HANDSAW QUIVER VARIETIES AND FINITE W-ALGEBRAS

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ABSTRACT. Following Braverman-Finkelberg-Feigin-Rybnikov (arXiv:1008.3655), we study the convolution algebra of a handsaw quiver variety, a.k.a. a parabolic Laumon space, and a finite W-algebra of type A. This is a finite analog of the AGT conjecture on 4-dimensional supersymmetric Yang-Mills theory with surface operators. Our new observation is that the \mathbb{C}^* -fixed point set of a handsaw quiver variety is isomorphic to a graded quiver variety of type A, which was introduced by the author in connection with the representation theory of a quantum affine algebra. As an application, simple modules of the W-algebra are described in terms of IC sheaves of graded quiver varieties of type A, which were known to be related to Kazhdan-Lusztig polynomials. This gives a new proof of a conjecture by Brundan-Kleshchev on composition multiplicities on Verma modules, which was proved by Losev, in a wider context, by a different method.

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Introduction

Alday, Gaiotto and Tachikawa proposed a conjecture relating an N=2 4-dimensional supersymmetric Yang-Mills theory and a 2-dimensional conformal field theory [1]. This AGT conjecture predicts an existence of an action of a Virasoro algebra on the direct sum of the equivariant cohomology group of framed moduli spaces of SU(2)-instantons on \mathbb{R}^4 with various instanton numbers.

Rigorously speaking, we need to use the Uhlenbeck partial compactification and its equivariant intersection cohomology group, but we will ignore this point in the introduction.

There are lots of variants of this conjecture. When SU(2) is replaced by a compact simple Lie group K, the Virasoro algebra is replaced by the W-algebra associated with the Langlands dual of the corresponding affine Lie algebra (((Lie K) $^{\mathbb{C}}$) $^{\mathrm{aff}}$) L . If instantons have singularities along x-axis (with surface operators in physics terminology), the W-algebra is replaced by its

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variant associated with a nilpotent orbit in $((\text{Lie }K)^{\mathbb{C}})^{L1}$, describing the type of singularities. A proof of this conjecture for K = SU(n) without surface operators will be given by Maulik-Okounkov [18].

In a finite analog of the AGT conjecture with surface operators, one considers spaces of based maps from \mathbb{P}^1 to a partial flag variety G/P in the geometric side, while in the representation theory side the finite W-algebra $W(\mathfrak{g}, e)$ associated with the principal nilpotent element e in the Lie algebra of the Levi subgroup of P appears. This conjecture is proved recently by Braverman, Feigin, Finkelberg and Rybnikov for G = GL(N) with arbitrary P [6], using Brundan-Kleshchev's presentation of the finite W-algebra via generators and relations [7].

The AGT conjecture is close to the author's study of quiver varieties and their relation to representation theory of Kac-Moody Lie algebras [19, 20] and their quantum loop algebras [22]. In fact, quiver varieties (of an affine type) are framed moduli spaces of Γ -equivariant U(N)-instantons on \mathbb{R}^4 , where Γ is a finite subgroup of SU(2). In this theory, Γ cannot be the trivial group, as their is no corresponding affine Kac-Moody Lie algebra under the McKay correspondence. Hence it cannot be applied to the setting of the AGT conjecture².

The space of based maps has a quiver description in type A, hence is called a handsaw quiver variety by Finkelberg-Rybnikov [10]. A trivial, but crucial new observation in this paper, is that the \mathbb{C}^* -fixed point set of a handsaw quiver variety is isomorphic to a graded quiver variety of type A, which is also the \mathbb{C}^* -fixed point set of a quiver variety. This observation is useful as we can apply various results in graded quiver varieties to handsaw quiver varieties.

As an example of such applications, we give a formula of the composition multiplicity of a simple module in a Verma module of the finite W-algebra of type A in terms of intersection cohomology groups of graded quiver varieties. On the other hand, the same intersection cohomology groups control the composition multiplicity of representations of quantum affine algebras of type A [22] and Yangians [30]. Then this composition multiplicity is equal to a certain Kazhdan-Lusztig polynomial evaluated at 1 thanks to Arakawa's result [2]. This gives a new proof of a conjecture by Brundan-Kleshchev [8, Conj. 7.17], which was proved earlier by Losev [17, Th. 4.1, Th. 4.3], in a wider context, by a different method.

Other applications, such as a handsaw version of a geometric realization of tensor products [23], which gives the Miura transform of a finite W-algebra, generalizations to chainsaw quiver varieties, will be discussed in a subsequent publication.

The paper is organized as follows: in $\S 1$ we recall Brundan-Kleshchev's results on the representation theory of finite W-algebras. In $\S 2$ we introduce handsaw quiver varieties and explain their basic properties. In $\S 3$ we give a self-contained proof of the fact that handsaw quiver varieties are isomorphic to the parabolic Laumon spaces. In $\S 4$ we study torus fixed points in handsaw quiver varieties and compute Betti numbers as an application. In $\S 5$ we introduce a subvariety in the product of two handsaw quiver varieties, which we call a Hecke correspondence. It will give generators of the finite W-algebra. In $\S 6$ we introduce a Steinberg type variety, and define its convolution algebra. We modify the construction in [6] to define a homomorphism from the finite W-algebra to the convolution algebra. In $\S 7$ we identify standard modules of the convolution algebra with Verma modules of the finite W-algebra. In the course of the proof, we give a geometric interpretation of the Gelfand-Tsetlin character of standard

¹This is only a rough idea, and a precise formulation of the conjecture is not understood yet even in physics literature. The author thanks Yuji Tachikawa for an explanation of this problem.

²One exception is that it is known that the Heisenberg algebra acts on the cohomology of U(1)-instanton moduli spaces on \mathbb{R}^4 (= Hilbert schemes of points in \mathbb{C}^2) (see [21]). It suggests that the W-algebra for \mathfrak{gl}_1 should be the Heisenberg algebra.

modules in the same spirit as a geometric interpretation of q-characters via graded quiver varieties. In §8 we give the composition multiplicity formula in terms of intersection cohomology sheaves of graded quiver varieties.

Notation. For a linear map $\alpha: V \to W$, its transpose $W^* \to V^*$ is denoted by ${}^t\alpha$.

Let X be a variety endowed with an action of a connected reductive group G. The equivariant Borel-Moore homology group of X is denoted by

$$H_*^G(X)$$
.

where the coefficient field is \mathbb{C} . It is a module over the equivariant cohomology $H_G^*(\mathrm{pt})$ of a single point, which is isomorphic to the algebra of G-invariant polynomial functions on the Lie algebra \mathfrak{g} of G. We denote $H_G^*(\operatorname{pt})$ by S(G), and its field of fractions by S(G).

The character of \mathbb{C}^* given by the identity map is denoted by q. We usually omit the symbol \otimes for a tensor product of q and a representation or an equivariant vector bundle.

1. Shifted Yangians and finite W-algebras

We fix notation on shifted Yangian and finite W-algebras following [8] in this section. We slightly change terminologies following [22] so that analogy with quantum affine algebras will be apparent.

1(i). **Pyramid.** We fix a pyramid π of level l, that is, a sequence of positive integers q_1, \ldots, q_l such that

$$q_1 \le \dots \le q_k, \quad q_{k+1} \ge \dots \ge q_l$$

for some fixed integer $0 \le k \le l$. We set $n = \max(q_1, \dots q_l)$. We define a sequence $0 \le p_1 \le l$

 $p_2 \leq \cdots \leq p_n = l$ so that p_i is the number of entries q_c with $q_c \geq n - i + 1$. We draw the pyramid as a diagram consisting of q_i boxes in the i^{th} column. See Figure 1. Then p_i is the i^{th} row length, that is, the number of boxes in the i^{th} row.

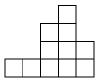


FIGURE 1. Pyramid for $\pi = (1, 1, 3, 4, 2)$

Let N be the total number of boxes, i.e., $N = \sum p_i = \sum q_i$. For 0 < i < n we set

$$s_{i+1,i} \stackrel{\text{def.}}{=} \#\{c = 1, \dots, k \mid n - q_c = i\},\$$

 $s_{i,i+1} \stackrel{\text{def.}}{=} \#\{c = k + 1, \dots, l \mid n - q_c = i\}.$

We then define the shift matrix $\sigma = (s_{ij})_{1 \leq i,j \leq n}$ associated with π so that $s_{ii} = 0$ and

$$s_{ij} + s_{jk} = s_{ik}$$

whenever |i-j|+|j-k|=|i-k|.

Note that the pyramid is recovered from the shift matrix σ and the level $l = p_n$.

