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A new proof of the Bott periodicity theorem

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Abstract

We give a simplification of the proof of the Bott periodicity theorem presented by Aguilar and Prieto. These methods are extended to provide a new proof of the real Bott periodicity theorem. The loop spaces of the groups O and U are identified by considering the fibers of explicit quasifibrations with contractible total spaces. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [1], Aguilar and Prieto gave a new proof of the complex Bott periodicity theorem based on ideas of McDuff [4]. The idea of the proof is to use an appropriate restriction of the exponential map to construct an explicit quasifibration with base space U and contractible total space. The fiber of this map is seen to be $BU \times \mathbb{Z}$. This proof is compelling because it is more elementary and simpler than previous proofs. In this paper we present a streamlined version of the proof by Aguilar and Prieto, which is simplified by the introduction of coordinate free vector space notation and a more convenient filtration for application of the Dold–Thom theorem. These methods are then extended to prove the real Bott periodicity theorem.

2. Preliminaries

We shall review the necessary facts about quasifibrations that will be used in the proof of the Bott periodicity theorem, as well as prove a technical result on the behavior of the

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classical groups under linear isometries. The latter will be essential for our applications of the Dold–Thom theorem. A surjective map $p: X \to Y$ is a quasifibration if for every $y \in Y$ and $x \in p^{-1}(y)$, the natural map

$$\pi_i\big(X, p^{-1}(y); x\big) \to \pi_i(Y; y)$$

is an isomorphism for every i. It follows immediately that if F is a fiber of p, then there is a long exact homotopy sequence associated to p.

If X is contractible, we obtain a map of the quasifibration sequence to the path space fibration.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] \\ \Omega Y \arrow[r] & PY \arrow[r] & Y \end{tikzcd}$$

It follows from the long exact homotopy sequences and the five lemma that $F \simeq \Omega Y$.

The definition of a quasifibration does not lend itself to easy verification. The following theorem of Dold and Thom [2] gives a more practical program. Recall that for a map $p: X \to Y$, a subset $S \subseteq Y$ is said to be *distinguished* if for every open $U \subseteq S$, the map $p^{-1}(U) \to U$ is a quasifibration.

Theorem 2.1. *Suppose $p: X \to Y$ is a surjection. Suppose that X is endowed with an increasing filtration $\{F_i Y\}$, such that the following conditions hold.*

(1) *$F_n Y - F_{n-1} Y$ is distinguished for every n.*

(2) *For every n there exists a neighborhood N of $F_{n-1}Y$ in $F_n Y$ and a deformation $h: N \times I \to N$ such that $h_0 = \mathrm{Id}$ and $h_1(N) \subseteq F_{n-1}Y$.*

(3) *This deformation is covered by a deformation $H: p^{-1}(N) \times I \to p^{-1}(N)$ such that $H_0 = \mathrm{Id}$, and for every $y \in N$, the induced map*

$$H_1: p^{-1}(y) \to p^{-1}\big(h_1(y)\big)$$

is a weak homotopy equivalence.

Then p is a quasifibration.

Let Λ be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $\mathcal{I}(W, V)$ denote the space of linear isometries from W to V, where W and V are (possibly countably infinite dimensional) inner product spaces over Λ topologized as the unions of their finite dimensional subspaces. Let $G(W)$ be $O(W)$, $U(W)$, or $Sp(W)$, where $G(W)$ is the space finite type linear automorphisms of W. We define a continuous map

$$\Gamma_{W,V}: \mathcal{I}(W, V) \to \mathrm{Map}\big(G(W), G(V)\big).$$

Writing $\Gamma_{W,V}(\phi) = \phi_*$, if $X \in G(W)$, then $\phi_*(X): V \to V$ is defined by

$$\phi_*(X) = \phi X \phi^{-1} \oplus I_{\phi(W)^\perp}$$

under the orthogonal decomposition $V = \phi(W) \oplus \phi(W)^{\perp}$. Let $\mathcal{U}$ and $\mathcal{V}$ be countably infinite dimensional Λ inner product spaces. In [3, II.1.5] it is proven that $\mathcal{I}(\mathcal{U}, \mathcal{V})$ is contractible. So we have the following lemmas.

Lemma 2.2. *Let $\phi, \phi' \in \mathcal{I}(\mathcal{U}, \mathcal{V})$. Then the induced maps*

$$\phi_*, \phi'_* : G(\mathcal{U}) \to G(\mathcal{V})$$

are homotopic.

Lemma 2.3. *Let $\phi \in \mathcal{I}(\mathcal{U}, \mathcal{V})$. Then ϕ_* is a homotopy equivalence.*

3. Complex Bott periodicity

The existence of the fiber sequence

$$U \to EU \to BU$$

yields that $\Omega BU \simeq U$. We aim to prove the following theorem, which implies that $\Omega^2 BU \simeq BU \times \mathbb{Z}$.

Theorem 3.1. *Let U denote the infinite unitary group. There exists a quasifibration sequence*

$$BU \times \mathbb{Z} \to E \to U$$

such that E is contractible. Consequently, $\Omega U \simeq BU \times \mathbb{Z}$.

Fix a complex infinite dimensional inner product space $\mathcal{U} \cong \mathbb{C}^\infty$. For $W \subset \mathcal{U}$, a finite dimensional complex subspace, let $U(W \oplus W)$ denote complex linear isometries of $W \oplus W$. If $V \subseteq W$, then there is a natural map $i_{V,W} : U(V \oplus V) \to U(W \oplus W)$ given by

$$i_{V,W}(X) = X \oplus I_{(W-V)\oplus(W-V)},$$

where $W - V$ denotes the orthogonal complement of V in W. Then

$$U = \varinjlim_W U(W \oplus W),$$

where the colimit is taken over all finite dimensional subspaces $W \subset \mathcal{U}$.

Let $H(W \oplus W)$ denote the hermitian linear transformations of $W \oplus W$. Define

$$E(W) = \bigl\{A \mid \sigma(A) \subseteq I = [0, 1]\bigr\} \subseteq H(W \oplus W),$$

where $\sigma(A)$ is the spectrum of A. Define

$$p_W : E(W) \to U(W \oplus W)$$

by $p_W(A) = \exp(2\pi \mathrm{i} A)$. Analogous to U, define a map $E(V) \to E(W)$ for $V \subseteq W$ by sending A to $A \oplus \pi_{(W-V)\oplus 0}$. Here, π_Y denotes orthogonal projection onto the subspace Y.

It will become apparent that this map is defined so that, upon stabilization, the fibers are $BU \times \mathbb{Z}$. Then the following diagram commutes, since $\mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^0$.

$$\begin{tikzcd} E(V) \arrow[r] \arrow[d, "p_V"'] & E(W) \arrow[d, "p_W"] \\ U(V \oplus V) \arrow[r] & U(W \oplus W) \end{tikzcd}$$

So taking colimits we obtain

$$p : E \to U,$$

where $E = \varinjlim_W E(W)$. We claim that this map is a quasifibration. E is clearly contractible, by the contracting homotopy $h_t(A) = tA$.

To fix notation define

$$BU_n(Y) = \{V \mid V \subseteq Y,\ \dim_{\mathbb{C}} V = n\}$$

for any $Y \subset \mathcal{U} \oplus \mathcal{U}$. For $V \subseteq W \subset \mathcal{U}$, there is a natural map $BU_n(V \oplus V) \to BU_m(W \oplus W)$ given by sending V' to $V' \oplus ((W - V) \oplus 0)$. Letting $BU(Y) = \coprod_n BU_n(Y)$, define $BU \times \mathbb{Z} = \varinjlim_W BU(W \oplus W)$.

Lemma 3.2. *Let $X \in U(W \oplus W)$. Then $p_W^{-1}(X) \cong BU(\ker(X - I))$.*

Proof. Define $\phi : p_W^{-1}(X) \to BU(\ker(X - I))$ by $\phi(A) = \ker(A - I)$. In order for this map to make sense, we must verify that $\ker(A - I) \subseteq \ker(X - I)$. Suppose $Av = v$. Then

$$Xv = \exp(2\pi\mathrm{i}A)v = \sum_n \frac{(2\pi\mathrm{i})^n}{n!} A^n v = \mathrm{e}^{2\pi\mathrm{i}} v = v$$

so $v \in \ker(X - I)$. Suppose the spectral decomposition of X is

$$X = \pi_V + \sum_i \lambda_i \pi_{V_i},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$, $\lambda_i \neq 1$, and $\pi_{V'}$ denotes orthogonal projection onto the subspace V' of $W \oplus W$. Note that $V = \ker(X - I)$ and since X is unitary, $|\lambda_i| = 1$ for all i and $W \oplus W = V \oplus \bigoplus_i V_i$. Suppose that $A \in p_W^{-1}(X)$. Then A, being hermitian, possesses a spectral decomposition

$$A = \pi_{V'} + 0 \cdot \pi_{V''} + \sum_i \mu_i \pi_{W_i},$$

where $W \oplus W = V' \oplus V'' \oplus \bigoplus_i W_i$. Since

$$\exp(2\pi\mathrm{i}A) = \pi_{V' \oplus V''} + \sum_i \mathrm{e}^{2\pi\mathrm{i}\mu_i} \pi_{W_i} = X$$

we conclude that $V' \oplus V'' = V$, $V_i = W_i$, and the eigenvalues μ_i are uniquely determined by the non-unital eigenvalues λ_i of X. It is clear then that $\phi(A) = V'$ possesses a continuous inverse $\psi : BU(V) \to p_W^{-1}(X)$ given by

$$\psi(V') = \pi_{V'} + \sum_i \mu_i \pi_{V_i}. \qquad \square$$

We shall now prove that $p : E \to U$ is a quasifibration. Define a filtration of U by letting

$$F_n U = \{X \mid \dim_{\mathbb{C}}(\ker(X - I)^{\perp}) \leqslant n\} \subseteq U.$$

Let $B_n = F_n U - F_{n-1} U$. The following lemma proves that B_n is distinguished.

