(X)$  which parameterizes subschemes supported on  $\sigma(C) \cup F$ .

## REFERENCES

- [1] Mina Aganagic, Albrecht Klemm, Marcos Mariño, and Cumrun Vafa. The topological vertex. *Comm. Math. Phys.*, 254(2):425–478, 2005. arXiv:hep-th/0305132.
- [2] Kai Behrend. Donaldson-Thomas type invariants via microlocal geometry. *Ann. of Math. (2)*, 170(3):1307–1338, 2009. arXiv:math/0507523.
- [3] Spencer Bloch and Andrei Okounkov. The character of the infinite wedge representation. *Adv. Math.*, 149(1):1–60, 2000. arXiv:alg-geom/9712009.



- [4] Jérémie Bouttier, Guillaume Chapuy, and Sylvie Corteel. From Aztec diamonds to pyramids: steep tilings. [arXiv:math/1407.0665](#).
- [5] Jim Bryan. The Donaldson-Thomas theory of  $K3 \times E$  via the topological vertex. [arXiv:math/1504.02920](#).
- [6] Jim Bryan and Martijn Kool. Donaldson-Thomas invariants of local elliptic surfaces via the topological vertex. In preparation.
- [7] Jim Bryan, Georg Oberdieck, Rahul Pandharipande, and Qizheng Yin. Curve counting on abelian surfaces and threefolds. [arXiv:math/1506.00841](#).
- [8] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [9] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. I. *Compos. Math.*, 142(5):1263–1285, 2006. [arXiv:math.AG/0312059](#).
- [10] A. Okounkov and R. Pandharipande. Gromov-Witten theory, Hurwitz theory, and completed cycles. *Ann. of Math. (2)*, 163(2):517–560, 2006. [arXiv:math.AG/0204305](#).
- [11] Andrei Okounkov. Infinite wedge and random partitions. *Selecta Math. (N.S.)*, 7(1):57–81, 2001. [arXiv:9907127](#).
- [12] Andrei Okounkov, Nikolai Reshetikhin, and Cumrun Vafa. Quantum Calabi-Yau and classical crystals. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 597–618. Birkhäuser Boston, Boston, MA, 2006. [arXiv:hep-th/0309208](#).

J.BRYAN@MATH.UBC.CA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, ROOM 121, 1984 MATHEMATICS ROAD, VANCOUVER, B.C., CANADA V6T 1Z2

M.KOOL@UU.NL, MATHEMATICAL INSTITUTE, UTRECHT UNIVERSITY, BUDAPEST-LAAN 6, 3584 CD, UTRECHT, THE NETHERLANDS

B.JYU@OREGON.EDU, DEPARTMENT OF MATHEMATICS, FENTON HALL, UNIVERSITY OF OREGON, EUGENE, OR 97403-1222 USA