TRACE IDENTITIES FOR THE TOPOLOGICAL VERTEX

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ABSTRACT. The topological vertex is a universal series which can be regarded as an object in combinatorics, representation theory, geometry, or physics. It encodes the combinatorics of 3D partitions, the action of vertex operators on Fock space, the Donaldson-Thomas theory of toric Calabi-Yau threefolds, or the open string partition function of $\mathbb{C}^3$.

We prove several identities in which a sum over terms involving the topological vertex is expressed as a closed formula, often a product of simple terms, closely related to Fourier expansions of Jacobi forms. We use purely combinatorial and representation theoretic methods to prove our formulas, but we discuss applications to the Donaldson-Thomas invariants of elliptically fibered Calabi-Yau threefolds at the end of the paper.

1. INTRODUCTION

The topological vertex $\mathsf{V}_{\lambda\mu\nu} = \mathsf{V}_{\lambda\mu\nu}(p)$ is a universal formal Laurent series in p depending on a triple of partitions (λ, μ, ν). It can be considered as an object in combinatorics, representation theory, geometry, or physics. In combinatorics, $\mathsf{V}_{\lambda\mu\nu}$ is the generating function for the number of 3D partitions with asymptotic legs of type (λ, μ, ν) (see Definition 2). In representation theory, $\mathsf{V}_{\lambda\mu\nu}$ is given as the matrix coefficients of a certain vertex operator on Fock space [11]. In geometry, $\mathsf{V}_{\lambda\mu\nu}$ is the basic building block for computing the Donaldson-Thomas/Gromov-Witten invariants of toric Calabi-Yau threefolds [8]; in this incarnation, it can be realized as the generating function for the Euler characteristics of certain Hilbert schemes of curves in $\mathbb{C}^3$ (see § 6). The topological vertex was first discovered in physics as an open string partition function in type IIA string theory on $\mathbb{C}^3$ [1]. An explicit expression for $\mathsf{V}_{\lambda\mu\nu}$ in terms of Schur functions was given by [11] (see section 3).

In this paper we prove several "trace identities" in which a sum over certain combinations of the vertex is expressed as a closed formula, often a product of simple terms. The products are closely related to the Fourier expansions of Jacobi forms. Applications of these identities are used to

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compute the Donaldson-Thomas partition functions of certain Calabi-Yau threefolds in terms of Jacobi forms [5, 4, 6].

2. Definitions and the main result.

In this section we give the combinatorial definition of the vertex and we state our main identities.

Definition 1. Let (λ, μ, ν) be a triple of ordinary partitions. A *3D partition π asymptotic to* (λ, μ, ν) is a subset

$$\pi \subset (\mathbb{Z}_{\geq 0})^3$$

satisfying

(1) if any of $(i+1, j, k)$, $(i, j+1, k)$, and $(i, j, k+1)$ is in π, then (i, j, k) is also in π, and
(2) (a) $(j, k) \in \lambda$ if and only if $(i, j, k) \in \pi$ for all $i \gg 0$,
 (b) $(k, i) \in \mu$ if and only if $(i, j, k) \in \pi$ for all $j \gg 0$,
 (c) $(i, j) \in \nu$ if and only if $(i, j, k) \in \pi$ for all $k \gg 0$.

where we regard ordinary partitions as finite subsets of $(\mathbb{Z}_{\geq 0})^2$ via their diagram.

Intuitively, π is a pile of boxes in the positive octant of 3-space. Condition (1) means that the boxes are stacked stably with gravity pulling them in the $(-1, -1, -1)$ direction; condition (2) means that the pile of boxes is infinite along the coordinate axes with cross-sections asymptotically given by λ, μ, and ν.

The subset $\{(i, j, k) : (j, k) \in \lambda\} \subset \pi$ will be called the *leg* of π in the i direction, and the legs in the j and k directions are defined analogously. Let

$$\xi_\pi(i, j, k) = 1 - \#\text{ of legs of } \pi \text{ containing } (i, j, k).$$

We define the renormalized volume of π by

$$|\pi| = \sum_{(i,j,k) \in \pi} \xi_\pi(i, j, k).$$

Note that $|\pi|$ can be negative.