Lemma 3.3. *$p^{-1}(B_n) \to B_n$ is a Serre fibration.*

Proof. Suppose we are presented with the following commutative diagram.

$$\begin{tikzcd} \{0\} \times I^k \arrow[r, "\alpha"] \arrow[d] & p^{-1}(B_n) \arrow[d] \\ I^{k+1} \arrow[r, "\beta"'] & B_n \end{tikzcd}$$

We wish to give a lift of this diagram. By compactness, there exists a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{tikzcd} \{0\} \times I^k \arrow[r, "\alpha'"] \arrow[d] & E(W) \cap p^{-1}(B_n) \arrow[r] \arrow[d] & p^{-1}(B_n) \arrow[d] \\ I^{k+1} \arrow[r, "\beta'"'] & U(W \oplus W) \cap B_n \arrow[r] & B_n \end{tikzcd}$$

Now, let $A(0, t_1, \ldots, t_k) = \alpha'(t_1, \ldots, t_k)$ and $X(t_0, \ldots, t_k) = \beta'(t_0, \ldots, t_k)$. Then we may write spectral decompositions, for $t \in I^k, I^{k+1}$, respectively, as

$$\begin{aligned} A(t) &= \pi_{W_0(t)} + \sum_l \mu_l(t) \pi_{W_l(t)}, \\ X(t) &= \pi_{V_0(t)} + \sum_l \lambda_l(t) \pi_{V_l(t)}, \end{aligned}$$

where $\mathrm{e}^{2\pi \mathrm{i} \mu_l(t)} = \lambda_l(t)$, $W_0(t) \subseteq V_0(t)$, and $W_l(t) = V_l(t)$ for all $t \in I^k$. Consider, for an n-dimensional complex subspace W of $\mathcal{U}$, the homogeneous space

$$\begin{aligned} \mathrm{Perp}_{i,j}(W \oplus W) &= \{(V', V'') \mid V', V'' \subseteq W \oplus W,\ V' \perp V'', \\ &\qquad \dim_{\mathbb{C}} V' = i,\ \dim_{\mathbb{C}} V'' = j\} \\ &\cong U_{2n}/U_i \times U_j \times U_{2n-(i+j)}. \end{aligned}$$

There is a natural mapping

$$P : \mathrm{Perp}_{i,j}(W \oplus W) \to BU_{i+j}(W \oplus W)$$

given by $P(V', V'') = V' \oplus V''$. Under the isomorphism $BU_{i+j}(W \oplus W) \cong U_{2n}/U_{i+j} \times U_{2n-(i+j)}$, we see that P is the natural projection map, and therefore is a fibration. Let $\alpha'' : I^k \to \mathrm{Perp}_{i,j}(W \oplus W)$ where $i = \dim W_0(0)$ and $j = \dim(V_0(0) - W_0(0))$ be given by $\alpha''(t) = (W_0(t), V_0(t) - W_0(t))$, and let $\beta'' : I^{k+1} \to BU_{i+j}(W \oplus W)$ be given by

$\beta''(t) = V_0(t)$. Our filtration is defined so that these maps make sense. Then, since P is a fibration, there exists a lift ω'' making the diagram below commute.

$$\begin{tikzcd} \{0\} \times I^k \arrow[r, "\alpha''"] \arrow[d] & \mathrm{Perp}_{i,j}(W \oplus W) \arrow[d, "P"] \\ I^{k+1} \arrow[r, "\beta''"'] \arrow[ur, "\omega''"] & BU_{i+j}(W \oplus W) \end{tikzcd}$$

Let $\mu_l(t) \in (0,1)$ be the unique solutions to $\mathrm{e}^{2\pi \mathrm{i}\mu_l(t)} = \lambda_l(t)$, and write $\omega''(t) = (W_0'(t), V_0(t) - W_0'(t))$. Then letting $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ be defined by

$$\omega'(t) = \pi_{W_0'(t)} + \sum_l \mu_l(t)\pi_{V_l(t)}$$

we obtain a lift to our original diagram. $\square$

Define

$$\overline{BU}_{V,W} = \varinjlim_{W' \geqslant W} BU\big(V \oplus \big(W' - W\big) \oplus \big(W' - W\big)\big)$$

for W finite dimensional and $V \subseteq W \oplus W$. It is clear that $\overline{BU}_{V,W} \cong BU \times \mathbb{Z}$, by a (non-canonical) choice of isometry $V \oplus W^{\perp} \oplus W^{\perp} \cong \mathcal{U} \oplus \mathcal{U}$. Then if $X \in U(W \oplus W)$, $p^{-1}(X) \cong \overline{BU}_{\ker(X-I),W}$.

Define a neighborhood N_n of $F_{n-1}U$ in F_nU to be

$$N_n = \{X \in F_nU\colon \dim_{\mathbb{C}} \mathrm{Eig}_{\mathrm{e}^{2\pi \mathrm{i}[1/3,2/3]}} X < n\} \subseteq F_nU,$$

where $\mathrm{Eig}_S X$ is the direct sum of the eigenspaces of X corresponding to eigenvalues in S. In other words, N_n is simply the space of unitary matrices with "extra eigenvalues" in a neighborhood of 1 that we shall deform to 1, pushing the matrix into $F_{n-1}U$. Let $f : I \to I$ be defined by

$$f(x) = \begin{cases} 1, & x \geqslant \frac{2}{3}, \\ 3x - 1, & \frac{1}{3} \leqslant x \leqslant \frac{2}{3}, \\ 0, & x \leqslant \frac{1}{3}. \end{cases}$$

Clearly $f \simeq \mathrm{Id}$ rel ∂I. Let H be such a homotopy. Then, since H fixes ∂I, there exists an $h : S^1 \times I \to S^1$ such that the following diagram commutes.

$$\begin{tikzcd} I \arrow[r, "H_t"] \arrow[d, "\mathrm{e}^{2\pi \mathrm{i}(\cdot)}"'] & I \arrow[d, "\mathrm{e}^{2\pi \mathrm{i}(\cdot)}"] \\ S^1 \arrow[r, "h_t"'] & S^1 \end{tikzcd}$$

Then for $A \in E$, where $A = \sum_i \mu_i \pi_{W_i}$, define a new hermitian matrix $H_t(A)$ where for $t \in I$,

$$H_t(A) = \sum_i H_t(\mu_i)\pi_{W_i}.$$

We may similarly define $h_t : U \to U$. Observe that the $h_t : N_n \to N_n$ satisfy $h_1 = \mathrm{Id}$ and $h_0(N_n) \subseteq F_{n-1}U$. Furthermore, h_t is covered by $H_t : p^{-1}(N_n) \to p^{-1}(N_n)$.

Consider the induced map on fibers $H_0 : p^{-1}(X) \to p^{-1}(h_0(X))$. We need only prove that this map is a weak equivalence to complete the proof that p is a quasifibration. This follows from the following lemma.

Lemma 3.4. *Suppose $V \subseteq V' \subseteq W \oplus W$ and $V'' \subseteq V' - V$. Then the map $\overline{BU}_{V,W} \to \overline{BU}_{V',W}$ given by sending Y to $Y \oplus V''$ is a weak equivalence.*

Proof. If C is a pointed compact space, then the induced map on reduced K-theory

$$\widetilde{K}_{\mathbb{C}}(C) \cong \left[C, \overline{BU}_{V,W}\right] \to \left[C, \overline{BU}_{V',W}\right] \cong \widetilde{K}_{\mathbb{C}}(C)$$

is just the addition of a trivial bundle, so induces an isomorphism. In particular, letting $C = S^i$, we get an isomorphism of homotopy groups. □

4. Real Bott periodicity

The same methods used in the complex case lend themselves to computing the iterated loop spaces of BO as well.

Theorem 4.1. *The loops of BO may be identified as follows.*

$$\begin{aligned}
\Omega BO &\simeq O,\\
\Omega O &\simeq O/U,\\
\Omega O/U &\simeq U/Sp,\\
\Omega U/Sp &\simeq BSp \times \mathbb{Z},\\
\Omega BSp &\simeq Sp,\\
\Omega Sp &\simeq Sp/U,\\
\Omega Sp/U &\simeq U/O,\\
\Omega U/O &\simeq BO \times \mathbb{Z}.
\end{aligned}$$

We shall prove this theorem one loop at a time by constructing quasifibrations with contractible total spaces. Note that $\Omega BO \simeq O$ and $\Omega BSp \simeq Sp$ are obvious.

4.1. $\Omega O \simeq O/U$

Let $\mathcal{U} \cong \mathbb{C}^\infty$ be an infinite dimensional complex inner product space. For finite dimensional complex $W \subset \mathcal{U}$, let $O(W)$ denote the real linear isometries of W. Define

$$E(W) = \left\{A \mid \sigma(A) \subseteq [-i, i]\right\} \subseteq \mathfrak{o}(W),$$

where $\mathfrak{o}(W)$ is the lie algebra of $O(W)$; it consists of skew symmetric real linear transformations. Observe that $E(W)$ is contractible. Define

$$p_W : E(W) \to O(W)$$

by $p_W(A) = -\exp(\pi A)$. If $V \subseteq W$ then we have maps $O(V) \to O(W)$ given by sending X to $X \oplus I_{W-V}$, and $E(V) \to E(W)$ given by sending A to $A \oplus i$ where i is viewed as a skew symmetric real transformation of $W - V$. Upon taking colimits over finite dimensional subspaces of $\mathcal{U}$, these maps yield a map $p : E \to O$. We claim this map is a quasifibration onto SO, with fiber O/U.

We need a convenient way to think about O/U. For any finite dimensional $W \subset \mathcal{U}$, let $CX(W)$ denote the space of complex structures on W, that is, the space of real linear isometries $J : W \to W$ such that $J^2 = -I$.

