DONALDSON-THOMAS INVARIANTS OF LOCAL ELLIPTIC SURFACES VIA THE TOPOLOGICAL VERTEX

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex proved in [4], we derive product formulas for the partition functions. The connected version of the partition function is written in terms of Jacobi forms. In the special case where the elliptic surface is a K3 surface, we get a new derivation of the Katz-Klemm-Vafa formula.

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1. Introduction

Let $p : S \to B$ be a non-trivial elliptic surface over a complex smooth projective curve B. We assume p has a section and all singular fibres are irreducible rational nodal curves.

In this paper, we study the Donaldson-Thomas (DT) invariants of $X = \mathrm{Tot}(K_S)$, i.e. the total space of the canonical bundle K_S. This is a non-compact Calabi-Yau threefold. Let β be an effective curve class on S. Consider the Hilbert scheme

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X \, : \, [Z] = \beta, \ \chi(\mathcal{O}_Z) = n\}$$

of proper subschemes $Z \subset X$ with homology class β and holomorphic Euler characteristics n. The DT invariants of X can be defined as

$$\mathsf{DT}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X), \nu) := \sum_{k \in \mathbb{Z}} k \, e(\nu^{-1}(k)),$$

where $e(\cdot)$ denotes topological Euler characteristic and $\nu : \mathrm{Hilb}^{\beta,n}(X) \to \mathbb{Z}$ is Behrend's constructible function [1]. We also consider an unweighted Euler characteristic version of these invariants

$$\widehat{\mathsf{DT}}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X)).$$

We choose a section $B \subset S$ and focus on the primitive classes $\beta = B + dF$, where B is the class of the chosen section and F the class of the fiber. We define the partition functions by

$$\widehat{\mathsf{DT}}(X) = \sum_{d=0}^{\infty} \sum_{n \in \mathbb{Z}} \widehat{\mathsf{DT}}_{B+dF,n}(X) p^n q^d,$$

$$\mathsf{DT}(X) = \sum_{d=0}^{\infty} \sum_{n \in \mathbb{Z}} \mathsf{DT}_{B+dF,n}(X) y^n q^d.$$

We also consider the partition functions for the invariants for multiples of the fiber class:

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \widehat{\mathsf{DT}}_{dF,n}(X) p^n q^d,$$

$$\mathsf{DT}_{\mathsf{fib}}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \mathsf{DT}_{dF,n}(X) y^n q^d.$$

The main result of this paper are closed product formulas for the partition functions $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$. Assuming a general conjecture about the Behrend function, we also determine $\mathsf{DT}(X)$ and $\mathsf{DT}_{\mathsf{fib}}(X)$.

We use the notation

$$M(p,q) = \prod_{m=1}^{\infty} (1-p^m q)^{-m}$$

and the shorthand $M(p) = M(p,1)$.

Theorem 1. *Let $e(S)$ and $e(B)$ denote the topological Euler characteristic of the elliptic surface and the base. Then*

$$\widehat{\mathsf{DT}}(X) = \left\{ M(p) \prod_{d=1}^{\infty} \frac{M(p,q^d)}{(1-q^d)} \right\}^{e(S)} \left\{ \frac{1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})} \prod_{d=1}^{\infty} \frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)} \right\}^{e(B)}$$

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = \left\{ M(p) \prod_{d=1}^{\infty} M(p,q^d) \right\}^{e(S)} \left\{ \prod_{d=1}^{\infty} \frac{1}{(1-q^d)} \right\}^{e(B)}$$

The formula for $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ was previously proved using wall-crossing methods by Toda[1].

The ratio $\widehat{\mathsf{DT}}(X)/\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ can be considered as the generating function for the connected invariants in the classes $B + dF$. This series has a particularly nice form and can be written in terms of classical Jacobi forms. Consider the Dedekind eta function and the Jacobi theta function

$$\eta = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1-q^k),$$

$$\Theta = (p^{\frac{1}{2}} - p^{-\frac{1}{2}}) \prod_{k=1}^{\infty} \frac{(1-pq^k)(1-p^{-1}q^k)}{(1-q^k)^2}.$$

Corollary 2. *The partition function of the connected invariants is given as follows*

$$\frac{\widehat{\mathsf{DT}}(X)}{\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)} = \left(q^{-\frac{1}{24}}\eta\right)^{-e(S)} \Theta^{-e(B)}.$$

[1] After applying the PT/DT correspondence [2], this is essentially [15, Thm 6.9]

In the case where $S \to \mathbb{P}^1$ is an elliptically fibred $K3$ surface, the above series specializes to the well-known Katz-Klemm-Vafa formula. Because X is non-compact, the connected series is required to obtain the KKV formula. Our result provides a new derivation of the KKV formula for primitive classes. The KKV formula was proved in all curve classes in [14]. Ours is the first derivation of the KKV formula, which does not depend on the Kawai-Yoshioka formula [8].

Our results can be extended to apply to the usual (Behrend function weighted) Donaldson-Thomas invariants if we assume a general conjecture which we formulate in Section 8. Our conjecture relates the Behrend function at subschemes with embedded points to the value of the Behrend function at the underlying Cohen-Macaulay subscheme and may be of independent interest.

Theorem 3. *Assume that Conjecture 18 holds, then*

$$\mathsf{DT}(X) = (-1)^{\chi(\mathcal{O}_S)}\widehat{\mathsf{DT}}(X)$$

and

$$\mathsf{DT}_{\mathsf{fib}}(X) = \widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$$

under the change of variables

$$y = -p.$$

A similar phenomenon to the above is known to hold when X is a toric Calabi-Yau threefold.

We expect that the method of computation that we introduce in this paper should apply to other elliptically fibered geometries. Indeed, it has already found applications to the calculation of DT generating functions on $K3\times E$, where E is an elliptic curve [3], and to abelian threefolds [5]. Even though the geometry under consideration is not toric, we combine $\mathbb{C}^*$-localization, motivic methods, formal methods, and $(\mathbb{C}^*)^3$-localization to end up with expressions that only depend on the topological vertex $\mathsf{V}_{\lambda\mu\nu}$, and the topological Euler characteristics $e(B)$, and $e(S)$. The outline of our method is as follows:

- The $\mathbb{C}^*$ action on X induces an action on $\operatorname{Hilb}(X)$ whose Euler characteristic localizes to the $\mathbb{C}^*$-fixed locus. In section 3 we show that any $\mathbb{C}^*$-invariant subscheme has a maximal Cohen-Macaulay subscheme which is a curve of a special form which we call a partition thickened comb curve (definition 6). This curve is determined by data consisting of points $x_i \in B$ labelled by integer partitions $\lambda^{(i)}$. This gives rise to a morphism ρ to $\operatorname{Sym} B$ taking the value $\sum_i |\lambda^{(i)}| x_i$ on such a curve (see theorem 7).

- In section 4, we pushforward the Euler characteristic measure to $\operatorname{Sym} B$ via the map ρ. We show that $\rho_*(1)$, the pushforward measure, has nice multiplicative properties which allows us to compute the weighted Euler characteristic over $\operatorname{Sym} B$ using a general result about symmetric products (lemma 28).
- To compute the pushforward measure $\rho_*(1)$ explicitely, we must compute the Euler characteristics of the fibers of ρ. These fibers are strata in the Hilbert scheme parameterizing subschemes whose maximal Cohen-Macaulay subscheme is a fixed partition thickened comb curve C. The moduli of this stratum comes solely from the ways that embedded points and zero dimensional components can be added to C. We write this stratum as a product of "local" Hilbert schemes using a fpqc cover of X which includes the formal neighborhoods of the nodal points of C_{red}, the formal neighborhoods of the components of C_{red} (minus the nodes), and the complement of C. This is done in section 5.
- After further pushforwards to further symmetric products, we reduce our expression for $\rho_*(1)$ (and hence our expression for $\widehat{\mathsf{DT}}(X)$) to an expression only involving the Euler characteristics of the local Hilbert schemes of formal neighborhoods of points. Since these formal neighborhoods are isomorphic to $\widehat{\mathbb{C}}^3_0 = \operatorname{Spec} \mathbb{C}[[r,s,t]]$, they admit a $T = (\mathbb{C}^*)^3$ action. T-localization then allows us to write these Euler characteristics in terms of the topological vertex (see section 6).
- Finally, using the trace formulas for the topologocal vertex proved in [4], we write our expression for $\widehat{\mathsf{DT}}(X)$ as the closed product formula given in Theorem 1.

2. Definitions, notation, and conventions

Let $p : S \to B$ be an elliptic surface over a smooth projective curve B. We assume:

(1) S is a non-trivial fibration,
(2) p has a section $B \subset S$,
(3) all singular fibers of p are irreducible rational nodal curves.

We note that the number of singular fibers is equal to $e(S)$.

We write F_x for the fibre $p^{-1}(x)$ over a point $x \in B$. We choose a section $B \subset S$ and denote its class in $H_2(S)$ by B as well. We denote the class of the fiber by $F \in H_2(S)$.

For brevity, we define

$$\operatorname{Hilb}^{d,n}(X) := \operatorname{Hilb}^{B+dF,n}(X),$$
$$\widehat{\mathsf{DT}}_{d,n}(X) := \widehat{\mathsf{DT}}_{B+dF,n}(X).$$

Since we are dealing with generating functions and our calculations involve motivic methods on the Hilbert schemes, it is useful to introduce the following notation. We define

$$\operatorname{Hilb}^{d,\bullet}(X) := \sum_{n\in\mathbb{Z}} \operatorname{Hilb}^{d,n}(X)\, p^n,$$

where we view the right hand side as a formal Laurent series whose coefficients elements in the Grothendieck ring of varieties, i.e. $K_0(\operatorname{Var}_{\mathbb{C}})(\!(p)\!)$.

Convention 3.1. *When an index is replaced by a bullet, we will multiply by the appropriate variable and sum over the index. We regard the result as a formal power (or Laurent) series whose coefficients lie in $K_0(\operatorname{Var}_{\mathbb{C}})$ and we extend operations of the Grothendieck group (addition, multiplication, Euler characteristic) to the series in the obvious way.*

For example

$$\operatorname{Hilb}^{\bullet,\bullet}(X) = \sum_{d=0}^{\infty}\sum_{n\in\mathbb{Z}} \operatorname{Hilb}^{d,n}(X) q^d p^n \in K_0(Var)((p))[[q]],$$

so that we can write

$$\widehat{\mathsf{DT}}(X) = e(\operatorname{Hilb}^{\bullet,\bullet}(X)).$$

It is notationally convenient to treat an Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, the measurable functions are constructible functions, and the measure of a set is given by its Euler characteristic. In this language we have

$$\widehat{\mathsf{DT}}_{d,n}(X) = \int_{\operatorname{Hilb}^{d,n}(X)} 1\, de, \qquad \mathsf{DT}_{d,n}(X) = \int_{\operatorname{Hilb}^{d,n}(X)} \nu\, de,$$

and following the bullet convention we have

$$\widehat{\mathsf{DT}}(X) = \int_{\operatorname{Hilb}^{\bullet,\bullet}(X)} 1\, de, \qquad \mathsf{DT}(X) = \int_{\operatorname{Hilb}^{\bullet,\bullet}(X)} \nu\, de.$$

We will also need notation for subsets of the Hilbert scheme which parameterize those subschemes obtained by adding embedded points and/or zero dimensional components to some fixed Cohen-Macaulay curve.

Definition 4. Let $U \subset X$ be an open set (possibly in the fpcq topology) and let $C \subset U$ be a (not necessarily reduced) Cohen-Macaulay subscheme of dimension 1 which we assume is the restriction of some $\overline{C} \subset X$ to U. We define

$$\mathrm{Hilb}^n(U,C) = \{Z \subset U \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n\}.$$

Via the inclusion $U \subset X$, $\mathrm{Hilb}^n(Y,C)$ can be viewed as a constructible subscheme of $\mathrm{Hilb}(X)$. It parameterizes subschemes which roughly speaking are obtained from C by adding n embedded points and/or zero dimensional components.

3. The reduction to partition thickened comb curves

The action of $\mathbb{C}^*$ on the fibers of X lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1\, de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\, de.$$

The main result of this section is a classification of the subschemes parameterized by $\mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*}$, namely the $\mathbb{C}^*$-invariant subschemes. We find that the maximal Cohen-Macaulay subscheme of a $\mathbb{C}^*$-invariant subscheme is determined by a point in $\mathrm{Sym}^d(B)$ along with some discrete data (a collection of integer partitions). We begin with some notation.

Definition 5. Let $T = \mathrm{Tot}(K_S|_B)$ and let $p : X \to T$ be the elliptic fibration induced by the elliptic fibration $p : S \to B$. We say that a subscheme $C \subset X$ is a **comb curve** if $C = B \cup p^{-1}(Z)$ where $Z \subset T$ is a 0-dimensional subscheme which is set-theoretically supported on B.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ be an integer partition. Then λ determines a 0-dimensional subscheme $Z_\lambda \subset \mathrm{Spec}\, \mathbb{C}[r,s]$ given by the monomial ideal

$$I_\lambda = (r^{\lambda_1}, r^{\lambda_2}s, \ldots, r^{\lambda_l}s^{l-1}, r^l). \tag{1}$$

In terms of λ as a Young diagram, we note $(\rho,\sigma) \in \lambda$ if and only if $r^\rho s^\sigma \notin I_\lambda$. Convention check!