Definition 2. The topological vertex $\mathsf{V}_{\lambda\mu\nu}$ is defined to be

$$\mathsf{V}_{\lambda\mu\nu} = \sum_{\pi} p^{|\pi|}$$

where the sum is taken over all 3D partitions π asymptotic to (λ, μ, ν). We regard $\mathsf{V}_{\lambda\mu\nu}$ as a formal Laurent series in p. Note that $\mathsf{V}_{\lambda\mu\nu}$ is clearly cyclically symmetric in the indices, and reflection about the $i = j$ plane yields

$$\mathsf{V}_{\lambda\mu\nu} = \mathsf{V}_{\mu'\lambda'\nu'}$$

where $'$ denotes conjugate partition:

$$\lambda' = \{(i,j) : (j,i) \in \lambda\}.$$

This definition of topological vertex differs from the vertex $C(\lambda,\mu,\nu)$ of the physics literature by a normalization factor (and we use the variable p instead of q). Our $\mathsf{V}_{\lambda\mu\nu}$ is equal to $P(\lambda,\mu,\nu)$ defined by Okounkov, Reshetikhin, and Vafa [11, eqn 3.16]. They derive an explicit formula for $\mathsf{V}_{\lambda\mu\nu} = P(\lambda,\mu,\nu)$ in terms of Schur functions [11, eqns 3.20 and 3.21].

The *rows* or *parts* of λ are the integers $\lambda_j = \min\{i \mid (i,j) \notin \lambda\}$, for $j \geq 0$. We use the notation

$$|\lambda| = \sum_j \lambda_j, \quad \|\lambda\|^2 = \sum_j \lambda_j^2.$$

Let $\square$ denote the partition with a single part of size 1.

We also use the notation

$$M(p,q) = \prod_{m=1}^{\infty} (1-p^m q)^{-m}$$

and the shorthand $M(p) = M(p,1)$.

We can now state our main result.

Theorem 3. *The following identities hold as formal power series in q whose coefficients are formal Laurent series in p:*

(1)

$$\sum_\lambda q^{|\lambda|} p^{\|\lambda'\|^2} \mathsf{V}_{\lambda'\lambda\varnothing} = M(p) \prod_{d=1}^{\infty} (1-q^d)^{-1} M(p,q^d)$$

(2)

$$\sum_\lambda q^{|\lambda|} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = (1-p)^{-1} \prod_{d=1}^{\infty} \frac{(1-q^d)}{(1-pq^d)(1-p^{-1}q^d)}$$

(3)

$$\sum_\lambda q^{|\lambda|} p \frac{\mathsf{V}_{\square\square\lambda}}{\mathsf{V}_{\varnothing\varnothing\lambda}} = \prod_{m=1}^{\infty} (1-q^m)^{-1} \cdot \left\{ 1 + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k}) q^q \right\}$$

(4)

$$\sum_\lambda q^{|\lambda|} p^{\|\lambda'\|^2} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} \mathsf{V}_{\lambda'\lambda\varnothing} = (1-p)^{-1} M(p) \prod_{d=1}^{\infty} \frac{M(p,q^d)}{(1-pq^d)(1-p^{-1}q^d)}$$

The sums in the left hand sides of the above formulas run over all partitions.

We call these formulas "trace formulas" since the left hand side can be expressed as the traces of certain operators on Fock space. This will be made explicit in section 5.

We prove Formula (1) in section 3 using the orthogonality properties of skew Schur functions. Formulas (2) and (3) are proved in section 4 using a theorem of Bloch-Okounkov [2]. The most difficult identity to prove is equation (4) which we do in section 5. There we prove that the left hand side of equation (4) is given as the trace of a certain product of operators on Fock space. To compute the trace, we use a trick which involves an "infinite number" of permutations of the operators.