Proposition 4.2. *Let $W \subset \mathcal{U}$ be finite dimensional. Then $O/U(W) \cong CX(W)$.*

Proof. $O(W)$ acts transitively on $CX(W)$ by conjugation, with stabilizer $U(W)$. □

The fiber of p is therefore identified in the following lemma.

Lemma 4.3. *For $X \in SO(W)$, $p_W^{-1}(X) \cong CX(\ker(X - I))$.*

Proof. Regarding $\mathfrak{o}(W) \subseteq \mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$, we see that if $A \in p^{-1}(X)$ then

$$A = i\pi_{V'} - i\pi_{V''} + \sum_j \mu_j \pi_{W_j},$$

where $\mu_j \in (-i, i)$. If we regard $O(W) \subseteq U(W \otimes_{\mathbb{R}} \mathbb{C})$, then we may write

$$X = \pi_V + \sum_j \lambda_j \pi_{V_j},$$

where $\lambda_j \neq 1$. Thus, $V = V' \oplus V'' = \ker(X - I) \otimes_{\mathbb{R}} \mathbb{C}$, $V_j = W_j$ and μ_j is completely determined by λ_j for all j. We conclude that $A(\ker(X - I)) \subseteq \ker(X - I)$, and $A^2|_{\ker(X-I)} = -I_{\ker(X-I)}$. Therefore $A \in CX(\ker(X - I))$. Conversely, given $J \in CX(\ker(X - I))$, let $A = J + \sum_j \mu_j \pi_{V_j}$. Then $A \in p_W^{-1}(X)$. □

Define

$$\overline{O/U}_{V,W} = \varinjlim_{W' \geqslant W} O/U\big(V \oplus \big(W' - W\big)\big)$$

for $V \subseteq W \subset \mathcal{U}$ where W is a complex space and V is a real even dimensional subspace. Then it is clear that for $X \in SO(W)$, we have $p^{-1}(X) \cong \overline{(O/U)}_{\ker(X-I),W}$. Define a filtration on SO by letting

$$F_n SO = \big\{X \in SO\colon \dim_{\mathbb{R}} \ker(X - I)^{\perp} \leqslant 2n\big\}.$$

We wish to show that $B_n = F_n SO - F_{n-1} SO$ is distinguished. Observe that B_n is the set of $X \in SO$ such that $\dim \ker(X - I)^{\perp} = 2n$. We claim that $p^{-1}(B_n) \to B_n$ is actually a Serre fibration. The proof of this is completely analogous to the proof of Lemma 3.3; it amounts to observing that the natural map $O_m/U_n \times O_{m-2n} \to O_m/O_{2n} \times O_{m-2n}$ is a fibration.

We define a neighborhood N_n of $F_{n-1}SO$ in $F_n SO$ by

$$N_n = \{X \mid \dim_{\mathbb{R}} \mathrm{Eig}_{e^{2\pi i[1/4,3/4]}} X < 2n\} \subseteq F_n SO.$$

Let $f:[-i,i]\to[-i,i]$ be defined by

$$f(x)=\begin{cases}-i, & \operatorname{Im}(x)<-\frac{1}{2},\\ 2x, & -\frac{1}{2}\leqslant \operatorname{Im}(x)\leqslant\frac{1}{2},\\ i, & \operatorname{Im}(x)>\frac{1}{2}.\end{cases}$$

Then $f\simeq \mathrm{Id}\,\mathrm{rel}\{-i,i\}$. Let H be such a homotopy, and let $h:S^1\times I\to S^1$ be such that the following diagram commutes for all $t\in I$.

$$\begin{tikzcd}
{[-i,i]} \arrow[r, "H_t"] \arrow[d, "-e^{\pi(\cdot)}"'] & {[-i,i]} \arrow[d, "-e^{\pi(\cdot)}"] \\
S^1 \arrow[r, "h_t"'] & S^1
\end{tikzcd}$$

Then H and h induce the deformations of N_n into $F_{n-1}SO$ as required in the Dold–Thom theorem. The fact that H_0 induces weak equivalences on fibers follows from the following lemma, which is proved by Lemma 2.3.

Lemma 4.4. *Let $V\subseteq V'$ be even dimensional real subspaces of a finite dimensional complex space $W\subset\mathcal{U}$. Then the map $f:\overline{O/U}_{V,W}\to\overline{O/U}_{V',W}$ given by sending A to $A\oplus J$ for some fixed complex structure J on $V'-V$ is a homotopy equivalence.*

4.2. $\Omega O/U\simeq U/Sp$

Let $\mathcal{U}\cong\mathbb{H}^\infty$ be an infinite dimensional quaternionic inner product space. For finite dimensional $W\subset\mathcal{U}$, $O(W)$ is the space of real linear isometries of W, and $U(W)$ is the space of complex linear isometries of W. Then $O/U=\varinjlim_W O/U(W)$. Define

$$E(W)=\bigl\{A\mid A\text{ is conjugate linear and }\sigma(A)\subseteq[-i,i]\bigr\}\subseteq\mathfrak{o}(W).$$

Note that $\mathfrak{u}(W)^\perp\subseteq\mathfrak{o}(W)$ is the collection of all skew symmetric conjugate linear transformations of W. This implies that every coset $[X]\in SO/U(W)$ has a representative $X\in O(W)$ such that $X=\exp(A)$ for some skew symmetric conjugate linear transformation A. Also observe that $E(W)$ is contractible. Define

$$p_W:E(W)\to O/U(W)$$

by $p_W(A)=i\exp(\frac{1}{2}\pi A)$. If $V\subseteq W$ then we have maps $O/U(V)\to O/U(W)$ given by sending $[X]$ to $[X\oplus I_{W-V}]$, and $E(V)\to E(W)$ given by sending A to $A\oplus j$ where j is viewed as a conjugate linear skew-symmetric transformation of $W-V$. Upon taking colimits over finite dimensional quaternionic subspaces of $\mathcal{U}$, we obtain $p:E\to O/U$, which we wish to show is a quasifibration over SO/U, with fiber U/Sp.

For $W\subset\mathcal{U}$, let $QS(W)$ denote the space of quaternionic structures on W. These are the conjugate linear isometries J of W such that $J^2=-I$.

Proposition 4.5. *Let $W\subset\mathcal{U}$ be a finite dimensional quaternionic subspace. Then $U/Sp(W)\cong QS(W)$.*

Proof. $U(W)$ acts transitively on $QS(W)$ with stabilizer $Sp(W)$. $\square$

With the intent of understanding the coset representatives of $O/U(W)$, we give the following two lemmas.

Lemma 4.6. *Suppose that $Y = \exp(A)$, where $A \in \mathfrak{o}(W)$ is conjugate linear. Then $Yi = iY^{-1}$.*

Proof.

$$-iYi = -i\exp(A)i = \exp(-iAi) = \exp(-A) = Y^{-1}. \qquad \square$$

Lemma 4.7. *Suppose that $Y, Z \in O(W)$ satisfy $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$. Then there is an $X \in U(W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

Proof. Suppose that there is an $X \in U(W)$ such that $Y = XZ$. Observe that

$$Z^{-1}X^{-1}i = Y^{-1}i = iY = iXZ = XZ^{-1}i$$

and therefore $XZX = Z$. But then $Y^2 = XZXZ = Z^2$.

Conversely, suppose that $Y^2 = Z^2$. Then $Y = (Y^{-1}Z)Z$, so we need only show that $Y^{-1}Z \in U(W)$. But $YZ^{-1} = Y^{-1}Z$, so $Y^{-1}Zi = iYZ^{-1} = iY^{-1}Z$. $\square$

We shall say that $X \in SO(W)$ is a *special representative* of the equivalence class $[X] \in SO/U(W)$ if $X = \exp(A)$ for some conjugate linear $A \in \mathfrak{o}(W)$. Observe that by the previous two lemmas, any two special representatives are in the same equivalence class if and only if they have identical squares.

Lemma 4.8. *Every $[X] \in SO/U(W)$ has a special representative.*

Proof. $SO/U(W)$ is geodesically complete, and the geodesics γ of $SO/U(W)$ all take the form $\gamma(t) = [Y\exp(tB)]$ for $Y \in SO(W)$ and $B \in \mathfrak{u}(W)^{\perp}$ (see, for example, [5, VI.2.15]). $\square$

Lemma 4.9. *Suppose that $W \subset \mathcal{U}$ is a finite dimensional quaternionic space. Let X be a special representative of the class $[X] \in SO/U(W)$. Then $p_W^{-1}([X]) = U/Sp(\ker(X^2 - I))$.*

Proof. We claim that if $A \in p_W^{-1}([X])$, then A defines a quaternionic structure on $\ker(X^2 - I)$, that is, $A(\ker(X^2 - I)) \subseteq \ker(X^2 - I)$, and $A^2 = -I$. If $A \in E(W)$ we may regard A as an element of $\mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$, and write a spectral decomposition

$$A = i\pi_{W'} - i\pi_{W''} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (-i, i)$. Regarding $X \in U(W \otimes_{\mathbb{R}} \mathbb{C})$, its spectral decomposition is

$$X = \pi_{V'} - \pi_{V''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''}),$$

where $|\lambda_l| = 1$ and $\mathrm{Im}(\lambda_j) < 0$. If $p_W(A) = [i\exp(\frac{1}{2}\pi A)] = [X]$, then we have $-\exp(\pi A) = X^2$, so $V' \oplus V'' = W' \oplus W'' = \ker(X^2 - I) \otimes_{\mathbb{R}} \mathbb{C}$. So $A \in QS(\ker(X^2 - I))$. Conversely, suppose that J is a quaternionic structure on $\ker(X^2 - I)$. Then, regarding J as an element of $\mathfrak{u}(\ker(X^2 - I) \otimes_{\mathbb{R}} \mathbb{C})$ we obtain a spectral decomposition $J = i\pi_{W'} - i\pi_{W''}$. Let

$$A = i\pi_{W'} - i\pi_{W''} + \sum_l \mu_l \pi_{V_l' \oplus V_l''},$$

where $\mu_l \in (-i, i)$ are the unique solutions in the given range to the equation $-\mathrm{e}^{\pi\mu_l} = \lambda_l^2$. Then $(i\exp(\frac{1}{2}\pi A))^2 = X^2$, so $A \in p_W^{-1}([X])$. $\square$

For $V \subseteq W \subset \mathcal{U}$, let

$$\overline{U/Sp}_{V,W} = \varinjlim_{W' \geqslant W} U/Sp\bigl(V \oplus \bigl(W - W'\bigr)\bigr).$$

Then for a special representative $X \in SO(W)$, $p^{-1}([X])$ may be canonically identified with $\overline{U/Sp}_{\ker(X^2-I),W}$. Of course, $\overline{U/Sp}_{V,W} \cong U/Sp$.

