DONALDSON-THOMAS INVARIANTS OF LOCAL ELLIPTIC SURFACES VIA THE TOPOLOGICAL VERTEX

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ABSTRACT. We compute the Donaldson-Thomas invariants of a local elliptic surface with section. We introduce a new computational technique which is a mixture of motivic and toric methods. This allows us to write the partition function for the invariants in terms of the topological vertex. Utilizing identities for the topological vertex proved in [4], we derive product formulas for the partition functions. The connected version of the partition function is written in terms of Jacobi forms. In the special case where the elliptic surface is a K3 surface, we get a new derivation of the Katz-Klemm-Vafa formula.

1. INTRODUCTION

Let $p : S \to B$ be a non-trivial elliptic surface over a complex smooth projective curve B. We assume p has a section and all singular fibres are irreducible rational nodal curves.

We are interested in the Donaldson-Thomas (DT) invariants of $X = \mathrm{Tot}(K_S)$, i.e. the total space of the canonical bundle K_S. This is a non-compact Calabi-Yau threefold. Let β be an effective curve class on S. Consider the Hilbert scheme

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X \ : \ [Z] = \beta, \ \chi(\mathcal{O}_Z) = n\}$$

of proper subschemes $Z \subset X$ with homology class β and holomorphic Euler characteristics n. The DT invariants of X can be defined as

$$\mathsf{DT}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X), \nu) := \sum_{k \in \mathbb{Z}} k \, e(\nu^{-1}(k)),$$

where $e(\cdot)$ denotes topological Euler characteristic and $\nu : \mathrm{Hilb}^{\beta,n}(X) \to \mathbb{Z}$ is Behrend's constructible function [2]. We consider an Euler characteristic version of these invariants

$$\widehat{\mathsf{DT}}_{\beta,n}(X) := e(\mathrm{Hilb}^{\beta,n}(X)).$$

We choose a section $B \subset S$ and focus on the primitive classes $\beta = B + dF$, where B is the class of the chosen section and F the class of the fibre. We define the partition functions by

$$\widehat{\mathsf{DT}}(X) = \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \widehat{\mathsf{DT}}_{B+dF,n}(X) p^n q^d,$$

$$\mathsf{DT}(X) = \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \mathsf{DT}_{B+dF,n}(X) y^n q^d.$$

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We also consider the partition functions for the invariants for multiples of the fiber class:

$$\widehat{\mathsf{DT}}_{\mathrm{fib}}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \widehat{\mathsf{DT}}_{dF,n}(X)p^n q^d,$$
$$\mathsf{DT}_{\mathrm{fib}}(X) = \sum_{d\geq 0}\sum_{n\in\mathbb{Z}} \mathsf{DT}_{dF,n}(X)y^n q^d.$$

The main result of this paper are closed product formulas for the partition functions $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$. Assuming a general conjecture, we also determine $\mathsf{DT}(X)$ and $\mathsf{DT}_{\mathrm{fib}}(X)$.

We use the notation

$$M(p,q) = \prod_{m=1}^{\infty}(1-p^m q)^{-m}$$

and the shorthand $M(p) = M(p,1)$.

Theorem 1.

$$\widehat{\mathsf{DT}}(X) = \left\{ M(p)\prod_{d=1}^{\infty}\frac{M(p,q^d)}{(1-q^d)}\right\}^{e(S)} \left\{ \frac{1}{(p^{\frac{1}{2}}-p^{-\frac{1}{2}})}\prod_{d=1}^{\infty}\frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)}\right\}^{e(B)}$$
$$\widehat{\mathsf{DT}}_{\mathrm{fib}}(X) = \left\{ M(p)\prod_{d=1}^{\infty}M(p,q^d)\right\}^{e(S)} \left\{ \prod_{d=1}^{\infty}\frac{1}{(1-q^d)}\right\}^{e(B)}$$

The ratio $\widehat{\mathsf{DT}}(X)/\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$ can be considered as the generating function for the connected invariants in the classes $B+dF$. This series has a particularly nice form and can be written in terms of classical Jacobi forms. Consider the Dedekind eta function and the Jacobi theta function

$$\eta = q^{\frac{1}{24}}\prod_{k=1}^{\infty}(1-q^k),$$
$$\Theta = (p^{\frac{1}{2}}-p^{-\frac{1}{2}})\prod_{k=1}^{\infty}\frac{(1-pq^k)(1-p^{-1}q^k)}{(1-q^k)^2}.$$

Corollary 2. *The partition function of the connected invariants is given as follows*

$$\frac{\widehat{\mathsf{DT}}(X)}{\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)} = \left(q^{-\frac{1}{24}}\eta\right)^{-e(S)}\Theta^{-e(B)}.$$

In the case $S\to\mathbb{P}^1$ is an elliptically fibred $K3$ surface, the above series specializes to the well-known Katz-Klemm-Vafa formula. Because X is non-compact, the connected series is required to obtain the KKV formula. Our result provides a new derivation of the KKV formula. The KKV formula was recently proved in *all* curve classes in [13]. This is the first derivation of the KKV formula, which does not depend on the Kawai-Yoshioka formula [8].

The most important result of this paper is perhaps not the formula, but rather the method of calculation. This approach has found further applications to the calculation of DT generating functions on $K3\times E$, where E is an elliptic curve [3], and abelian 3-folds [5]. Even though the geometry under consideration is not toric, we combine $\mathbb{C}^*$-localization, motivic methods, formal methods, and $(\mathbb{C}^*)^3$-localization to end up with expressions that only depend on $\mathsf{V}_{\lambda\mu\nu}$, $e(B)$, and $e(S)$. Here is a rough sketch:

(A) The action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space[1] $\mathrm{Hilb}^{B+dF,\bullet}(X)$. Therefore, we only have to understand the fixed locus $\mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*}$. Pushforward along $X \to S \to B$ induces a morphism

$$\rho_d : \mathrm{Hilb}^{B+dF,\bullet}(X)^{\mathbb{C}^*} \to \mathrm{Sym}^d(B).$$

This map is constructed in Section 3. The fibres of ρ_d decompose into components according to the shape of the underlying Cohen-Macaulay curve. This leads to a decomposition over 2D partitions $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$.

(B) The Euler characteristics of the fibres of ρ_d define a constructible function f_d on $\mathrm{Sym}^d(B)$. In Section 4, we show that if f_d satisfies a certain product formula, then $\widehat{\mathsf{DT}}(X)$ satisfies a corresponding product formula. This follows from general power structure arguments reviewed in Appendix A.2.

(C) A component Σ of a fibre of ρ_d indexed by λ can be further broken down by taking a certain fpqc cover of the underlying (now fixed) Cohen-Macaulay curve Z_{CM} determined by λ. This cover consists of formal neighbourhoods $\widehat{X}_x$ around the singular points x of the reduced support of Z_{CM} and "tubular neighbourhoods" along the reduced support of Z_{CM} after removing the singularities. Since Z_{CM} is already fixed, gluing is automatic. Hence restriction to the elements of the cover gives a bijection morphism of Σ to local Hilbert schemes on the elements of the cover. In Section 5, we show this leads to the product formula for f_d in (B).

(D) On the formal neighbourhoods $\widehat{X}_x$, we have an action of $\mathbb{C}^{*3}$. This allows us to express their contributions to the generating function in terms of the topological vertex. The contributions of the tubular neighbourhoods along the *punctured* section and fibres can also be expressed in terms of the topological vertex (utilizing a map to $\mathrm{Sym}^n(F)$ which records the location and multiplicity of the embedded points). This is worked out in Section 6.

Our results can be extended to apply to the usual (Behrend function weighted) Donaldson-Thomas invariants if we assume a general conjecture which we formulate in Section 7. The basic results (assuming Conjecture 20) are

$$\mathsf{DT}(X) = (-1)^{\chi(\mathcal{O}_S)}\widehat{\mathsf{DT}}(X)$$

and

$$\mathsf{DT}_{\mathrm{fib}}(X) = \widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$$

under the change of variables

$$y = -p.$$

A similar phenomenon to the above is known to hold when X is a toric Calabi-Yau threefold.

2. Definitions, notation, and conventions

Let $p : S \to B$ be an elliptic surface over a smooth projective curve B. Besides assuming S is not a product, we require:

(1) p has a section $B \subset S$,
(2) all singular fibres of p are irreducible rational nodal curves.

[1]The bullet indicates that we take the union of $\mathrm{Hilb}^{B+dF,n}(X)$ over all n, see Convention 2.1g.

We write F_x for the fibre $p^{-1}(\{x\})$ for all closed points $x \in B$. We choose a section $B \subset S$ and denote its class in $H_2(S)$ by B as well. We denote the class of the fibre by $F \in H_2(S)$.

For brevity, we define

$$\begin{aligned} \operatorname{Hilb}^{d,n}(X) &:= \operatorname{Hilb}^{B+dF,n}(X), \\ \widehat{\mathsf{DT}}_{d,n}(X) &:= \widehat{\mathsf{DT}}_{B+dF,n}(X). \end{aligned}$$

Since we are dealing with generating functions and our calculations involve cut-paste methods on the moduli space, it is useful to introduce the following notation. We define

$$\operatorname{Hilb}^{d,\bullet}(X) := \sum_{n\in\mathbb{Z}} \operatorname{Hilb}^{d,n}(X)\, p^n,$$

where we view the right hand side as a formal Laurent series whose coefficients elements in the Grothendieck ring of varieties, i.e. $K_0(\mathrm{Var}_{\mathbb{C}})(\!(p)\!)$.

Convention 2.1. *When an index is replaced by a bullet, we will sum over the index, multiplying by the appropriate variable. We regard the result as a formal power (or Laurent) series whose coefficients lie $K_0(\mathrm{Var}_{\mathbb{C}})$ and we extend operations of the Grothendieck group (addition, multiplication, Euler characteristic) to the series in the obvious way.*

For example

$$\operatorname{Hilb}^{\bullet,\bullet}(X) = \sum_{d=0}^{\infty}\sum_{n\in\mathbb{Z}} \operatorname{Hilb}^{d,n}(X) q^d p^n,$$

so that we can write

$$\widehat{\mathsf{DT}}(X) = e(\operatorname{Hilb}^{\bullet,\bullet}(X)).$$

It is notationally convenient to treat an Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, the measurable functions are constructible functions, and the measure of a set is given by its Euler characteristic. In this language we have

$$\widehat{\mathsf{DT}}_{\beta,n}(X) = \int_{\operatorname{Hilb}^{\beta,n}(X)} 1\, de, \qquad \mathsf{DT}_{\beta,n}(X) = \int_{\operatorname{Hilb}^{\beta,n}(X)} \nu\, de,$$

and following the bullet convention we have

$$\widehat{\mathsf{DT}}(X) = \int_{\operatorname{Hilb}^{\bullet,\bullet}(X)} 1\, de, \qquad \mathsf{DT}(X) = \int_{\operatorname{Hilb}^{\bullet,\bullet}(X)} \nu\, de.$$

We will also need notation for subsets of the Hilbert scheme which parameterize those subschemes obtained by adding embedded points to some fixed Cohen-Macaulay curve.