3. THE TOPOLOGICAL VERTEX AND SCHUR FUNCTIONS

Okounkov-Reshetikhin-Vafa derived a formula for the topological vertex in terms of skew Schur functions. Translating their formulas [11, 3.20& 3.21] into our notation, we get:

$$\mathsf{V}_{\lambda\mu\nu}(p) = M(p)p^{-\frac{1}{2}(\|\lambda\|^2+\|\mu'\|^2+\|\nu\|^2)}s_{\nu'}(p^{-\rho})\sum_{\eta} s_{\lambda'/\eta}(p^{-\nu-\rho})s_{\mu/\eta}(p^{-\nu'-\rho}). \tag{5}$$

Here, $s_{\alpha/\beta}(x_1, x_2, \dots)$ is the skew Schur function (see for example [7, § 5]) and

$$\rho = \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\right)$$

so that $p^{-\nu-\rho}$ is notation for the variable list

$$p^{-\nu-\rho} = \left(p^{-\nu_1+\frac{1}{2}}, p^{-\nu_2+\frac{3}{2}}, \dots\right).$$

We prove equation (1) as follows. Using equation (5) we see

$$\mathsf{V}_{\lambda'\lambda\varnothing} = M(p)p^{-\|\lambda'\|^2}\sum_{\eta} s_{\lambda/\eta}(p^{-\rho})^2$$

and so (using orthogonality of skew Schur functions [7, 28(a) pg 94] in the second line below) we see

$$\begin{aligned}\sum_{\lambda} q^{|\lambda|} p^{\|\lambda'\|^2} \mathsf{V}_{\lambda'\lambda\varnothing} &= M(p) \sum_{\lambda,\eta} q^{|\lambda|} (s_{\lambda/\eta}(p^{\frac{1}{2}}, p^{\frac{3}{2}}, \dots))^2 \\ &= M(p) \prod_{d=1}^{\infty} \left((1-q^d)^{-1} \prod_{j,k=1}^{\infty} (1 - q^d p^{i-\frac{1}{2}+j-\frac{1}{2}})^{-1} \right) \\ &= M(p) \prod_{d=1}^{\infty} (1-q^d)^{-1} \prod_{m=1}^{\infty} (1-q^d p^m)^{-m} \\ &= M(p) \prod_{d=1}^{\infty} (1-q^d)^{-1} M(p, q^d).\end{aligned}$$

We also use equation (5) to derive the following key formulas:

Lemma 4. *The following hold:*

$$\begin{aligned}p^{\frac{1}{2}} \frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}} \\ p \frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= 1 - \left(\sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}} \right) \left(\sum_{j=1}^{\infty} p^{\lambda_j-j+\frac{1}{2}} \right)\end{aligned}$$

Proof. Applying equation (5) to $\mathsf{V}_{\lambda\square\varnothing}/\mathsf{V}_{\lambda\varnothing\varnothing} = \mathsf{V}_{\square\varnothing\lambda}/\mathsf{V}_{\varnothing\varnothing\lambda}$ we see that

$$p^{\frac{1}{2}} \frac{\mathsf{V}_{\lambda\square\varnothing}}{V_{\lambda\varnothing\varnothing}} = s_{\square}(p^{-\lambda-\rho}) = s_{\square}(p^{-\lambda_1+\frac{1}{2}}, p^{-\lambda_2+\frac{3}{2}}, \dots) = \sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}}.$$

Similarly,

$$\begin{aligned}p \frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} = p \frac{\mathsf{V}_{\square\square\lambda}}{\mathsf{V}_{\varnothing\varnothing\lambda}} &= \sum_{\eta} s_{\square/\eta}(p^{-\lambda-\rho}) s_{\square/\eta}(p^{-\lambda'-\rho}) \\ &= 1 + s_{\square}(p^{-\lambda-\rho}) s_{\square}(p^{-\lambda'-\rho}).\end{aligned}$$

In general we have the following relation (see [11, Eqn (3.10)])[1]

$$s_{\lambda/\mu}(p^{\nu+\rho}) = (-1)^{|\lambda|-|\mu|} s_{\lambda'/\mu'}(p^{-\nu'-\rho})$$

[1]There is a typo in equation 3.10 in [11] — the exponent on the right hand side should be $-\nu' - \rho$.

so in particular $s_{\square}(p^{\nu+\rho}) = -s_{\square}(p^{-\nu'-\rho})$ and thus

$$\begin{aligned} p\frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= 1 - s_{\square}(p^{-\lambda-\rho})s_{\square}(p^{\lambda+\rho}) \\ &= 1 - \left(\sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}}\right)\left(\sum_{j=1}^{\infty} p^{\lambda_j-j+\frac{1}{2}}\right) \end{aligned}$$

which proves the lemma.

