Sunday Aug. 29. 2021

Step 1: Notation for $\mathbb{P}^1$

$\mathbb{P}^1$ x_0 x_1 $x = x_0/x_1$ $y = x_1/x_0$ $p_\infty = V(x_1)$ $p_0 = V(x_0)$

$U_0 = \operatorname{Spec} \mathbb{C}[x]$ $U_1 = \operatorname{Spec} \mathbb{C}[y]$ $U_{01} = \operatorname{Spec} \mathbb{C}[x, \frac{1}{x}]$

$\mathcal{U}$ is a fixed vector space of dimension n.

d will later be the rank of our bundle.

Step II: $GL(n)$ / Strømme

$n = \dim \mathcal{U} = \operatorname{rk} \mathcal{E}$

non-negative.
split-framed by $\mathcal{U} \xrightarrow{\theta} \mathcal{E}$
θ iso @ P_∞

$d = \dim A = \deg \mathcal{E}$

$A = H^0(\mathcal{E}(-1)) \qquad H^0(\mathcal{E}) \xrightarrow{\sim} A \oplus \mathcal{U}$

"bundle-injective"

$$0 \to A(-1) \xrightarrow{\begin{pmatrix} x_0 I_A + x_1 \alpha \\ x_1 j \end{pmatrix}} \begin{matrix} A \\ \oplus \\ \mathcal{U} \end{matrix} \xrightarrow{(\beta \ \ \theta)} \mathcal{E} \to 0$$

Ask Jim: Can we change this sign?

$$\begin{tikzcd}
0 \arrow[r] & A(-1) \arrow[r, "x_0 I_A + x_1 \alpha"] \arrow[d] & A \arrow[r] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{U} \arrow[r, "\theta"] & \mathcal{E} \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

$\beta: A \to \mathcal{E}$ given by $A \to \begin{matrix} A \\ \oplus \\ \mathcal{U} \end{matrix}$ on global sections

$\theta: \mathcal{U} \to \mathcal{E}$ given by $\mathcal{U} \to \begin{matrix} A \\ \oplus \\ \mathcal{U} \end{matrix}$ on global sections.

purple: ingredients present on both sides

blue: geometry data

green: linear algebra data

orange: output of both sides

blue tells us green:

$A = H^0(\mathcal{E}(-1))$

α and j from Strømme.

green tells us blue:

$\mathcal{E} = \operatorname{coker}(A(-1) \to A \oplus \mathcal{U})$.

III: Isotropic Grassmannian ② and ②'

geometry: GL info plus $\gamma : \mathcal{U}^\vee \to \mathcal{E}(-p_\infty)$ such that

equivalently, $\mathcal{U}^\vee \to A$.

don't you want $\gamma : \mathcal{U}^\vee \to \mathcal{E}$? Maybe with the condition that it factors $\mathcal{U}^\vee \to \mathcal{E}(-p_\infty) \to \mathcal{E}$

yes, that's what I meant.

$$J = \begin{pmatrix} 0 & \pm I \\ I & 0 \end{pmatrix}$$

$$0 \to \mathcal{E}^\vee \xrightarrow{\begin{pmatrix} \pm\gamma^\vee \\ \theta^\vee \end{pmatrix}} \begin{matrix} \mathcal{U} \\ \oplus \\ \mathcal{U}^\vee \end{matrix} \xrightarrow{(\theta \ \ \gamma)} \mathcal{E} \to 0$$

γ is 0 @ p_∞.

is an exact sequence

Equivalently:

$\begin{pmatrix} \pm\gamma^\vee \\ \theta^\vee \end{pmatrix}$ is bundle-injective and $\gamma\theta^\vee = \mp(\gamma\theta^\vee)^\vee : \mathcal{E}^\vee \to \mathcal{E}$

Mystery: This forces a $\mp$ structure on A. And given such a structure, we should get a highly contractible choice of geometric guys.

