(A) The action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space[1] $\mathrm{Hilb}^{B+dF,\bullet}(X)$. Therefore, we only have to understand the fixed locus $\mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*}$. Pushforward along $X \to S \to B$ induces a morphism

$$\rho_d : \mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*} \to \mathrm{Sym}^d(B).$$

This map is constructed in Section **??** . The fibres of ρ_d decompose into components according to the shape of the underlying Cohen-Macaulay curve. This leads to a decomposition over 2D partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$.

(B) The Euler characteristics of the fibres of ρ_d define a constructible function f_d on $\mathrm{Sym}^d(B)$. In Section **??**, we show that if f_d satisfies a certain product formula, then $\widehat{\mathsf{DT}}(X)$ satisfies a corresponding product formula. This follows from general power structure arguments reviewed in Appendix **??**.

(C) A component Σ of a fibre of ρ_d indexed by λ can be further broken down by taking a certain fpqc cover of the underlying (now fixed) Cohen-Macaulay curve Z_{CM} determined by λ. This cover consists of formal neighbourhoods $\widehat{X}_x$ around the singular points x of the reduced support of Z_{CM} and "tubular neighbourhoods" along the reduced support of Z_{CM} after removing the singularities. Since Z_{CM} is already fixed, gluing is automatic. Hence restriction to the elements of the cover gives a bijection morphism of Σ to local Hilbert schemes on the elements of the cover. In Section **??**, we show this leads to the product formula for f_d in (B).

(D) On the formal neighbourhoods $\widehat{X}_x$, we have an action of $\mathbb{C}^{*3}$. This allows us to express their contributions to the generating function in terms of the topological vertex. The contributions of the tubular neighbourhoods along the *punctured* section and fibres can also be expressed in terms of the topological vertex (utilizing a map to $\mathrm{Sym}^n(F)$ which records the location and multiplicity of the embedded points). This is worked out in Section **??**.

In this section, we obtain (Theorem 4) expressions for $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$ in terms of the topological vertex $\mathsf{V}_{\lambda\mu\nu}(p)$, $e(B)$, and N (the number of nodal fibres). The theorem follows by expressing $g(a)$ and $h(b)$ of Proposition 12 in terms of the topological vertex.

0.1. **Point contributions.** Following the conventions of [**?**], we denote by

$$\mathsf{V}_{\lambda\mu\nu} = \sum_{\pi} p^{|\pi|},$$

the topological vertex. Here the sum is over all 3D partitions π with outgoing legs λ, μ, ν and $|\pi|$ denotes renormalized volume (see Definitions (1) and (2) in [**?**]). For a 2D partition $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$, we write λ' for the corresponding transposed partition and

$$|\lambda| := \sum_{k=0}^{\infty} \lambda_k,$$
$$\|\lambda\|^2 := \sum_{k=0}^{\infty} \lambda_k^2.$$

[1] The bullet indicates that we take the union of $\mathrm{Hilb}^{B+dF,n}(X)$ over all n, see Convention **??**.

Proposition 1. *Let F_x be a smooth fibre and F_y a singular fibre with singularity z. Then for any $\lambda \vdash a$, $\mu \vdash b$*

$$\begin{aligned} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda) &= \mathsf{V}_{\lambda\Box\varnothing}, \\ p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu) &= \mathsf{V}_{\mu\Box\varnothing}, \\ p^{-\|\mu\|^2} e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu) &= \mathsf{V}_{\mu\mu'\varnothing}. \end{aligned}$$

Proof. Recall that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!].$$

Therefore, $\mathbb{C}^{*3}$ acts on each of these schemes and their moduli spaces

$$\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda, \ \mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu, \ \mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu.$$

The coordinates can be chosen such that the action of the last factor of $\mathbb{C}^{*3}$ corresponds to $x_3 \mapsto t_3 x_3$. This component acts trivially since we are already on the $\mathbb{C}^*$-fixed locus. The $\mathbb{C}^{*3}$-fixed locus consists of isolated reduced points corresponding to monomial ideals with asymptotics $(\lambda, \varnothing, \varnothing)$, $(\mu, \varnothing, \varnothing)$, $(\mu, \mu', \varnothing)$ respectively[2]. These monomial ideals are exactly what the topological vertex counts.

Finally, note that the generating functions $e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)$, $e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)$, $e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)$ all start with 1. On the other hand, from the definition

$$\begin{aligned} \mathsf{V}_{\lambda\Box\varnothing} &= p^{-\lambda_0} + \cdots, \\ \mathsf{V}_{\mu\Box\varnothing} &= p^{-\mu_0} + \cdots, \\ \mathsf{V}_{\mu\mu'\varnothing} &= p^{-\sum_{k=0}^\infty \mu_k^2} + \cdots, \end{aligned}$$

where $\cdots$ stands for higher order terms in p. The proposition follows. □

0.2. **Fibre contribution.** Let F_x be a smooth fibre and F_y a singular fibre. Recall the formal neighbourhoods $\widehat{X}^\circ_{F_x^\circ}$, $\widehat{X}^\circ_{F_y^\circ}$ of Section **??**.

Proposition 2. *For any $\lambda \vdash a$ and $\mu \vdash b$, we have*

$$\begin{aligned} e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F_x^\circ})^{\mathbb{C}^*}_\lambda) &= \frac{1}{\mathsf{V}_{\lambda\varnothing\varnothing}}, \\ e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F_y^\circ})^{\mathbb{C}^*}_\mu) &= \frac{1}{\mathsf{V}_{\mu\varnothing\varnothing}}. \end{aligned}$$

Proof. ******** □

0.3. **Putting it together.** Combining Proposition 12 with Propositions 1, 2 gives:

Proposition 3. *For any $a, b > 0$*

$$\begin{aligned} g(a) &= (1-p) \sum_{\lambda \vdash a} \frac{\mathsf{V}_{\lambda\Box\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}, \\ h(b) &= \frac{1-p}{M(p)} \sum_{\mu \vdash b} p^{\|\mu\|^2} \frac{\mathsf{V}_{\mu\Box\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}} \mathsf{V}_{\mu\mu'\varnothing}. \end{aligned} \tag{1}$$

Putting all our results together, we obtain formulas for the partition functions in terms of the vertex:

---

[2]The transpose in μ' occurs, because we follow the orientation convention of [**?**].

Theorem 4.

$$\widehat{\mathsf{DT}}(X) = \frac{1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \left((1-p) \sum_{\lambda} q^{|\lambda|} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \right)^{e(B)-e(S)} \left((1-p) \sum_{\mu} q^{|\mu|} p^{\|\mu\|^2} \frac{\mathsf{V}_{\mu\square\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}} \mathsf{V}_{\mu\mu'\varnothing} \right)^{e(S)}$$

$$\widehat{\mathsf{DT}}_{\mathsf{fib}}(X) = \left(\sum_{\lambda} q^{|\lambda|} \right)^{e(B)-e(S)} \left(\sum_{\mu} q^{|\mu|} p^{\|\mu\|^2} \mathsf{V}_{\mu\mu'\varnothing} \right)^{e(S)}.$$

Proof. Inserting the equations for $g(a)$, $h(b)$ of Proposition 3 into (35) gives the formula for $\widehat{\mathsf{DT}}(X)$. Similar reasoning... □

Go back and put in the necessary stuff so that the $\widehat{\mathsf{DT}}_{\mathsf{fib}}$ computation isn't just hand waving.

Corollary 5. *Theorem **??** is true.*

Proof. We apply the main theorem of [**?**]. In particular, we substitute [**?**, Eqns (2)&(4)] into the formula for $\widehat{\mathsf{DT}}(X)$ and we substitute [**?**, Eqn (1)], as well as the well-known formula

$$\sum_{\lambda} q^{|\lambda|} = \prod_{d=1}^{\infty} \frac{1}{(1-q^d)}$$

into the formula for $\widehat{\mathsf{DT}}_{\mathsf{fib}}(X)$. □

Proof with additional assumption $d > 2g-2$: Since $d > 2g-2$ the Abel-Jacobi map

$$\mathrm{Sym}^d(B) \to \mathrm{Pic}^d(B)$$

is a projective bundle with fibres linear systems of dimension $d-g$. We immediately deduce that all horizontal arrows of the commutative diagram are surjective. At the level of fibres, we have inclusions of linear systems

$$|\epsilon| \hookrightarrow |p^*\epsilon| \hookrightarrow |p^*\epsilon(B)|, \tag{2}$$

where $\epsilon \in \mathrm{Pic}^d(B)$. We want to prove that all these linear systems have the same dimension (Claim). The first map is clearly an isomorphism since

$$H^0(S, p^*\epsilon) \cong H^0(B, \epsilon \otimes p_*\mathcal{O}_S) = H^0(B, \epsilon)$$

where we used the projection formula and $p_*\mathcal{O}_S \cong \mathcal{O}_B$. Surjectivity of the second map of (2) requires more work and is proved below. Before we prove Claim, we show it implies the proposition. Choose Poincaré line bundles

$$\begin{aligned} &\mathcal{P} \text{ on } \mathrm{Pic}^d(B) \times B, \\ &\mathcal{Q} \text{ on } \mathrm{Pic}^{dF}(S) \times S, \\ &\mathcal{R} \text{ on } \mathrm{Pic}^{B+dF}(S) \times S. \end{aligned}$$

In each case, denote projection to the base Picard by π_{Pic}. Since $d > 2g-2$, the sheaf $\pi_*\mathcal{P}$ is locally free because the dimensions of $|\epsilon|$, $\epsilon \in \mathrm{Pic}^d(B)$ do not jump. Using the isomorphism $\mathrm{Pic}^d(B) \cong \mathrm{Pic}^{dF}(S)$, we see that pull-back of $\mathcal{P}$ along

$$\mathrm{Pic}^{dF}(S) \times S \to \mathrm{Pic}^d(B) \times B$$

equals $\mathcal{Q}$ (up to tensoring by a line bundle pulled-back from S, which we get rid of by redefining $\mathcal{Q}$). We write this as $p^*\mathcal{P} \cong \mathcal{Q}$. Pushing down to $\mathrm{Pic}^{dF}(S)$, using the projection formula and $p_*\mathcal{O}_S \cong \mathcal{O}_B$, we obtain

$$\pi_{\mathrm{Pic}*}\mathcal{P} \cong \pi_{\mathrm{Pic}*}\mathcal{Q}$$

on $\mathrm{Pic}^{dF}(S) \cong \mathrm{Pic}^d(B)$. On $\mathrm{Pic}^{dF}(S) \times S$ we can form $\mathcal{Q}(B)$ by pulling back $\mathcal{O}_S(B)$ along projection to the second factor. Then

$$\mathcal{R} \cong \mathcal{Q}(B),$$

on $\mathrm{Pic}^{B+dF}(S) \times S \cong \mathrm{Pic}^{dF}(S) \times S$ (again, after possibly redefining $\mathcal{R}$). Multiplying by the section defining B and pushing down gives a morphism

$$\pi_{\mathrm{Pic}*}\mathcal{Q} \hookrightarrow (\pi_{\mathrm{Pic}*}\mathcal{Q})(B) \cong \pi_{\mathrm{Pic}*}\mathcal{R}. \tag{3}$$

Claim implies that the dimension of the fibres of $|p^*\epsilon(B)|$, $\epsilon \in \mathrm{Pic}^d(B)$ do not jump. Therefore $(\pi_{\mathrm{Pic}*}\mathcal{Q})(B) \cong \pi_{\mathrm{Pic}*}\mathcal{R}$ is locally free. Since (3) is a morphism of locally free sheaves, it suffices to check we have induced isomorphisms on the fibres. This is exactly the content of Claim. We conclude all Abel-Jacobi maps are projective bundles

$$\begin{aligned}\mathbb{P}(\pi_{\mathrm{Pic}*}\mathcal{P}) &\to \mathrm{Pic}^d(B),\\ \mathbb{P}(\pi_{\mathrm{Pic}*}\mathcal{Q}) &\to \mathrm{Pic}^{dF}(S),\\ \mathbb{P}(\pi_{\mathrm{Pic}*}\mathcal{R}) &\to \mathrm{Pic}^{B+dF}(S).\end{aligned}$$

and

$$\mathbb{P}(\pi_{\mathrm{Pic}*}\mathcal{P}) \cong \mathbb{P}(\pi_{\mathrm{Pic}*}\mathcal{Q}) \cong \mathbb{P}\big((\pi_{\mathrm{Pic}*}\mathcal{Q})(B)\big) \cong \mathbb{P}(\pi_{\mathrm{Pic}*}\mathcal{R}).$$

What is left is the Claim: the second map of (2) is an isomorphism for all $\epsilon \in \mathrm{Pic}^d(C)$. For any line bundle δ on B, the Leray spectral sequence yields the following short exact sequence

$$0 \to H^1(B,\delta) \xrightarrow{\alpha} H^1(S,p^*\delta) \longrightarrow H^0(B,\delta\otimes R^1p_*\mathcal{O}_S) \to 0.$$

Therefore, if

$$\deg(\delta) - \deg(R^1p_*\mathcal{O}_S)^\vee < 0, \tag{4}$$

then α is an isomorphism. Assume this is the case. The long exact cohomology sequence associated to

$$0 \to p^*\delta(-B) \to p^*\delta \to \mathcal{O}_B \otimes p^*\delta \to 0$$

is

$$\cdots \to H^1(S,p^*\delta) \xrightarrow{\alpha} H^1(B,\delta) \to H^2(S,p^*\delta(-B)) \to H^2(S,p^*\delta) \to 0,$$

and since α is a surjection, we get an isomorphism of the last two terms. We apply Serre duality to that isomorphism and we use the fact that $K_S = p^*(K_B \otimes L)$ where $L = \big(R^1p_*\mathcal{O}_S\big)^\vee$ [**?**, Thm. 12.1] to obtain

$$H^0(S,p^*(\delta^{-1}\otimes K_B\otimes L)(B)) \cong H^0(S,p^*(\delta^{-1}\otimes K_B\otimes L)). \tag{5}$$

Let $\epsilon \in \mathrm{Pic}^d(B)$ and take

$$\delta = K_B \otimes L \otimes \epsilon^{-1}.$$

Then δ satisfies (4) if and only if $d > 2g-2$, which we assumed. The lemma follows from (5).