Define a filtration of SO/U by

$$F_n SO/U = \bigl\{[X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker\bigl(X^2 - I\bigr)^{\perp} \leqslant 2n\bigr\}.$$

We are implicitly using the fact that any two special representatives of the same coset have identical squares in making this definition. Then the same argument used for the previous spaces works for our present situation, to prove that $p^{-1}(F_n SO/U - F_{n-1}SO/U) \to F_n SO/U - F_{n-1}SO/U$ is a Serre fibration. The key point is that $U_m/Sp_n \times U_{m-2n} \to U_m/U_{2n} \times U_{m-2n}$ is a fibration. Therefore $F_n U/Sp - F_{n-1}U/Sp$ is distinguished.

Just as in the previous section, one may define a neighborhood N_n of $F_{n-1}SO/U$ in $F_n SO/U$ by

$$N_n = \bigl\{[X] \mid X \text{ is a special representative, } \dim \mathrm{Eig}_{\mathrm{e}^{\pi \mathrm{i}[1/2,3/2]}} X^2 < 2n\bigr\}.$$

Let f and H_t be defined as in the previous section. These yield the deformations required by the Dold–Thom theorem. One verifies that the induced maps on fibers are weak equivalences by the same methods in the previous section, by the following consequence of Lemma 2.3.

Lemma 4.10. *Suppose that $V \subseteq V' \subseteq W$ where V and V' are even dimensional complex spaces and W is a finite dimensional quaternionic subspace of $\mathcal{U}$. Fix a quaternionic structure J on $V' - V$. Then the map $\overline{U/Sp}_{V,W} \to \overline{U/Sp}_{V',W}$ given by sending A to $A \oplus J$ is a homotopy equivalence.*

4.3. $\Omega U/Sp \simeq BSp \times \mathbb{Z}$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be a countably infinite dimensional quaternionic inner product space. For finite dimensional $W \subset \mathcal{U}$, $U(W \oplus W)$ is the collection of complex linear isometries of

$W \oplus W$, and $Sp(W \oplus W)$ is the subgroup of quaternion linear isometries of $W \oplus W$. Then $U/Sp = \varinjlim_W U/Sp(W \oplus W)$. Define

$$E(W) = \{A \mid jA = Aj,\ \sigma(A) \subseteq I\} \subseteq H(W \oplus W),$$

where $H(W \oplus W)$ is the collection of all complex linear transformations of $W \oplus W$ which are hermitian. Observe that $\mathfrak{sp}(W \oplus W)^{\perp} = \{A \in \mathfrak{u}(W \oplus W):\ Aj = -jA\}$. Define a map $p_W : E(W) \to U/Sp(W \oplus W)$ by $p_W(A) = [\exp(\pi \mathrm{i} A)]$. Then, analogous to the previous section, we have the following two lemmas which allow us to understand a system of coset representatives of U/Sp. The proofs are nearly identical to those of Lemmas 4.6 and 4.7, respectively.

Lemma 4.11. *Let $W \subset \mathcal{U}$ be finite dimensional. If $A \in H(W \oplus W)$ has the property that $Aj = jA$, then $X = \exp(\mathrm{i}A)$ has the property that $Xj = jX^{-1}$.*

Lemma 4.12. *Suppose that $Y, Z \in U(W \oplus W)$ have the property that $-jYj = Y^{-1}$ and $-jZj = Z^{-1}$. Then there exists an $X \in Sp(W \oplus W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

We shall call $X \in U(W \oplus W)$ such that $X = \exp(\pi \mathrm{i} A)$ for some $A \in E(W)$ a *special representative* for the class $[X] \in U/Sp(W \oplus W)$. Note that the previous two lemmas ensure that two special representatives are in the same equivalence class if and only if they have the same squares. An argument similar to that of Lemma 4.8 ensures that every coset of $U/Sp(W \oplus W)$ has a special representative. Define, for a quaternionic space Y,

$$BSp(Y) = \coprod_n \{V \mid V \text{ is a quaternionic supspace of } Y,\ \dim V_{\mathbb{H}} = n\}.$$

For $V \subseteq W \subset \mathcal{U}$, $BSp(V \oplus V) \to BSp(W \oplus W)$ is given by sending Y to $Y \oplus (W - V) \oplus 0$, so that $BSp \times \mathbb{Z} = \varinjlim_W BSp(W \oplus W)$. The fiber of p_W can now be identified.

Lemma 4.13. *Let $W \subset \mathcal{U}$ be finite dimensional. If X is a special representative for $[X] \in U/Sp(W)$, then $p_W^{-1}([X]) \cong BSp(\ker(X^2 - I))$.*

Proof. Suppose $A \in E(W)$. Write the spectral decomposition of A as

$$A = \pi_{W_0} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (0,1)$ and W_0 and W_l are complex subspaces of $W \oplus W$. These are actually quaternionic subspaces because if $Av = \mu v$, then $Ajv = jAv = j\mu v = \mu jv$, since μ must be real. Similarly, write the spectral decomposition of the special representative X as

$$X = \pi_{V'} - \pi_{V''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''}),$$

where $\mathrm{Im}(\lambda_l) > 0$ and $|\lambda_l| = 1$. Now, $p_W(A) = [X]$ if and only if $W_0 \subseteq V' \oplus V'' = \ker(X^2 - I)$, $W_l = V_l' \oplus V_l''$, and $\mu_l \in (0,1)$ is the unique solution of $\mathrm{e}^{2\pi \mathrm{i} \mu_l} = \lambda_l^2$. It is then clear that the map $p_W^{-1}([X]) \to BSp(\ker(X^2 - I))$ given by sending A to $\ker(A - I)$ is a homeomorphism. □

For $V \subseteq W$, define $E(V) \to E(W)$ by sending A to $A \oplus \pi_{(W-V)\oplus 0}$, and $U/Sp(V \oplus V) \to U/Sp(W \oplus W)$ by sending $[X]$ to $[X \oplus I_{(W-V)\oplus(W-V)}]$. Taking colimits over W we obtain $p : E \to U/Sp$, which we shall see is a quasifibration. Define

$$\overline{BSp}_{V,W} = \varinjlim_{W' \geqslant W} BSp\big(V \oplus \big(W' - W\big) \oplus \big(W' - W\big)\big),$$

where $V \subseteq W \oplus W \subset \mathcal{U} \oplus \mathcal{U}$. Upon stabilization, the above lemma yields that for a special representative $X \in Sp(W)$, $p^{-1}([X]) = \overline{BSp}_{\ker(X^2-I),W}$. Define a filtration of U/Sp by

$$F_n U/Sp = \big\{[X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker\big(X^2 - I\big)^{\perp} \leqslant 2n\big\}.$$

Then the techniques used in the previous sections go through in this instance to prove that $p^{-1}(F_n U/Sp - F_{n-1} U/Sp) \to F_n U/Sp - F_{n-1} U/Sp$ is a Serre fibration, hence $F_n U/Sp - F_{n-1} U/Sp$ is distinguished. Techniques completely analogous to those used in the previous sections provide the neighborhoods and deformations required by the Dold–Thom theorem. By a proof similar to that of Lemma 3.4, one obtains the following lemma, which verifies that the induced maps on fibers are homotopy equivalences.

Lemma 4.14. *Suppose that we have finite dimensional quaternionic spaces $V \subseteq V' \subseteq W \oplus W$. Let $V'' \subseteq V' - V$. Then the natural map $\overline{BSp}_{V,W} \to \overline{BSp}_{V',W}$ given by sending X to $X \oplus V''$ is a homotopy equivalence.*

4.4. $\Omega Sp \simeq Sp/U$

Let $\mathcal{U} \cong \mathbb{H}^{\infty}$ be a countably infinite dimensional quaternionic inner product space. For finite dimensional $W \subset \mathcal{U}$, $Sp(W)$ is the space of quaternionic isometries of W. Then $Sp = \varinjlim_W Sp(W)$. Define

$$E(W) = \big\{A \mid \sigma(A) \subseteq [-1, 1],\ Aj = -jA\big\} \subseteq H(W),$$

where $H(W)$ is the space of all complex linear hermitian operators on W. Define $p_W : E(W) \to Sp(W)$ by $p_W(A) = -\exp(\pi \mathrm{i} A)$. We need a convenient model for $Sp/U(W)$.

Lemma 4.15. *Let $W \subset \mathcal{U}$ be a finite dimensional quaternionic subspace. Then there is an isomorphism*

$$Sp/U(W) \cong \{V \mid V \text{ is a complex subspace of } W,\ W = V \oplus jV\}.$$

Proof. $Sp(W)$ acts transitively on this space, with stabilizer $U(W)$. □

With this in mind we may identify the fiber of p_W.

