# DONALDSON-THOMAS INVARIANTS OF LOCAL ELLIPTIC SURFACES VIA THE TOPOLOGICAL VERTEX

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex proved in [4], we derive product formulas for the partition functions. The connected version of the partition function is written in terms of Jacobi forms. In the special case where the elliptic surface is a K3 surface, we get a new derivation of the Katz-Klemm-Vafa formula.

## 1. INTRODUCTION

Let $p : S \to B$ be a non-trivial elliptic surface over a complex smooth projective curve $B$. We assume $p$ has a section and all singular fibres are irreducible rational nodal curves.

We are interested in the Donaldson-Thomas (DT) invariants of $X = \mathrm{Tot}(K_S)$, i.e. the total space of the canonical bundle $K_S$. This is a non-compact Calabi-Yau threefold. Let $\beta$ be an effective curve class on $S$. Consider the Hilbert scheme

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X \,:\, [Z] = \beta,\ \chi(\mathcal{O}_Z) = n\}$$

of proper subschemes $Z \subset X$ with homology class $\beta$ and holomorphic Euler characteristics $n$. The DT invariants of $X$ can be defined as

$$\mathsf{DT}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X), \nu) := \sum_{k\in\mathbb{Z}} k\, e(\nu^{-1}(k)),$$

where $e(\cdot)$ denotes topological Euler characteristic and $\nu : \mathrm{Hilb}^{\beta,n}(X) \to \mathbb{Z}$ is Behrend's constructible function [2]. We consider an Euler characteristic version of these invariants

$$\widehat{\mathsf{DT}}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X)).$$

We choose a section $B \subset S$ and focus on the primitive classes $\beta = B + dF$, where $B$ is the class of the chosen section and $F$ the class of the fibre. We define the partition functions by

$$\widehat{\mathsf{DT}}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \widehat{\mathsf{DT}}_{B+dF,n}(X) p^n q^d,$$
$$\mathsf{DT}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \mathsf{DT}_{B+dF,n}(X) y^n q^d.$$

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We also consider the partition functions for the invariants for multiples of the fiber class:

$$\widehat{\mathsf{DT}}_{\mathrm{fib}}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \widehat{\mathsf{DT}}_{dF,n}(X)p^n q^d,$$
$$\mathsf{DT}_{\mathrm{fib}}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \mathsf{DT}_{dF,n}(X)y^n q^d.$$

The main result of this paper are closed product formulas for the partition functions $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$. Assuming a general conjecture, we also determine $\mathsf{DT}(X)$ and $\mathsf{DT}_{\mathrm{fib}}(X)$.

We use the notation

$$M(p,q) = \prod_{m=1}^{\infty}(1-p^m q)^{-m}$$

and the shorthand $M(p) = M(p,1)$.

**Theorem 1.**

$$\widehat{\mathsf{DT}}(X) = \left\{ M(p)\prod_{d=1}^{\infty}\frac{M(p,q^d)}{(1-q^d)}\right\}^{e(S)} \left\{\frac{1}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})}\prod_{d=1}^{\infty}\frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)}\right\}^{e(B)}$$

$$\widehat{\mathsf{DT}}_{\mathrm{fib}}(X) = \left\{ M(p)\prod_{d=1}^{\infty}M(p,q^d)\right\}^{e(S)} \left\{\prod_{d=1}^{\infty}\frac{1}{(1-q^d)}\right\}^{e(B)}$$

The ratio $\widehat{\mathsf{DT}}(X)/\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$ can be considered as the generating function for the connected invariants in the classes $B+dF$. This series has a particularly nice form and can be written in terms of classical Jacobi forms. Consider the Dedekind eta function and the Jacobi theta function

$$\eta = q^{\frac{1}{24}}\prod_{k=1}^{\infty}(1-q^k),$$
$$\Theta = (p^{\frac{1}{2}}-p^{-\frac{1}{2}})\prod_{k=1}^{\infty}\frac{(1-pq^k)(1-p^{-1}q^k)}{(1-q^k)^2}.$$

**Corollary 2.** *The partition function of the connected invariants is given as follows*

$$\frac{\widehat{\mathsf{DT}}(X)}{\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)} = \left(q^{-\frac{1}{24}}\eta\right)^{-e(S)}\Theta^{-e(B)}.$$

In the case $S\to\mathbb{P}^1$ is an elliptically fibred $K3$ surface, the above series specializes to the well-known Katz-Klemm-Vafa formula. Because $X$ is non-compact, the connected series is required to obtain the KKV formula. Our result provides a new derivation of the KKV formula. The KKV formula was recently proved in *all* curve classes in [13]. This is the first derivation of the KKV formula, which does not depend on the Kawai-Yoshioka formula [8].

The most important result of this paper is perhaps not the formula, but rather the method of calculation. This approach has found further applications to the calculation of DT generating functions on $K3\times E$, where $E$ is an elliptic curve [3], and abelian 3-folds [5]. Even though the geometry under consideration is not toric, we combine $\mathbb{C}^*$-localization, motivic methods, formal methods, and $(\mathbb{C}^*)^3$-localization to end up with expressions that only depend on $\mathsf{V}_{\lambda\mu\nu}$, $e(B)$, and $e(S)$. Here is a rough sketch:

(A) The action of $\mathbb{C}^*$ on the fibres of $X$ lifts to the moduli space[1] $\mathrm{Hilb}^{B+dF,\bullet}(X)$. Therefore, we only have to understand the fixed locus $\mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*}$. Push-forward along $X \to S \to B$ induces a morphism

$$\rho_d : \mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*} \to \mathrm{Sym}^d(B).$$

This map is constructed in Section 3. The fibres of $\rho_d$ decompose into components according to the shape of the underlying Cohen-Macaulay curve. This leads to a decomposition over 2D partitions $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$.

(B) The Euler characteristics of the fibres of $\rho_d$ define a constructible function $f_d$ on $\mathrm{Sym}^d(B)$. In Section 4, we show that if $f_d$ satisfies a certain product formula, then $\widehat{\mathsf{DT}}(X)$ satisfies a corresponding product formula. This follows from general power structure arguments reviewed in Appendix A.2.

(C) A component $\Sigma$ of a fibre of $\rho_d$ indexed by $\lambda$ can be further broken down by taking a certain fpqc cover of the underlying (now fixed) Cohen-Macaulay curve $Z_{\mathrm{CM}}$ determined by $\lambda$. This cover consists of formal neighbourhoods $\widehat{X}_x$ around the singular points $x$ of the reduced support of $Z_{\mathrm{CM}}$ and "tubular neighbourhoods" along the reduced support of $Z_{\mathrm{CM}}$ after removing the singularities. Since $Z_{\mathrm{CM}}$ is already fixed, gluing is automatic. Hence restriction to the elements of the cover gives a bijection morphism of $\Sigma$ to local Hilbert schemes on the elements of the cover. In Section 5, we show this leads to the product formula for $f_d$ in (B).

(D) On the formal neighbourhoods $\widehat{X}_x$, we have an action of $\mathbb{C}^{*3}$. This allows us to express their contributions to the generating function in terms of the topological vertex. The contributions of the tubular neighbourhoods along the *punctured* section and fibres can also be expressed in terms of the topological vertex (utilizing a map to $\mathrm{Sym}^n(F)$ which records the location and multiplicity of the embedded points). This is worked out in Section 6.

Our results can be extended to apply to the usual (Behrend function weighted) Donaldson-Thomas invariants if we assume a general conjecture which we formulate in Section 7. The basic results (assuming Conjecture 20) are

$$\mathsf{DT}(X) = (-1)^{\chi(\mathcal{O}_S)}\widehat{\mathsf{DT}}(X)$$

and

$$\mathsf{DT}_{\mathrm{fib}}(X) = \widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$$

under the change of variables

$$y = -p.$$

A similar phenomenon to the above is known to hold when $X$ is a toric Calabi-Yau threefold.

## 2. Definitions, notation, and conventions

Let $p : S \to B$ be an elliptic surface over a smooth projective curve $B$. Besides assuming $S$ is not a product, we require:

(1) $p$ has a section $B \subset S$,
(2) all singular fibres of $p$ are irreducible rational nodal curves.

[1] The bullet indicates that we take the union of $\mathrm{Hilb}^{B+dF,n}(X)$ over all $n$, see Convention 2.1g.

We write $F_x$ for the fibre $p^{-1}(\{x\})$ for all closed points $x \in B$. We choose a section $B \subset S$ and denote its class in $H_2(S)$ by $B$ as well. We denote the class of the fibre by $F \in H_2(S)$.

For brevity, we define

$$\mathrm{Hilb}^{d,n}(X) := \mathrm{Hilb}^{B+dF,n}(X),$$
$$\widehat{\mathsf{DT}}_{d,n}(X) := \widehat{\mathsf{DT}}_{B+dF,n}(X).$$

Since we are dealing with generating functions and our calculations involve cut-paste methods on the moduli space, it is useful to introduce the following notation. We define

$$\mathrm{Hilb}^{d,\bullet}(X) := \sum_{n\in\mathbb{Z}} \mathrm{Hilb}^{d,n}(X)\, p^n,$$

where we view the right hand side as a formal Laurent series whose coefficients elements in the Grothendieck ring of varieties, i.e. $K_0(\mathrm{Var}_{\mathbb{C}})(\!(p)\!)$.

**Convention 2.1.** *When an index is replaced by a bullet, we will sum over the index, multiplying by the appropriate variable. We regard the result as a formal power (or Laurent) series whose coefficients lie $K_0(\mathrm{Var}_{\mathbb{C}})$ and we extend operations of the Grothendieck group (addition, multiplication, Euler characteristic) to the series in the obvious way.*

For example

$$\mathrm{Hilb}^{\bullet,\bullet}(X) = \sum_{d=0}^{\infty}\sum_{n\in\mathbb{Z}} \mathrm{Hilb}^{d,n}(X) q^d p^n,$$

so that we can write

$$\widehat{\mathsf{DT}}(X) = e(\mathrm{Hilb}^{\bullet,\bullet}(X)).$$

It is notationally convenient to treat an Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, the measurable functions are constructible functions, and the measure of a set is given by its Euler characteristic. In this language we have

$$\widehat{\mathsf{DT}}_{\beta,n}(X) = \int_{\mathrm{Hilb}^{\beta,n}(X)} 1\, de, \qquad \mathsf{DT}_{\beta,n}(X) = \int_{\mathrm{Hilb}^{\beta,n}(X)} \nu\, de,$$

and following the bullet convention we have

$$\widehat{\mathsf{DT}}(X) = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)} 1\, de, \qquad \mathsf{DT}(X) = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)} \nu\, de.$$

We will also need notation for subsets of the Hilbert scheme which parameterize those subschemes obtained by adding embedded points to some fixed Cohen-Macaulay curve.

**Definition 3.** Let $Y \subset X$ be an open set (possibly in the fpcq topology) and let $C \subset Y$ be a Cohen-Macaulay subscheme of dimension 1 which we assume is the restriction of some $\overline{C} \subset X$ to $Y$. We define

$$\mathrm{Hilb}^n(Y,C) = \{Z \subset Y \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n\}.$$

Via the inclusion $Y \subset X$, $\mathrm{Hilb}^n(Y,C)$ can be viewed as a constructible subscheme of $\mathrm{Hilb}(X)$. It parameterizes subschemes which roughly speaking are obtained from $C$ by adding $n$ embedded points.