Jim's original: In this subsection we prove the following lemma and corollary, which will tell us what is the reduced support of all curves in the class $\beta = B + dF$.

Lemma 6. *For any line bundle ϵ on B, multiplication by the canonical section of $\mathcal{O}_S(B)$ induces an isomorphism*

$$H^0(S,p^*(\epsilon)(B)) \cong H^0(S,p^*(\epsilon)).$$

Corollary 7. *Let $\beta = B + dF \in H_2(S)$. Then the Chow variety of curves in the class β is isomorphic to $\mathrm{Sym}^d(B)$ where a point $\sum_i d_i x_i \in \mathrm{Sym}^d(B)$ corresponds to the curve $B + \sum_i d_i F_{x_i}$.*

Proof. The corollary follows immediately from the lemma since the Chow variety is the space of effective divisors and the lemma implies that any effective divisor in the class β is a union of the section B with an effective divisor pulled back from the base.

To prove Lemma 6 we proceed as follows. For any line bundle δ on B, the Leray spectral sequence yields the short exact sequence:

I think arrows Leray go other way around. Anyway... last two terms iso when $2g-2 < d$.

$$0 \to H^0(B, \delta \otimes R^1 p_* \mathcal{O}_S) \to H^1(S, p^*\delta) \xrightarrow{\alpha} H^1(B, \delta) \to 0,$$

in particular, α is a surjection.

Then the long exact cohomology sequence associated to

$$0 \to p^*\delta \otimes \mathcal{O}_S(-B) \to p^*\delta \to \mathcal{O}_B \otimes p^*\delta \to 0$$

is

$$\cdots \to H^1(S, p^*\delta) \overset{\alpha}{\to} H^1(B, \delta) \to H^2(S, p^*\delta \otimes \mathcal{O}_S(-B)) \to H^2(S, p^*\delta) \to 0,$$

and since α is a surjection, we get an isomorphism of the last two terms. We apply Serre duality to that isomorphism and we use the fact that $K_S = p^*(K_B \otimes L)$ where $L = \left(R^1 p_* \mathcal{O}_S\right)^\vee$ [**?**, Thm. 12.1] to obtain

$$H^0(S, p^*(\delta^{-1} \otimes K_B \otimes L)(B)) \cong H^0(S, p^*(\delta^{-1} \otimes K_B \otimes L)).$$

Letting $\delta = K_B \otimes L \otimes \epsilon^{-1}$, the lemma is proved. $\square$

0.4. **Fpqc cover.** The idea is to use an appropriate cover of X and calculate on pieces of the cover. We first give a complex analytic definition of the cover to aid the intuition and then give the actual "algebro-geometric cover":

(1) The reduced support $B \cup F_x \cup F_y$ has three singular points[3]: $x, y \in B$ and $z \in F_y^{\mathrm{sing}}$. We take small open balls around these points.
(2) Consider the punctured curve $B^\circ := B \setminus \{x, y\}$ and let $X^\circ := X \setminus (F_x \cup F_y)$. We take a tubular neighbourhood of $B^\circ \subset X^\circ$.
(3) Consider the punctured curve $F_x^\circ := F_x \setminus \{x\}$ and let $X^\circ := X \setminus B$. We take a tubular neighbourhood of $F_x^\circ \subset X^\circ$.
(4) Consider the punctured curve $F_y^\circ := F_y \setminus \{y, z\}$ and let $X^\circ := X \setminus (B \cup \{z\})$. We take a tubular neighbourhood of $F_y^\circ \subset X^\circ$.
(5) Finally, we take $W = X \setminus (B \cup F_x \cup F_y)$.

In order to work in algebraic geometry, in (1) we take the formal neighbourhood $\widehat{X}_x$ of $\{x\}$ in X. Denote the local ring at x by $(R, \mathfrak{m})$. By $\widehat{X}_x$ we mean the (non-noetherian) scheme

$$\mathrm{Spec} \varprojlim R/\mathfrak{m}^n$$

and *not* the formal scheme

$$\mathrm{Spf} \varprojlim R/\mathfrak{m}^n.$$

Similarly in (2) and (3), let $\widehat{X}_y$ be the formal neighbourhood of $\{y\}$ in X and $\widehat{X}_z$ the formal neighbourhood of $\{z\}$ in X. Even though $\widehat{X}_x$ is non-noetherian, the morphism $\widehat{X}_x \to X$ has a good property: it is fpqc so can be used as part of a cover [**?**, Vistoli, Sect. 2.3.2].

[3]Recall that $x, y \in B$ in the base can be viewed as points on S and X via the sections $B \subset S \subset X$.

Flatness of this map follows from the fact that formal completion is an exact operation [**?**, Tag 0BNH] [**?**, Prop. 10.14].

In (2) we consider $B^\circ := B \setminus \{x, y\}$, $X^\circ := X \setminus (F_x \cup F_y)$ and let $\widehat{X}^\circ_{B^\circ}$ be the formal neighbourhood of F°_x in X°. For (3) and (4) the formal neighbourhoods $\widehat{X}^\circ_{F^\circ_x}$ and $\widehat{X}^\circ_{F^\circ_y}$ are defined analogously. Note that the definition of X° in (2)–(4) varies. Finally in (5) we take $W = X \setminus (B \cup F_x \cup F_y)$. Then

$$\mathfrak{U} = \{\widehat{X}_x \to X, \widehat{X}_y \to X, \widehat{X}_z \to X, \widehat{X}^\circ_{B^\circ} \to X, \widehat{X}^\circ_{F^\circ_x} \to X, \widehat{X}^\circ_{F^\circ_y} \to X, W \subset X\}$$

is an fpqc cover of X. Consequently the data of a quasi-coherent sheaf on X is equivalent to the data of quasi-coherent sheaves on each of the opens of $\mathfrak{U}$ and gluing isomorphisms between the restrictions on the overlaps. Technically: quasi-coherent sheaves on X form a stack with respect to the fpqc topology [**?**, Vistoli, Thm. 4.23].

0.5. **Local moduli spaces.** We now introduce moduli spaces of closed subschemes of dimension ≤ 1 on the pieces of the cover $\mathfrak{U}$. Assume the coordinates on

$$\widehat{X}_x \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$$

are chosen such that $x_1 = x_3 = 0$ corresponds to the intersection $\widehat{X}_x \times_X B$ and $x_2 = x_3 = 0$ corresponds to $\widehat{X}_x \times_X F_x$. Define

$$\begin{aligned}&\operatorname{Hilb}^{(1,d),n}(\widehat{X}_x) := \\ &\big\{ I_Z \subset \mathcal{O}_{\widehat{X}_x} \;:\; [Z] = [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.\end{aligned}$$

Here the equation

$$[Z] = [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x]$$

means Z is supported along

$$(\widehat{X}_x \times_X B) \cup (\widehat{X}_x \times_X F_x)$$

with multiplicity 1 along $\widehat{X}_x \times_X B$ and multiplicity d along $\widehat{X}_x \times_X F_x$ and Z_{CM} denotes the maximal Cohen-Macaulay subcurve of Z. The ideal sheaves fit into a short exact sequence

$$0 \longrightarrow I_Z \longrightarrow I_{Z_{\mathrm{CM}}} \longrightarrow Q \longrightarrow 0,$$

where Q is 0-dimensional. The Hilbert scheme $\operatorname{Hilb}^{(1,d),n}(\widehat{X}_y)$ is defined likewise replacing the point x by y. For $\widehat{X}_z$, we define

$$\operatorname{Hilb}^{d,n}(\widehat{X}_z) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}_z} \;:\; [Z] = d[\widehat{X}_z \times_X F_y] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

Each of $\widehat{X}_x, \widehat{X}_y, \widehat{X}_z$ has an action of $\mathbb{C}^*$ compatible with the fibre scaling on X. This action lifts to the moduli space. Since each of these formal neighbourhoods is isomorphic to $\operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$, the bigger torus $\mathbb{C}^{*3}$ acts on it and this action lifts to the moduli space. The existence of these "extra actions" will be used in Section **??**.

Next consider $\widehat{X}^\circ_{B^\circ}$, i.e. the formal neighbourhood of the punctured zero section $B^\circ \subset X^\circ$. Define

$$\operatorname{Hilb}^{1,n}(\widehat{X}^\circ_{B^\circ}) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{B^\circ}} \;:\; [Z] = [\widehat{X}^\circ_{B^\circ} \times_X B] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

For $\widehat{X}^\circ_{F_x^\circ}$, $\widehat{X}^\circ_{F_y^\circ}$ we define

$$\begin{aligned}
\mathrm{Hilb}^{d,n}(\widehat{X}^\circ_{F_x^\circ}) &:= \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{F_x^\circ}} \ : \ [Z] = d[\widehat{F}_x^\circ] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}, \\
\mathrm{Hilb}^{d,n}(\widehat{X}^\circ_{F_y^\circ}) &:= \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{F_y^\circ}} \ : \ [Z] = d[\widehat{F}_y^\circ] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.
\end{aligned}$$

Finally for W we define

$$\mathrm{Hilb}^{0,n}(W) := \big\{ I_Z \subset \mathcal{O}_W \ : \ \dim(Z) = 0 \text{ and } h^0(\mathcal{O}_Z) = n \big\}.$$

On $\widehat{X}^\circ_{B^\circ}$, $\widehat{X}^\circ_{F_x^\circ}$, $\widehat{X}^\circ_{F_y^\circ}$, and W we have an action of $\mathbb{C}^*$ compatible with the fibre scaling on X. These actions lift to the moduli space. However, *unlike* for $\widehat{X}_x$, $\widehat{X}_y$, $\widehat{X}_z$, no additional tori act.

As before, we use the notation $\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ for the union of all $\mathrm{Hilb}^{(1,d),n}(\widehat{X}_x)$ and similarly for all other moduli spaces of this section. Like in Section **??**, the components of the $\mathbb{C}^*$-fixed locus of $\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ are indexed by 2D partitions

$$\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)^{\mathbb{C}^*} = \bigsqcup_{\lambda \vdash d} \mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda.$$

Proposition 8. *Consider the stratum $\Sigma(x, y, \lambda, \mu)$, where $|\lambda| = a$ and $|\mu| = b$. Restriction from X to the elements of the cover $\mathfrak{U}$ induces a morphism*

$$\begin{aligned}
\Sigma(x, y, \lambda, \mu) \longrightarrow\, & \mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda \times \mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu \times \mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu \times \\
& \mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})^{\mathbb{C}^*} \times \mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F_x^\circ})^{\mathbb{C}^*}_\lambda \times \mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F_y^\circ})^{\mathbb{C}^*}_\mu \times \\
& \mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*},
\end{aligned} \tag{6}$$

which is a bijection on closed points.

Proof. Since pull-back works in families, restriction indeed defines a morphism. For the rest of the proof, we work on closed points only.

Since $\mathfrak{U}$ is an fpqc cover, fpqc descent implies that any ideal sheaf $I_Z \subset \mathcal{O}_X$ is entirely determined by its restriction along the morphisms of the elements of $\mathfrak{U}$. This proves injectivity.

Conversely, given local ideal sheaves in the image of (6), their restrictions to overlaps only depend on the underlying Cohen-Macaulay curve and not on the embedded points. Since we chose the strata such that the underlying Cohen-Macaulay curve is already fixed, there are no further gluing conditions and fpqc descent implies surjectivity. $\square$

Remark 9. Note that the argument of Proposition 8 produces a bijective morphism — we do *not* claim (6) is an isomorphism of schemes. However, a bijective morphism induces an equality of (topological) Euler characteristic, which is what we use.