Definition 3. Let $Y \subset X$ be an open set (possibly in the fpcq topology) and let $C \subset Y$ be a Cohen-Macaulay subscheme of dimension 1 which we assume is the restriction of some $\overline{C} \subset X$ to Y. We define

$$\operatorname{Hilb}^n(Y, C) = \{Z \subset Y \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n\}.$$

Via the inclusion $Y \subset X$, $\operatorname{Hilb}^n(Y, C)$ can be viewed as a constructible subscheme of $\operatorname{Hilb}(X)$. It parameterizes subschemes which roughly speaking are obtained from C by adding n embedded points.

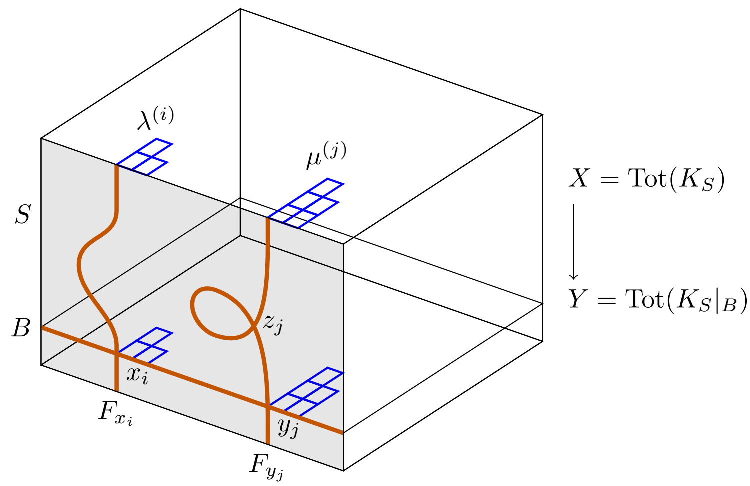

FIGURE 1. Cohen-Macaulay curve Z_{CM} with underlying reduced support in orange and thickenings $\lambda^{(i)}$ along smooth fibers F_{x_i}, $\mu^{(j)}$ along singular fibers F_{y_j}, and multiplicity one along the section B.

3. THE $\mathbb{C}^*$-FIXED LOCUS

The action of $\mathbb{C}^*$ on the fibres of X lifts to the moduli space $\mathrm{Hilb}^{d,\bullet}(X)$. Therefore

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)} 1\, de = \int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\, de.$$

In order to understand $\mathrm{Hilb}^{d,n}(X)$, we first study $Z_{\mathrm{CM}} \subset Z$, the maximal Cohen-Macaulay subscheme of any $\mathbb{C}^*$-invariant subschemes $Z \subset X$. We find that such subschemes are determined by a point in $\mathrm{Sym}^d(B)$ along with some discrete data (a collection of integer partitions). This is given by the following two propositions and is illustrated in Figure 1.

Proposition 4. *A closed points Z of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ correspond to a finite nesting of closed subschemes of S*

$$Z_0 \supset Z_1 \supset \cdots \supset Z_l,$$

satisfying

$$\sum_{k=0}^{l} [Z_k] = B + dF \in H_2(S).$$

Proof. Using projection $\pi : X \to S$, a quasi-coherent sheaf on X can be viewed as a quasi-coherent sheaf $\mathcal{F}$ on S together with a morphism $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$. A $\mathbb{C}^*$-equivariant structure on $\mathcal{F}$ translates into a $\mathbb{Z}$-grading

$$\pi_* \mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k,$$

such that $\mathcal{F} \otimes K_S^{-1} \to \mathcal{F}$ is graded, i.e.

$$\mathcal{F}_k \otimes K_S^{-1} \longrightarrow \mathcal{F}_{k-1}.$$

Here $\mathcal{F}_k$ has weight k and K_S weight 1 under the $\mathbb{C}^*$-action. The structure sheaf $\mathcal{O}_X$ corresponds to

$$\pi_* \mathcal{O}_X = \bigoplus_{k=0}^{\infty} K_S^{-k}.$$

Therefore a $\mathbb{C}^*$-equivariant morphism $\mathcal{F} \to \mathcal{O}_X$ corresponds to a graded sheaf $\mathcal{F}$ as above together with maps $F_i \to K_S^i$ for all i such that

$$\begin{tikzcd}
\mathcal{F}_k \otimes K_S^{-1} \arrow[r] \arrow[d] & \mathcal{F}_{k-1} \arrow[d] \\
K_S^k \otimes K_S^{-1} \arrow[r, equal] & K_S^{k-1}
\end{tikzcd}$$

commute for all $k \leq 0$ and the composition $\mathcal{F}_1 \otimes K_S^{-1} \to \mathcal{F}_0 \to \mathcal{O}_S$ is to zero.

It is useful to re-define $\mathcal{G}_k := \mathcal{F}_{-k} \otimes K_S^k$. Then a $\mathbb{C}^*$-equivariant morphism $\mathcal{F} \to \mathcal{O}_X$ is equivalent to the following data:

- quasi-coherent sheaves $\{\mathcal{G}_k\}_{k \in \mathbb{Z}}$ on S,
- morphisms $\{\mathcal{G}_k \to \mathcal{G}_{k+1}\}_{k \in \mathbb{Z}}$,
- morphisms $\mathcal{G}_k \to \mathcal{O}_S$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}_{-1} \arrow[r] \arrow[d] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r, equal] & \cdots
\end{tikzcd}$$

In the case of interest to us $\mathcal{G} \to \mathcal{O}_X$ is an ideal sheaf $I_Z \hookrightarrow \mathcal{O}_X$ cutting out $Z \subset X$. In the above language, this means $\mathcal{G}_k = 0$ for $k < 0$, the morphisms $\mathcal{G}_k \to \mathcal{O}_S$ are injective, and the morphisms $\mathcal{G}_k \to \mathcal{G}_{k+1}$ are injective. Therefore $\mathcal{G}_k = I_{Z_k \subset S}$ is an ideal sheaf cutting out $Z_k \subset S$ and

$$I_{Z_k \subset S} \subset I_{Z_{k+1} \subset S},$$

for all k. $\square$

Let $\mathrm{Hilb}^{B+dF}(S)$ be the Hilbert scheme of effective divisors on S with class

$$B + dF \in H_2(S).$$

By Lemma 22 of the Appendix A.1, pull-back along p and adding the section B induces an isomorphism

$$\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{B+dF}(S).$$

For any reduced curve $C \subset S$ defined by ideal sheaf $I_{C \subset S}$ and $d > 0$, we denote by dC the Gorenstein curve defined by the ideal sheaf $I_{C \subset S}^d$, the dth power of $I_{C \subset S}$. We combine Lemma 22 with a (family version of) Proposition 4 to conclude the following:

Proposition 5. *There exists a morphism*

$$\rho_d : \mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*} \longrightarrow \mathrm{Sym}^d(B),$$

which at the level of closed points can be can be described as follows. Let $Z \in \mathrm{Hilb}^{d,n}(X)^{\mathbb{C}^*}$ and let $Z_{\mathrm{CM}} \subset Z$ be the maximal Cohen-Macaulay subcurve of Z. Since Z_{CM} is $\mathbb{C}^*$-fixed, its ideal sheaf decomposes

$$I_{Z_{\mathrm{CM}}} = \bigoplus_{k=0}^{\infty} I_{Z_k \subset S} \otimes K_S^{-k},$$

where

$$Z_0 = B \cup \lambda_0^{(1)} F_{x_1} \cup \cdots \cup \lambda_0^{(l)} F_{x_l}$$

for some distinct closed points $x_i \in B$ *and* $\lambda_0^{(i)} > 0$, *and*

$$Z_k = \lambda_k^{(1)} F_{x_1} \cup \cdots \cup \lambda_k^{(l)} F_{x_l}.$$

for some $\lambda_k^{(i)} \leq \lambda_{k-1}^{(i)}$. *Here* $\lambda^{(i)} = (\lambda_0^{(i)} \geq \lambda_1^{(i)} \geq \cdots)$ *define 2D partitions satisfying*

$$\sum_{i=1}^{l} |\lambda^{(i)}| = d.$$

See Figure 1 for an illustration. The map ρ_d *sends* Z *to*

$$\sum_{i=1}^{l} |\lambda^{(i)}| x_i \in \mathrm{Sym}^d(B).$$

Remark 6. The morphism of this proposition is perhaps somewhat surprising. Since we are on a 3-fold, the map which sends a closed subscheme of $Z \in \mathrm{Hilb}^{d,n}(X)$ to its underlying Cohen-Macaulay curve Z_{CM} is *not* a morphism. Nevertheless, the map ρ_d which records the location of the fibres in Z_{CM} and their multiplicities is a morphism.

Proof. The description of ρ_d at the level of closed points is clear. We construct ρ_d as a morphism from Proposition 4 and Lemma 22 of Appendix A.1.

Let T be an arbitrary base scheme of finite type and let

$$\mathcal{Z} \subset X \times T$$

be a $\mathbb{C}^*$-fixed and T-flat closed subscheme. Assume for each $t \in T$ the fibre $\mathcal{Z}_t$ has class $B + dF \in H_2(S)$ and $\chi(\mathcal{O}_{\mathcal{Z}_t}) = n$. Since $\mathcal{Z}$ is $\mathbb{C}^*$-fixed, Proposition 4 implies that its ideal sheaf decomposes[2]

$$I_{\mathcal{Z}} = \bigoplus_{k=0}^{\infty} I_{\mathcal{Z}_k \subset S \times T} \otimes K_S^{-k},$$

where K_S is pulled-back along $S \times T \to S$ and

$$\mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots.$$

Then each $\mathcal{Z}_k \subset S \times T$ is T-flat as well. The maximal CM subschemes $\mathcal{Z}_{k,\mathrm{CM}} \subset \mathcal{Z}_k \subset S \times T$ are also T-flat and induces morphisms

$$\begin{aligned} &T \longrightarrow \mathrm{Hilb}^{B+d_0F}(S), \\ &T \longrightarrow \mathrm{Hilb}^{d_kF}(S), \quad \text{for} \;\; k > 0 \end{aligned}$$

where $\sum_k d_k = d$. Adding divisors gives a morphism $T \longrightarrow \mathrm{Hilb}^{B+dF}(S)$. By Lemma 22, we obtain a morphism $T \to \mathrm{Sym}^d(B)$. This morphism corresponds to a T-flat family for $\mathrm{Sym}^d(B)$. We have defined ρ_d as a morphism. $\square$

[2]The arguments leading to Proposition 4 hold equally well for T-flat families over a base T.

