Proposition 0.1. *Let $Z \hookrightarrow X$ be a regular embedding of $\mathbb{C}$-schemes of finite type, and let $\mathcal{I}$ be the ideal sheaf of Z. Let $\widehat{X}_Z$ be the formal neighbourhood and $N_{Z/X}$ the normal bundle of Z in X. Denote by $\widehat{N_{Z/X}}_Z$ the formal neighbourhood of the zero-section Z in $N_{Z/X}$. Then there exists an isomorphism of sheaves of ring $\mathcal{O}_{\widehat{X}_Z} \cong \mathcal{O}_{\widehat{N_{Z/X}}_Z}$ commuting with the structure maps to $\mathcal{O}_Z$ if and only if for each $r \geq 1$ there exists a morphism of sheaves of abelian groups*

$$s : \mathcal{O}_X/\mathcal{I}^r \longrightarrow \mathcal{O}_X/\mathcal{I}^{r+1}$$

with the following properties:

(1) *s is a section of the quotient map $\pi : \mathcal{O}_X/\mathcal{I}^{r+1} \longrightarrow \mathcal{O}_X/\mathcal{I}^r$,*
(2) *for each $p \in X$ at the level of stalks at p we have $s(1) = 1$ and for any $i, j, a, b \geq 0$ such that $i + a = j + b$, $x \in \mathcal{I}_p^i/\mathcal{I}_p^{i+1}$, $y \in \mathcal{I}_p^j/\mathcal{I}_p^{j+1}$ we have*

$$s^a(x) \cdot s^b(y) = \begin{cases} s^{a-j}(x \cdot y) & \text{if } a \geq j \\ 0 & \text{otherwise} \end{cases},$$

where $s^a = s \circ \cdots \circ s$ denotes the a-fold composition of s, $s^a(x) \cdot s^b(y) \in \mathcal{O}_{X,p}/\mathcal{I}_p^{a+i}$, and $x \cdot y \in \mathcal{I}^{i+j}/\mathcal{I}^{i+j+1}$.

Proof. In this proof we abbreviate $N := N_{Z/X}$, $\widehat{X} := \widehat{X}_Z$, and $\widehat{N} := \widehat{N_{Z/X}}_Z$. By definition $N = \mathbf{Spec}\ \mathrm{Sym}^\bullet \mathcal{I}/\mathcal{I}^2$. Since the embedding is regular we have an isomorphism [Ful, B.7.1]

$$\mathrm{Sym}^\bullet \mathcal{I}/\mathcal{I}^2 \cong \bigoplus_{i \geq 0} \mathcal{I}^i/\mathcal{I}^{i+1},$$

where $\mathcal{I}^0 := \mathcal{O}_X$. We denote the ideal inside $\bigoplus_{i \geq 0} \mathcal{I}^i/\mathcal{I}^{i+1}$ generated by $\mathcal{I}/\mathcal{I}^2$ by $\langle \mathcal{I} \rangle$. Then $\langle \mathcal{I} \rangle$ is the ideal of the zero-section $Z \hookrightarrow N$.

We first prove the "only if" part. Throughout we often use the canonical isomorphism

$$\mathcal{O}_{\widehat{X}}/\widehat{\mathcal{I}}^r \cong \mathcal{O}_X/\mathcal{I}^r,$$

for all $r \geq 1$ [Har, Cor. II.9.8]. Since the isomorphism $\mathcal{O}_{\widehat{X}} \cong \mathcal{O}_{\widehat{N}}$ commutes with the structure maps to $\mathcal{O}_Z$ it induces isomorphisms

$$\mathcal{O}_{\widehat{X}}/\widehat{\mathcal{I}}^r \cong \mathcal{O}_{\widehat{N}/\widehat{\langle \mathcal{I} \rangle}^r},$$

for all $r \geq 1$. We obtain the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X/\mathcal{I}^{r+1} \arrow[r] \arrow[d, "\cong"'] & \mathcal{O}_X/\mathcal{I}^r \arrow[d, "\cong"'] \\
\mathcal{O}_{\widehat{X}}/\widehat{\mathcal{I}}^{r+1} \arrow[r] \arrow[d, "\cong"'] & \mathcal{O}_{\widehat{X}}/\widehat{\mathcal{I}}^r \arrow[d, "\cong"'] \\
\mathcal{O}_{\widehat{N}}/\widehat{\langle\mathcal{I}\rangle}^{r+1} \arrow[r] \arrow[d, "\cong"'] & \mathcal{O}_{\widehat{N}}/\widehat{\langle\mathcal{I}\rangle}^r \arrow[d, "\cong"'] \\
\bigoplus_{i=0}^r \mathcal{I}^i/\mathcal{I}^{i+1} \arrow[r] & \bigoplus_{i=0}^{r-1} \mathcal{I}^i/\mathcal{I}^{i+1}.
\end{tikzcd}$$

The bottom map clearly has a section. It induces the desired section

$$s : \mathcal{O}_X/\mathcal{I}^r \longrightarrow \mathcal{O}_X/\mathcal{I}^{r+1}$$

by the diagram. Property (2) is easily verified for these sections.

