$\Omega^2(BSP)$ **NOTES**

RAVI VAKIL FOR JIM BRYAN

CONTENTS

These are Ravi's notes, for Jim.

1. INITIAL SETTING OF NOTATION

$\boxed{x} = \boxed{x_0}/\boxed{x_1}$. $\boxed{p_\infty} = \infty = V(x_0)$. (So caution: I think it is not $V(x_1) = [1, 0]$.)

$\boxed{z} = y = t = x_1/x_0$ is the coordinate near ∞.

1.1. *The space* V.

We start with a vector space $\boxed{U = V}$ with an alternating form. (I expect everything I type here to work fine for O in place of Sp with change of signs, but I stick to the symplectic case for concreteness and to avoid confusion.) Choosing a splitting $V = L \oplus L^*$ is called a polarization or Lagrangian splitting. Jim says it is a "Weinstein normal form".

More generally, we can work over an arbitrary base, and all statements will behave well with respect to base change. So most generally, we work over $\operatorname{Spec} \mathbb{Z}$, and V is a free sheaf on $\operatorname{Spec} \mathbb{Z}$ of rank $\boxed{\boxed{2n}}$, etc. Caution: $\dim U = n$ in earlier discussions. I will continue to use vector space language for simplicity.

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Note that $\boxed{\boxed{\dim LG(V) = n(n+1)/2}}$. (In the orthogonal case, it is $\boxed{\boxed{\dim OG(V) = n(n-1)/2}}$.)

2. Background on the symplectic affine Grassmannian

2.1. *The loop space of* V.

The loop space of V is

$$\boxed{H} = \boxed{V((z))} := V[[z]] \oplus z^{-1}V[z^{-1}].$$

We define $\boxed{H^+} = V[[z]]$ and $\boxed{H^-} = z^{-1}V[z^{-1}]$. H has a residue pairing:

$$\boxed{\langle f(z), g(z) \rangle} := \mathrm{Res}_{z=0}\, \omega(f(-z), g(z))\ dz.$$

This residue pairing is skew-symmetric and nondegenerate. H is thought of as some sort of infinite-dimensional symplectic Hilbert space.

Question: What's the reason for the sign in pairing? I think in our pairing (see later on), there is no sign.

2.2. *The Lagrangian Sato Grassmannian.*

The Lagrangian Sato Grassmannian parametrizes "Lagrangians" of the loop space that are close to H^+. It is contained in the index zero component of the (usual — not symplectic) Sato Grassmannian (not defined here). The Lagrangian Sato Grassmannian is often denoted LGr_∞ (or $LGr(H)$). I will use temporary notation $\boxed{\mathrm{LgStGr}}$ to remind us of its meaning. (This is a macro which can be easily changed.)

The Nth truncation is denoted $\boxed{\mathrm{LgStGr}^{(N)}} = \mathrm{LgStGr}^{(N)}(H)$, and is isomorphic to

$$LG(N \dim V, 2N \dim V).$$

We can show that LgStGr_∞ is the limit of $LGr(nN, 2nN)$ in our naive homotopy category. Thus its homotopy type is $LGr(\infty, 2\infty) = U/O = \Omega(Sp)$.

Remark: This will likely be relevant for us, because we will be "increasing V" which will be changing the symplectic affine Grassmannian, but the ambient Lagrangian Sato Grassmannian (even for finite V) will be what we want.

2.3. *The symplectic affine Grassmannian.*

The symplectic affine Grassmannian is the z-stable locus inside the Lagrangian Sato Grassmannian. The symplectic affine Grassmannian is unfortunately often denoted Gr or

$Gr_{Sp_{2n}}$ or Gr_{Sp}. I will use temporary notation $\boxed{\text{SpAfGr}}$ to remind us. This is a temporary macro which can be easily changed.

Fact: $\text{SpAfGr}_{Sp(V)}$ is homotopic to $\Omega(Sp(V))$.

Vague argument (that I've not thought through): $LSp(V)$ parametrizes maps from loops to $Sp(V)$ — smooth maps from S^1. $\boxed{L^+} = L^+(V)$ is the subset where the map extends holomorphically to the disk. Then show that $L/L^+ \cong LGr$, by describing a transitive action of L, and identifying the stabilizer as L^+. But L^+ is contractible. (Presumably I should look at [PS] for more. It might be in a later paper of Nadler, perhaps [Na], see one of the references of [Z].) (There is an argument of this sort in the algebraic category, involving algebraic loops; and also in the continuous category.)

2.4. *Truncations of the symplectic affine Grassmannian.* The Nth truncation of the symplectic affine Grassmannian is called $\boxed{\text{SpAfGr}_{Sp}^{(N)}} = Gr^{(N)}$. (Caution: I am worried that the terminology Gr gets used both for the affine and the Sato case.)

Here are some interesting facts from [Z]. I'm not sure, but I think

$$\dim \text{SpAfGr}^{(N)} = \boxed{\boxed{N(2n-1)}}. \tag{1}$$

(The orthogonal case would be $\boxed{\boxed{N(2n-2)}}$.) This perhaps could be unwound from [Z, S 2.3]. (Perhaps investigate S. Kumar, Kac–Moody Groups, their Flag Varieties and Representation Theory, around §10.2.) Also from [Z, Thm. 2.1.21]: singularities are normal, Cohen-Macaulay, Gorenstein, and rational singularities. He refers to Faltings [Z, ref. 24], Pappas-Rapoport [Z, ref. 60], Beilinson-Drinfeld [Z, ref. 11], and Zhu [Z, ref. 80].

