TRACE IDENTITIES FOR THE TOPOLOGICAL VERTEX

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ABSTRACT.

1. INTRODUCTION

The topological vertex $\mathsf{V}_{\lambda\mu\nu} = \mathsf{V}_{\lambda\mu\nu}(p)$ is a universal formal Laurent series in p depending on a triple of partitions (λ, μ, ν). It can be considered as an object in combinatorics, representation theory, geometry, or physics. In combinatorics, $\mathsf{V}_{\lambda\mu\nu}$ is the generating function for the number of 3D partitions with asymptotic legs of type (λ, μ, ν) (see Definition 2). In representation theory, $\mathsf{V}_{\lambda\mu\nu}$ is given as the matrix coefficients of a certain vertex operator on Fock space (see ???). In geometry, $\mathsf{V}_{\lambda\mu\nu}$ is the basic building block for computing the Donaldson-Thomas/Gromov-Witten invariants of toric Calabi-Yau threefolds; in this incarnation, it can be realized as the generating function for the Euler characteristics of certain Hilbert schemes of curves in $\mathbb{C}^3$ (see ???). The topological vertex was first discovered in physics as an open string partition function in type IIA string theory on $\mathbb{C}^3$ [1]. An explicit expression for $\mathsf{V}_{\lambda\mu\nu}$ in terms of Schur functions was given by [7] (see section 3).

In this paper we prove several "trace identities" in which a sum over certain combinations of the vertex is expressed as a closed formula, often a product of simple terms. The products are closely related to the Fourier expansions of Jacobi forms. Applications of these identities are used to compute the Donaldson-Thomas partition functions of certain Calabi-Yau threefolds in terms of Jacobi forms [4, 3, 5].

2. DEFINITIONS AND THE MAIN RESULT.

In this section we give the combinatorial definition of the vertex and we state our main identities.

Definition 1. Let (λ, μ, ν) be a triple of ordinary partitions. A *3D partition π asymptotic to (λ, μ, ν)* is a subset

$$\pi \subset (\mathbb{Z}_{\geq 0})^3$$

satisfying

(1) if any of $(i+1, j, k)$, $(i, j+1, k)$, and $(i, j, k+1)$ is in π, then (i, j, k) is also in π, and
(2) (a) $(j, k) \in \lambda$ if and only if $(i, j, k) \in \pi$ for all $i \gg 0$,
 (b) $(k, i) \in \mu$ if and only if $(i, j, k) \in \pi$ for all $j \gg 0$,
 (c) $(i, j) \in \nu$ if and only if $(i, j, k) \in \pi$ for all $k \gg 0$.

where we regard ordinary partitions as finite subsets of $(\mathbb{Z}_{\geq 0})^2$ via their diagram.

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Intuitively, π is a pile of boxes in the positive octant of 3-space. Condition (1) means that the boxes are stacked stably with gravity pulling them in the $(-1,-1,-1)$ direction; condition (2) means that the pile of boxes is infinite along the coordinate axes with cross-sections asymptotically given by λ, μ, and ν.

The subset $\{(i,j,k) : (j,k) \in \lambda\} \subset \pi$ will be called the *leg* of π in the i direction, and the legs in the j and k directions are defined analogously. Let

$$\xi_\pi(i,j,k) = 1 - \#\text{ of legs of } \pi \text{ containing } (i,j,k).$$

We define the renormalized volume of π by

$$|\pi| = \sum_{(i,j,k)\in\pi} \xi_\pi(i,j,k).$$

Note that $|\pi|$ can be negative.

Definition 2. The topological vertex $\mathsf{V}_{\lambda\mu\nu}$ is defined to be

$$\mathsf{V}_{\lambda\mu\nu} = \sum_\pi p^{|\pi|}$$

where the sum is taken over all 3D partitions π asymptotic to (λ,μ,ν). We regard $\mathsf{V}_{\lambda\mu\nu}$ as a formal Laurent series in p. Note that $\mathsf{V}_{\lambda\mu\nu}$ is clearly cyclically symmetric in the indices, and reflection about the $i=j$ plane yields

$$\mathsf{V}_{\lambda\mu\nu} = \mathsf{V}_{\mu'\lambda'\nu'}$$

where $'$ denotes conjugate partition:

$$\lambda' = \{(i,j) : (j,i) \in \lambda\}.$$

This definition of topological vertex differs from the vertex $C(\lambda,\mu,\nu)$ of the physics literature by a normalization factor (and we use the variable p instead of q). Our $\mathsf{V}_{\lambda\mu\nu}$ is equal to $P(\lambda,\mu,\nu)$ defined by Okounkov, Reshetikhin, and Vafa [7, eqn 3.16]. They derive an explicit formula for $\mathsf{V}_{\lambda\mu\nu} = P(\lambda,\mu,\nu)$ in terms of Schur functions [7, eqns 3.20 and 3.21].