1(ii). **Generators and relations.** Suppose that a shift matrix $\sigma = (s_{ij})$ is given. Let $I = \{1, 2, \ldots, n-1\}$, $\widetilde{I} = I \sqcup \{n\}$. The shifted Yangian $Y_n(\sigma)$ is a \mathbb{C} -graded algebra with generators $D_i^{(r)}$ $(i \in \widetilde{I}, r > 0)$, $E_i^{(r)}$ $(i \in I, r > s_{i,i+1})$, $F_i^{(r)}$ $(i \in I, r > s_{i+1,i})$, subjected to the following relations:

$$[D_i^{(r)}, D_j^{(s)}] = 0,$$

(1.2)
$$[E_i^{(r)}, F_j^{(s)}] = \delta_{ij} \sum_{t=0}^{r+s-1} \widetilde{D}_i^{(t)} D_{i+1}^{(r+s-1-t)},$$

(1.3)
$$[D_i^{(r)}, E_j^{(s)}] = (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} D_i^{(t)} E_j^{(r+s-1-t)},$$

(1.4)
$$[D_i^{(r)}, F_j^{(s)}] = (\delta_{i,j+1} - \delta_{ij}) \sum_{t=0}^{r-1} F_j^{(r+s-1-t)} D_i^{(t)},$$

$$[E_i^{(r)}, E_i^{(s+1)}] - [E_i^{(r+1)}, E_i^{(s)}] = E_i^{(r)} E_i^{(s)} + E_i^{(s)} E_i^{(r)},$$

$$[F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] = F_i^{(r)} F_i^{(s)} + F_i^{(s)} F_i^{(r)},$$

$$[E_i^{(r)}, E_{i+1}^{(s+1)}] - [E_i^{(r+1)}, E_{i+1}^{(s)}] = -E_i^{(r)} E_{i+1}^{(s)},$$

$$[F_i^{(r+1)}, F_{i+1}^{(s)}] - [F_i^{(r)}, F_{i+1}^{(s+1)}] = -F_{i+1}^{(s)} F_i^{(r)},$$

(1.9)
$$[E_i^{(r)}, E_i^{(s)}] = 0 \quad \text{if } |i - j| > 1,$$

$$[F_i^{(r)}, F_j^{(s)}] = 0 \quad \text{if } |i - j| > 1,$$

$$[E_i^{(r)}, [E_i^{(s)}, E_i^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_i^{(t)}]] = 0 \quad \text{if } |i - j| = 1,$$

$$[F_i^{(r)}, [F_i^{(s)}, F_i^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_i^{(t)}]] = 0 \quad \text{if } |i - j| = 1,$$

where $D_i^{(0)}=1$ and $\widetilde{D}_i^{(r)}$ is determined recursively by the relation $\sum_{t=0}^r D_i^{(t)} \widetilde{D}_i^{(r-t)}=\delta_{r0}$. It is known that two shifted Yangians $Y_n(\sigma)$, $Y_n(\dot{\sigma})$ are isomorphic under an explicit ho-

It is known that two shifted Yangians $Y_n(\sigma)$, $Y_n(\dot{\sigma})$ are isomorphic under an explicit homomorphism [8, §2.3] when σ and $\dot{\sigma}$ satisfy $s_{i,i+1} + s_{i+1,i} = \dot{s}_{i,i+1} + \dot{s}_{i+1,i}$. This condition is satisfied when σ and $\dot{\sigma}$ come from pyramids π , $\dot{\pi}$ which have the same row lengths.

1(iii). Rees algebra. Let us define higher root elements inductively by

$$\begin{split} E_{i,i+1}^{(r)} &\stackrel{\text{def.}}{=} E_i^{(r)}, \quad E_{i,j}^{(r)} &\stackrel{\text{def.}}{=} [E_{i,j-1}^{(r-s_{j-1,j})}, E_{j-1}^{(s_{j-1,j}+1)}], \\ F_{i,i+1}^{(r)} &\stackrel{\text{def.}}{=} F_i^{(r)}, \quad F_{i,j}^{(r)} &\stackrel{\text{def.}}{=} [F_{j-1}^{(s_{j,j-1}+1)}, F_{i,j-1}^{(r-s_{j,j-1})}]. \end{split}$$

We introduce a filtration $F_0Y_n(\sigma) \subset F_1Y_n(\sigma) \subset \cdots$ on $Y_n(\sigma)$ so that $F_dY_n(\sigma)$ is the span of all monomials in $F_{i,j}^{(r)}$, $F_{i,j}^{(r)}$, $D_i^{(r)}$ of degree $\leq d$, where we assign degree r for all $F_{i,j}^{(r)}$, $F_{i,j}^{(r)}$, $D_i^{(r)}$. Then $Y_n(\sigma)$ is a filtered algebra.

Let $Y_n^{\hbar}(\sigma)$ be the associated Rees algebra, i.e.,

$$Y_n^{\hbar}(\sigma) \stackrel{\text{def.}}{=} \bigoplus_{d>0} \hbar^d F_d Y_n(\sigma).$$

It is a graded algebra over $\mathbb{C}[\hbar]$. We denote $\hbar^r E_i^{(r)} \in \hbar^r F_r Y_n(\sigma)$, considered as an element in $Y_n^{\hbar}(\sigma)$, by $E_i^{(r)}$.

The higher root vector in $Y_n^{\hbar}(\sigma)$ is given by

(1.13)
$$\hbar E_{i,j}^{(r)} = \left[E_{i,j-1}^{(r-s_{j-1,j})}, E_{j-1}^{(s_{j-1,j}+1)} \right]$$

in $Y_n^{\hbar}(\sigma)$.

The specialization $Y_n^{\hbar}(\sigma)/\hbar Y_n^{\hbar}(\sigma) = Y_n^{\hbar}(\sigma) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}$ at $\hbar = 0$ is the graded algebra gr $Y_n(\sigma) = \bigoplus_{d \geq 0} F_d Y_n(\sigma)/F_{d-1} Y_n(\sigma)$, where we set $F_{-1} Y_n(\sigma) = 0$. It is known that gr $Y_n(\sigma)$ is the free commutative algebra on generators $\operatorname{gr}_r E_{i,j}^{(r)}(s_{i,j} < r)$, $\operatorname{gr}_r F_{i,j}^{(r)}(s_{j,i} < r)$, $\operatorname{gr}_r D_i^{(r)}(0 < r)$ ([7, Th. 5.2]).

Remark 1.14. In [6] the Rees algebra with respect to the shifted Kazhdan filtration $F'_dY_n(\sigma) \stackrel{\text{def.}}{=} F_{d+1}Y_n(\sigma)$ is considered. However it seems to the author that one of the defining relations [6, (3.2)] is not compatible with this grading.

1(iv). **Finite** W-algebra. By [7] the finite W-algebra $W(\mathfrak{gl}_N, e)$ associated with a nilpotent matrix e whose Jordan type is given by the partition (p_n, \ldots, p_1) is isomorphic to the quotient of $Y_n(\sigma)$ divided by the two sided ideal generated by $\{D_1^{(r)} \mid r > p_1\}$. Following [8] we denote this quotient by $W(\pi)$, where π is the pyramid corresponding to σ and p_1 (or p_n).

The isomorphism $Y_n(\sigma) \cong Y_n(\dot{\sigma})$ mentioned above descends to quotients, so the W-algebra depends only on the conjugacy class of e up to an isomorphism.

The filtration $F_{\bullet}Y_n(\sigma)$ induces a filtration on $W(\pi)$, which is the same as the Kazhdan filtration [7]. We denote by $W^{\hbar}(\pi)$ the corresponding Rees algebra.

It is known that $W^{\hbar}(\pi)/\hbar W^{\hbar}(\pi)$ is isomorphic to the polynomial algebra in variables $\operatorname{gr}_r E_{i,j}^{(r)}$ $(s_{i,j} < r \le S_{ij}), \operatorname{gr}_r F_{i,j}^{(r)}$ $(s_{j,i} < r \le S_{j,i}), \operatorname{gr}_r D_i^{(r)}$ $(s_{i,i} < r \le S_{i,i})$ where $S_{i,j} = s_{i,j} + p_{\min(i,j)}$ ([7, Th. 6.2]). It is further isomorphic to the coordinate ring of the Slodowy slice to the nilpotent orbit through e.

When e = 0 (i.e., $p_1 = \cdots = p_n = 1$), it is known that $W(\mathfrak{gl}_N, e)$ is the universal enveloping algebra $U(\mathfrak{gl}_N)$ of \mathfrak{gl}_N . On the other hand, if e is regular nilpotent (i.e., $p_1 = \cdots = p_{n-1} = 0$, $p_n = N$), $W(\mathfrak{gl}_N, e)$ is the center of $U(\mathfrak{gl}_N)$.

1(v). Center of $W(\pi)$. Let

(1.15)
$$D_{i}(u) \stackrel{\text{def.}}{=} \sum_{r=0}^{\infty} D_{i}^{(r)} u^{-r},$$

$$C_{i}(u) \equiv \sum_{r=0}^{\infty} C_{i}^{(r)} u^{-r} \stackrel{\text{def.}}{=} D_{1}(u) D_{2}(u-1) \cdots D_{i}(u-i+1),$$

$$Z_{N}(u) \stackrel{\text{def.}}{=} u^{p_{1}} (u-1)^{p_{2}} \cdots (u-n+1)^{p_{n}} C_{n}(u).$$

We follow the notation in [8], and the above are denoted by $d_i(u)$, $a_i(u)$, $A_n(u)$ respectively in [6].

It was shown that the elements $C_n^{(1)}$, $C_n^{(2)}$, ... are algebraically independent and generate the center $Z(Y_n(\sigma))$ of the shifted Yangian $Y_n(\sigma)$ [8, Th. 2.6].

It was also shown that $Z_N(u)$ is a polynomial of degree $N = p_1 + \cdots + p_n$ in $W(\pi)$, and the center $Z(W(\pi))$ of $W(\pi)$ is a polynomial algebra in generators $Z_N^{(1)}, \ldots, Z_N^{(N)}$, the coefficients of $Z_N(u)$ [8, Th. 6.10]. (See also [8, Lem. 3.7] to identify $Z_N(u)$ with the above definition).