Definition 6. Let $C = B \cup p^{-1}(Z)$ be a comb curve, let $x_1, \ldots, x_n \in B \subset T$ be the points where Z is supported, an let (r_i, s_i) be formal local coordinates on T about each point x_i so that s_i vanishes on $S \cap T$ and r_i vanishes on $R_i \cap T$ where $R_i = \mathrm{Tot}(K_S|_{F_{x_i}})$. We say that C is a **partition thickened comb curve** if there exists partitions $\lambda^{(1)}, \ldots, \lambda^{(n)}$ such that Z is given by $Z_{\lambda^{(i)}}$ in the local coordinates (r_i, s_i) about x_i. We denote such a curve by $B \cup_i \left(\lambda^{(i)} F_{x_i}\right)$. We say that a subscheme $Z \subset X$ is a **partition thickened comb curve with points (PCP)** if the maximal Cohen-Macaulay subscheme $Z_{CM} \subset Z$ is a partition thickened comb curve, in other words,

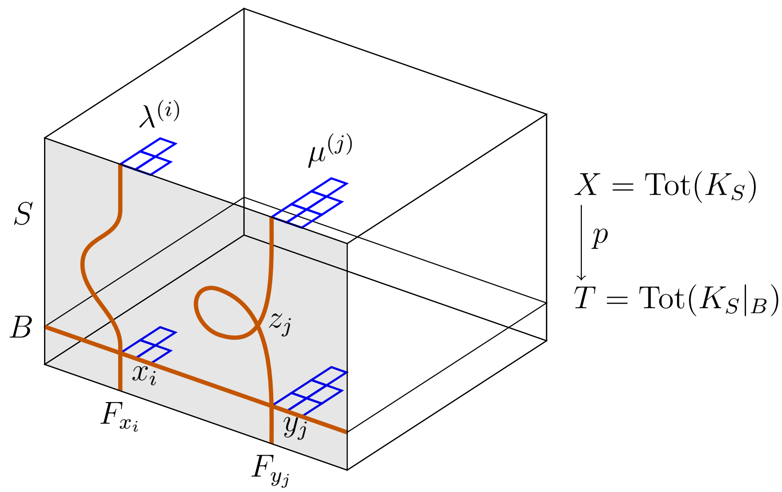

FIGURE 1. A partition thickened comb curve $C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right) \cup_j \left(\mu^{(j)} F_{y_j}\right)$.

Z is obtained from a partition thickened comb curve by adding embedded points and/or zero dimensional components. We denote by

$$\mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X) \subset \mathrm{Hilb}^{d,n}(X)$$

the locus in the Hilbert scheme parameterizing partition thickened comb curves with points.

In the next section it will be important to notationally distinguish between singular and smooth fibers. See figure 1 for an illustration of a partition thickened comb curve with smooth fibers $\{F_{x_i}\}$ thickened by partitions $\{\lambda^{(i)}\}$ and nodal fibers $\{F_{y_j}\}$ thickened by partitions $\{\mu^{(j)}\}$.

The main result of this section is the following

Theorem 7. *If a subscheme $Z \subset X$ in the class $[Z] = B + dF$ is $\mathbb{C}^*$-invariant, then it is a partition thickened comb curve with points. That is*

$$\mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*} \subset \mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X) \subset \mathrm{Hilb}^{d,n}(X).$$

Moreover, $\mathbb{C}^*$ acts on $\mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X)$ and there exists a morphism

$$\rho_d : \mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X) \to \mathrm{Sym}^d(B) \tag{2}$$

such that if $[Z] \in \mathrm{Hilb}^{d,n}_{\mathsf{PCP}}(X)$ where the maximal Cohen-Macaulay subscheme of Z is $B \cup_i \left(\lambda^{(i)} F_{x_i}\right)$, then

$$\rho_d([Z]) = \sum_i |\lambda^{(i)}|\, x_i.$$

Insert Martijn's proof but adapted to the theorem as stated and maybe streamlined a bit.

Proof. Adapt and streamline Martijn's proof for the theorem as stated. □

4. Push-forward to the symmetric product

From the $\mathbb{C}^*$-equivariant inclusions in Theorem 7 and $\mathbb{C}^*$ localization of Euler characteristic, we have

$$\widehat{\mathsf{DT}}(X) = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)} 1\, de = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)^{\mathbb{C}^*}} 1\, de = \int_{\mathrm{Hilb}^{\bullet,\bullet}_{\mathsf{PCP}}(X)} 1\, de.$$

We compute these Euler characteristics by pushing forward along the map ρ_d constructed in Theorem 7. That is we use

$$\int_{\mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)} 1\, de = \int_{\mathrm{Sym}^d(B)} (\rho_d)_*(1)\, de,$$

where $(\rho_d)_*(1)$ is the $\mathbb{Z}((p))$-valued constructible function on $\mathrm{Sym}^d(B)$ given by pushing forward the Euler characteristic measure [9]. We denote $(\rho_d)_*(1)$ by f_d so by definition, the value of f_d at a point $\boldsymbol{ax} = \sum_i a_i x_i \in \mathrm{Sym}^d(B)$ is

$$f_d(\boldsymbol{ax}) = \int_{\rho_d^{-1}(\boldsymbol{ax})} 1\, de.$$

We will show that f_d has some nice multiplicative properties. Let $B^{\mathrm{sing}} \subset B$ be the points over which the fibers of $S \to B$ are singular. Note that $\# B^{\mathrm{sing}} = e(S)$. Let $B^{\mathrm{sm}} = B - B^{\mathrm{sing}}$.

Proposition 8. *Let $x_1, \dots, x_n \in B^{\mathrm{sm}}$ and $y_1, \dots, y_m \in B^{\mathrm{sing}}$ and let $a_1, \dots, a_n, b_1, \dots, b_m$ be positive integers summing to d. Let $\boldsymbol{ax}$ and $\boldsymbol{by}$ denote $\sum_i a_i x_i$ and $\sum_j b_j y_j$ respectively. Then there exists $F_1, F_2 \in \mathbb{Z}((p))$ and $g, h : \mathbb{N} \to \mathbb{Z}((p))$ such that*

$$f_d(\boldsymbol{ax} + \boldsymbol{by}) = F_1^{e(B)} \cdot F_2^{e(S)} \cdot G(\boldsymbol{ax}) \cdot H(\boldsymbol{by})$$

where

$$G(\boldsymbol{ax}) = \prod_{i=1}^{n} g(a_i), \quad H(\boldsymbol{ax}) = \prod_{j=1}^{n} h(b_j).$$

This proposition follows from proposition 13 which will be stated and proved in the next section.

Corollary 9.

$$\widehat{\mathsf{DT}}(X) = F_1^{e(B)} \cdot F_2^{e(S)} \cdot \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)-e(S)} \cdot \left(\sum_{b=0}^{\infty} h(b) q^b \right)^{e(S)}$$

where we have set $g(0) = h(0) = 1$.

Proof. We apply proposition 8 to the computation of $\widehat{\mathsf{DT}}(X)$ as follows:

$$\begin{aligned}\widehat{\mathsf{DT}}(X) &= \int_{\mathrm{Hilb}^{\bullet,\bullet}_{\mathsf{PCP}}(X)} 1\, de \\ &= \int_{\mathrm{Sym}^{\bullet}(B)} f_{\bullet}\, de \\ &= F_1^{e(B)} \cdot F_2^{e(S)} \cdot \int_{\mathrm{Sym}^{\bullet}(B^{\mathrm{sm}})} G\, de \cdot \int_{\mathrm{Sym}^{\bullet}(B^{\mathrm{sing}})} H\, de.\end{aligned}$$

Applying Lemma 28 to this last equation yields the corollary. □

To prove Proposition 8 and explicitly compute F_1, F_2, g, and h, we need a good understanding of the strata $\rho_d^{-1}(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y}) \subset \mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)$.

We define $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ to be the locus of points $Z \in \mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)$, for which the maximal Cohen-Macaulay subcurve $Z_{\mathrm{CM}} \subset Z$ is given by

$$C = B \bigcup_{i=1}^{n} \left(\lambda^{(i)} F_{x_i}\right) \bigcup_{j=1}^{m} \left(\mu^{(j)} F_{y_j}\right)$$

The bullet reminds us that we are multiplying by $p^{\chi(\mathcal{O}_Z)}$ and summing over all possible holomorphic Euler characteristics (recall Convention 3.1). We regard $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ as an element in $K_0(Var_{\mathbb{C}}((p))$.

Theorem 7 gives the following decomposition of the fibers of ρ_d into components[2]

$$\rho_d^{-1}(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y}) = \bigsqcup_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \bigsqcup_{\boldsymbol{\mu} \vdash \boldsymbol{b}} \Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{3}$$

where

$$\begin{aligned} \boldsymbol{a} &= (a_1, \ldots, a_n), \quad & \boldsymbol{b} &= (b_1, \ldots, b_m), \\ \boldsymbol{x} &= (x_1, \ldots, x_n), \quad & \boldsymbol{y} &= (y_1, \ldots, y_m), \\ \boldsymbol{\lambda} &= (\lambda^{(1)}, \ldots, \lambda^{(n)}), \quad & \boldsymbol{\mu} &= (\mu^{(1)}, \ldots, \mu^{(n)}), \end{aligned}$$

and the meaning of $\boldsymbol{\lambda} \vdash \boldsymbol{a}$ and $\boldsymbol{\mu} \vdash \boldsymbol{b}$ is that $\lambda^{(i)} \vdash a_i$ and $\mu^{(j)} \vdash b_j$ for all i and j.

In the next section, we will see that the Euler characteristic of $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ does not depend on the exact location of the points $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$, but only on their number n and m and the partitions $\lambda^{(i)}$ and $\mu^{(j)}$.

[2]We use the term component somewhat loose: it means a subset which is both open and closed. We do not care whether it is connected.

5. Restriction to formal neighbourhoods

In the previous two sections we reduced our consideration to the strata $\Sigma^{\bullet}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\mu})$ of $\mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)$ which parameterize subschemes Z whose maximal Cohen-Macaulay subscheme $Z_{\mathrm{CM}}\subset Z$ is the partition thickened comb curve

$$C=B\cup_i(\lambda^{(i)}F_{x_i})\cup_j(\mu^{(j)}F_{x_j}).$$

In this section we use an fpqc open cover to express $\Sigma^{\bullet}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\mu})$ as a product of "local" Hilbert schemes. We then use this product to compute the Euler characteristic of $\Sigma^{\bullet}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\mu})$. The main result of this section is Proposition 13.

5.1. **The fpqc cover.** The reduced support of C is $B\cup_i F_{x_i}\cup_j F_{y_j}$ which is a nodal curve with nodes at $(x_1,\dots,x_n)$, $(y_1,\dots,y_n)$ and $(z_1,\dots,z_m)$ where z_j is the node of the nodal fiber F_{y_j} (see figure 1).

We wish to make an open cover of X which is compatible with the support of C. Intuitively, one can think of the complex analytic open cover consisting of small balls around the points $\{x_i,y_j,z_j\}$, a small tubular neighborhood of $B-\{x_i,y_j\}$, small tubular neighborhoods of $F_{x_i}-\{x_i\}$ and $F_{y_j}-\{y_j,z_j\}$, and finally the complement of C.

To work algebraically, we use formal neighborhoods instead of tubular neighborhoods and balls. This also has an addition advantage over an analytic cover: the intersection of these open sets is very small in the sense that no embedded points or zero-dimensional components can occur inside the intersections of distinct open sets in the cover.

Definition 10. We define $\mathfrak{U}$, a cover of X, to be the collection of the following sets, open in the fpqc topology:

(1) W, the complement of the support of C.
(2) $\widehat{X}_{x_i}$, for $i=1,\dots,n$, the formal neighborhoods of the points x_i.
(3) $\widehat{X}_{y_j}$, for $i=1,\dots,m$, the formal neighborhoods of the points y_j.
(4) $\widehat{X}_{z_j}$, for $i=1,\dots,m$, the formal neighborhoods of the points z_j.
(5) $\widehat{X}_{B^\circ}$, the formal neighborhood of $B^\circ\subset X$ where $B^\circ=B-\{x_1,\dots,x_n,y_1\cdots,y_m\}$.
(6) $\widehat{X}_{F^\circ_{x_i}}$, for $i=1,\dots,n$ the formal neighborhood of $F^\circ_{x_i}\subset X$ where

$$F^\circ_{x_i}=F_{x_i}-\{x_i\}.$$

(7) $\widehat{X}_{F^\circ_{y_j}}$, for $j=1,\dots,m$ the formal neighborhood of $F^\circ_{y_j}\subset X$ where

$$F^\circ_{y_j}=F_{y_j}-\{y_j,z_j\}.$$

The subschemes B°, $F^\circ_{x_i}$, and $F^\circ_{y_j}$ are not closed in X but they are locally closed. To define $\widehat{X}_{V^\circ}$, the formal neighborhood of a locally closed subscheme V°, we take any open $X^\circ \subset X$ such that $V^\circ \subset X^\circ$ is closed and then define $\widehat{X}_{V^\circ}$ to be $\widehat{X}^\circ_{V^\circ}$.