The *rows* or *parts* of λ are the integers $\lambda_j = \min\{i \mid (i,j) \notin \lambda\}$, for $j \geq 0$. We use the notation

$$|\lambda| = \sum_j \lambda_j, \quad ||\lambda||^2 = \sum_j \lambda_j^2.$$

Let $\square$ denote the partition with a single part of size 1.

We also use the notation

$$M(p,q) = \prod_{m=1}^{\infty} (1-p^m q)^{-m}$$

and the shorthand $M(p) = M(p,1)$.

We can now state our main result.

Theorem 3. *The following identities hold:*

$$\sum_{\lambda} q^{|\lambda|} = \prod_{m=1}^{\infty} (1-q^m)^{-1} \tag{1}$$

$$\sum_{\lambda} q^{|\lambda|} p^{||\lambda||^2} \mathsf{V}_{\lambda'\lambda\varnothing} = M(p) \prod_{d=1}^{\infty} (1-q^d)^{-1} M(p,q^d) \tag{2}$$

$$\sum_{\lambda} q^{|\lambda|} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = (1-p)^{-1} \prod_{d=1}^{\infty} \frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)} \tag{3}$$

$$\sum_{\lambda} q^{|\lambda|} p \frac{\mathsf{V}_{\square\square\lambda}}{\mathsf{V}_{\varnothing\varnothing\lambda}} = \prod_{m=1}^{\infty} (1-q^m)^{-1} \cdot \left\{ 1 + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k}) q^q \right\} \tag{4}$$

(5)

$$\sum_{\lambda} q^{|\lambda|} p^{||\lambda||^2} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \mathsf{V}_{\lambda'\lambda\varnothing} = (1-p)^{-1} M(p) \prod_{d=1}^{\infty} \frac{M(p,q^d)}{(1-pq^d)(1-p^{-1}q^d)}$$

We call these formulas "trace formulas" since the left hand side can be expressed as the traces of certain operators on Fock space. This will be made explicit in section ???.

We note that formula (1) is elementary and well known. We prove Formula (2) in section 3 using the orthogonality properties of skew Schur functions. Formulas (3) and (4) are proved in section 4 using a theorem of Bloch-Okounkov [2]. The most difficult identity to prove is equation (5) which we do in section ???. There we prove that the left hand side of equation (5) is given as the trace of a certain product of operators on Fock space. To compute the trace, we introduce a new trick which involves an "infinite number" of permutations of the operators.

3. The topological vertex and Schur functions

Okounkov-Reshetikhin-Vafa derived a formula for the topological vertex in terms of skew Schur functions. Translating their formulas [7, 3.20& 3.21] into our notation, we get:

$$\mathsf{V}_{\lambda\mu\nu}(p) = M(p) p^{-\frac{1}{2}(||\lambda||^2 + ||\mu||^2 + ||\nu||^2)} s_{\nu'}(p^{-\rho}) \sum_{\eta} s_{\lambda'/\eta}(p^{-\nu-\rho}) s_{\mu/\eta}(p^{\nu'-\rho}). \tag{6}$$

Here, $s_{\alpha/\beta}(x_1, x_2, \dots)$ is the skew Schur function (see for example [6, § 5]) and

$$\rho = \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\right)$$

so that $p^{-\nu-\rho}$ is notation for the variable list

$$p^{-\nu-\rho} = \left(p^{-\nu_1+\frac{1}{2}}, p^{-\nu_2+\frac{3}{2}}, \dots\right).$$