4. PUSH-FORWARD TO THE SYMMETRIC PRODUCT

In the previous section we constructed a morphism (Proposition 5)

$$\rho_d : \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*} \longrightarrow \mathrm{Sym}^d(B). \tag{1}$$

We obtain

$$\int_{\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}} 1\, de = \int_{\mathrm{Sym}^d(B)} \rho_{d*}(1)\, de,$$

where $f_d := \rho_{d*}(1)$ is the $\mathbb{Z}((p))$-valued constructible function on $\mathrm{Sym}^d(B)$ given by pushing forward the Euler characteristic measure[9]. Its value at a closed point $\mathfrak{a} \in \mathrm{Sym}^d(B)$ is

$$f_d(\mathfrak{a}) = \int_{\rho_d^{-1}(\mathfrak{a})} 1\, de.$$

It turns out that the constructible function $f_d : \mathrm{Sym}^d(B) \to \mathbb{Z}(\!(p)\!)$ satisfies two multiplicative properties. The first one is described as follows. Denote by $B^{\mathrm{sm}} \subset B$ the open subset over which the fibres are smooth and by B^{sing} the N points over which the fibres are singular. We can consider the restrictions of f_d to $\mathrm{Sym}^d(B^{\mathrm{sm}}) \subset \mathrm{Sym}^d(B)$ and $\mathrm{Sym}^d(B^{\mathrm{sing}}) \subset \mathrm{Sym}^d(B)$. Denote by $M(p)$ the MacMahon function.

Proposition 7. *Let $d_1, d_2 \geq 0$ be such that $d_1 + d_2 = d$. Then*

$$f_d(\mathfrak{a}+\mathfrak{b}) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \cdot f_{d_1}(\mathfrak{a}) \cdot f_{d_2}(\mathfrak{b}),$$

for any $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$.

We prove this proposition in Section 5.3. The following product formula is an immediate consequence of this result

$$\widehat{\mathsf{DT}}(X) = \frac{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}{M(p)^{e(X)}} \left(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sm}})} f_d\, de \right) \cdot \left(\sum_{d\geq 0} q^d \int_{\mathrm{Sym}^d(B^{\mathrm{sing}})} f_d\, de \right). \tag{2}$$

The restricted constructible functions $f_d : \mathrm{Sym}^d(B^{\mathrm{sm}}) \to \mathbb{Z}(\!(p)\!)$ and $f_d : \mathrm{Sym}^d(B^{\mathrm{sing}}) \to \mathbb{Z}(\!(p)\!)$ satisfy further multiplicative properties:

Proposition 8. *There exist functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ and $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ taking values in formal Laurent series $\mathbb{Z}(\!(p)\!)$, such that $g(0) = 1$, $h(0) = 1$, and*

$$f_d(\mathfrak{a}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{i=1}^{l} g(a_i),$$
$$f_d(\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_{j=1}^{m} h(b_j),$$

for all $\mathfrak{a} = \sum_{i=1}^{l} a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$, and $\mathfrak{b} = \sum_{j=1}^{m} b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$, where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.

We prove this proposition in Section 5.3. Together with Lemma 23 of Appendix A.2, Proposition 8 and equation (2) imply

$$\widehat{\mathsf{DT}}(X) = \frac{M(p)^N}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)-N} \cdot \left(\sum_{b=0}^{\infty} h(b) q^b \right)^{N}. \tag{3}$$

Our goal is to prove Propositions 7 and 8, and find formulae for $g(a)$, $h(b)$. This requires a better understanding of the strata

$$\rho_d^{-1}(\mathfrak{a}+\mathfrak{b}) \subset \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*},$$

for all $\mathfrak{a} \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}})$ and $\mathfrak{b} \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}})$ with $d_1 + d_2 = d$. Suppose

$$\mathfrak{a} = \sum_{i=1}^{l} a_i x_i \in \mathrm{Sym}^{d_1}(B^{\mathrm{sm}}),$$
$$\mathfrak{b} = \sum_{j=1}^{m} b_j y_j \in \mathrm{Sym}^{d_2}(B^{\mathrm{sing}}),$$

where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points. Proposition 5 gives a decomposition of $\rho_d^{-1}(\mathfrak{a}+\mathfrak{b})$ into components[3]

$$\bigsqcup_{\substack{\lambda^{(1)} \vdash a_1 \\ \cdots \\ \lambda^{(l)} \vdash a_l}} \bigsqcup_{\substack{\mu^{(1)} \vdash b_1 \\ \cdots \\ \mu^{(m)} \vdash b_m}} \Sigma(x_1, \ldots, x_l, y_1, \ldots, y_m, \lambda^{(1)}, \ldots, \lambda^{(l)}, \mu^{(1)}, \ldots, \mu^{(m)}). \tag{4}$$

We abbreviate these components by $\Sigma(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$. Therefore $\Sigma(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$ is the stratum of points $Z \in \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$, for which the maximal Cohen-Macaulay subcurve $Z_{\mathrm{CM}} \subset Z$ is determined by the data $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$ as in Proposition 5. Note that these strata have a natural scheme structure: the fibres of ρ_d are closed subschemes of $\mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ and these strata are components of them. We are interested in the Euler characteristics of these strata. In the next section, we will see that the Euler characteristic of $\Sigma(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$ does *not* depend on the exact location of the points $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$, but only on their number m and n and the partitions $\lambda^{(i)}$ and $\mu^{(j)}$.

5. Restriction to formal neighbourhoods

In the previous two sections we reduced our consideration to the strata $\Sigma(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{\lambda}; \boldsymbol{\mu})$ of $Z \in \mathrm{Hilb}^{d,\bullet}(X)^{\mathbb{C}^*}$ for which the maximal Cohen-Macaulay subcurve $Z_{\mathrm{CM}} \subset Z$ is determined by the data $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}$. In this section we break down this stratum further by cutting it up in pieces covered by formal neighbourhoods. For notational simplicity, we first consider the case where the base point is

$$ax + by \in \mathrm{Sym}^d(B),$$

with $x \in B^{\mathrm{sm}}$, $y \in B^{\mathrm{sing}}$, and $d = a + b$. We show how to compute $e(\Sigma(x, y, \lambda, \mu))$. Once this case is established, it is not hard to generalize to arbitrary $e(\Sigma(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{\lambda}; \boldsymbol{\mu}))$. This leads to a proof of Propositions 7 and 8, and a geometric characterization of the functions $g(a)$, $h(b)$ of Section 4.

5.1. **Fpqc cover.** The idea is to use an appropriate cover of X and calculate on pieces of the cover. We first give a complex analytic definition of the cover to aid the intuition and then give the actual "algebro-geometric cover":

(1) The reduced support $B \cup F_x \cup F_y$ has three singular points[4]: $x, y \in B$ and $z \in F_y^{\mathrm{sing}}$. We take small open balls around these points.

[3] We use the term component somewhat loose: it means a subset which is both open and closed. We do not care whether it is connected.

[4] Recall that $x, y \in B$ in the base can be viewed as points on S and X via the sections $B \subset S \subset X$.

(2) Consider the punctured curve $B^\circ := B \setminus \{x,y\}$ and let $X^\circ := X \setminus (F_x \cup F_y)$. We take a tubular neighbourhood of $B^\circ \subset X^\circ$.
(3) Consider the punctured curve $F_x^\circ := F_x \setminus \{x\}$ and let $X^\circ := X \setminus B$. We take a tubular neighbourhood of $F_x^\circ \subset X^\circ$.
(4) Consider the punctured curve $F_y^\circ := F_y \setminus \{y,z\}$ and let $X^\circ := X \setminus (B \cup \{z\})$. We take a tubular neighbourhood of $F_y^\circ \subset X^\circ$.
(5) Finally, we take $W = X \setminus (B \cup F_x \cup F_y)$.

In order to work in algebraic geometry, in (1) we take the formal neighbourhood $\widehat{X}_x$ of $\{x\}$ in X. Denote the local ring at x by $(R, \mathfrak{m})$. By $\widehat{X}_x$ we mean the (non-noetherian) scheme

$$\operatorname{Spec} \varprojlim R/\mathfrak{m}^n$$

and *not* the formal scheme

$$\operatorname{Spf} \varprojlim R/\mathfrak{m}^n.$$

Similarly in (2) and (3), let $\widehat{X}_y$ be the formal neighbourhood of $\{y\}$ in X and $\widehat{X}_z$ the formal neighbourhood of $\{z\}$ in X. Note that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \operatorname{Spec} \mathbb{C}[\![x_1,x_2,x_3]\!].$$

Even though $\widehat{X}_x$ is non-noetherian, the morphism $\widehat{X}_x \to X$ has a good property: it is fpqc so can be used as part of a cover [7, Vistoli, Sect. 2.3.2]. Flatness of this map follows from the fact that formal completion is an exact operation [14, Tag 0BNH] [1, Prop. 10.14].

In (2) we consider $B^\circ := B \setminus \{x,y\}$, $X^\circ := X \setminus (F_x \cup F_y)$ and let $\widehat{X}^\circ_{B^\circ}$ be the formal neighbourhood of F_x° in X°. For (3) and (4) the formal neighbourhoods $\widehat{X}^\circ_{F_x^\circ}$ and $\widehat{X}^\circ_{F_y^\circ}$ are defined analogously. Note that the definition of X° in (2)–(4) varies. Finally in (5) we take $W = X \setminus (B \cup F_x \cup F_y)$. Then

$$\mathfrak{U} = \{\widehat{X}_x \to X, \widehat{X}_y \to X, \widehat{X}_z \to X, \widehat{X}^\circ_{B^\circ} \to X, \widehat{X}^\circ_{F_x^\circ} \to X, \widehat{X}^\circ_{F_y^\circ} \to X, W \subset X\}$$

is an fpqc cover of X. Consequently the data of a quasi-coherent sheaf on X is equivalent to the data of quasi-coherent sheaves on each of the opens of $\mathfrak{U}$ and gluing isomorphisms between the restrictions on the overlaps. Technically: quasi-coherent sheaves on X form a stack with respect to the fpqc topology [7, Vistoli, Thm. 4.23].