Here is a different way of getting a perfect pairing on A, using the $\mathbb{C}[x]$-module Q.

$$0 \to U \xrightarrow{\theta} \mathcal{E} \to Q \to 0 \quad (\text{def. of } Q)$$

Now consider:
$$\begin{tikzcd} A(-1) \arrow[r, "x_0 I - x_1 \alpha"] \arrow[d, "x_1 j"'] & A \arrow[d, "\beta"] \\ U \arrow[r, "\theta"] & \mathcal{E} \end{tikzcd}$$
which is exact. $\therefore$

$$\begin{tikzcd} 0 \arrow[r] & A(-1) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \operatorname{coker}(x_0 I - x_1 \alpha) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & U \arrow[r] & \mathcal{E} \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

Also dualizing $0 \to \mathcal{U} \xrightarrow{\theta} \mathcal{E} \to \mathcal{Q} \to 0$

we get $0 \to \mathcal{E}^\vee \xrightarrow{\theta^\vee} \mathcal{U}^\vee \to \mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}) \to 0$

(now using the additional structure):

$$\begin{tikzcd}
\mathcal{E}^\vee \arrow[r, "\theta^\vee"] \arrow[d, "\pm\gamma^\vee"'] & \mathcal{U}^\vee \arrow[d, "\gamma"] \\
\mathcal{U} \arrow[r, "\theta"] & \mathcal{E}
\end{tikzcd}$$

is exact so

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^\vee \arrow[r, "\theta^\vee"] \arrow[d, "\pm\gamma^\vee"'] & \mathcal{U}^\vee \arrow[r] \arrow[d, "\gamma"] & \mathcal{E}xt^1(\mathcal{Q},\mathcal{O}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{U} \arrow[r, "\theta"] & \mathcal{E} \arrow[r] & \mathcal{Q} \arrow[r] & 0
\end{tikzcd}$$

Hence:

$$\begin{tikzcd}
0 \arrow[r] & A^\vee \arrow[r] & A^\vee(1) \arrow[r] & \mathcal{E}xt^1(\mathcal{Q},\mathcal{O}) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}^\vee \arrow[r, "\theta^\vee"] \arrow[u, "\beta^\vee"] \arrow[d, "\pm\gamma^\vee"'] & \mathcal{U}^\vee \arrow[r] \arrow[u, "x_1 j^\vee"'] \arrow[d, "\gamma"] & \mathcal{E}xt^1(\mathcal{Q},\mathcal{O}) \arrow[r] \arrow[u] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{U} \arrow[r, "\theta"] & \mathcal{E} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \\
0 \arrow[r] & A(-1) \arrow[r] \arrow[u, "x_1 j"] & A \arrow[r, "x_0 I - x_1 \alpha"'] \arrow[u, "\beta"'] & \mathcal{Q} \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

$\Rightarrow 0 \to A^\vee(-1) \to A^\vee \to \mathcal{Q}(-1) \to 0$

Taking H^0: $A^\vee \xrightarrow{\sim} H^0(\mathcal{Q}(-1))$

$$\Big\downarrow \sim \quad \times x_1$$

Taking H^0: $A \xrightarrow{\sim} H^0(\mathcal{Q})$

Aug. 30, 2021. I think this might work. Start with preliminary steps I and II from a few pages earlier.

ISOTROPIC GRASSMANNIAN

Fix n, d. $U = \mathbb{C}^n$

$\mathbb{C}$ not important; should work over $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{2}]$.

I think it works over $\mathbb{Z}$ but 8-fold periodicity collapses to 4-fold.

It's Sp/P where P is a parabolic homotopic to GL

Q: IG is not Sp/GL (dimensions are wrong). Why is this an ok model? Does it differ just by affines?

yes.

We wish to describe the moduli space parametrizing:

self-dual

$$0 \to \mathcal{E}^\vee \xrightarrow{\begin{bmatrix} \pm\gamma^\vee \\ \theta^\vee \end{bmatrix}} \begin{matrix} U \\ \oplus \\ U^\vee \end{matrix} \xrightarrow{[\theta, \gamma]} \mathcal{E} \to 0 \quad \text{on } \mathbb{P}^1$$

$\mathcal{E}$: rank n, degree d

$$J = \begin{pmatrix} 0 & \pm I \\ I & 0 \end{pmatrix}$$

Note: $\mathcal{E}$ is non-negative. $\pm\theta\gamma^\vee + \gamma\theta^\vee = 0.$ $\gamma|_{P_\infty} = 0 \Rightarrow \theta|_{P_\infty}$ is iso.