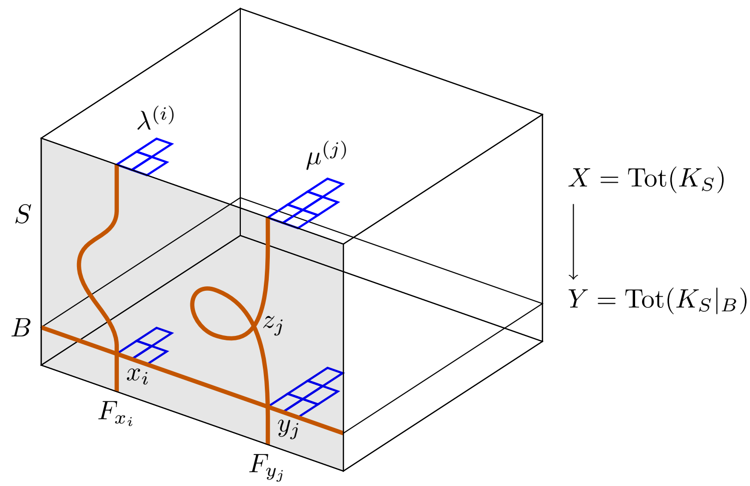


FIGURE 1. Cohen-Macaulay curve $Z_{\mathrm{CM}}$ with underlying reduced support in orange and thickenings $\lambda^{(i)}$ along smooth fibers $F_{x_i}$, $\mu^{(j)}$ along singular fibers $F_{y_j}$, and multiplicity one along the section $B$.

## 3. THE $\mathbb{C}^*$-FIXED LOCUS

The action of $\mathbb{C}^*$ on the fibres of $X$ lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1\, de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\, de.$$

In order to understand $\mathrm{Hilb}^{d,n}(X)$, we first study $Z_{\mathrm{CM}} \subset Z$, the maximal Cohen-Macaulay subscheme of any $\mathbb{C}^*$-invariant subschemes $Z \subset X$. We find that such subschemes are determined by a point in $\mathrm{Sym}^d(B)$ along with some discrete data (a collection of integer partitions). This is given by the following two propositions and is illustrated in Figure 1.

**Proposition 4.** *A closed points $Z$ of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ correspond to a finite nesting of closed subschemes of $S$*

$$Z_0 \supset Z_1 \supset \cdots \supset Z_l,$$

*satisfying*

$$\sum_{k=0}^{l} [Z_k] = B + dF \in H_2(S).$$

*Proof.* Using projection $\pi : X \to S$, a quasi-coherent sheaf on $X$ can be viewed as a quasi-coherent sheaf $\mathcal{F}$ on $S$ together with a morphism $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$. A $\mathbb{C}^*$-equivariant structure on $\mathcal{F}$ translates into a $\mathbb{Z}$-grading

$$\pi_* \mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k,$$

such that $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$ is graded, i.e.

$$\mathcal{F}_k \otimes K_S^{-1} \longrightarrow \mathcal{F}_{k-1}.$$

Here $\mathcal{F}_k$ has weight $k$ and $K_S$ weight 1 under the $\mathbb{C}^*$-action. The structure sheaf $\mathcal{O}_X$ corresponds to

$$\pi_*\mathcal{O}_X = \bigoplus_{k=0}^{\infty} K_S^{-k}.$$

Therefore a $\mathbb{C}^*$-equivariant morphism $\mathcal{F} \to \mathcal{O}_X$ corresponds to a graded sheaf $\mathcal{F}$ as above together with maps $F_i \to K_S^i$ for all $i$ such that

$$\begin{tikzcd} \mathcal{F}_k \otimes K_S^{-1} \arrow[r] \arrow[d] & \mathcal{F}_{k-1} \arrow[d] \\ K_S^k \otimes K_S^{-1} \arrow[r, equal] & K_S^{k-1} \end{tikzcd}$$

commute for all $k \leq 0$ and the composition $\mathcal{F}_1 \otimes K_S^{-1} \to \mathcal{F}_0 \to \mathcal{O}_S$ is to zero.

It is useful to re-define $\mathcal{G}_k := \mathcal{F}_{-k} \otimes K_S^k$. Then a $\mathbb{C}^*$-equivariant morphism $\mathcal{F} \to \mathcal{O}_X$ is equivalent to the following data:

- quasi-coherent sheaves $\{\mathcal{G}_k\}_{k\in\mathbb{Z}}$ on $S$,
- morphisms $\{\mathcal{G}_k \to \mathcal{G}_{k+1}\}_{k\in\mathbb{Z}}$,
- morphisms $\mathcal{G}_k \to \mathcal{O}_S$ such that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{G}_{-1} \arrow[r] \arrow[d] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \cdots \end{tikzcd}$$

In the case of interest to us $\mathcal{G} \to \mathcal{O}_X$ is an ideal sheaf $I_Z \hookrightarrow \mathcal{O}_X$ cutting out $Z \subset X$. In the above language, this means $\mathcal{G}_k = 0$ for $k < 0$, the morphisms $\mathcal{G}_k \to \mathcal{O}_S$ are injective, and the morphisms $\mathcal{G}_k \to \mathcal{G}_{k+1}$ are injective. Therefore $\mathcal{G}_k = I_{Z_k\subset S}$ is an ideal sheaf cutting out $Z_k \subset S$ and

$$I_{Z_k\subset S} \subset I_{Z_{k+1}\subset S},$$

for all $k$. □

Let $\mathrm{Hilb}^{B+dF}(S)$ be the Hilbert scheme of effective divisors on $S$ with class

$$B + dF \in H_2(S).$$

By Lemma 22 of the Appendix A.1, pull-back along $p$ and adding the section $B$ induces an isomorphism

$$\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{B+dF}(S).$$

For any reduced curve $C \subset S$ defined by ideal sheaf $I_{C\subset S}$ and $d > 0$, we denote by $dC$ the Gorenstein curve defined by the ideal sheaf $I_{C\subset S}^d$, the $d$th power of $I_{C\subset S}$. We combine Lemma 22 with a (family version of) Proposition 4 to conclude the following:

**Proposition 5.** *There exists a morphism*

$$\rho_d : \mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*} \longrightarrow \mathrm{Sym}^d(B),$$

*which at the level of closed points can be can be described as follows. Let $Z \in \mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*}$ and let $Z_{\mathrm{CM}} \subset Z$ be the maximal Cohen-Macaulay subcurve of $Z$. Since $Z_{\mathrm{CM}}$ is $\mathbb{C}^*$-fixed, its ideal sheaf decomposes*

$$I_{Z_{\mathrm{CM}}} = \bigoplus_{k=0}^{\infty} I_{Z_k\subset S} \otimes K_S^{-k},$$

*where*

$$Z_0 = B \cup \lambda_0^{(1)} F_{x_1} \cup \cdots \cup \lambda_0^{(l)} F_{x_l}$$

*for some distinct closed points* $x_i \in B$ *and* $\lambda_0^{(i)} > 0$*, and*

$$Z_k = \lambda_k^{(1)} F_{x_1} \cup \cdots \cup \lambda_k^{(l)} F_{x_l}.$$

*for some* $\lambda_k^{(i)} \leq \lambda_{k-1}^{(i)}$*. Here* $\lambda^{(i)} = (\lambda_0^{(i)} \geq \lambda_1^{(i)} \geq \cdots)$ *define 2D partitions satisfying*

$$\sum_{i=1}^{l} |\lambda^{(i)}| = d.$$

*See Figure 1 for an illustration. The map* $\rho_d$ *sends* $Z$ *to*

$$\sum_{i=1}^{l} |\lambda^{(i)}| x_i \in \mathrm{Sym}^d(B).$$

**Remark 6.** The morphism of this proposition is perhaps somewhat surprising. Since we are on a 3-fold, the map which sends a closed subscheme of $Z \in \mathrm{Hilb}^{d,n}(X)$ to its underlying Cohen-Macaulay curve $Z_{\mathrm{CM}}$ is *not* a morphism. Nevertheless, the map $\rho_d$ which records the location of the fibres in $Z_{\mathrm{CM}}$ and their multiplicities is a morphism.

*Proof.* The description of $\rho_d$ at the level of closed points is clear. We construct $\rho_d$ as a morphism from Proposition 4 and Lemma 22 of Appendix A.1.

Let $T$ be an arbitrary base scheme of finite type and let

$$\mathcal{Z} \subset X \times T$$

be a $\mathbb{C}^*$-fixed and $T$-flat closed subscheme. Assume for each $t \in T$ the fibre $\mathcal{Z}_t$ has class $B + dF \in H_2(S)$ and $\chi(\mathcal{O}_{\mathcal{Z}_t}) = n$. Since $\mathcal{Z}$ is $\mathbb{C}^*$-fixed, Proposition 4 implies that its ideal sheaf decomposes[2]

$$I_{\mathcal{Z}} = \bigoplus_{k=0}^{\infty} I_{\mathcal{Z}_k \subset S \times T} \otimes K_S^{-k},$$

where $K_S$ is pulled-back along $S \times T \to S$ and

$$\mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots.$$

Then each $\mathcal{Z}_k \subset S \times T$ is $T$-flat as well. The maximal CM subschemes $\mathcal{Z}_{k,\mathrm{CM}} \subset \mathcal{Z}_k \subset S \times T$ are also $T$-flat and induces morphisms

$$\begin{aligned} &T \longrightarrow \mathrm{Hilb}^{B+d_0F}(S), \\ &T \longrightarrow \mathrm{Hilb}^{d_kF}(S), \ \ \text{for} \ \ k > 0 \end{aligned}$$

where $\sum_k d_k = d$. Adding divisors gives a morphism $T \longrightarrow \mathrm{Hilb}^{B+dF}(S)$. By Lemma 22, we obtain a morphism $T \to \mathrm{Sym}^d(B)$. This morphism corresponds to a $T$-flat family for $\mathrm{Sym}^d(B)$. We have defined $\rho_d$ as a morphism. $\square$

[2]The arguments leading to Proposition 4 hold equally well for $T$-flat families over a base $T$.

## 4. PUSH-FORWARD TO THE SYMMETRIC PRODUCT

In the previous section we constructed a morphism (Proposition 5)

$$\rho_d : \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*} \longrightarrow \mathrm{Sym}^d(B). \tag{1}$$

We obtain

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\, de = \int_{\mathrm{Sym}^d(B)} \rho_{d*}(1)\, de,$$

where $f_d := \rho_{d*}(1)$ is the $\mathbb{Z}((p))$-valued constructible function on $\mathrm{Sym}^d(B)$ given by pushing forward the Euler characteristic measure[9]. Its value at a closed point $\mathfrak{a} \in \mathrm{Sym}^d(B)$ is

$$f_d(\mathfrak{a}) = \int_{\rho_d^{-1}(\mathfrak{a})} 1\, de.$$

It turns out that the constructible function $f_d : \mathrm{Sym}^d(B) \to \mathbb{Z}(\!(p)\!)$ satisfies two multiplicative properties. The first one is described as follows. Denote by $B^{\mathrm{sm}} \subset B$ the open subset over which the fibres are smooth and by $B^{\mathrm{sing}}$ the $N$ points over which the fibres are singular. We can consider the restrictions of $f_d$ to $\mathrm{Sym}^d(B^{\mathrm{sm}}) \subset \mathrm{Sym}^d(B)$ and $\mathrm{Sym}^d(B^{\mathrm{sing}}) \subset \mathrm{Sym}^d(B)$. Denote by $M(p)$ the MacMahon function.