4. Applications of a theorem of Bloch-Okounkov

We summarize a result of Bloch-Okounkov [2] and use it to prove equations (2) and (3).

We define the following theta function

$$\Theta(p,q) = \eta(q)^{-3}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}p^{n+\frac{1}{2}}$$

which, by the Jacobi triple product formula is given by

$$\Theta(p,q) = (p^{\frac{1}{2}} - p^{-\frac{1}{2}})\prod_{m=1}^{\infty}\frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}.$$

We suppress the q from the notation: $\Theta(p) = \Theta(p,q)$, and we note that

$$\Theta(p) = -\Theta(p^{-1}).$$

Theorem 5 (Bloch-Okounkov [2])**.** *Define the n point correlation function by the formula*

$$F(p_1,\dots,p_n) = \prod_{m=1}^{\infty}(1-q^m)\sum_{\lambda}q^{\lambda}\prod_{k=1}^{n}\left(\sum_{i=1}^{\infty}p_k^{\lambda_i-i+\frac{1}{2}}\right).$$

Then

$$F(p) = \frac{1}{\Theta(p)}$$

and

$$F(p_1,p_2) = \frac{1}{\Theta(p_1p_2)}\left(p_1\frac{d}{dp_1}\log(\Theta(p_1)) + p_2\frac{d}{dp_2}\log(\Theta(p_2))\right).$$

In [2], formulas for the general n variable function are given, but we will only need the cases of $n=1$ and $n=2$.

Using this theorem, we will prove equations (2) and (3) of the main theorem.

4.1. **Proofs of equations** (2) **and** (3)**.**
We apply Lemma 4 and Theorem 5:

$$\begin{aligned}\sum_\lambda (1-p)q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= (p^{-\frac{1}{2}}-p^{\frac{1}{2}})\sum_\lambda q^{|\lambda|}\sum_{i=1}^\infty p^{-\lambda_i+i-\frac{1}{2}}\\ &= (p^{-\frac{1}{2}}-p^{\frac{1}{2}})\prod_{m=1}^\infty (1-q^m)^{-1}F(p^{-1})\\ &= (p^{-\frac{1}{2}}-p^{\frac{1}{2}})\prod_{m=1}^\infty (1-q^m)^{-1}\frac{1}{-\Theta(p)}\\ &= \prod_{m=1}^\infty \frac{(1-q^m)}{(1-pq^m)(1-p^{-1}q^m)}\end{aligned}$$

which proves equation (2).
Again we apply Lemma 4 and Theorem 5:

$$\begin{aligned}\sum_\lambda q^{|\lambda|}p\frac{\mathsf{V}_{\lambda\square\square}}{\mathsf{V}_{\lambda\varnothing\varnothing}} &= \sum_\lambda q^{|\lambda|}\left\{1-\left(\sum_{i=1}^\infty p^{-\lambda_i+i-\frac{1}{2}}\right)\left(\sum_{j=1}^\infty p^{\lambda_j-j+\frac{1}{2}}\right)\right\}\\ &= \prod_{m=1}^\infty (1-q^m)^{-1}\left(1-F(p,p^{-1})\right).\end{aligned}$$

From Theorem 5, we see that

$$F(p,p^{-1}) = \lim_{(p_1,p_2)\to(p,p^{-1})}\frac{1}{\Theta(p_1p_2)}\left(p_1\frac{d}{dp_1}\log(\Theta(p_1))+p_2\frac{d}{dp_2}\log(\Theta(p_2))\right).$$

To evaluate this limit, we simplify the above expression. A short computation shows that

$$p\frac{d}{dp}\log(\Theta(p)) = \frac{1}{2}\frac{p+1}{p-1}+\sum_{m=1}^\infty\sum_{k=1}^\infty\left(-p^k+p^{-k}\right)q^{mk}.$$