Remark 10. It is important to relate holomorphic Euler characteristic of domain and target in (6). For any subscheme Z in the domain $\Sigma(x, y, \lambda, \mu)$, denote the corresponding maximal Cohen-Macaulay curve of its elements by Z_{CM} (Theorem **??**). Then

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{Z_{\mathrm{CM}}}) + \chi(I_{Z_{\mathrm{CM}}}/I_Z).$$

Recall that Z_{CM} is entirely determined by the data x, y, λ, μ, where $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$ and $\mu = (\mu_0 \geq \mu_1 \geq \cdots)$ are 2D partitions (equation (**??**)). An easy calculation shows

$$\chi(\mathcal{O}_{Z_{\mathrm{CM}}}) = \chi(\mathcal{O}_B) - \lambda_0 - \mu_0.$$

We conclude

$$\chi(\mathcal{O}_Z) = \frac{e(B)}{2} - \lambda_0 - \mu_0 + \chi(I_{Z_{\mathrm{CM}}}/I_Z). \tag{7}$$

Proposition 8 allows us to calculate

$$f_d(ax+by) = e(\rho_d^{-1}(ax+by)) = \sum_{\lambda\vdash a}\sum_{\mu\vdash b} e(\Sigma(x,y,\lambda,\mu)).$$

By Proposition 8 and (7)

$$\begin{aligned} f_d(ax+by) = p^{\frac{e(B)}{2}} & e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^{\circ}_{B^\circ})^{\mathbb{C}^*})\, e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*})\times \\ & \sum_{\lambda\vdash a}\sum_{\mu\vdash b} p^{-\lambda_0-\mu_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)\times \\ & e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^{\circ}_{F^\circ_x})^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^{\circ}_{F^\circ_y})^{\mathbb{C}^*}_\mu). \end{aligned} \tag{8}$$

Before we proceed, we calculate $e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*})$ and $e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^{\circ}_{B^\circ})^{\mathbb{C}^*})$. The first follows from a formula of J. Cheah [**?**]

$$e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) = M(p)^{e(W)}. \tag{9}$$

For the second we use the following proposition:

Proposition 11. *Let $x_1,\dots,x_l \in B$ be any number of distinct closed points. Define*

$$B^\circ := B\setminus\{x_1,\dots,x_l\},$$
$$X^\circ := X\setminus \bigsqcup_{i=1}^{l} F_{x_i}.$$

Let $\widehat{X}^{\circ}_{B^\circ}$ be the formal neighbourhood of B° in X°. Define $\mathrm{Hilb}^{1,n}(\widehat{X}^{\circ}_{B^\circ})$ to be the Hilbert scheme of subschemes $Z\subset \widehat{X}^{\circ}_{B^\circ}$, such that $Z_{\mathrm{CM}} = B^\circ$ and $\chi(I_{Z_{\mathrm{CM}}}/I_Z) = n$. , where Z_{CM} denotes the maximal Cohen-Macaulay subscheme contained in Z (Proposition 17). Then

$$e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^{\circ}_{B^\circ})) = \left(\frac{M(p)}{1-p}\right)^{e(B^\circ)}.$$

Proof. Pick any $y\in B^\circ$ and let $\widehat{X}_y \cong \operatorname{Spec}\mathbb{C}[\![x_1,x_2,x_3]\!]$ be the formal neighbourhood of y in X°. Denote by

$$\mathrm{Hilb}^{1,n}(\widehat{X}^{\circ}_y)$$

the Hilbert scheme of subschemes $Z\subset\widehat{X}^{\circ}_y$, such that $Z_{\mathrm{CM}} = \{x_1 = x_3 = 0\}$ and $\chi(I_{Z_{\mathrm{CM}}}/I_Z) = n$.

We have projections

$$X^\circ \longrightarrow S^\circ \longrightarrow B^\circ.$$

These map induces a morphism

$$\mathrm{Hilb}^{1,n}(\widehat{X}^{\circ}_{B^\circ}) \longrightarrow \mathrm{Sym}^n(B^\circ).$$

The fibre over a point $\mathfrak{a} = \sum_i a_i y_i$, with all $y_i\in B^\circ$ distinct, equals

$$\prod_i \mathrm{Hilb}^{1,a_i}(\widehat{X}^{\circ}_{y_i}).$$

This follows by using an appropriate fpqc cover of B° similar to Proposition 8. Since B is reduced and smooth, $\mathrm{Hilb}^{1,a_i}(\widehat{X}^\circ_{y_i})$ only depends on a_i and not on the point $y_i \in B^\circ$. Therefore Lemma **??** of Appendix **??** implies

$$e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})) = \left(\sum_{a=0}^{\infty} e(\mathrm{Hilb}^{1,a}(\widehat{X}^\circ_y))p^a\right)^{e(B^\circ)}.$$

The formal neighbourhood $\widehat{X}_y$ has an action of $\mathbb{C}^{*3}$ and this action lifts to $\mathrm{Hilb}^{1,a}(\widehat{X}_y)$. The fixed locus consists of a finite number of points counted by the topological vertex[4]

$$\mathsf{V}_{\square\varnothing\varnothing} = \frac{M(p)}{1-p}. \qquad \square$$

The proof follows.

Using (8), (9), and Proposition 11 gives

$$\begin{aligned} f_d(ax+by) = &\frac{M(p)^{e(X)}}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})^{e(B)}} \times \\ &(1-p)\sum_{\lambda\vdash a} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})^{\mathbb{C}^*}_\lambda) \times \\ &\frac{1-p}{M(p)}\sum_{\mu\vdash b} p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})^{\mathbb{C}^*}_\mu). \end{aligned}$$

0.6. **Geometric characterization of $g(a)$ and $h(b)$.** The arguments of the preceding two sections are straightforwardly modified to any stratum $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$. Fix a smooth fibre F_x and a singular fibre F_y. Denote the singular point of F_y by z. Let $\widehat{X}_x$, $\widehat{X}_z$ be the formal neighbourhoods of x, z in X. Define $\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)$, $\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)$ as in Section 0.5. Like in Section 0.5, we also consider the "tubular" formal neighbourhoods $\widehat{X}^\circ_{F^\circ_x}$, $\widehat{X}^\circ_{F^\circ_y}$ and corresponding Hilbert schemes $\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})$, $\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})$. A straightforward generalization of the calculation of $f_d(ax+by)$ yields:

Proposition 12. *For any $a,b>0$ define*

$$\begin{aligned} g(a) &:= (1-p)\sum_{\lambda\vdash a} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})^{\mathbb{C}^*}_\lambda), \\ h(b) &:= \frac{1-p}{M(p)}\sum_{\mu\vdash b} p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})^{\mathbb{C}^*}_\mu), \end{aligned}$$

and let $g(0):=1$, $h(0):=1$. Then

$$f_d(\mathfrak{a}+\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_i g(a_i) \cdot \prod_j h(b_j),$$

for any $\mathfrak{a} = \sum_i a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$ and $\mathfrak{b} = \sum_j b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$, where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.

We immediately deduce:

Corollary 13. *Propositions 25 and 26 are true for $g(a)$ and $h(b)$ defined in Proposition 12.*

[4]Discussed in general in Section **??**.

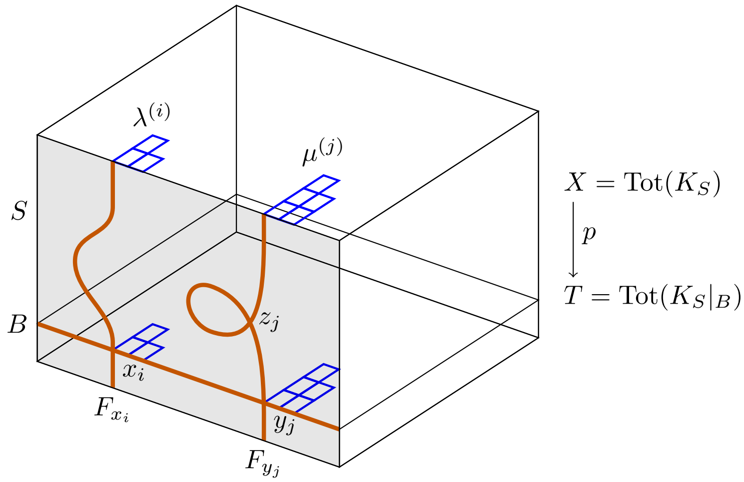

FIGURE 1. A partition thickened comb curve $C = B \cup_i \left(\lambda^{(i)} F_{x_i}\right) \cup_j \left(\mu^{(j)} F_{y_j}\right)$.

1. THE $\mathbb{C}^*$-FIXED LOCUS

The action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1\, de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\, de.$$

In order to understand $\mathrm{Hilb}^{d,n}(X)$, we first study $Z_{\mathrm{CM}} \subset Z$, the maximal Cohen-Macaulay subscheme of any $\mathbb{C}^*$-invariant subschemes $Z \subset X$. We find that such subschemes are determined by a point in $\mathrm{Sym}^d(B)$ along with some discrete data (a collection of integer partitions). This is given by the following two propositions and is illustrated in Figure **??**.

Proposition 14. *A closed points Z of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ correspond to a finite nesting of closed subschemes of S*

$$Z_0 \supset Z_1 \supset \cdots \supset Z_l,$$

satisfying

$$\sum_{k=0}^{l} [Z_k] = B + dF \in H_2(S).$$

Proof. Using projection $\pi : X \to S$, a quasi-coherent sheaf on X can be viewed as a quasi-coherent sheaf $\mathcal{F}$ on S together with a morphism $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$. A $\mathbb{C}^*$-equivariant structure on $\mathcal{F}$ translates into a $\mathbb{Z}$-grading

$$\pi_* \mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k,$$

such that $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$ is graded, i.e.

$$\mathcal{F}_k \otimes K_S^{-1} \longrightarrow \mathcal{F}_{k-1}.$$

Here $\mathcal{F}_k$ has weight k and K_S weight 1 under the $\mathbb{C}^*$-action. The structure sheaf $\mathcal{O}_X$ corresponds to

$$\pi_*\mathcal{O}_X = \bigoplus_{k=0}^{\infty} K_S^{-k}.$$

Therefore a $\mathbb{C}^*$-equivariant morphism $\mathcal{F} \to \mathcal{O}_X$ corresponds to a graded sheaf $\mathcal{F}$ as above together with maps $F_i \to K_S^i$ for all i such that

$$\begin{tikzcd}
\mathcal{F}_k \otimes K_S^{-1} \arrow[r] \arrow[d] & \mathcal{F}_{k-1} \arrow[d] \\
K_S^k \otimes K_S^{-1} \arrow[r, equal] & K_S^{k-1}
\end{tikzcd}$$

commute for all $k \leq 0$ and the composition $\mathcal{F}_1 \otimes K_S^{-1} \to \mathcal{F}_0 \to \mathcal{O}_S$ is to zero.

It is useful to re-define $\mathcal{G}_k := \mathcal{F}_{-k} \otimes K_S^k$. Then a $\mathbb{C}^*$-equivariant morphism $\mathcal{F} \to \mathcal{O}_X$ is equivalent to the following data:

- quasi-coherent sheaves $\{\mathcal{G}_k\}_{k\in\mathbb{Z}}$ on S,
- morphisms $\{\mathcal{G}_k \to \mathcal{G}_{k+1}\}_{k\in\mathbb{Z}}$,
- morphisms $\mathcal{G}_k \to \mathcal{O}_S$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}_{-1} \arrow[r] \arrow[d] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \cdots
\end{tikzcd}$$

In the case of interest to us $\mathcal{G} \to \mathcal{O}_X$ is an ideal sheaf $I_Z \hookrightarrow \mathcal{O}_X$ cutting out $Z \subset X$. In the above language, this means $\mathcal{G}_k = 0$ for $k < 0$, the morphisms $\mathcal{G}_k \to \mathcal{O}_S$ are injective, and the morphisms $\mathcal{G}_k \to \mathcal{G}_{k+1}$ are injective. Therefore $\mathcal{G}_k = I_{Z_k\subset S}$ is an ideal sheaf cutting out $Z_k \subset S$ and

$$I_{Z_k\subset S} \subset I_{Z_{k+1}\subset S},$$

for all k. $\square$

Let $\mathrm{Hilb}^{B+dF}(S)$ be the Hilbert scheme of effective divisors on S with class

$$B + dF \in H_2(S).$$

By Lemma **??** of the Appendix **??**, pull-back along p and adding the section B induces an isomorphism

$$\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{B+dF}(S).$$

This allows us to see the curves on S.

For any reduced curve $C \subset S$ defined by ideal sheaf $I_{C\subset S}$ and $d > 0$, we denote by dC the Gorenstein curve defined by the ideal sheaf $I^d_{C\subset S}$, the dth power of $I_{C\subset S}$. We combine Lemma **??** with a (family version of) Proposition 14 to conclude the following:

Proposition 15. *There exists a morphism*

$$\rho_d : \mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*} \longrightarrow \mathrm{Sym}^d(B),$$

which at the level of closed points can be can be described as follows. Let $Z \in \mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*}$ and let $Z_{\mathrm{CM}} \subset Z$ be the maximal Cohen-Macaulay subcurve of Z. Since Z_{CM} is $\mathbb{C}^*$-fixed, its ideal sheaf decomposes

$$I_{Z_{\mathrm{CM}}} = \bigoplus_{k=0}^{\infty} I_{Z_k\subset S} \otimes K_S^{-k},$$

where

$$Z_0 = B \cup \lambda_0^{(1)} F_{x_1} \cup \cdots \cup \lambda_0^{(l)} F_{x_l}$$

for some distinct closed points $x_i \in B$ and $\lambda_0^{(i)} > 0$, and

$$Z_k = \lambda_k^{(1)} F_{x_1} \cup \cdots \cup \lambda_k^{(l)} F_{x_l}.$$

for some $\lambda_k^{(i)} \leq \lambda_{k-1}^{(i)}$. Here $\lambda^{(i)} = (\lambda_0^{(i)} \geq \lambda_1^{(i)} \geq \cdots)$ define 2D partitions satisfying

$$\sum_{i=1}^{l} |\lambda^{(i)}| = d.$$

See Figure **??** *for an illustration. The map ρ_d sends Z to*

$$\sum_{i=1}^{l} |\lambda^{(i)}| x_i \in \mathrm{Sym}^d(B).$$

Remark 16. The morphism of this proposition is perhaps somewhat surprising. Since we are on a 3-fold, the map which sends a closed subscheme of $Z \in \mathrm{Hilb}^{d,n}(X)$ to its underlying Cohen-Macaulay curve Z_{CM} is *not* a morphism. Nevertheless, the map ρ_d which records the location of the fibres in Z_{CM} and their multiplicities is a morphism.

Proof. The description of ρ_d at the level of closed points is clear. We construct ρ_d as a morphism from Proposition 14 and Lemma **??** of Appendix **??**.