Lemma 4.16. *Let $W \subset \mathcal{U}$ be a finite dimensional quaternionic subspace. For $X \in Sp(W)$, $p_W^{-1}(X) \cong Sp/U(\ker(X - I))$.*

Proof. For $A \in E(W)$, write the spectral decomposition of A

$$A = \pi_{W'} - \pi_{W''} + \sum_l (\mu_l \pi_{W_l'} - \mu_l \pi_{W_l''}),$$

where $\mu_l \in (0,1)$ and $jW' = W''$ and $jW_l' = W_l''$. The latter conditions are seen to be necessary since if $Av = \mu v$, then $Ajv = -jAv = -j\mu v = -\mu jv$. Similarly, write X as

$$X = \pi_V - \pi_{V_0} + \sum_l \bigl(\lambda_l \pi_{V_l'} + \overline{\lambda_l} \pi_{V_l''}\bigr),$$

where $|\lambda_l| = 1$, $\mathrm{Im}(\lambda_l) < 0$, V and V_0 are quaternionic subspaces of W, and $jV_l' = V_l''$. This condition is required since if $Xv = \lambda v$, then $Xjv = jXv = j\lambda v = \overline{\lambda} jv$. So $p_W(A) = X$ if and only if $W' \oplus W'' = V$, $W_l' = V_l'$, $W_l'' = V_l''$, and $\mu_l \in (0,1)$ are the unique solutions to the equation $-\mathrm{e}^{\pi \mathrm{i} \mu_l} = \lambda_l$. It follows immediately that $p_W^{-1}(X) = Sp/U(\ker(X - I))$. □

Let Y be a quaternionic vector space, and define $Y^{\mathbb{C}} = \{v \mid iv = vi\} \subseteq Y$. For $V \subseteq W$, define maps $E(V) \to E(W)$ by sending A to $A \oplus \pi_{(W-V)^{\mathbb{C}}}$. Taking the colimit over all $W \subset \mathcal{U}$ yields $p : E \to Sp$. The proof that this is a quasifibration is completely analogous to the previous sections. Since $E(W)$ is contractible for all W, E is contractible, and the previous lemma implies that the fiber of p is Sp/U.

4.5. $\Omega Sp/U \simeq U/O$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be an infinite dimensional quaternionic space endowed with a real inner product such that multiplication by i and multiplication by j are real isometries. For a finite dimensional right quaternionic subspace $W \subset \mathcal{U}$, regard $Sp(W)$ as the collection of real isometries X of W that are right quaternion linear, in the sense that for all $\alpha \in \mathbb{H}$, $X(v\alpha) = (Xv)\alpha$. The elements of $Sp(W)$ may be regarded as matrices with quaternion coefficients. Then $U(W)$ is the subgroup of $Sp(W)$ consisting of all X which are left complex linear, in the sense that $X(iv) = iX(v)$. Let $W^{\mathbb{R}}$ be the real subspace of W given by $\{v \mid vi = iv \text{ and } vj = jv\}$. The Lie algebra of Sp is given by

$$\mathfrak{sp}(W) = \mathfrak{o}\bigl(W^{\mathbb{R}}\bigr) \oplus iS\bigl(W^{\mathbb{R}}\bigr) \oplus jS\bigl(W^{\mathbb{R}}\bigr) \oplus kS\bigl(W^{\mathbb{R}}\bigr),$$

where $S(X)$ denotes symmetric linear transformations of a space X. The Lie subalgebra corresponding to $\mathfrak{u}(W)$ is $\mathfrak{o}(W^{\mathbb{R}}) \oplus iS(W^{\mathbb{R}})$. We let

$$E(W) = \bigl\{ jA + kB \mid \sigma(A),\ \sigma(B) \subseteq [-1,1] \bigr\} \subseteq jS\bigl(W^{\mathbb{R}}\bigr) \oplus kS\bigl(W^{\mathbb{R}}\bigr).$$

Define $p_W : E(W) \to Sp/U(W)$ by $p_W(A) = [i \exp(\frac{1}{2}\pi A)]$. We identify U/O in the following proposition.

Proposition 4.17. *Let W be a finite dimensional quaternionic inner product space. Then there is an isomorphism*

$$U/O(W) \cong \{V \mid V \text{ is a right complex subspace of } W,\ W = V \oplus iV = V \oplus Vj\}.$$

Proof. $U(W)$ acts transitively on this space, with stabilizer $O(W)$. $\square$

To understand the coset representatives of $U/O(W)$, we give the following two lemmas. Their proofs are completely analogous to the proofs of Lemmas 4.6 and 4.7.

Lemma 4.18. *Let $W \subset \mathcal{U}$ be a right quaternionic subspace of finite dimension. Suppose that $A \in \mathfrak{sp}(W)$ has the property that $Ai = -iA$. Then $X = \exp(A)$ has the property that $Xi = iX^{-1}$.*

Lemma 4.19. *Suppose that $W \subset \mathcal{U}$ is a right quaternionic subspace of finite dimension. If $Y, Z \in Sp(W)$ possess the property that $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$, then there exists an $X \in U(W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

We shall call an $X \in Sp(W)$ such that there exists an $A \in \mathfrak{sp}(W)$ such that $Ai = -iA$, yielding $X = \exp(A)$ a *special representative* of $[X] \in Sp/U(W)$. The above two lemmas imply that two special representatives are in the same coset if and only if they have identical squares. The argument of Lemma 4.8 shows that any coset has a special representative. With this knowledge we may proceed to identify the fiber of p_W.

Lemma 4.20. *Let $W \subset \mathcal{U}$ be a finite dimensional right quaternionic subspace. For a special representative X of $[X] \in Sp/U(W)$, we have $p_W^{-1}([X]) \cong U/O(\ker(X^2 - I))$.*

Proof. Suppose $A \in E(W)$. Being careful to write our eigenvalues on the right since A is a right skew-hermitian operator, we may express a spectral decomposition of A as

$$A = \pi_{W'} i - \pi_{W''} i + \sum_l (\pi_{W_l'} i\mu_l - \pi_{W_l''} i\mu_l),$$

where $\mu_l \in (0,1)$, W', W'', W_l', and W_l'' are right complex spaces, $iW' = W''$, $W'j = W''$, $iW_l' = W_l''$, and $W_l'j = W_l''$. For if $Av = vi\mu$, then $Aiv = -iAv = -ivi\mu = iv(-i\mu)$ and $A(vj) = (Av)j = vi\mu j = vj(-i\mu)$. Similarly, write the spectral decomposition of the special representative X as

$$X = \pi_{V'} - \pi_{V''} + \pi_{V_0'} i - \pi_{V_0''} i + \sum_l \bigl(\pi_{V_l'} \lambda_l + \pi_{V_l''} \overline{\lambda_l} - \pi_{\widetilde{V}_l'} \lambda_l - \pi_{\widetilde{V}_l''} \overline{\lambda}_l\bigr),$$

where $|\lambda_l| = 1$, $\mathrm{Im}(\lambda_l^2) < 0$, $\mathrm{Im}(\lambda_l) > 0$, V' and V'' are quaternionic spaces, $iV_l' = V_l''$, $V_l'j = V_l''$. $i\widetilde{V}_l' = \widetilde{V}_l''$, and $\widetilde{V}_l'j = \widetilde{V}_l''$. For if $Xv = v\lambda$, then $Xiv = iX^{-1}v = iv\overline{\lambda}$, and $Xvj = v\lambda j = vj\overline{\lambda}$. Now, if $-\exp(\pi \mathrm{i} A) = X^2$, we see that $\mu_l \in (0,1)$ are the unique solutions to $-\mathrm{e}^{\pi \mathrm{i}\mu_l} = \lambda_l^2$, $W' \oplus W'' = V' \oplus V'' = \ker(X^2 - I)$, $W_l' = V_l' \oplus \widetilde{V}_l'$, and $W_l'' = V_l'' \oplus \widetilde{V}_l''$. The result follows immediately. $\square$

For $V \subseteq W$, define $i_{V,W} : E(V) \to E(W)$ by

$$i_{V,W}(A) = A \oplus (\pi_{(k+1)(W-V)^{\mathbb{R}}} - \pi_{(i-j)(W-V)^{\mathbb{R}}}).$$

Taking the colimit over $W \subset \mathcal{U}$, we obtain a map $p : E \to Sp/U$, which, by repeating the techniques of the previous sections, is a quasifibration with fiber U/O.

4.6. $\Omega U/O = BO \times \mathbb{Z}$

Let $\mathcal{U} \cong \mathbb{C}^\infty$. Fix a complex conjugation $c : \mathcal{U} \to \mathcal{U}$. For the purposes of this section, all finite dimensional complex subspaces of $\mathcal{U}$ are assumed to be closed under the conjugation map c. For a complex finite dimensional $W \subset \mathcal{U}$, the *real subspace* of W is defined to be $W^{\mathbb{R}} = \{v \in W \colon v = \overline{v}\}$. $U(W \oplus W)$ is the collection of complex isometries of $W \oplus W$, and $O(W \oplus W)$ is the collection of all $X \in U(W \oplus W)$ such that $X = \overline{X}$. Define

$$E(W) = \left\{A \mid \overline{A} = A,\ \sigma(A) \subseteq [0,1]\right\} \subseteq H(W \oplus W).$$

Define $p_W : E(W) \to U/O(W)$ by $p_W(A) = [\exp(\pi \mathrm{i} A)]$. Observe that we have the following two lemmas, whose proofs are analogous to those of Lemmas 4.6 and 4.7.

Lemma 4.21. *Let W be a finite dimensional complex space. Then if $A \in \mathfrak{u}(W \oplus W)$ has the property that $\overline{A} = -A$, then $X = \exp(A)$ has the property that $X^{-1} = \overline{X}$.*

Lemma 4.22. *Suppose that W is a finite dimensional complex space. Then if $Y, Z \in U(W \oplus W)$ have the property that $Y^{-1} = \overline{Y}$ and $Z^{-1} = \overline{Z}$, then there exists an $X \in O(W \oplus W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

If $X \in U(W \oplus W)$, and $X = \exp(A)$ for some $A \in \mathfrak{u}(W \oplus W)$ such that $\overline{A} = -A$, then we shall say that X is a *special representative* of $[X] \in U/O(W \oplus W)$. Evidently two special representatives represent the same equivalence class if and only if they have identical squares. The argument of Lemma 4.8 implies that every coset has a special representative. The following lemma identifies the fiber of p_W.

