A TOPOLOGICAL VERTEX IDENTITY AND THE KATZ-KLEMM-VAFA FORMULA

JIM BRYAN, MARTIJN KOOL AND BENJAMIN YOUNG

ABSTRACT. Motivated by a new calculation of the Katz-Klemm-Vafa formula (primitive case), the first two authors conjectured a product formula for a certain generating function involving the topological vertex. In this paper we prove this formula using the infinite wedge formalism. The method is by writing the generating function as a trace and then combining standard commutation relations of vertex operators with cyclicity of trace. This trick was picked up by the third author from a paper of J. Bouttier, G. Chapuy and S. Corteel. The techniques of this paper are useful for calculating generating functions of Donaldson-Thomas or stable pair invariants in numerous geometric settings.

1. INTRODUCTION

Acknowledgements. Paul Johnson, PIMS, ...

2. REVIEW OF THE INFINITE WEDGE FORMALISM

We start with a brief review of the infinite wedge formalism, various operators and their commutation relations of vertex operators appearing in the work of A. Okounkov, R. Pandharipande [**?**] and Okounov, N. Reshetikhin [**?**]. See also [**?**] and [**?**]. This section is transcribed from these references. We include it in order to establish our sign conventions and for the readers convenience.

Let V be the complex vector space spanned by $\underline{k}$, where $k \in \mathbb{Z} + \frac{1}{2}$. By definition, the infinite wedge space $\Lambda^{\frac{\infty}{2}} V$ is the complex vector space spanned by vectors

$$v_S := \underline{s_1} \wedge \underline{s_2} \wedge \cdots$$

where $S = \{s_1 > s_2 > \cdots\} \subset \mathbb{Z} + \frac{1}{2}$ for which both

$$S_+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right),\ S_- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \setminus S$$

are finite. The subspace spanned v_S for which $|S_+| = |S_-|$ is known as the zero charge space and denoted by $\Lambda_0^{\frac{\infty}{2}} V$. The collection of subset $S = \{s_1 > s_2 > \cdots\} \subset \mathbb{Z} + \frac{1}{2}$ for which $|S_+| = |S_-|$ is in natural bijection with the collection of plane partitions $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots\} \subset \mathbb{Z}_{\geq 0}$ via the mapping [**?**, 2.1.3]

$$\lambda \mapsto \mathfrak{S}(\lambda) = \left\{\lambda_i - i + \frac{1}{2}\right\}_i \subset \mathbb{Z} + \frac{1}{2}.$$

These are known as modified Frobenius coordinates. We denote partitions by $\lambda, \mu, \nu, \eta, \ldots$ and define

$$|\lambda\rangle := v_{\mathfrak{S}(\lambda)}.$$

Date: December 29, 2014.

In particular, the vacuum vector is given by

$$|\varnothing\rangle := \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots .$$

Denote by $\langle\cdot|\cdot\rangle$ the complex inner product for which

$$\langle\lambda|\mu\rangle = \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ is the Kronecker delta.

For each $k \in \mathbb{Z} + \frac{1}{2}$ one defines the operator (on $\Lambda^{\frac{\infty}{2}} V$)

$$\psi_k := \underline{k} \wedge \cdot$$

and its adjoint is denoted by ψ_k^*. These operators satisfy the anti-commutation relations

$$\psi_k\psi_l + \psi_l\psi_k = \psi_k^*\psi_l^* + \psi_l^*\psi_k^* = 0,$$
$$\psi_k\psi_l^* + \psi_l^*\psi_k = \delta_{kl}.$$

These operators can be combined to

$$\psi(a) := \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k a^k,\ \psi^*(a) := \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k^* a^{-k}.$$

Next consider the operators

$$\alpha_n := \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k+n}\psi_k^*,\ n \in \mathbb{Z}.$$

These satisfy the Heisenberg commutation relations $[\alpha_n, \alpha_m] = -n\delta_{n,-m}$ and

$$[\alpha_n, \psi(a)] = a^{-n}\psi(a),\ [\alpha_n, \psi^*(a)] = -a^{-n}\psi^*(a).$$

We are interested in the vertex operators

$$\Gamma_\pm(q) := \exp\Big(\sum_{n\geq 1} \frac{q^n}{n!}\alpha_\pm\Big).$$

These acts on $\Lambda_0^{\frac{\infty}{2}} V$ as follows

$$\Gamma_-(q)|\mu\rangle = \sum_{\lambda\succ\mu} q^{|\lambda|-|\mu|}|\lambda\rangle,$$
$$\Gamma_+(q)|\lambda\rangle = \sum_{\lambda\succ\mu} q^{|\lambda|-|\mu|}|\mu\rangle.$$

Here $|\lambda| := \sum_i \lambda_i$ is the size of the partition and $\lambda \succ \mu$ means λ interlaces μ, i.e.

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots .$$

Equivalently, the skew diagram $\lambda \setminus \mu$ is a disjoint union of horizontal strips (see [**?**] for details). Therefore we think of $\Gamma_-(q)|\mu\rangle$ as adding horizontal strips from μ and $\Gamma_+(q)|\lambda\rangle$ as removing horizontal strips from λ.

We will use the following commutation relations from Okounkov and Reshetikhin [**?**]

$$\Gamma_+(a)\psi(b) = (1 - ab^{-1})^{-1}\psi(b)\Gamma_+(a),$$
$$\Gamma_-(a)\psi(b) = (1 - ab)^{-1}\psi(b)\Gamma_-(a),$$
$$\Gamma_+(a)\psi^*(b) = (1 - ab^{-1})\psi^*(b)\Gamma_+(a),$$
$$\Gamma_-(a)\psi^*(b) = (1 - ab)\psi^*(b)\Gamma_-(a)$$

$$\Gamma_+(a)\Gamma_-(b) = (1-ab)\Gamma_-(b)\Gamma_+(a)$$

3. Operators and commutation relations

Define our bold face operators with their commutation relations derived from previous section. Reference to Bloch-Okounkov. Possibly derivation MacMahon and 2 leg DT=PT as warm-up examples?

4. Calculation

The disconnected series (known identity) first using the trace trick. Our new formula second.

Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2