Let T be an arbitrary base scheme of finite type and let

$$\mathcal{Z} \subset X \times T$$

be a $\mathbb{C}^*$-fixed and T-flat closed subscheme. Assume for each $t \in T$ the fibre $\mathcal{Z}_t$ has class $B + dF \in H_2(S)$ and $\chi(\mathcal{O}_{\mathcal{Z}_t}) = n$. Since $\mathcal{Z}$ is $\mathbb{C}^*$-fixed, Proposition 14 implies that its ideal sheaf decomposes[5]

$$I_{\mathcal{Z}} = \bigoplus_{k=0}^{\infty} I_{\mathcal{Z}_k \subset S \times T} \otimes K_S^{-k},$$

where K_S is pulled-back along $S \times T \to S$ and

$$\mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots.$$

Then each $\mathcal{Z}_k \subset S \times T$ is T-flat as well. The maximal CM subschemes $\mathcal{Z}_{k,\mathrm{CM}} \subset \mathcal{Z}_k \subset S \times T$ are also T-flat and induces morphisms

$$\begin{aligned} &T \longrightarrow \mathrm{Hilb}^{B+d_0F}(S), \\ &T \longrightarrow \mathrm{Hilb}^{d_kF}(S), \ \text{ for } \ k > 0 \end{aligned}$$

where $\sum_k d_k = d$. Adding divisors gives a morphism $T \longrightarrow \mathrm{Hilb}^{B+dF}(S)$. By Lemma **??**, we obtain a morphism $T \to \mathrm{Sym}^d(B)$. This morphism corresponds to a T-flat family for $\mathrm{Sym}^d(B)$. We have defined ρ_d as a morphism. $\square$

In the above proposition, each $Z_k \subset S$ contains a maximal Cohen-Macaulay (in fact, Gorenstein) subcurve D_k such that $Z_k \setminus D_k$ is 0-dimensional. For $k = 0$, D_0 is the

[5]The arguments leading to Proposition 14 hold equally well for T-flat families over a base T.

scheme-theoretic union of the section B and thickenings of certain distinct fibres $F_{x_1}, \ldots, F_{x_n}$. Denoting the orders of thickenings by $\lambda_0^{(1)}, \ldots, \lambda_0^{(n)} > 0$, we obtain[6]

$$D_0 = B \cup \lambda_0^{(1)} F_{x_1} \cup \cdots \cup \lambda_0^{(n)} F_{x_n}.$$

This statement follows from Corollary 7 of the appendix. Next, for all $i = 1, \ldots, n$ and $k \geq 1$, there are $\lambda_k^{(i)} \leq \lambda_{k-1}^{(i)}$ such that

$$D_k = \lambda_k^{(1)} F_{x_1} \cup \cdots \cup \lambda_k^{(n)} F_{x_n}.$$

We conclude:

Proposition 17. *To each closed point Z of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ correspond distinct closed points $x_1, \ldots, x_n \in B$ for some n and (finite) 2D partitions $\lambda^{(1)}, \ldots, \lambda^{(n)}$ such that*

$$\sum_{i=1}^{n} |\lambda^{(i)}| = d.$$

The maximal Cohen-Macaulay subcurve of Z is given by the scheme-theoretic union of the zero section B and the schemes with ideal sheaves

$$\bigoplus_{k=0}^{\infty} \mathcal{O}_S(-\lambda_k^{(i)} F_{x_i}) \otimes K_S^{-k},$$

for all $i = 1, \ldots, n$.

Note that in the notation of this proposition, the morphism ρ_d in (**??**) maps Z to

$$\sum_{i=1}^{n} |\lambda^{(i)}| x_i \in \mathrm{Sym}^d(B),$$

where $|\lambda^{(i)}|$ denotes the size of the 2D partition $\lambda^{(i)}$. This leads to the following proposition:

Proposition 18. *For each $h > 0$, there exists a stratification*

$$\mathrm{Hilb}^{h,\bullet}(X)^{\mathbb{C}^*} = \coprod_{n=1}^{\infty} \coprod_{\substack{\lambda^{(1)}, \ldots, \lambda^{(n)} \mathrm{s.t.} \\ \sum_{\alpha=1}^{n} |\lambda^{(\alpha)}| = h}} \mathrm{Hilb}^{h,\bullet}_{\lambda^{(1)}, \ldots, \lambda^{(n)}}(X)^{\mathbb{C}^*},$$

where $\mathrm{Hilb}^{h,\bullet}_{\lambda^{(1)}, \ldots, \lambda^{(n)}}(X)^{\mathbb{C}^*}$ is the locally closed subset of subschemes $Z \subset X$ with maximal Cohen-Macaulay curve defined by the scheme-theoretic union of B and schemes with ideal sheaves of the form (**??**) *for some distinct fibre $F_{x_1}, \ldots, F_{x_n} \subset S$.*

In this proposition, the number of points n and the position of the points $x_1, \ldots, x_n \in B$ can still vary freely. We now want to refine the stratification by fixing the position of these points.

[6]For any reduced curve C on a surface S with ideal sheaf $I_C \subset \mathcal{O}_S$ and $d > 0$, we denote by dC the scheme defined by the ideal sheaf $I_C^d \subset \mathcal{O}_S$.

2. Smoothness at FoLoMo Cohen-Macaulay curves

In this section we denote by $p : X \to Y$ an elliptically fibred threefold with section $\sigma : Y \hookrightarrow X$, where X and Y are smooth (not necessarily projective). Let $B \subset Y$ be a smooth projective curve, which we can view as a curve in X via the section. We denote its homology class on X by B as well. Let

$$\beta := B + dF \in H_2(X),$$

where F denotes the class of the fibre and $d \geq 0$. Let $0 \leq \ell \leq d$ and define

$$n := 1 - g_B - \ell, \tag{10}$$

where g_B is the arithmetic genus of B. As usual, we denote by $\mathrm{Hilb}^{\beta,n}(X)$ the Hilbert scheme of closed subschemes $C \subset X$ with $[C] = \beta$ and $\chi(\mathcal{O}_C) = n$.

We now present a way of producing certain Cohen-Macaulay curves in $\mathrm{Hilb}^{\beta,n}(X)$. Denote by $\mathrm{Hilb}^d(Y)$ the Hilbert scheme of 0-dimensional length d subschemes of Y and consider the stratum

$$\Sigma_\ell := \left\{ Z \in \mathrm{Hilb}^d(Y) \ : \ \ell(Z \cap B) = \ell \right\},$$

where $\ell(Z \cap B)$ denotes the length of the scheme theoretic intersection $Z \cap B$. We also use $\mathrm{Hilb}^B(Y)$ —the Hilbert scheme of effective divisors on Y with class B. Consider the morphism

$$\begin{aligned} \Sigma_\ell \times \mathrm{Hilb}^B(Y) &\to \mathrm{Hilb}^{\beta,n}(X) \\ (Z, B') &\mapsto B' \cup p^*Z, \end{aligned} \tag{11}$$

where $B' \cup p^*Z$ denotes the scheme theoretic union. We refer to curves of this form as *untwisted Cohen-Macaulay curves*. For any such $C = B' \cup p^*Z$ we have a short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{B'} \oplus \mathcal{O}_{p^*Z} \to \mathcal{O}_{B' \cap p^*Z} \to 0,$$

where $\ell(B \cap p^*Z) = \ell$. From the short exact sequence we deduce

$$\begin{aligned} \chi(\mathcal{O}_C) &= \chi(\mathcal{O}_{B'}) + \chi(\mathcal{O}_{p^*Z}) - \ell \\ &= 1 - g_B + \chi(\mathcal{O}_Z \otimes Rp_*\mathcal{O}_X) - \ell \\ &= 1 - g_B - \ell \\ &= n, \end{aligned}$$

where we use that $Rp_*\mathcal{O}_X$ is a complex of rank 0 because X is elliptically fibred and n is given by (10). Our starting point is the following smoothness result.

Proposition 19. *$\Sigma_\ell \subset \mathrm{Hilb}^d(Y)$ is locally closed and smooth of dimension $2d - \ell$.*

Proof. The constructible function

$$\mathrm{Hilb}^d(Y) \to \mathbb{Z}, \ Z \mapsto \ell(Z \cap B)$$

is upper semicontinuous, which shows Σ_ℓ is locally closed .

More detail? The flattening stratification for $\mathcal{O}_{\mathcal{Z} \cap (\mathrm{Hilb}^d(Y) \times Y)}$ on $\mathrm{Hilb}^d(Y) \times Y \to \mathrm{Hilb}^d(Y)$ exactly gives the desired stratification b/c restrictions to fibres are 0-dimensional.

The stratum Σ_ℓ has an open subset consisting of configurations of d distinct points, ℓ of which lie on B. This locus is smooth of dimension $2d - \ell$. We now show that for any $Z \in \Sigma_\ell$ we have

$$\hom_Y(I_Z, \mathcal{O}_Z) = 2d - \ell. \tag{12}$$

Since Z is 0-dimensional, it suffices to prove this for the case $Y = \operatorname{Spec} \mathbb{C}[\![x, y]\!]$. Then Y has the standard $\mathbb{C}^{*2}$-action, which also acts on Σ_ℓ. Any element $I = I_1 \in \mathrm{Hilb}^d(Y)$

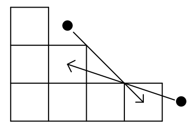

FIGURE 2. The partition $\lambda = (3, 2, 1, 1)$ corresponds to the ideal $I_Z = (y^3, xy^2, x^2y, x^4)$. The two Haiman arrows correspond to the box $(1, 0)$.

moves in a flat family I_t to its initial ideal $\mathrm{in}(I) = I_0 \in \mathrm{Hilb}^d(Y)$, which is a monomial ideal (w.r.t. some chosen order). This process is called Groebner degeneration [**?**, Sect. 18.2]. For the standard lexicographic order $x > y$, it is easy to see that if $I_1 \in \Sigma_\ell$, then $I_t \in \Sigma_\ell$ for all t. In particular $I_0 \in \Sigma_\ell$. Therefore any $\mathbb{C}^{*2}$ orbit in Σ_ℓ contains a monomial ideal in its closure. Consequently we only have to prove (12) at $\mathbb{C}^{*2}$ fixed points.

Suppose $I_Z \in \mathrm{Hilb}^d(Y)$ is a $\mathbb{C}^{*2}$ fixed point, i.e. I_Z is a monomial ideal. There is a pictorial way of describing a basis of $2d$ element of the vector space $\mathrm{Hom}_Y(I_Z, \mathcal{O}_Z)$ due to M. Haiman [**?**, Sect. 18.2], which we now describe. Let $\lambda \subset \mathbb{Z}^2$ be the partition corresponding to I_Z defined by the requirement that $(\alpha, \beta) \in \lambda$ if and only if $x^\alpha y^\beta \notin I_Z$. In particular, we write $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$ where $(\alpha, \beta) \notin I_Z$ if and only if $0 \leq \beta \leq \lambda_\alpha - 1$. We denote the transpose of λ by λ'. Let $(\alpha, \beta) \in \lambda$ be a box in λ. To it we associate two arrows, which we refer to as *Haiman arrows*: one with tail (α, λ_α) and head $(\lambda'_\beta - 1, \beta)$ and one with tail (λ'_β, β) and head $(\alpha, \lambda_\alpha - 1)$. For an example, see Figure 2. Two arrows are identified if one translates to the other whilst keeping the tail outside λ and the head inside λ. To each Haiman arrow we assign an element of $\mathrm{Hom}(I_Z, \mathcal{O}_Z)$ as follows. Suppose we translate the arrow to the unique position for which the tail emanates from a homogeneous generator g of I_Z. The corresponding morphism $I_Z \to \mathcal{O}_Z$ is defined by sending g to the element of $\mathcal{O}_Z$ corresponding to the head of the arrow and all other homogenous generators to zero. This describes a basis of $2d$ elements for $\mathrm{Hom}_Y(I_Z, \mathcal{O}_Z)$.

Is this originally due to Haiman? Is it ok if we call them Haiman arrows?

Let $D := \mathbb{C}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers. We have to figure out which Haiman arrows correspond to first order deformations that move Z out of Σ_ℓ. A morphism $\phi \in \mathrm{Hom}_Y(I_Z, \mathcal{O}_Z)$ corresponds to the following first order deformation

$$I_{Z_\phi} = \{f + \epsilon \cdot g \;:\; f \in I_Z,\; g \in \mathbb{C}[\![x, y]\!] \text{ and } \phi(f) = [g] \in \mathcal{O}_Z\} \subset \mathbb{C}[\![x, y]\!] \otimes_{\mathbb{C}} D.$$

Note that each $Z_\phi \subset Y \times D$ comes from a (not necessarily unique) global deformation $\widetilde{Z}_\phi \subset Y \times \mathbb{C}$. The scheme theoretic intersection of $\widetilde{Z}_\phi$ with $B \times \mathbb{C}$ and $B \times \mathbb{C}^*$ are given by the following ideals

$$(I_{\widetilde{Z}_\phi} + (y))/(y) \subset \mathbb{C}[\![x, y, \epsilon]\!]/(y),\; (I_{\widetilde{Z}_\phi} + (y))/(y) \subset \mathbb{C}[\![x, y, \epsilon, \epsilon^{-1}]\!]/(y).$$

Using the explicit description of Haiman arrows given above, we see that the family over $B \times \mathbb{C}^*$ lies outside Σ_ℓ precisely when the tail of the arrow is located at (α, λ_α) with $0 \leq \alpha \leq \ell(\lambda) - 1$ and the head of the arrow is located at $(\ell(\lambda) - 1, 0)$. Since there are precisely ℓ such arrows we obtain (12).

As an illustration of this principle we give the two deformations corresponding to the Haiman arrows of Figure 2

$$I_{Z_{\phi_1}} = (y^3, xy^2 + \epsilon x^3, x^2y, x^4), I_{Z_{\phi_2}} = (y^3, xy^2, x^2y, x^4 + \epsilon xy).$$

The first deformation leads to scheme theoretic intersection $(\epsilon x^3, x^4)$ with B while the second to scheme theoretic intersection (x^4) with B. Therefore ϕ_1 is normal to Σ_4, whereas ϕ_2 is tangent to Σ_4. $\square$

Among all untwisted CM curves on X, we identify a special class of curves:

Definition 20. We say that an element $Z \in \Sigma_\ell \subset \mathrm{Hilb}^d(Y)$ is *formally locally monomial with respect to* B, if at each reduced point $P \in \mathrm{Supp}(Z)$ we can choose coordinates x, y on the formal neighborhood

$$\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[\![x, y]\!]$$

such that

- $I_{B \subset Y} \subset \widehat{\mathcal{O}}_{Y,P}$ is the ideal (y),
- $I_Z \subset \widehat{\mathcal{O}}_{Y,P}$ is a monomial ideal in x, y.