Reminder of additional structures determined by this.

$A := H^0(\mathcal{E}(-1))$

$$0 \to U \xrightarrow{\theta} \mathcal{E} \to \mathcal{a} \to 0$$

Now consider the <u>commutative</u> diagram:

$$\begin{tikzcd}
A^\vee \arrow[r, "\overbrace{x_0 I - x_1\alpha^\vee}^{\sigma^\vee}"] & A^\vee(1) \\
\mathcal{E}^\vee \arrow[u, "-\beta^\vee"] \arrow[r, "\theta^\vee"] \arrow[d, "\mp\gamma^\vee"'] & \mathcal{U}^\vee \arrow[u, "x_1 j^\vee"'] \arrow[d, "\gamma"] \\
\mathcal{U} \arrow[r, "\theta"] & \mathcal{E} \\
A(-1) \arrow[u, "x_1 j"] \arrow[r, "\underbrace{x_0 I - x_1\alpha}_{\sigma}"'] & A \arrow[u, "-\beta"']
\end{tikzcd}$$

Notice that I changed the signs on the three green arrows in order to make the squares commute.

Notice also that the diagram is self-dual, <u>except</u> for the red $\mp$.

Finally, notice that the horizontal maps are all sheaf-injective, and hence the cokernels are hence isomorphic

$$\begin{tikzcd}
0 \arrow[r] & A^\vee \arrow[r, "\sigma^\vee"] & A^\vee(1) \arrow[r] & \operatorname{coker}(\sigma^\vee) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}^\vee \arrow[u, "-\beta^\vee"] \arrow[r, "\theta^\vee"] \arrow[d, "\mp\gamma^\vee"'] & \mathcal{U}^\vee \arrow[u, "x_1 j^\vee"'] \arrow[d, "\gamma"] \arrow[r] & \operatorname{coker}\theta^\vee \arrow[u, "\sim"'] \arrow[d, "\sim"] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{U} \arrow[r, "\theta"] & \mathcal{E} \arrow[r] & \boxed{\operatorname{coker}\theta} \arrow[r] & 0 \\
0 \arrow[r] & A(-1) \arrow[u, "x_1 j"] \arrow[r, "\sigma"] & A \arrow[u, "-\beta"'] \arrow[r] & \operatorname{coker}(\sigma) \arrow[u, "\sim"'] \arrow[r] & 0
\end{tikzcd}$$

Now taking H^0 of the bottom row gives an isomorphism $A \xrightarrow{\sim} H^0(\operatorname{coker}(\sigma))$

Applying $H^0(\cdot \otimes \mathcal{O}(-1))$ to the top row gives $A^\vee \xrightarrow{\sim} H^0(\operatorname{coker}(\sigma^\vee))$.

↳ (a bit sloppily stated)

We thus have an <u>isomorphism</u> $\phi : A \to A^\vee$. This induces a map

$$\begin{tikzcd}
A^\vee \arrow[rr, "x_0 I - x_1 \alpha^\vee"] & & A^\vee(1) \\
\check{\mathcal{E}} \arrow[u, "-\beta^\vee"'] \arrow[d, "{\color{red}\mp}\gamma^\vee"'] \arrow[rr, "\theta^\vee"] & & \mathcal{U}^\vee \arrow[u, "x_1 j^\vee"'] \arrow[d, "\gamma"] \\
\mathcal{U} \arrow[rr, "\theta"] & & \mathcal{E} \\
A(-1) \arrow[u, "x_1 j"] \arrow[rr, "x_0 I - x_1 \alpha"] \arrow[uuu, bend left=60, "x_1 \phi"] & & A \arrow[u, "-\beta"'] \arrow[uuu, bend right=60, "x_1 \phi"']
\end{tikzcd}$$