We consider the formal neighborhoods which are members of the cover $\mathfrak{U}$ not as formal schemes, but as the associated non-finite-type schemes. The scheme maps

$$\widehat{X}_{x_i} \to X, \quad \widehat{X}_{B^\circ} \to X, \dots$$

are open in the fpqc topology [**?**, find exact reference] so $\mathfrak{U}$ forms an fpqc cover. For $U \in \mathfrak{U}$, we suppress the associated map $U \to X$ from the notation and use the usual restriction notation to denote the pullback, for example:

$$U|_C := U \times_X C.$$

5.2. **Writing $\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ as a product of local Hilbert schemes.** Let

$$C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right) \cup_j \left(\mu^{(j)} F_{y_j}\right).$$

Recall that the component

$$\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \subset \operatorname{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X)$$

parameterizes subschemes $Z \subset X$ whose minimal Cohen-Macaulay subscheme is C, i.e. Z is obtained from C by adding embedded points and/or 0-dimensional components. We apply the definition of $\operatorname{Hilb}^n(U, C)$ (Definition 4) to the open sets in our cover $\mathfrak{U}$. The following lemma writes the stratum $\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ as a product of those schemes.

Lemma 11. *The following equation holds in $K_0(Var_{\mathbb{C}})((p))$:*

$$\begin{aligned}
\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= p^{\chi(\mathcal{O}_C)} \cdot \prod_{U \in \mathfrak{U}} \operatorname{Hilb}^\bullet(U, C|_U) \\
&= p^{\chi(\mathcal{O}_C)} \cdot \operatorname{Hilb}^\bullet(W) \cdot \operatorname{Hilb}^\bullet(\widehat{X}_{B^\circ}, B^\circ) \\
&\quad \cdot \prod_{i=1}^{n} \operatorname{Hilb}^\bullet(\widehat{X}_{x_i}, \widehat{C}_{x_i}) \cdot \operatorname{Hilb}^\bullet(\widehat{X}_{F^\circ_{x_i}}, \lambda^{(i)} F^\circ_{x_i}) \\
&\quad \cdot \prod_{j=1}^{m} \operatorname{Hilb}^\bullet(\widehat{X}_{y_j}, \widehat{C}_{y_j}) \cdot \operatorname{Hilb}^\bullet(\widehat{X}_{z_j}, \widehat{C}_{z_j}) \cdot \operatorname{Hilb}^\bullet(\widehat{X}_{F^\circ_{y_j}}, \mu^{(j)} F^\circ_{y_j})
\end{aligned}$$

Note that we have implicitly introduced notation for our curve restricted to the various open sets: $\widehat{C}_{x_i} := C|_{\widehat{X}_{x_i}}$, $\mu^{(j)} F^\circ_{y_j} := C|_{\widehat{X}_{F^\circ_{y_j}}}$, etc.

Proof. Let Z be a subscheme corresponding to a point in $\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$. Our fpqc cover $\mathfrak{U}$ has the property that every 0-dimensional component or embedded point of Z is contained in a unique open set in the cover. In other words, the restriction of Z to any intersection of sets in the cover $\mathfrak{U}$ has no embedded points or 0-dimensional components. fpqc decent then tells us that Z is uniquely determined by its restriction to the open sets of the cover. This yields a constructible bijective morphism from the product of Hilbert schemes on the open sets to the Hilbert scheme on X which induces the equality in the Grothendieck group. Finally, we verify that the powers of p correctly match up each side. For a subscheme with Z maximal Cohen-Macaulay subscheme C, we have

$$\begin{aligned}\chi(\mathcal{O}_Z) &= \chi(\mathcal{O}_C) + \operatorname{length}(I_C/I_Z) \\ &= \chi(\mathcal{O}_C) + \sum_{U\in\mathfrak{U}} \operatorname{length}(I_C/I_Z|_U)\end{aligned}$$

therefore the $p^{\chi(\mathcal{O}_Z)}$ term on the left hand side of the equation correctly matches the

$$p^{\chi(\mathcal{O}_C)} \cdot \prod_{U\in\mathfrak{U}} p^{\operatorname{length}(I_C/I_Z|_U)}$$

term on the right hand side. □

Lemma 12. *Let*

$$C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right) \cup_j \left(\mu^{(j)} F_{y_j}\right)$$

then

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_B) - \sum_{i=1}^{n} \lambda_1^{(i)} - \sum_{j=1}^{m} \mu_1^{(j)}.$$

Proof. Since $\lambda^{(i)} F_{x_i} = p^{-1}(Z_{\lambda^{(i)}})$ and p is an elliptic fibration, $\chi(\mathcal{O}_{F_{x_i}}) = 0$ and similarly we have $\chi(\mathcal{O}_{F_{y_j}}) = 0$. Note that $B \cap \lambda^{(i)} F_{x_i}$ and $B \cap \mu^{(j)} F_{y_j}$ are zero dimensional subschemes of length $\lambda_1^{(i)}$ and $\mu_1^{(j)}$ respectively (c.f. equation (1)). The lemma then follows from the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_B \oplus_i \mathcal{O}_{\lambda^{(i)} F_{x_i}} \oplus_j \mathcal{O}_{\mu^{(j)} F_{y_j}} \to \oplus_i \mathcal{O}_{B\cap\lambda^{(i)} F_{x_i}} \oplus_j \mathcal{O}_{B\cap\mu^{(j)} F_{y_j}} \to 0.$$

□

5.3. Formal coordinates and reduction to Hilbert schemes of $\operatorname{Spec} \mathbb{C}[[r, s, t]]$.

Let λ, μ, ν be integer partitions which we also regard as subsets in $(\mathbb{Z}_{\geq 0})^2$ by their diagram as in [4]. Consider the subscheme

$$Z_{\lambda\mu\nu} = Z_{\lambda\varnothing\varnothing} \cup Z_{\varnothing\mu\varnothing} \cup Z_{\varnothing\varnothing\nu} \subset \mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[r, s, t]$$

defined by the monomial ideal

$$I_{\lambda\mu\nu} = I_{\lambda\varnothing\varnothing} \cap I_{\varnothing\mu\varnothing} \cap I_{\varnothing\varnothing\nu}$$

Convention check! where

$$\begin{aligned} r^\rho s^\sigma t^\tau \in I_{\lambda\varnothing\varnothing} &\iff (\sigma,\tau) \notin \lambda, \\ r^\rho s^\sigma t^\tau \in I_{\varnothing\mu\varnothing} &\iff (\tau,\rho) \notin \mu, \\ r^\rho s^\sigma t^\tau \in I_{\varnothing\varnothing\nu} &\iff (\rho,\sigma) \notin \nu. \end{aligned}$$

Let $\widehat{Z}_{\lambda\mu\nu}$ be the restriction of $Z_{\lambda\mu\nu}$ to $\widehat{\mathbb{C}}^3_0 = \operatorname{Spec}\mathbb{C}[[r,s,t]]$. Let

$$\operatorname{Hilb}^n(\lambda,\mu,\nu) := \operatorname{Hilb}^n(\widehat{\mathbb{C}}^3_0, \widehat{Z}_{\lambda\mu\nu})$$

be the Hilbert scheme parameterizing the subschemes obtained by adding a length n embedded point to $Z_{\lambda\mu\nu}$ at the origin (see Definition 4). We note that the permutations $(r,s,t) \mapsto (t,r,s)$ and $(r,s,t) \mapsto (s,r,t)$ induce the isomorphisms

$$\operatorname{Hilb}^n(\lambda,\mu,\nu) \cong \operatorname{Hilb}^n(\nu,\lambda,\mu), \quad \operatorname{Hilb}^n(\lambda,\mu,\nu) \cong \operatorname{Hilb}^n(\mu',\lambda',\nu').$$

We define $\widetilde{\mathsf{V}}_{\lambda\mu\nu} \in \mathbb{Z}[[p]]$ by

$$\widetilde{\mathsf{V}}_{\lambda\mu\nu} = e\left(\operatorname{Hilb}^\bullet(\lambda,\mu,\nu)\right)$$

and note the symmetries

$$\widetilde{\mathsf{V}}_{\lambda\mu\nu} = \widetilde{\mathsf{V}}_{\nu\lambda\mu} = \widetilde{\mathsf{V}}_{\mu'\lambda'\nu'}.$$

We now choose formal local coordinates at x_i, y_j, and z_j so that we can identify $\operatorname{Hilb}^n(\widehat{X}_{x_i}, \widehat{C}_{x_i})$, $\operatorname{Hilb}^n(\widehat{X}_{y_j}, \widehat{C}_{y_j})$, and $\operatorname{Hilb}^n(\widehat{X}_{z_j}, \widehat{C}_{z_j})$ in terms of $\operatorname{Hilb}^n(\lambda,\mu,\nu)$ for appropriate choices of λ,μ,ν.

Recall that $S \subset X$ is the elliptic surface and $T = \operatorname{Tot}(K_S|_B)$. For any point $p \in B$, let $R_p = \operatorname{Tot}(K_S|F_p)$. We choose isomorphisms

$$\widehat{X}_{x_i} \cong \widehat{X}_{y_j} \cong \widehat{X}_{z_j} \cong \operatorname{Spec}\mathbb{C}[[r,s,t]]$$

such that on $\widehat{X}_p$ when p is x_i or y_j

$$R_p = \{r=0\}, \quad S = \{s=0\}, \quad T = \{t=0\}$$

and when p is z_j

$$R_p = \{rt=0\}, \quad S = \{s=0\}.$$

Note that at x_i or y_j, the curve B is given by $\{s=t=0\}$ and the fiber F_{x_i} or F_{y_j} is given by $\{s=r=0\}$. At the point z_j, the fiber is a nodal curve and is given by $\{s=rt=0\}$.

With these choices, we have the identifications:

$$\begin{aligned}
\operatorname{Hilb}^n\left(\widehat{X}_{x_i},\widehat{C}_{x_i}\right) &\cong \operatorname{Hilb}^n\left(\Box,\varnothing,\lambda^{(i)}\right)\\
\operatorname{Hilb}^n\left(\widehat{X}_{y_j},\widehat{C}_{y_j}\right) &\cong \operatorname{Hilb}^n\left(\Box,\varnothing,\mu^{(j)}\right)\\
\operatorname{Hilb}^n\left(\widehat{X}_{z_j},\widehat{C}_{z_j}\right) &\cong \operatorname{Hilb}^n\left(\mu^{(j)\prime},\varnothing,\mu^{(j)}\right)
\end{aligned}$$

Here $\Box$ is the unique partition of size 1 (whose diagram is a single box), $\varnothing$ is the empty partition, and prime denotes conjugate partition.

An immediate consequence of the above and the symmetries of $\widetilde{\mathsf{V}}$ is

$$\begin{aligned}
e\left(\operatorname{Hilb}^\bullet\left(\widehat{X}_{x_i},\widehat{C}_{x_i}\right)\right) &= \widetilde{\mathsf{V}}_{\lambda^{(i)}\Box\varnothing}\\
e\left(\operatorname{Hilb}^\bullet\left(\widehat{X}_{y_j},\widehat{C}_{y_j}\right)\right) &= \widetilde{\mathsf{V}}_{\mu^{(j)}\Box\varnothing}\\
e\left(\operatorname{Hilb}^\bullet\left(\widehat{X}_{z_j},\widehat{C}_{z_j}\right)\right) &= \widetilde{\mathsf{V}}_{\mu^{(j)}\mu^{(j)\prime}\varnothing}
\end{aligned} \tag{4}$$

We also choose formal local coordinates at all other points. For each point in B°, choose local coordinates (r,s,t) such that $T=\{t=0\}$ and $S=\{s=0\}$. For each point in $F^\circ_{x_i}$ or $F^\circ_{y_j}$, choose local coordinates (r,s,t) such that $S=\{s=0\}$ and $R_{x_i}=\{r=0\}$ or $R_{y_j}=\{r=0\}$ respectively. For each point in W, choose any formal local coordinates (r,s,t).

Consider the following constructible "support" morphisms[3]

$$\begin{aligned}
\sigma_W &: \operatorname{Hilb}^\bullet(W)\to \operatorname{Sym}^\bullet(W)\\
\sigma_{B^\circ} &: \operatorname{Hilb}^\bullet(\widehat{X}_{B^\circ},B^\circ)\to \operatorname{Sym}^\bullet(B^\circ)\\
\sigma_{F^\circ_{x_i}} &: \operatorname{Hilb}^\bullet(\widehat{X}_{F^\circ_{x_i}},\lambda^{(i)}F^\circ_{x_i})\to \operatorname{Sym}^\bullet(F^\circ_{x_i})\\
\sigma_{F^\circ_{y_j}} &: \operatorname{Hilb}^\bullet(\widehat{X}_{F^\circ_{y_j}},\mu^{(j)}F^\circ_{y_j})\to \operatorname{Sym}^\bullet(F^\circ_{y_j})
\end{aligned}$$

which to each subscheme Z representing a point in a Hilbert scheme associates the location and length of the embedded points or zero dimensional components of Z.

[3]A constructible morphism is a map which is regular on each piece of a decomposition of its domain into locally closed subsets. Because we work with Euler characteristics and the Grothendieck group, we need only work with constructible morphisms.