We call an untwisted CM curve $B \cup p^*Z$ *formally locally monomial* (FoLoMo in short) when Z is formally locally monomial with respect to B.

The main result of this appendix is the following:

Theorem 21. *Let $p : X \to Y$ be an elliptically fibred threefold with section. Let $B \subset Y$ be a smooth projective curve, $\beta = B + dF$, and $n = 1 - g_B - \ell$ for some $0 \le \ell \le d$. Then the Zariski tangent space at FoLoMo CM curves in* $\mathrm{Hilb}^{\beta,n}(X)$ *has dimension*

$$2d - \ell + h^0(N_{B/X}).$$

Before giving the proof, we discuss our main application.

Corollary 22. *Let the setup be as in Theorem 21. Assume in addition $B \in \mathrm{Hilb}^B(Y)$ is a smooth point and $h^0(N_{B/X}) = h^0(N_{B/Y})$. Then* $\mathrm{Hilb}^{\beta,n}(X)$ *is smooth of dimension* $2d - \ell + h^0(N_{B/Y})$ *at FoLoMo CM curves.*

Proof. Consider the morphism (32)

$$\Sigma_\ell \times \mathrm{Hilb}^B(Y) \to \mathrm{Hilb}^{\beta,n}(X).$$

By Proposition 19 and the assumption on B, the domain is smooth in a neighbourhood of any point of the form (Z, B). The Zariski tangent space at such points has dimension $2d - \ell + h^0(N_{B/Y})$ again by Proposition 19. If in addition Z is FoLoMo with respect to B, then $\mathrm{Hilb}^{\beta,n}(X)$ has Zariski tangent space of the same dimension at $C = B \cup p^*Z$ by Theorem 21 and the assumption $h^0(N_{B/X}) = h^0(N_{B/Y})$. Hence our map is a local isomorphism at all points (Z, B) with Z FoLoMo with respect to B. □

Remark 23. The assumption $h^0(N_{B/X}) = h^0(N_{B/Y})$ is satisfied for $X = \mathrm{Tot}(K_S)$, where S is any elliptically fibered surface considered in this paper; i.e. with section B and N rational nodal fibres. Define $Y = \mathrm{Tot}(K_S|_B)$, then we have a morphism $p : X \to Y$ with section. Moreover $N_{B/S}$ is the dual of the fundamental line bundle which has degree $-\chi(\mathcal{O}_S) = -N/12 < 0$. Therefore

$$H^0(N_{B/X}) = H^0(N_{B/S}) \oplus H^0(N_{B/Y}) = H^0(N_{B/Y}).$$

In Section **??** we actually need the value of $h^0(N_{B/Y})$ which we now compute. Since X is Calabi-Yau, $N_{B/Y} \cong N^*_{B/S} \otimes K_B$. By Riemann-Roch $\chi(N_{B/S}) = 1 - g_B - \chi(\mathcal{O}_S)$ and Serre duality implies

$$h^0(N_{B/Y}) = \chi(\mathcal{O}_S) + g_B - 1.$$

Remark 24. It is not unreasonable to expect that Theorem 21 holds at all untwisted CM curves $B' \cup p^*Z$ with $Z \in \Sigma_\ell$ and $Y \in \mathrm{Hilb}^B(Y)$. Suppose this is the case. Also suppose $\mathrm{Hilb}^d(Y)$ is irreducible and smooth. Then we get an irreducible smooth Zariski open subset of $\mathrm{Hilb}^{\beta,n}(X)$ isomorphic to $\Sigma_\ell \times \mathrm{Hilb}^B(Y)$. Geometrically: the untwisted

CM curves form an irreducible smooth Zariski open in $\mathrm{Hilb}^{\beta,n}(X)$ and the closure gives a component of dimension $2d-\ell+h^0(N_{B/X})$. Elements of the boundary are obtained by families $Z_t \in \mathrm{Hilb}^d(Y)$ for which $Z_t \in \Sigma_\ell$ when $t \neq 0$ and $Z_0 \notin \Sigma_\ell$. This gives rise to families $C_t = B \cap p^*Z_t \in \mathrm{Hilb}^{\beta,n}(X)$, which pick up embedded points at $t=0$. At such a boundary curve C_0 we may move into other components of $\mathrm{Hilb}^{\beta,n}(X)$ by realizing C_0 as the limit of a family of *twisted CM curves* D_t, where the limit $D_0 = C_0$ picks up an embedded point.

Proof of Theorem 21. Let $Z \in Y^{[n]}$ be a FoLoMo subscheme and consider $C = B \cup p^*Z$.

Step 1: Let U_a be an affine open cover of Y such that each U_a contains at most one point of $\mathrm{Supp}(Z)$. Let $Z_a := Z \cap U_a$, $B_a := B \cap U_a$, $d_a = \ell(Z_a)$, $\ell_a = \ell(Z_a \cap B_a)$, $X_a := p^*U_a$, and $C_a := C \cap X_a$. We will prove

$$\mathrm{Hom}_{X_a}(I_{C_a}, \mathcal{O}_{C_a}) \cong \mathbb{C}^{2d_a-\ell_a} \oplus \Gamma(U_a, N_{B_a/X_a}). \tag{13}$$

Moreover we will see that the elements of $\mathbb{C}^{2d_a-\ell_a}$ restrict to 0 on overlaps $U_a \cap U_b$. Therefore, gluing of morphisms proves the theorem.

Step 2: We are reduced to work over the opens $X_a \to U_a$. In the case $\mathrm{Supp}(Z_a)$ is empty (13) is clear, so we assume $\mathrm{Supp}(Z_a)$ is non-empty, in which case it contains a single point which we denote by P. For notational convenience we drop all subscripts a, remembering that X means X_a, C means C_a, U means U_a etc.

We construct a Koszul resolution of $C = B \cup p^*Z$. Let $S := p^*B$, which is an elliptic surface over B. We need a divisor $M \subset X$ which intersects both S and U transversally at $P \in U \subset X$. Since we work locally on the base, this can be achieved by taking a divisor Δ transverse to $B \subset U$ at P (obtained after possibly shrinking U) and pulling it back to X. Since Z if FoLoMo with respect to B, we have formal coordinates x, y at $P \in U$ such that formally locally

$$I_Z = (x^{\alpha_1}y^{\beta_1}, \ldots, x^{\alpha_r}y^{\beta_r}, x^\ell),$$

for some minimal set of monomial generators ordered by $\alpha_1 > \cdots > \alpha_r \geq \ell$. The standard minimal Koszul resolution of I_Z can be slightly modified to give a Koszul resolution of I_C as follows

$$\begin{aligned} &0 \longrightarrow R^* \xrightarrow{\mathsf{M}} G^* \xrightarrow{\mathsf{N}} I_C \longrightarrow 0, \\ &G := \bigoplus_{i=1}^{r} \mathcal{O}_X(\alpha_i M + \beta_i S) \oplus \mathcal{O}_X(\ell M + U), \\ &R := \bigoplus_{i=1}^{r-1} \mathcal{O}_X(\alpha_{i+1} M + \beta_i S) \oplus \mathcal{O}_X(\ell M + S + U), \end{aligned} \tag{14}$$

where, in matrix notation, we have

$$\mathsf{N} := (x^{\alpha_1}y^{\beta_1} \ \ldots \ x^{\alpha_r}y^{\beta_r} \ x^{\ell}z),$$

$$\mathsf{M} := \begin{pmatrix} x^{\alpha_2-\alpha_1} & 0 & & 0 & 0 \\ -y^{\beta_1-\beta_2} & x^{\alpha_3-\alpha_2} & & 0 & 0 \\ 0 & -y^{\beta_2-\beta_1} & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & x^{\alpha_r-\alpha_{r-1}} & 0 \\ 0 & 0 & & -y^{\beta_{r-1}-\beta_r} & x^{\ell-x_r}z \\ 0 & 0 & & 0 & -y \end{pmatrix}.$$

Here z is the local coordinate at $P \in U \subset X$ defining the section $U \subset X$.

The second exact sequence we use is

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_B \oplus \mathcal{O}_{p^*Z} \longrightarrow \mathcal{O}_{B\cap p^*Z} \longrightarrow 0.$$

We conclude that $\mathrm{Hom}_X(I_C, \mathcal{O}_C)$ equals the kernel of

$$\Phi \oplus \Psi : \Gamma(G|_B) \oplus \Gamma(G|_{p^*Z}) \longrightarrow \Gamma(R|_B) \oplus \Gamma(R|_{p^*Z}) \oplus \Gamma(G|_{B\cap p^*Z}),$$

where the $\Phi = \mathsf{M}|_B^t \oplus \mathsf{M}|_{p^*Z}^t$ and Ψ is "difference restricted to $B \cap p^*Z$".

Step 3: We first calculate the kernel of

$$\Gamma(G|_B) \oplus \Gamma(G|_{p^*Z}) \xrightarrow{\Phi} \Gamma(R|_B) \oplus \Gamma(R|_{p^*Z}).$$

The matrix $\mathsf{M}|_B^t$ is obtained by putting $y = z = 0$, which are local equations defining $B \subset U \subset X$ at P. From the explicit form of the matrix we deduce at once that the kernel of $\Phi|_{\Gamma(G|_B)}$ equals

$$\Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(\mathcal{O}_B(\ell M + U)).$$

The kernel of $\Phi|_{\Gamma(p^*\mathcal{O}_Z\otimes G)}$ is more complicated. Recall that $S = p^*B$ and $M = p^*\Delta$, where $B, \Delta \subset U$ intersect transversally at P. We want to compute the kernel of

$$\begin{array}{ccc} \begin{smallmatrix} \bigoplus_{i=1}^r \Gamma(X,p^*\mathcal{O}_Z(\alpha_i M+\beta_i S)) \\ \oplus\Gamma(X,p^*\mathcal{O}_Z(\ell M+U)) \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} \bigoplus_{i=1}^{r-1} \Gamma(X,p^*\mathcal{O}_Z(\alpha_{i+1} M+\beta_i S)) \\ \oplus\Gamma(X,p^*\mathcal{O}_Z(\ell M+S+U)) \end{smallmatrix} \\ \wr\| & & \wr\| \\ \begin{smallmatrix} \bigoplus_{i=1}^r \Gamma(U,\mathcal{O}_Z(\alpha_i \Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta)\otimes p_*\mathcal{O}_X(U)) \end{smallmatrix} & & \begin{smallmatrix} \bigoplus_{i=1}^{r-1} \Gamma(U,\mathcal{O}_Z(\alpha_{i+1}\Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta+B)\otimes p_*\mathcal{O}_X(U)) \end{smallmatrix}, \end{array} \tag{15}$$

where

$$p_*\mathcal{O}_X(U) \cong p_*\mathcal{O}_X = \mathcal{O}_U,$$

because X is elliptically fibred.

Before computing the kernel of (15), we consider a slightly easier problem. If we remove the divisor U from G and R and z from the matrices M and N in (14), then we get the Koszul resolution of I_{p^*Z}. In this case the analog of (15) is

$$\begin{smallmatrix} \bigoplus_{i=1}^r \Gamma(U,\mathcal{O}_Z(\alpha_i\Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta)) \end{smallmatrix} \longrightarrow \begin{smallmatrix} \bigoplus_{i=1}^{r-1} \Gamma(U,\mathcal{O}_Z(\alpha_{i+1}\Delta+\beta_i B)) \\ \oplus\Gamma(U,\mathcal{O}_Z(\ell\Delta+B)) \end{smallmatrix}, \tag{16}$$

which has kernel $\mathrm{Hom}_U(I_Z, \mathcal{O}_Z)$. Therefore a basis for the kernel of (16) is formed by the $2d$ Haiman arrows described in the proof of Proposition 19. The Haiman arrows correspond to morphisms

$$\mathcal{O}_U(-\alpha_i\Delta - \beta_i B) \to \mathcal{O}_Z \tag{17}$$

$$\mathcal{O}_U(-\ell\Delta) \to \mathcal{O}_Z, \tag{18}$$

where, in a neighborhood of the point $P \in U$, the generator maps to the monomial indicated by the head of the arrow. The maps on X are obtained by applying p^* to these morphism.

Since $p_*\mathcal{O}_X(U) \cong \mathcal{O}_U$, the kernel of (15) has *the same* basis of Haiman arrows with one minor (but crucial) modification. Under the vertical isomorphisms of (15), Haiman arrows of the form (17) map to

$$\mathcal{O}_X(-\alpha_i M - \beta_i S) = p^*\mathcal{O}_U(-\alpha_i\Delta - \beta_i B) \to p^*\mathcal{O}_Z$$

as before. However Haiman arrows of the form (18) map to

$$\mathcal{O}_X(-\ell M - U) = p^*\mathcal{O}_U(-\ell\Delta) \otimes \mathcal{O}_X(-U) \to p^*\mathcal{O}_Z \otimes \mathcal{O}_X(-U) \hookrightarrow p^*\mathcal{O}_Z. \tag{19}$$

Due to the extra $\mathcal{O}_X(-U) \hookrightarrow \mathcal{O}_X$, these latter Haiman arrows all restrict to zero on $B \cap p^*Z$.

In conclusion

$$\ker\Phi = \Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(\mathcal{O}_B(\ell M + U)) \oplus \mathbb{C}^{2d-\ell} \oplus z\cdot\mathbb{C}^\ell, \tag{20}$$

where $z\cdot\mathbb{C}^\ell$ is spanned by Haiman arrows of the form (19).