2.5. *Singularities of the truncation.* Apparently $\text{SpAfGr}^{(N)}$ is singular in codimension $\dim V$, with possible reference [MOV].

3. Introduction to moduli of vector bundles with symplectic structure: A, j, α, $X(d)$

Now let $\boxed{\Omega^2_{alg}(BSp(2n))}$ be the moduli space of vector bundles on $\mathbb{P}^1$, framed at p_∞ by V, with its form.

Let $\boxed{\Omega^2_{alg}(BSp(2n))^{[N]}}$ parametrize those bundles $\mathcal{F}$ such that $\mathcal{F}(N)$ is globally generated. For example, $\Omega^2_{alg}(BSp(2n))^{[N]}$ is empty if $N < 0$.

3.1. *Basic facts about this space.* The space $\Omega^2_{alg}(BSp(2n))$ is an Artin stack.

The space $\Omega^2_{alg}(BSp(2n))$ is an $Sp(2n)$ bundle over the "unframed" moduli space, which is also thus an Artin stack. This latter space is smooth, because the automorphisms/deformation/obstructions of a principle bundle E is given by the cohomology of the adjoint bundle $ad(E)$, so automorphisms are $H^0(ad(E))$, deformations are $H^1(ad(E))$, and obstructions are $H^2(ad(E))$, which in this case are zero. (The adjoint bundle is the "twisted Lie algebra bundle", as I think Jim was telling me.)

The space $\Omega^2_{alg}(BSp(2n))$ is the union of an increasing sequence of open substacks $\Omega^2_{alg}(BSp(2n))^{[N]}$.

Each $\Omega^2_{alg}(BSp(2n))^{[N]}$ is quasicompact and finite type.

$\Omega^2_{alg}(BSp(2n))^{[0]}$ is a reduced point.

The space of bundles trivialized in a formalized neighborhood of p_∞ is apparently, as an ind-scheme, the affine symplectic Grassmannian SpAfGrSp$(2n)$.

3.2. *Sketch of why these things are true.* These are all well-known facts, but we will also end reproving them, so we can have confidence in these statements. Here is the architecture of our argument.

First: $\Omega^2_{alg}(BSp(2n))$ and $\Omega^2_{alg}(BSp(2n))^{[N]}$ are stacks in the usual (smooth etc.) topology (not yet obviously algebraic stacks).

(ii) We construct a finite type affine scheme (explicitly, with generators and relations), that will parametrize the same things as $\Omega^2_{alg}(BSp(2n))^{[N]}$, with in addition a Zariski-splitting. This is an affine bundle over $\Omega^2_{alg}(BSp(2n))^{[N]}$. Thus $\Omega^2_{alg}(BSp(2n))^{[N]}$ is an algebraic stack (i.e., an Artin stack).

We have open embeddings of these truncations, and their union is the entire space, so $\Omega^2_{alg}(BSp(2n))$ is an algebraic stack.

$\Omega^2_{alg}(BSp(V))$ parametrizes the following.

- $\boxed{\mathcal{F}}$ is a rank $2n$ vector bundle on $\mathbb{P}^1$. (Caution: earlier the bundle on $\mathbb{P}^1$ was considered to be rank n.)
- We have an identification $\mathcal{F}|_{p_\infty} \xrightarrow{\sim} V$
- $\boxed{\phi_{\mathcal{F}}} : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee$ satisfying $\phi_{\mathcal{F}}^\vee = -\phi_{\mathcal{F}}$, and $\phi_{\mathcal{F}}|_{p_\infty} = \phi_V$ (where ϕ_V comes from the alternating form). Or equivalently: $\boxed{\psi_{\mathcal{F}}} : \mathcal{F} \otimes \mathcal{F} \to \mathcal{O}$, with appropriate hypotheses.

3.3. *Comparison to topology.* I think we quote Cohen-Lupercio-Segal or someone else to show that smooth maps to $BSp(V)$ are homotopic to holomorphic maps to $BSp(V)$. Then by GAGA (some justification needed) this is the same as algebraic maps to $BSp(V)$.

3.4. $\boxed{\Omega^2_{d,alg}(BSp(V))}$ parametrizes the same, with the additional requirement that $\mathcal{F}(dp_\infty)$ is generated by global sections. (Equivalently, when you write $\mathcal{F}$ as a direct sum of line bundles, the summands are all of degree between $-d$ and d inclusive. This interpretation is *not helpful*.)

$\boxed{d = N}$: Jim and I used d, and the affine Grassmannian people use N.

We define $\boxed{\mathcal{E}} = \mathcal{F}(dp_\infty)$ for convenience. This bundle is rank $2n$ and

We then have $\boxed{\phi_\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\vee(2d)$, and

$$\boxed{\psi_\mathcal{E}} = \psi_\mathcal{F}(2d) : \mathcal{E} \otimes \mathcal{E} \to \mathcal{O}(2dp_\infty). \tag{2}$$

3.5. **Claim. —** *We have an induced isomorphism* $\mathcal{E}|_{p_\infty} \xrightarrow{\sim} V$.

(Proof omitted.)