5.2. **Local moduli spaces.** We now introduce moduli spaces of closed subschemes of dimension ≤ 1 on the pieces of the cover $\mathfrak{U}$. Assume the coordinates on

$$\widehat{X}_x \cong \operatorname{Spec} \mathbb{C}[\![x_1,x_2,x_3]\!]$$

are chosen such that $x_1 = x_3 = 0$ corresponds to the intersection $\widehat{X}_x \times_X B$ and $x_2 = x_3 = 0$ corresponds to $\widehat{X}_x \times_X F_x$. Define

$$\begin{aligned}&\operatorname{Hilb}^{(1,d),n}(\widehat{X}_x) := \\ &\big\{ I_Z \subset \mathcal{O}_{\widehat{X}_x} \ : \ [Z] = [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.\end{aligned}$$

Here the equation

$$[Z] = [\widehat{X}_x \times_X B] + d[\widehat{X}_x \times_X F_x]$$

means Z is supported along

$$(\widehat{X}_x \times_X B) \cup (\widehat{X}_x \times_X F_x)$$

with multiplicity 1 along $\widehat{X}_x \times_X B$ and multiplicity d along $\widehat{X}_x \times_X F_x$ and Z_{CM} denotes the maximal Cohen-Macaulay subcurve of Z. The ideal sheaves fit into a short exact sequence

$$0 \longrightarrow I_Z \longrightarrow I_{Z_{\mathrm{CM}}} \longrightarrow Q \longrightarrow 0,$$

where Q is 0-dimensional. The Hilbert scheme $\mathrm{Hilb}^{(1,d),n}(\widehat{X}_y)$ is defined likewise replacing the point x by y. For $\widehat{X}_z$, we define

$$\mathrm{Hilb}^{d,n}(\widehat{X}_z) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}_z} \ : \ [Z] = d[\widehat{X}_z \times_X F_y] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

Each of $\widehat{X}_x, \widehat{X}_y, \widehat{X}_z$ has an action of $\mathbb{C}^*$ compatible with the fibre scaling on X. This action lifts to the moduli space. Since each of these formal neighbourhoods is isomorphic to $\operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$, the bigger torus $\mathbb{C}^{*3}$ acts on it and this action lifts to the moduli space. The existence of these "extra actions" will be used in Section 6.

Next consider $\widehat{X}^\circ_{B^\circ}$, i.e. the formal neighbourhood of the punctured zero section $B^\circ \subset X^\circ$. Define

$$\mathrm{Hilb}^{1,n}(\widehat{X}^\circ_{B^\circ}) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{B^\circ}} \ : \ [Z] = [\widehat{X}^\circ_{B^\circ} \times_X B] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

For $\widehat{X}^\circ_{F_x^\circ}$, $\widehat{X}^\circ_{F_y^\circ}$ we define

$$\mathrm{Hilb}^{d,n}(\widehat{X}^\circ_{F_x^\circ}) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{F_x^\circ}} \ : \ [Z] = d[\widehat{F}_x^\circ] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\},$$
$$\mathrm{Hilb}^{d,n}(\widehat{X}^\circ_{F_y^\circ}) := \big\{ I_Z \subset \mathcal{O}_{\widehat{X}^\circ_{F_y^\circ}} \ : \ [Z] = d[\widehat{F}_y^\circ] \text{ and } h^0(I_{Z_{\mathrm{CM}}}/I_Z) = n \big\}.$$

Finally for W we define

$$\mathrm{Hilb}^{0,n}(W) := \big\{ I_Z \subset \mathcal{O}_W \ : \ \dim(Z) = 0 \text{ and } h^0(\mathcal{O}_Z) = n \big\}.$$

On $\widehat{X}^\circ_{B^\circ}$, $\widehat{X}^\circ_{F_x^\circ}$, $\widehat{X}^\circ_{F_y^\circ}$, and W we have an action of $\mathbb{C}^*$ compatible with the fibre scaling on X. These actions lift to the moduli space. However, *unlike* for $\widehat{X}_x$, $\widehat{X}_y$, $\widehat{X}_z$, no additional tori act.

As before, we use the notation $\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ for the union of all $\mathrm{Hilb}^{(1,d),n}(\widehat{X}_x)$ and similarly for all other moduli spaces of this section. Like in Section 3, the components of the $\mathbb{C}^*$-fixed locus of $\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)$ are indexed by 2D partitions

$$\mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)^{\mathbb{C}^*} = \bigsqcup_{\lambda \vdash d} \mathrm{Hilb}^{(1,d),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda.$$

Proposition 9. *Consider the stratum $\Sigma(x, y, \lambda, \mu)$, where $|\lambda| = a$ and $|\mu| = b$. Restriction from X to the elements of the cover $\mathfrak{U}$ induces a morphism*

$$\begin{aligned} \Sigma(x,y,\lambda,\mu) \longrightarrow\ & \mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda \times \mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu \times \mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu \times \\ & \mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})^{\mathbb{C}^*} \times \mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F_x^\circ})^{\mathbb{C}^*}_\lambda \times \mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F_y^\circ})^{\mathbb{C}^*}_\mu \times \\ & \mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}, \end{aligned} \tag{5}$$

which is a bijection on closed points.

Proof. Since pull-back works in families, restriction indeed defines a morphism. For the rest of the proof, we work on closed points only.

Since $\mathfrak{U}$ is an fpqc cover, fpqc descent implies that any ideal sheaf $I_Z \subset \mathcal{O}_X$ is entirely determined by its restriction along the morphisms of the elements of $\mathfrak{U}$. This proves injectivity.

Conversely, given local ideal sheaves in the image of (5), their restrictions to overlaps only depend on the underlying Cohen-Macaulay curve and not on the embedded points. Since we chose the strata such that the underlying Cohen-Macaulay curve is already fixed, there are no further gluing conditions and fpqc descent implies surjectivity. $\square$

Remark 10. Note that the argument of Proposition 9 produces a bijective morphism — we do *not* claim (5) is an isomorphism of schemes. However, a bijective morphism induces an equality of (topological) Euler characteristic, which is what we use.

Remark 11. It is important to relate holomorphic Euler characteristic of domain and target in (5). For any subscheme Z in the domain $\Sigma(x,y,\lambda,\mu)$, denote the corresponding maximal Cohen-Macaulay curve of its elements by Z_{CM} (Proposition 5). Then

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{Z_{\mathrm{CM}}}) + \chi(I_{Z_{\mathrm{CM}}}/I_Z).$$

Recall that Z_{CM} is entirely determined by the data x,y,λ,μ, where $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$ and $\mu = (\mu_0 \geq \mu_1 \geq \cdots)$ are 2D partitions (equation (4)). An easy calculation shows

$$\chi(\mathcal{O}_{Z_{\mathrm{CM}}}) = \chi(\mathcal{O}_B) - \lambda_0 - \mu_0.$$

We conclude

$$\chi(\mathcal{O}_Z) = \frac{e(B)}{2} - \lambda_0 - \mu_0 + \chi(I_{Z_{\mathrm{CM}}}/I_Z). \tag{6}$$

Proposition 9 allows us to calculate

$$f_d(ax+by) = e(\rho_d^{-1}(ax+by)) = \sum_{\lambda \vdash a} \sum_{\mu \vdash b} e(\Sigma(x,y,\lambda,\mu)).$$

By Proposition 9 and (6)

$$\begin{aligned} f_d(ax+by) = p^{\frac{e(B)}{2}} e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})^{\mathbb{C}^*})\, e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) \times \\ \sum_{\lambda \vdash a} \sum_{\mu \vdash b} p^{-\lambda_0-\mu_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu) \times \\ e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})^{\mathbb{C}^*}_\mu). \end{aligned} \tag{7}$$

Before we proceed, we calculate $e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*})$ and $e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})^{\mathbb{C}^*})$. The first follows from a formula of J. Cheah [6]

$$e(\mathrm{Hilb}^{0,\bullet}(W)^{\mathbb{C}^*}) = M(p)^{e(W)}. \tag{8}$$

For the second we use the following proposition:

Proposition 12. *Let $x_1, \ldots, x_l \in B$ be any number of distinct closed points. Define*

$$B^\circ := B \setminus \{x_1, \ldots, x_l\},$$
$$X^\circ := X \setminus \bigsqcup_{i=1}^{l} F_{x_i}.$$

Let $\widehat{X}^\circ_{B^\circ}$ be the formal neighbourhood of B° in X°. Define $\mathrm{Hilb}^{1,n}(\widehat{X}^\circ_{B^\circ})$ to be the Hilbert scheme of subschemes $Z \subset \widehat{X}^\circ_{B^\circ}$, such that $Z_{\mathrm{CM}} = B^\circ$ and $\chi(I_{Z_{\mathrm{CM}}}/I_Z) = n$. Then

$$e(\mathrm{Hilb}^{1,\bullet}(\widehat{X}^\circ_{B^\circ})) = \left(\frac{M(p)}{1-p}\right)^{e(B^\circ)}.$$

Proof. Pick any $y \in B^\circ$ and let $\widehat{X}_y \cong \operatorname{Spec}\mathbb{C}[\![x_1,x_2,x_3]\!]$ be the formal neighbourhood of y in X°. Denote by

$$\operatorname{Hilb}^{1,n}(\widehat{X}_y^\circ)$$

the Hilbert scheme of subschemes $Z \subset \widehat{X}_y^\circ$, such that $Z_{\mathrm{CM}} = \{x_1 = x_3 = 0\}$ and $\chi(I_{Z_{\mathrm{CM}}}/I_Z) = n$.