1(vi). Admissible modules and ℓ -weight spaces. Let \mathfrak{c} be the abelian Lie algebra generated by $D_1^{(1)}, \ldots, D_n^{(1)}$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the basis of \mathfrak{c}^* dual to $D_1^{(1)}, \ldots, D_n^{(1)}$. Let \leq be the dominance order on \mathfrak{c}^* defined by

$$\alpha \ge \beta \iff \alpha - \beta \in \sum_{i} \mathbb{Z}_{\ge 0}(\varepsilon_i - \varepsilon_{i+1}).$$

A $Y_n(\sigma)$ -module M is said to be *admissible* if we have a generalized weight space decomposition $M = \bigoplus_{\alpha \in \mathfrak{c}^*} M_{\alpha}$, where

$$M_{\alpha} = \left\{ m \in M \mid (D_i^{(1)} - \langle \alpha, D_i^{(1)} \rangle)^N m = 0 \text{ for } i \in \widetilde{I} \text{ and sufficiently large } N \right\}$$

satisfying the conditions

- (1) each M_{α} is finite-dimensional,
- (2) the set of all $\alpha \in \mathfrak{c}^*$ such that $M_{\alpha} \neq 0$ is contained in a finite union of sets of the form $D(\beta) = \{\alpha \in \mathfrak{c}^* \mid \alpha \leq \beta\}$ for $\beta \in \mathfrak{c}^*$.

Since $D_i^{(r)}$ $(i \in \widetilde{I}, r > 0)$ form a commutative subalgebra of $Y_n(\sigma)$, an admissible module M has a finer decomposition $M = \bigoplus M_{\Psi}$ where

$$\Psi = (\Psi_1(u), \dots, \Psi_n(u)), \quad \Psi_i(u) = 1 + \sum_{r=1}^{\infty} \Psi_i^{(r)} u^{-r} \in 1 + u^{-1} \mathbb{C}[[u^{-1}]],$$

$$M_{\Psi} = \left\{ m \in M \mid (D_i^{(r)} - \Psi_i^{(r)})^N m = 0 \text{ for } i \in \widetilde{I}, r > 0 \text{ and sufficiently large } N \right\}.$$

We call M_{Ψ} the ℓ -weight space following [22, §1.3] (or the Gelfand-Tsetlin subspace following [8, §5]) with the ℓ -weight Ψ .

It turns out that it is more appropriate to consider another sequence of generators instead of $D_i^{(r)}$ for the W-algebra. We introduce

(1.16)
$$\mathsf{A}_{i}(u) \stackrel{\text{def.}}{=} u^{p_{1}}(u-1)^{p_{2}} \cdots (u-(i-1))^{p_{i}} C_{i}(u)$$

following [12, 6]. Then it is a polynomial in u in $W(\pi)$ [12, Lem. 2.1]. Instead of $D_i^{(r)}$, we can define an ℓ -weight space as a simultaneous generalized eigenspace for operators $A_i(u)$ and the corresponding ℓ -weight as an \widetilde{I} -tuple of rational functions $Q = (Q_1(u), Q_2(u), \ldots, Q_n(u))$ as generalized eigenvalues of $A_i(u)/A_{i-1}(u)$ ($i \in \widetilde{I}$), where $A_{-1}(u) = 1$.

Following [25, Def. 2.8] we say an ℓ -weight Q is ℓ -dominant if every $Q_i(u)$ is a polynomial.

1(vii). **Gelfand-Tsetlin character.** The generating function of dimensions of ℓ -weight spaces of an admissible $Y_n(\sigma)$ -module M is called the *Gelfand-Tsetlin character* after [8, §5.2]:

$$\operatorname{ch} M \stackrel{\operatorname{def.}}{=} \sum_{\Psi} \dim M_{\Psi} [\Psi].$$

This is an element of the completed group algebra, consisting of formal sums satisfying the conditions corresponding to the admissibility. See [8, §5.2] for detail.

For a $W(\pi)$ -module M, we use the second convention for ℓ -weights and write

$$\operatorname{ch} M = \sum_{Q} \dim M_{Q} [Q].$$

This definition was motivated by q-characters of finite dimensional representations of quantum affine algebras, introduced in [14, 11], and further studied in [22, §13.5]. Later in §7(ii) we

will show that this is not just an analogy: both Gelfand-Tsetlin and q-characters are expressed in terms of a common geometric object, a graded quiver variety.

1(viii). ℓ -highest weight module. A vector v in a $Y_n(\sigma)$ -module M is called an ℓ -highest weight vector with ℓ -highest weight Ψ if

(1)
$$E_i^{(r)}v = 0$$
 for $i \in I$, $r > s_{i,i+1}$,
(2) $D_i^{(r)}v = \Psi_i^{(r)}v$ for $i \in \widetilde{I}$, $r > 0$,

(2)
$$D_i^{(r)}v = \Psi_i^{(r)}v \text{ for } i \in \widetilde{I}, r > 0,$$

where $\Psi_i^{(r)}$ is a coefficient of $\Psi_i(u)$ as above.

An ℓ -highest weight module is a $Y_n(\sigma)$ -module M generated by an ℓ -highest weight vector. From the triangular decomposition of $Y_n(\sigma)$, we can construct the universal ℓ -highest weight module $M(\sigma, \Psi)$ with ℓ -highest weight Ψ [8, §5].

We consider the quotient

$$W(\pi) \otimes_{Y_n(\sigma)} M(\sigma, \Psi).$$

By [8, Th. 6.1], it is nonzero if and only if $u^{p_i}\Psi_i(u) \in \mathbb{C}[u]$ for $i \in \widetilde{I}$. When it is nonzero, we call it a Verma module. We set $P_i(u) = (u-i+1)^{p_i}\Psi_i(u-i+1)$ and call $P = (P_1(u), \ldots, P_n(u))$ the *Drinfeld polynomial* of the Verma module $M(P) \stackrel{\text{def.}}{=} W(\pi) \otimes_{Y_n(\sigma)} M(\sigma, \Psi)$. It has the unique simple quotient, which is denoted by L(P). It is known that both M(P) and L(P) are admissible $[8, \S 6.1]$.

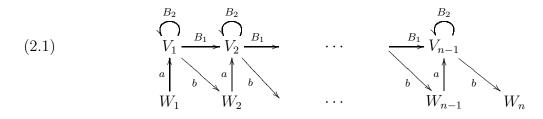
This convention is compatible with our definition of ℓ -weights. The ℓ -highest weight vector v as above has the ℓ -weight P in the above second sense. Note that it is ℓ -dominant since every $P_i(u)$ is a polynomial.

2. Handsaw Quiver Varieties

In this section, we introduce handsaw quiver varieties following [10], and explain various their properties. Handsaw quiver varieties are isomorphic to Laumon spaces (see §3), and most of results are known in the context of Laumon spaces. But we present them in terms of handsaw quivers so that they can be used in later sections. Proofs are usually the same as ones of ordinary quiver varieties and are not given. The readers are supposed to be familiar with [20, §3].

2(i). **Definition.** Let $n \in \mathbb{Z}_{>0}$ be as in the previous section. Recall $I = \{1, 2, \dots, n-1\}$, $\widetilde{I} = I \sqcup \{n\}.$

We take I and \widetilde{I} -graded vector spaces $V = \bigoplus_{i \in I} V_i$, $W = \bigoplus_{i \in \widetilde{I}} W_i$ and consider the representation of the following handsaw quiver



where

$$B_1 \in \bigoplus_{i=1}^{n-2} \operatorname{Hom}(V_i, V_{i+1}), \quad B_2 \in \bigoplus_{i=1}^{n-1} \operatorname{End}(V_i),$$

$$a \in \bigoplus_{i=1}^{n-1} \operatorname{Hom}(W_i, V_i), \quad b \in \bigoplus_{i=1}^{n-1} \operatorname{Hom}(V_i, W_{i+1}).$$

We denote by \mathbf{M} the vector space of all quadruples (B_1, B_2, a, b) as above. When we want to emphasize graded dimensions of V, W, we denote it by $\mathbf{M}(\mathbf{v}, \mathbf{w})$, where $\mathbf{v} \in \mathbb{Z}[I]$, $\mathbf{w} \in \mathbb{Z}[\widetilde{I}]$ are dimension vectors given by $\mathbf{v} = \sum_{i \in I} \dim V_i \alpha_i$, $\mathbf{w} = \sum_{i \in \widetilde{I}} \dim W_i \alpha_i$, where α_i is the i^{th} coordinate vector of $\mathbb{Z}[I]$ or $\mathbb{Z}[\widetilde{I}]$.

In a later relation to the finite W-algebra $W(\pi)$, we have

$$\dim W_i = p_i$$

We introduce the following notation:

$$\mathbf{L}(W, V) \stackrel{\text{def.}}{=} \bigoplus_{i} \text{Hom}(W_i, V_i), \quad \mathbf{E}(V, W) \stackrel{\text{def.}}{=} \bigoplus_{i} \text{Hom}(V_i, W_{i+1}).$$

We also write $\bigoplus_i \operatorname{Hom}(V_i, V_{i+1})$ and $\bigoplus_i \operatorname{End}(V_i)$ as $\mathbf{L}(V, V)$, $\mathbf{E}(V, V)$ respectively by setting $V_n = 0$.