3.6. *Zariski-framing.*

Define $\boxed{A} := H^0(\mathcal{E}(-p_\infty))$. $\mathcal{E}$ has rank $2n$ and degree $2nd$. Thus $\dim A = 2n(d-1) + 2n = 2nd$. $\boxed{\boxed{\dim A = 2nd}}$

3.7. We now consider the Zariski-framed moduli space, which doesn't yet have a name. I temporarily dub it $\boxed{X(d)}$. $\dim X(d) = (\dim A)(\dim U) = (2dn)(2n) = 4dn^2$. $\boxed{\boxed{\dim X(d) = 4dn^2}}$

3.8. The data of the Zariski-framed bundle $\mathcal{E}$ is the data of A, plus $\boxed{\alpha} : A \to A$, and $\boxed{j} : A \to U$, along with an *open condition* on j and α.

The following matrix (where we are treating elements of A and U as column vectors) is required to be full rank for all $x \in \mathbb{C}$.

$$\begin{array}{|c|} \hline x\mathrm{Id} - \alpha \\ \hline j \\ \hline \end{array}$$

This matrix has $\dim A + \dim U$ rows and $\dim A^\vee$ columns. Better: the top block's rows are parametrized by A; the bottom block's rows are parametrized by U; and the colums are parametrized by $\mathbb{A}^\vee$. I might write this as:

$$\begin{array}{|c|c}
\hline
x\mathrm{Id} - \alpha & A \\
\hline
j & U \\
\hline
A^\vee &
\end{array}$$

For each $x \in \mathbb{C}$, the locus where the matrix is not full rank is codimension $2n + 1$. Thus as x varies, the locus where the matrix is not full rank is codimension $\boxed{\boxed{2n}}$. (That's not quite rigorous.)

3.9. $\dot{j\alpha}$ **and** $\vec{\dot{j\alpha}}$**.**

For future use, define $\dot{j\alpha} : A \to tU[[t]]$ by

$$\boxed{\dot{j\alpha}} = jt + j\alpha t^2 + j\alpha^2 t^3 + \cdots = jt/(1-\alpha)$$

This is the generating function for $j\alpha^{k-1}$.

We also define

$$\boxed{\vec{\dot{j\alpha}}} := \begin{pmatrix} j \\ j\alpha \\ \vdots \\ j\alpha^{2d} \end{pmatrix} : A \to U^{\oplus 2d}$$

so roughly $\vec{\dot{j\alpha}} = \dot{j\alpha} \pmod{t}^{2d+1}$.

3.10. **Claim. —** $\vec{\dot{j\alpha}}$ *is an injection* $A \hookrightarrow U^{\oplus 2d}$.

Jim says this is the same as as the open condition on j and α; type this in soon.

3.11. *Sketch of proof.*

Consider $0 \to \mathcal{E}(-2d-1) \to \mathcal{E}(-1) \to \mathcal{E}(-1)/\mathcal{E}(-2d-1) \to 0$. Now $\mathcal{E}$ is the direct sum of line bundles with degrees at most 2d, so $H^0(\mathcal{E}(-2d-1) = 0$ from which

$$H^0(\mathcal{E}(-1)) \to H^0(\mathcal{E}(-1)/\mathcal{E}(-2d-1))$$

is an injection. $\square$

Now A has rank $2dn$, and the right side has rank $4dn$.

3.12. *Recovering* (A, j, α) *from this subset of* $tU \oplus \cdots \oplus t^{2d}U \oplus t^{2d+1}U$ *(take one).*

A is just the subset.

j is just $[t]j\alpha$.

We can almost get α, but not quite. We can recover it from the image in $tU \oplus \cdots t^{2d+2}U$: simply truncate the first and slide left. The key missing piece (that we fill in later): how can we get $j\alpha^{2d}a$ if we know $j\alpha^{s}a$ for $0 \leq s < 2d$?

4. Mixing in the inner product

4.1. **Relationships between** (α, j) **and the pairing** $(\psi_{\mathcal{E}}/\phi_{\mathcal{E}})$**.**

We unpack (2). Induced by $H^0(\mathcal{F}(d))) \otimes H^0(\mathcal{F}(d)) \to H^0(\mathcal{O}(2d))$ is

$$\boxed{\psi_{A\oplus U}} : \underbrace{(A \oplus U)}_{\dim = (d+1)(2n)} \otimes (A \oplus U) \to \mathbb{C}[x_0, x_1]_{2d}$$

where the subscript means "homogeneous of degree 2d." Equivalently,

$$\boxed{\phi_{A\oplus U}} : A \oplus U \to (A^\vee \oplus U^\vee)[x_0, x_1]_{2d}$$

satisfying

$$\phi_{A\oplus U} = -\phi^\vee_{A\oplus U}, \tag{3}$$

satisfying three conditions:

(V) "vanishes on $(x - \alpha, -j)$" (on either factor, but (3) means we need only the first factor) (closed condition) and

(ND) "nondegenerate on the quotient for all $x \in \mathbb{C}$" (open condition)

(BP) $[x_0^{2d}]\psi_{A\oplus U}((a_1, u_1), (a_2, u_2)) = \psi_U(u_1, u_2)$ (base point condition, hence nondegeneracy at $x = \infty$)

The following is then tautological.

4.2. **Claim. —** *This precisely parametrizes our space. Hence our space* $X(d)$ *is a quasiaffine variety.*

We unpack this. (Notice that with this description, it will not be obvious that $X(d)$ is smooth!)

We write the components of ϕ out explicitly as follows.