**Proposition 7.** *Let $d_1, d_2 \geq 0$ be such that $d_1 + d_2 = d$. Then*

$$f_d(\mathfrak{a} + \mathfrak{b}) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \cdot f_{d_1}(\mathfrak{a}) \cdot f_{d_2}(\mathfrak{b}),$$

*for any $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$.*

We prove this proposition in Section 5.3. The following product formula is an immediate consequence of this result

$$\widehat{\mathsf{DT}}(X) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \left( \sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sm}})} f_d\, de \right) \cdot \left( \sum_{d \geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sing}})} f_d\, de \right). \tag{2}$$

The restricted constructible functions $f_d : \mathrm{Sym}^d(B^{\mathrm{sm}}) \to \mathbb{Z}(\!(p)\!)$ and $f_d : \mathrm{Sym}^d(B^{\mathrm{sing}}) \to \mathbb{Z}(\!(p)\!)$ satisfy further multiplicative properties:

**Proposition 8.** *There exist functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ and $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ taking values in formal Laurent series $\mathbb{Z}(\!(p)\!)$, such that $g(0) = 1$, $h(0) = 1$, and*

$$f_d(\mathfrak{a}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{i=1}^{l} g(a_i),$$

$$f_d(\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{j=1}^{m} h(b_j),$$

*for all $\mathfrak{a} = \sum_{i=1}^{l} a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$, and $\mathfrak{b} = \sum_{j=1}^{m} b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$, where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.*

We prove this proposition in Section 5.3. Together with Lemma 23 of Appendix A.2, Proposition 8 and equation (2) imply

$$\widehat{\mathsf{DT}}(X) = \frac{M(p)^N}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \left( \sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)-N} \cdot \left( \sum_{b=0}^{\infty} h(b) q^b \right)^{N}. \tag{3}$$

Our goal is to prove Propositions 7 and 8, and find formulae for $g(a)$, $h(b)$. This requires a better understanding of the strata

$$\rho_d^{-1}(\mathfrak{a}+\mathfrak{b}) \subset \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*},$$

for all $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$ with $d_1 + d_2 = d$. Suppose

$$\mathfrak{a} = \sum_{i=1}^{l} a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}}),$$
$$\mathfrak{b} = \sum_{j=1}^{m} b_j y_j \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}}),$$

where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points. Proposition 5 gives a decomposition of $\rho_d^{-1}(\mathfrak{a}+\mathfrak{b})$ into components[3]

$$\bigsqcup_{\substack{\lambda^{(1)} \vdash a_1 \\ \cdots \\ \lambda^{(l)} \vdash a_l}} \bigsqcup_{\substack{\mu^{(1)} \vdash b_1 \\ \cdots \\ \mu^{(m)} \vdash b_m}} \Sigma(x_1, \ldots, x_l, y_1, \ldots, y_m, \lambda^{(1)}, \ldots, \lambda^{(l)}, \mu^{(1)}, \ldots, \mu^{(m)}). \tag{4}$$

We abbreviate these components by $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$. Therefore $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$ is the stratum of points $Z \in \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$, for which the maximal Cohen-Macaulay subcurve $Z_{\mathrm{CM}} \subset Z$ is determined by the data $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$ as in Proposition 5. Note that these strata have a natural scheme structure: the fibres of $\rho_d$ are closed subschemes of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ and these strata are components of them. We are interested in the Euler characteristics of these strata. In the next section, we will see that the Euler characteristic of $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$ does *not* depend on the exact location of the points $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$, but only on their number $m$ and $n$ and the partitions $\lambda^{(i)}$ and $\mu^{(j)}$.

## 5. Restriction to formal neighbourhoods

In the previous two sections we reduced our consideration to the strata $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$ of $Z \in \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ for which the maximal Cohen-Macaulay subcurve $Z_{\mathrm{CM}} \subset Z$ is determined by the data $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$. In this section we break down this stratum further by cutting it up in pieces covered by formal neighbourhoods. For notational simplicity, we first consider the case where the base point is

$$ax + by \in \mathrm{Sym}^d(B),$$

with $x \in B^{\mathrm{sm}}$, $y \in B^{\mathrm{sing}}$, and $d = a + b$. We show how to compute $e(\Sigma(x, y, \lambda, \mu))$. Once this case is established, it is not hard to generalize to arbitrary $e(\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu}))$. This leads to a proof of Propositions 7 and 8, and a geometric characterization of the functions $g(a)$, $h(b)$ of Section 4.

5.1. **Fpqc cover.** The idea is to use an appropriate cover of $X$ and calculate on pieces of the cover. We first give a complex analytic definition of the cover to aid the intuition and then give the actual "algebro-geometric cover":

(1) The reduced support $B \cup F_x \cup F_y$ has three singular points[4]: $x, y \in B$ and $z \in F_y^{\mathrm{sing}}$. We take small open balls around these points.

[3]We use the term component somewhat loose: it means a subset which is both open and closed. We do not care whether it is connected.

[4]Recall that $x, y \in B$ in the base can be viewed as points on $S$ and $X$ via the sections $B \subset S \subset X$.

(2) Consider the punctured curve $B^\circ := B \setminus \{x, y\}$ and let $X^\circ := X \setminus (F_x \cup F_y)$. We take a tubular neighbourhood of $B^\circ \subset X^\circ$.
(3) Consider the punctured curve $F_x^\circ := F_x \setminus \{x\}$ and let $X^\circ := X \setminus B$. We take a tubular neighbourhood of $F_x^\circ \subset X^\circ$.
(4) Consider the punctured curve $F_y^\circ := F_y \setminus \{y, z\}$ and let $X^\circ := X \setminus (B \cup \{z\})$. We take a tubular neighbourhood of $F_y^\circ \subset X^\circ$.
(5) Finally, we take $W = X \setminus (B \cup F_x \cup F_y)$.

In order to work in algebraic geometry, in (1) we take the formal neighbourhood $\widehat{X}_x$ of $\{x\}$ in $X$. Denote the local ring at $x$ by $(R, \mathfrak{m})$. By $\widehat{X}_x$ we mean the (non-noetherian) scheme

$$\operatorname{Spec} \varprojlim R/\mathfrak{m}^n$$

and *not* the formal scheme

$$\operatorname{Spf} \varprojlim R/\mathfrak{m}^n.$$

Similarly in (2) and (3), let $\widehat{X}_y$ be the formal neighbourhood of $\{y\}$ in $X$ and $\widehat{X}_z$ the formal neighbourhood of $\{z\}$ in $X$. Note that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!].$$

Even though $\widehat{X}_x$ is non-noetherian, the morphism $\widehat{X}_x \to X$ has a good property: it is fpqc so can be used as part of a cover [7, Vistoli, Sect. 2.3.2]. Flatness of this map follows from the fact that formal completion is an exact operation [14, Tag 0BNH] [1, Prop. 10.14].

In (2) we consider $B^\circ := B \setminus \{x, y\}$, $X^\circ := X \setminus (F_x \cup F_y)$ and let $\widehat{X}^\circ_{B^\circ}$ be the formal neighbourhood of $F_x^\circ$ in $X^\circ$. For (3) and (4) the formal neighbourhoods $\widehat{X}^\circ_{F_x^\circ}$ and $\widehat{X}^\circ_{F_y^\circ}$ are defined analogously. Note that the definition of $X^\circ$ in (2)–(4) varies. Finally in (5) we take $W = X \setminus (B \cup F_x \cup F_y)$. Then

$$\mathfrak{U} = \{\widehat{X}_x \to X, \widehat{X}_y \to X, \widehat{X}_z \to X, \widehat{X}^\circ_{B^\circ} \to X, \widehat{X}^\circ_{F_x^\circ} \to X, \widehat{X}^\circ_{F_y^\circ} \to X, W \subset X\}$$

is an fpqc cover of $X$. Consequently the data of a quasi-coherent sheaf on $X$ is equivalent to the data of quasi-coherent sheaves on each of the opens of $\mathfrak{U}$ and gluing isomorphisms between the restrictions on the overlaps. Technically: quasi-coherent sheaves on $X$ form a stack with respect to the fpqc topology [7, Vistoli, Thm. 4.23].

5.2. **Local moduli spaces.** We now introduce moduli spaces of closed subschemes of dimension $\leq 1$ on the pieces of the cover $\mathfrak{U}$. Assume the coordinates on

$$\widehat{X}_x \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$$

are chosen such that $x_1 = x_3 = 0$ corresponds to the intersection $\widehat{X}_x \times_X B$ and $x_2 = x_3 = 0$ corresponds to $\widehat{X}_x \times_X F_x$. Define

$$\operatorname{Hilb}^{(1,d),n}(\widehat{X}_x) :=$$
$$\big\{ I_Z \subset \mathcal{O}_{\widehat{X}_x} \ : \ [Z] = [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

Here the equation

$$[Z] = [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x]$$

means $Z$ is supported along

$$(\widehat{X}_x \times_X B) \cup (\widehat{X}_x \times_X F_x)$$

with multiplicity 1 along $\widehat{X}_x \times_X B$ and multiplicity $d$ along $\widehat{X}_x \times_X F_x$ and $Z_{\mathrm{CM}}$ denotes the maximal Cohen-Macaulay subcurve of $Z$. The ideal sheaves fit into a short exact sequence

$$0 \longrightarrow I_Z \longrightarrow I_{Z_{\mathrm{CM}}} \longrightarrow Q \longrightarrow 0,$$

where $Q$ is 0-dimensional. The Hilbert scheme $\mathrm{Hilb}^{(1,d),n}(\widehat{X}_y)$ is defined likewise replacing the point $x$ by $y$. For $\widehat{X}_z$, we define

$$\mathrm{Hilb}^{d,n}(\widehat{X}_z) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}_z} \ : \ [Z] = d[\widehat{X}_z \times_X F_y] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

Each of $\widehat{X}_x, \widehat{X}_y, \widehat{X}_z$ has an action of $\mathbb{C}^*$ compatible with the fibre scaling on $X$. This action lifts to the moduli space. Since each of these formal neighbourhoods is isomorphic to $\operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$, the bigger torus $\mathbb{C}^{*3}$ acts on it and this action lifts to the moduli space. The existence of these "extra actions" will be used in Section 6.