Lemma 4.23. *Let $W \subset \mathcal{U}$ be a finite dimensional complex space. If $X \in U(W \oplus W)$ is a special representative for $[X] \in U/O(W \oplus W)$, then $p_W^{-1}([X]) \cong BO(\ker(X^2 - I)^{\mathbb{R}})$.*

Proof. If $A \in E(W)$, then A admits a spectral decomposition

$$A = \pi_{W_0} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (0,1)$. We claim that the spaces W_l are closed under the conjugation in W. Indeed, if $Av = \mu v$, then $A\overline{v} = \overline{Av} = \overline{\mu v} = \mu \overline{v}$. The special representative X has a spectral decomposition

$$X = \pi_{V_0'} - \pi_{V_0''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''}),$$

where $\mathrm{Im}(\lambda_l) > 0$. We claim that V_l', V_l'' are closed under conjugation. Indeed, if $Xv = \lambda v$ then $X\overline{v} = \overline{X^{-1}v} = \overline{\overline{\lambda}v} = \lambda\overline{v}$. So if $\exp(2\pi \mathrm{i} A) = X^2$, then the eigenvalues $\mu_l \in (0,1)$ must be the unique solutions to the equation $\mathrm{e}^{2\pi \mathrm{i} \mu_l} = \lambda_l^2$. Also $W_l = V_l' \oplus V_l''$ for all $l \neq 0$ and $W_0 \subseteq V_0' \oplus V_0''$ is simply a subspace closed under conjugation. Define $\phi : p_W^{-1}([X]) \to$

$BO(\ker(X^2 - I)^{\mathbb{R}})$ by $\phi(A) = \ker(A - I)^{\mathbb{R}}$. This map clearly has a continuous inverse ψ, namely, for a real subspace, $V \subseteq \ker(X^2 - I)^{\mathbb{R}}$, let $W_0 = V \oplus iV$. Then define

$$\psi(A) = \pi_{W_0} + \sum_l \mu_l \pi_{W_l}. \qquad \square$$

For $V \subseteq W \subset \mathcal{U}$, complex finite dimensional subspaces closed under conjugation, define $U/O(V \oplus V) \to U/O(W \oplus W)$ by sending X to $X \oplus I_{(W-V)\oplus(W-V)}$. Define $E(V) \to E(W)$ by sending A to $A \oplus \pi_{(W-V)\oplus 0}$. Taking the colimit over W, we obtain a map $p: E \to U/O$, which, by arguments completely analogous to those given in the previous sections, is a quasifibration. For $V \subseteq W$, let $BO(V^{\mathbb{R}} \oplus V^{\mathbb{R}}) \to BO(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$ be defined by sending Y to $Y \oplus (W-V)^{\mathbb{R}} \oplus 0$, and define $BO \times \mathbb{Z} = \varinjlim_W BO(W^{\mathbb{R}} \oplus W^{\mathbb{R}})$. Upon stabilization the previous lemma yields that $p^{-1}([X]) \simeq BO \times \mathbb{Z}$, which completes the proof of the real Bott periodicity theorem.

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Update

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Erratum

Addendum to "A new proof of the Bott periodicity theorem" [Topology Appl. 119 (2002) 167–183] ☆

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Abstract

A. Elmendorf has found an error in the approach to Lemmas 2.2 and 2.3 of "A new proof of the Bott periodicity theorem" (Topology and its Applications, 2002, 167–183). There are also errors in the definitions of the maps in Sections 4.2 and 4.5. In this paper we supply corrections to these errors. We also sketch a major simplification of the argument proving real Bott periodicity, unifying the eight quasifibrations appearing in the real case, using Clifford algebras.

MSC: primary 55R45; secondary 55R65

Keywords: Clifford modules; Bott periodicity; Homotopy groups; Spaces and groups of matrices

In Section 1 we make a correction to the definition of a mapping used in Lemmas 2.2 and 2.3 of [2]. The original error was pointed out to the author by Tony Elmendorf. We also correct some flaws in the definition of the maps p_W of Sections 4.2 and 4.5 of [2].

We also take this opportunity to explain how each of the eight quasifibrations arising in the approach to real Bott periodicity given in [2] may be unified, in the context of Clifford algebras. This has the added benefit of explaining real Bott periodicity in terms of the periodicity of Clifford modules, and directly links our approach to work of Atiyah et al. [1]. Each of the quasifibrations of [2] is the instance of a general quasifibration relating certain spaces of Clifford structures. So while we are providing corrections to Sections 4.2

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and 4.5, we are also inviting the reader to skip Section 4 of [2] altogether in favor of the Clifford algebra approach given in this note.

In Section 2, we introduce the spaces of Clifford extensions $X(n,\mathcal{U})$, and explain how they may be identified with the various homogeneous spaces which appear in the real Bott periodicity theorem. In Section 3, we outline a proof following the methods of [2] that there is a quasifibration

$$X(n+1,\mathcal{U}) \to E(n,\mathcal{U}) \xrightarrow{p} X(n,\mathcal{U}).$$

These spaces $E(n,\mathcal{U})$ will be contractible, thus proving Bott periodicity. Each of the separate arguments of [2] are special cases of this general argument. Section 2 is independent of [2]. Section 3 may be read as a terse proof of the real and complex Bott periodicity theorems, with the exception of occasional references to specific arguments given in [2].

1. Corrections to [2]

1.1. The definition of $\Gamma_{W,V}$ in Section 2

Tony Elmendorf has pointed out to the author that the definition of the map

$$\Gamma_{W,V} : \mathcal{I}(W,V) \to \mathrm{Map}\big(G(W), G(V)\big)$$

preceding Lemma 2.2 is not sound. Here W and V are countably infinite dimensional inner product spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The space $\mathcal{I}(W,V)$ is the space of linear isometries from W to V. The spaces $G(W)$ and $G(V)$ are the groups of finite type isometric linear automorphisms of W and V, respectively. These groups are isomorphic to O, U, or Sp, depending on the ground ring.

The problem is that if V is infinite dimensional, then given an infinite subspace V_0 of V, the containment

$$V_0 \oplus V_0^{\perp} \subseteq V \tag{1.1}$$

is *not* necessarily an equality. Elmendorf points out that if one takes $V = \mathbb{R}^{\infty}$ with orthonormal basis $\{e_i\}$, then for the subspace V_0 spanned by $\{e_i + e_{i+1}\}$, the containment (1.1) is not an equality. Of course, (1.1) is an equality if V_0 is finite dimensional. The definition of $\Gamma_{W,V}$ given in [2] incorrectly relied on (1.1) always being an equality.

We give a correct definition of $\Gamma_{W,V}$. The finite type assumption implies that given an element X of $G(W)$, there exists a finite dimensional subspace $W_0 \subseteq W$ and a transformation $X_0 \in G(W_0)$, so that

$$X = X_{W_0} \oplus I_{W_0^{\perp}}$$

under the orthogonal decomposition $W = W_0 \oplus W_0^{\perp}$. Then, given a linear isometry $\phi : W \to V$, the induced element $\phi_*(X)$ is given by

$$\phi_*(X) = \phi_{W_0} X \phi_{W_0}^{-1} \oplus I_{\phi(W_0)^{\perp}}$$

under the orthogonal decomposition $V = \phi(W_0) \oplus \phi(W_0)^{\perp}$. The definition of $\phi_*(X)$ is easily seen to be independent of the choice of W_0. With this definition of ϕ_*, Lemmas 2.2 and 2.3 hold.

1.2. The definition of the map p_W of Section 4.2

The definition of the map $p_W : E(W) \to O/U(W)$ preceding Proposition 4.5 is incorrect, as it is not compatible with the proof of Lemma 4.9. Here $E(W)$ was defined by

$$E(W) = \{A \mid A \text{ is conjugate linear and } \sigma(A) \subseteq [-i, i]\} \subseteq \mathfrak{o}(W).$$

We recall the statement of Lemma 4.9 for the reader's convenience.

Lemma 4.9 of [2]. *Suppose that $W \subset \mathcal{U}$ is a finite dimensional quaternionic space. Let X be a special representative of the class $[X] \in SO/U(W)$. Then $p_W^{-1}([X]) = U/Sp(\ker(X^2 - I))$.*

In the proof of Lemma 4.9, it is used that $p_W(A)$ is a special representative of $[X]$, but the factor of i in the definition of p_W makes this assertion false.

The map $p_W : E(W) \to O/U(W)$ should be defined by

$$p_W(A) = \left[j \exp\left(\frac{\pi}{2} A\right)\right]$$

which we are regarding as an element of the *right* coset space $O/U(W)$.

The following lemma is proved by the same algebraic manipulations that prove Lemma 4.6 of [2].

Lemma 1.1. *Suppose that Y and Z in $O(W)$ satisfy $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$. Then there is an $X \in U(W)$ such that $jY = XZ$ if and only if $-Y^2 = Z^2$.*

The proof of Lemma 4.9 of [2] then proceeds as written, since our new definition of p_W combined with Lemma 1.1 implies that $p_W(A) = X$ if and only if $-\exp(\pi A) = X^2$.

1.3. The definition of the map p_W of Section 4.5

In the sentence immediately following the proof of Proposition 4.17 of [2], "$U/O(W)$" should be replaced with "$Sp/U(W)$".