Thus

$$\begin{aligned}
F(p,p^{-1}) = &\lim_{\substack{(p_1,p_2)\to\\(p,p^{-1})}} \left((p_1p_2)^{\frac{1}{2}}-(p_1p_2)^{-\frac{1}{2}}\right)^{-1}\prod_{m=1}^{\infty}\frac{(1-q^m)^2}{(1-(p_1p_2)q^m)(1-(p_1p_2)^{-1}q^m)}\\
&\cdot\left\{\frac{1}{2}\cdot\frac{p_1+1}{p_1-1}+\frac{1}{2}\cdot\frac{p_2+1}{p_2-1}+\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(-p_1^k-p_2^k+p_1^{-k}+p_2^{-k}\right)q^{mk}\right\}\\
=&\lim_{\substack{(p_1,p_2)\to\\(p,p^{-1})}}\frac{-(p_1p_2)^{\frac{1}{2}}}{1-p_1p_2}\cdot\left\{\frac{p_1p_2-1}{(p_1-1)(p_2-1)}+\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(1-p_1^kp_2^k)(p_1^{-k}+p_2^{-k})q^{mk}\right\}\\
=&\lim_{\substack{(p_1,p_2)\to\\(p,p^{-1})}}(p_1p_2)^{\frac{1}{2}}\left\{\frac{1}{(1-p_1)(1-p_2)}-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1-(p_1p_2)^k}{1-p_1p_2}(p_1^{-k}+p_2^{-k})q^{mk}\right\}\\
=&\frac{1}{(1-p)(1-p^{-1})}-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}k(p^k+p^{-k})q^{mk}.
\end{aligned}$$

Therefore

$$1-F(p,p^{-1})=1+\frac{p}{(1-p)^2}+\sum_{d=1}^{\infty}\sum_{k|d}k(p^k+p^{-k})q^d$$

which finishes the proof of equation (3).

5. VERTEX OPERATORS AND THE PROOF OF EQUATION (4)

There are several sources for vertex operators and the infinite wedge formalism. For consistency, we will follow the notation and conventions of [10, Appendix A].

Let V be the vector space with basis $\{\underline{k}\}$, $k\in\mathbb{Z}+\frac{1}{2}$. We define *Fock space* $\Lambda^{\frac{\infty}{2}}V$ to be the vector space spanned by vectors

$$v_S=\underline{s_1}\wedge\underline{s_2}\wedge\dots$$

where $S=\{s_1>s_2>\dots\}\subset\mathbb{Z}+\frac{1}{2}$ is any subset such that the sets

$$S_+=S\cap\left(\mathbb{Z}+\frac{1}{2}\right)_{>0}\quad\text{and}\quad S_-=S^c\cap\left(\mathbb{Z}+\frac{1}{2}\right)_{<0}$$

are both finite. Let $(\cdot,\cdot)$ be the inner product on $\Lambda^{\frac{\infty}{2}}V$ such that the basis $\{v_S\}$ is orthonormal.

For any $k\in\mathbb{Z}+\frac{1}{2}$ let ψ_k be the operator

$$\psi_k(f)=\underline{k}\wedge f$$

and let ψ_k^* be its adjoint.

For any partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$, we define the vector

$$v_\lambda = \underline{(\lambda_1 - \frac{1}{2})} \wedge \underline{(\lambda_2 - \frac{3}{2})} \wedge \dots$$

Let $\Lambda_0^{\frac{\infty}{2}} V \subset \Lambda^{\frac{\infty}{2}} V$ be the subspace spanned by the vectors $\{v_\lambda\}$ where λ runs over all partitions. We call this *charge zero Fock space.*

The *energy operator*

$$H = \sum_{k>0} k \left(\psi_k \psi_k^* + \psi_{-k}^* \psi_{-k} \right)$$

acts on the basis v_λ by

$$H v_\lambda = |\lambda| v_\lambda$$

and so the operator q^H acts by

$$q^H v_\lambda = q^{|\lambda|} v_\lambda$$

where q is a formal parameter.

For $n \in \mathbb{Z}$, $n \neq 0$ define

$$\alpha_n = \sum_k \psi_{k-n} \psi_k^*$$

and observe that $\alpha_n^* = \alpha_{-n}$.

Following [10], we define the *vertex operators* $\Gamma_\pm(\mathbf{x})$ which are operators on $\Lambda_0^{\frac{\infty}{2}} V$ over the coefficient ring given by symmetric functions in an infinite set of variables $\mathbf{x} = (x_1, x_2, x_3, \dots)$. Let $\mathbf{s} = (s_1, s_2, \dots)$

$$s_k = \frac{1}{k} \sum_{i=1}^{\infty} x_i^k$$

be the power sum basis for the ring of symmetric functions and let [2]

$$\Gamma_\pm(\mathbf{x}) = \exp\left(\sum_{n=1}^{\infty} s_n \alpha_{\pm n} \right).$$

Observe that $\Gamma_\pm^* = \Gamma_\mp$.