Step 4: Finally, we study the kernel of Ψ restricted to the kernel of Φ (20). Among the Haiman arrows spanning $\mathbb{C}^{2d-\ell}$ in (20), there is a subspace $\mathbb{C}^{2d-2\ell}$ spanned by arrows which map to zero on restriction to $B \cap p^*Z$. These are the arrows corresponding to the boxes $(\alpha, \beta) \in \lambda$ with $\beta > 0$. As already observed in Step 3, the entire subspace $z\cdot\mathbb{C}^\ell$ restricts to zero on $B \cap p^*Z$. This leaves us with a subspace

$$\mathbb{C}^\ell \subset \mathbb{C}^{2n-\ell}$$

spanned by Haiman arrows which may not restrict to zero on $B \cap p^*Z$. These are the arrows with tail in (α, λ_α) with $0 \leq \alpha \leq \ell - 1$ and head in $(\ell-1, 0)$.

The space $\mathbb{C}^\ell$ discussed above splits up further as

$$\mathbb{C}^\ell = \mathbb{C}^{\alpha_r} \oplus \mathbb{C}^{\ell-\alpha_r},$$

where the first component is spanned by arrows with tail in (α, λ_α) for $0 \leq \alpha \leq \alpha_r - 1$ and the second by arrows with tail in (α, λ_α) for $\alpha_r \leq \alpha \leq \ell - 1$. In Step 3 we proved

$$\Gamma(\mathcal{O}_B(\alpha_i M + \beta_i S)) \cap \ker\Phi = 0,$$

for all $i = 1, \ldots, r-1$. We deduce at once that $\mathbb{C}^{\alpha_r} \cap \ker\Psi = 0$.

So far the kernel consists of $\mathbb{C}^{2d-\ell}$ to which we have to add the kernels of the following two maps

$$\Gamma(\mathcal{O}_B(\ell M + U)) \oplus \Gamma(p^*\mathcal{O}_Z(\ell M + U)) \cap \ker\Phi \to \Gamma(\mathcal{O}_{B\cap p^*Z}(\ell M + U)), \tag{21}$$

$$\Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(p^*\mathcal{O}_Z(\alpha_r S + U)) \cap \ker\Phi \to \Gamma(\mathcal{O}_{B\cap p^*Z}(\alpha_r S + U)). \tag{22}$$

The kernel of (21) is easy because

$$\Gamma(\mathcal{O}_B(\ell M + U)) \oplus \Gamma(p^*\mathcal{O}_Z(\ell M + U)) \cap \ker\Phi = \Gamma(\mathcal{O}_B(\ell M + U)) \oplus z\cdot\mathbb{C}^\ell.$$

and $z\cdot\mathbb{C}^\ell$ restricts to zero on $Z \cap p^*Z$. Therefore the kernel of (21) is the kernel of

$$\Gamma(\mathcal{O}_B(\ell M + U)) \cong \Gamma(N_{B/S}(\ell P)) \to \Gamma(N_{B/S}(\ell P)|_{\ell P}),$$

where ℓP denotes the ℓ times thickening of P in B. This gives $\Gamma(N_{B/S})$.

Finally we calculate the kernel of (22), which we claim is $\Gamma(N_{B/Y})$. Note that

$$\Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \Gamma(p^*\mathcal{O}_Z(\alpha_r M + S)) \cap \ker\Phi = \Gamma(\mathcal{O}_B(\alpha_r M + S)) \oplus \mathbb{C}^{\ell-\alpha_r},$$

where $\mathbb{C}^{\ell-\alpha_r}$ was introduced above. In the case $\alpha_r = \ell$ the same reasoning as above gives kernel $\Gamma(N_{B/U})$ and we are done. Now the case $\alpha_r < \ell$. For any $i \geq 0$, we think of

$$\Gamma(\mathcal{O}_B((\alpha_r - i)M + S))$$

as the $\mathcal{O}_B$-module generated by a single morphism

$$\mathcal{O}_X(-\alpha_r M - S) \to \mathcal{O}_B(-iP) \subset \mathcal{O}_B,$$

which does not factor through $\mathcal{O}_B(-(i+1)P)$. (Recall that we work on an *affine* open U.) For any $i = 0, \dots, \alpha_r - 1$, the generator of $\Gamma(\mathcal{O}_B((\alpha_r - i)M + S))$ lands in a box $(\alpha, 0)$ with $0 \leq \alpha \leq \alpha_r - 1$. However all arrows of $\mathbb{C}^{\ell-\alpha_r}$ land in $(\ell - 1, 0)$ and they *cannot* be translated so their heads lie in $(\alpha, 0)$ with $0 \leq \alpha \leq \alpha_r - 1$. We conclude that non-zero elements of $\Gamma(\mathcal{O}_B((\alpha_r - i)M + S))$ do not occur in the kernel for any $i = 0, \dots, \alpha_r - 1$. Therefore the kernel factors through

$$\Gamma(N_{B/U}) \cong \Gamma(\mathcal{O}_B(S)) \subset \Gamma(\mathcal{O}_B(\alpha_r + S)).$$

The generators of $\Gamma(N_{B/U}((\alpha_r - \ell - 1)P)$ automatically restrict to zero on $B \cap p^* Z$. This leaves us with $\ell - \alpha_r$ generators landing in $(\alpha, 0)$ with $\alpha_r \leq \alpha \leq \ell - 1$, none of which restricts to zero. We can take any linear combination of these generators, which then *determines* the remaining arrows of $\mathbb{C}^{\ell-\alpha_r}$ uniquely upon restriction. (Up to translation, the Haiman arrows corresponding to a basis of $\mathbb{C}^{\ell-\alpha_r}$ exactly land in $(\alpha, 0)$ with $\alpha_r \leq \alpha \leq \ell - 1$.) Therefore the kernel of (22) is isomorphic to $\Gamma(N_{B/U})$.

In conclusion we find the following kernel

$$\mathbb{C}^{2d-\ell} \oplus \Gamma(N_{B/S}) \oplus \Gamma(N_{B/U}) \cong \mathbb{C}^{2d-\ell} \oplus \Gamma(N_{B/X}). \tag{23}$$

Recall that we work on opens $X_a \to U_a$. Since the elements of the first factor of (23) restrict to zero on overlaps $X_a \cap X_b$, we proved what we claimed in Step 1. $\square$

3. Stuff we gathered

3.1. **Normal bundle.** First order deformations of $B \subset X$ are given by $H^0(N_{B/X})$. We now calculate this. Let $\pi : X \to S$ and $p : S \to B$. Facts: (adjunction, canonical bundle of an elliptic surface)

$$N_{B/X} \cong N_{B/S} \oplus K_S|_B$$
$$K_B \cong K_S|_B \otimes N_{B/S} \cong K_S|_B(S)$$
$$K_S \cong p^*(K_B \otimes L),$$

where $L = (R^1 p_* \mathcal{O}_X)^\vee$ and

$$\deg L = \chi(\mathcal{O}_S) = \frac{e(S)}{12} = \frac{N}{12} > 0,$$

where N is the number of nodal fibres. Combining gives

$$N_{B/S} \cong K_B \otimes K_S^{-1}|_B \cong L^\vee,$$

which has negative degree so

$$H^0(N_{B/S}) = 0.$$

Next

$$H^0(K_S|_B) \cong H^1(K_S|_B^{-1} \otimes K_B)^* \cong H^1(N_{B/S})^*.$$

By Riemann-Roch and $H^0(N_{B/S}) = 0$ we get

$$h^1(N_{B/S}) = -\chi(N_{B/S}) = -\chi(L^\vee) = -(1 - g - \chi(\mathcal{O}_S)) = \chi(\mathcal{O}_S) - 1 + g.$$

In conclusion

$$h^0(N_{X/B}) = \chi(\mathcal{O}_S) - 1 + g = \frac{e(S)}{12} - \frac{e(B)}{2}.$$

Question: Do we not need Appendix A.1 anymore? I don't think so.

3.2. **Impact on Behrend function.** How does this affect the overall sign of the Behrend function? The section B cannot move in the surface direction and in the fibre direction we can move it by sections of

$$H^0(K_S|_B) \cong \mathbb{C}^{\chi(\mathcal{O}_S)-1+g}.$$

Aside: $K_S|_F = \mathcal{O}_F$ for each fibre F. Let $\mathrm{Hilb}^{B+\bullet F}(X)_{\mathrm{CM}}$ be the Hilbert scheme of *naked* CM curved in class $B + \bullet F$. In the last section we hopefully see that locally formally a neighborhood of $C \in \mathrm{Hilb}^{B+\bullet F}(X)_{\mathrm{CM}}^{\mathbb{C}^*}$ inside $\mathrm{Hilb}^{B+\bullet F}(X)_{\mathrm{CM}}$ is of the form

I'm now being really sketchy!

$$H^0(K_S|_B) \times M,$$

where M is some space (which below we hopefully identify with some locus in $Y^{[n]}$ for $Y = K_S|_B$). Hence for any $C \in \mathrm{Hilb}^{B+\bullet F}(X)_{\mathrm{CM}}^{\mathbb{C}^*}$ we have

$$\nu(C) = (-1)^{\chi(\mathcal{O}_S)-1+g} \cdot \nu_M(C) = (-1)^{\frac{e(S)}{12} - \frac{e(B)}{2}} \cdot \nu_M(C).$$

Suppose C is described by partitions $\lambda^{(i)}$ with

$$\sum_i |\lambda^{(i)}| = d.$$

We also want to prove M is *smooth* of dimension

$$2d - \sum_i \lambda_0^{(i)},$$

where $\lambda^{(i)} = (\lambda_0^{(i)} \geq \lambda_1^{(i)} \geq \cdots)$.

Don't we also get Behrend signs $(-1)^{\lambda_0^{(i)}}$? Those seem bad!

3.3. **Hilbert schemes on $\mathbb{C}^2$.** Let $X = \mathbb{C}^2$ and

$$L_k := \{Z \in X^n \;:\; \ell(Z \cap \{y=0\}) = k\} \subset X^{[n]}.$$

Why normal?

Ref? I learned this from Dori Bejleri.

Claim: L_k is locally closed and smooth of dimension $2n-k$. Proof: since L_k is normal and $T = \mathbb{C}^{*2}$ acts on it, it suffices to prove smoothness at T-fixed points. At the T-fixed points we can write a basis for the tangent space by pairs of arrows with tail just outside and head just inside as described by Haiman. I'm not going to write this out now formally. Just remember how each arrow defines a first order deformation of the monomial ideal ("plus ϵ times the monomial the head points at"). The corresponding global deformation can easily be seen to move outside of L_k for precisely k arrows with head in the bottom row. This gives the result.

3.4. **Key iso.** Let $C \in \mathrm{Hilb}^{B+\bullet F}(X)^{\mathbb{C}^*}_{\mathrm{CM}}$ be described by partitions $\lambda^{(i)}$ with

$$\sum_i |\lambda^{(i)}| = d.$$

Claim: there is a formal neighborhood of C inside $\mathrm{Hilb}^{B+\bullet F}(X)_{\mathrm{CM}}$ which maps to $H^0(N_{B/X})$ with fibre

$$L_k,$$

where $k := \sum \lambda_0^{(i)}$ and L_k is smooth by the previous subsection.

Let's look at the case where $C = B \cup F_\lambda$, where $E := F_\lambda$ is a single thickened fibre with cross-section λ. We have a couple of useful short exact sequences

$$0 \to I_E/I_C \to \mathcal{O}_C \to \mathcal{O}_E \to 0 \tag{24}$$

$$0 \to I_B/I_C \to \mathcal{O}_C \to \mathcal{O}_B \to 0 \tag{25}$$

$$0 \to I_C \to I_E \to I_E/I_C \to 0 \tag{26}$$

$$0 \to I_C \to I_B \to I_B/I_C \to 0, \tag{27}$$

where

$$I_E/I_C \cong \mathcal{O}_B(-\lambda_0)$$
$$G := I_B/I_C,$$

where G is an ideal sheaf of a fat point of length λ_0 inside $\mathcal{O}_E$:

$$0 \to G \to \mathcal{O}_E \to \mathcal{O}_{\lambda_0 p} \to 0. \tag{28}$$

Two more significant short exact sequences

$$0 \to \mathcal{O}_C \to \mathcal{O}_B \oplus \mathcal{O}_E \to \mathcal{O}_{\lambda_0 p} \to 0 \tag{29}$$

$$0 \to \mathcal{O}_B(-\lambda_0) \to \mathcal{O}_B \to \mathcal{O}_{\lambda_0 p} \to 0. \tag{30}$$

We can build *many* double complexes from these seven short exact sequences by applying $\mathrm{Hom}(I_C, \cdot)$, $\mathrm{Hom}(\cdot, \mathcal{O}_C)$ etc.

It seems significant to apply $\mathrm{Hom}(I_E, \cdot)$ to (29). This gives

$$0 \to \mathrm{Hom}(I_E, G) \to \mathrm{Hom}(I_E, \mathcal{O}_E) \xrightarrow{\alpha} \mathrm{Hom}(I_E, \mathcal{O}_{\lambda_0 p}) \to \cdots \tag{31}$$

The last hom space seems isomorphic to $\mathbb{C}^{\lambda_0}$ and we believe there is a splitting of the map α from the local description. It seems $\mathrm{Hom}(I_E, G)$ can be interpreted as the space of deformations of E which keep the length of $E \cap B$ fixed! We want to prove there exists a short exact sequence

$$0 \to \mathrm{Hom}(I_E, G) \to \mathrm{Hom}(I_C, \mathcal{O}_C) \to H^0(N_{B/X}) \to 0.$$

The good news: we can construct an injection $\mathrm{Hom}(I_E, G) \hookrightarrow \mathrm{Hom}(I_C, \mathcal{O}_C)$ as follows. From (25) and (26) we get the following double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{O}_B(-\lambda_0), G) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{O}_B(-\lambda_0), \mathcal{O}_C) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{O}_B(-\lambda_0), \mathcal{O}_B) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(I_E, G) \arrow[r] \arrow[d] & \mathrm{Hom}(I_E, \mathcal{O}_C) \arrow[r] \arrow[d] & \mathrm{Hom}(I_E, \mathcal{O}_B) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(I_C, G) \arrow[r] \arrow[d] & \mathrm{Hom}(I_C, \mathcal{O}_C) \arrow[r] \arrow[d] & \mathrm{Hom}(I_C, \mathcal{O}_B) \arrow[r] \arrow[d] & \cdots \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Note: $\mathrm{Hom}(\mathcal{O}_B(-\lambda_0), G) = 0$ from the local description, so indeed we get the required injection!