$\gamma^\vee$ $\quad$ $\mp\gamma$

$\pm \theta \gamma^\vee + \gamma \theta^\vee = 0.$

$$\begin{aligned}
A &\xrightarrow{\sim} H^0(\operatorname{coker}(\sigma)) \\
&\cong H^0(\operatorname{coker} \theta) \\
&\cong H^0(\operatorname{coker} \theta^\vee) \\
&\cong H^0(\operatorname{coker}(\sigma^\vee)) \\
&\cong_{\cdot x_1^{-1}} H^0(\operatorname{coker}(\sigma^\vee)(-1)) \\
&\cong A^\vee
\end{aligned}$$

making the entire diagram (cube) commute. But we could have done the same thing with the <u>dual</u> diagram, and we would get the same thing, except we need to change the middle vertical arrows by a sign: $\mp$.

Conclusion: $\boxed{\phi^\vee = \mp \phi}$.

Wow. If this is right that's a cool way to do it! – JB

Now we don't have to do the exercise! :)

That's our induced pairing on A!

It remains to reverse this. We need to construct:

$$\begin{tikzcd}
& \mathcal{E}^\vee \arrow[rr, hook, "\theta^\vee"] \arrow[dl, "\mp\gamma^\vee"'] \arrow[dd, "-\beta^\vee"] & & \mathcal{U}^\vee \arrow[dl, "\gamma"'] \arrow[dd, "x, j^\vee"] \\
\mathcal{U} \arrow[rr, hook, "\theta"] & & \mathcal{E} & \\
& A^\vee \arrow[rr, hook, "\sigma^\vee"] & & A^\vee(1) \\
A(-1) \arrow[uu, "x, j"] \arrow[ur, "x, \phi"] \arrow[rr, hook, "\sigma"] & & A \arrow[uu, "-\beta"'] \arrow[ur, "x, \phi"'] &
\end{tikzcd}$$

starting from $A, \mathcal{U}, \quad \phi: A \to A^\vee$ with $\phi^\vee = \mp\phi$.

and then making choices that are an open subset of a linear space.

It remains to reverse this. We need to construct:

$$\begin{tikzcd}
 & \mathcal{E}^\vee \arrow[rr, "\theta^\vee"] \arrow[dl, "\mp\gamma^\vee"'] \arrow[dd, "-\beta^\vee"' near start] & & \mathcal{U}^\vee \arrow[dl, "\gamma"'] \arrow[dd, "x_1 j^\vee"] \\
\mathcal{U} \arrow[rr, "\theta" near end, crossing over] & & \mathcal{E} & \\
 & A^\vee \arrow[rr, "\sigma^\vee" near start] & & A^\vee(1) \\
A(-1) \arrow[uu, "x_1 j"] \arrow[ur, "x_1 \phi"] \arrow[rr, "\sigma"] & & A \arrow[uu, "-\beta"' near end, crossing over] \arrow[ur, "x_1 \phi"'] &
\end{tikzcd}$$

Can we see why d must be even for the case where $\phi^\vee = -\phi$? Other than the existence of ϕ itself?

In the end, I couldn't see this directly — only in these two ways: (i) the argument here, and (ii) the fact that $\mathbb{Z} \cong A_1(IG) \longrightarrow A_1(G(n,2n)) \xrightarrow{\sim} A_1(\mathbb{P}^{\cdot}) = \mathbb{Z}$

$IG \hookrightarrow G(n,2n) \overset{\text{plucke.}}{\hookrightarrow} \mathbb{P}^{\cdot}$

is index 2. In more traditional notation: replace A_1 by H_2, except I would have to think about how to prove it.

starting from $A, \mathcal{U}, \phi : A \to A^\vee$ with $\phi^\vee = \mp \phi$.

and then making choices that are an open subset of a linear space.

But we just pick j and α as before, thereby determining σ (just defined as $x_0 I - x_1 \alpha$), β, θ. Finally, $\gamma = j^\vee \circ \phi^{-1} \circ (-\beta)$.