Using equation (6) and orthogonality of skew Schur functions [6, 28(a),page 94] we prove equation (2) as follows:

$$\mathsf{V}_{\lambda'\lambda\varnothing} = M(p) p^{-||\lambda||^2} \sum_{\eta} s_{\lambda/\eta}(p^{-\rho})^2$$

so

$$\begin{aligned}
\sum_{\lambda} q^{|\lambda|}p^{||\lambda||^2}\mathsf{V}_{\lambda'\lambda\varnothing} &= M(p)\sum_{\lambda,\eta} q^{|\lambda|}(s_{\lambda/\eta}(p^{\frac{1}{2}},p^{\frac{3}{2}},\dots))^2 \\
&= M(p)\prod_{d=1}^{\infty}\left((1-q^d)^{-1}\prod_{j,k=1}^{\infty}(1-q^dp^{i-\frac{1}{2}+j-\frac{1}{2}})^{-1}\right) \\
&= M(p)\prod_{d=1}^{\infty}(1-q^d)^{-1}\prod_{m=1}^{\infty}(1-q^dp^m)^{-m} \\
&= M(p)\prod_{d=1}^{\infty}(1-q^d)^{-1}M(p,q^d).
\end{aligned}$$

We also use equation 6 to derive the following key formulas:

Lemma 4. *The following hold:*

$$\begin{aligned}
p^{\frac{1}{2}}\ \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}} \\
p\ \frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= 1-\left(\sum_{i=1}^{\infty}p^{-\lambda_i+i-\frac{1}{2}}\right)\left(\sum_{j=1}^{\infty}p^{\lambda_j-j+\frac{1}{2}}\right)
\end{aligned}$$

Proof. Applying equation (6) to $\mathsf{V}_{\lambda\square\varnothing}/\mathsf{V}_{\lambda\varnothing\varnothing} = \mathsf{V}_{\square\varnothing\lambda}/\mathsf{V}_{\varnothing\varnothing\lambda}$ we see that the left hand side of the first equation is given by

$$s_{\square}(p^{-\lambda-\rho}) = s_{\square}(p^{-\lambda_1+\frac{1}{2}},p^{-\lambda_2+\frac{3}{2}},\dots) = \sum_{i=1}^{\infty}p^{-\lambda_i+i-\frac{1}{2}}.$$

Similarly,

$$\begin{aligned}
p\frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = p\frac{\mathsf{V}_{\square\square\lambda}}{\mathsf{V}_{\varnothing\varnothing\lambda}} &= \sum_{\eta} s_{\square/\eta}(p^{-\lambda-\rho})s_{\square/\eta}(p^{-\lambda'-\rho}) \\
&= 1+s_{\square}(p^{-\lambda-\rho})s_{\square}(p^{-\lambda'-\rho}).
\end{aligned}$$

In general we have the following relation (see [7, Eqn (3.10)])[1]

$$s_{\lambda/\mu}(p^{\nu+\rho}) = (-1)^{|\lambda|-|\mu|}s_{\lambda'/\mu'}(p^{-\nu'-\rho})$$

so in particular $s_{\square}(p^{\nu+\rho}) = -s_{\square}(p^{-\nu'-\rho})$ and thus

$$\begin{aligned}
p\frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= 1-s_{\square}(p^{-\lambda-\rho})s_{\square}(p^{\lambda+\rho}) \\
&= 1-\left(\sum_{i=1}^{\infty}p^{-\lambda_i+i-\frac{1}{2}}\right)\left(\sum_{j=1}^{\infty}p^{\lambda_j-j+\frac{1}{2}}\right)
\end{aligned}$$

which proves the lemma.

[1]There is a typo in equation 3.10 in [7] — the exponent on the right hand side should be $\nu'-\rho$.

4. APPLICATIONS OF A THEOREM OF BLOCH-OKOUNKOV

We summarize a result of Bloch-Okounkov [2] and use it apply it to proving more of our trace formulas.

We define the following theta function

$$\Theta(p,q)=\eta(q)^{-3}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}p^{n+\frac{1}{2}}$$

which, by the Jacobi triple product formula is given by

$$\Theta(p,q)=(p^{\frac{1}{2}-p^{-\frac{1}{2}}})\prod_{m=1}^{\infty}\frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}.$$

We often supress the q from the notation: $\Theta(p)=\Theta(p,q)$, and we note that

$$\Theta(p)=-\Theta(p^{-1}).$$

Theorem 5 (Bloch-Okounkov [2])**.** *Define the n point correlation function by the formula*

$$F(p_1,\dots,p_n)=\prod_{m=1}^{\infty}(1-q^m)\sum_{\lambda}q^{\lambda}\prod_{k=1}^{n}\left(\sum_{i=1}^{\infty}p_k^{\lambda_i-i+\frac{1}{2}}\right).$$

Then

$$F(p)=\frac{1}{\Theta(p)}$$

and

$$F(p_1,p_2)=\frac{1}{\Theta(p_1p_2)}\left(p_1\frac{d}{dp_1}\log(\Theta(p_1))+p_2\frac{d}{dp_2}\log(\Theta(p_2))\right).$$

In [2], formulas for the general n variable function is given, but we will only need the cases of $n=1$ and $n=2$.