Consider a point p in W, B°, $F^\circ_{x_i}$, or $F^\circ_{y_j}$. Then using the formal local coordinates chosen above, we see that

$$\begin{aligned}
\sigma_W^{-1}(np) &\cong \mathrm{Hilb}^n\left(\varnothing,\varnothing,\varnothing\right)\\
\sigma_{B^\circ}^{-1}(np) &\cong \mathrm{Hilb}^n\left(\square,\varnothing,\varnothing\right)\\
\sigma_{F^\circ_{x_i}}^{-1}(np) &\cong \mathrm{Hilb}^n\left(\varnothing,\varnothing,\lambda^{(i)}\right)\\
\sigma_{F^\circ_{y_j}}^{-1}(np) &\cong \mathrm{Hilb}^n\left(\varnothing,\varnothing,\mu^{(j)}\right)
\end{aligned}$$

Moreover, the preimages of the support morphisms clearly satisfy the following multiplicative property:

$$\sigma_U^{-1}\left(\sum_i n_i p_i\right) = \prod_i \sigma_U^{-1}\left(n_i p_i\right).$$

Pushing foward the Euler characteristic measure along the support maps, applying Lemma 28, and using the symmetries of $\widetilde{\mathsf{V}}$ we find the following formulas:

$$\begin{aligned}
e\left(\mathrm{Hilb}^\bullet(W)\right) &= \int_{\mathrm{Sym}^\bullet(W)} \left(\sigma_W\right)_*(1)\,de = \left(\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}\right)^{e(W)} \qquad (5)\\
e\left(\mathrm{Hilb}^\bullet\left(\widehat{X}_{B^\circ}, B^\circ\right)\right) &= \int_{\mathrm{Sym}^\bullet(B^\circ)} \left(\sigma_{B^\circ}\right)_*(1)\,de = \left(\widetilde{\mathsf{V}}_{\square\varnothing\varnothing}\right)^{e(B^\circ)}\\
e\left(\mathrm{Hilb}^\bullet\left(\widehat{X}_{F^\circ_{x_i}}, \lambda^{(i)}F^\circ_{x_i}\right)\right) &= \int_{\mathrm{Sym}^\bullet(F^\circ_{x_i})} \left(\sigma_{F^\circ_{x_i}}\right)_*(1)\,de = \left(\widetilde{\mathsf{V}}_{\lambda^{(i)}\varnothing\varnothing}\right)^{e(F^\circ_{x_i})}\\
e\left(\mathrm{Hilb}^\bullet\left(\widehat{X}_{F^\circ_{y_j}}, \mu^{(j)}F^\circ_{y_j}\right)\right) &= \int_{\mathrm{Sym}^\bullet(F^\circ_{y_j})} \left(\sigma_{F^\circ_{y_j}}\right)_*(1)\,de = \left(\widetilde{\mathsf{V}}_{\mu^{(j)}\varnothing\varnothing}\right)^{e(F^\circ_{y_j})}
\end{aligned}$$

We are now ready to prove the main result of this section:

Proposition 13. *Recall that $f_d = (\rho_d)_*(1) \in \mathbb{Z}((p))$ is the push forward of the Euler characteristic measure by the map ρ_d. As before, let $x_1,\dots,x_n \in B^{\mathrm{sm}}, y_1,\dots,y_m \in B^{\mathrm{sing}}$ and let $a_1,\dots,a_n,b_1,\dots,b_m$ be positive integers summing to d. Then*

$$f_d\left(\boldsymbol{ax}+\boldsymbol{by}\right) = \left(p^{\frac{1}{2}}\frac{\widetilde{\mathsf{V}}_{\square\varnothing\varnothing}}{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}}\right)^{e(B)} \cdot \widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}^{e(S)} \cdot \prod_{i=1}^n g(a_i)\prod_{j=1}^m h(b_j)$$

where

$$g(a) = \sum_{\lambda \vdash a} p^{-\lambda_1} \frac{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing} \widetilde{\mathsf{V}}_{\square\lambda\varnothing}}{\widetilde{\mathsf{V}}_{\square\varnothing\varnothing} \widetilde{\mathsf{V}}_{\lambda\varnothing\varnothing}},$$

$$h(b) = \sum_{\mu \vdash b} p^{-\mu_1} \frac{\widetilde{\mathsf{V}}_{\mu'\mu\varnothing} \widetilde{\mathsf{V}}_{\square\mu\varnothing}}{\widetilde{\mathsf{V}}_{\square\varnothing\varnothing} \widetilde{\mathsf{V}}_{\mu\varnothing\varnothing}}.$$

Note that this proves Proposition 8 and provides the values of the unknowns g, h *(as above) and* F_1, F_2*:*

$$F_1 = p^{\frac{1}{2}} \frac{\widetilde{\mathsf{V}}_{\square\varnothing\varnothing}}{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}}, \quad F_2 = \widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}.$$

Proof. We apply, in order, equation (3), Lemma 11, equations (4) and (5), and Lemma 12 to compute:

$$\begin{aligned}
f_d(\boldsymbol{ax} + \boldsymbol{by}) &= e\left(\rho_d^{-1}(\boldsymbol{ax} + \boldsymbol{by})\right) \\
&= \sum_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \sum_{\boldsymbol{\mu} \vdash \boldsymbol{b}} e\left(\Sigma^\bullet(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})\right) \\
&= p^{\chi(\mathcal{O}_C)} \cdot e(\mathrm{Hilb}^\bullet(W)) \cdot e\left(\mathrm{Hilb}^\bullet(\widehat{X}_{B^\circ}, B^\circ\right) \cdot \sum_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \sum_{\boldsymbol{\mu} \vdash \boldsymbol{b}} \\
&\quad \cdot \prod_{i=1}^n e\left(\mathrm{Hilb}^\bullet(\widehat{X}_{x_i}, \widehat{C}_{x_i})\right) \cdot e\left(\mathrm{Hilb}^\bullet(\widehat{X}_{F^\circ_{x_i}}, \lambda^{(i)} F^\circ_{x_i}\right) \\
&\quad \cdot \prod_{j=1}^m e\left(\mathrm{Hilb}^\bullet(\widehat{X}_{y_j}, \widehat{C}_{y_j})\right) \cdot e\left(\mathrm{Hilb}^\bullet(\widehat{X}_{z_j}, \widehat{C}_{z_j})\right) \cdot e\left(\mathrm{Hilb}^\bullet(\widehat{X}_{F^\circ_{y_j}}, \mu^{(j)} F^\circ_{y_j}\right) \\
&= p^{\chi(\mathcal{O}_C)} \cdot \widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}^{e(W)} \cdot \widetilde{\mathsf{V}}_{\square\varnothing\varnothing}^{e(B^\circ)} \cdot \sum_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \sum_{\boldsymbol{\mu} \vdash \boldsymbol{b}} \\
&\quad \cdot \prod_{i=1}^n \widetilde{\mathsf{V}}_{\square\lambda^{(i)}\varnothing} \cdot \widetilde{\mathsf{V}}_{\lambda^{(i)}\varnothing\varnothing}^{e(F^\circ_{x_i})} \cdot \prod_{j=1}^m \widetilde{\mathsf{V}}_{\square\mu^{(j)}\varnothing} \cdot \widetilde{\mathsf{V}}_{\mu^{(j)\prime}\mu^{(j)}\varnothing} \cdot \widetilde{\mathsf{V}}_{\mu^{(j)}\varnothing\varnothing}^{e(F^\circ_{y_j})} \\
&= p^{\chi(\mathcal{O}_B)} \cdot \widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}^{e(W)} \cdot \widetilde{\mathsf{V}}_{\square\varnothing\varnothing}^{e(B^\circ)} \\
&\quad \cdot \prod_{i=1}^n \sum_{\lambda^{(i)} \vdash a_i} p^{-\lambda_1^{(i)}} \widetilde{\mathsf{V}}_{\square\lambda^{(i)}\varnothing} \cdot \widetilde{\mathsf{V}}_{\lambda^{(i)}\varnothing\varnothing}^{e(F^\circ_{x_i})} \\
&\quad \cdot \prod_{j=1}^m \sum_{\mu^{(j)} \vdash b_j} p^{-\mu_1^{(j)}} \widetilde{\mathsf{V}}_{\square\mu^{(j)}\varnothing} \cdot \widetilde{\mathsf{V}}_{\mu^{(j)\prime}\mu^{(j)}\varnothing} \cdot \widetilde{\mathsf{V}}_{\mu^{(j)}\varnothing\varnothing}^{e(F^\circ_{y_j})}
\end{aligned}$$

We note that $e(F_{x_i}) = 0$ and $e(F_{y_j}) = 1$ so that $e(F^\circ_{x_i}) = -1$ and $e(F^\circ_{y_j}) = -1$. Also, since $e(B^\circ) = e(B) - n - m$, we have

$$\begin{aligned} e(W) &= e(S) - e(B^\circ) - \sum_i e(F_{x_i}) - \sum_j e(F_{y_j}) \\ &= e(S) - e(B) + n. \end{aligned}$$

The above equations allow us to redistribute the terms of $f_d(\boldsymbol{ax}+\boldsymbol{by})$ as follows:

$$\begin{aligned} f_d(\boldsymbol{ax}+\boldsymbol{by}) = p^{\chi(\mathcal{O}_B)} \cdot \widetilde{\mathsf{V}}^{e(S)}_{\varnothing\varnothing\varnothing} \cdot \left(\frac{\widetilde{\mathsf{V}}_{\square\varnothing\varnothing}}{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}}\right)^{e(B)} \\ \cdot \prod_{i=1}^{n} \sum_{\lambda^{(i)} \vdash a_i} p^{\lambda^{(i)}_1} \cdot \frac{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}\widetilde{\mathsf{V}}_{\square\lambda^{(i)}\varnothing}}{\widetilde{\mathsf{V}}_{\square\varnothing\varnothing}\widetilde{\mathsf{V}}_{\lambda^{(i)}\varnothing\varnothing}} \\ \cdot \prod_{j=1}^{m} \sum_{\mu^{(j)} \vdash b_j} p^{\mu^{(j)}_1} \cdot \frac{\widetilde{\mathsf{V}}_{\mu^{(j)\prime}\mu^{(j)}\varnothing}\widetilde{\mathsf{V}}_{\square\mu^{(j)}\varnothing}}{\widetilde{\mathsf{V}}_{\square\varnothing\varnothing}\widetilde{\mathsf{V}}_{\mu^{(j)}\varnothing\varnothing}} \end{aligned}$$

Noting that $\chi(\mathcal{O}_B) = e(B)/2$, we see the above proves the proposition. □

6. Reduction to the topological vertex and exact valuation of the series

In this section, we express $\widehat{\mathsf{DT}}(X)$ in terms of the topological vertex, and then use the trace formulas of [4] to obtain a closed formula for $\widehat{\mathsf{DT}}(X)$.

6.1. $\widetilde{\mathsf{V}}_{\lambda\mu\nu}$ in terms of $\mathsf{V}_{\lambda\mu\nu}$. Recall that the coefficients of the series $\widetilde{\mathsf{V}}_{\lambda\mu\nu} \in \mathbb{Z}[[p]]$ is given by the Euler characteristics of the local Hilbert schemes:

$$\widetilde{\mathsf{V}}_{\lambda\mu\nu} = e\left(\mathrm{Hilb}^\bullet(\widehat{\mathbb{C}}^3_0, \widehat{Z}_{\lambda\mu\nu})\right).$$

We can compute the Euler characteristics using the $T = (\mathbb{C}^*)^3$ action on the Hilbert schemes induced by the T action on $\mathbb{C}^3$. An ideal $I \subset \mathbb{C}[r,s,t]$ is T-invariant if and only if it is generated by monomials. Moreover, there is a bijection between monomial ideals and 3D partitions (see § 6.3 of [4]) where a monomial ideal $I \subset \mathbb{C}[r,s,t]$ corresponds to a 3D parition $\pi \in (\mathbb{Z}_{\geq 0})^3$ by

$$(\rho,\sigma,\tau) \in \pi \quad \Longleftrightarrow \quad r^\rho s^\sigma t^\tau \notin I.$$

The subschemes represented by points in $\mathrm{Hilb}^\bullet(\widehat{\mathbb{C}}^3_0, \widehat{Z}_{\lambda\mu\nu})^T$ are given by monomial ideals corresponding to 3D-partitions asymptotic to $(\lambda\mu\nu)$, see

[4, Defn 1]. Consequently,

$$\begin{aligned}\widetilde{\mathsf{V}}_{\lambda\mu\nu} &= e\left(\mathrm{Hilb}^\bullet(\widehat{\mathbb{C}}^3_0, \widehat{Z}_{\lambda\mu\nu})^T\right)\\ &= \sum_\pi p^{n(\pi)}\end{aligned}$$

where the sum runs over all 3D partitions asymptotic to $(\lambda\mu\nu)$ and $n(\pi)$ is the number of boxes in π which are not contained in any of the legs. We see that $\widetilde{\mathsf{V}}_{\lambda\mu\nu}$ differs from the usual topological vertex by an overall normalization:

$$\mathsf{V}_{\lambda\mu\nu} = p^{|\pi_{min}|}\widetilde{\mathsf{V}}_{\lambda\mu\nu}$$

where $\mathsf{V}_{\lambda\mu\nu}$ is the usual topological vertex [4, Defn 2] and $|\pi_{min}|$ is the normalized volume of the minimal 3D partition asymptotic to $(\lambda\mu\nu)$.