We define

$$\mu(B_1, B_2, a, b) = [B_1, B_2] + ab \in \mathbf{E}(V, V),$$

and consider an affine variety $\mu^{-1}(0)$. It is acted by the group $G = \prod_{i \in I} GL(V_i)$.

In order to consider quotients of $\mu^{-1}(0)$ by G, we introduce the following conditions coming from the geometric invariant theory (cf. [20, §3.ii]):

Definition 2.2. (1) A point $(B_1, B_2, a, b) \in \mathbf{M}$ is called *stable* if there is no proper *I*-graded subspace $S = \bigoplus S_i$ of V stable under B_1 , B_2 and containing a(W).

(2) A point $(B_1, B_2, a, b) \in \mathbf{M}$ is called *costable* if there is no nonzero *I*-graded subspace *S* of *V* stable under B_1, B_2 and contained in Ker *b*.

These conditions are invariant under the G-action. We set

$$\mathcal{Q} \equiv \mathcal{Q}(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \{ (B_1, B_2, a, b) \in \mu^{-1}(0) \mid \text{stable} \} / G,$$

$$\mathcal{Q}_0 \equiv \mathcal{Q}_0(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \mu^{-1}(0) / \!\!/ G,$$

$$\mathcal{Q}_0^{\text{reg}} \equiv \mathcal{Q}_0^{\text{reg}}(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \{ (B_1, B_2, a, b) \in \mu^{-1}(0) \mid \text{stable and costable} \} / G,$$

where $/\!\!/$ denote the affine quotient. When we want to emphasize \mathbf{v} , \mathbf{w} as we will do later, we use the notation $\mathcal{Q}(\mathbf{v}, \mathbf{w})$, and \mathcal{Q} otherwise. These are called *handsaw quiver varieties*.

For a stable $(B_1, B_2, a, b) \in \mu^{-1}(0)$, the corresponding point in \mathcal{Q} is denoted by $[B_1, B_2, a, b]$. A (closed) point in $\mathcal{Q}_0(\mathbf{v}, \mathbf{w})$ is represented by a closed G-orbit in $\mu^{-1}(0)$. When (B_1, B_2, a, b) has a closed orbit, the corresponding point in \mathcal{Q}_0 is denoted by $[B_1, B_2, a, b]$.

We have a projective morphism $\pi: \mathcal{Q} \to \mathcal{Q}_0$. The second space $\mathcal{Q}_0^{\text{reg}}$ is a (possibly empty) open subscheme in both \mathcal{Q} and \mathcal{Q}_0 so that π is an isomorphism between them.

We denote $\pi^{-1}(0)$ by \mathcal{O} , where 0 is the closed orbit consisting of $(B_1, B_2, a, b) = (0, 0, 0, 0)$. This is a projective variety, and will played an important role later.

We have an action of \mathbb{C} on \mathbf{M} by the translation of B_2 : $B_2 \mapsto B_2 - \sum z \operatorname{id}_{V_i}$ $(z \in \mathbb{C})$. It preserves the equation $\mu = 0$ and commutes with the G-action. Therefore we have an induced

 \mathbb{C} -action on \mathcal{Q} , $\mathcal{Q}_0^{\text{reg}}$. We can normalize a point so that $\operatorname{tr} B_2 = 0$. Therefore we have a decomposition $\mathcal{Q} = \mathcal{Q}^0 \times \mathbb{C}$, etc, where \mathcal{Q}^0 is the space of normalized points.

2(ii). \mathbb{C}^* -equivariant sheaves on the projective plane. In this subsection, we describe \mathscr{Q} as a fixed point locus of the framed moduli space of torsion free sheaves (E, φ) on \mathbb{P}^2 of rank r with $c_2(E) = n$ with respect to a \mathbb{C}^* -action, in other words an ordinary quiver variety of Jordan type [21, Ch. 2]. The quiver variety in question corresponds to the original ADHM description of instantons on S^4 [5].

Let V, W be complex vector spaces of dimension n, r respectively. Let

$$\widetilde{\mathbf{M}} \stackrel{\mathrm{def.}}{=} \mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W).$$

An element in $\widetilde{\mathbf{M}}$ is a quadruple (B_1, B_2, a, b) according to the above decomposition. The group $\mathrm{GL}(V)$ acts naturally on $\widetilde{\mathbf{M}}$. We define $\mu(B_1, B_2, a, b) = [B_1, B_2] + ab$ as before.

We define the stability and costability conditions exactly as above, where $S \subset V$ is an arbitrary subspace, not necessarily graded.

These conditions are invariant under the GL(V)-action. We set

$$M(n,r) \stackrel{\text{def.}}{=} \{(B_1, B_2, a, b) \in \mu^{-1}(0) \mid \text{stable}\}/\text{GL}(V),$$

$$M_0(n,r) \stackrel{\text{def.}}{=} \mu^{-1}(0) /\!\!/ \text{GL}(V),$$

$$M_0^{\text{reg}}(n,r) \stackrel{\text{def.}}{=} \{(B_1, B_2, a, b) \in \mu^{-1}(0) \mid \text{stable and costable}\}/\text{GL}(V).$$

The framed moduli space of locally free sheaves is $M^{\text{reg}}(n,r)$, and M(n,r), $M_0(n,r)$ are its Gieseker and Uhlenbeck partial compactification respectively.

We choose and fix a homomorphism $\rho_W \colon \mathbb{C}^* \to \mathrm{GL}(W)$. We define a \mathbb{C}^* -action on $\widetilde{\mathbf{M}}$ by

$$(2.3) (B_1, B_2, a, b) \mapsto (t_1 B_1, B_2, a \rho_W(t_1)^{-1}, t_1 \rho_W(t_1) b) \quad t_1 \in \mathbb{C}^*.$$

The equation $\mu(B_1, B_2, a, b) = 0$ and the (co)stability are preserved under the \mathbb{C}^* -action, and hence we have an induced \mathbb{C}^* -actions on M(n, r), $M_0(n, r)$, $M^{\text{reg}}(n, r)$.

Take a point $x \in M(n,r)$ and its representative (B_1, B_2, a, b) . Then x is fixed by the \mathbb{C}^* -action if and only if there exists a $\rho(t_1) \in GL(V)$ such that

$$(2.4) (t_1B_1, B_2, a\rho_W(t_1)^{-1}, t_1\rho_W(t_1)b) = (\rho(t_1)^{-1}B_1\rho(t_1), \rho(t_1)^{-1}B_2\rho(t_1), \rho(t_1)^{-1}a, b\rho(t_1))$$

for any $t_1 \in \mathbb{C}^*$. By the freeness of the GL(V)-action on the stable locus, $\rho(t_1)$ is uniquely determined by t_1 . In particular, the map $t_1 \mapsto \rho(t_1)$ is a homomorphism. Its conjugacy class is independent of the choice of the representative of x.

We decompose V, W as $\bigoplus V_i$, $\bigoplus W_i$ where $t_1 \in \mathbb{C}^*$ acts by t_1^i on V_i , W_i . Then (2.4) means that (B_1, B_2, a, b) is a representation of the handsaw quiver (2.1), where we shift the index i suitably so that it runs from 1 to n. Thus \mathscr{Q} is a fixed point locus in M(n, r) with a given conjugacy class of a homomorphism $\rho \colon \mathbb{C}^* \to \mathrm{GL}(V)$. The same is true for $\mathscr{Q}^{\mathrm{reg}}$.

We do not compare \mathcal{Q}_0 with $M_0(n,r)^{\mathbb{C}^*}$ directly, as we are only intereseted in the image of \mathcal{Q} under π in practice.

This relation allows us to apply results on M(n,r), $M_0^{\text{reg}}(n,r)$ to \mathcal{Q} , $\mathcal{Q}_0^{\text{reg}}$. For example, \mathcal{Q} is smooth as M(n,r) is so. And the action of G on the open locus consisting of stable points in $\mu^{-1}(0)$ is free, and the quotient by G is a principal G-bundle, etc.

Remark 2.5. A quiver variety of type A_n [19], which is associated with a quiver

$$V_{1} \xrightarrow{B_{2}} V_{2} \xrightarrow{B_{2}} \cdots \xrightarrow{B_{2}} V_{n}$$

$$\downarrow \downarrow \uparrow a \qquad \downarrow \downarrow \uparrow a \qquad \downarrow \downarrow \uparrow a$$

$$W_{1} \qquad W_{2} \qquad \cdots \qquad W_{n}$$

is also a fixed component of M(n,r) with respect to a \mathbb{C}^* -action, but a different one:

$$(B_1, B_2, a, b) \mapsto (t_1 B_1, t_1^{-1} B_2, a \rho_W(t_1)^{-1}, \rho_W(t_1) b) \quad t_1 \in \mathbb{C}^*.$$

Note also that we exchange the stability and costability from those in [19, 20]. This change is not essential, as they are exchanged if we replace all linear maps by their transposes.

Remark 2.6. In [10] a chainsaw quiver variety is introduced as a fixed point locus of $\mathfrak{M}(n,r)$ with respect to the cyclic group action defined by the same formula as (2.3) for a homomorphism $\rho_W \colon \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(W)$. Many of results in this section and those in §4 are generalizations to handsaw quiver varieties.