Define

$$\boxed{T_A} : A \to A^\vee[x_0, x_1]_{2d} \qquad T_A = T_{A,0}x_0^{2d} + \cdots + T_{A,2d}x_1^{2d} \qquad \text{so } T_{A,i}^\vee = -T_{A,i} \qquad T_{A,0} = 0$$

$$\boxed{T_U} : U \to U^\vee[x_0, x_1]_{2d} \qquad T_U = T_{U,0}x_0^{2d} + \cdots + T_{U,2d}x_1^{2d} \qquad \text{so } T_{U,i}^\vee = -T_{U,i} \qquad T_{U,0} = \phi_U$$

$$\boxed{T_{AU}} : A \to U^\vee[x_0, x_1]_{2d} \qquad T_{AU} = T_{AU,0}x_0^{2d} + \cdots + T_{AU,2d}x_1^{2d} \qquad T_{AU,0} = 0$$

and $\boxed{T_{UA}} = -T_{AU}^\vee : U \to A^\vee[x_0, x_1]_{2d}$

The final column gives the third condition (BP).

Condition (V) translates to

$$\phi_{A\oplus U}((x_0 - x_1\alpha)a, -x_1 j a) = 0 \in A^\vee \oplus U^\vee$$

for any $x = x_0/x_1 \in \mathbb{C}$, $a \in A$.

The $A^\vee$ component of this gives:

$$0 = \sum x_0^{2d-1-i}x_1^i \left(T_{A,i}(x_0 a - x_1\alpha(a)) + T_{AU,i}^\vee(j(a)x_1)\right)$$

The $U^\vee$ component of this gives:

$$0 = \sum x_0^{2d-1-i}x_1^i \left(T_{AU,i}(x_0 a - x_1\alpha(a)) + T_{U,i}(-j(a)x_1)\right)$$

We thus have two sequences of $2d+2$ conditions from these. From the coefficients of $x_0^{2d-1-i}x_1^{i+1}$ (where i runs from -1 to $2d$):

$$0 = T_{A,i+1} - T_{A,i} \circ \alpha - T_{AU,i}^\vee \circ j : A \to A^\vee \tag{4}$$

and

$$0 = T_{AU,i+1} - T_{AU,i} \circ \alpha + T_{U,i} \circ j : A \to U^\vee \tag{5}$$

The case $i = -1$ is the previously-stated relations $T_{A,0} = 0$ and $T_{AU,0} = 0$.

Thus the closed condition (V) means that if we choose $T_{U,i}$ freely for $i = 1, \ldots, 2d$, then the $T_{AU,i}$ and $T_{A,i}$ are determined inductively from the cases $i = 0$ through $2d - 1$. The two remaining conditions when $i = 2d$ give us relations.

4.3. *Checking dimensions so far.* Now our choices of j and α give us $(\dim A)(\dim U) = \dim X(d)$ dimensions already $T_{U,i} : U \to U^\vee$ must satisfy $T_{U,i} = -T_{U,i}^\vee$, so the dimension of choices of one of them is $1 + \cdots + (2n-1) = (2n-1)(2n)/2 = n(2n-1)$ (roughly $(\dim U)^2/2$). Thus the total number of choices is for all of them is $2dn(2n-1) = 4dn^2 - 2dn$.

So our new conditions for $i = 2d$ must give us precisely this many relations, and also cut out something smooth!

4.4. **Towards the two conditions.**

Recall $z = x_1/x_0$. Define $\boxed{T_{AU}} := \sum T_{AU,i} z^i : A \to U^\vee[[z]]$, and similarly for $\boxed{T_U} : U \to U^\vee[[z]]$ and $\boxed{T_A} : A \to A^\vee$ (and $\boxed{T_{UA}} = -T_{AU}^\vee : U \to A^\vee[[z]]$). These are all actually polynomials of degree (at most) $2d$, but I wish to deliberately consider them as power series.

4.5. **Claim. —** $T_{AU} = -T_U \rlap{j}\alpha$ *and* $T_{UA} = \rlap{j}\alpha^\vee T_U$.

Proof. From (5)

$$0 = T_{AU,i+1} - T_{AU,i} \circ \alpha + T_{U,i} \circ j : A \to U^\vee$$

we have $T_{AU} - T_{AU}\alpha z + T_U jz = 0$ from which $T_{AU}(1 - \alpha z) = -T_U jz$ from which $T_{AU} = -T_U jz/(1-\alpha z) = -T_U \rlap{j}\alpha$. □

4.6. **Claim. —** $T_A = T_{AU}^\vee \rlap{j}\alpha = \rlap{j}\alpha^\vee T_U \rlap{j}\alpha$.

Proof. From (4)

$$0 = T_{A,i+1} - T_{A,i} \circ \alpha - T_{AU,i}^\vee \circ j : A \to A^\vee$$

we have $T_A - T_A \alpha z - T_{AU}^\vee jz = 0$ from which $T_A = T_{AU}^\vee \rlap{j}\alpha = \rlap{j}\alpha^\vee T_U \rlap{j}\alpha$. □

4.7. **First closed condition.**

From condition (5), $T_{AU} = T_U \rlap{j}\alpha$,

$$-T_{AU} = \left(T_{U,0} + T_{U,1} z + \cdots + T_{U,2d} z^{2d}\right)\left(jz + j\alpha z^2 + \cdots\right).$$

The coefficient of z^{2d+1} is:

$$-T_{AU,2d+1} = T_{U,0} j \alpha^{2d} + T_{U,1} j \alpha^{2d-1} + \cdots + T_{U,2d} j \alpha^0$$

Indeed, more generally, for $s \geq 0$

$$-T_{AU,2d+1+s} = -T_{AU,2d+1} \circ \alpha^s$$

so our countably many conditions ensuring that T_{AU} is a polynomial of degree $2d$ is just one condition:

$$\boxed{T_{U,0} j \alpha^{2d} + T_{U,1} j \alpha^{2d-1} + \cdots + T_{U,2d} j \alpha^0 = 0 \in \mathrm{Hom}(A, U^\vee).} \tag{6}$$

This is a linear condition of dimension $(\dim A)(\dim U)$.