We have projections

$$X^\circ \longrightarrow S^\circ \longrightarrow B^\circ.$$

These map induces a morphism

$$\operatorname{Hilb}^{1,n}(\widehat{X}_{B^\circ}^\circ) \longrightarrow \operatorname{Sym}^n(B^\circ).$$

The fibre over a point $\mathfrak{a} = \sum_i a_i y_i$, with all $y_i \in B^\circ$ distinct, equals

$$\prod_i \operatorname{Hilb}^{1,a_i}(\widehat{X}_{y_i}^\circ).$$

Since B is reduced and smooth, $\operatorname{Hilb}^{1,a_i}(\widehat{X}_{y_i}^\circ)$ only depends on a_i and not on the point $y_i \in B^\circ$. Therefore Lemma 23 of Appendix A.2 implies

$$e(\operatorname{Hilb}^{1,\bullet}(\widehat{X}_{B^\circ}^\circ)) = \left(\sum_{a=0}^{\infty} e(\operatorname{Hilb}^{1,a}(\widehat{X}_y^\circ))p^a\right)^{e(B^\circ)}.$$

The formal neighbourhood $\widehat{X}_y$ has an action of $\mathbb{C}^{*3}$ and this action lifts to $\operatorname{Hilb}^{1,a}(\widehat{X}_y)$. The fixed locus consists of a finite number of points counted by the topological vertex[5]

$$\mathsf{V}_{\Box\varnothing\varnothing} = \frac{M(p)}{1-p}. \qquad \Box$$

Using (7), (8), and Proposition 12 gives

$$\begin{aligned} f_d(ax+by) = {} & \frac{M(p)^{e(X)}}{(p^{\frac12}-p^{-\frac12})^{e(B)}} \times \\ & (1-p)\sum_{\lambda\vdash a} p^{-\lambda_0} e(\operatorname{Hilb}^{(1,a),\bullet}(\widehat{X}_x)_\lambda^{\mathbb{C}^*})\, e(\operatorname{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ}^\circ)_\lambda^{\mathbb{C}^*})\times \\ & \frac{1-p}{M(p)}\sum_{\mu\vdash b} p^{-\mu_0} e(\operatorname{Hilb}^{(1,b),\bullet}(\widehat{X}_y)_\mu^{\mathbb{C}^*})\, e(\operatorname{Hilb}^{b,\bullet}(\widehat{X}_z)_\mu^{\mathbb{C}^*})\, e(\operatorname{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ}^\circ)_\mu^{\mathbb{C}^*}). \end{aligned}$$

5.3. **Geometric characterization of $g(a)$ and $h(b)$.** The arguments of the preceding two sections are straightforwardly modified to any stratum $\Sigma(\boldsymbol{x};\boldsymbol{y};\boldsymbol{\lambda};\boldsymbol{\mu})$. Fix a smooth fibre F_x and a singular fibre F_y. Denote the singular point of F_y by z. Let $\widehat{X}_x$, $\widehat{X}_z$ be the formal neighbourhoods of x, z in X. Define $\operatorname{Hilb}^{(1,a),\bullet}(\widehat{X}_x)$, $\operatorname{Hilb}^{b,\bullet}(\widehat{X}_z)$ as in Section 5.2. Like in Section 5.2, we also consider the "tubular" formal neighbourhoods $\widehat{X}_{F_x^\circ}^\circ$, $\widehat{X}_{F_y^\circ}^\circ$ and corresponding Hilbert schemes $\operatorname{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ}^\circ)$, $\operatorname{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ}^\circ)$. A straightforward generalization of the calculation of $f_d(ax+by)$ yields:

[5]Discussed in general in Section 6.

Proposition 13. *For any $a, b > 0$ define*

$$g(a) := (1-p) \sum_{\lambda \vdash a} p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)\, e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}^\circ_{F^\circ_x})^{\mathbb{C}^*}_\lambda),$$

$$h(b) := \frac{1-p}{M(p)} \sum_{\mu \vdash b} p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)\, e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}^\circ_{F^\circ_y})^{\mathbb{C}^*}_\mu),$$

and let $g(0) := 1$, $h(0) := 1$. Then

$$f_d(\mathfrak{a}+\mathfrak{b}) = \frac{M(p)^{e(X)}}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}} \cdot \prod_i g(a_i) \cdot \prod_j h(b_j),$$

for any $\mathfrak{a} = \sum_i a_i x_i \in \mathrm{Sym}^d(B^{\mathrm{sm}})$ and $\mathfrak{b} = \sum_j b_j y_j \in \mathrm{Sym}^d(B^{\mathrm{sing}})$, where $x_i \in B^{\mathrm{sm}}$ and $y_j \in B^{\mathrm{sing}}$ are collections of distinct closed points.

We immediately deduce:

Corollary 14. *Propositions 7 and 8 are true for $g(a)$ and $h(b)$ defined in Proposition 13.*

6. Reduction to the topological vertex

In this section, we obtain (Theorem 18) expressions for $\widehat{\mathsf{DT}}(X)$ and $\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$ in terms of the topological vertex $\mathsf{V}_{\lambda\mu\nu}(p)$, $e(B)$, and N (the number of nodal fibres). The theorem follows by expressing $g(a)$ and $h(b)$ of Proposition 13 in terms of the topological vertex.

6.1. **Point contributions.**

Following the conventions of [4], we denote by

$$\mathsf{V}_{\lambda\mu\nu} = \sum_\pi p^{|\pi|},$$

the topological vertex. Here the sum is over all 3D partitions π with outgoing legs λ, μ, ν and $|\pi|$ denotes renormalized volume (see Definitions (1) and (2) in [4]). For a 2D partition $\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)$, we write λ' for the corresponding transposed partition and

$$|\lambda| := \sum_{k=0}^{\infty} \lambda_k,$$

$$\|\lambda\|^2 := \sum_{k=0}^{\infty} \lambda_k^2.$$

Proposition 15. *Let F_x be a smooth fibre and F_y a singular fibre with singularity z. Then for any $\lambda \vdash a$, $\mu \vdash b$*

$$p^{-\lambda_0} e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda) = \mathsf{V}_{\lambda\square\varnothing},$$

$$p^{-\mu_0} e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu) = \mathsf{V}_{\mu\square\varnothing},$$

$$p^{-\|\mu\|^2} e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu) = \mathsf{V}_{\mu\mu'\varnothing}.$$

Proof. Recall that

$$\widehat{X}_x \cong \widehat{X}_y \cong \widehat{X}_z \cong \mathrm{Spec}\, \mathbb{C}[\![x_1, x_2, x_3]\!].$$

Therefore, $\mathbb{C}^{*3}$ acts on each of these schemes and their moduli spaces

$$\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda,\ \mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu,\ \mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu.$$

The coordinates can be chosen such that the action of the last factor of $\mathbb{C}^{*3}$ corresponds to $x_3 \mapsto t_3 x_3$. This component acts trivially since we are already on the $\mathbb{C}^*$-fixed locus.

The $\mathbb{C}^{*3}$-fixed locus consists of isolated reduced points corresponding to monomial ideals with asymptotics $(\lambda, \varnothing, \varnothing)$, $(\mu, \varnothing, \varnothing)$, $(\mu, \mu', \varnothing)$ respectively[6]. These monomial ideals are exactly what the topological vertex counts.

Finally, note that the generating functions $e(\mathrm{Hilb}^{(1,a),\bullet}(\widehat{X}_x)^{\mathbb{C}^*}_\lambda)$, $e(\mathrm{Hilb}^{(1,b),\bullet}(\widehat{X}_y)^{\mathbb{C}^*}_\mu)$, $e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_z)^{\mathbb{C}^*}_\mu)$ all start with 1. On the other hand, from the definition

$$\begin{aligned} \mathsf{V}_{\lambda\square\varnothing} &= p^{-\lambda_0} + \cdots, \\ \mathsf{V}_{\mu\square\varnothing} &= p^{-\mu_0} + \cdots, \\ \mathsf{V}_{\mu\mu'\varnothing} &= p^{-\sum_{k=0}^{\infty}\mu_k^2} + \cdots, \end{aligned}$$

where $\cdots$ stands for higher order terms in p. The proposition follows. □

6.2. **Fibre contribution.** Let F_x be a smooth fibre and F_y a singular fibre. Recall the formal neighbourhoods $\widehat{X}_{F_x^\circ}$, $\widehat{X}_{F_y^\circ}$ of Section 5.

Proposition 16. *For any $\lambda \vdash a$ and $\mu \vdash b$, we have*

$$\begin{aligned} e(\mathrm{Hilb}^{a,\bullet}(\widehat{X}_{F_x^\circ})^{\mathbb{C}^*}_\lambda) &= \frac{1}{\mathsf{V}_{\lambda\varnothing\varnothing}}, \\ e(\mathrm{Hilb}^{b,\bullet}(\widehat{X}_{F_y^\circ})^{\mathbb{C}^*}_\mu) &= \frac{1}{\mathsf{V}_{\mu\varnothing\varnothing}}. \end{aligned}$$

Proof. ******** □

6.3. **Putting it together.** Combining Proposition 13 with Propositions 15, 16 gives:

Proposition 17. *For any $a, b > 0$*

$$\begin{aligned} g(a) &= (1-p)\sum_{\lambda\vdash a}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}, \\ h(b) &= \frac{1-p}{M(p)}\sum_{\mu\vdash b} p^{\|\mu\|^2}\frac{\mathsf{V}_{\mu\square\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}}\mathsf{V}_{\mu\mu'\varnothing}. \end{aligned} \tag{9}$$

Putting all our results together, we obtain formulas for the partition functions in terms of the vertex:

Theorem 18.

$$\widehat{\mathsf{DT}}(X) = \frac{1}{(p^{\frac{1}{2}} - p^{-\frac{1}{2}})^{e(B)}}\left((1-p)\sum_\lambda q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}\right)^{e(B)-N}\left((1-p)\sum_\mu q^{|\mu|}p^{\|\mu\|^2}\frac{\mathsf{V}_{\mu\square\varnothing}}{\mathsf{V}_{\mu\varnothing\varnothing}}\mathsf{V}_{\mu\mu'\varnothing}\right)^N$$

$$\widehat{\mathsf{DT}}_{\mathrm{fib}}(X) = \left(\sum_\lambda q^{|\lambda|}\right)^{e(B)-N}\left(\sum_\mu q^{|\mu|}p^{\|\mu\|^2}\mathsf{V}_{\mu\mu'\varnothing}\right)^N.$$

Proof. Inserting the equations for $g(a)$, $h(b)$ of Proposition 17 into (3) gives the formula for $\widehat{\mathsf{DT}}(X)$. Similar reasoning… □

Go back and put in the necessary stuff so that the $\widehat{\mathsf{DT}}_{\mathrm{fib}}$ computation isn't just hand waving.

Corollary 19. *Theorem 1 is true.*

[6]The transpose in μ' occurs, because we follow the orientation convention of [4].