Lemma 14. *The following hold:* Convention check!

$$\mathsf{V}_{\lambda\varnothing\varnothing} = \widetilde{\mathsf{V}}_{\lambda\varnothing\varnothing}, \quad \mathsf{V}_{\square\lambda\varnothing} = p^{-\lambda_1}\widetilde{\mathsf{V}}_{\square\lambda\varnothing}, \quad \mathsf{V}_{\mu'\mu\varnothing} = p^{-||\mu||^2}\widetilde{\mathsf{V}}_{\mu'\mu\varnothing}$$

where $||\mu||^2 := \sum_{j=1}^\infty \mu_j^2$.

Proof. The normalized volume of a 3D-partition asymptotic to $(\lambda\mu\nu)$ is defined (see [4, page 2]) by

$$|\pi| = \sum_{(\rho,\sigma,\tau)\in\pi} (1 - \#\text{ of legs of }\pi\text{ containing }(\rho,\sigma,\tau)).$$

For π_{min}, the minimal 3D-partition asymptotic to $(\square\lambda\varnothing)$, the only cubes contributing to $|\pi_{min}|$ are those contained in both the $\square$-leg and the λ-leg. They intersect exactly in the cubes corresponding to the first part of λ, namely λ_1. Thus $|\pi_{min}| = -\lambda_1$ in this case.

For the case of $(\lambda\varnothing\varnothing)$ every cube is in the λ-leg and so $|\pi_{min}| = 0$. For the case of $(\mu'\mu\varnothing)$, each cube in the intersection of the μ-leg and the μ'-leg contribute -1 and all other cubes contribute 0. This intersection is a stack of squares of side lengths $\mu_1, \mu_2, \ldots$ and hence

$$|\pi_{min}| = -\sum_{j=1}^\infty \mu_j^2.$$

$\square$

6.2. **Putting it together and applying the trace formulas.** Substituting the values of F_1, F_2, g, and h from proposition 13 into corollary 9, and then substituting in the formulas from lemma 14 we obtain the following:

$$\widehat{\mathsf{DT}}(X) = \left(p^{\frac{1}{2}}\frac{\mathsf{V}_{\square\varnothing\varnothing}}{\mathsf{V}_{\varnothing\varnothing\varnothing}}\right)^{e(B)} \cdot \mathsf{V}_{\varnothing\varnothing\varnothing}^{e(S)} \cdot \left(\sum_{\lambda}\frac{\mathsf{V}_{\varnothing\varnothing\varnothing}\mathsf{V}_{\square\lambda\varnothing}}{\mathsf{V}_{\square\varnothing\varnothing}\mathsf{V}_{\lambda\varnothing\varnothing}}q^{|\lambda|}\right)^{e(B)-e(S)}$$
$$\cdot\left(\sum_{\mu}p^{||\mu||^2}\frac{\mathsf{V}_{\mu'\mu\varnothing}\mathsf{V}_{\square\mu\varnothing}}{\mathsf{V}_{\square\varnothing\varnothing}\mathsf{V}_{\mu\varnothing\varnothing}}q^{|\mu|}\right)^{e(S)}.$$

From the Okounkov-Reshitikhin-Vafa formula for the topological vertex [13][4, eqn 5], we get

$$\mathsf{V}_{\varnothing\varnothing\varnothing} = M(p), \quad \mathsf{V}_{\square\varnothing\varnothing} = \frac{M(p)}{1-p}$$

which we substitute into the above to find

$$\widehat{\mathsf{DT}}(X) = \left(\frac{1}{p^{-\frac{1}{2}}-p^{\frac{1}{2}}}\right)^{e(B)}\left(\sum_{\lambda}(1-p)\frac{\mathsf{V}_{\square\lambda\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}q^{|\lambda|}\right)^{e(B)-e(S)}$$
$$\cdot\left(\sum_{\mu}(1-p)p^{||\mu||^2}\mathsf{V}_{\mu'\mu\varnothing}\frac{\mathsf{V}_{\square\mu\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}}q^{|\mu|}\right).$$

Applying slight rearrangments of [4, eqns (2)&(4)] we see that

$$\widehat{\mathsf{DT}}(X) = \left(p^{-\frac{1}{2}}-p^{\frac{1}{2}}\right)^{-e(B)}\cdot\left(\prod_{d=1}^{\infty}\frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)}\right)^{e(B)-e(S)}$$
$$\cdot\left(M(p)\prod_{d=1}^{\infty}\frac{M(p,q^d)}{(1-pq^d)(1-p^{-1}q^d)}\right)^{e(S)}.$$

Noting that $e(B)$ is even, the above expression is easily seen to be equivalent to the formula for $\widehat{\mathsf{DT}}(X)$ in theorem 1. Since the formula for $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ was previously proven by Toda, we may now regard the proof of theorem 1 complete. In the next section, we will outline the proof of the formula for $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ using our methods.

7. The case of $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$

The formula for $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ given in theorem 1 follows from a wall-crossing computation of Toda [15, Thm 6.9] along with the PT/DT correspondence [2]. However in this section we describe how to adapt our computation of $\widehat{\mathsf{DT}}(X)$ to the easier case of $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$. Our approach yields a proof which is independent of Toda's.

Definition 15. We say that a subscheme $C \subset X$ is a **partition thickened fiber curve** if it is of the form

$$C = \cup_i \left(\lambda^{(i)} F_{x_i}\right)$$

where we are using the notation of definition 6. We say that a subscheme $Z \subset X$ is a **partition thickened fiber curve with points (PFP)** if the maximal Cohen-Macaulay subscheme $Z_{CM} \subset Z$ is a partition thickened fiber curve. We denote by

$$\mathrm{Hilb}^{dF,n}_{\mathsf{PFP}}(X) \subset \mathrm{Hilb}^{dF,n}(X)$$

the locus in the Hilbert scheme parameterizing partition thickened fiber curves with points.

Our proof of theorem 7 is easily adapted to prove the following:

Theorem 16. *If a subscheme $Z \subset X$ in the class $[Z] = dF$ is $\mathbb{C}^*$-invariant, then it is a partition thickened fiber curve with points. That is*

$$\mathrm{Hilb}^{dF,n}(X)^{\mathbb{C}^*} \subset \mathrm{Hilb}^{dF,n}_{\mathsf{PFP}}(X) \subset \mathrm{Hilb}^{dF,n}(X).$$

Moreover, $\mathbb{C}^*$ acts on $\mathrm{Hilb}^{dF,n}_{\mathsf{PFP}}(X)$ and there exists a morphism

$$\rho^{\mathsf{fib}}_d : \mathrm{Hilb}^{dF,\bullet}_{\mathsf{PFP}}(X) \to \mathrm{Sym}^d(B)$$

such that if $[Z] \in \mathrm{Hilb}^{dF,n}_{\mathsf{PFP}}(X)$ where the maximal Cohen-Macaulay subscheme of Z is $\cup_i \left(\lambda^{(i)} F_{x_i}\right)$, then

$$\rho^{\mathsf{fib}}_d([Z]) = \sum_i |\lambda^{(i)}| x_i.$$

It follows that the preimages of points under the map ρ^{fib}_d breaks into components:

$$\left(\rho^{\mathsf{fib}}_d\right)^{-1}(\boldsymbol{ax} + \boldsymbol{by}) = \bigsqcup_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \bigsqcup_{\boldsymbol{\mu} \vdash \boldsymbol{b}} \Sigma^{\bullet}_{\mathsf{fib}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$$

where we have adopted the same notation as in section 4. Note that the strata

$$\Sigma^{\bullet}_{\mathsf{fib}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \subset \mathrm{Hilb}^{dF,\bullet}(X)$$

parameterize subschemes whose maximal Cohen-Macaulay subscheme is

$$C = \cup_{i=1}^n \left(\lambda^{(i)} F_{x_i}\right) \cup_{j=1}^m \left(\mu^{(j)} F_{y_j}\right).$$

Since the only nodes of C_{red} are $z_1, \dots, z_m$ (adopting the notation of section 5) our fpcq cover is simplier in this case:

$$\mathfrak{U}_{\mathsf{fib}} = \left\{ \widehat{X}_{z_1}, \dots, \widehat{X}_{z_m}, \widehat{X}_{F^\circ_{y_1}}, \dots, \widehat{X}_{F^\circ_{y_m}}, \widehat{X}_{F_{x_1}}, \dots, \widehat{X}_{F_{x_n}}, W \right\}$$

where our notation is the same as in definition 10 except that

$$F^{\circ}_{y_j} = F_{y_j} - \{z_j\}, \quad W = X - C_{red}.$$

With virtually the same proof, we obtain the following analog of lemma 11:

Lemma 17. *The following equation holds in* $K_0(Var_{\mathbb{C}})((p))$*:*

$$\Sigma^{\bullet}_{\mathsf{fib}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\mu}) = p^{\chi(\mathcal{O}_C)} \cdot \mathrm{Hilb}^{\bullet}(W) \cdot \prod_{i=1}^{n} \mathrm{Hilb}^{\bullet}\left(\widehat{X}_{F_i}, \lambda^{(i)} F_{x_i}\right)$$
$$\cdot \prod_{j=1}^{m} \mathrm{Hilb}^{\bullet}\left(\widehat{X}_{F^{\circ}_{y_j}}, \mu^{(j)} F^{\circ}_{y_j}\right) \cdot \mathrm{Hilb}^{\bullet}\left(\widehat{X}_{z_j}, \widehat{C}_{z_j}\right).$$

We choose the same set of formal local coordinates at each point as we did in section 5 and by the same reasoning as in that section, we find

$$e\left(\mathrm{Hilb}^{\bullet}(\widehat{X}_{z_j}, \widehat{C}_{z_j})\right) = \widetilde{\mathsf{V}}_{\mu^{(j)\prime}\mu^{(j)}\varnothing},$$
$$e\left(\mathrm{Hilb}^{\bullet}(W)\right) = \widetilde{\mathsf{V}}^{e(W)}_{\varnothing\varnothing\varnothing},$$
$$e\left(\mathrm{Hilb}^{\bullet}(\widehat{X}_{F_{x_i}}, \lambda^{(i)} F_{x_i})\right) = \widetilde{\mathsf{V}}^{e(F_{x_i})}_{\lambda^{(i)}\varnothing\varnothing},$$
$$e\left(\mathrm{Hilb}^{\bullet}(\widehat{X}_{F^{\circ}_{y_j}}, \mu^{(j)} F^{\circ}_{y_j})\right) = \widetilde{\mathsf{V}}^{e(F^{\circ}_{y_j})}_{\mu^{(j)}\varnothing\varnothing}.$$

Now since $\chi(\mathcal{O}_C) = 0$, $e(F_{x_i}) = e(F^{\circ}_{y_j}) = 0$, and

$$e(W) = e(S) - m,$$

we have that

$$f^{\mathsf{fib}}_d(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y}) := e\left(\left(\rho^{\mathsf{fib}}_d\right)^{-1}(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y})\right)$$
$$= \sum_{\boldsymbol{\lambda}\vdash\boldsymbol{a}} \sum_{\boldsymbol{\mu}\vdash\boldsymbol{b}} e\left(\Sigma^{\bullet}_{\mathsf{fib}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a},\boldsymbol{b})\right)$$
$$= \sum_{\boldsymbol{\lambda}\vdash\boldsymbol{a}} \sum_{\boldsymbol{\mu}\vdash\boldsymbol{b}} \widetilde{\mathsf{V}}^{e(S)}_{\varnothing\varnothing\varnothing} \prod_{j=1}^{m} \frac{\widetilde{\mathsf{V}}_{\mu^{(j)\prime}\mu^{(j)}\varnothing}}{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}}.$$

We may rewrite the above as

$$f^{\mathsf{fib}}_d(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y}) = \widetilde{\mathsf{V}}^{e(S)}_{\varnothing\varnothing\varnothing} \prod_{i=1}^{n} g_{\mathsf{fib}}(a_i) \prod_{j=1}^{m} h_{\mathsf{fib}}(b_j)$$

where

$$g_{\mathsf{fib}}(a) = \sum_{\lambda\vdash a} 1 \qquad h_{\mathsf{fib}}(b) = \sum_{\mu\vdash b} \frac{\widetilde{\mathsf{V}}_{\mu'\mu\varnothing}}{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}}.$$

We then get the following result, analogous to corollary 9, with a similar proof:

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = \widetilde{\mathsf{V}}^{e(S)}_{\varnothing\varnothing\varnothing} \cdot \left(\sum_{\lambda} q^{|\lambda|}\right)^{e(B)-e(S)} \cdot \left(\sum_{\mu} \frac{\widetilde{\mathsf{V}}_{\mu'\mu\varnothing}}{\widetilde{\mathsf{V}}_{\varnothing\varnothing\varnothing}} q^{|\mu|}\right)^{e(S)}.$$

Sustituting in the equations in lemma 14, using the well known generating function for 2D-partitions, and applying [4, eqn (1)] we get

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = M(p)^{e(S)} \cdot \left(\prod_{d=1}^{\infty} (1-q^d)^{-1}\right)^{e(B)-e(S)} \cdot \left(\prod_{d=1}^{\infty} (1-q^d)^{-1} M(p,q^d)\right)^{e(S)}$$

which is easily seen to be equivalent to the formula for $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ given in theorem 1. □

8. Putting in the Behrend function

The aim of this section is to prove theorem 3, which says that up to an overall sign, the partition functions $\widehat{\mathsf{DT}}(X)$ and $\mathsf{DT}(X)$ are equal after the simple change of variables $y = -p$. In order to do this we will need to assume a conjecture about the Behrend function which we formulate below for general Calabi-Yau threefolds and may be of independent interest.