4.8. **Corollary. —** *The first closed condition* (6) *can be rewritten in the following two useful ways.*

(a)

$$j\alpha^{2d} = -T_{U,0}^{-1}T_{U,1}j\alpha^{2d-1} - \cdots - T_{U,0}^{-1}T_{U,2d}j\alpha^0 \in \mathrm{Hom}(A, U^\vee).$$

(b)

$$T_U j\alpha^{2d} + \begin{pmatrix} T_{U,2d-1} & \dots & T_{U,1} & T_{U,0} \end{pmatrix} \vec{j\alpha} = 0 \in \mathrm{Hom}(A, U^\vee)$$

4.9. **Second closed condition.**

From condition (4), it is best to use the form $T_{A,2d+1} = T_{AU}^\vee \vec{j\alpha}$. The analogous argument gives the single condition

$$T_{AU,0}^\vee j\alpha^{2d} + T_{AU,1}^\vee j\alpha^{2d-1} + \cdots + T_{AU,2d}^\vee j\alpha^0 = 0 \in \mathrm{Hom}(A, A^\vee)$$

or equivalently

$$(\alpha^\vee)^{2d} j^\vee T_{AU,0} + (\alpha^\vee)^{2d-1} j^\vee T_{AU,1} + \cdots + (\alpha^\vee)^0 j^\vee T_{AU,2d} = 0 \in \mathrm{Hom}(A, A^\vee)$$

from which

$$\boxed{\left((\alpha^\vee)^{2d-1}(j^\vee T_{U,0}j)\alpha^0 + (\alpha^\vee)^{2d-2}(j^\vee T_{U,0}j)\alpha^1 + \cdots + (\alpha^\vee)^0(j^\vee T_{U,0}j)\alpha^{2d-1}\right) +} \tag{7}$$

$$\boxed{\left((\alpha^\vee)^{2d-2}(j^\vee T_{U,1}j)\alpha^0 + \cdots + (\alpha^\vee)^0(j^\vee T_{U,1}j)\alpha^{2d-2}\right) + \cdots + \left(j^\vee T_{U,2d-1}j\right) = 0}$$

4.10. **Corollary. —** *This can be rewritten as:*

$$\vec{j\alpha}^\vee \begin{pmatrix} T_{U,2d-1} & T_{U,2d-2} & T_{U,2d-3} & \dots & T_{U,1} & T_{U,0} \\ T_{U,2d-2} & T_{U,2d-3} & T_{U,2d-4} & \dots & T_{U,0} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{U,1} & T_{U,0} & 0 & \dots & 0 & 0 \\ T_{U,0} & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \vec{j\alpha} = 0 \in \mathrm{Hom}(A, A^\vee) \tag{8}$$

Note that this $\boxed{T\mathrm{-matrix}}$ is invertible and alternating, and thus gives a symplectic structure to $U^{\oplus 2d}$.

4.11. **Theorem. —** *Recall that we given* d, U, *and* $T_U : U \to U^\vee$ *(invertible, signed).* $X(d)$ *is paramatrized by precisely the following information. We need* $T_{U,1}, \dots, T_{U,2d} : U \to U^\vee$ *with* $T_{U,i}^\vee = -T_{U,i}$ *(and define* $T_{U,0} = T_U$*). Then* $X(d)$ *consists of the space parametrizing half-dimensional subspaces* $\boxed{A'}$ *of* $U^{\oplus 2d}$ *such that (i)* A' *is "(maximal) isotropic with respect to the* T*-matrix" (see Corollary 4.10), such that furthermore (ii) under the linear map* $\alpha' : U^{2d} \to U^{2d}$ *defined by*

$$\boxed{\alpha'} : (u_0, u_1, \dots, u_{2d-2}, u_{2d-1}) \mapsto (u_1, u_2, \dots, u_{2d-1}, -T_{U,0}^{-1}T_{U,1}u_{2d-1} - \cdots - T_{U,0}^{-1}T_{U,2d}u_0)$$

A' maps to A'. (iii) then we have an open condition, given by the induced j and α (ensuring that we build a vector bundle). (iv) We have an open condition (ND) on nondegeneracy of the inner product we've built on that vector bundle.

Note that α' is given by the matrix

$$-T_{U,0}^{-1}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ T_{U,2d} & T_{U,2d-1} & T_{U,2d-2} & \dots & T_{U,2} & T_{U,1} \end{pmatrix}$$

Proof. We have shown how to get from $X(d)$ to the above data. We now do the reverse. We recover A as A'. We recover α as α'. □

4.12. **Proposition. —** *The open condition (iii) on j and α is vacuous.*

I separated this from the main theorem because I fear I am making a mistake. But it looks to me like we are describing a projective morphism from $X(d)$ to an affine space parametrizing the $T_{U,i}$'s. Also, Jim (when meeting in Fort Collins) agreed (for reasons I didn't understand) that this condition will be automatic.

Proof. The open condition is that $\ker(\alpha - x\mathrm{Id}) \cap \ker(j) = 0$ for all $x \in \mathbb{C}$. Here we are interpreting α as a map $A \to A$, and j as a map $A \to U$.

