1. DEFINITIONS OF $P_\alpha(Q)$ AND $R_\alpha(Q)$

Let $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots)$ be a partition. We define a Laurent polynomial $P_\alpha(Q)$ as follows

$$P_\alpha(Q) = \sum_{k=1}^{\infty} (Q^{-\alpha_k+k-1} - Q^{-\alpha_k+k}).$$

Note that this is a Laurent *polynomial* since the sum telescopes for large k. Its leading term is $Q^{-\alpha_1}$. It can be expressed in terms of the topological vertex as follows:

$$P_\alpha(Q) = (1-Q)\frac{V_{\alpha,\square,\emptyset}}{V_{\alpha,\emptyset,\emptyset}}.$$

A closely related quantity shows up in Okounkov's papers. For example in Okounkov-Pandharipande (math/0204305) page 33, the eigenvalues of the operator $\mathcal{E}_0(z)$ (acting on $|\alpha\rangle$) is given by $e(\alpha, z)$ which is equal to $P_\alpha(Q)$ upto a factor of $(Q^{-1/2} - Q^{1/2})$ under the substitution $Q = e^{-z}$.

The second important quantity we consider is $R_\alpha(Q)$. We can define it as follows:

$$R_\alpha(Q) = \sum_\eta Q^{|\alpha|-|\eta|} \left(S_{\alpha/\eta}(\mathbf{Q})\right)^2$$

where $S_{\alpha/\eta}$ is the skew Schur function and $S_{\alpha/\eta}(\mathbf{Q}) = S_{\alpha/\eta}(1, Q, Q^2, Q^3, \dots)$. In terms of the (PT) vertex, R_α is the vertex $V_{\emptyset,\alpha,\alpha'}$ normalized to be a series starting with 1.

$R_\alpha(Q)$ is a rational function in Q. These functions satisfy the following symmetry relation

$$P_\alpha(Q) = P_{\alpha'}(Q^{-1}) \qquad R_\alpha(Q) = R_{\alpha'}(Q^{-1})$$

where α' is the conjugate partition.

2. THE MAIN CONJECTURE, RELATED PROPOSTIONS, EQUIVALENT FORMULATIONS

Proposition 1. *The follow product formulas hold.*

$$\sum_\alpha u^{|\alpha|} = \prod_{i=1}^{\infty} (1-u^i)^{-1}$$

$$\sum_\alpha R_\alpha(Q) u^{|\alpha|} = \prod_{i=1}^{\infty} \left((1-u^i)^{-1} \cdot \prod_{m=1}^{\infty} (1-u^i Q^m)^{-m} \right)$$

$$\sum_\alpha P_\alpha(Q) u^{|\alpha|} = \prod_{i=1}^{\infty} (1-u^i)(1-Qu^i)^{-1}(1-Q^{-1}u^i)^{-1}.$$

The first formula is elementary and well known, the second formula follows from the orthogonality properties of skew Schur functions. The third formula follows easily from Theorem 6.5 of Bloch-Okounkov (9712009v2). Our main conjecture, extensively checked with Maple (it's certainly true) is

Conjecture 2. *The following formula holds*

$$\sum_\alpha P_{\alpha'}(Q) R_\alpha(Q) u^{|\alpha|} = \prod_{i=1}^{\infty} \left((1-Qu^i)^{-1}(1-Q^{-1}u^i)^{-1} \prod_{m=1}^{\infty} (1-Q^m u^i)^{-m} \right).$$

Using the formulas in the proposition, the above formula is equivalent to the following weirdly symmetric looking formula

$$\sum_\alpha P_{\alpha'}(Q) R_\alpha(Q) u^{|\alpha|} = \sum_{\alpha,\mu} P_{\alpha'}(Q) R_\mu(Q) u^{|\alpha|+|\mu|}.$$

Note the presense of the transpose partition α' in the subscript of P on the left hand sides of the conjecture formulas. The conjecture is false without it.