2(iii). Tangent space. We consider the complex

(2.7)
$$\mathbf{L}(V,V) \xrightarrow{\sigma} \mathbf{E}(V,V) \oplus \mathbf{L}(V,V) \\ \oplus \\ \mathbf{L}(W,V) \oplus \mathbf{E}(V,W) \xrightarrow{\tau} \mathbf{E}(V,V),$$

where σ and τ are defined by

$$\sigma(\xi) = \begin{pmatrix} \xi B_1 - B_1 \xi \\ \xi B_2 - B_2 \xi \\ \xi a \\ -b \xi \end{pmatrix}, \quad \tau \begin{pmatrix} C_1 \\ C_2 \\ A \\ B \end{pmatrix} = [B_1, C_2] + [C_1, B_2] + aB + Ab.$$

This σ is the differential of G-action and τ is the differential of the map μ . It is the \mathbb{C}^* -fixed part of the complex considered in [26, (2.13)]. It follows that σ is injective and τ is surjective, and the tangent space of \mathcal{Q} at $[B_1, B_2, a, b]$ is isomorphic to the middle cohomology group of the above complex.

As an application we have

$$\dim \mathcal{Q} = \dim \mathbf{L}(W, V) + \dim \mathbf{E}(V, W) = \sum_{i} \dim V_{i} (\dim W_{i} + \dim W_{i+1}).$$

As we remarked above, we have a natural principal G-bundle over \mathcal{Q} from the construction. Since V_i is a representation of G, we have an induced vector bundle over \mathcal{Q} , which we also denote by V_i for brevity. We also consider W_i as a trivial bundle. Then $\mathbf{L}(V,V)$, $\mathbf{E}(V,V)$, etc are vector bundles over \mathcal{Q} , and τ , σ are vector bundle homomorphisms. Therefore $\ker \tau / \operatorname{Im} \sigma$ is also a vector bundle, which is isomorphic to the tangent bundle of \mathcal{Q} .

2(iv). **Stratification.** According to a local structure at a closed orbit $x = [B_1, B_2, a, b]$, we have a natural stratification of $\mathcal{Q}_0(\mathbf{v}, \mathbf{w})$, similar to one for a quiver variety [19, §6].

As a first step of the stratification, we have $V = V' \oplus V''$ such that the stabilizer subgroup acts trivially on V''. The restriction of (B_1, B_2, a, b) to V'' defines a point in $\mathscr{Q}_0^{\text{reg}}(\mathbf{v}'', \mathbf{w})$, where \mathbf{v}'' is the dimension vector of V''. The restriction of (B_1, B_2, a, b) to V' is a datum with W = 0. Thus we have a = 0 = b. We take a homomorphism $\mathbb{C}^* \to \prod \operatorname{GL}(V_i')$ given by $t \mapsto \prod t^i \operatorname{id}_{V_i'}$, and consider the limit $t \mapsto 0$. Then the closedness of the orbit implies

 $B_2|_{V'}=0$. Therefore we only need to consider the closed orbit $B_1|_{V'_i}\in \operatorname{End}(V'_i)$ with respect to the $\operatorname{GL}(V_i)$ -action. Closed orbits of $\operatorname{End}(V'_i)$ under the conjugation are classified by their eigenvalues with multiplicities, and hence points in the symmetric product $S^{\dim V'_i}\mathbb{C}$. We thus have

$$\mathcal{Q}_0(\mathbf{v}, \mathbf{w}) = \left| \quad \mathcal{Q}_0^{\text{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) \times S^{v_1'} \mathbb{C} \times S^{v_2'} \mathbb{C} \times \cdots \times S^{v_{n-1}'} \mathbb{C}, \right|$$

where $\mathbf{v}' = (v'_1, v'_2, \dots, v'_{n-1}).$

We denote by $S^{\mathbf{v}'}\mathbb{C}$ the product $S^{v'_1}\mathbb{C} \times S^{v'_2}\mathbb{C} \times \cdots \times S^{v'_{n-1}}\mathbb{C}$, and consider it as the configuration space of unordered colored points, where the set of colors is I.

We refine the above stratification by further decomposing the symmetric product part $S^{\mathbf{v}'}\mathbb{C}$ (see [16, §1.3]):

$$S^{\mathbf{v}'}\mathbb{C} = \bigsqcup_{\Gamma} S_{\Gamma}^{\mathbf{v}'}\mathbb{C},$$

where Γ is a collection $\{\mathbf{v}'_1, \dots, \mathbf{v}'_m\}$ of $\mathbb{Z}_{\geq 0}[I] \setminus \{0\}$ with the ordering disregarded, satisfying $\mathbf{v}'_1 + \dots + \mathbf{v}'_m = \mathbf{v}'$. Here the number m is not fixed. The corresponding stratum $S_{\Gamma}^{\mathbf{v}'}\mathbb{C}$ is defined as

$$S_{\Gamma}^{\mathbf{v}'}\mathbb{C} = \left\{ \sum_{\alpha=1}^{m} \mathbf{v}_{\alpha}' x_{\alpha} \middle| x_{\alpha} \neq x_{\beta} \text{ for } \alpha \neq \beta \right\},$$

where $\mathbf{v}'_{\alpha}x_{\alpha}$ is a point x_{α} whose multiplicity for a color $i \in I$ is given by the i^{th} -component of \mathbf{v}'_{α} . We thus get

(2.8)
$$\mathcal{Q}_0(\mathbf{v}, \mathbf{w}) = | \mathcal{Q}_0^{\text{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) \times S_{\Gamma}^{\mathbf{v}'} \mathbb{C}.$$

We have

(2.9)
$$\dim \mathcal{Q}_0^{\text{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) \times S_{\Gamma}^{\mathbf{v}'} \mathbb{C} = \sum_i (\dim V_i - \dim V_i') (\dim W_i + \dim W_{i+1}) + m.$$

From [22, §3.2], which is applicable thanks to §2(ii), a local structure of a neighborhood of $x \in \mathcal{Q}_0(\mathbf{v}, \mathbf{w})$ and $\pi^{-1}(x)$ is determined by the stratum $\mathcal{Q}_0^{\text{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) \times S_{\Gamma}^{\mathbf{v}'}\mathbb{C}$ to which x is contained. Moreover it is described as a local structure of a \mathbb{C}^* -fixed subvariety of products of several copies of quiver varieties of Jordan type around 0 and $\pi^{-1}(0)$. As a particular result, we find that $\pi^{-1}(x)$ is isomorphic to the product

$$\mathscr{O}(\mathbf{v}'_1, \mathbf{w}) \times \cdots \times \mathscr{O}(\mathbf{v}'_m, \mathbf{w})$$

of various central fibers. This was observed in [16, Prop. 2.1.2] in terms of Laumon spaces.

2(v). Group action. Let $G_{\mathbf{w}} = \prod_{i \in \widetilde{I}} GL(W_i)$. It acts on \mathbf{M} by conjugation. It preserves the equation $\mu = 0$ and commutes with the G-action which we have used to define quotients. Therefore we have induced $G_{\mathbf{w}}$ -actions on \mathcal{Q} and \mathcal{Q}_0 .

We also have a \mathbb{C}^* -action induced from

$$(B_1, B_2, a, b) \mapsto (B_1, tB_2, a, tb).$$

We set $\mathbb{G}_{\mathbf{w}} = \mathbb{C}^* \times G_{\mathbf{w}}$.

3. Based maps and Laumon space

Let \mathbb{P}^1 be the projective line and z its inhomogeneous coordinate.

The handsaw variety \mathcal{Q} is isomorphic to the parabolic Laumon space, that is the moduli space of flags

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = W \otimes \mathcal{O}_{\mathbb{P}^1}$$

of locally free sheaves over \mathbb{P}^1 such that rank $E_i = \sum_{j \leq i} \dim W_j$, $\langle c_1(E_i), [\mathbb{P}^1] \rangle = -\dim V_i$, and at $z = \infty \in \mathbb{P}^1$ the infinity of \mathbb{P}^1 equal to $0 \subset W_1 \subset W_1 \oplus W_2 \subset \cdots \subset W_1 \oplus W_2 \oplus \cdots \oplus W_{n-1} \subset W$. We omit the word 'parabolic' from the Laumon space for brevity hereafter.

Remark that the inclusion $E_i \subset E_{i+1}$ is an injective homomorphism of sheaves, and E_i is not necessarily a subbundle of E_{i+1} , where a stronger condition that E_{i+1}/E_i is locally free is required. If we impose this stronger condition, we get the space of based rational maps from \mathbb{P}^1 to the partial flag variety of W. Here based means that ∞ is mapped to the specific point $0 \subset W_1 \subset W_1 \oplus W_2 \subset \cdots \subset W_1 \oplus W_2 \oplus \cdots \oplus W_{n-1} \subset W$. This (possibly empty) open subscheme of Laumon space is isomorphic to \mathcal{Q}^{reg} .

This result is proved in [10, §2.3], and is mentioned that it goes back to Strømme [29]. It also follows from the ADHM description (see §2(ii)) together with Atiyah's observation [3] that the space of based rational maps is a \mathbb{C}^* -fixed point in a framed instanton moduli space by the observation in §2(ii). Since these proofs are combination of results scattered in various literature, we give a self-contained argument in this subsection for the sake of a reader.