Suppose $(u_0, \dots, u_{2d-1}) \in \ker(\alpha - x\mathrm{Id}) \cap \ker(j)$. Then $u_0 = 0$ (the $\ker(j)$ condition).

$$\alpha - x\mathrm{Id} : (u_0, u_1, \dots, u_{2d-2}, u_{2d-1}) \mapsto$$
$$(u_1 - xu_0, u_2 - xu_1, \dots, u_{2d-1} - xu_{2d-2}, -T_{U,0}^{-1}T_{U,1}u_{2d-1} - \cdots - T_{U,0}^{-1}T_{U,2d}u_0 - xu_{2d-1})$$

I think this is automatic! Solving successively, we get $u_1 = 0$, then $u_2 = 0$, and so forth.

□

4.13. **Proposition. —** *The condition (iv) is vacuous.*

Proof. The locus where the form is degenerate should be interpreted as the vanishing of a section of a particular (determinant) line bundle. But this line bundle is the trivial bundle, and the section is nonzero at p_∞. □

We now write Theorem 4.11 in its new form.

4.14. **Theorem. —** *Recall that we given* d, U, *and* $T_U : U \to U^\vee$ *(invertible, signed).* $X(d)$ *is paramatrized by precisely the following information. We need* $T_{U,1}, \ldots, T_{U,2d} : U \to U^\vee$ *with* $T_{U,i}^\vee = -T_{U,i}$ *(and define* $T_{U,0} = T_U$*). Then* $X(d)$ *consists of the space parametrizing half-dimensional subspaces* $\boxed{A'}$ *of* $U^{\oplus 2d}$ *such that (i)* A' *is "maximal isotropic with respect to the* T*-matrix" (see Corollary 4.10), such that furthermore (ii) under the linear map* $\alpha' : U^{2d} \to U^{2d}$ *defined by*

$$\boxed{\alpha'} : (u_0, u_1, \ldots, u_{2d-2}, u_{2d-1}) \mapsto (u_1, u_2, \ldots, u_{2d-1}, -T_{U,0}^{-1}T_{U,1}u_{2d-1} - \cdots - T_{U,0}^{-1}T_{U,2d}u_0),$$

i.e., given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -T_{U,0}^{-1}T_{U,2d} & -T_{U,0}^{-1}T_{U,2d-1} & -T_{U,0}^{-1}T_{U,2d-2} & \ldots & -T_{U,0}^{-1}T_{U,2} & -T_{U,0}^{-1}T_{U,1} \end{pmatrix}, \tag{9}$$

A' *maps to* A'.

We will clean it up further below.

4.15. **Affine Grassmannian.** The fiber over $T_{U,i} = 0$ is literally Nth truncation of the symplectic affine Grassmannian.

4.16. **Aside: Torus action.**

The space has a torus action, which we can presumably interpret in terms of actions only on the Zariski-lifting sections, although I haven't checked (yet).

The action is as follows. $T_{U,i} \mapsto t^i T_{U,i}$. For the half-dimensional subspaces, we map $\tau : (u_0, \ldots, u_{2d-1}) \mapsto (t^0 u_0, \ldots, t^{2d-1}u_{2d-1})$.

We verify that (i) continues to hold, by expanding

$$\begin{pmatrix} u_0 & tu_1 & \cdots & t^{2d-1}u_{2d-1} \end{pmatrix} \begin{pmatrix} t^{2d-1}T_{U,2d-1} & t^{2d-2}T_{U,2d-2} & t^{2d-3}T_{U,2d-3} & \ldots & tT_{U,1} & T_{U,0} \\ t^{2d-2}T_{U,2d-2} & t^{2d-3}T_{U,2d-3} & t^{2d-4}T_{U,2d-4} & \ldots & T_{U,0} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ tT_{U,1} & T_{U,0} & 0 & \ldots & 0 & 0 \\ T_{U,0} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0' \\ tu_1' \\ \ldots \\ t^{2d-1}u_{2d-1}' \end{pmatrix}$$

When we expand this out, by inspecting (7), we see that we get t^{2d-1} times what we had before, so we still get 0.

We see that (ii) holds too; put informally, $\alpha' \circ \tau = t\tau \circ \alpha$.

4.17. *Limit of the torus action.* Sending $t \to 0$ the resulting points in the closed condition seem to be the symplectic affine Grassmannian. More precisely: if we take the limits of conditions (i) and (ii) (the closed conditions), we describe the symplectic affine Grassmannian. Caution: we haven't shown that the flat limit of this family of varieties (satisfying (i) and (ii)) is the symplectic affine Grassmannian; the actual limit might be smaller.

4.18. **Dimension check.** I'm expecting to see $\dim X(d) = (\dim A)(\dim U) = (2dn)(2n) = 4dn^2$, computed in another way.

From (1) the truncated symplectic affine Grassmannian is, I think, $d(2n-1)$. This should be a family over the space of choices of $T_{U,i}$ ($i = 1, \ldots, 2d$), with generic fiber dimension at least that of the fiber over zero. The dimension of choice of each $T_{U,i}$ is, I think, $(2n)(2n+1)/2$; this is something I find confusing (and we get the *same* answer in the orthogonal case). This underlying affine space then has dimension $(2d)(2n+1)n = 4dn^2 + 2dn$. Thus the map from $X(d)$ to this affine space is not surjective. The fibers over zero is the truncated symplectic affine Grassmannian, so the fiber over 0 is of dimension $2dn - d$.

(I would expect that the only global functions on $X(d)$ would be those coming from the $T_{U,i}$.)

4.19. **Example** $d = 1$, $n = 1$.