Next consider $\widehat{X}^\circ_{B^\circ}$, i.e. the formal neighbourhood of the punctured zero section $B^\circ \subset X^\circ$. Define

$$\mathrm{Hilb}^{1,n}(\widehat{X}^\circ_{B^\circ}) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{B^\circ}} \ : \ [Z] = [\widehat{X}^\circ_{B^\circ} \times_X B] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

For $\widehat{X}^\circ_{F_x^\circ}$, $\widehat{X}^\circ_{F_y^\circ}$ we define

$$\mathrm{Hilb}^{d,n}(\widehat{X}^\circ_{F_x^\circ}) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{F_x^\circ}} \ : \ [Z] = d[\widehat{F}_x^\circ] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\},$$
$$\mathrm{Hilb}^{d,n}(\widehat{X}^\circ_{F_y^\circ}) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{F_y^\circ}} \ : \ [Z] = d[\widehat{F}_y^\circ] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

Finally for $W$ we define

$$\mathrm{Hilb}^{0,n}(W) := \big\{ I_Z \subset \mathcal{O}_W \ : \ \dim(Z) = 0 \text{ and } h^0(\mathcal{O}_Z) = n \big\}.$$

On $\widehat{X}^\circ_{B^\circ}$, $\widehat{X}^\circ_{F_x^\circ}$, $\widehat{X}^\circ_{F_y^\circ}$, and $W$ we have an action of $\mathbb{C}^*$ compatible with the fibre scaling on $X$. These actions lift to the moduli space. However, *unlike* for $\widehat{X}_x$, $\widehat{X}_y$, $\widehat{X}_z$, no additional tori act.

As before, we use the notation $\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ for the union of all $\mathrm{Hilb}^{(1,d),n}(\widehat{X}_x)$ and similarly for all other moduli spaces of this section. Like in Section 3, the components of the $\mathbb{C}^*$-fixed locus of $\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ are indexed by 2D partitions

$$\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)^{\mathbb{C}^*} = \bigsqcup_{\lambda \vdash d} \mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda.$$

**Proposition 9.** *Consider the stratum $\Sigma(x, y, \lambda, \mu)$, where $|\lambda| = a$ and $|\mu| = b$. Restriction from $X$ to the elements of the cover $\mathfrak{U}$ induces a morphism*

$$\begin{aligned} \Sigma(x,y,\lambda,\mu) \longrightarrow\ & \mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda \times \mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu \times \mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu \times \\ & \mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})^{\mathbb{C}^*} \times \mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F_x^\circ})^{\mathbb{C}^*}_\lambda \times \mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F_y^\circ})^{\mathbb{C}^*}_\mu \times \\ & \mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}, \end{aligned} \tag{5}$$

*which is a bijection on closed points.*

*Proof.* Since pull-back works in families, restriction indeed defines a morphism. For the rest of the proof, we work on closed points only.

Since $\mathfrak{U}$ is an fpqc cover, fpqc descent implies that any ideal sheaf $I_Z \subset \mathcal{O}_X$ is entirely determined by its restriction along the morphisms of the elements of $\mathfrak{U}$. This proves injectivity.

Conversely, given local ideal sheaves in the image of (5), their restrictions to overlaps only depend on the underlying Cohen-Macaulay curve and not on the embedded points. Since we chose the strata such that the underlying Cohen-Macaulay curve is already fixed, there are no further gluing conditions and fpqc descent implies surjectivity. $\square$

**Remark 10.** Note that the argument of Proposition 9 produces a bijective morphism — we do *not* claim (5) is an isomorphism of schemes. However, a bijective morphism induces an equality of (topological) Euler characteristic, which is what we use.

**Remark 11.** It is important to relate holomorphic Euler characteristic of domain and target in (5). For any subscheme $Z$ in the domain $\Sigma(x, y, \lambda, \mu)$, denote the corresponding maximal Cohen-Macaulay curve of its elements by $Z_{\mathrm{CM}}$ (Proposition 5). Then

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{Z_{\mathrm{CM}}}) + \chi(I_{Z_{\mathrm{CM}}}/I_Z).$$

Recall that $Z_{\mathrm{CM}}$ is entirely determined by the data $x, y, \lambda, \mu$, where $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$ and $\mu = (\mu_0 \geq \mu_1 \geq \cdots)$ are 2D partitions (equation (4)). An easy calculation shows

$$\chi(\mathcal{O}_{Z_{\mathrm{CM}}}) = \chi(\mathcal{O}_B) - \lambda_0 - \mu_0.$$

We conclude

$$\chi(\mathcal{O}_Z) = \frac{e(B)}{2} - \lambda_0 - \mu_0 + \chi(I_{Z_{\mathrm{CM}}}/I_Z). \tag{6}$$

Proposition 9 allows us to calculate

$$f_d(ax+by) = e(\rho_d^{-1}(ax+by)) = \sum_{\lambda \vdash a} \sum_{\mu \vdash b} e(\Sigma(x, y, \lambda, \mu)).$$

By Proposition 9 and (6)

$$\begin{aligned} f_d(ax+by) = p^{\frac{e(B)}{2}} & e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^{\circ}_{B^{\circ}})^{\mathbb{C}^*})\, e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) \times \\ & \sum_{\lambda \vdash a} \sum_{\mu \vdash b} p^{-\lambda_0 - \mu_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_{\lambda})\, e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_{\mu}) \times \\ & e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_{\mu})\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^{\circ}_{F^{\circ}_x})^{\mathbb{C}^*}_{\lambda})\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^{\circ}_{F^{\circ}_y})^{\mathbb{C}^*}_{\mu}). \end{aligned} \tag{7}$$

Before we proceed, we calculate $e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*})$ and $e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^{\circ}_{B^{\circ}})^{\mathbb{C}^*})$. The first follows from a formula of J. Cheah [6]

$$e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) = M(p)^{e(W)}. \tag{8}$$

For the second we use the following proposition:

**Proposition 12.** *Let $x_1, \ldots, x_l \in B$ be any number of distinct closed points. Define*

$$B^{\circ} := B \setminus \{x_1, \ldots, x_l\},$$
$$X^{\circ} := X \setminus \bigsqcup_{i=1}^{l} F_{x_i}.$$

*Let $\widehat{X}^{\circ}_{B^{\circ}}$ be the formal neighbourhood of $B^{\circ}$ in $X^{\circ}$. Define $\mathrm{Hilb}^{1,n}(\widehat{X}^{\circ}_{B^{\circ}})$ to be the Hilbert scheme of subschemes $Z \subset \widehat{X}^{\circ}_{B^{\circ}}$, such that $Z_{\mathrm{CM}} = B^{\circ}$ and $\chi(I_{Z_{\mathrm{CM}}}/I_Z) = n$. Then*

$$e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^{\circ}_{B^{\circ}})) = \left(\frac{M(p)}{1-p}\right)^{e(B^{\circ})}.$$

*Proof.* Pick any $y \in B^\circ$ and let $\widehat{X}_y \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$ be the formal neighbourhood of $y$ in $X^\circ$. Denote by

$$\mathrm{Hilb}^{1,n}(\widehat{X}^\circ_y)$$

the Hilbert scheme of subschemes $Z \subset \widehat{X}^\circ_y$, such that $Z_{\mathrm{CM}} = \{x_1 = x_3 = 0\}$ and $\chi(I_{Z_{\mathrm{CM}}}/I_Z) = n$.

We have projections

$$X^\circ \longrightarrow S^\circ \longrightarrow B^\circ.$$

These map induces a morphism

$$\mathrm{Hilb}^{1,n}(\widehat{X}^\circ_{B^\circ}) \longrightarrow \mathrm{Sym}^n(B^\circ).$$

The fibre over a point $\mathfrak{a} = \sum_i a_i y_i$, with all $y_i \in B^\circ$ distinct, equals

$$\prod_i \mathrm{Hilb}^{1,a_i}(\widehat{X}^\circ_{y_i}).$$

Since $B$ is reduced and smooth, $\mathrm{Hilb}^{1,a_i}(\widehat{X}^\circ_{y_i})$ only depends on $a_i$ and not on the point $y_i \in B^\circ$. Therefore Lemma 23 of Appendix A.2 implies

$$e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})) = \left( \sum_{a=0}^{\infty} e(\mathrm{Hilb}^{1,a}(\widehat{X}^\circ_y)) p^a \right)^{e(B^\circ)}.$$

The formal neighbourhood $\widehat{X}_y$ has an action of $\mathbb{C}^{*3}$ and this action lifts to $\mathrm{Hilb}^{1,a}(\widehat{X}_y)$. The fixed locus consists of a finite number of points counted by the topological vertex[5]

$$\mathsf{V}_{\square\varnothing\varnothing} = \frac{M(p)}{1-p}. \qquad \square$$

Using (7), (8), and Proposition 12 gives

$$\begin{aligned} f_d(ax+by) = {} & \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \times \\ & (1-p) \sum_{\lambda \vdash a} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})^{\mathbb{C}^*}_\lambda) \times \\ & \frac{1-p}{M(p)} \sum_{\mu \vdash b} p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})^{\mathbb{C}^*}_\mu). \end{aligned}$$

5.3. **Geometric characterization of $g(a)$ and $h(b)$.** The arguments of the preceding two sections are straightforwardly modified to any stratum $\Sigma(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$. Fix a smooth fibre $F_x$ and a singular fibre $F_y$. Denote the singular point of $F_y$ by $z$. Let $\widehat{X}_x$, $\widehat{X}_z$ be the formal neighbourhoods of $x$, $z$ in $X$. Define $\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)$, $\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)$ as in Section 5.2. Like in Section 5.2, we also consider the "tubular" formal neighbourhoods $\widehat{X}^\circ_{F^\circ_x}$, $\widehat{X}^\circ_{F^\circ_y}$ and corresponding Hilbert schemes $\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})$, $\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})$. A straightforward generalization of the calculation of $f_d(ax+by)$ yields:

[5]Discussed in general in Section 6.

**Proposition 13.** *For any $a, b > 0$ define*

$$g(a) := (1-p)\sum_{\lambda \vdash a} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})^{\mathbb{C}^*}_\lambda),$$

$$h(b) := \frac{1-p}{M(p)}\sum_{\mu \vdash b} p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})^{\mathbb{C}^*}_\mu),$$

*and let $g(0) := 1$, $h(0) := 1$. Then*

$$f_d(\mathfrak{a}+\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_i g(a_i) \cdot \prod_j h(b_j),$$

*for any $\mathfrak{a} = \sum_i a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$ and $\mathfrak{b} = \sum_j b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$, where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.*

We immediately deduce:

**Corollary 14.** *Propositions 7 and 8 are true for $g(a)$ and $h(b)$ defined in Proposition 13.*

## 6. Reduction to the topological vertex

In this section, we obtain (Theorem 18) expressions for $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$ in terms of the topological vertex $\mathsf{V}_{\lambda\mu\nu}(p)$, $e(B)$, and $N$ (the number of nodal fibres). The theorem follows by expressing $g(a)$ and $h(b)$ of Proposition 13 in terms of the topological vertex.