It is also possible to describe $\pi: \mathcal{Q} \to \mathcal{Q}_0$ as follows. Consider the inclusion $E_i \subset W \otimes \mathcal{O}_{\mathbb{P}^1}$. The quotient $W \otimes \mathcal{O}_{\mathbb{P}^1}/E_i$ is not necessarily locally free, so we take its locally free part $(W \otimes \mathcal{O}_{\mathbb{P}^1}/E_i)_{l.f.}$ and define

$$\tilde{E}_i = \operatorname{Ker} \left[W \otimes \mathcal{O}_{\mathbb{P}^1} \to (W \otimes \mathcal{O}_{\mathbb{P}^1}/E_i)_{l.f.} \right].$$

Then we have $E_i \subset \tilde{E}_i \subset W \otimes \mathcal{O}_{\mathbb{P}^1}$ and $W \otimes \mathcal{O}_{\mathbb{P}^1}/\tilde{E}_i$ is locally free. We have $\tilde{E}_i \subset \tilde{E}_{i+1}$ and $\tilde{E}_{i+1}/\tilde{E}_i$ is locally free from the construction. Therefore $0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset E_n$ defines a point in \mathscr{Q}^{reg} . The length of \tilde{E}_i/E_i defines a point in the symmetric product $S^n\mathbb{C}$ where $n = \langle c_1(\tilde{E}_i) - c_1(E_i), [\mathbb{P}^1] \rangle$. The identification of this construction and π is left as an exercise for the reader.

3(i). From a handsaw quiver variety to a Laumon space. For $i \in I$ we consider homomorphisms

(3.1)
$$\alpha_i \colon (W_1 \oplus W_2 \oplus \cdots \oplus W_i \oplus V_i) \otimes \mathcal{O}_{\mathbb{P}^1} \to V_i \otimes \mathcal{O}_{\mathbb{P}^1}(1),$$
$$\beta_i \colon (W_1 \oplus W_2 \oplus \cdots \oplus W_i \oplus V_i) \otimes \mathcal{O}_{\mathbb{P}^1} \to (W_1 \oplus W_2 \oplus \cdots \oplus W_{i+1} \oplus V_{i+1}) \otimes \mathcal{O}_{\mathbb{P}^1},$$

defined by

$$\alpha_i \stackrel{\text{def.}}{=} (B_1^{i-1}a, B_1^{i-2}a, \dots, a, z - B_2), \qquad \beta_i \stackrel{\text{def.}}{=} \bigoplus_{j \le i} \operatorname{id}_{W_j} \oplus \begin{pmatrix} b \\ B_1 \end{pmatrix}.$$

The equation $\mu = 0$ is equivalent to $B_1 \alpha_i = \alpha_{i+1} \beta_i$.

We consider linear maps $\alpha_i|_z$, $\beta_i|_z$ between fibers of vector bundles at a point $z \in \mathbb{P}^1$.

Lemma 3.2. (1) (B_1, B_2, a, b) is stable if and only if $\alpha_i|_z$ is surjective for all i. (2) (B_1, B_2, a, b) is costable if and only if $\operatorname{Ker} \alpha_i|_z \cap \operatorname{Ker} \beta_i|_z$ is zero for all i.

Proof. (1) We first prove the \Rightarrow direction.

We set ${}^{i}S_{i}$ as the image of $\alpha_{i}|_{z}$ and consider it as a subspace of V_{i} . We set all other ${}^{i}S_{j}$ $(j \neq i)$ as V_{i} , and define a graded subspace ${}^{i}S = \bigoplus_{j=1}^{i} S_{j}$. It contains a(W). We have

$$B_2 \alpha_i = \alpha_i \begin{pmatrix} z \operatorname{id} & 0 \\ -B_1^{i-1} a & \cdots & -a & B_2 \end{pmatrix}.$$

This implies ${}^{i}S$ is invariant under B_2 .

We show that iS is invariant under B_1 by the induction on i. It is clear for i=1. Suppose that it is true for i-1. Then the stability of (B_1, B_2, a, b) implies that ${}^{i-1}S = V$. It means that $\alpha_{i-1}|_z$ is surjective. Therefore $B_1\alpha_{i-1} = \alpha_i\beta_{i-1}$ implies $B_1(V_{i-1}) \subset \operatorname{Im} \alpha_i$. Thus iS is also invariant under B_1 . It implies the surjectivity of $\alpha_i|_z$ as we have already noticed.

We next prove the \Leftarrow direction. Suppose that (B_1, B_2, a, b) is not stable and take a proper I-graded subspace $S = \bigoplus S_i$ of V stable under B_1 , B_2 and containing a(W). We consider the annihilator S^{\perp} of S in V^* . Then it is invariant under tB_1 , tB_2 and contained in Ker ta . Consider a nonzero component S_i^{\perp} . Since it is preserved by tB_2 , there exists an eigenvector $0 \neq \phi$ of tB_2 in S_i^{\perp} . If the eigenvalue is z, ${}^t\alpha_i$ has kernel at the point $z \in \mathbb{C} \subset \mathbb{P}^1$. This contradicts the surjectivity of α_i at z. We thus have $S^{\perp} = 0$, i.e., S = V.

(2) Note first that

$$\operatorname{Ker} \alpha_i|_z \cap \operatorname{Ker} \beta_i|_z \cong \operatorname{Ker} \begin{bmatrix} z - B_2 \\ b \\ B_1 \end{bmatrix} : V_i \to V_i \oplus W_{i+1} \oplus V_{i+1}.$$

Let us show the \Rightarrow direction. Suppose that $\operatorname{Ker} \alpha_i|_z \cap \operatorname{Ker} \beta_i|_z$ is nonzero. It means that $S_i \stackrel{\text{def.}}{=} \operatorname{Ker}(z - B_2) \cap \operatorname{Ker} b \cap \operatorname{Ker} B_1$ (in V_i) is nonzero. Setting other spaces $S_j = 0$, we define a graded subspace $S = \bigoplus_j S_j$. It is contained in $\operatorname{Ker} b$ and is invariant under B_1 , B_2 . The costability implies $S_i = 0$. This is a contradiction.

We next prove \Leftarrow direction. Suppose that (B_1, B_2, a, b) is not costable and take a nonzero I-graded subspace $S = \bigoplus S_i$ of V stable under B_1 , B_2 and contained in Ker b. We prove $S_i = 0$ by an induction from above.

Suppose $S_{n-1} \neq 0$. Since it is invariant under B_2 , we have an eigenvector $0 \neq v \in S_{n-1}$ with the eigenvalue z. Then Ker $\alpha_{n-1}|_z \cap \text{Ker } \beta_{n-1}|_z$ is nonzero. This is a contradiction, and hence we must have $S_{n-1} = 0$.

Suppose $S_j = 0$ for j > i and $S_i \neq 0$. Then the same argument as above shows Ker $\alpha_i|_z \cap$ Ker $\beta_i|_z$ is nonzero. This contradicts the assumption, and hence we have $S_i = 0$.

Suppose that (B_1, B_2, a, b) is stable. Thanks to the above lemma, $E_i \stackrel{\text{def.}}{=} \operatorname{Ker} \alpha_i$ is a vector bundle of rank $\sum_{j \leq i} \dim W_j$ with $\langle c_1(E_i), [\mathbb{P}^1] \rangle = -\dim V_i$. Its fiber at $z = \infty$ is identified with $W_1 \oplus \cdots \oplus W_i$.

The homomorphism β_i induces $E_i \to E_{i+1}$. From the proof of Lemma 3.2(2) the corresponding linear map between fibers at a point z is not injective only when z is an eigenvalue of B_2 . Thus it can happen only at finitely many points in \mathbb{P}^1 . Therefore $E_i \to E_{i+1}$ is injective as a sheaf homomorphism. We thus get a flag of locally free sheaves, that is a point in a Laumon space. It is independent on the choice of a point in a G-orbit, and hence a map from \mathcal{Q} to the Laumon space. It also gives a map from \mathcal{Q}^{reg} to the space of based maps by Lemma 3.2(2). We have an universal family on the space \mathcal{Q} , and therefore the above construction gives a morphism from \mathcal{Q} to the Laumon space.

3(ii). From a Laumon space to a handsaw quiver variety. Let us describe the inverse morphism.

From the definition of E_i , we have a short exact sequence

$$0 \to E_i \to (W_1 \oplus \cdots \oplus W_i \oplus V_i) \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\alpha_i} V_i \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

We thus have

$$V_i \cong H^1(\mathbb{P}^1, E_i(-1)).$$

Using this observation, we take this as a definition of V_i for the inverse morphism. Note that $H^0(\mathbb{P}^1, E_i(-1)) = 0$ as E_i is a subsheaf of a trivial sheaf $E_n = W \otimes \mathcal{O}_{\mathbb{P}^1}$. Hence dim V_i is $-\langle c_1(E_i), \mathbb{P}^1 \rangle$ as expected.

We next study $H^1(\mathbb{P}^1, E_i(-2))$ and show that it is isomorphic to $W_1 \oplus \cdots \oplus W_i \oplus V_i$. This is expected from the same short exact sequence.

Considering a short exact sequence

$$0 \to E_i(-2) \to E_i(-1) \to E_i|_{z=\infty} \to 0$$

we get

$$(3.3) 0 \to W_1 \oplus \cdots \oplus W_i \xrightarrow{\varphi} H^1(\mathbb{P}^1, E_i(-2)) \xrightarrow{\psi} V_i = H^1(\mathbb{P}^1, E_i(-1)) \to 0.$$

Since E_i is a subsheaf of $W \otimes \mathcal{O}_{\mathbb{P}^1}$, we have a homomorphism

$$H^1(\mathbb{P}^1, E_i(-2)) \to W \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong W.$$

When we compose it with φ , it gives the inclusion

$$W_1 \oplus \cdots \oplus W_i \to W$$
.