Even this case is complicated. $\dim X(d)$ is supposed to be 4, mapped to $\mathbb{A}^6$ (parametrizing $T_{U,1}$ and $T_{U,2}$). The fiber over 0 is the SAG, which is 1-dimensional.

4.20. **Behavior under increase of** d **and** n.

When d increases, we are keeping U, but increasing A by $2n = \dim U$. we are multiplying the sections by x_0 (so this involves the choice of a second point in our genus 0 curve). We get a map $\boxed{A_d} \to \boxed{A_{d+1}}$,

Here is how to increase $d+1$. $d' = d+1$. $T'_{U,2d'-1} = T'_{U,2d'} = 0$, and $T'_{U,i} = T_{U,i}$ otherwise (if $0 \le i \le 2d$).

To get the corresponding A' from A, as a subset of $U^{\oplus}2d'$. Start with your subset of $U^{\oplus}2d'$. Add 0 to the front, and U to the end.

$X(d) \to X(d+1)$ is of course increasingly connected, as can be seen from its incarnation as an open embedding.

Here is how to increase n by 1. You increase U by a 2-dimensional vector space (with form), call it U_1, so $U_{n+1} = U_n \oplus U_1$. Given $A_n \subset U_n^{2d}$, we get $A_{n+1} \subset U_n^{2d} \oplus U_1^{2d}$ as $(A_n, 0) \oplus (0, 0^{\oplus d} \oplus U_1^{\oplus d})$.

It is not yet clear why $X_n(d) \to X_{n+1}(d)$ is increasingly connected; I've not thought it through.

4.21. **Cleaning up Theorem 4.14 further.**

We can do a symplectic Gramm-Schmidt change of basis to make the T-matrix antidiagonal, and clean out the non-T_0-terms.

Let's find the change of basis $\boxed{P}$. We will need to invert 2. Let M be the block matrix of (8) (the "T-matrix"), and let M_0 be the block matrix that is the same but only with the antidiagonal entries (the "T_0-matrix" perhaps). I will tell you the matrix g such $P^T M P = M_0$. P will be a lower-triangular matrix, where each entry ℓ below the diagonal will be called P_ℓ, so $A_0 = 1$. Here is the $d = 2$ case as an example.

$$\begin{pmatrix} I & A_1^T & A_2^T & A_3^T \\ 0 & I & A_1^T & A_2^T \\ 0 & 0 & I & A_1^T \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} T_3 & T_2 & T_1 & T_0 \\ T_2 & T_1 & T_0 & 0 \\ T_1 & T_0 & 0 & 0 \\ T_0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ A_1 & I & 0 & 0 \\ A_2 & A_1 & I & 0 \\ A_3 & A_2 & A_1 & I \end{pmatrix}$$

When you expand this out, each entry ℓ steps above the diagonal is

$$\sum_{i+j+k=\ell} A_i^T T_k A_j$$

The sum has $\binom{\ell}{2}$ terms. Note that $A_i^T T_k A_j + A_j^T T_k A_i$ is skew (equal to the negative of its transpose). Hence recursively in ℓ, we get

$$A_\ell^T T_0 + T_0 A_\ell = -T_\ell - \tilde{\sum_{i+j+k=\ell}} A_i^T T_k A_j$$

where the sum omits the three obvious terms. We can get a choice-free solution by taking both terms on the left to be equal:

$$A_\ell := -\frac{1}{2} T_0^{-1} \left(\sum_{i+j+k=\ell; i,j<\ell} A_i^T T_k A_j \right).$$

We note that our A_ℓ satisfy $T_0 A_\ell T_0^{-1} = A_\ell^T$, i.e., A_ℓ is a self–adjoint endomorphisms with respect to the bilinear form defined by T_0. I think they are the opposite of the Lie algebra of the symplectic group. So this really is a canonical choice.

Here are the first values.

$$A_1 = -\tfrac{1}{2}\, T_0^{-1}\, T_1$$

$$A_2 = \frac{3}{8} T_0^{-1} T_1 T_0^{-1} T_1 - \frac{1}{2} T_0^{-1} T_2.$$

$$A_3 = -\tfrac{1}{8}\, T_0^{-1}\, T_1\, T_0^{-1}\, T_1\, T_0^{-1}\, T_1 + \tfrac{1}{4}\, T_0^{-1} \big(T_1\, T_0^{-1}\, T_2 + T_2\, T_0^{-1}\, T_1\big) - \tfrac{1}{2}\, T_0^{-1}\, T_3.$$

$$\begin{aligned} A_4 = -\frac{1}{2}\, T_0^{-1} \Big(& T_4 - \tfrac{1}{2}\big(T_3\, T_0^{-1} T_1 + T_1\, T_0^{-1} T_3\big) - T_2\, T_0^{-1} T_2 + \tfrac{1}{4}\, T_1\, T_0^{-1} T_2\, T_0^{-1} T_1 \\ & + \tfrac{3}{8}\, T_2\, T_0^{-1} T_1\, T_0^{-1} T_1 + \tfrac{3}{8}\, T_1\, T_0^{-1} T_1\, T_0^{-1} T_2 - \tfrac{1}{8}\, T_1\, T_0^{-1} T_1\, T_0^{-1} T_1\, T_0^{-1} T_1 \\ & - \tfrac{1}{2}\big(T_1\, T_0^{-1} T_3 + T_3\, T_0^{-1} T_1\big)\Big). \end{aligned}$$

I bet I can figure out a formula for all the coefficients.