Let Y be any quasi-projective Calabi-Yau threefold. Let $C \subset Y$ be a (not necessarily reduced) Cohen-Macaulay curve with proper support. Assume that the singularities of C_{red} are locally toric[4]. Recall that by definition 4,

$$\mathrm{Hilb}^n(Y,C) = \{Z \subset Y \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n\}.$$

Note that $\mathrm{Hilb}^n(Y,C) \subset \mathrm{Hilb}(Y)$ and let ν denote the Behrend function on $\mathrm{Hilb}(Y)$. Our conjecture is the following:

Conjecture 18.

$$\int_{\mathrm{Hilb}^n(Y,C)} \nu \, de = (-1)^n \nu([C]) \int_{\mathrm{Hilb}^n(Y,C)} de$$

where $\nu([C])$ is the value of the Behrend function at the point $[C] \in \mathrm{Hilb}(Y)$.

Remark 19. Conceivably, the condition that C_{red} has locally toric singularities could be weakened, although we do not have any evidence for this case. Our conjecture is true for Y a (globally) toric Calabi-Yau. This follows from the computations in [10].

[4]This means that formally locally C_{red} is either smooth, nodal, or the union of the three coordinate axes. That is at $p \in C_{\mathrm{red}} \subset Y$ the ideal $\widehat{I}_{C_{\mathrm{red}}} \subset \widehat{\mathcal{O}}_{Y,p}$ is given by (x_1, x_2), (x_1, x_2x_3), or (x_1x_2, x_2x_3, x_1x_3) for some isomorphism $\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[[x_1, x_2, x_3]]$.

One could also make the much stronger conjecture that

$$\nu([Z]) = (-1)^n \nu([C])$$

for all $[Z] \in \mathrm{Hilb}^n(Y, c)$. This would of course imply our conjecture as stated. However, we do not know if this stronger version holds, even in the case where Y is $\mathbb{C}^3$ and C is empty. In this case, this stronger conjecture says that the Behrend function on $\mathrm{Hilb}^n(\mathbb{C}^3)$ is the constant function $(-1)^n$.

8.1. **Proof of theorem 3.** The Behrend function of any scheme is invariant under automorphisms. In particular, it is constant on the orbits of the $\mathbb{C}^*$ action on $\mathrm{Hilb}(X)$. We thus have

$$\mathsf{DT}(X) = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)} \nu\, de = \int_{\mathrm{Hilb}^{\bullet,\bullet}(X)^{\mathbb{C}^*}} \nu\, de = \int_{\mathrm{Hilb}^{\bullet,\bullet}_{\mathsf{PCP}}(X)} \nu\, de$$

and so

$$\mathsf{DT}(X) = \int_{\mathrm{Sym}^{\bullet}(B)} (\rho_{\bullet})_*(\nu)\, de.$$

Let $f^{\nu}_d = (\rho_d)_*(\nu)$ so that in the notation of section 4, we have

$$f^{\nu}_d(\boldsymbol{ax} + \boldsymbol{by}) = \int_{\rho_d^{-1}(\boldsymbol{ax}+\boldsymbol{by})} \nu\, de.$$

Recall that for the partition function $\mathsf{DT}(X)$, the variable tracking the holomorphic Euler characteristic is y rather than p so $f^{\nu}_d(\boldsymbol{ax} + \boldsymbol{by}) \in \mathbb{Z}((y))$.

By equation (3) we have

$$f^{\nu}_d(\boldsymbol{ax} + \boldsymbol{by}) = \sum_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \sum_{\boldsymbol{\mu} \vdash \boldsymbol{b}} \int_{\Sigma^{\bullet}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\mu})} \nu\, de.$$

Recall that $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ parameterizes subschemes $Z \subset X$ such that the maximal Cohen-Macaulay subscheme of Z is the partition thickened comb curve

$$C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right) \cup_j \left(\mu^{(j)} F_{y_j}\right)$$

In other words, $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is essentially equal to the local Hilbert scheme $\mathrm{Hilb}^{\bullet}(X, C)$ (definition 4) except that the later is indexed by the length of I_C/I_Z whereas $\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is indexed by $\chi(\mathcal{O}_Z)$. Consequently

$$\Sigma^{\bullet}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = y^{\chi(\mathcal{O}_C)}\, \mathrm{Hilb}^{\bullet}(X, C)$$

and so

$$f^{\nu}_d(\boldsymbol{ax} + \boldsymbol{by}) = \sum_{\boldsymbol{\lambda} \vdash \boldsymbol{a}} \sum_{\boldsymbol{\mu} \vdash \boldsymbol{b}} y^{\chi(\mathcal{O}_C)} \sum_{n=0}^{\infty} y^n \int_{\mathrm{Hilb}^n(X,C)} \nu\, de.$$

We apply conjecture 18 to the above and also use $\nu([C]) = (-1)^{\chi(\mathcal{O}_S)-\chi(\mathcal{O}_C)}$, the highly non-trivial result given in corollary 23 and proved in the next section.

$$\begin{aligned} f_d^{\nu}(\boldsymbol{ax}+\boldsymbol{by}) &= \sum_{\boldsymbol{\lambda}\vdash \boldsymbol{a}}\sum_{\boldsymbol{\mu}\vdash \boldsymbol{b}} y^{\chi(\mathcal{O}_C)} \sum_{n=0}^{\infty} y^n(-1)^n \nu([C]) \int_{\mathrm{Hilb}^n(X,C)} de \\ &= (-1)^{\chi(\mathcal{O}_S)} \sum_{\boldsymbol{\lambda}\vdash \boldsymbol{a}}\sum_{\boldsymbol{\mu}\vdash \boldsymbol{b}} \sum_{n=0}^{\infty} (-y)^{\chi(\mathcal{O}_C)+n} \int_{\mathrm{Hilb}^n(X,C)} de. \end{aligned}$$

After the substitution $-y = p$, we find that the above is equivalent to

$$f_d^{\nu}(\boldsymbol{ax}+\boldsymbol{by}) = (-1)^{\chi(\mathcal{O}_S)} f_d(\boldsymbol{ax}+\boldsymbol{by}).$$

It follows that

$$\mathsf{DT}(X) = (-1)^{\chi(\mathcal{O}_S)} \widehat{\mathsf{DT}}(X)$$

after the substitution $p = -y$ as asserted by theorem 3.

The case of $\mathsf{DT}_{\mathsf{fib}}(X)$ (previously shown by Toda) proceeds similarly except that it doesn't require the difficult deformation result of next section. Indeed, in this case, we only need to know that the value of the Behrend function at a partition thicked fiber curve is 1:

$$\nu\left(\left[\cup_i \left(\lambda^{(i)} F_{x_i}\right)\right]\right) = 1.$$

This follows from the fact that subschemes in X of the form $p^{-1}(Z)$ where Z is a zero-dimensional subscheme of T form a component of $\mathrm{Hilb}(X)$ which is isomorphic to the Hilbert scheme of points on T and hence smooth and even dimensional.

9. Smoothness and infinitesimal deformations

In this section we show that the locus of partition thickened comb curves lies in the non-singular locus of $\mathrm{Hilb}(X)$ and we compute the dimension of $\mathrm{Hilb}(X)$ at those points. As a corollary, we determine the value of the Behrend function at the points of the Hilbert scheme corresponding to partition thickened comb curves. This is the key technical result require in section 8 to promote our computation of $\widehat{\mathsf{DT}}(X)$ to the Behrend function version $\mathsf{DT}(X)$.

We begin by stating the three main results of this section.

Theorem 20. *Let $B \subset T$ be a smooth curve contained in a smooth surface T. Define $V_l \subset \mathrm{Hilb}^d(T)$ to be the locus of points parameterizing dimension 0 subschemes $Z \subset T$ of length n such that the length of the scheme theoretic intersection $Z \cap B$ is l. Then V_l is locally closed and smooth of dimension $2d - l$.*

Theorem 21. *Let $\lambda^{(1)}, \dots, \lambda^{(n)}$ be partitions and let $C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right)$ be a partition thickened comb curve. The the Zariski tangent space of* $\mathrm{Hilb}(X)$ *at the point* $[C]$*, which is given by* $\mathrm{Hom}(I_C, \mathcal{O}_C) \cong \mathrm{Ext}^1_0(I_C, I_C)$*, has dimension*

$$h^0(N_{B/T}) + \sum_{i=1}^{n} \left(2|\lambda^{(i)}| - \lambda_1^{(i)}\right).$$

Theorem 22. *The locus of partition thickened comb curves is contained in the non-singular locus of* $\mathrm{Hilb}(X)$.

Corollary 23. *The value of the Berhend function at* $[C] \in \mathrm{Hilb}(X)$ *for a partition thickened comb curve* $C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right)$ *is given by*

$$\nu([C]) = (-1)^{\chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)}$$

Proof. By [1], the Behrend function on a smooth scheme V is $(-1)^{\dim V}$ and so by theorems 22 and 21

$$\nu([C]) = (-1)^{h^0(N_{B/X})} \prod_{i=1}^{n} (-1)^{\lambda_1^{(i)}}.$$

Lemma 26 and lemma 12 say that

$$h^0(N_{B/X}) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_B), \quad \chi(\mathcal{O}_C) = \chi(\mathcal{O}_B) - \sum_i \lambda_1^{(i)}$$

which, when substituted into the above prove the corollary. □

The most difficult of the above results is theorem 21 and its proof occupies the majority of this section.

Our method for computing the dimension of deformation spaces is an adaption of Haiman's method for computing infinitesimal deformations of 0-dimensional subschemes on a surface [7]. Indeed, the proof of Theorem 20 follows directly using Haiman's argument. For Theorem 21, we use Haiman's method to study local deformations of C in the formal neighborhoods of the points x_i, but we use the global geometry to keep track of which local deformations extend.

9.1. **Notation and setup for proof of Theorem 21.** For notational simplicity we first treat the case where there is a single parititon thickened fiber $F = F_x$ at $x \in B$, that is

$$C = B \cup \lambda F$$

where $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l)$ is an integer partition of length l.

Consider the divisors

$$S, \quad R = \mathrm{Tot}(K_S|_F), \quad T = \mathrm{Tot}(K_S|_B)$$

and let (r,s,t) be formal local coordinates at x such that

$$R=\{r=0\}\,,\quad S=\{s=0\}\,,\quad T=\{t=0\}\,.$$

The formal local ring $\widehat{\mathcal{O}}_{X,x}\cong\mathbb{C}[[r,s,t]]$ has a basis as a $\mathbb{C}$-vector space given by monomials $\{r^\rho s^\sigma t^\tau\}$ for $(\rho,\sigma,\tau)\in(\mathbb{Z}_{\geq 0})^3$. We visualize these basis vectors as unit cubes in the positive octant of $\mathbb{R}^3$ with the monomial $r^\rho s^\sigma t^\tau$ corresponding to the cube whose corner closest to the origin is at (ρ,σ,τ).

9.2. **Exact sequences.** The ideal sheaf I_C has a locally free resolution of the form

$$0\to\oplus_\beta R_\beta\to\oplus_\alpha G_\alpha\to I_C\to 0 \tag{6}$$

where G_α (the "generators") and R_β (the "relations") are of the form $\mathcal{O}(-\rho R-\sigma S-\tau T)$.

Indeed, we can explicitly take the collection of (ρ,σ,τ) for G_α to be

$$\{(\lambda_1,0,1),(\lambda_2,1,0),(\lambda_3,2,0),\dots,(\lambda_l,l-1,0),(0,l,0)\}\,.$$

We also have the sequence

$$0\to\mathcal{O}_C\to\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}\to\mathcal{O}_{\lambda_1 x}\to 0$$

where $\lambda_1 x=B\cap\lambda F$ is the length λ_1 subscheme of B supported at x.

By standard homological algebra, we have that $\operatorname{Hom}(I_C,\mathcal{O}_C)$ is H^0 of the complex

$$\operatorname{Hom}\left([\oplus_\beta R_\beta\to\oplus_\alpha G_\alpha]\,,[\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}\to\mathcal{O}_{\lambda_1 x}]\right).$$

Namely, we have that $\operatorname{Hom}(I_C,\mathcal{O}_C)$ is given by the kernel of the map

$$\operatorname{Hom}(\oplus_\alpha G_\alpha,\mathcal{O}_B\oplus\mathcal{O}_{\lambda F})\xrightarrow{\Phi_1\oplus\Phi_2}\operatorname{Hom}(\oplus_\alpha G_\alpha,\mathcal{O}_{\lambda_1 x})\oplus\operatorname{Hom}(\oplus_\beta R_\beta,\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}).$$

This identification of $\operatorname{Hom}(I_C,\mathcal{O}_C)$ has a straight forward interpretation: a homomorphism $I_C\to\mathcal{O}_C$ is determined by what it is on each of the generators of I_C, considered as maps to $\mathcal{O}_B$ and to $\mathcal{O}_{\lambda F}$. To be in the kernel of Φ_1 just means that these maps should agree on $B\cap\lambda F$ and to be in the kernel of Φ_2 means that the images must obey the module relations. We will make this combinatorially more explicit by studying the restriction of the homomorphisms $\oplus_\alpha G_\alpha\to\mathcal{O}_B\oplus\mathcal{O}_{\lambda F}$ to $\widehat{X}_x\cong\operatorname{Spec}\mathbb{C}[[r,s,t]]$.