Significant special case. When S is the rational elliptic surface we have $H^0(N_{B/X}) = H^1(N_{B/X}) = 0$ in which case we get $\mathrm{Hom}(I_B, \mathcal{O}_B) = \mathrm{Ext}^1(I_B, \mathcal{O}_B) = 0$ (see next subsection for relating $\mathrm{Ext}^i(I_B, \mathcal{O}_B)$ to $H^i(N_{B/X})$)! In this case we really expect our injection

$$\mathrm{Hom}(I_E, G) \hookrightarrow \mathrm{Hom}(I_C, \mathcal{O}_C)$$

to be an isomorphism. If so, our earlier split exact sequence (31) proves $\mathrm{Hom}(I_C, \mathcal{O}_C)$ has dimension

$$\hom(I_E, \mathcal{O}_E) - \lambda_0 = \hom(I_\lambda, \mathcal{O}_\lambda) - \lambda_0 = 2|\lambda| - \lambda_0,$$

where $\mathcal{O}_\lambda \subset \mathbb{C}^2$ is the zero-dimensional structure sheaf from which $\mathcal{O}_E$ pulls-back.

In this special case of the rational elliptic surface we have a lot more vanishing and we hope that the big double complex yields the surjection. The easiest way to get the surjection is when

$$\mathrm{Ext}^1(G, \mathcal{O}_B) \overset{?}{\cong} 0$$
$$\mathrm{Ext}^1(O_B(-\lambda_0), G) \overset{?}{\cong} 0,$$

and then use $\mathrm{Hom}(I_C, \mathcal{O}_B) \cong \mathrm{Ext}^1(G, \mathcal{O}_B)$ (which I get from $\mathrm{Hom}(I_B, \mathcal{O}_B) = \mathrm{Ext}^1(I_B, \mathcal{O}_B) = 0$). Sadly I cannot prove the two previous Ext groups are indeed zero.

Comment: there does not seem to be a simple map from $\mathrm{Hom}(I_C, \mathcal{O}_C) \to \mathrm{Hom}(I_B, \mathcal{O}_B)$ simply by playing with $\mathcal{O}_C \to \mathcal{O}_B$ or $I_C \subset I_B$.

3.5. **Miscalleneous.** Significant fact: $\mathrm{Hom}(I_C, \mathcal{O}_C) \cong \mathrm{Ext}^1(I_C, I_C)_0$ by PT1, Section 2.2 for naked CM curves.

Some remarks. By cutting out I_B from a rank 2 vector bundle we get maps

$$0 \to \mathrm{Hom}(I_B, \mathcal{O}_B) \to H^0(B, \mathcal{O}_B(S)) \oplus H^0(B, \mathcal{O}_B(Y)) \to H^0(B, \mathcal{O}_B(S+Y)) \to \cdots$$

remember: $Y := K_S|_B$. Here

$$\begin{aligned}\mathcal{O}_B(S) &\cong \mathcal{O}_S(S)|_B \cong K_X(S)|_B \cong K_S|_B \\ \mathcal{O}_B(Y) &\cong \mathcal{O}_B(B) \cong N_{B/S},\end{aligned}$$

and $N_{B/S} \otimes K_S|_B \cong K_B$. We get

$$0 \to \mathrm{Hom}(I_B, \mathcal{O}_B) \to H^0(N_{B/X}) \to H^0(K_B) \to \cdots.$$

The first map is an isomorphism because both terms are first order deformations of B as a subscheme of X. Not sure we need this.

3.6. **Direct approach.** We have the following commutative diagram

$$\begin{tikzcd} X := K_S \arrow[r, "\varpi"] \arrow[d, "\pi"'] & Y := K_S|_B \arrow[d, "q"] \\ S \arrow[r, "p"'] & B. \end{tikzcd}$$

Note: π_* and q_* are exact (because π, q are affine morphisms), but I don't think I need this now. I do need ϖ^*, p^* are exact (because ϖ, p are flat morphisms). Let $\mathrm{Hilb}^n(Y)$ be the Hilbert scheme of 0-dim subschemes of length n on Y. By exactness of ϖ^* we can pull-back

$$0 \to I_Z \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0$$

on Y to a short exact sequence

$$0 \to \varpi^* I_Z \to \mathcal{O}_X \to \varpi^* \mathcal{O}_Z \to 0$$

on X. Hence $\varpi^*\mathcal{O}_Z$ is a structure sheaf and $\varpi^* I_Z$ is its ideal sheaf. Let $\mathrm{Hilb}^0(X, [nF])$ be the Hilbert scheme of Cohen-Macaulay curves in homology class $[nF]$. Then at the set level we have a map

$$\mathrm{Hilb}^n(Y) \to \mathrm{Hilb}^0(X, [nF]),\ (I_Z \subset \mathcal{O}_Y) \mapsto (\varpi^* I_Z \subset \mathcal{O}_X).$$

This works in families ($(\varpi \times 1_B)^*$ is exact because ϖ is flat) so this indeed gives a morphism. At the level of closed points this is clearly injective.

The tangent space at $\varpi^* I_Z \in \mathrm{Hilb}^0(X, [nF])$ is

$$\begin{aligned}\mathrm{Hom}_X(\varpi^* I_Z, \varpi^* \mathcal{O}_Z) &\cong \mathrm{Hom}_Y(I_Z, \varpi_* \varpi^* \mathcal{O}_Z) \\ &\cong \mathrm{Hom}_Y(I_Z, \mathcal{O}_Z \otimes \varpi^* \mathcal{O}_X) \\ &\cong \mathrm{Hom}(I_Z, \mathcal{O}_Z),\end{aligned}$$

which is the tangent space at $I_Z \in \mathrm{Hilb}^n(Y)$. I used: (1) ordinary adjunction for Hom, (2) projection formula, (3) $\varpi_* \mathcal{O}_X \cong \mathcal{O}_Y$ (how do we prove this last one? I know $\varpi_*\mathcal{O}_X$ is a line bundle so should go similar to $p_* \mathcal{O}_S \cong \mathcal{O}_B$). So indeed deformations match! We see there cannot be any twistings but only deformations induced from moving Z in the surface.

The next part may not be needed but I find it instructive. There is a spectral sequence

$$\mathrm{Ext}^i_Y(I_Z, R^j \varpi_* \mathcal{O}_Z) \cong \mathrm{Ext}^i_Y(I_Z, \mathcal{O}_Z \otimes R^j \varpi_* \mathcal{O}_X) \Rightarrow \mathrm{Ext}^{i+j}_X(\varpi^* I_Z, \varpi^* \mathcal{O}_Z).$$

Note: $R^0 \varpi_* \mathcal{O}_X \cong \mathcal{O}_Y$, $R^1 \varpi_* \mathcal{O}_X$ is a line bundle (the fibres of ϖ have genus 1), and $R^{>1} \varpi_* \mathcal{O}_X = 0$ (the fibres are 1-dimensional). Therefore the terms of the spectral sequence are zero unless $j = 0, 1$ which leads to an injection

$$\mathrm{Ext}^1_Y(I_Z, \mathcal{O}_Z) \hookrightarrow \mathrm{Ext}^1_X(\varpi^* I_Z, \varpi^* \mathcal{O}_Z).$$

This injections maps the obstruction class for deforming Z to the obstruction class for deforming $\varpi^* Z$. In other words, we can deform Z if and only if we can deform $\varpi^* Z$. So indeed we have an isomorphism of schemes

$$\mathrm{Hilb}^n(Y) \cong \mathrm{Hilb}^0(X, [nF]).$$

From now on it gets sketchy... perhaps we'd like to deal with the section along the following lines... Let $\mathrm{Hilb}^\chi(X, [B + nF])$ be the Hilbert scheme of 1-dim subschemes with homology class $[B + nF]$ and $\chi(\mathcal{O}_Z) = \chi$ ($[B]$ the class of some section). We want to construct a morphism

$$\mathrm{Hilb}^\chi(X, [B + nF]) \to H^0(N_{B/X}), \tag{32}$$

which only remembers the section. I'm not quite sure how to do this rigorously but this sounds doable. Anyway, we want to know what $\mathrm{Hilb}^\chi(X, [B + nF])$ looks like near an element of the form $B \cup C_0$, where C_0 is a CM curve in class $[nF]$ and B a section in class $[B]$. Claim: We want to say that the scheme theoretic *fibre* of (32) over $B \in H^0(N_{B/X})$ near $B \cup C_0$ is isomorphic to the following locally closed subset (with its induced scheme structure)

$$\{C \in \mathrm{Hilb}^0(X, [nF]) \ : \ \chi(\mathcal{O}_{B\cup C}) = \chi\} \subset \mathrm{Hilb}^0(X, [nF]). \tag{33}$$

The rough idea should be that by looking in the fibre of the map (32) we do not allow B to move. Moreover an ideal sheaf of a CM curve is reflexive, hence determined entirely by its restriction to the complement of B. That's why the *fibre* near $C_0 \cup B$ should be isomorphic to (33).

Let $L_k \subset \mathrm{Hilb}^n(Y)$ be the locally closed subscheme of elements Z such that $Z \cap B$ has length k. Via our isomorphism $\mathrm{Hilb}^n(Y) \cong \mathrm{Hilb}^0(X, [nF])$ the locus (33) is clearly isomorphic to

$$L_{\chi(\mathcal{O}_B)-\chi},$$

which is *smooth*. Hence, near $B \cup C_0 \in \mathrm{Hilb}^\chi(X, [B + nF])$, the morphism (32) is a smooth morphism over a smooth base. Conclusion: near $C_0 \cup B$ we have that $\mathrm{Hilb}^\chi(X, [B + dF])$ is smooth (and we can calculate the dimension). I'm using the fact that a smooth morphism over a smooth base has a smooth total space.

3.7. **Koszul calculations.** Let $X = K_S$, $B \subset S \subset X$, and $Y := K_S|_B$. For any point on B, I use local coordinates x, y, z in a formal neighbourhood of that point such that $x = z = 0$ defines the section B. Denote by $p : X \to Y$ the projection to Y. Besides the divisors $S, Y \subset X$, we have $\tilde{F} \subset X$, where $\tilde{F} = K_S|_F$ for any fibre $F \subset S$.

Let $C = B \cup p^* Z$, where $Z \subset Y$ is a 0-dimensional scheme. Assume $I_{Z\subset Y} = (z^k, y^l)$ (so the partition is a rectangle). We have the following Koszul resolution of I_C

$$0 \to \mathcal{O}_X(-kS-l\tilde{F}) \oplus \mathcal{O}_X(-S-Y-l\tilde{F}) \to \mathcal{O}_X(-kS) \oplus \mathcal{O}_X(-S-l\tilde{F}) \oplus \mathcal{O}_X(-Y-l\tilde{F}) \to I_C \to 0,$$

where the second map is $(z^k, y^l z, xy^l)$ and the first map is

$$M := \begin{pmatrix} y^l & 0 \\ -z^{k-1} & x \\ 0 & -z \end{pmatrix}.$$

A local computation shows this is indeed a resolution. The second short exact sequence we use is

$$0 \to \mathcal{O}_C \to \mathcal{O}_B \oplus \mathcal{O}_{p^*Z} \to \mathcal{O}_P \to 0,$$

where $P := B \cap p^*Z$. Using the obvious double complex, we conclude that $\mathrm{Hom}_X(I_C, \mathcal{O}_C)$ is given as the intersection of the kernels of the following two maps

$$\begin{array}{c}\Gamma(\mathcal{O}_B(kS)) \oplus \Gamma(\mathcal{O}_B(S+l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(Y \oplus l\tilde{F})) \\ \Gamma(\mathcal{O}_{p^*Z}(kS)) \oplus \Gamma(\mathcal{O}_{p^*Z}(S+l\tilde{F})) \oplus \Gamma(\mathcal{O}_{p^*Z}(Y \oplus l\tilde{F}))\end{array} \xrightarrow{f} \begin{array}{c}\Gamma(\mathcal{O}_B(kS+l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(S+l\tilde{F}+Y)) \\ \Gamma(\mathcal{O}_{p^*Z}(kS+l\tilde{F})) \oplus \Gamma(\mathcal{O}_{p^*Z}(S+l\tilde{F}+Y))\end{array},$$

and

$$\begin{array}{c}\Gamma(\mathcal{O}_B(kS)) \oplus \Gamma(\mathcal{O}_B(S+l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(Y \oplus l\tilde{F})) \\ \Gamma(\mathcal{O}_{p^*Z}(kS)) \oplus \Gamma(\mathcal{O}_{p^*Z}(S+l\tilde{F})) \oplus \Gamma(\mathcal{O}_{p^*Z}(Y \oplus l\tilde{F}))\end{array} \xrightarrow{g} \Gamma(\mathcal{O}_P(kS) \oplus \Gamma(\mathcal{O}_P(S+l\tilde{F})) \oplus \Gamma(\mathcal{O}_P(l\tilde{F}+Y)),$$

where f is given by

We will show that the constructible function $f_d : \mathrm{Sym}^d(B) \to \mathbb{Z}(\!(p)\!)$ satisfies two multiplicative properties. The first one is described as follows. Denote by $B^{\mathrm{sm}} \subset B$ the open subset over which the fibres are smooth and by B^{sing} the $e(S)$ points over which the fibres are singular. We can consider the restrictions of f_d to $\mathrm{Sym}^d(B^{\mathrm{sm}}) \subset \mathrm{Sym}^d(B)$ and $\mathrm{Sym}^d(B^{\mathrm{sing}}) \subset \mathrm{Sym}^d(B)$.