4.22. *In the new basis, understanding the shift.*

We now consider the shift matrix (9), in the new basis. We compute PNP^{-1}.

$$\begin{pmatrix} I & 0 & 0 & 0 \\ A_1 & I & 0 & 0 \\ A_2 & A_1 & I & 0 \\ A_3 & A_2 & A_1 & I \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -T_{U,0}^{-1} T_{U,2d} & -T_{U,0}^{-1} T_{U,2d-1} & -T_{U,0}^{-1} T_{U,2d-2} & \dots & -T_{U,0}^{-1} T_{U,2} & -T_{U,0}^{-1} T_{U,1} \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ B_1 & I & 0 & 0 \\ B_2 & B_1 & I & 0 \\ B_3 & B_2 & B_1 & I \end{pmatrix}$$

Here the matrix with the B's is the inverse of the matrix with the A's, so $A_1 + B_1 = 0$, $A_2 + A_1 B_1 + B_2 = 0$, $A_3 + A_2 B_1 + A_1 B_2 + B_3 = 0$, etc. The first two multiplied together give:

$$\begin{pmatrix} 0 & I & 0 & 0 \\ 0 & A_1 & I & 0 \\ 0 & A_2 & A_1 & I \\ -T_0^{-1} T_4 & A_3 - T_0^{-1} T_3 & A_2 - T_0^{-1} T_2 & A_1 - T_0^{-1} T_1 \end{pmatrix}$$

Then when multiplying it out it is easy to see the first three rows, but the fourth is a mess:

$$\begin{pmatrix} -A_1 & I & 0 & 0 \\ -A_2 & 0 & I & 0 \\ -A_3 & 0 & 0 & I \\ ? & ? & ? & ? \end{pmatrix}$$

But in any case we are now considering maximal isotropic subspaces in $U^{\oplus}2d$ (with the usual form) along with generally chosen A_1, A_2, A_3, ..., A_{2d-1}, T_{2d}, subject to the condition that A is closed under the transformation:

$$(u_0, u_1, u_2, ..., u_{2d-1}) \mapsto (u_1 - A_1 u_0, u_2 - A_2 u_0, ..., u_{2d-1} - A_{2d-1} u_0, ???)$$

I have no fucking idea why this is smooth from this point of view, let alone why there is a map to $LG(U)$, or why this map is highly connected.

4.23. *What next.* Now to understand this space, we conder $LG(U^{\oplus 2d})$, with this T_0-pairing, parametrizing maximal isotropic subspaces $A \subset U^{\oplus 2d}$. Over this space, we have the tautological sequence $0 \to A \to U^{\oplus 2d} \to Q \to 0$ of vector bundles. Let's find the $T_1, \dots, T_{2d}$ so that (ii) holds. We have a trivial vector bundle T parametrizing canddiate $T_1, \dots, T_{2d}$ (roughly speaking, each elements of $\wedge^2 U$) which map to $\mathrm{Hom}(A, Q)$ — basically, they give a map from A to U. We are consider $\ker(T, \mathrm{Hom}(A, Q))$.

4.24. **Daniel Litt's gambit.**

Daniel Litt points out that there is good reason for thinking that the case $d = 1$ may suffice. So here is the above discussion, in the special case $d = 1$. From 4.22:

We compute PNP^{-1}.

$$\begin{pmatrix} I & 0 \\ A_1 & I \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -T_{U,0}^{-1}T_{U,2} & -T_{U,0}^{-1}T_{U,1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_1 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_2 - T_0^{-1}T_2 & A_1 - T_0^{-1}T_1 \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_1 & I \end{pmatrix}$$

$$= \begin{pmatrix} -A_1 & I \\ A_2 - T_0^{-1}T_2 - A_1^2 + T_0^{-1}T_1 A_1 & A_1 - T_0^{-1}T_1 \end{pmatrix}$$

For convenience let $S_i = T_0^{-1}T_i$. Then $A_1 = -\frac{1}{2}S_1$ and $A_2 = \frac{3}{8}S_1^2 - \frac{1}{2}S_2$. Then the transition matrix is:

$$\begin{pmatrix} -\frac{1}{2}S_1 & I \\ \frac{3}{8}S_1^2 - \frac{1}{2}S_2 - S_2 + \frac{1}{4}S_1^2 - \frac{1}{2}S_1^2 & -\frac{1}{2}S_1 - S_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}S_1 & I \\ \frac{1}{8}S_1^2 - \frac{3}{2}S_2 & -\frac{3}{2}S_1 \end{pmatrix}$$

So the space is an isotropic subspace of $A \subset U_0 \oplus U_1$ along with an endormorphism

$$\alpha : (u_0, u_1) \mapsto (u_1 - \frac{1}{2}S_1 u_0, -\frac{3}{2}S_2 u_0 + \frac{1}{8}S_1^2 u_0 - \frac{3}{2}S_1 u_1).$$

5. To do next

Is $X_n(d) \to X_{n+1}(d)$ increasingly connected?

5.1. **Describe the map to the Lagrangian Grassmannian.**

Question: Is there a variation of the definition of the Nth truncation of the symplectic affine Grassmannian that yields a bigger subvariety of the Nth truncation of the Lagrangian Sato Grassmannian that is actually smooth? Gerd Faltings, Algebraic loop groups and moduli spaces of bundles. J. Eur. Math. Soc. 5 (2003), no. 1, pp. 41–68. Ulrich Gortz, On the flatness of local models for the symplectic group, Adv. Math. 176 (2003), 89–115. look around Prop 5.1.

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