The matrix coefficients of the vertex operators in the $\{v_\lambda\}$ basis are given by skew Schur functions:

$$(\Gamma_-(\mathbf{x}) v_\mu, v_\lambda) = (v_\mu, \Gamma_+(\mathbf{x}) v_\lambda) = s_{\lambda/\mu}(\mathbf{x}). \tag{6}$$

[2]In [10], the argument of $\Gamma_\pm$ is $\mathbf{s}$, and the dependence on the underlying set of variables $\mathbf{x}$ is left implicit. We prefer to make $\mathbf{x}$ the explicit argument.

Orthogonality of the skew Schur functions then gives rise to the following commutation equation:

$$\Gamma_+(\mathbf{x})\Gamma_-(\mathbf{y}) = \prod_{i,j}(1-x_jy_j)^{-1}\Gamma_-(\mathbf{y})\Gamma_+(\mathbf{x}),$$

in particular

$$\Gamma_+(up^{-\rho})\Gamma_-(vp^{-\rho}) = M(p,uv)\Gamma_-(vp^{-\rho})\Gamma_+(up^{-\rho}) \tag{7}$$

where recall that $up^{-\rho} = (up^{\frac{1}{2}}, up^{\frac{3}{2}}, up^{\frac{5}{2}}, \dots)$.

We let

$$\psi(z) = \sum_i z^i\psi_i \quad \text{and} \quad \psi^*(w) = \sum_j w^{-j}\psi_j^*.$$

The commutation relations of these operators with the vertex operators is given by

$$\begin{aligned} \Gamma_\pm(\mathbf{x})\psi(z) &= \prod_{i=1}^{\infty}(1-x_iz^{\pm 1})^{-1}\ \psi(z)\Gamma_\pm(\mathbf{x}) \\ \Gamma_\pm(\mathbf{x})\psi^*(w) &= \prod_{i=1}^{\infty}(1-x_iw^{\pm 1})\ \psi^*(w)\Gamma_\pm(\mathbf{x}). \end{aligned} \tag{8}$$

We use operators $\mathcal{E}_r$ introduced by Okounkov-Pandharipande in [9, § 2.2.4]. For $r \in \mathbb{Z}$, let[3]

$$\mathcal{E}_r(p) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} p^{-k+\frac{r}{2}}\ \psi_{k-r}\psi_k^*.$$

Our variable p is related to the variable z in [9] by $p = e^{-z}$.

From [9, Eqns 2.9 and 0.18], we see that $\mathcal{E}_0$ is the diagonal operator given by

$$\mathcal{E}_0(p)v_\lambda = \left(\sum_{i=1}^{\infty} p^{-\lambda_i+i-\frac{1}{2}}\right) v_\lambda. \tag{9}$$

We define

$$\mathcal{E}(a,p) = \sum_{r\in\mathbb{Z}} a^{-r}\mathcal{E}_r(p)$$

where a is a formal parameter. A short computation shows that

$$\mathcal{E}(a,p) = \psi(ap^{\frac{1}{2}})\psi^*(ap^{-\frac{1}{2}}).$$

From equation (8) we get

$$\Gamma_\pm(\mathbf{x})\mathcal{E}(a,p) = \prod_{i=1}^{\infty}\frac{(1-a^{\mp 1}p^{\mp\frac{1}{2}}x_i)}{(1-a^{\mp 1}p^{\pm\frac{1}{2}}x_i)}\ \mathcal{E}(a,p)\Gamma_\pm(\mathbf{x}).$$

[3] We avoid the use of normal ordering by allowing coefficients which are Laurent series in $p^{\frac{1}{2}}$.