We now turn to the "if" part. We first construct an isomorphism of *sheaves of abelian groups* $\mathcal{O}_{\widehat{N}} \cong \mathcal{O}_{\widehat{X}}$ commuting with the structure maps to $\mathcal{O}_Z$. This part does not use property (2). By definition

$$\mathcal{O}_{\widehat{X}} := \varprojlim \mathcal{O}_X/\mathcal{I}^i$$
$$\mathcal{O}_{\widehat{N}} := \varprojlim \mathcal{O}_N/\langle\mathcal{I}\rangle^i \cong \prod_{i=0}^{\infty} \mathcal{I}^i/\mathcal{I}^{i+1},$$

where $\mathcal{I}^0 := \mathcal{O}_X$. Define the following morphism of sheaves of abelian groups (e.g. on the level of stalks)

$$\begin{aligned} \prod_{i=0}^{\infty} \mathcal{I}^i/\mathcal{I}^{i+1} &\longrightarrow \varprojlim \mathcal{O}_X/\mathcal{I}^i \\ \{x_i\}_{i=0}^{\infty} &\mapsto \Big\{ \sum_{j=0}^{i} s^{i-j}(x_j) \Big\}_{i=0}^{\infty}, \end{aligned} \tag{1}$$

where $s^{i-j} = s \circ \cdots \circ s$ denotes the $(i-j)$-fold composition of s. This is well-defined because each s is a section and for any $x \in \mathcal{I}^i/\mathcal{I}^{i+1} \subset \mathcal{O}_X/\mathcal{I}^{i+1}$ we have $\pi(x) = 0$. Injectivity of this map is immediate. For given $\{y_i\}_{i=0}^{\infty} \in \varprojlim \mathcal{O}_X/\mathcal{I}^i$ it is easy to solve

$$\Big\{ y_i = \sum_{j=0}^{i} s^{i-j}(x_j) \Big\}_{i=0}^{\infty},$$

for $x_i \in \mathcal{O}_X/\mathcal{I}^{i+1}$. The projective limit property implies $x_i \in \mathcal{I}^i/\mathcal{I}^{i+1} \subset \mathcal{O}_X/\mathcal{I}^{i+1}$. This proves surjectivity.

It is not hard to check that condition (2) is equivalent to (1) being a *ring homorphism*. □

Let $\pi : S \to B$ be a flat morphism of smooth varieties with $\dim(B) = 1$ (but S of any dimension) and $F \subset S$ any fibre. There is no reason why $\mathbb{C}^*$ has to act on F. Surprisingly the previous proposition can be used to construct an action of $\mathbb{C}^*$ on the *formal neighbourhood* $\widehat{S}_F$ of $F \subset S$. E.g. when $\pi : S \to B$ is an elliptic surface this gives $\mathbb{C}^*$-actions on the formal neighbourhood of the fibres even though the fibres themselves need not have natural $\mathbb{C}^*$-actions.

Corollary 0.2. *Let $\pi : S \to B$ be a flat morphism of smooth varieties with $\dim(B) = 1$ and let F be any fibre. Denote by $\widehat{\mathbb{C}}_0$ the formal neighbourhood of $\{0\} \subset \mathbb{C}$. Then*

$$\widehat{S}_F \cong F \times \widehat{\mathbb{C}}_0.$$

In particular, the natural scaling action of $\mathbb{C}^*$ on $\widehat{\mathbb{C}}_0$ induces an action of $\mathbb{C}^*$ on $\widehat{S}_F$.