Proof. We apply the main theorem of [4]. In particular, we substitute [4, Eqns (2)&(4)] into the formula for $\widehat{\mathsf{DT}}(X)$ and we substitute [4, Eqn (1)], as well as the well-known formula

$$\sum_{\lambda} q^{|\lambda|} = \prod_{d=1}^{\infty} \frac{1}{(1-q^d)}$$

into the formula for $\widehat{\mathsf{DT}}_{\mathrm{fib}}(X)$. $\square$

7. PUTTING IN THE BEHREND FUNCTION

The aim of this section is to show that the partition functions $\widehat{\mathsf{DT}}(X)$ and $\mathsf{DT}(X)$ are equal after the simple change of variables $y = -p$. In order to do this we will need to assume a conjecture about the Behrend function which we formulate below for general Calabi-Yau threefolds.

Let Y be any quasi-projective Calabi-Yau threefold. Let $C \subset Y$ be a (not necessarily reduced) Cohen-Macaulay curve with proper support. Assume that the singularities of C_{red} are locally toric[7]. Define

$$\mathrm{Hilb}^{C,n}(Y) = \{Z \subset Y \textit{ such that } C \subset Z \textit{ and } I_C/I_Z \textit{ has finite length } n\}.$$

Note that $\mathrm{Hilb}^{C,n}(Y) \subset \mathrm{Hilb}(Y)$ and let ν denote the Behrend function on $\mathrm{Hilb}(Y)$. Our conjecture is the following:

Conjecture 20.

$$\int_{\mathrm{Hilb}^{C,n}(Y)} \nu \, de = (-1)^n \nu([C]) \int_{\mathrm{Hilb}^{C,n}(Y)} de$$

where $\nu([C])$ is the value of the Behrend function at the point $[C] \in \mathrm{Hilb}(Y)$.

Remark 21. Conceivably, the condition that C_{red} has locally toric singularities could be weakened, although we do not have any evidence for this case. Our conjecture is true for Y a (globally) toric Calabi-Yau. This follows from the computations in [10].

One could also make the much stronger conjecture that

$$\nu([Z]) = (-1)^n \nu([C])$$

for all $[Z] \in \mathrm{Hilb}^{C,n}(Y)$. This would of course imply our conjecture as stated. However, we do not know if this stronger version holds, even in the case where Y is $\mathbb{C}^3$ and C is empty. In this case, this stronger conjecture says that the Behrend function on $\mathrm{Hilb}^n(\mathbb{C}^3)$ is the constant function $(-1)^n$.

APPENDIX A. ODDS AND ENDS

A.1. **Curves on elliptic surfaces.** Let $p : S \to B$ be an elliptic surface with section $B \subset S$. In this appendix we allow any type of singular fibres. We assume S is not a product, which implies

$$p^* : \mathrm{Pic}^0(B) \xrightarrow{\cong} \mathrm{Pic}^0(S)$$

is an isomorphism [11, VII.1.1]. For any $\beta \in H_2(S)$, we denote by $\mathrm{Hilb}^{\beta}(S)$ the Hilbert scheme of effective divisors on S in class β.

[7]This means that formally locally C_{red} is either smooth, nodal, or the union of the three coordinate axes. That is at $p \in C_{\mathrm{red}} \subset Y$ the ideal $\widehat{I}_{C_{\mathrm{red}}} \subset \widehat{\mathcal{O}}_{Y,p}$ is given by (x_1, x_2), (x_1, x_2x_3), or (x_1x_2, x_2x_3, x_1x_3) for some isomorphism $\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[[x_1, x_2, x_3]]$.

Denote by $B \in H_2(S)$ the class of the section $B \subset S$ and by $F \in H_2(S)$ the class of the fibre. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sym}^d(B) \arrow[r] \arrow[d, "p^*"] & \mathrm{Pic}^d(B) \arrow[d, "p^*", "\cong"'] \\
\mathrm{Hilb}^{dF}(S) \arrow[r] \arrow[d, "+B"] & \mathrm{Pic}^{dF}(S) \arrow[d, "\otimes \mathcal{O}_S(B)", "\cong"'] \\
\mathrm{Hilb}^{B+dF}(S) \arrow[r] & \mathrm{Pic}^{B+dF}(S).
\end{tikzcd}$$

The horizontal arrows are Abel-Jacobi maps. The vertical arrows are induced by pull-back and adding the section $B \subset S$.

Lemma 22. *The above maps induce an isomorphism*

$$\mathrm{Sym}^d(B) \xrightarrow{\cong} \mathrm{Hilb}^{B+dF}(S).$$

Proof. Clearly p^* gives an isomorphism $\mathrm{Sym}^d(B) \cong \mathrm{Hilb}^{dF}(S)$ and $+B$ defines a closed embedding $\mathrm{Hilb}^{dF}(S) \hookrightarrow \mathrm{Hilb}^{B+dF}(S)$. Since $\mathrm{Sym}^d(B)$ is smooth and $\mathrm{Hilb}^{B+dF}(S)$ is reduced (by [12, Lect. 25]), it suffices to show

$$\mathrm{Sym}^d(B) \to \mathrm{Hilb}^{B+dF}(S)$$

is surjective on closed points.

For surjectivity, suppose D' is an effective divisor with class $B + dF$ which does *not* lie in the image. Firstly, we note that for any fibre F we have $D' \cdot F = 1$. Therefore D' contains a section $B' \subset S$ as an effective summand. Moreover $B \neq B'$ or else D' would lie in the image. Next, we take any D in the image and compare D and D'. Then

$$\mathcal{O}_S(D - D') \in \mathrm{Pic}^0(S) \cong \mathrm{Pic}^0(B).$$

Therefore after re-arranging we find that there are distinct fibres F_{x_i}, F_{y_j} and $a_i \geq 0$, $b_j \geq 0$ such that

$$B + \sum_i a_i F_{x_i} \sim_{\mathrm{lin}} B' + \sum_j b_j F_{y_j},$$

where $\sim_{\mathrm{lin}}$ denotes linear equivalence. Hence there exists a pencil $\{C_t\}_{t \in \mathbb{P}^1}$ of effective divisors such that

$$C_0 = B + \sum_i a_i F_{x_i},\ C_\infty = B' + \sum_j b_j F_{y_j}.$$

Now fix a smooth fibre F. Then $C_t \cdot F = 1$ for any $t \in \mathbb{P}^1$, so we get a morphism

$$\mathbb{P}^1 \longrightarrow F,\ t \mapsto C_t \cap F.$$

But F is a smooth elliptic curve so this map is constant. We conclude

$$B \cap F = C_0 \cap F = C_\infty \cap F = B' \cap F.$$

Since F was chosen arbitrary, we deduce that $B = B'$ which is a contradiction. $\square$

A.2. **Weighted Euler characteristics of symmetric products.** In this section we prove the following formula for the weighted Euler characteristic of symmetric products.

Lemma 23. *Let B be a scheme of finite type over $\mathbb{C}$ and let $e(B)$ be its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$ be any function with $g(0) = 1$. Let $f_d : \operatorname{Sym}^d(B) \to \mathbb{Z}((p))$ be the constructible function defined by*

$$f_d(\mathfrak{a}) = \prod_i g(a_i),$$

for all $\mathfrak{a} = \sum_i a_i x_i \in \operatorname{Sym}^d(B)$ where $x_i \in B$ are distinct closed points. Then

$$\sum_{d=0}^{\infty} q^d \int_{\operatorname{Sym}^d(B)} f_d \, de = \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)}.$$

Remark 24. In the special case where $g = f_d \equiv 1$, the lemma recovers MacDonald's formula:

$$\sum_{d=0}^{\infty} e(\operatorname{Sym}^d(B))\, q^d = \frac{1}{(1-q)^{e(B)}}.$$

The lemma is essentially a consequence of the existence of a power structure on the Grothendieck group of varieties definited by symmetric products and the compatibility of the Euler characteristic homomorphism with that power structure []. For convenience's sake, we provide a direct proof here.

Ref? Bryan-Young?

Proof. The dth symmetric product admits a stratification with strata labelled by partitions of d. Associated to any partition of d is a unique tuple $(m_1, m_2, \dots)$ of non-negative integers with $\sum_{j=1}^{\infty} j m_j = d$. The stratum labelled by $(m_1, m_2, \dots)$ parameterizes collections of points where there are m_j points of multiplicity j. The full stratification is given by:

$$\operatorname{Sym}^d(B) = \bigsqcup_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left\{ \left(\prod_{j=1}^{\infty} B^{m_j} \right) - \Delta \right\} \Big/ \prod_{j=1}^{\infty} \sigma_{m_j}$$

where by convention, B^0 is a point, Δ is the large diagonal, and σ_m is the mth symmetric group. Note that the function f_d is constant on each stratum and has value $\prod_{j=1}^{\infty} g(j)^{m_j}$. Note also that the action of $\prod_{j=1}^{\infty} \sigma_{m_j}$ on each stratum is free.

For schemes over $\mathbb{C}$, topological Euler characteristic is additive under stratification and multiplicative under maps which are (topological) fibrations. Thus

$$\int_{\operatorname{Sym}^d(B)} f_d \; de = \sum_{\substack{(m_1, m_2, \dots) \\ \sum_{j=1}^{\infty} j m_j = d}} \left(\prod_{j=1}^{\infty} g(j)^{m_j} \right) \frac{e(B^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \dots}.$$

For any natural number N, the projection $B^N - \Delta \to B^{N-1} - \Delta$ has fibers of the form $B - \{N-1 \text{ points}\}$. The fibers have constant Euler characteristic given by $e(B) - (N-1)$ and consequently, $e(B^N - \Delta) = (e(B) - (N-1)) \cdot e(B^{N-1} - \Delta)$. Thus by induction, we find $e(B^N - \Delta) = e(B) \cdot (e(B) - 1) \cdots (e(B) - (N-1))$ and so

$$\frac{e(B^{\sum_j m_j} - \Delta)}{m_1!\, m_2!\, m_3! \cdots} = \binom{e(B)}{m_1, m_2, m_3, \cdots}$$

where the right hand side is the generalized multinomial coefficient.