### 6.1. Point contributions.

Following the conventions of [4], we denote by

$$\mathsf{V}_{\lambda\mu\nu} = \sum_\pi p^{|\pi|},$$

the topological vertex. Here the sum is over all 3D partitions $\pi$ with outgoing legs $\lambda, \mu, \nu$ and $|\pi|$ denotes renormalized volume (see Definitions (1) and (2) in [4]). For a 2D partition $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$, we write $\lambda'$ for the corresponding transposed partition and

$$|\lambda| := \sum_{k=0}^{\infty} \lambda_k,$$

$$\|\lambda\|^2 := \sum_{k=0}^{\infty} \lambda_k^2.$$

**Proposition 15.** *Let $F_x$ be a smooth fibre and $F_y$ a singular fibre with singularity $z$. Then for any $\lambda \vdash a$, $\mu \vdash b$*

$$p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda) = \mathsf{V}_{\lambda\square\varnothing},$$

$$p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu) = \mathsf{V}_{\mu\square\varnothing},$$

$$p^{-\|\mu\|^2} e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu) = \mathsf{V}_{\mu\mu'\varnothing}.$$

*Proof.* Recall that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \mathrm{Spec}\, \mathbb{C}[\![x_1, x_2, x_3]\!].$$

Therefore, $\mathbb{C}^{*3}$ acts on each of these schemes and their moduli spaces

$$\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda,\ \mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu,\ \mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu.$$

The coordinates can be chosen such that the action of the last factor of $\mathbb{C}^{*3}$ corresponds to $x_3 \mapsto t_3 x_3$. This component acts trivially since we are already on the $\mathbb{C}^*$-fixed locus.

The $\mathbb{C}^{*3}$-fixed locus consists of isolated reduced points corresponding to monomial ideals with asymptotics $(\lambda, \varnothing, \varnothing)$, $(\mu, \varnothing, \varnothing)$, $(\mu, \mu', \varnothing)$ respectively[6]. These monomial ideals are exactly what the topological vertex counts.

Finally, note that the generating functions $e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)$, $e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)$, $e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)$ all start with 1. On the other hand, from the definition

$$\begin{aligned} \mathsf{V}_{\lambda\square\varnothing} &= p^{-\lambda_0} + \cdots, \\ \mathsf{V}_{\mu\square\varnothing} &= p^{-\mu_0} + \cdots, \\ \mathsf{V}_{\mu\mu'\varnothing} &= p^{-\sum_{k=0}^{\infty}\mu_k^2} + \cdots, \end{aligned}$$

where $\cdots$ stands for higher order terms in $p$. The proposition follows. □

6.2. **Fibre contribution.** Let $F_x$ be a smooth fibre and $F_y$ a singular fibre. Recall the formal neighbourhoods $\widehat{X}_{F_x^\circ}$, $\widehat{X}_{F_y^\circ}$ of Section 5.

**Proposition 16.** *For any $\lambda \vdash a$ and $\mu \vdash b$, we have*

$$\begin{aligned} e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ})^{\mathbb{C}^*}_\lambda) &= \frac{1}{\mathsf{V}_{\lambda\varnothing\varnothing}}, \\ e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ})^{\mathbb{C}^*}_\mu) &= \frac{1}{\mathsf{V}_{\mu\varnothing\varnothing}}. \end{aligned}$$

*Proof.* ******** □

6.3. **Putting it together.** Combining Proposition 13 with Propositions 15, 16 gives:

**Proposition 17.** *For any $a, b > 0$*

$$\begin{aligned} g(a) &= (1-p)\sum_{\lambda\vdash a} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}, \\ h(b) &= \frac{1-p}{M(p)} \sum_{\mu\vdash b} p^{\|\mu\|^2} \frac{\mathsf{V}_{\mu\square\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}} \mathsf{V}_{\mu\mu'\varnothing}. \end{aligned} \tag{9}$$

Putting all our results together, we obtain formulas for the partition functions in terms of the vertex:

**Theorem 18.**

$$\widehat{\mathsf{DT}}(X) = \frac{1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \left( (1-p)\sum_\lambda q^{|\lambda|} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \right)^{e(B)-N} \left( (1-p)\sum_\mu q^{|\mu|} p^{\|\mu\|^2} \frac{\mathsf{V}_{\mu\square\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}} \mathsf{V}_{\mu\mu'\varnothing} \right)^N$$

$$\widehat{\mathsf{DT}}_{\mathrm{fib}}(X) = \left( \sum_\lambda q^{|\lambda|} \right)^{e(B)-N} \left( \sum_\mu q^{|\mu|} p^{\|\mu\|^2} \mathsf{V}_{\mu\mu'\varnothing} \right)^N.$$

*Proof.* Inserting the equations for $g(a)$, $h(b)$ of Proposition 17 into (3) gives the formula for $\widehat{\mathsf{DT}}(X)$. Similar reasoning... □

Go back and put in the necessary stuff so that the $\widehat{\mathsf{DT}}_{\mathrm{fib}}$ computation isn't just hand waving.

**Corollary 19.** *Theorem 1 is true.*

[6] The transpose in $\mu'$ occurs, because we follow the orientation convention of [4].

*Proof.* We apply the main theorem of [4]. In particular, we substitute [4, Eqns (2)&(4)] into the formula for $\widehat{\mathsf{DT}}(X)$ and we substitute [4, Eqn (1)], as well as the well-known formula

$$\sum_{\lambda} q^{|\lambda|} = \prod_{d=1}^{\infty} \frac{1}{(1-q^d)}$$

into the formula for $\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$. □

## 7. PUTTING IN THE BEHREND FUNCTION

The aim of this section is to show that the partition functions $\widehat{\mathsf{DT}}(X)$ and $\mathsf{DT}(X)$ are equal after the simple change of variables $y = -p$. In order to do this we will need to assume a conjecture about the Behrend function which we formulate below for general Calabi-Yau threefolds.

Let $Y$ be any quasi-projective Calabi-Yau threefold. Let $C \subset Y$ be a (not necessarily reduced) Cohen-Macaulay curve with proper support. Assume that the singularities of $C_{\mathrm{red}}$ are locally toric[7]. Define

$$\mathrm{Hilb}^{C,n}(Y) = \{Z \subset Y \textit{ such that } C \subset Z \textit{ and } I_C/I_Z \textit{ has finite length } n\}.$$

Note that $\mathrm{Hilb}^{C,n}(Y) \subset \mathrm{Hilb}(Y)$ and let $\nu$ denote the Behrend function on $\mathrm{Hilb}(Y)$. Our conjecture is the following:

**Conjecture 20.**

$$\int_{\mathrm{Hilb}^{C,n}(Y)} \nu \, de = (-1)^n \nu([C]) \int_{\mathrm{Hilb}^{C,n}(Y)} de$$

*where $\nu([C])$ is the value of the Behrend function at the point $[C] \in \mathrm{Hilb}(Y)$.*

**Remark 21.** Conceivably, the condition that $C_{\mathrm{red}}$ has locally toric singularities could be weakened, although we do not have any evidence for this case. Our conjecture is true for $Y$ a (globally) toric Calabi-Yau. This follows from the computations in [10].

One could also make the much stronger conjecture that

$$\nu([Z]) = (-1)^n \nu([C])$$

for all $[Z] \in \mathrm{Hilb}^{C,n}(Y)$. This would of course imply our conjecture as stated. However, we do not know if this stronger version holds, even in the case where $Y$ is $\mathbb{C}^3$ and $C$ is empty. In this case, this stronger conjecture says that the Behrend function on $\mathrm{Hilb}^n(\mathbb{C}^3)$ is the constant function $(-1)^n$.

## APPENDIX A. ODDS AND ENDS

A.1. **Curves on elliptic surfaces.** Let $p : S \to B$ be an elliptic surface with section $B \subset S$. In this appendix we allow any type of singular fibres. We assume $S$ is not a product, which implies

$$p^* : \mathrm{Pic}^0(B) \xrightarrow{\cong} \mathrm{Pic}^0(S)$$

is an isomorphism [11, VII.1.1]. For any $\beta \in H_2(S)$, we denote by $\mathrm{Hilb}^{\beta}(S)$ the Hilbert scheme of effective divisors on $S$ in class $\beta$.

---

[7]This means that formally locally $C_{\mathrm{red}}$ is either smooth, nodal, or the union of the three coordinate axes. That is at $p \in C_{\mathrm{red}} \subset Y$ the ideal $\widehat{I}_{C_{\mathrm{red}}} \subset \widehat{\mathcal{O}}_{Y,p}$ is given by $(x_1, x_2)$, $(x_1, x_2x_3)$, or $(x_1x_2, x_2x_3, x_1x_3)$ for some isomorphism $\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[[x_1, x_2, x_3]]$.

Denote by $B \in H_2(S)$ the class of the section $B \subset S$ and by $F \in H_2(S)$ the class of the fibre. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sym}^d(B) \arrow[r] \arrow[d, "p^*"] & \mathrm{Pic}^d(B) \arrow[d, "p^*", "\cong"'] \\
\mathrm{Hilb}^{dF}(S) \arrow[r] \arrow[d, "+B"] & \mathrm{Pic}^{dF}(S) \arrow[d, "\otimes \mathcal{O}_S(B)", "\cong"'] \\
\mathrm{Hilb}^{B+dF}(S) \arrow[r] & \mathrm{Pic}^{B+dF}(S).
\end{tikzcd}$$

The horizontal arrows are Abel-Jacobi maps. The vertical arrows are induced by pull-back and adding the section $B \subset S$.

**Lemma 22.** *The above maps induce an isomorphism*

$$\mathrm{Sym}^d(B) \xrightarrow{\cong} \mathrm{Hilb}^{B+dF}(S).$$

*Proof.* Clearly $p^*$ gives an isomorphism $\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{dF}(S)$ and $+B$ defines a closed embedding $\mathrm{Hilb}^{dF}(S) \hookrightarrow \mathrm{Hilb}^{B+dF}(S)$. Since $\mathrm{Sym}^d(B)$ is smooth and $\mathrm{Hilb}^{B+dF}(S)$ is reduced (by [12, Lect. 25]), it suffices to show

$$\mathrm{Sym}^d(B) \to \mathrm{Hilb}^{B+dF}(S)$$

is surjective on closed points.

For surjectivity, suppose $D'$ is an effective divisor with class $B + dF$ which does *not* lie in the image. Firstly, we note that for any fibre $F$ we have $D' \cdot F = 1$. Therefore $D'$ contains a section $B' \subset S$ as an effective summand. Moreover $B \neq B'$ or else $D'$ would lie in the image. Next, we take any $D$ in the image and compare $D$ and $D'$. Then

$$\mathcal{O}_S(D - D') \in \mathrm{Pic}^0(S) \cong \mathrm{Pic}^0(B).$$

Therefore after re-arranging we find that there are distinct fibres $F_{x_i}$, $F_{y_j}$ and $a_i \geq 0$, $b_j \geq 0$ such that

$$B + \sum_i a_i F_{x_i} \sim_{\mathrm{lin}} B' + \sum_j b_j F_{y_j},$$

where $\sim_{\mathrm{lin}}$ denotes linear equivalence. Hence there exists a pencil $\{C_t\}_{t \in \mathbb{P}^1}$ of effective divisors such that

$$C_0 = B + \sum_i a_i F_{x_i},\ C_\infty = B' + \sum_j b_j F_{y_j}.$$

Now fix a smooth fibre $F$. Then $C_t \cdot F = 1$ for any $t \in \mathbb{P}^1$, so we get a morphism

$$\mathbb{P}^1 \longrightarrow F,\ t \mapsto C_t \cap F.$$

But $F$ is a smooth elliptic curve so this map is constant. We conclude

$$B \cap F = C_0 \cap F = C_\infty \cap F = B' \cap F.$$

Since $F$ was chosen arbitrary, we deduce that $B = B'$ which is a contradiction. $\square$

A.2. **Weighted Euler characteristics of symmetric products.** In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

**Lemma 23.** *Let $B$ be a scheme of finite type over $\mathbb{C}$ and let $e(B)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$ be any function with $g(0) = 1$. Let $f_d : \operatorname{Sym}^d(B) \to \mathbb{Z}((p))$ be the constructible function defined by*

$$f_d(\mathfrak{a}) = \prod_i g(a_i),$$

*for all $\mathfrak{a} = \sum_i a_i x_i \in \operatorname{Sym}^d(B)$ where $x_i \in B$ are distinct closed points. Then*

$$\sum_{d=0}^{\infty} q^d \int_{\operatorname{Sym}^d(B)} f_d \, de = \left( \sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)}.$$

**Remark 24.** In the special case where $g = f_d \equiv 1$, the lemma recovers MacDonald's formula:

$$\sum_{d=0}^{\infty} e(\operatorname{Sym}^d(B))\, q^d = \frac{1}{(1-q)^{e(B)}}.$$

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties definited by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure []. For convenience's sake, we provide a direct proof here.