For $\mathbf{x} = up^{-\rho} = (up^{\frac{1}{2}}, up^{\frac{3}{2}}, up^{\frac{5}{2}}, \dots)$ the above simplifies to

$$\begin{aligned}\Gamma_+(up^{-\rho})\mathcal{E}(a,p) &= (1-au)\mathcal{E}(a,p)\Gamma_+(up^{-\rho})\\ \mathcal{E}(a,p)\Gamma_-(up^{-\rho}) &= (1-a^{-1}u)\Gamma_-(up^{-\rho})\mathcal{E}(a,p).\end{aligned} \tag{10}$$

Finally, it follows from equation (6) that

$$\Gamma_\pm(\mathbf{x})q^H = q^H\Gamma_\pm(q^{\pm 1}\mathbf{x}). \tag{11}$$

We now write the left hand side of equation (4) in the main theorem as a trace of operators on charge zero Fock space.

Lemma 6.

$$\begin{aligned}\sum_\lambda q^{|\lambda|}p^{\|\lambda'\|^2}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}\mathsf{V}_{\lambda'\lambda\varnothing} &= p^{-\frac{1}{2}}\operatorname{Tr}\left(\mathcal{E}_0(p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right)\\ &= p^{-\frac{1}{2}}\operatorname{Coeff}_{a^0}\left\{\operatorname{Tr}\left(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right)\right\}.\end{aligned}$$

Proof. The following computation uses, in order, the commutation relation for Γ_+ and Γ_- (equation (7)), the definition of trace, the formula for $\mathcal{E}_0(p)$ (equation (9)), Lemma 4, equation (6), and finally Lemma 4 again:

$$\begin{aligned}&p^{-\frac{1}{2}}\operatorname{tr}\left(\mathcal{E}_0(p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right)\\ &= p^{-\frac{1}{2}}M(p)\operatorname{tr}\left(\mathcal{E}_0(p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^H\right)\\ &= p^{-\frac{1}{2}}M(p)\sum_\lambda\left(v_\lambda, \mathcal{E}_0(p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^Hv_\lambda\right)\\ &= p^{-\frac{1}{2}}\sum_\lambda q^{|\lambda|}\left(\sum_{i=1}^\infty p^{-\lambda_i+i-\frac{1}{2}}\right)\left(v_\lambda, \Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})v_\lambda\right)\\ &= M(p)\sum_\lambda q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}\left(\Gamma_+(p^{-\rho})V_\lambda, \Gamma_+(p^{-\rho})V_\lambda\right)\\ &= \sum_\lambda q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}M(p)\sum_\eta\left(s_{\lambda/\eta}(p^{-\rho})\right)^2\\ &= \sum_\lambda q^{|\lambda|}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}p^{\|\lambda'\|^2}V_{\lambda'\lambda\varnothing}.\end{aligned}$$

□

While the operator $\mathcal{E}_0(p)$ does not have good commutation relations with the vertex operators, the operator $\mathcal{E}(a,p)$ does. Hence we first replace $\mathcal{E}_0(p)$ with the more general $\mathcal{E}(a,p)$, compute the trace, and then specialize to the a^0 coefficient.

Lemma 7.

$$\operatorname{tr}\left(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H\right) = \\ \frac{1}{p^{-\frac{1}{2}}-p^{\frac{1}{2}}} M(p) \prod_{m=1}^{\infty} \frac{(1-q^m a)(1-q^{m-1}a^{-1})(1-q^m)M(p,q^m)}{(1-pq^m)(1-p^{-1}q^m)}.$$

Proof. Our strategy is the following. We use the cyclic invariance of trace along with the commutation relations for Γ_+ to move the operator Γ_+ past the other operators cyclically to the right until the operators are back to their original positions, but with new arguments. We perform this operation a countable number of times, eventually making the Γ_+ operator disappear[4]. We then employ the same strategy moving Γ_- cyclically to the left a countable number of times until it disappears and we are left with a term which we can evaluate with the Okounkov-Bloch theorem (theorem 5).