Therefore we compose this with the projection $W \to W_1 \oplus \cdots \oplus W_i$ to get a splitting of (3.3). Therefore we have

$$H^1(\mathbb{P}^1, E_i(-2)) \cong W_1 \oplus \cdots \oplus W_i \oplus V_i.$$

We define $B_1: V_i \to V_{i+1}$ as an induced homomorphism

$$H^1(\mathbb{P}^1, E_i(-1)) \to H^1(\mathbb{P}^1, E_{i+1}(-1))$$

from the inclusion $E_i \to E_{i+1}$.

The multiplication of the inhomogeneous coordinate z induces a homomorphism

$$W_1 \oplus \cdots \oplus W_i \oplus V_i = H^1(\mathbb{P}^1, E_i(-2)) \xrightarrow{\zeta} V_i = H^1(\mathbb{P}^1, E_i(-1)).$$

We set its restriction to W_i and V_i as a and $-B_2$ respectively. Other components for W_1 , ..., W_{i-1} come from the same homomorphisms for E_{i-1} together with B_1 . Then by induction we see that they are $B_1^{i-1}a$, ..., B_1a . In view of the construction in §3(i) this is natural as $\zeta = \alpha_i|_{z=0}$. In fact, E_i is recovered as

Ker
$$(\zeta - z\psi : H^1(\mathbb{P}^1, E_i(-2)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to H^1(\mathbb{P}^1, E_i(-1)) \otimes \mathcal{O}_{\mathbb{P}^1})$$
,

as one can show using the resolution of the diagonal

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathcal{O}_{\Delta} \to 0.$$

(This is a basis of the well-known theorem: the derived category of coherent sheaves on \mathbb{P}^1 is equivalent to that of representations of the Kronecker quiver.) The above definition has been given so that $\zeta - z\psi$ coincides with α_i in the previous subsection.

We next consider the commutative diagram

$$H^{1}(\mathbb{P}^{1}, E_{i}(-2)) \longrightarrow H^{1}(\mathbb{P}^{1}, E_{i+1}(-2))$$

$$\cong \downarrow \qquad \qquad \downarrow \cong$$

$$W_{1} \oplus \cdots \oplus W_{i} \oplus V_{i} \longrightarrow W_{1} \oplus \cdots \oplus W_{i} \oplus W_{i+1} \oplus V_{i+1},$$

where the upper arrow is induced homomorphism from the inclusion $E_i \to E_{i+1}$. The restriction of the lower arrow to $W_1 \oplus \cdots \oplus W_i$ is the inclusion to $W_1 \oplus \cdots \oplus W_i \oplus W_{i+1}$. And the component from $V_i \to V_{i+1}$ is given by B_1 . These come from consideration of (3.3) and the same sequence for E_{i+1} . Let us set the remaining component $V_i \to W_{i+1}$ as b.

We have just defined all linear maps B_1 , B_2 , a, b. They satisfy the equation $\mu = 0$. This follows from the commutativity of the diagram

$$H^{1}(\mathbb{P}^{1}, E_{i}(-2)) \longrightarrow H^{1}(\mathbb{P}^{1}, E_{i+1}(-2))$$

$$\downarrow \qquad \qquad \downarrow$$

$$H^{1}(\mathbb{P}^{1}, E_{i}(-1)) \longrightarrow H^{1}(\mathbb{P}^{1}, E_{i+1}(-1)),$$

where the vertical arrows are given by the multiplication of z.

The stability of (B_1, B_2, a, b) follows from Lemma 3.2(1), as we have already noticed that E_i is recovered as Ker α_i .

If the original $E_1 \subset \cdots \subset E_n$ is a flag of subbundles, then (B_1, B_2, a, b) is costable by Lemma 3.2(2).

We thus have maps from the Laumon space and the based map space to \mathcal{Q} and \mathcal{Q}^{reg} respectively. These are morphisms thanks to a universal family on \mathcal{Q} . They are inverse to morphisms constructed in the previous subsection, as we have checked during the above proof.

4. Fixed points

In this section we study the fixed point set with respect to the action of the maximal torus $\mathbb{T}_{\mathbf{w}}$ of $\mathbb{G}_{\mathbf{w}}$. One of applications is a formula of Betti numbers of $\mathscr{O}(\mathbf{v},\mathbf{w})$, which will be used in a later section. This study is well known in the context of Laumon spaces as in the previous section, but we present it here for completeness.

4(i). **Parametrization.** We fix a decomposition $W_i = \bigoplus_{\alpha} W_i^{\alpha}$ of W_i into sum of 1-dimensional subspaces, and consider the torus T_i of $GL(W_i)$ preserving it. Here α runs from 1 to dim W_i . We set $\mathbb{T}_{\mathbf{w}} = \mathbb{C}^* \times \prod_i T_i$. This is an example of an abelian reductive subgroup of $\mathbb{G}_{\mathbf{w}}$ considered in §2(v). It is maximal among such subgroups.

Let us describe $\mathbb{T}_{\mathbf{w}}$ -fixed points in \mathscr{Q} . Take a point $x \in \mathscr{Q}$ and its representative (B_1, B_2, a, b) . It is fixed by $\prod_i T_i$ if and only if we have a direct sum decomposition

$$V = \bigoplus_{i,\alpha} V_i^{\alpha}, \qquad V_i^{\alpha} = \bigoplus_{j \in I} V_{i;j}^{\alpha}$$

such that V_i^{α} with W_i^{α} is invariant under (B_1, B_2, a, b) . Here W_i^{α} is considered as an \widetilde{I} -graded vector space by setting its i^{th} -component as W_i^{α} and all other components as 0. Then each summand of (B_1, B_2, a, b) corresponding to W_i^{α} is a \mathbb{C}^* -equivariant rank 1-torsion free sheaf on \mathbb{P}^2 , in other words, a \mathbb{C}^* -equivariant ideal sheaf on \mathbb{C}^2 by $\S 2(\text{ii})$.

It is fixed further by the remaining \mathbb{C}^* if and only if the ideal sheaf is $\mathbb{C}^* \times \mathbb{C}^*$ -equivariant, where the action is given by

$$(B_1, B_2, a, b) \mapsto (t_1 B_1, t B_2, a, t_1 t b).$$

It is a monomial ideal and is parametrized by a Young diagram (see e.g., [21, Chapter 5]).

In summary, $\mathbb{T}_{\mathbf{w}}$ -fixed points in \mathcal{Q} are parametrized by a tuple of Young diagrams Y_i^{α} indexed by $i \in \widetilde{I}$ and $\alpha = 1, \ldots, \dim W_i$. We denote by \vec{Y} the tuple of Young diagrams, and consider it as a point in \mathcal{Q} .

When we draw Young diagrams on the plane, it is natural to shift Y_i^{α} to the right from the convention in [21, Chapter 5] so that the x-coordinate of the box at the bottom left corner is i. See Figure 2. The constraint is that the total number of boxes whose x-coordinates are i is equal to $\dim V_i$.

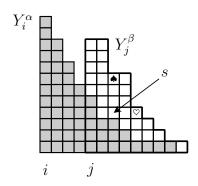


FIGURE 2. A tuple of Young diagrams

Take a box s in the plane as in the figure. We define its leg length $l_{Y_i^{\beta}}(s)$ and arm length $a_{Y_i^{\beta}}(s)$ with respect to the Young diagram Y_j^{β} as follows. We put a \heartsuit mark on the rightmost box in Y_j^{β} in the same row with s. If there is no row in Y_j^{β} , we put a mark on the box whose x-coordinate is (j-1). Then $l_{Y_j^{\beta}}(s)$ is the difference of the x-coordinates of the marked box and s. In the above example, it is equal to 2. Changing rows with columns, we define the arm length in the same way, where we put a \spadesuit mark for the relevant one in the figure. Thus we have $a_{Y_j^{\beta}}(s) = 3$ in the example. These may have negative values if s is not in the Young diagram. In the above example, the leg and arm lengths with respect to Y_i^{α} are $l_{Y_i^{\alpha}}(s) = -2$, $a_{Y^{\alpha}}(s) = -1$ respectively.

4(ii). Character of the tangent space. Let \vec{Y} be a T-fixed point in \mathcal{Q} as above. The tangent space $T_{\vec{Y}}\mathscr{Q}$ of \mathscr{Q} at \vec{Y} is a \mathbb{T} -module. We denote by e_i^{α} the character of $\mathbb{T} \to \mathbb{C}^*$, corresponding to the summand W_i^{α} .

Proposition 4.1. The character of the \mathbb{T} -module $T_{\vec{V}}\mathscr{Q}$ is given by

$$\operatorname{ch} T_{\vec{Y}} \mathcal{Q} = \sum_{(i,\alpha),(j,\beta)} e_j^{\beta} (e_i^{\alpha})^{-1} \left(\sum_{\substack{s \in Y_i^{\alpha} \\ l_{Y_j^{\beta}}(s) = 0}} t^{a_{Y_i^{\alpha}}(s) + 1} + \sum_{\substack{s \in Y_j^{\beta} \\ l_{Y_i^{\alpha}}(s) = -1}} t^{-a_{Y_j^{\beta}}(s)} \right).$$

This is a direct consequence of the formula of the tangent space of M(n,r) at the torus fixed point \vec{Y} (see e.g., [27, Th. 2.11]), as \mathcal{Q} is an union of components of the fixed point locus

 $M(n,r)^{\mathbb{C}^*}$. More precisely, the character of the $\mathbb{C}^* \times \mathbb{T}$ -module $T_{\vec{Y}}M(n,r)$ is given by

$$\sum_{(i,\alpha),(j,\beta)} e_j^{\beta} \left(e_i^{\alpha} \right)^{-1} \left(\sum_{s \in Y_i^{\alpha}} t_1^{-l_{Y_j^{\beta}}(s)} t^{a_{Y_i^{\alpha}}(s)+1} + \sum_{s \in Y_j^{\beta}} t_1^{l_{Y_i^{\alpha}}(s)+1} t^{-a_{Y_j^{\beta}}(s)} \right).$$

And we take its coefficient of t_1^0 to get the above formula.