Ref? Bryan-Young?

*Proof.* The $d$th symmetric product admits a stratification with strata labelled by partitions of $d$. Associated to any partition of $d$ is a unique tuple $(m_1, m_2, \dots)$ of non-negative integers with $\sum_{j=1}^{\infty} j m_j = d$. The stratum labelled by $(m_1, m_2, \dots)$ parameterizes collections of points where there are $m_j$ points of multiplicity $j$. The full stratification is given by:

$$\operatorname{Sym}^d(B) = \bigsqcup_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left\{ \left( \prod_{j=1}^{\infty} B^{m_j} \right) - \Delta \right\} \Big/ \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, $B^0$ is a point, $\Delta$ is the large diagonal, and $\sigma_m$ is the $m$th symmetric group. Note that the function $f_d$ is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

For schemes over $\mathbb{C}$, topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\operatorname{Sym}^d(B)} f_d \; de = \sum_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left( \prod_{j=1}^{\infty} g(j)^{m_j} \right) \frac{e(B^{\sum_j m_j} - \Delta)}{m_1! \, m_2! \, m_3! \dots}.$$

For any natural number $N$, the projection $B^N - \Delta \to B^{N-1} - \Delta$ has fibers of the form $B - \{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(B) - (N-1)$ and consequently, $e(B^N - \Delta) = (e(B) - (N-1)) \cdot e(B^{N-1} - \Delta)$. Thus by induction, we find $e(B^N - \Delta) = e(B) \cdot (e(B) - 1) \cdots (e(B) - (N-1))$ and so

$$\frac{e(B^{\sum_j m_j} - \Delta)}{m_1! \, m_2! \, m_3! \cdots} = \binom{e(B)}{m_1, m_2, m_3, \cdots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\begin{aligned}\sum_{d=0}^{\infty} q^d \int_{\mathrm{Sym}^d(B)} f_d \, de &= \sum_{(m_1,m_2,\dots)} \prod_{j=1}^{\infty} \left(g(j)q^j\right)^{m_j} \binom{e(B)}{m_1, m_2, m_3, \dots} \\ &= \left(1 + \sum_{j=1}^{\infty} g(j)q^j\right)^{e(B)}\end{aligned}$$

which proves the lemma. □

## Appendix B. Smoothness at FoLoMo Cohen-Macaulay curves

In this section we denote by $p : X \to Y$ an elliptically fibred threefold with section $\sigma : Y \hookrightarrow X$, where $X$ and $Y$ are smooth (not necessarily projective). Let $B \subset Y$ be a smooth projective curve, which we can view as a curve in $X$ via the section. We denote its homology class on $X$ by $B$ as well. Let

$$\beta := B + dF \in H_2(X),$$

where $F$ denotes the class of the fibre and $d \geq 0$. Let $0 \leq \ell \leq d$ and define

$$n := 1 - g_B - \ell, \tag{10}$$

where $g_B$ is the arithmetic genus of $B$. As usual, we denote by $\mathrm{Hilb}^{\beta,n}(X)$ the Hilbert scheme of closed subschemes $C \subset X$ with $[C] = \beta$ and $\chi(\mathcal{O}_C) = n$.

We now present a way of producing certain Cohen-Macaulay curves in $\mathrm{Hilb}^{\beta,n}(X)$. Denote by $\mathrm{Hilb}^d(Y)$ the Hilbert scheme of 0-dimensional length $d$ subschemes of $Y$ and consider the stratum

$$\Sigma_\ell := \left\{ Z \in \mathrm{Hilb}^d(Y) \ : \ \ell(Z \cap B) = \ell \right\},$$

where $\ell(Z \cap B)$ denotes the length of the scheme theoretic intersection $Z \cap B$. We also use $\mathrm{Hilb}^B(Y)$ —the Hilbert scheme of effective divisors on $Y$ with class $B$. Consider the morphism

$$\begin{aligned}\Sigma_\ell \times \mathrm{Hilb}^B(Y) &\to \mathrm{Hilb}^{\beta,n}(X) \\ (Z, B') &\mapsto B' \cup p^*Z,\end{aligned} \tag{11}$$

where $B' \cup p^*Z$ denotes the scheme theoretic union. We refer to curves of this form as *untwisted Cohen-Macaulay curves*. For any such $C = B' \cup p^*Z$ we have a short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{B'} \oplus \mathcal{O}_{p^*Z} \to \mathcal{O}_{B' \cap p^*Z} \to 0,$$

where $\ell(B \cap p^*Z) = \ell$. From the short exact sequence we deduce

$$\begin{aligned}\chi(\mathcal{O}_C) &= \chi(\mathcal{O}_{B'}) + \chi(\mathcal{O}_{p^*Z}) - \ell \\ &= 1 - g_B + \chi(\mathcal{O}_Z \otimes Rp_*\mathcal{O}_X) - \ell \\ &= 1 - g_B - \ell \\ &= n,\end{aligned}$$

where we use that $Rp_*\mathcal{O}_X$ is a complex of rank 0 because $X$ is elliptically fibred and $n$ is given by (10). Our starting point is the following smoothness result.

**Proposition 25.** $\Sigma_\ell \subset \mathrm{Hilb}^d(Y)$ *is locally closed and smooth of dimension* $2d - \ell$.

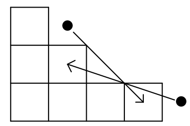

FIGURE 2. The partition $\lambda = (3,2,1,1)$ corresponds to the ideal $I_Z = (y^3, xy^2, x^2y, x^4)$. The two Haiman arrows correspond to the box $(1,0)$.

*Proof.* The constructible function

$$\mathrm{Hilb}^d(Y) \to \mathcal{Z},\ Z \mapsto \ell(Z \cap B)$$

is upper semicontinuous, which shows $\Sigma_\ell$ is locally closed .

More detail? The flattening stratification for $\mathcal{O}_{\mathcal{Z}\cap(\mathrm{Hilb}^d(Y)\times Y)}$ on $\mathrm{Hilb}^d(Y)\times Y \to \mathrm{Hilb}^d(Y)$ exactly gives the desired stratification b/c restrictions to fibres are 0-dimensional.

The stratum $\Sigma_\ell$ has an open subset consisting of configurations of $d$ distinct points, $\ell$ of which lie on $B$. This locus is smooth of dimension $2d - \ell$. We now show that for any $Z \in \Sigma_\ell$ we have

$$\hom_Y(I_Z, \mathcal{O}_Z) = 2d - \ell. \tag{12}$$

Since $Z$ is 0-dimensional, it suffices to prove this for the case $Y = \operatorname{Spec} \mathbb{C}[\![x,y]\!]$. Then $Y$ has the standard $\mathbb{C}^{*2}$-action, which also acts on $\Sigma_\ell$. Any element $I = I_1 \in \mathrm{Hilb}^d(Y)$ moves in a flat family $I_t$ to its initial ideal $\mathrm{in}(I) = I_0 \in \mathrm{Hilb}^d(Y)$, which is a monomial ideal (w.r.t. some chosen order). This process is called Groebner degeneration [**?**, Sect. 18.2]. For the standard lexicographic order $x > y$, it is easy to see that if $I_1 \in \Sigma_\ell$, then $I_t \in \Sigma_\ell$ for all $t$. In particular $I_0 \in \Sigma_\ell$. Therefore any $\mathbb{C}^{*2}$ orbit in $\Sigma_\ell$ contains a monomial ideal in its closure. Consequently we only have to prove (12) at $\mathbb{C}^{*2}$ fixed points.

Suppose $I_Z \in \mathrm{Hilb}^d(Y)$ is a $\mathbb{C}^{*2}$ fixed point, i.e. $I_Z$ is a monomial ideal. There is a pictorial way of describing a basis of $2d$ element of the vector space $\mathrm{Hom}_Y(I_Z, \mathcal{O}_Z)$ due to M. Haiman [**?**, Sect. 18.2], which we now describe. Let $\lambda \subset \mathbb{Z}^2$ be the partition corresponding to $I_Z$ defined by the requirement that $(\alpha,\beta) \in \lambda$ if and only if $x^\alpha y^\beta \notin I_Z$. In particular, we write $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$ where $(\alpha,\beta) \notin I_Z$ if and only if $0 \leq \beta \leq \lambda_\alpha - 1$. We denote the transpose of $\lambda$ by $\lambda'$. Let $(\alpha,\beta) \in \lambda$ be a box in $\lambda$. To it we associate two arrows, which we refer to as *Haiman arrows*: one with tail $(\alpha, \lambda_\alpha)$ and head $(\lambda'_\beta - 1, \beta)$ and one with tail $(\lambda'_\beta, \beta)$ and head $(\alpha, \lambda_\alpha - 1)$. For an example, see Figure B. Two arrows are identified if one translates to the other whilst keeping the tail outside $\lambda$ and the head inside $\lambda$. To each Haiman arrow we assign an element of $\mathrm{Hom}(I_Z, \mathcal{O}_Z)$ as follows. Suppose we translate the arrow to the unique position for which the tail emanates from a homogeneous generator $g$ of $I_Z$. The corresponding morphism $I_Z \to \mathcal{O}_Z$ is defined by sending $g$ to the element of $\mathcal{O}_Z$ corresponding to the head of the arrow and all other homogenous generators to zero. This describes a basis of $2d$ elements for $\mathrm{Hom}_Y(I_Z, \mathcal{O}_Z)$.

Is this originally due to Haiman? Is it ok if we call them Haiman arrows?