Putting it together and applying the generalized multinomial theorem, we find

$$\begin{aligned}\sum_{d=0}^{\infty} q^d \int_{\mathrm{Sym}^d(B)} f_d \; de &= \sum_{(m_1, m_2, \dots)} \prod_{j=1}^{\infty} \left(g(j) q^j\right)^{m_j} \binom{e(B)}{m_1, m_2, m_3, \dots} \\ &= \left(1 + \sum_{j=1}^{\infty} g(j) q^j\right)^{e(B)}\end{aligned}$$

which proves the lemma. □

APPENDIX B. STUFF WE GATHERED

B.1. **Normal bundle.** First order deformations of $B \subset X$ are given by $H^0(N_{B/X})$. We now calculate this. Let $\pi : X \to S$ and $p : S \to B$. Facts: (adjunction, canonical bundle of an elliptic surface)

$$\begin{aligned} N_{B/X} &\cong N_{B/S} \oplus K_S|_B \\ K_B &\cong K_S|_B \otimes N_{B/S} \cong K_S|_B(S) \\ K_S &\cong p^*(K_B \otimes L), \end{aligned}$$

where $L = (R^1 p_* \mathcal{O}_X)^\vee$ and

$$\deg L = \chi(\mathcal{O}_S) = \frac{e(S)}{12} = \frac{N}{12} > 0,$$

where N is the number of nodal fibres. Combining gives

$$N_{B/S} \cong K_B \otimes K_S^{-1}|_B \cong L^\vee,$$

Question: Do we not need Appendix A.1 anymore? I don't think so.

which has negative degree so

$$H^0(N_{B/S}) = 0.$$

Next

$$H^0(K_S|_B) \cong H^1(K_S|_B^{-1} \otimes K_B)^* \cong H^1(N_{B/S})^*.$$

By Riemann-Roch and $H^0(N_{B/S}) = 0$ we get

$$h^1(N_{B/S}) = -\chi(N_{B/S}) = -\chi(L^\vee) = -(1 - g - \chi(\mathcal{O}_S)) = \chi(\mathcal{O}_S) - 1 + g.$$

In conclusion

$$h^0(N_{X/B}) = \chi(\mathcal{O}_S) - 1 + g = \frac{e(S)}{12} - \frac{e(B)}{2}.$$

B.2. **Hilbert schemes on $\mathbb{C}^2$.** Let $X = \mathbb{C}^2$ and

$$L_k := \{Z \in X^n \ : \ \ell(Z \cap \{y = 0\}) = k\} \subset X^{[n]}.$$

Why normal?

Ref? I learned this from Dori Bejleri.

Claim: L_k is locally closed and smooth of dimension $2n - k$. Proof: since L_k is normal and $T = \mathbb{C}^{*2}$ acts on it, it suffices to prove smoothness at T-fixed points. At the T-fixed points we can write a basis for the tangent space by pairs of arrows with tail just outside and head just inside as described by Haiman. I'm not going to write this out now formally. Just remember how each arrow defines a first order deformation of the monomial ideal ("plus ϵ times the monomial the head points at"). The corresponding global deformation can easily be seen to move outside of L_k for precisely k arrows with head in the bottom row. This gives the result.

B.3. **Key iso.** Let $C \in \mathrm{Hilb}^{B+\bullet F}(X)^{\mathbb{C}^*}_{\mathrm{CM}}$ be described by partitions $\lambda^{(i)}$ with

$$\sum_i |\lambda^{(i)}| = d.$$

Claim: there is a formal neighborhood of C inside $\mathrm{Hilb}^{B+\bullet F}(X)_{\mathrm{CM}}$ which maps to $H^0(N_{B/X})$ with fibre

$$L_k,$$

where $k := \sum \lambda_0^{(i)}$ and L_k is smooth by the previous subsection.

Let's look at the case where $C = B \cup F_\lambda$, where $E := F_\lambda$ is a single thickened fibre with cross-section λ. We have a couple of useful short exact sequences

$$0 \to I_E/I_C \to \mathcal{O}_C \to \mathcal{O}_E \to 0 \tag{10}$$

$$0 \to I_B/I_C \to \mathcal{O}_C \to \mathcal{O}_B \to 0 \tag{11}$$

$$0 \to I_C \to I_E \to I_E/I_C \to 0 \tag{12}$$

$$0 \to I_C \to I_B \to I_B/I_C \to 0, \tag{13}$$

where

$$I_E/I_C \cong \mathcal{O}_B(-\lambda_0)$$
$$G := I_B/I_C,$$

where G is an ideal sheaf of a fat point of length λ_0 inside $\mathcal{O}_E$:

$$0 \to G \to \mathcal{O}_E \to \mathcal{O}_{\lambda_0 p} \to 0. \tag{14}$$

Two more significant short exact sequences

$$0 \to \mathcal{O}_C \to \mathcal{O}_B \oplus \mathcal{O}_E \to \mathcal{O}_{\lambda_0 p} \to 0 \tag{15}$$

$$0 \to \mathcal{O}_B(-\lambda_0) \to \mathcal{O}_B \to \mathcal{O}_{\lambda_0 p} \to 0. \tag{16}$$

We can build *many* double complexes from these seven short exact sequences by applying $\mathrm{Hom}(I_C, \cdot)$, $\mathrm{Hom}(\cdot, \mathcal{O}_C)$ etc.

It seems significant to apply $\mathrm{Hom}(I_E, \cdot)$ to (15). This gives

$$0 \to \mathrm{Hom}(I_E, G) \to \mathrm{Hom}(I_E, \mathcal{O}_E) \xrightarrow{\alpha} \mathrm{Hom}(I_E, \mathcal{O}_{\lambda_0 p}) \to \cdots \tag{17}$$

The last hom space seems isomorphic to $\mathbb{C}^{\lambda_0}$ and we believe there is a splitting of the map α from the local description. It seems $\mathrm{Hom}(I_E, G)$ can be interpreted as the space of deformations of E which keep the length of $E \cap B$ fixed! We want to prove there exists a short exact sequence

$$0 \to \mathrm{Hom}(I_E, G) \to \mathrm{Hom}(I_C, \mathcal{O}_C) \to H^0(N_{B/X}) \to 0.$$

The good news: we can construct an injection $\mathrm{Hom}(I_E, G) \hookrightarrow \mathrm{Hom}(I_C, \mathcal{O}_C)$ as follows. From (11) and (12) we get the following double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{O}_B(-\lambda_0), G) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{O}_B(-\lambda_0), \mathcal{O}_C) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{O}_B(-\lambda_0), \mathcal{O}_B) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(I_E, G) \arrow[r] \arrow[d] & \mathrm{Hom}(I_E, \mathcal{O}_C) \arrow[r] \arrow[d] & \mathrm{Hom}(I_E, \mathcal{O}_B) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(I_C, G) \arrow[r] \arrow[d] & \mathrm{Hom}(I_C, \mathcal{O}_C) \arrow[r] \arrow[d] & \mathrm{Hom}(I_C, \mathcal{O}_B) \arrow[r] \arrow[d] & \cdots \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Note: $\mathrm{Hom}(\mathcal{O}_B(-\lambda_0), G) = 0$ from the local description, so indeed we get the required injection!

Significant special case. When S is the rational elliptic surface we have $H^0(N_{B/X}) = H^1(N_{B/X}) = 0$ in which case we get $\mathrm{Hom}(I_B, \mathcal{O}_B) = \mathrm{Ext}^1(I_B, \mathcal{O}_B) = 0$ (see next subsection for relating $\mathrm{Ext}^i(I_B, \mathcal{O}_B)$ to $H^i(N_{B/X})$)! In this case we really expect our injection

$$\mathrm{Hom}(I_E, G) \hookrightarrow \mathrm{Hom}(I_C, \mathcal{O}_C)$$

to be an isomorphism. If so, our earlier split exact sequence (17) proves $\mathrm{Hom}(I_C, \mathcal{O}_C)$ has dimension

$$\hom(I_E, \mathcal{O}_E) - \lambda_0 = \hom(I_\lambda, \mathcal{O}_\lambda) - \lambda_0 = 2|\lambda| - \lambda_0,$$

where $\mathcal{O}_\lambda \subset \mathbb{C}^2$ is the zero-dimensional structure sheaf from which $\mathcal{O}_E$ pulls-back.

In this special case of the rational elliptic surface we have a lot more vanishing and we hope that the big double complex yields the surjection. The easiest way to get the surjection is when

$$\mathrm{Ext}^1(G, \mathcal{O}_B) \overset{?}{\cong} 0$$
$$\mathrm{Ext}^1(O_B(-\lambda_0), G) \overset{?}{\cong} 0,$$

and then use $\mathrm{Hom}(I_C, \mathcal{O}_B) \cong \mathrm{Ext}^1(G, \mathcal{O}_B)$ (which I get from $\mathrm{Hom}(I_B, \mathcal{O}_B) = \mathrm{Ext}^1(I_B, \mathcal{O}_B) = 0$). Sadly I cannot prove the two previous Ext groups are indeed zero.

Comment: there does not seem to be a simple map from $\mathrm{Hom}(I_C, \mathcal{O}_C) \to \mathrm{Hom}(I_B, \mathcal{O}_B)$ simply by playing with $\mathcal{O}_C \to \mathcal{O}_B$ or $I_C \subset I_B$.

B.4. **Miscalleneous.** Significant fact: $\mathrm{Hom}(I_C, \mathcal{O}_C) \cong \mathrm{Ext}^1(I_C, I_C)_0$ by PT1, Section 2.2 for naked CM curves.

Some remarks. By cutting out I_B from a rank 2 vector bundle we get maps

$$0 \to \mathrm{Hom}(I_B, \mathcal{O}_B) \to H^0(B, \mathcal{O}_B(S)) \oplus H^0(B, \mathcal{O}_B(Y)) \to H^0(B, \mathcal{O}_B(S+Y)) \to \cdots$$

remember: $Y := K_S|_B$. Here

$$\mathcal{O}_B(S) \cong \mathcal{O}_S(S)|_B \cong K_X(S)|_B \cong K_S|_B$$
$$\mathcal{O}_B(Y) \cong \mathcal{O}_B(B) \cong N_{B/S},$$

and $N_{B/S} \otimes K_S|_B \cong K_B$. We get

$$0 \to \mathrm{Hom}(I_B, \mathcal{O}_B) \to H^0(N_{B/X}) \to H^0(K_B) \to \cdots.$$

The first map is an isomorphism because both terms are first order deformations of B as a subscheme of X. Not sure we need this.