We first cyclically commute the operator Γ_+ to the right using equations (7), (10), and (11):

$$\begin{aligned} &\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) \\ &= M(p)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})\Gamma_+(p^{-\rho})q^H) \\ &= M(p)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H\Gamma_+(qp^{-\rho})) \\ &= M(p)\operatorname{tr}(\Gamma_+(qp^{-\rho})\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H) \\ &= M(p)(1-qa)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(qp^{-\rho})\Gamma_-(p^{-\rho})q^H). \end{aligned}$$

Cyclically commuting Γ_+ to the right a second time we get:

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \\ M(p)(1-qa)M(p,q)(1-q^2a)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(q^2p^{-\rho})\Gamma_-(p^{-\rho})q^H).$$

After performing N iterations of this strategy, we arrive at

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \\ \prod_{d=1}^{N} M(p,q^{d-1})(1-q^d a)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(q^N p^{-\rho})\Gamma_-(p^{-\rho})q^H).$$

It follows from Equation (6) that

$$\Gamma_\pm(q^N p^{-\rho}) \equiv \mathrm{Id} \mod q^N.$$

[4]The third author thanks Guillaume Chapuy and Sylvie Corteel for teaching him this trick at a conference lunch in 2014. Bouttier, Chapuy, and Corteel used the trick in the paper [3] in the proof of theorem 12 therein.

So the above two equations imply that the equation

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \prod_{d=1}^{\infty} M(p,q^{d-1})(1-q^d a)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H)$$

holds to all orders in q and is hence true as a formal power series in q.

We now apply the same strategy commuting Γ_- to the left:

$$\begin{aligned}\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(p^{-\rho})q^H) &= (1-a^{-1})\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(qp^{-\rho})q^H)\\ &= (1-a^{-1})(1-a^{-1}q)\operatorname{tr}(\mathcal{E}(a,p)\Gamma_-(q^2p^{-\rho})q^H)\\ &= \dots\\ &= \prod_{d=1}^{\infty}(1-a^{-1}q^{d-1})\operatorname{tr}(\mathcal{E}(a,p)q^H)\end{aligned}$$

and so we have proved

$$\operatorname{tr}(\mathcal{E}(a,p)\Gamma_+(p^{-\rho})\Gamma_-(p^{-\rho})q^H) = \prod_{d=1}^{\infty} M(p,q^{d-1})(1-q^d a)(1-a^{-1}q^{d-1})\operatorname{tr}(\mathcal{E}(a,p)q^H).$$

From the definition of $\mathcal{E}_r(p)$ we see that its matrix entries are all off-diagonal if $r \neq 0$. Therefore

$$\begin{aligned}\operatorname{tr}(\mathcal{E}(a,p)q^H) &= \operatorname{tr}(\mathcal{E}_0(p)q^H)\\ &= \sum_{\lambda} q^{|\lambda|}\sum_{i=1}^{\infty} p^{-\lambda+i-\frac{1}{2}}\\ &= (p^{-\frac{1}{2}}-p^{\frac{1}{2}})^{-1}\prod_{m=1}^{\infty}\frac{(1-q^m)}{(1-pq^m)(1-p^{-1}q^m)}\end{aligned}$$

where the last equality follows from the computation in the proof of equation (2) in § 4.1. Combining this with the previous computations finishes the proof of the lemma. □

Combining Lemmas 6 and 7, we get

$$\begin{aligned}\sum_{\lambda} q^{|\lambda|}p^{\|\lambda'\|^2}\frac{\mathsf{V}_{\lambda\square\varnothing}}{\mathsf{V}_{\lambda\varnothing\varnothing}}\mathsf{V}_{\lambda'\lambda\varnothing} &= \frac{1}{1-p}M(p)\prod_{m=1}^{\infty}\frac{M(p,q^m)}{(1-pq^m)(1-p^{-1}q^m)}\\ &\quad\cdot \operatorname{Coeff}_{a^0}\left\{\prod_{m=1}^{\infty}(1-q^m a)(1-q^m a^{-1})(1-q^m)\right\}.\end{aligned}$$

By the Jacobi triple product identity, we have

$$\prod_{m=1}^{\infty}(1-q^m a)(1-q^m a^{-1})(1-q^m) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-a)^n$$

whose a^0 coefficient is 1. Plugging into the previous equation we finish the proof of equation (4).

6. Geometry and Applications

$\mathsf{V}_{\lambda\mu\nu}(p)$ as generating function for Euler characteristics of Hilbert schemes on $\mathbb{C}^3$.

short description of local contributions to DT invariants in threefolds with elliptic fibrations.

(1) multiple of smooth fiber
(2) multiple of nodal fiber
(3) multiple of smooth fiber attached to section
(4) multiple of nodal fiber attached to section
(5) multiple of smooth fiber attached to node

connection of formulas with Jacobi forms.

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