Proof. Let $p \in B$ be a closed point with ideal sheaf $\mathcal{I}_p$. Then

$$\mathcal{O}_B/\mathcal{I}_p^r \cong \mathbb{C}[\![t]\!]/(t)^r$$

and the natural quotient map

$$\mathcal{O}_B/\mathcal{I}_p^{r+1} \longrightarrow \mathcal{O}_B/\mathcal{I}_p^r$$

is just the natural quotient map

$$\mathbb{C}[\![t]\!]/(t)^{r+1} \longrightarrow \mathbb{C}[\![t]\!]/(t)^r.$$

This has an obvious section

$$s : \sum_{i=0}^{r-1} a_i t^i \in \mathbb{C}[\![t]\!]/(t)^r \mapsto \sum_{i=0}^{r-1} a_i t^i \in \mathbb{C}[\![t]\!]/(t)^{r+1}.$$

These sections satisfy property (2) of Proposition 0.1. Let $\mathcal{I}_F$ be the ideal sheaf of $F \subset X$. Note that $\mathcal{I}_p \cong \mathcal{O}_B(-p)$ and $\mathcal{I}_F \cong \mathcal{O}_S(-F)$ so

$$\pi^*\mathcal{I}_p^r \cong \mathcal{I}_F^r.$$

Since π is flat

$$\pi^*\big(\mathcal{O}_B/\mathcal{I}_p^r\big) \cong \pi^*\mathcal{O}_B/\pi^*\mathcal{I}_p^r \cong \mathcal{O}_S/\mathcal{I}_F^r.$$

Therefore $\pi^*(s)$ are sections of the natural quotient maps

$$\mathcal{O}_S/\mathcal{I}_F^{r+1} \longrightarrow \mathcal{O}_S/\mathcal{I}_F^r$$

satisfying property (2). By Proposition 0.1

$$\widehat{S}_F \cong \widehat{N_{F/S}}_F.$$

Since π is flat

$$N_{F/S} \cong \pi^* N_{\{p\}/B} \cong \mathcal{O}_F.$$

We claim this implies

$$F \times \widehat{\mathbb{C}}_0 \cong \widehat{N_{F/S}}_F.$$

This follows by showing that for each closed point $p \in F$ we have an isomorphism

$$(\mathcal{O}_{X,p}/\mathcal{I}_p)[\![z]\!] \cong \prod_{i=0}^{\infty} \mathcal{I}_p^i/\mathcal{I}_p^{i+1}.$$

This can be seen by writing $\mathcal{I}_p = (s)$ for some $s \in \mathcal{O}_{X,p}^*$ and sending

$$\sum_{i=0}^{\infty} a_i z^i \mapsto \left\{ \sum_{i=0}^{\infty} a_i s^i \right\}_{i=0}^{\infty}.$$

The corollary follows. $\square$

A less surprising corollary is the following.

Corollary 0.3. *Let $Z \hookrightarrow S$ be a regular embedding of $\mathbb{C}$-schemes of finite type, let $\mathcal{I}$ be the ideal sheaf of Z, and let $\mathcal{L}$ be a line bundle on S. Let $X = \mathrm{Tot}(\mathcal{L})$ and denote the formal completion of S resp. X along Z by $\widehat{S}_Z$, $\widehat{X}_Z$. Suppose*

$$\widehat{S}_Z \cong \widehat{N_{Z/S}}_Z,$$

and the isomorphism commutes with the structure maps to $\mathcal{O}_Z$. Then

$$\widehat{X}_Z \cong \widehat{N_{Z/X}}_Z,$$

and the isomorphism commutes with the structure maps to $\mathcal{O}_Z$.

Proof. The ideal sheaf of $Z \hookrightarrow X$ is

$$\mathcal{J} := \mathcal{I} \oplus \mathcal{L}^* \oplus \mathcal{L}^{*2} \oplus \cdots .$$

Therefore

$$\mathcal{O}_X/\mathcal{J}^r \cong \mathcal{O}_S/\mathcal{I}^r \oplus \mathcal{L}^*/\mathcal{I}^{r-1}\mathcal{L}^* \oplus \cdots \oplus \mathcal{L}^{*(r-1)}/\mathcal{I}\mathcal{L}^{*(r-1)}. \tag{2}$$

By Proposition 0.1 we have sections

$$\mathcal{O}_S/\mathcal{I}^r \longrightarrow \mathcal{O}_S/\mathcal{I}^{r+1}$$

for all $r \geq 1$ satisfying property (2). Using (2) we easily construct induced sections

$$\mathcal{O}_X/\mathcal{J}^r \longrightarrow \mathcal{O}_X/\mathcal{J}^{r+1}$$

satisfying property (2). The corollary follows from Proposition 0.1. □

References

[Ful] W. Fulton, *Intersection theory*, Springer-Verlag (1998).
[Har] R. Hartshorne, *Algebraic geometry*, Springer-Verlag (1977).