B.5. **Direct approach.** We have the following commutative diagram

$$\begin{tikzcd} X := K_S \arrow[r, "\varpi"] \arrow[d, "\pi"'] & Y := K_S|_B \arrow[d, "q"] \\ S \arrow[r, "p"'] & B. \end{tikzcd}$$

Note: π_* and q_* are exact (because π, q are affine morphisms), but I don't think I need this now. I do need ϖ^*, p^* are exact (because ϖ, p are flat morphisms). Let $\mathrm{Hilb}^n(Y)$ be the Hilbert scheme of 0-dim subschemes of length n on Y. By exactness of ϖ^* we can pull-back

$$0 \to I_Z \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0$$

on Y to a short exact sequence

$$0 \to \varpi^* I_Z \to \mathcal{O}_X \to \varpi^* \mathcal{O}_Z \to 0$$

on X. Hence $\varpi^* \mathcal{O}_Z$ is a structure sheaf and $\varpi^* I_Z$ is its ideal sheaf. Let $\mathrm{Hilb}^0(X, [nF])$ be the Hilbert scheme of Cohen-Macaulay curves in homology class $[nF]$. Then at the set level we have a map

$$\mathrm{Hilb}^n(Y) \to \mathrm{Hilb}^0(X, [nF]),\ (I_Z \subset \mathcal{O}_Y) \mapsto (\varpi^* I_Z \subset \mathcal{O}_X).$$

This works in families ($(\varpi \times 1_B)^*$ is exact because ϖ is flat) so this indeed gives a morphism. At the level of closed points this is clearly injective.

The tangent space at $\varpi^* I_Z \in \mathrm{Hilb}^0(X, [nF])$ is

$$\begin{aligned} \mathrm{Hom}_X(\varpi^* I_Z, \varpi^* \mathcal{O}_Z) &\cong \mathrm{Hom}_Y(I_Z, \varpi_* \varpi^* \mathcal{O}_Z) \\ &\cong \mathrm{Hom}_Y(I_Z, \mathcal{O}_Z \otimes \varpi^* \mathcal{O}_X) \\ &\cong \mathrm{Hom}(I_Z, \mathcal{O}_Z), \end{aligned}$$

which is the tangent space at $I_Z \in \mathrm{Hilb}^n(Y)$. I used: (1) ordinary adjunction for Hom, (2) projection formula, (3) $\varpi_* \mathcal{O}_X \cong \mathcal{O}_Y$ (how do we prove this last one? I know $\varpi_* \mathcal{O}_X$ is a line bundle so should go similar to $p_* \mathcal{O}_S \cong \mathcal{O}_B$). So indeed deformations match! We see there cannot be any twistings but only deformations induced from moving Z in the surface.

The next part may not be needed but I find it instructive. There is a spectral sequence

$$\mathrm{Ext}^i_Y(I_Z, R^j \varpi_* \mathcal{O}_Z) \cong \mathrm{Ext}^i_Y(I_Z, \mathcal{O}_Z \otimes R^j \varpi_* \mathcal{O}_X) \Rightarrow \mathrm{Ext}^{i+j}_X(\varpi^* I_Z, \varpi^* \mathcal{O}_Z).$$

Note: $R^0 \varpi_* \mathcal{O}_X \cong \mathcal{O}_Y$, $R^1 \varpi_* \mathcal{O}_X$ is a line bundle (the fibres of ϖ have genus 1), and $R^{>1} \varpi_* \mathcal{O}_X = 0$ (the fibres are 1-dimensional). Therefore the terms of the spectral sequence are zero unless $j = 0, 1$ which leads to an injection

$$\mathrm{Ext}^1_Y(I_Z, \mathcal{O}_Z) \hookrightarrow \mathrm{Ext}^1_X(\varpi^* I_Z, \varpi^* \mathcal{O}_Z).$$

This injections maps the obstruction class for deforming Z to the obstruction class for deforming $\varpi^* Z$. In other words, we can deform Z if and only if we can deform $\varpi^* Z$. So indeed we have an isomorphism of schemes

$$\mathrm{Hilb}^n(Y) \cong \mathrm{Hilb}^0(X, [nF]).$$

B.6. **Koszul calculations.** Let $X = K_S$, $B \subset S \subset X$, and $Y := K_S|_B$. For any point on B, I use local coordinates x, y, z in a formal neighbourhood of that point such that $x = z = 0$ defines the section B. Denote by $p : X \to Y$ the projection to Y. Besides the divisors $S, Y \subset X$, we have $\tilde{F} \subset X$, where $\tilde{F} = K_S|_F$ for any fibre $F \subset S$.

Let $C = B \cup p^* Z$, where $Z \subset Y$ is a 0-dimensional scheme. Assume $I_{Z \subset Y} = (z^k, y^l)$ (so the partition is a rectangle). We have the following Koszul resolution of I_C

$$0 \to \mathcal{O}_X(-kS - l\tilde{F}) \oplus \mathcal{O}_X(-S - Y - l\tilde{F}) \to \mathcal{O}_X(-kS) \oplus \mathcal{O}_X(-S - l\tilde{F}) \oplus \mathcal{O}_X(-Y - l\tilde{F}) \to I_C \to 0,$$

where the second map is $(z^k, y^l z, xy^l)$ and the first map is

$$M := \begin{pmatrix} y^l & 0 \\ -z^{k-1} & x \\ 0 & -z \end{pmatrix}.$$

A local computation shows this is indeed a resolution. The second short exact sequence we use is

$$0 \to \mathcal{O}_C \to \mathcal{O}_B \oplus \mathcal{O}_{p^*Z} \to \mathcal{O}_P \to 0,$$

where $P := B \cap p^*Z$. Using the obvious double complex, we conclude that $\mathrm{Hom}_X(I_C, \mathcal{O}_C)$ is given as the intersection of the kernels of the following two maps

$$\begin{array}{c} \Gamma(\mathcal{O}_B(kS)) \oplus \Gamma(\mathcal{O}_B(S + l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(Y \oplus l\tilde{F})) \\ \Gamma(\mathcal{O}_{p^*Z}(kS)) \oplus \Gamma(\mathcal{O}_{p^*Z}(S + l\tilde{F})) \oplus \Gamma(\mathcal{O}_{p^*Z}(Y \oplus l\tilde{F})) \end{array} \stackrel{f}{\to} \begin{array}{c} \Gamma(\mathcal{O}_B(kS + l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(S + l\tilde{F} + Y)) \\ \Gamma(\mathcal{O}_{p^*Z}(kS + l\tilde{F})) \oplus \Gamma(\mathcal{O}_{p^*Z}(S + l\tilde{F} + Y)) \end{array},$$

and

$$\begin{array}{c} \Gamma(\mathcal{O}_B(kS)) \oplus \Gamma(\mathcal{O}_B(S + l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(Y \oplus l\tilde{F})) \\ \Gamma(\mathcal{O}_{p^*Z}(kS)) \oplus \Gamma(\mathcal{O}_{p^*Z}(S + l\tilde{F})) \oplus \Gamma(\mathcal{O}_{p^*Z}(Y \oplus l\tilde{F})) \end{array} \stackrel{g}{\to} \Gamma(\mathcal{O}_P(kS) \oplus \Gamma(\mathcal{O}_P(S + l\tilde{F})) \oplus \Gamma(\mathcal{O}_P(l\tilde{F} + Y)),$$

where f is given by

$$M^T = \left(\begin{array}{ccc} y^l & -z^{k-1} & 0 \\ 0 & x & -z \end{array} \right).$$

and g is given by taking the difference and restricting to P. Restricting M^T to B we get many zeroes and the kernel of f to the part involving B equals

$$0 \oplus \Gamma(\mathcal{O}_B(S + l\tilde{F})) \oplus \Gamma(\mathcal{O}_B(Y + l\tilde{F})).$$

The kernel of F restricted to the part involving p^*Z has a more complicated kernel. Certainly $\Gamma(\mathcal{O}_{p^*Z}(kS))$ lies in this kernel because $y^l = 0$ on p^*Z. Furthermore, this kernel contains all elements $(\beta, \gamma) \in \Gamma(p^*\mathcal{O}_Z(S + l\tilde{F})) \oplus \Gamma(p^*\mathcal{O}_Z(Y + l\tilde{F}))$ satisfying the equations

$$-z^{k-1}\beta = 0, \ x\beta = z\gamma.$$

This means $\beta = z\beta'$ and $\gamma = x\beta' + z^{k-1}\gamma'$. We observe that elements in the kernel of f restricted to the part involving p^*Z automatically restrict to zero on P. Therefore we should compute the kernels of

$$\Gamma(\mathcal{O}_B(S + l\tilde{F})) \to \Gamma(\mathcal{O}_P(S + l\tilde{F}))$$
$$\Gamma(\mathcal{O}_B(Y + l\tilde{F})) \to \Gamma(\mathcal{O}_P(Y + l\tilde{F})),$$

which give $\Gamma(N_{B/Y})$ and $\Gamma(N_{B/S})$! Next we should calculate the kernel of

$$\Gamma(\mathcal{O}_{p^*Z}(mS)) \to \Gamma(\mathcal{O}_P(mS)),$$

which equals $\Gamma(I_{P \subset p^*Z}(mS))$. In turn this can be computed using the short exact sequence

$$0 \to K \to I_{P \subset p^*Z} \to \mathcal{O}_{p^*Z}(-Y) \to 0,$$

where K is a zero-dimensional sheaf of length $n - l$. It is not hard to see that $\Gamma(\mathcal{O}_{p^*Z}(mS - Y)) = 0$ so we obtain

$$\Gamma(I_{P \subset p^*Z}(mS)) \cong \mathbb{C}^{n-l},$$

where $n = kl$. The part which is left is $(\beta, \gamma) \in \Gamma(p^*\mathcal{O}_Z(S + l\tilde{F})) \oplus \Gamma(p^*\mathcal{O}_Z(Y + l\tilde{F}))$ satisfying

$$\beta = z\beta', \ \gamma = x\beta' + z^{k-1}\gamma'.$$

The solutions $\beta = z\beta'$, $\gamma = x\beta'$ are given by the space

$$\Gamma(\mathcal{O}_{p^*Z}(l\tilde{F})) \cong \Gamma(\mathcal{O}_{p^*Z}) \cong \mathbb{C}^n.$$

At this stage we have got $H^0(N_{B/Y}) \oplus \mathbb{C}^{2n-l}$ worth of deformations. So what about the solutions $\gamma = z^{k-1}\gamma'$? Solutions of this form which are *not* of the form $\beta = z\beta'$, $\gamma = x\beta'$ are given by

$$\Gamma(\mathcal{O}_{lF}(Y + l\tilde{F} - (m-1)S)) - \Gamma(\mathcal{O}_{lF}(Y + l\tilde{F} - (m-1)S - Y)).$$

This amounts to

$$\Gamma(\mathcal{O}_{lF}(B)) - \Gamma(\mathcal{O}_{lF}) = \mathbb{C}^l - \mathbb{C}^l,$$

so there is nothing we do not already have. In conclusion

$$\mathrm{Hom}(I_C, \mathcal{O}_C) \cong H^0(N_{B/Y}) \oplus \mathbb{C}^{2n-l}.$$

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