Let $D := \mathbb{C}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers. We have to figure out which Haiman arrows correspond to first order deformations that move $Z$ out of $\Sigma_\ell$. A morphism $\phi \in \mathrm{Hom}_Y(I_Z, \mathcal{O}_Z)$ corresponds to the following first order deformation

$$I_{Z_\phi} = \{f + \epsilon \cdot g \ : \ f \in I_Z,\ g \in \mathbb{C}[\![x,y]\!] \text{ and } \phi(f) = [g] \in \mathcal{O}_Z\} \subset \mathbb{C}[\![x,y]\!] \otimes_{\mathbb{C}} D.$$

Note that each $Z_\phi \subset Y \times D$ comes from a (not necessarily unique) global deformation $\widetilde{Z}_\phi \subset Y \times \mathbb{C}$. The scheme theoretic intersection of $\widetilde{Z}_\phi$ with $B \times \mathbb{C}$ and $B \times \mathbb{C}^*$ are given by the following ideals

$$(I_{\widetilde{Z}_\phi} + (y))/(y) \subset \mathbb{C}[\![x,y,\epsilon]\!]/(y),\ (I_{\widetilde{Z}_\phi} + (y))/(y) \subset \mathbb{C}[\![x,y,\epsilon,\epsilon^{-1}]\!]/(y).$$

Using the explicit description of Haiman arrows given above, we see that the family over $B \times \mathbb{C}^*$ lies outside $\Sigma_\ell$ precisely when the tail of the arrow is located at $(\alpha, \lambda_\alpha)$ with $0 \le \alpha \le \ell(\lambda) - 1$ and the head of the arrow is located at $(\ell(\lambda) - 1, 0)$. Since there are precisely $\ell$ such arrows we obtain (12).

As an illustration of this principle we give the two deformations corresponding to the Haiman arrows of Figure B

$$I_{Z_{\phi_1}} = (y^3, xy^2 + \epsilon x^3, x^2y, x^4), I_{Z_{\phi_2}} = (y^3, xy^2, x^2y, x^4 + \epsilon xy).$$

The first deformation leads to scheme theoretic intersection $(\epsilon x^3, x^4)$ with $B$ while the second to scheme theoretic intersection $(x^4)$ with $B$. Therefore $\phi_1$ is normal to $\Sigma_4$, whereas $\phi_2$ is tangent to $\Sigma_4$. □

Among all untwisted CM curves on $X$, we identify a special class of curves:

**Definition 26.** We say that an element $Z \in \Sigma_\ell \subset \mathrm{Hilb}^d(Y)$ is *formally locally monomial with respect to* $B$, if at each reduced point $P \in \mathrm{Supp}(Z)$ we can choose coordinates $x, y$ on the formal neighborhood

$$\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[\![x, y]\!]$$

such that

- $I_{B\subset Y} \subset \widehat{\mathcal{O}}_{Y,P}$ is the ideal $(y)$,
- $I_Z \subset \widehat{\mathcal{O}}_{Y,P}$ is a monomial ideal in $x, y$.

We call an untwisted CM curve $B \cup p^*Z$ *formally locally monomial* (FoLoMo in short) when $Z$ is formally locally monomial with respect to $B$.

The main result of this appendix is the following:

**Theorem 27.** *Let $p : X \to Y$ be an elliptically fibred threefold with section. Let $B \subset Y$ be a smooth projective curve, $\beta = B + dF$, and $n = 1 - g_B - \ell$ for some $0 \le \ell \le d$. Then the Zariski tangent space at FoLoMo CM curves in $\mathrm{Hilb}^{\beta,n}(X)$ has dimension*

$$2d - \ell + h^0(N_{B/X}).$$

Before giving the proof, we discuss our main application.

**Corollary 28.** *Let the setup be as in Theorem 27. Assume in addition $B \in \mathrm{Hilb}^B(Y)$ is a smooth point and $h^0(N_{B/X}) = h^0(N_{B/Y})$. Then $\mathrm{Hilb}^{\beta,n}(X)$ is smooth of dimension $2d - \ell + h^0(N_{B/Y})$ at FoLoMo CM curves.*

*Proof.* Consider the morphism (11)

$$\Sigma_\ell \times \mathrm{Hilb}^B(Y) \to \mathrm{Hilb}^{\beta,n}(X).$$

By Proposition 25 and the assumption on $B$, the domain is smooth in a neighbourhood of any point of the form $(Z, B)$. The Zariski tangent space at such points has dimension $2d - \ell + h^0(N_{B/Y})$ again by Proposition 25. If in addition $Z$ is FoLoMo with respect to $B$, then $\mathrm{Hilb}^{\beta,n}(X)$ has Zariski tangent space of the same dimension at $C = B \cup p^*Z$ by Theorem 27 and the assumption $h^0(N_{B/X}) = h^0(N_{B/Y})$. Hence our map is a local isomorphism at all points $(Z, B)$ with $Z$ FoLoMo with respect to $B$. □

**Remark 29.** The assumption $h^0(N_{B/X}) = h^0(N_{B/Y})$ is satisfied for $X = \mathrm{Tot}(K_S)$, where $S$ is any elliptically fibered surface considered in this paper; i.e. with section $B$ and $N$ rational nodal fibres. Define $Y = \mathrm{Tot}(K_S|_B)$, then we have a morphism $p : X \to Y$

with section. Moreover $N_{B/S}$ is the dual of the fundamental line bundle which has degree $-\chi(\mathcal{O}_S) = -N/12 < 0$. Therefore

$$H^0(N_{B/X}) = H^0(N_{B/S}) \oplus H^0(N_{B/Y}) = H^0(N_{B/Y}).$$

In Section **??** we actually need the value of $h^0(N_{B/Y})$ which we now compute. Since $X$ is Calabi-Yau, $N_{B/Y} \cong N^*_{B/S} \otimes K_B$. By Riemann-Roch $\chi(N_{B/S}) = 1 - g_B - \chi(\mathcal{O}_S)$ and Serre duality implies

$$h^0(N_{B/Y}) = \chi(\mathcal{O}_S) + g_B - 1.$$

*Proof of Theorem 27.* Let $Z \in Y^{[n]}$ be a FoLoMo subscheme and consider $C = B \cup p^*Z$.

**Step 1:** Let $U_a$ be an affine open cover of $Y$ such that each $U_a$ contains at most one point of $\mathrm{Supp}(Z)$. Let $Z_a := Z \cap U_a$, $B_a := B \cap U_a$, $d_a = \ell(Z_a)$, $\ell_a = \ell(Z_a \cap B_a)$, $X_a := p^*U_a$, and $C_a := C \cap X_a$. We will prove

$$\mathrm{Hom}_{X_a}(I_{C_a}, \mathcal{O}_{C_a}) \cong \mathbb{C}^{2d_a - \ell_a} \oplus \Gamma(U_a, N_{B_a/X_a}). \tag{13}$$

Moreover we will see that the elements of $\mathbb{C}^{2d_a-\ell_a}$ restrict to 0 on overlaps $U_a \cap U_b$. Therefore, gluing of morphisms proves the theorem.

**Step 2:** We are reduced to work over the opens $X_a \to U_a$. In the case $\mathrm{Supp}(Z_a)$ is empty (13) is clear, so we assume $\mathrm{Supp}(Z_a)$ is non-empty, in which case it contains a single point which we denote by $P$. For notational convenience we drop all subscripts $a$, remembering that $X$ means $X_a$, $C$ means $C_a$, $U$ means $U_a$ etc.

We construct a Koszul resolution of $C = B \cup p^*Z$. Let $S := p^*B$, which is an elliptic surface over $B$. We need a divisor $M \subset X$ which intersects both $S$ and $U$ transversally at $P \in U \subset X$. Since we work locally on the base, this can be achieved by taking a divisor $\Delta$ transverse to $B \subset U$ at $P$ (obtained after possibly shrinking $U$) and pulling it back to $X$. Since $Z$ if FoLoMo with respect to $B$, we have formal coordinates $x, y$ at $P \in U$ such that formally locally

$$I_Z = (x^{\alpha_1}y^{\beta_1}, \ldots, x^{\alpha_r}y^{\beta_r}, x^\ell),$$

for some minimal set of monomial generators ordered by $\alpha_1 > \cdots > \alpha_r \geq \ell$. The standard minimal Koszul resolution of $I_Z$ can be slightly modified to give a Koszul resolution of $I_C$ as follows

$$\begin{aligned}
&0 \longrightarrow R^* \xrightarrow{M} G^* \xrightarrow{N} I_C \longrightarrow 0, \\
&G := \bigoplus_{i=1}^{r} \mathcal{O}_X(\alpha_i M + \beta_i S) \oplus \mathcal{O}_X(\ell M + U), \\
&R := \bigoplus_{i=1}^{r-1} \mathcal{O}_X(\alpha_{i+1} M + \beta_i S) \oplus \mathcal{O}_X(\ell M + S + U),
\end{aligned} \tag{14}$$

where, in matrix notation, we have

$$N := (x^{\alpha_1}y^{\beta_1}\ \ldots\ x^{\alpha_r}y^{\beta_r}\ x^{\ell}z),$$

$$M := \begin{pmatrix} x^{\alpha_2-\alpha_1} & 0 & & 0 & 0 \\ y^{\beta_1-\beta_2} & x^{\alpha_3-\alpha_2} & & 0 & 0 \\ 0 & y^{\beta_2-\beta_1} & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & x^{\alpha_r-\alpha_{r-1}} & 0 \\ 0 & 0 & & y^{\beta_{r-1}-\beta_r} & x^{\ell-x_r}z \\ 0 & 0 & & 0 & y \end{pmatrix}.$$

Here $z$ is the local coordinate at $P \in U \subset X$ defining the section $U \subset X$.

The second exact sequence we use is

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_B \oplus \mathcal{O}_{p^*Z} \longrightarrow \mathcal{O}_{B\cap p^*Z} \longrightarrow 0.$$

We conclude that $\mathrm{Hom}_X(I_C, \mathcal{O}_C)$ equals the kernel of

$$\Phi \oplus \Psi : \Gamma(G|_B) \oplus \Gamma(G|_{p^*Z}) \longrightarrow \Gamma(R|_B) \oplus \Gamma(R|_{p^*Z}) \oplus \Gamma(G|_{B\cap p^*Z}),$$

where the $\Phi = M|_B^t \oplus M|_{p^*Z}^t$ and $\Psi$ is "difference restricted to $B \cap p^*Z$".

**Step 3:** We first calculate the kernel of

$$\Gamma(G|_B) \oplus \Gamma(G|_{p^*Z}) \xrightarrow{\Phi} \Gamma(R|_B) \oplus \Gamma(R|_{p^*Z}).$$

The matrix $M|_B^t$ is obtained by putting $y = z = 0$, which are local equations defining $B \subset U \subset X$ at $P$. From the explicit form of the matrix we deduce at once that the kernel of $\Phi|_{\Gamma(G|_B)}$ equals

$$\Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(\mathcal{O}_B(\ell M + U)).$$

The kernel of $\Phi|_{\Gamma(p^*\mathcal{O}_Z\otimes G)}$ is more complicated. Recall that $S = p^*B$ and $M = p^*\Delta$, where $B, \Delta \subset U$ intersect transversally at $P$. We want to compute the kernel of

$$\begin{array}{ccc} \begin{array}{c}\bigoplus_{i=1}^{r}\Gamma(X,p^*\mathcal{O}_Z(\alpha_i M+\beta_i S)) \\ \oplus\Gamma(X,p^*\mathcal{O}_Z(\ell M+U))\end{array} & \longrightarrow & \begin{array}{c}\bigoplus_{i=1}^{r-1}\Gamma(X,p^*\mathcal{O}_Z(\alpha_{i+1} M+\beta_i S)) \\ \oplus\Gamma(X,p^*\mathcal{O}_Z(\ell M+S+U))\end{array} \\ \wr\| & & \wr\| \\ \begin{array}{c}\bigoplus_{i=1}^{r}\Gamma(U,\mathcal{O}_Z(\alpha_i \Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta)\otimes p_*\mathcal{O}_X(U))\end{array} & & \begin{array}{c}\bigoplus_{i=1}^{r-1}\Gamma(U,\mathcal{O}_Z(\alpha_{i+1} \Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta+B)\otimes p_*\mathcal{O}_X(U))\end{array} \end{array}, \tag{15}$$

where

$$p_*\mathcal{O}_X(U) \cong p_*\mathcal{O}_X = \mathcal{O}_U,$$

because $X$ is elliptically fibred.