Using this theorem, we will prove equations (3) and (4) of the main theorem.

4.1. **Proofs of equations** (3) **and** (4)**.**

We apply Lemma 4 and Theorem 5:

$$\begin{aligned}\sum_{\lambda}(1-p)q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}&=(p^{-\frac{1}{2}}-p^{\frac{1}{2}})\sum_{\lambda}q^{|\lambda|}\sum_{i=1}^{\infty}p^{-\lambda_i+i-\frac{1}{2}}\\&=(p^{-\frac{1}{2}}-p^{\frac{1}{2}})\prod_{m=1}^{\infty}(1-q^m)^{-1}F(p^{-1})\\&=(p^{-\frac{1}{2}}-p^{\frac{1}{2}})\prod_{m=1}^{\infty}(1-q^m)^{-1}\frac{1}{-\Theta(p)}\\&=\prod_{m=1}^{\infty}\frac{(1-q^m)}{(1-pq^m)(1-p^{-1}q^m)}\end{aligned}$$

which proves equation (3).

Again we apply Lemma 4 and Theorem 5:

$$\begin{aligned}\sum_{\lambda} q^{|\lambda|} p \frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= \sum_{\lambda} q^{|\lambda|} \left\{ 1 - \left(\sum_{i=1}^{\infty} p^{-\lambda_i + i - \frac{1}{2}} \right) \left(\sum_{j=1}^{\infty} p^{-\lambda_j - j + \frac{1}{2}} \right) \right\} \\ &= \prod_{m=1}^{\infty} (1-q^m)^{-1} \left(1 - F(p, p^{-1}) \right).\end{aligned}$$

From Theorem 5, we see that

$$F(p,p^{-1}) = \lim_{(p_1,p_2)\to(p,p^{-1})} \frac{1}{\Theta(p_1 p_2)} \left(p_1 \frac{d}{dp_1} \log(\Theta(p_1)) + p_2 \frac{d}{dp_2} \log(\Theta(p_2)) \right).$$

To evaluate this limit, we simplify the above expression. A short computation shows that

$$p \frac{d}{dp} \log(\Theta(p)) = \frac{1}{2} \frac{p+1}{p-1} + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \left(-p^k + p^{-k} \right) q^{mk}.$$

Thus

$$\begin{aligned}F(p,p^{-1}) = &\lim_{\substack{(p_1,p_2)\to \\ (p,p^{-1})}} \left((p_1p_2)^{\frac{1}{2}} - (p_1p_2)^{-\frac{1}{2}} \right)^{-1} \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-(p_1p_2)q^m)(1-(p_1p_2)^{-1}q^m)} \\ &\cdot \left\{ \frac{1}{2} \cdot \frac{p_1+1}{p_1-1} + \frac{1}{2} \cdot \frac{p_2+1}{p_2-1} + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \left(-p_1^k - p_2^k + p_1^{-k} + p_2^{-k} \right) q^{mk} \right\} \\ = &\lim_{\substack{(p_1,p_2)\to \\ (p,p^{-1})}} \frac{-(p_1p_2)^{\frac{1}{2}}}{1-p_1p_2} \cdot \left\{ \frac{p_1p_2 - 1}{(p_1-1)(p_2-1)} + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} (1 - p_1^k p_2^k)(p_1^{-k} + p_2^{-k}) q^{mk} \right\} \\ = &\lim_{\substack{(p_1,p_2)\to \\ (p,p^{-1})}} (p_1p_2)^{\frac{1}{2}} \left\{ \frac{1}{(1-p_1)(1-p_2)} - \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{1-(p_1p_2)^k}{1-p_1p_2} (p_1^{-k} + p_2^{-k}) q^{mk} \right\} \\ = &\frac{1}{(1-p)(1-p^{-1})} - \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} k(p^k + p^{-k}) q^{mk}.\end{aligned}$$

Therefore

$$1 - F(p,p^{-1}) = 1 + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k}) q^d$$

which finishes the proof of equation (4).

5. VERTEX OPERATORS AND THE PROOF OF EQUATION (5)

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