The definition of the map $p_W : E(W) \to Sp/U(W)$ of Section 4.5 suffers the same deficiency as in Section 4.2, and this deficiency is fixed in exactly the same manner. Namely, the map p_W is not defined correctly to make the proof of Lemma 4.20 work correctly. We recall the statement of this lemma.

Lemma 4.20 of [2]. *Let $W \subset \mathcal{U}$ be a finite dimensional right quaternionic subspace. For a special representative X of $[X] \in Sp/U(W)$, we have $p_W^{-1}([X]) \cong U/O(\ker(X^2 - I))$.*

The definition of the map p_W immediately preceding Proposition 4.17 of [2] should altered to read

$$p_W(A) = \left[j \exp\left(\frac{\pi}{2} A \right) \right]$$

which we are regarding as an element of the *right* coset space $Sp/U(W)$.

One has the following lemma, analogous to Lemma 1.1. (Recall that in Section 4.5 of [2], the group $Sp(W)$ was defined to be the collection of all *right* quaternion linear isometries of W, and $U(W)$ was the subgroup of right quaternion linear, left complex linear isometries.)

Lemma 1.2. *Suppose that Y and Z in $Sp(W)$ satisfy $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$. Then there is an $X \in U(W)$ such that $jY = XZ$ if and only if $-Y^2 = Z^2$.*

Then, in the proof of Lemma 4.20, the new definition of p_W together with Lemma 1.2, implies that $p_W(A) = [X]$ if and only if $-\exp(\pi A) = X^2$, and the rest of the proof proceeds as written.

2. Spaces of Clifford structures

We now explain how the ad hoc methods of Section 4 of [2] may be united in the context of Clifford algebras. Fix a real inner product space W. Let C_n be the Clifford algebra generated by $\mathbb{R}^n$ with the standard metric. It is a real algebra on generators $e_1, \ldots, e_n$ subject to the relations

$$e_i^2 = -1,$$
$$e_i e_j = -e_j e_i, \quad i \neq j.$$

Define a *C_n-structure* on W to be an (ungraded) C_n-module structure over $\mathbb{R}$ such that the generators e_i act by isometries. If W is given a C_n-structure, let $O_{C_n}(W) \subseteq O(W)$ be the collection of isometries of W which preserve the C_n-structure.

Suppose that W is given a C_{n-1} structure. A C_n *extension* is a C_n-structure which restricts to the given C_{n-1}-structure under the inclusion $C_{n-1} \hookrightarrow C_n$. Observe that to give a C_n-extension is to give an isometry e_n of W such that

$$e_n^2 = -I_W,$$
$$e_i e_n = -e_n e_i, \quad 0 \leqslant i < n.$$

Let $X(n, W)$ be the space of C_n-extensions on W, thought of as a subspace of $O(W)$. The group $O_{C_{n-1}}(W)$ acts on $X(n, W)$ by means of conjugation. Given $Y \in O_{C_{n-1}}(W)$, and $e_n \in X(n, W)$, the action is given by

$$Y : e_n \mapsto Y e_n Y^{-1}.$$

Clearly, the stabilizer of e_n in $O_{C_{n-1}}(W)$ is $O_{C_n}(W)$, so the e_n orbit is given by

$$X(n, W)_{e_n} = O_{C_{n-1}}(W)/O_{C_n}(W_{e_n})$$

where W_{e_n} is given the C_n-structure corresponding to the C_n-extension e_n.

Given a C_n-structure on W, the module W breaks up into an orthogonal direct sum of irreducible C_n-submodules

$$W = W_1 \oplus \cdots \oplus W_k.$$

We define $\dim_{C_n}(W)$ to be the number k.

If e_n and f_n are two C_n-extensions for which the C_n-modules W_{e_n} and W_{f_n} are isomorphic, then there exists an isometry $Y \in O_{C_{n-1}}(W)$ so that

$$Ye_n = f_nY.$$

It follows that f_n is in the orbit of e_n. If $n \not\equiv 3 \pmod 4$, then C_n has only one isomorphism class of irreducible modules. Thus, we have

Lemma 2.1. *If $n \not\equiv 3 \pmod 4$, then given any C_n-extension e_n, we have*

$$X(n, W)_{e_n} = X(n, W).$$

Suppose that we have $n \equiv 3 \pmod 4$. Then the various e_n-orbits correspond to the path components of $X(n, W)$. Define a volume element $\omega \in C_n$ by

$$\omega = e_1 \cdots e_n.$$

Then $\omega^2 = 1$, and W breaks up as the orthogonal direct sum of its $+1$ and -1 eigenspaces under ω-multiplication.

$$W = W^+ \oplus W^-.$$

Let $\mathcal{U}$ be a (countable infinite dimensional) real inner product space with a C_n-structure which contains countably many copies of each irreducible C_n-module as a direct summand. We shall call such a $\mathcal{U}$ a *complete C_n-universe*. Define spaces

$$X(n,\mathcal{U}) = \varinjlim X(n, W)$$

where the colimit is taken over finite dimensional C_n-submodules W of $\mathcal{U}$ by extending by the given C_n-extension e_n.

We introduce one last bit of notation. Suppose that $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $\mathbb{K}(n)$ denote the algebra of $n \times n$ matrices with entries in $\mathbb{K}$. Let $\pi_1 \in \mathbb{K}(n)$ be the projection onto the first component. Its matrix has a 1 in the $(1, 1)$-position, and zeroes elsewhere. We shall denote the image $\pi_1(W)$ by W/n.

Table 1 explains why the spaces $X(n,\mathcal{U})$ are important. They are the various loop spaces of $BO \times \mathbb{Z}$. Note that our use of the complete universe is necessary so that $X(3,\mathcal{U}) = BSp \times \mathbb{Z}$ and $X(7,\mathcal{U}) = BO \times \mathbb{Z}$.

We remark that this analysis carries over to the complex case to simultaneously prove complex Bott periodicity. One just replaces all real inner product spaces with complex inner product spaces, and the Clifford algebras C_n with their complex analogs $C_n^{\mathbb{C}}$. The corresponding spaces $X^{\mathbb{C}}(n,\mathcal{U})$ are also given in Table 1.

Observe that there are Morita equivalence homeomorphisms

$$X(n, W) \approx X(n + 8, 16W)$$

Table 1
The spaces $X(n,\mathcal{U})$

n	C_n	$O_{C_n}(W)$	$X(n,W)$	$X(n,\mathcal{U})$
0	$\mathbb{R}$	$O(W)$	–	–
1	$\mathbb{C}$	$U(W)$	$O(W)/U(W)$	O/U
2	$\mathbb{H}$	$Sp(W)$	$U(W)/Sp(W)$	U/Sp
3	$\mathbb{H}\oplus\mathbb{H}$	$Sp(W^+)\times Sp(W^-)$	$BSp(W)$	$BSp\times\mathbb{Z}$
4	$\mathbb{H}(2)$	$Sp(W/2)$	$Sp(W^-)$	Sp
5	$\mathbb{C}(4)$	$U(W/4)$	$Sp(W/2)/U(W/4)$	Sp/U
6	$\mathbb{R}(8)$	$O(W/8)$	$U(W/4)/O(W/8)$	U/O
7	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$O(W/8^+)\times O(W/8^-)$	$BO(W/8)$	$BO\times\mathbb{Z}$
8	$\mathbb{R}(16)$	$O(W/16)$	$O(W/8^-)$	O
n	$C_n^{\mathbb{C}}$	$U_{C_n^{\mathbb{C}}}(W)$	$X^{\mathbb{C}}(n,W)$	$X^{\mathbb{C}}(n,\mathcal{U})$
0	$\mathbb{C}$	$U(W)$	–	–
1	$\mathbb{C}\oplus\mathbb{C}$	$U(W^+)\times U(W^-)$	$BU(W)$	$BU\times\mathbb{Z}$
2	$\mathbb{C}(2)$	$U(W/2)$	$U(W^-)$	U

which will yield Bott periodicity. We also remark that we may extend the definition of our spaces of Clifford extensions to $X(-n,W)$ for $n\geqslant 0$. If $C_{p,q}$ is the Clifford algebra generated by $\mathbb{R}^{p+q}$ with the standard inner product of type (p,q), then for a space W with a $C_{0,n+1}$-structure, we define $X(-n,W)$ to be the space of $C_{1,n+1}$-extensions on W.

One could also work with $\mathbb{Z}/2$-graded modules instead of ungraded modules. Everything we have done would go through with a degree shift. Note that graded C_n-modules are the same thing as ungraded $C_{n,1}$-modules.

3. The general quasifibration

We will prove the following theorem, which is Bott periodicity.

Theorem 3.1. *Let $\mathcal{U}$ be a complete C_{n+1}-universe. Then there exists a quasifibration*

$$X(n+1,\mathcal{U})\to E(n,\mathcal{U})\xrightarrow{p}X(n,\mathcal{U})$$

whose total space is contractible. Therefore there is a weak equivalence

$$\Omega X(n,\mathcal{U})\simeq X(n+1,\mathcal{U}).$$

The quasifibration p of Theorem 3.1 is the colimit of a collection of maps

$$p_W: E(n,W)\to X(n,W)$$

for each finite dimensional C_{n+1}-submodule W of $\mathcal{U}$. Define $E(n,W)$ as space of skew-symmetric transformations

$$E(n,W)=\bigl\{A\in\mathfrak{o}(W)\colon\ \sigma(A)\subseteq[-i,i],\ e_nA=-Ae_n,\ e_iA=Ae_i,\ 1\leqslant i<n\bigr\}.$$

Here $\sigma(A)$ is the spectrum of A, thinking of it as an element of $\mathfrak{u}(W\otimes_{\mathbb{R}}\mathbb{C})$. Note that the commutation relations we have imposed on elements of $E(n,W)$ force them to lie in the orthogonal complement of the Lie algebra $\mathfrak{o}_{C_n}(W)$ in $\mathfrak{o}_{C_{n-1}}(W)$.