Before computing the kernel of (15), we consider a slightly easier problem. If we remove the divisor $U$ from $G$ and $R$ and $z$ from the matrices $M$ and $N$ in (14), then we get the Koszul resolution of $I_{p^*Z}$. In this case the analog of (15) is

$$\begin{array}{c}\bigoplus_{i=1}^{r}\Gamma(U,\mathcal{O}_Z(\alpha_i \Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta))\end{array} \longrightarrow \begin{array}{c}\bigoplus_{i=1}^{r-1}\Gamma(U,\mathcal{O}_Z(\alpha_{i+1} \Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta+B))\end{array}, \tag{16}$$

which has kernel $\mathrm{Hom}_U(I_Z, \mathcal{O}_Z)$. Therefore a basis for the kernel of (16) is formed by the $2d$ Haiman arrows described in the proof of Proposition 25. The Haimain arrows correspond to morphisms

$$\mathcal{O}_U(-\alpha_i\Delta - \beta_i B) \to \mathcal{O}_Z \tag{17}$$

$$\mathcal{O}_U(-\ell\Delta) \to \mathcal{O}_Z, \tag{18}$$

where, in a neighborhood of the point $P \in U$, the generator maps to the monomial indicated by the head of the arrow. The maps on $X$ are obtained by applying $p^*$ to these morphism.

Since $p_*\mathcal{O}_X(U) \cong \mathcal{O}_U$, the kernel of (15) has *the same* basis of Haiman arrows with one minor (but crucial) modification. Under the vertical isomorphisms of (15), Haiman arrows of the form (17) map to

$$\mathcal{O}_X(-\alpha_i M - \beta_i S) = p^*\mathcal{O}_U(-\alpha_i \Delta - \beta_i B) \to p^*\mathcal{O}_Z$$

as before. However Haiman arrows of the form (18) map to

$$\mathcal{O}_X(-\ell M - U) = p^*\mathcal{O}_U(-\ell\Delta) \otimes \mathcal{O}_X(-U) \to p^*\mathcal{O}_Z \otimes \mathcal{O}_X(-U) \hookrightarrow p^*\mathcal{O}_Z. \tag{19}$$

Due to the extra $\mathcal{O}_X(-U) \hookrightarrow \mathcal{O}_X$, these latter Haiman arrows all restrict to zero on $B \cap p^*Z$.

In conclusion

$$\ker\Phi = \Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(\mathcal{O}_B(\ell M + U)) \oplus \mathbb{C}^{2d-\ell} \oplus z \cdot \mathbb{C}^\ell, \tag{20}$$

where $z \cdot \mathbb{C}^\ell$ is spanned by Haiman arrows of the form (19).

**Step 4:** Finally, we study the kernel of $\Psi$ restricted to the kernel of $\Phi$ (20). Among the Haiman arrows spanning $\mathbb{C}^{2d-\ell}$ in (20), there is a subspace $\mathbb{C}^{2d-2\ell}$ spanned by arrows which map to zero on restriction to $B \cap p^*Z$. These are the arrows corresponding to the boxes $(\alpha, \beta) \in \lambda$ with $\beta > 0$. As already observed in Step 3, the entire subspace $z \cdot \mathbb{C}^\ell$ restricts to zero on $B \cap p^*Z$. This leaves us with a subspace

$$\mathbb{C}^\ell \subset \mathbb{C}^{2n-\ell}$$

spanned by Haiman arrows which may not restrict to zero on $B \cap p^*Z$. These are the arrows with tail in $(\alpha, \lambda_\alpha)$ with $0 \leq \alpha \leq \ell - 1$ and head in $(\ell - 1, 0)$.

The space $\mathbb{C}^\ell$ discussed above splits up further as

$$\mathbb{C}^\ell = \mathbb{C}^{\alpha_r} \oplus \mathbb{C}^{\ell - \alpha_r},$$

where the first component is spanned by arrows with tail in $(\alpha, \lambda_\alpha)$ for $0 \leq \alpha \leq \alpha_r - 1$ and the second by arrows with tail in $(\alpha, \lambda_\alpha)$ for $\alpha_r \leq \alpha \leq \ell - 1$ . In Step 3 we proved

$$\Gamma(\mathcal{O}_B(\alpha_{i+1} M + \beta_i S)) \cap \ker\Phi = \varnothing,$$

for all $i = 1, \ldots, r-1$. We deduce at once that $\mathbb{C}^{\alpha_r} \cap \ker\Psi = 0$.

So far the kernel consists of $\mathbb{C}^{2d-\ell}$ to which we have to add the kernels of the following two maps

$$\Gamma(\mathcal{O}_B(\ell M + U)) \oplus \Gamma(p^*\mathcal{O}_Z(\ell M + U)) \cap \ker\Phi \to \Gamma(\mathcal{O}_{B\cap p^*Z}(\ell M + U)), \tag{21}$$

$$\Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(p^*\mathcal{O}_Z(\alpha_r S + U)) \cap \ker\Phi \to \Gamma(\mathcal{O}_{B\cap p^*Z}(\alpha_r S + U)). \tag{22}$$

The kernel of (21) is easy because

$$\Gamma(\mathcal{O}_B(\ell M + U)) \oplus \Gamma(p^*\mathcal{O}_Z(\ell M + U)) \cap \ker\Phi = \Gamma(\mathcal{O}_B(\ell M + U)) \oplus z \cdot \mathbb{C}^\ell.$$

and $z \cdot \mathbb{C}^\ell$ restricts to zero on $Z \cap p^*Z$. Therefore the kernel of (21) is the kernel of

$$\Gamma(\mathcal{O}_B(\ell M + U)) \cong \Gamma(N_{B/S}(\ell M)) \to \Gamma(N_{B/S}(\ell M)|_{\ell P}),$$

where $\ell P$ denotes the $\ell$ times thickening of $P$ in $B$. This gives $\Gamma(N_{B/S})$.

Finally we calculate the kernel of (22), which we claim is $\Gamma(N_{B/Y})$. Note that

$$\Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(p^*\mathcal{O}_Z(\alpha_r M + S)) \cap \ker\Phi = \Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \mathbb{C}^{\ell - \alpha_r},$$

where $\mathbb{C}^{\ell-\alpha_r}$ was introduced above. In the case $\alpha_r = \ell$ the same reasoning as above gives kernel $\Gamma(N_{B/U})$ and we are done. Now the case $\alpha_r < \ell$. For any $i \geq 0$, we think of

$$\Gamma(\mathcal{O}_B((\alpha_r - i)M + S))$$

as the $\mathcal{O}_B$-module generated by a single morphism

$$\mathcal{O}_X(-\alpha_r M - S) \to \mathcal{O}_B(-iP) \subset \mathcal{O}_B,$$

which does not factor through $\mathcal{O}_B(-(i+1)B)$. (Recall that we work on an *affine* open $U$.) For any $i = 0, \ldots, \alpha_r - 1$, the generator of $\Gamma(\mathcal{O}_B((\alpha_r - i)M + S))$ lands in a box $(\alpha, 0)$ with $0 \leq \alpha \leq \alpha_r - 1$. However all arrows of $\mathbb{C}^{\ell-\alpha_r}$ land in $(\ell - 1, 0)$ and they *cannot* be translated so their heads lie in $(\alpha, 0)$ with $0 \leq \alpha \leq \alpha_r - 1$. We conclude that non-zero elements of $\Gamma(\mathcal{O}_B((\alpha_r - i)M + S))$ do not occur in the kernel for any $i = 0, \ldots, \alpha_r - 1$. Therefore the kernel factors through

$$\Gamma(N_{B/U}) \cong \Gamma(\mathcal{O}_B(S)) \subset \Gamma(\mathcal{O}_B(\alpha_r + S)).$$

The generators of $\Gamma(N_{B/U}((\alpha_r - \ell - 1)P)$ automatically restrict to zero on $B \cap p^*Z$. This leaves us with $\ell - \alpha_r$ generators landing in $(\alpha, 0)$ with $\alpha_r \leq \alpha \leq \ell - 1$, none of which restricts to zero. We can take any linear combination of these generators, which then *determines* the remaining arrows of $\mathbb{C}^{\ell-\alpha_r}$ uniquely upon restriction. (Up to translation, the Haiman arrows corresponding to a basis of $\mathbb{C}^{\ell-\alpha_r}$ exactly land in $(\alpha, 0)$ with $\alpha_r \leq \alpha \leq \ell - 1$.) Therefore the kernel of (22) is isomorphic to $\Gamma(N_{B/U})$.

In conclusion we find the following kernel

$$\mathbb{C}^{2d-\ell} \oplus \Gamma(N_{B/S}) \oplus \Gamma(N_{B/U}) \cong \mathbb{C}^{2d-\ell} \oplus \Gamma(N_{B/X}). \tag{23}$$

Recall that we work on opens $X_a \to U_a$. Since the elements of the first factor of (23) restrict to zero on overlaps $X_a \cap X_b$, we proved what we claimed in Step 1. ☐

## References


[1] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[2] Kai Behrend. Donaldson-Thomas type invariants via microlocal geometry. *Ann. of Math. (2)*, 170(3):1307–1338, 2009. arXiv:math/0507523.

[3] Jim Bryan. The Donaldson-Thomas theory of $K3 \times E$ via the topological vertex. arXiv:math/1504.02920.

[4] Jim Bryan, Martijn Kool, and Ben Young. Trace identities for the topological vertex. arXiv:math/1603.05271.

[5] Jim Bryan, Georg Oberdieck, Rahul Pandharipande, and Qizheng Yin. Curve counting on abelian surfaces and threefolds. arXiv:math/1506.00841.

[6] Jan Cheah. On the cohomology of Hilbert schemes of points. *J. Algebraic Geom.*, 5(3):479–511, 1996.

[7] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[8] Toshiya Kawai and Kōta Yoshioka. String partition functions and infinite products. *Adv. Theor. Math. Phys.*, 4(2):397–485, 2000.

[9] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[10] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. I. *Compos. Math.*, 142(5):1263–1285, 2006. arXiv:math.AG/0312059.

[11] Rick Miranda. *The basic theory of elliptic surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.

[12] David Mumford. *Lectures on curves on an algebraic surface*. With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[13] R. Pandharipande and R. P. Thomas. The Katz–Klemm–Vafa conjecture for $K3$ surfaces. *Forum Math. Pi*, 4:e4, 111, 2016. arXiv:math/1404.6698.

[14] The Stacks Project Authors. *stacks project*. `http://stacks.math.columbia.edu`, *2016*.

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