↳ it is convenient to see this by "unwrapping" the cube in a different way than we did two pages earlier.

$$\begin{tikzcd}
0 \arrow[r] & A^\vee \arrow[r, "\sigma^\vee"] & A^\vee(1) \arrow[r] & \operatorname{coker}(\sigma^\vee) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}^\vee \arrow[r, "\theta^\vee"] \arrow[u, "-\beta^\vee"] \arrow[d, "\mp\gamma^\vee"'] & \mathcal{U}^\vee \arrow[r] \arrow[u, "x, j^\vee"'] \arrow[d, "\gamma"] & \operatorname{coker}\theta^\vee \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathcal{U} \arrow[r, "\theta"] & \mathcal{E} \arrow[r] & \boxed{\operatorname{coker}\theta} \arrow[r] & 0 \\
0 \arrow[r] & A(-1) \arrow[r, "\sigma"] \arrow[u, "x, j"] & A \arrow[r] \arrow[u, "-\beta"'] & \operatorname{coker}(\sigma) \arrow[r] \arrow[u, "\sim"] & 0
\end{tikzcd}$$

call this $Q^\vee$

Q

ϕ_{der} $\qquad$ $\phi_{der}^\vee = \mp\phi_{der}[1]$

$$\begin{tikzcd}
D(\mathbb{P}^1) \arrow[r, "R\mathcal{H}om(-,\mathcal{O})\otimes K_{\mathbb{P}^1}"] \arrow[d, "R\Gamma(-)"] & D(\mathbb{P}^1) \arrow[d, "R\Gamma(-)"] \\
D(\mathrm{Vect}_{\mathbb{C}}) \arrow[r, "R\mathrm{Hom}(-,\mathbb{C})[2]"'] & D(\mathrm{Vect}_{\mathbb{C}})
\end{tikzcd}$$

ϕ $\qquad$ $\phi^\vee$

Der. Cat. discussion

Let $Q_{der} = [\,A[-1] \xrightarrow{\sigma} A\,] \in D^b_{Coh}(\mathbb{P}^1)$

Proof:

$\phi_{der} : \quad Q_{der} \longrightarrow (Q_{der})^\vee$

with $\quad \phi_{der}^\vee = \mp\phi_{der}$

Want: $\phi : A \longrightarrow A^\vee$ with $\phi^\vee = \mp$

From which we recover ϕ_{der}

ladder: $\operatorname{Ext}^0(\mathcal{O}, Q) \xrightarrow{\sim} \operatorname{Ext}^1(Q, \mathcal{O})$

$\operatorname{Ext}^0(\mathcal{O}, Q) \xrightarrow{R\Gamma(\phi)} \operatorname{Ext}^1(Q, \mathcal{O})$

$A \longrightarrow A^\vee$

If: $R\Gamma$ and $R\mathcal{H}om(\cdot, \mathcal{O})$ commute.

Then: $\phi_{der} : Q_{der} \to Q_{der}^\vee \qquad \phi_{der}^\vee : Q_{de}^{\vee\vee} \to Q_{de}^\vee$

$\phi \simeq \Gamma(\phi_{der}) : A \to A^\vee \qquad \Gamma(\phi_{der}^\vee)\ \ A \to A^\vee$

$\Gamma(\phi_{der})^\vee \quad A \to A^\vee$

Figure out soon:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & A \arrow[d, "\phi^\vee"] \\ A^\vee \arrow[r, "\alpha^\vee"'] & A^\vee \end{tikzcd}$$

$\alpha^\vee \phi = \phi^\vee \alpha$

$\phi : Q \to Q^\vee$

Prop.

Given $A = \mathbb{C}^d$, $\alpha : A \to A$.

Then we have $[A(-1) \xrightarrow{\alpha} A] = Q_{der}$

Then isomorphisms $\phi_{der} : Q_{der} \to Q_{der}^\vee$

are determined by $\operatorname{coker}(A(-1) \to A) \to \operatorname{coker}(A^\vee \to A^\vee(1))$,

which are determined by $\Gamma\phi_{der} : A \to A^\vee$.

Pf: easy.