Define the map $p_W : E(n, W) \to X(n, W)$ by

$$p_W : A \mapsto -\exp\left(\frac{\pi}{2}A\right)e_n \exp\left(\frac{\pi}{2}A\right)^{-1}.$$

Observe that $e_{n+1}e_n$ may be regarded as an element of $E(n, W)$, and that we have

$$\begin{aligned} p_W(e_{n+1}e_n) &= -\exp\left(\frac{\pi}{2}e_{n+1}e_n\right)e_n \exp\left(\frac{\pi}{2}e_{n+1}e_n\right) \\ &= -\exp(\pi e_{n+1}e_n)e_n \\ &= e_n. \end{aligned}$$

The last equality follows from the fact that $(e_{n+1}e_n)^2 = -I$, so the eigenvalues of $e_{n+1}e_n$ are contained in $\{\pm i\}$.

For any C_{n+1}-space V contained in $W^\perp$, define inclusions $\iota_{W,V} : X(n, W) \hookrightarrow X(n, W \oplus V)$ and $\tilde{\iota}_{W,V} : E(n, W) \hookrightarrow E(n, W \oplus V)$ which for $f_n \in X(n, W)$ and $A \in E(n, W)$, are given by

$$\begin{aligned} &\iota_{W,V} : f_n \mapsto f_n \oplus e_n|_V, \\ &\tilde{\iota}_{W,V} : A \mapsto A \oplus e_{n+1}e_n|_V. \end{aligned}$$

These inclusions are compatible with p_W, so that we may define

$$p : E(n, \mathcal{U}) \to X(n, \mathcal{U})$$

to be the colimit of the maps p_W.

We now endeavor to identify the fiber of p_W. Note that for $A \in E(n, W)$, the matrix $Y = \exp(\frac{\pi}{2}A)$ has the properties:

$$\begin{aligned} &e_iY = Ye_i, \quad 1 \leqslant i < n, \\ &e_nY = Y^{-1}e_n. \end{aligned}$$

The first property implies that p_W takes values in $X(n, W)$. The second property allows us to apply the following trivial lemmas.

Lemma 3.2. *Suppose that Y and Z in $O(W)$ satisfy $e_nY = Y^{-1}e_n$ and $e_nZ = Z^{-1}e_n$. Then we have $-Ye_nY^{-1} = -Ze_nZ^{-1}$ if and only if $Y^2 = Z^2$.*

Lemma 3.3. *Given f_n in $X(n, W)$, we have $p_W(A) = f_n$ if and only if A satisfies* $\exp(\pi A) = f_ne_n$.

Proof. Given an element f_n of $X(n, W)$, such that $f_n = -Ye_nY^{-1}$ we may recover $Y^2 = f_ne_n$. Thus the lemma follows from Lemma 3.2. □

Lemma 3.4. *For f_n an element of $X(n, W)_{e_n}$, the fiber of p_W over f_n is given by*

$$p_W^{-1}(f_n) = X\big(n + 1, \ker(e_n - f_n)\big).$$

Here $\ker(e_n - f_n) \subset W$ *is a C_n-submodule with respect to the given C_n-structure on W.*

Proof. Regarding the matrix $f_n e_n$ as an element of $U(W \otimes_{\mathbb{R}} \mathbb{C})$, it has a spectral decomposition into a sum of projections

$$f_n e_n = -\pi_V + \sum_l \lambda_l \pi_{V_l}$$

where $V = \ker(f_n e_n + I) = \ker(e_n - f_n)$ and $\lambda_l \neq -1$. Let A be an element of $p_W^{-1}(f_n)$. By Lemma 3.3, we have $f_n e_n = \exp(\pi A)$. Regarding A as an element of $\mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$, it has a spectral decomposition

$$A = i\pi_{V'} - i\pi_{V''} + \sum_l \mu_l \pi_{V_l}$$

where μ_l are the unique elements of $(-i, i)$ for which $e^{\pi\mu_l} = \lambda_l$ and $V' \oplus V'' = V$. It follows that when restricted to V, $A^2 = -I$. One easily checks that given this and the commutation relations associated to being an element of $E(n, W)$, the transformation $f_{n+1} = e_n A$ is a C_{n+1}-extension on $V = \ker(e_n - f_n)$.

Conversely, given $f_{n+1} \in X(n+1, V)$, then $(f_{n+1}e_n)^2 = -I_V$, so on V the transformation $f_{n+1}e_n$ has a spectral decomposition of the form $i\pi_{V'} - i\pi_{V''}$ where $V = V' \oplus V''$. We then define the corresponding $A \in p_W^{-1}(f_n)$ by

$$A = i\pi_{V'} - i\pi_{V''} + \sum_l \mu_l \pi_{V_l}$$

where the μ_l are given as before. $\square$

Observe that elements of $X(n, \mathcal{U})$ may be regarded as C_n-extensions f_n on $\mathcal{U}$ for which there exists a finite dimensional subspace $W(f_n, e_n)$ such that

$$W(f_n, e_n)^\perp = \ker(e_n - f_n).$$

We shall say that such a C_n-structure f_n is *virtually equivalent* to e_n. Note that virtual equivalence is an equivalence relation. We have shown that the map $p : E(n, \mathcal{U}) \to X(n, \mathcal{U})$ has fibers

$$p^{-1}(f_n) = X\bigl(n+1, \ker(e_n - f_n)\bigr) = X\bigl(n+1, W(e_n, f_n)^\perp\bigr)$$

for f_n virtually equivalent to e_n.

Remark. The map p surjects onto the path component of e_n, using the fact that path components of $O_{C_{n-1}}(W)/O_{C_n}(W)$ are geodesically complete. If $f_n \in X(n, \mathcal{U})$ is in the image of p, then $\ker(e_n - f_n)$ will admit a C_{n+1}-extension which is the restriction of a C_{n+1} extension on $\mathcal{U}$ which is virtually equivalent to e_{n+1}. In fact, if $f_n = -Y e_n Y^{-1}$, for Y having the property that $e_i Y = Y e_i$ for $1 \leqslant i < n$ and $e_n Y = Y^{-1} e_n$, then $f_{n+1} = -Y e_{n+1} Y^{-1}$ is such a C_{n+1}-extension on $\mathcal{U}$, for which $\ker(e_n - f_n)$ and $W(e_n, f_n)$ are C_{n+1}-submodules. The space $X(n+1, \ker(e_n - f_n))$ is the space of C_{n+1}-extensions on $\ker(e_n - f_n)$ which are virtually equivalent to f_{n+1}.

We will apply the Dold–Thom theorem to prove that p is a quasifibration, thus completing the proof of Theorem 3.1. Define a filtration on $X(n, \mathcal{U})_{e_n}$ by setting

$$F_k X(n, \mathcal{U})_{e_n} = \bigl\{f_n\colon \dim_{C_{n+1}} W(f_n, e_n) \leqslant k\bigr\}.$$

The proof that the filtration annuli $F_kX(n,\mathcal{U}) - F_{k-1}X(n,\mathcal{U})$ are distinguished follows the same line of argument as Lemma 3.3 of [2]. The essential point is that for finite dimensional C_{n+1}-spaces W with a C_{n+1}-subspace V, the projection

$$O_{C_n}(W)/O_{C_{n+1}}(V) \times O_{C_n}\bigl(V^{\perp}\bigr) \to O_{C_n}(W)/O_{C_n}(V) \times O_{C_n}\bigl(V^{\perp}\bigr)$$

is a fibration.

We may define neighborhoods N_k of $F_{k-1}X(n,\mathcal{U})$ in $F_kX(n,\mathcal{U})$ by

$$N_k = \{f_n\colon \dim_{C_{n+1}} \mathrm{Eig}_{\exp(i\pi[-1/2,1/2])} f_n e_n < k\}$$

where the eigenspace is given the C_{n+1}-extension f_{n+1} as in the preceding remark.

Letting $f:[-i,i] \to [-i,i]$ be the function given by

$$f(x) = \begin{cases} -i, & \mathrm{Im}(x) < -1/2, \\ 2x, & -1/2 \leqslant \mathrm{Im}(x) \leqslant 1/2, \\ i, & \mathrm{Im}(x) > 1/2. \end{cases}$$

Then f is homotopic to Id rel $\{-i,i\}$. Let H be such a homotopy and define $h: S^1 \times I \to S^1$ so that the following diagram commutes.

$$\begin{tikzcd} {[-i,i]} \arrow[r, "H_t"] \arrow[d, "e^{\pi(\cdot)}"'] & {[-i,i]} \arrow[d, "e^{\pi(\cdot)}"] \\ S^1 \arrow[r, "h_t"'] & S^1 \end{tikzcd}$$

Then the functional calculus (see the discussion preceding Lemma 3.4 of [2]) gives a homotopy $H_t: E(n,\mathcal{U}) \to E(n,\mathcal{U})$ which covers $h_t: X(n,\mathcal{U}) \to X(n,\mathcal{U})$ by

$$H_t: A \mapsto H_t(A),$$

$$h_t: f_n \mapsto -h_t(f_n e_n)e_n.$$

The hypotheses of the Dold–Thom theorem require that the induced map $H_0: p^{-1}(f_n) \to p^{-1}(h_0(f_n))$ induces a homotopy equivalence on fibers. This follows from the following lemma.

Lemma 3.5. *Suppose that W and V are orthogonal finite-dimensional C_{n+1}-subspaces of $\mathcal{U}$. Then the map*

$$X\bigl(n+1, (V \oplus W)^{\perp}\bigr) \to X\bigl(n+1, W^{\perp}\bigr)$$

given by $f_{n+1} \mapsto f_{n+1} \oplus e_{n+1}|_V$ is a weak equivalence.

Proof. Since the spaces $X(n+1,\mathcal{V})$ are given as homogeneous spaces involving the groups O, U, or Sp (see Table 1), this theorem follows directly from Lemma 2.3 of [2]. □

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