Proposition 25. *Let $d_1, d_2 \geq 0$ be such that $d_1 + d_2 = d$. Then*

$$f_d(\mathfrak{a}+\mathfrak{b}) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \cdot f_{d_1}(\mathfrak{a}) \cdot f_{d_2}(\mathfrak{b}),$$

for any $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$.

We prove this proposition in Section **??**. The following product formula is an immediate consequence of this result

(34)

$$\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d\, de = \widehat{\mathsf{DT}}(X) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \left(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sm}})} f_d\, de \right) \cdot \left(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sing}})} f_d\, de \right).$$

The restricted constructible functions $f_d : \mathrm{Sym}^d(B^{\mathrm{sm}}) \to \mathbb{Z}(\!(p)\!)$ and $f_d : \mathrm{Sym}^d(B^{\mathrm{sing}}) \to \mathbb{Z}(\!(p)\!)$ satisfy further multiplicative properties:

Proposition 26. *There exist functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ and $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ taking values in formal Laurent series $\mathbb{Z}(\!(p)\!)$, such that $g(0) = 1$, $h(0) = 1$, and*

$$f_d(\mathfrak{a}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{i=1}^{l} g(a_i),$$

$$f_d(\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{j=1}^{m} h(b_j),$$

for all $\mathfrak{a} = \sum_{i=1}^n a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$, and $\mathfrak{b} = \sum_{j=1}^m b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$, where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.

We prove this proposition in Section 0.6. Together with Lemma **??** of Appendix **??**, Proposition 26 and equation (34) imply

(35)

$$\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B)} f_d\, de \widehat{\mathsf{DT}}(X) = \frac{M(p)^{e(S)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)-e(S)} \cdot \left(\sum_{b=0}^{\infty} h(b) q^b \right)^{e(S)}.$$

Our goal is to prove Propositions 25 and 26, and find formulae for $g(a)$, $h(b)$. This requires a better understanding of the strata

$$\rho_d^{-1}(\mathfrak{a}+\mathfrak{b}) \subset \mathrm{Hilb}^{d,\bullet}_{\mathsf{PCP}}(X),$$

for all $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$ with $d_1 + d_2 = d$. Suppose

$$\mathfrak{a} = \sum_{i=1}^{n} a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}}),$$
$$\mathfrak{b} = \sum_{j=1}^{m} b_j y_j \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}}),$$

where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.

$$M^T = \left(\begin{array}{ccc} y^l & -z^{k-1} & 0 \\ 0 & x & -z \end{array} \right).$$

and g is given by taking the difference and restricting to P. Restricting M^T to B we get many zeroes and the kernel of f to the part involving B equals

$$0 \oplus \Gamma(\mathcal{O}_B(S + l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(Y + l\tilde{F})).$$

The kernel of F restricted to the part involving p^*Z has a more complicated kernel. Certainly $\Gamma(\mathcal{O}_{p^*Z}(kS))$ lies in this kernel because $y^l = 0$ on p^*Z. Furthermore, this kernel contains all elements $(\beta, \gamma) \in \Gamma(p^*\mathcal{O}_Z(S + l\tilde{F})) \oplus \Gamma(p^*\mathcal{O}_Z(Y + l\tilde{F}))$ satisfying the equations

$$-z^{k-1}\beta = 0, \ x\beta = z\gamma.$$

This means $\beta = z\beta'$ and $\gamma = x\beta' + z^{k-1}\gamma'$. We observe that elements in the kernel of f restricted to the part involving p^*Z automatically restrict to zero on P. Therefore we should compute the kernels of

$$\Gamma(\mathcal{O}_B(S + l\tilde{F})) \to \Gamma(\mathcal{O}_P(S + l\tilde{F}))$$
$$\Gamma(\mathcal{O}_B(Y + l\tilde{F})) \to \Gamma(\mathcal{O}_P(Y + l\tilde{F})),$$

which give $\Gamma(N_{B/Y})$ and $\Gamma(N_{B/S})$! Next we should calculate the kernel of

$$\Gamma(\mathcal{O}_{p^*Z}(mS)) \to \Gamma(\mathcal{O}_P(mS)),$$

which equals $\Gamma(I_{P \subset p^*Z}(mS))$. In turn this can be computed using the short exact sequence

$$0 \to K \to I_{P \subset p^*Z} \to \mathcal{O}_{p^*Z}(-Y) \to 0,$$

where K is a zero-dimensional sheaf of length $n-l$. It is not hard to see that $\Gamma(\mathcal{O}_{p^*Z}(mS - Y)) = 0$ so we obtain

$$\Gamma(I_{P \subset p^*Z}(mS)) \cong \mathbb{C}^{n-l},$$

where $n = kl$. The part which is left is $(\beta, \gamma) \in \Gamma(p^*\mathcal{O}_Z(S + l\tilde{F})) \oplus \Gamma(p^*\mathcal{O}_Z(Y + l\tilde{F}))$ satisfying

$$\beta = z\beta', \ \gamma = x\beta' + z^{k-1}\gamma'.$$

The solutions $\beta = z\beta'$, $\gamma = x\beta'$ are given by the space

$$\Gamma(\mathcal{O}_{p^*Z}(l\tilde{F})) \cong \Gamma(\mathcal{O}_{p^*Z}) \cong \mathbb{C}^n.$$

At this stage we have got $H^0(N_{B/Y}) \oplus \mathbb{C}^{2n-l}$ worth of deformations. So what about the solutions $\gamma = z^{k-1}\gamma'$? Solutions of this form which are *not* of the form $\beta = z\beta'$, $\gamma = x\beta'$ are given by

$$\Gamma(\mathcal{O}_{lF}(Y + l\tilde{F} - (m-1)S)) - \Gamma(\mathcal{O}_{lF}(Y + l\tilde{F} - (m-1)S - Y)).$$

This amounts to

$$\Gamma(\mathcal{O}_{lF}(B)) - \Gamma(\mathcal{O}_{lF}) = \mathbb{C}^l - \mathbb{C}^l,$$

so there is nothing we do not already have. In conclusion

$$\mathrm{Hom}(I_C, \mathcal{O}_C) \cong H^0(N_{B/Y}) \oplus \mathbb{C}^{2n-l}.$$

martijn's old argument for the proposition in the "fiber contributions" subsection:

We start with the first equation. Let $F := F_x \subset S$ be a smooth fibre. Consider the auxiliary surface

$$\tilde{S} = B \times F$$

and let $Y = \mathrm{Tot}(K_{\tilde{S}})$. let $e \in F$ be any closed point and consider the embeddings

$$B \hookrightarrow \tilde{S},\ b \mapsto (b, e)$$
$$\tilde{S} \hookrightarrow \tilde{X},\ p \mapsto (p, 0).$$

Denote by $\widehat{X}_F$ the formal neighbourhood of F in X and by $\widehat{Y}_F$ the formal neighbourhood of

$$F \cong \{x\} \times F \subset \tilde{S} \subset Y,$$

where $\tilde{S} \subset Y$ denotes the zero section. In general $\widehat{X}_F$ and $\widehat{Y}_F$ are *not* isomorphic.

Denote by $\widehat{X}^{\circ}_{F^\circ}$ the formal neighbourhood of $F \setminus B$ in $X \setminus B$. Recall that through the section $B \subset S$, we can view x as an element of both B and F. We denote by $\widehat{Y}^{\circ}_{F^\circ}$ the formal neighbourhood of $(\{x\} \times F) \setminus (B \times \{x\})$ inside $Y \setminus (B \times \{x\})$. Since we have removed the section B, there exists an isomorphism

Provide more argument here?

$$\widehat{X}^{\circ}_{F^\circ} \cong \widehat{Y}^{\circ}_{F^\circ}. \tag{36}$$

We are interested in the moduli space $\mathrm{Hilb}^{a,\bullet}(\widehat{X}^{\circ}_{F^\circ})$ and the correspondingly defined moduli space $\mathrm{Hilb}^{a,\bullet}(\widehat{Y}^{\circ}_{F^\circ})$. Since $\widehat{X}^{\circ}_{F^\circ}$ and $\widehat{Y}^{\circ}_{F^\circ}$ have (compatible) $\mathbb{C}^*$-actions coming from scaling the fibres of X and Y, we can consider their $\mathbb{C}^*$-fixed loci and stratify them according to 2D partitions as in (**??**). By (36), we have an isomorphism

$$\mathrm{Hilb}^{a,\bullet}(\widehat{X}^{\circ}_{F^\circ})^{\mathbb{C}^*}_{\lambda} \cong \mathrm{Hilb}^{a,\bullet}(\widehat{Y}^{\circ}_{F^\circ})^{\mathbb{C}^*}_{\lambda}.$$

This observation allows us to work in the much simpler geometry of Y. This is only possible because we removed the section B from X and Y.

Let $F \cong \{x\} \times F \subset \tilde{S} \subset Y$ be as above. Denote by $\widehat{Y}_y$ the formal neighbourhood $y \in Y$, where y is the intersection of $F \cong \{x\} \times F$ and $B \times \{x\}$ inside the zero section $\tilde{S} \subset Y$. Let $\widehat{Y}_F$, $\widehat{Y}^{\circ}_{F^\circ}$ be the formal neighbourhoods introduced above. Then we have a cover

Strictly speaking: not clear these maps are flat since $\widehat{Y}_F$ non-noetherian! However: not an issue if you'd work with high enough truncation of $\widehat{Y}_F$.

$$\{\widehat{Y}_y \to \widehat{Y}_F, \widehat{Y}^{\circ}_{F^\circ} \to \widehat{Y}_F\}.$$

On these pieces, we introduce moduli spaces as in Section **??**

$$\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_y),\ \mathrm{Hilb}^{a,\bullet}(\widehat{Y}_F),\ \mathrm{Hilb}^{a,\bullet}(\widehat{Y}^{\circ}_{F^\circ}).$$

Similar to Proposition 8, restriction gives a bijective morphism on closed points

$$\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_F)^{\mathbb{C}^*}_{\lambda} \to \mathrm{Hilb}^{a,\bullet}(\widehat{Y}_y)^{\mathbb{C}^*}_{\lambda} \times \mathrm{Hilb}^{a,\bullet}(\widehat{Y}^{\circ}_{F^\circ})^{\mathbb{C}^*}_{\lambda}. \tag{37}$$

Recall that $\tilde{S} = B \times F$. Then F does not only act on $F \subset \tilde{S}$, but on any thickening $dF \subset \tilde{S}$. This is because

$$\mathcal{O}_{dF} = \mathcal{O}_{dx} \otimes \mathcal{O}_F,$$

where $dx \subset B$ denotes the d times thickening of the point $x \in B$. Moreover, F acts on the thickened curve defined by the ideal sheaf

$$\bigoplus_{k=0}^{\infty} \mathcal{O}_{\tilde{S}}(-\lambda_k F) \otimes K_{\tilde{S}}^{-k}.$$

The action of the elliptic curve F on itself is fixed-point-free, so it lifts to a free action on $\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_F)_\lambda^{\mathbb{C}^*}$. Since $e(F) = 0$, we deduce

$$e(\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_F)_\lambda^{\mathbb{C}^*}) = 1. \tag{38}$$

On $\widehat{Y}_y \cong \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$ we have an action of $\mathbb{C}^{*3}$, so

$$e(\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_y)_\lambda^{\mathbb{C}^*}) = e(\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_y)_\lambda^{\mathbb{C}^{*3}}) = \mathsf{V}_{\lambda\varnothing\varnothing}. \tag{39}$$

From equations (38), (37), (39), and (36) we conclude that

$$\begin{aligned} 1 = e(\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_F)_\lambda^{\mathbb{C}^*}) &= e(\mathrm{Hilb}^{a,\bullet}(\widehat{Y}_y)_\lambda^{\mathbb{C}^*})\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{Y}^\circ_{F^\circ})_\lambda^{\mathbb{C}^*}) \\ &= \mathsf{V}_{\lambda\varnothing\varnothing} \cdot e(\mathrm{Hilb}^{a,\bullet}(\widehat{Y}^\circ_{F^\circ})_\lambda^{\mathbb{C}^*}) \\ &= \mathsf{V}_{\lambda\varnothing\varnothing} \cdot e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ})_\lambda^{\mathbb{C}^*}). \end{aligned}$$

The equation for $e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})_\mu^{\mathbb{C}^*})$ can be deduced similarly. This time, the smooth fibre $F = F_x \subset S \subset X$ is replaced by the *smooth locus* of the singular fibre, i.e.

$$F' := F_y^{\mathrm{sm}} = F_y \setminus \{z\},$$

where z denotes the singularity of F_y. Note that

$$F' \cong \mathbb{P}^1 \setminus \{2 \text{ points}\} \cong \mathbb{C}^*.$$

Therefore, we again have a free action of F' on itself and $e(F') = 0$. The rest of the proof follows the same steps.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, ROOM 121, 1984 MATHEMATICS ROAD, VANCOUVER, B.C., CANADA V6T 1Z2

MATHEMATICAL INSTITUTE, UTRECHT UNIVERSITY, ROOM 502, BUDAPESTLAAN 6, 3584 CD UTRECHT, THE NETHERLANDS