9.3. **Combinatorics of Haiman arrows.** Restricted to the local ring $\widehat{\mathcal{O}}_{X,x}\cong\mathbb{C}[[r,s,t]]$, $\mathcal{O}_C$ is spanned over $\mathbb{C}$ by the monomials $r^\rho s^\sigma t^\tau$ where (ρ,σ,τ) are of the form $(\rho,0,0)$ or $(\rho,\sigma,\tau)_{(\rho,\sigma)\in\lambda}$ and I_C is spanned by the complementary monomials. As previously discussed, we view these monomials as cubes in the positive octant, see figure 2.

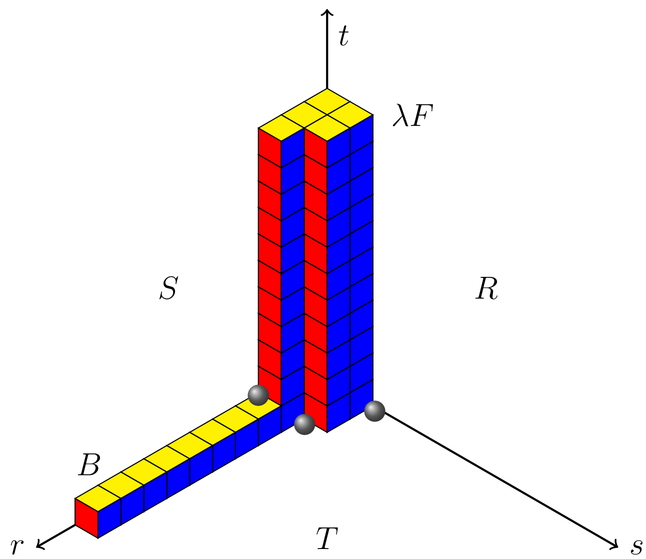

FIGURE 2. Monomials in the local ring $\widehat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[r,s,t]]$ are represented by cubes. Cubes shown are the monomials in $\widehat{\mathcal{O}}_{C,x}$. The gray balls are located at monomials which generate $\widehat{I}_{C,x}$

We call the cubes corresponding to $(\rho,0,0)$ and $(\rho,\sigma,\tau)_{(\rho,\sigma)\in\lambda}$ the B-cubes and λF-cubes respectively and the cubes in the union are called C-cubes. The complement of the C-cubes are the I_C-cubes.

A **Haiman arrow**

$$(\rho,\sigma,\tau)\to(\rho',\sigma',\tau')$$

is a vector whose tail (ρ,σ,τ) is an I_C-cube and whose head (ρ',σ',τ') is a C-cube.

The Haiman arrows form a basis for the $\mathbb{C}$-linear maps from $\widehat{I}_{C,x}$ to $\widehat{\mathcal{O}}_{C,x}$. We wish to determine a basis for $\operatorname{Hom}(I_C,\mathcal{O}_C)$ in terms of Haiman arrows.

The generators of I_C correspond to the cubes in the corners of the set of I_C cubes. They are located at (ρ,σ,τ) where (ρ,σ) are the corners just outside of λ and $\tau=0$ unless $\sigma=0$ in which case $\tau=1$ (they are indicated by the grey balls in figure 2). A generator at (ρ,σ,τ) corresponds to the image of $G_\alpha\to\mathcal{O}$ where $G_\alpha\cong\mathcal{O}(-\rho R-\sigma S-\tau T)$. The summands of

$$\operatorname{Hom}(\oplus_\alpha G_\alpha,\mathcal{O}_B\oplus\mathcal{O}_{\lambda F})\cong\oplus_\alpha H^0(B,G_\alpha^\vee\otimes\mathcal{O}_B)\oplus H^0(F,G_\alpha^\vee\otimes\mathcal{O}_{\lambda F})$$

are interpreted as follows. For $G_\alpha\cong\mathcal{O}(-\rho R-\sigma S-\tau T)$, a homomorphism in $\operatorname{Hom}(G_\alpha,\mathcal{O}_B)$ or $\operatorname{Hom}(G_\alpha,\mathcal{O}_{\lambda F})$ is determined by a Haiman arrow from (ρ,σ,τ) to (respectively) some B-cube or λF-cube (ρ',σ',τ'). The location of the head of such a Haiman arrow is determined by the order of vanishing of the corresponding section of $H^0(B, G_\alpha^\vee \otimes \mathcal{O}_B)$ or $H^0(F, G_\alpha^\vee \otimes \mathcal{O}_{\lambda F})$ — the head will occur at (ρ', σ', τ') if the corresponding section is order $r^{\rho'} s^{\sigma'} t^{\tau'}$.

We wish to determine a basis for $\mathrm{Hom}(I_C, \mathcal{O}_C) \cong \mathrm{Ker}(\Phi_1 \oplus \Phi_2)$ in terms of Haiman arrows. To be in the kernel of Φ_1 just means that a Haiman arrow whose head is both a B-cube and a λF-cube must arise as sections of both $H^0(B, G_\alpha^\vee \otimes \mathcal{O}_B)$ and $H^0(F, G_\alpha^\vee \otimes \mathcal{O}_{\lambda F})$. As for the kernel of Φ_2, the key observation is the following:

Remark 24. The equations defining the kernel of Φ_2 equate two Haiman arrows which are obtained from one another by translation through other Haiman arrows. Moreover, if a Haiman arrow can be translated so that its head passes into an octant with a negative coordinate (without its tail ever leaving the I_C-cubes) then in must be zero.

We now analyze the possible equivalence classes of Haiman arrows.

9.4. **Haiman arrows to λF-cubes.** Let $G_\alpha \cong \mathcal{O}(-\rho R - \sigma S - \tau T)$ be a generating line bundle and consider the sections $H^0(F, G_\alpha^\vee \otimes \mathcal{O}_{\lambda F})$. These correspond to the possible Haiman arrows $(\rho, \sigma, \tau) \to (\rho', \sigma', \tau')$ to λF-cubes. Since the normal bundle of F in X is trivial, $\mathcal{O}(R)$ and $\mathcal{O}(S)$ are trivial restricted to F. Thus

$$G_\alpha^\vee \otimes \mathcal{O}_{\lambda F} \cong \mathcal{O}_{\lambda F}(\rho R + \sigma S + \tau T) \cong \mathcal{O}_{\lambda F}(\tau x).$$

Since τ is either 0 or 1 for the generators G_α, the Haiman arrows correspond to

$$H^0(F, \mathcal{O}_{\lambda F}) \text{ if } \tau = 0, \quad H^0(F, \mathcal{O}_{\lambda F}(x)) \text{ if } \tau = 1.$$

In both cases, this space has a basis of sections which in the local coordinates are given by $\{r^{\rho'} s^{\sigma'} t^{\tau}\}_{(\rho', \sigma') \in \lambda}$. Note that the section we consider above are uniquely determined by their value on the formal neighborhood $\widehat{X}_x \cong \mathrm{Spec}\, \mathbb{C}[[r, s, t]]$, a property which uses crucially the fact that the genus of F is 1.

We have seen that the possible Haiman arrows to λF-cubes are given by

$$(\rho, \sigma, \tau) \to (\rho', \sigma', \tau')$$

where (ρ, σ, τ) is a generating cube, $\tau' = \tau$ and $(\rho', \sigma') \in \lambda$. In particular, the direction of the arrows is horizontal since there is no τ component in $(\rho - \rho', \sigma - \sigma', 0)$.

Since all the Haiman arrows to λF-cubes are horizontal, we view them from above in the (r, s) plane (see figure 3).

If the direction of the Haiman arrow is strictly southwest (i.e. it has strictly negative ρ and σ components), then by translating (see Remark 24) along the contour of λ to the edge of the s-axis, the arrow can be equated

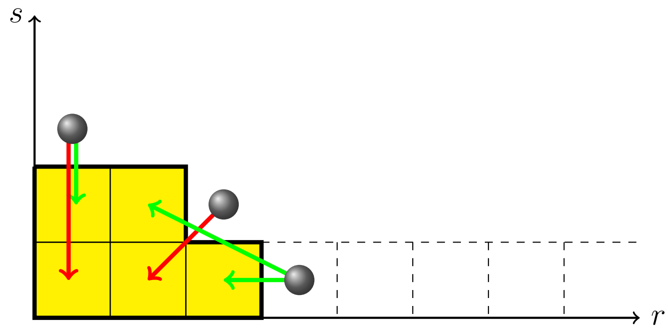

FIGURE 3. Examples of Haiman arrows to λF-cubes. The green arrows are non-zero and the red arrows are necessarily zero.

to an arrow whose head has a negative ρ component and is thus zero. There are no strictly Northeast pointing Haiman arrows, so all non-zero Haiman arrows must be weakly northwest pointing or weakly southeast pointing. Translating a weakly northwest pointing arrow as far to the northwest as possible, we find that its head will either cross the s-axis (and hence be 0) or it will be at the top of a column of λ and its tail just outside a row. Indeed, for each square in λ, there is exactly one equivalence class of weakly northwest pointing Haiman arrows represented by the arrow going from just outside the box's row to the top of the box's column. Similarly, there is one equivalence class of weakly southeast pointing Haiman arrows for each box in λ representated by the arrow going from just outside the top of the box's column to the end of the box's row. This accounts for exactly $2|\lambda|$ different equivalence classes of Haiman arrows to λF-cubes. However, λ_1 of these arrows have their head a B-cube, namely the southeast pointing arrows whose tails are just above the top of λ and whose head is the last square in the first row of λ[5].

We thus have exactly $2|\lambda| - \lambda_1$ distinct equivalence classes of Haiman arrows to λF-cubes which are not also arrows to B-cubes.

9.5. **Haiman arrows to B-cubes.** Any non-zero Haiman arrow to a B-cube must have a tail with coordinates $(\rho, 0, 1)$ or $(\rho, 1, 0)$ since if not, it could be translated (see Remark 24) to an arrow whose head has negative τ or σ coordinates by first translating sufficiently far in the positive ρ direction and then translating the tail so that it is just outside of the B-cubes. A

[5]Note that the northwest pointing Haiman arrows whose heads are in the first row of λ are necessarily strictly west pointing and hence originate at the generator whose τ component is 1. Therefore the head of these arrows also have τ component 1 and so they are not B-cubes.

Haiman arrow to a B-cube whose tail is $(\rho,0,1)$ or $(\rho,1,0)$ corresponds respectively to a section in $H^0(B,\mathcal{O}_B(\rho R+T))$ or $H^0(B,\mathcal{O}_B(\rho R+S))$. Since

$$\mathcal{O}_B(R)\cong\mathcal{O}_B(x),\quad \mathcal{O}_B(T)\cong N_{B/S},\quad \mathcal{O}_B(S)\cong N_{B/T},$$

we see that the Haiman arrows from $(\rho,0,1)$ or $(\rho,1,0)$ to B-cubes are given by

$$H^0(B,N_{B/S}(\rho x))\quad\text{or}\quad H^0(B,N_{B/T}(\rho x))$$

respectively. The head of such a Haiman arrow is $(\rho',0,0)$ where ρ' is the order of vanishing at x of the corresponding section.

Lemma 25. *Let $(\rho,0,1)\to(\rho',0,0)$ or $(\rho,1,0)\to(\rho',0,0)$ be a non-zero Haiman arrow. Then $\rho'\geq\rho$.*

Proof. Consider a Haiman arrow $(\rho,0,1)\to(\rho',0,0)$ with $\rho'<\rho$. Then this arrow can be translated so that its head is a λF-cube, however we saw in the previous subsection that Haiman arrows to λF-cubes must be horizontal and so this must be zero. Consider next a Haiman arrow $(\rho,1,0)\to(\rho',0,0)$ with $\rho'<\rho$. Then this arrow maybe translated so that it is a strictly southwest pointing Haiman arrow to an λF-cube which we showed in the previous subsection must be zero.[6] □

By the lemma, we conclude that the only sections of $H^0(B,N_{B/S}(\rho x))$ or $H^0(B,N_{B/T}(\rho x))$ which correspond to non-zero Haiman arrows vanish to order at least ρ at x, and thus they are necessarily in the image of the maps

$$H^0(B,N_{B/S})\to H^0(B,N_{B/S}(\rho x))$$
$$H^0(B,N_{B/T})\to H^0(B,N_{B/T}(\rho x))$$

By lemma 26, $H^0(B,N_{B/S})=0$. On the other hand, $H^0(B,N_{B/T})$ can be non-zero and these deformations do occur, they correspond to global deformations of B in the K_S direction.

In conclusion, we have completely classified all possible Haiman arrows upto equivalence and have thus constructed an explicit basis for $\operatorname{Hom}(I_C,\mathcal{O}_C)\cong\operatorname{Ker}(\Phi_1\oplus\Phi_2)$. They consist of the $2|\lambda|-\lambda_1$ Haiman arrows to λF-cubes which do not go to B-cubes and the $h^0(B,N_{B/T})=h^0(B,N_{B/X})$ dimensional space of Haiman arrows going to B-cubes. We've thus proved that

$$\dim\operatorname{Hom}(I_C,\mathcal{O}_C)=h^0(B,N_{B/T})+2|\lambda|-\lambda_1$$

for $C=B\cup\lambda F$. Our argument extends essentially word for word to the case where $C=B\cup_i(\lambda^{(i)}F_{x_i})$ has several partition thickened fibers.

[6] Are you still with us dear reader? We are deep in the weeds now, but are almost done.

Whether the fiber is smooth or nodal plays no role. We have thus proved Theorem 21. $\square$

9.6. **Proof of theorem 22.** Let $C = B \cup_{i=1}^{n} \left(\lambda^{(i)} F_{x_i}\right)$ be a partition thickened comb curve and let

$$d = \sum_{i=1}^{n} |\lambda^{(i)}|, \quad l = \sum_{i=1}^{n} \lambda_1^{(i)}.$$

To prove theorem 22 it will suffice to construct a flat family of distinct subschemes of X, containing C as a member, and over a base W which is smooth and of dimension

$$h^0(N_{B/T}) + 2d - l.$$

Indeed, theorem 21 then implies that the induced injective map $W \to \mathrm{Hilb}(X)$ is a local isomorphism and the assertion of theorem 22 follows. Let

$$H^0 := H^0(B, N_{B/T}),$$

and let

$$V_l \subset \mathrm{Hilb}^d(T)$$

be the strata defined in theorem 20. Let $W = H^0 \times V_l$ so by theorem 20, W is smooth and of dimension $h^0(N_{B/T}) + 2d - l$. We wish to construct a family over W of distinct subschemes.

Since $T = \mathrm{Tot}(N_{B/T})$, given $\theta \in H^0$, we get an automorphism of T, which we call Θ, given by $(p, v) \mapsto (p, v + \theta(p))$ where $p \in B$ and $v \in T|_p$.

We will construct a family of subschemes of X, flat over the base $H^0 \times V_l$, which over a point $(\theta, Z) \in H^0 \times V_l$ is the subscheme

$$C_\theta = \Theta(B) \cup p^{-1}(\Theta(Z)).$$

Clearly, all such subschemes are distinct, and moreover, every partition thickened comb curve is of the above form (with $\theta = 0$ so that $\Theta = Id$).

Formally, we construct a universal subscheme

$$\mathcal{C} \subset H^0 \times V_l \times X$$

flat over $H^0 \times V_l$ as follows. Consider the diagram:

$$\begin{tikzcd}
 & H^0 \times V_l \times X \arrow[d, shift left, "p"] & & \\
 & H^0 \times V_l \times T \arrow[u, shift left, "i"] \arrow[r, "\Theta"] \arrow[d, "\pi_V"] & H^0 \times V_l \times T \arrow[d, "\pi_H"] & \\
\mathcal{B} \arrow[r, hook] & H^0 \times T & V_l \times T & \mathcal{Z} \arrow[l, hook']
\end{tikzcd}$$

In the above diagram, $\mathcal{Z} \subset V_l \times T$ is the family subschemes of T by the universal subscheme over $\mathrm{Hilb}^d(T)$, $\mathcal{B} \subset H^0 \times T$ is the family of curves in T given by H^0, explicitly given by the set of points $(\theta, p, \theta(p))$. The maps π_V and π_H are the obvious projections and the maps p and i are the projection and the zero section of the elliptic fibration $X \to T$. We are also adopting the general abuse of notation that we drop factors of the identity map from the notation, that is if $f : A \to B$ we denote also by f the map $f \times Id_C : A \times C \to B \times C$.

Then the subscheme

$$\mathcal{C} = i(\pi_V^{-1}(\mathcal{B})) \cup (p \circ \Theta \circ \pi_H)^{-1}(\mathcal{Z}) \subset H^0 \times V_l \times X$$

is the desired universal subscheme over $H^0 \times V$. □

9.7. **Proof of theorem 20.** .

Appendix A. Odds and Ends

A.1. **Elliptic surfaces.** Let $p : S \to B$ be a non-trivial elliptic surface with section $B \subset S$. For simplicity, we assume that all singular fibers are irreducible nodal rational curves.

Let $X = \mathrm{Tot}(K_S)$ and let $T = \mathrm{Tot}(K_S|B)$, then we have

$$N_{B/X} \cong N_{B/S} \oplus N_{B/T}.$$

Lemma 26. *$h^0(N_{B/S}) = 0$ and $h^0(N_{B/T}) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_B)$.*

Proof. By a well known fact about elliptic surfaces (see [6, ???] or [11, III.1.1]),

$$K_S \cong p^*(K_B \otimes L)$$

where

$$L^\vee = R^1 p_* \mathcal{O}_S.$$

Consequently, $c_1(K_S)^2 = 0$ and so Hirzebruch-Riemann-Roch says

$$\chi(\mathcal{O}_S) = \frac{e(S)}{12} > 0$$

where positivity of $e(S)$ follows by pushing forward the Euler characteristic measure on S to B:

$$e(S) = \int_S de = \int_B p_*(1)\, de = \text{\# of singular fibers}.$$

On the other hand

$$\begin{aligned} \chi(\mathcal{O}_S) &= \chi(R^\bullet p_* \mathcal{O}_S) \\ &= \chi(\mathcal{O}_B) - \chi(L^\vee) \\ &= \deg(L). \end{aligned}$$

By adjunction
$$N_{B/S} \cong (K_S^\vee|_B) \otimes K_B \cong L^\vee.$$
Thus $\deg(N_{B/S}) = \deg(L^\vee) = -\chi(\mathcal{O}_S) < 0$ and so $h^0(N_{B/S}) = 0$.

Since
$$N_{B/T} = K_S|_B = K_B \otimes L$$
we see that
$$h^1(N_{B/T}) = h^1(K_B \otimes L) = h^0(L^\vee) = h^0(N_{B/S}) = 0,$$
and therefore
$$\begin{aligned} h^0(N_{B/T}) &= \chi(N_{B/T}) \\ &= \deg(K_B) + \deg(L) + 1 - g(B) \\ &= \chi(\mathcal{O}_S) + g - 1 \\ &= \chi(\mathcal{O}_S) - \chi(\mathcal{O}_B). \end{aligned}$$
□

By our assumption that S is not a product,
$$p^* : \mathrm{Pic}^0(B) \xrightarrow{\cong} \mathrm{Pic}^0(S)$$
is an isomorphism [11, VII.1.1]. For any $\beta \in H_2(S)$, we denote by $\mathrm{Hilb}^\beta(S)$ the Hilbert scheme of effective divisors on S in class β.

Denote by $B \in H_2(S)$ the class of the section $B \subset S$ and by $F \in H_2(S)$ the class of the fibre. Then we have the following commutative diagram
$$\begin{tikzcd} \mathrm{Sym}^d(B) \arrow[r] \arrow[d, "p^*"] & \mathrm{Pic}^d(B) \arrow[d, "p^*", "\cong"'] \\ \mathrm{Hilb}^{dF}(S) \arrow[r] \arrow[d, "+B"] & \mathrm{Pic}^{dF}(S) \arrow[d, "\otimes \mathcal{O}_S(B)", "\cong"'] \\ \mathrm{Hilb}^{B+dF}(S) \arrow[r] & \mathrm{Pic}^{B+dF}(S). \end{tikzcd}$$
The horizontal arrows are Abel-Jacobi maps. The vertical arrows are induced by pull-back and adding the section $B \subset S$.

Lemma 27. *The above maps induce an isomorphism*
$$\mathrm{Sym}^d(B) \xrightarrow{\cong} \mathrm{Hilb}^{B+dF}(S).$$

Proof. Clearly p^* gives an isomorphism $\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{dF}(S)$ and $+B$ defines a closed embedding $\mathrm{Hilb}^{dF}(S) \hookrightarrow \mathrm{Hilb}^{B+dF}(S)$. Since $\mathrm{Sym}^d(B)$ is smooth and $\mathrm{Hilb}^{B+dF}(S)$ is reduced (by [12, Lect. 25]), it suffices to show
$$\mathrm{Sym}^d(B) \to \mathrm{Hilb}^{B+dF}(S)$$

is surjective on closed points.

For surjectivity, suppose D' is an effective divisor with class $B + dF$ which does *not* lie in the image. Firstly, we note that for any fibre F we have $D' \cdot F = 1$. Therefore D' contains a section $B' \subset S$ as an effective summand. Moreover $B \neq B'$ or else D' would lie in the image. Next, we take any D in the image and compare D and D'. Then

$$\mathcal{O}_S(D - D') \in \operatorname{Pic}^0(S) \cong \operatorname{Pic}^0(B).$$

Therefore after re-arranging we find that there are distinct fibres F_{x_i}, F_{y_j} and $a_i \geq 0$, $b_j \geq 0$ such that

$$B + \sum_i a_i F_{x_i} \sim_{\text{lin}} B' + \sum_j b_j F_{y_j},$$

where $\sim_{\text{lin}}$ denotes linear equivalence. Hence there exists a pencil $\{C_t\}_{t \in \mathbb{P}^1}$ of effective divisors such that

$$C_0 = B + \sum_i a_i F_{x_i},\ C_\infty = B' + \sum_j b_j F_{y_j}.$$

Now fix a smooth fibre F. Then $C_t \cdot F = 1$ for any $t \in \mathbb{P}^1$, so we get a morphism

$$\mathbb{P}^1 \longrightarrow F,\ t \mapsto C_t \cap F.$$

But F is a smooth elliptic curve so this map is constant. We conclude

$$B \cap F = C_0 \cap F = C_\infty \cap F = B' \cap F.$$

Since F was chosen arbitrary, we deduce that $B = B'$ which is a contradiction. $\square$

A.2. **Weighted Euler characteristics of symmetric products.** In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

Lemma 28. *Let B be a scheme of finite type over $\mathbb{C}$ and let $e(B)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ be any function with $g(0) = 1$. Let $G : \operatorname{Sym}^d(B) \to \mathbb{Z}(\!(p)\!)$ be the constructible function defined by*

$$G(\boldsymbol{ax}) = \prod_i g(a_i),$$

for all $\boldsymbol{ax} = \sum_i a_i x_i \in \operatorname{Sym}^d(B)$ where $x_i \in B$ are distinct closed points. Then

$$\sum_{d=0}^{\infty} q^d \int_{\operatorname{Sym}^d(B)} f_d \, de = \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)}.$$

Remark 29. In the special case where $g = f_d \equiv 1$, the lemma recovers MacDonald's formula:

$$\sum_{d=0}^{\infty} e(\mathrm{Sym}^d(B))\, q^d = \frac{1}{(1-q)^{e(B)}}.$$

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties definited by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure. For convenience, we provide a direct proof here.

Proof. The dth symmetric product admits a stratification with strata labelled by partitions of d. Associated to any partition of d is a unique tuple $(m_1, m_2, \dots)$ of non-negative integers with $\sum_{j=1}^{\infty} j m_j = d$. The stratum labelled by $(m_1, m_2, \dots)$ parameterizes collections of points where there are m_j points of multiplicity j. The full stratification is given by:

$$\mathrm{Sym}^d(B) = \bigsqcup_{\substack{(m_1,m_2,\dots)\\ \sum_{j=1}^{\infty} j m_j = d}} \left\{ \left(\prod_{j=1}^{\infty} B^{m_j} \right) - \Delta \right\} \Big/ \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, B^0 is a point, Δ is the large diagonal, and σ_m is the mth symmetric group. Note that the function f_d is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

For schemes over $\mathbb{C}$, topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\mathrm{Sym}^d(B)} f_d \, de = \sum_{\substack{(m_1,m_2,\dots)\\ \sum_{j=1}^{\infty} j m_j = d}} \left(\prod_{j=1}^{\infty} g(j)^{m_j} \right) \frac{e(B^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \dots}.$$

For any natural number N, the projection $B^N - \Delta \to B^{N-1} - \Delta$ has fibers of the form $B - \{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(B) - (N-1)$ and consequently, $e(B^N - \Delta) = (e(B) - (N-1)) \cdot e(B^{N-1} - \Delta)$. Thus by induction, we find $e(B^N - \Delta) = e(B) \cdot (e(B) - 1) \cdots (e(B) - (N-1))$ and so

$$\frac{e(B^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \cdots} = \binom{e(B)}{m_1, m_2, m_3, \cdots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\sum_{d=0}^{\infty} q^d \int_{\mathrm{Sym}^d(B)} f_d \, de = \sum_{(m_1,m_2,\dots)} \prod_{j=1}^{\infty} \left(g(j)q^j\right)^{m_j} \binom{e(B)}{m_1, m_2, m_3, \dots}$$
$$= \left(1 + \sum_{j=1}^{\infty} g(j)q^j\right)^{e(B)}$$

which proves the lemma. □

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