Let us explain the meaning of an individual term for (i, α) , (j, β) in the sum. The complex (2.7) computing the tangent space of \mathscr{Q} decomposes according to $W = \bigoplus_{i,\alpha} W_i^{\alpha}$, $V = \bigoplus V_i^{\alpha}$:

$$\mathbf{L}(V_i^{\alpha}, V_j^{\beta}) \stackrel{\sigma}{\longrightarrow} \underbrace{\mathbf{E}(V_i^{\alpha}, V_j^{\beta}) \oplus \mathbf{L}(V_i^{\alpha}, V_j^{\beta})}_{\mathbf{L}(W_i^{\alpha}, V_j^{\beta}) \oplus \mathbf{E}(V_i^{\alpha}, W_j^{\beta})} \stackrel{\tau}{\longrightarrow} \mathbf{E}(V_i^{\alpha}, V_j^{\beta}).$$

Let us denote this by $C_{(i,\alpha),(j,\beta)}$, where we set the middle term to be of degree 0. Then $\operatorname{ch} T_{\vec{Y}} \mathscr{Q} = \sum_{(i,\alpha),(j,\beta)} \operatorname{ch} C_{(i,\alpha),(j,\beta)}$. Note that we have

(4.2)
$$\operatorname{rank} C_{(i,\alpha),(j,\beta)} = \dim V_{j;i}^{\beta} + \dim V_{i;j-1}^{\alpha}.$$

One of two terms vanishes according to either $i \geq j$ or i < j.

4(iii). Betti numbers of handsaw quiver varieties. Since \mathcal{Q} has a torus action with isolated fixed points, we can compute Betti numbers by decomposing it into (\pm)-attracting sets.

Let us choose $\lambda \colon \mathbb{C}^* \to \mathbb{T} = \mathbb{C}^* \times \prod_i T_i$ so that the weight of the \mathbb{C}^* -component is 1 and the difference of the weights of W_i^{α} and W_j^{β} -components is sufficiently large if $(i,\alpha) > (j,\beta)$. Here we put the lexicographic order on the set of indices (i,α) , i.e., $(i,\alpha) > (j,\beta)$ if and only if i > j or $i = j, \alpha > \beta$.

Under this choice of λ , every point in \mathcal{Q}_0 converges to 0 as $t \to 0$, and every point except 0 goes to infinity as $t \to \infty$ by the argument in [26, Th. 3.5].

For each fixed point \vec{Y} we consider (\pm) -attracting set:

$$S_{\vec{Y}} = \left\{ x \in \mathcal{Q} \, \middle| \, \lim_{t \to 0} \lambda(t) \cdot x = \vec{Y} \, \right\}, \quad U_{\vec{Y}} = \left\{ x \in \mathcal{Q} \, \middle| \, \lim_{t \to \infty} \lambda(t) \cdot x = \vec{Y} \, \right\}.$$

By the remark above, we have $\sqcup_{\vec{Y}} S_{\vec{Y}} = \mathcal{Q}$ and $\sqcup_{\vec{Y}} U_{\vec{Y}} = \mathcal{O}$. These are affine spaces whose dimensions are equal to sum of dimensions of positive and negative weight spaces of $T_{\vec{Y}}\mathcal{Q}$.

Let us calculate these dimensions by using the formula in Proposition 4.1. From our choice of λ , the sum of the dimensions of positive weight spaces is the number of terms with $(i, \alpha) < (j, \beta)$ or $(i, \alpha) = (j, \beta)$ and the power of t is positive. If $(i, \alpha) = (j, \beta)$, only the first sum in the parentheses survive, and hence the power is positive always. Therefore we have

$$\dim S_{\vec{Y}} = \sum_{(i,\alpha) \le (j,\beta)} \operatorname{rank} C_{(i,\alpha),(j,\beta)}, \qquad \dim U_{\vec{Y}} = \sum_{(i,\alpha) > (j,\beta)} \operatorname{rank} C_{(i,\alpha),(j,\beta)}.$$

By (4.2), we get

(4.3)
$$\dim S_{\vec{Y}} = \sum_{i} \dim V_{i} \dim W_{i+1} + \sum_{(i,\alpha)} \alpha \dim V_{i;i}^{\alpha},$$
$$\dim U_{\vec{Y}} = \sum_{i} \dim V_{i} \dim W_{i} - \sum_{(i,\alpha)} \alpha \dim V_{i;i}^{\alpha}.$$

Let us fix \mathbf{w} and consider Poincaré polynomials of $\mathcal{O}(\mathbf{v}, \mathbf{w})$ for various \mathbf{v} . To consider their generating function we introduce the following notation: Let $e^{\mathbf{v}}$ be an element of the group ring of $\mathbb{Z}[I]$ given by

$$e^{\mathbf{v}} \stackrel{\text{def.}}{=} e^{v_1 \alpha_1 + \dots + v_{n-1} \alpha_{n-1}},$$

where $\mathbf{v} = (v_1, \dots, v_{n-1})$. Then we have the following infinite product expansion.

Theorem 4.4.

$$\sum_{\mathbf{v}} P_t(\mathscr{O}(\mathbf{v}, \mathbf{w})) e^{\mathbf{v}} = \prod_{i \in I} \prod_{\alpha=1}^{\dim W_i} \prod_{j=i}^{n-1} \frac{1}{1 - t^{\dim W_i + \dots + \dim W_j - \alpha} e^{\alpha_i + \dots + \alpha_j}}.$$

When all dim $W_i = 1$, the right hand side is essentially equal to Lusztig's q-analog of Kostant's partition function, as first noticed by Kuznetsov [16].

Corollary 4.5. \mathcal{Q} is connected, provided it is nonempty.

Proof. We study a fixed point \vec{Y} with dim $U_{\vec{Y}} = 0$. From the second equality in (4.3) we have

$$\dim U_{\vec{Y}} = \sum_{(i,\alpha)} \sum_{j=i+1}^{n-1} \dim V_{i,j}^{\alpha} \dim W_j + \sum_{(i,\alpha)} \dim V_{i,i}^{\alpha} (\dim W_i - \alpha).$$

Suppose this vanishes. Since each term is nonnegative, both must vanish. The second term vanishes if and only if $Y_i^{\alpha} = \emptyset$ unless $\alpha = \dim W_i$, as $V_{i;i}^{\alpha} = 0$ implies $V_i^{\alpha} = 0$.

Next consider the first term. Let i^+ be the minimum among numbers j such that $W_j \neq 0$ and j > i. If there is no such numbers, we set $i^+ = n$. Then the first term vanishes if and only if $\dim V_{i;j}^{\alpha} = 0$ for $j \geq i_+$. Together with the above consideration, we find that only $V_i^{\dim W_i}$ contributes to V_j for $i \leq j < i^+$. Thus $Y_i^{\dim W_i}$ is determined from \mathbf{v} . This means the assertion.

4(iv). **Smallness.** We continue the setting of the previous subsection, and assume further the following condition:

$$\dim W_1 \le \dim W_2 \le \dots \le \dim W_n.$$

Then (4.3) implies

(4.7)
$$\dim \mathscr{O} = \max_{\vec{Y}} \dim U_{\vec{Y}} \le \frac{1}{2} \sum_{i} \dim V_{i} (\dim W_{i} + \dim W_{i+1}) - 1 = \frac{1}{2} \dim \mathscr{Q} - 1,$$

unless V = 0.

Theorem 4.8. Under (4.6) the map $\pi: \mathcal{Q} \to \mathcal{Q}_0$ is small.

This was proved in [16] when $\dim W_1 = \cdots = \dim W_n = 1$, and his proof works in general cases. We give a proof for completeness.

Proof. Since we need to treat various stratum, let us specify the dimension vector \mathbf{v} , \mathbf{w} . We consider the stratification (2.8).

Recall that the fiber of a point in the stratum $\mathscr{Q}_0^{\text{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) \times S_{\Gamma}^{\mathbf{v}'}\mathbb{C}$ is isomorphic to $\mathscr{O}(\mathbf{v}'_1, \mathbf{w}) \times \cdots \times \mathscr{O}(\mathbf{v}'_m, \mathbf{w})$. Then (4.7) applied to $\mathscr{Q}(\mathbf{v}'_\alpha, \mathbf{w})$ gives

$$(4.9) \quad \dim \pi^{-1}(x) \leq \frac{1}{2} \left(\dim \mathcal{Q}(\mathbf{v}'_1, \mathbf{w}) + \dots + \dim \mathcal{Q}(\mathbf{v}'_m, \mathbf{w}) \right) - m = \frac{1}{2} \dim \mathcal{Q}(\mathbf{v}', \mathbf{w}) - m$$

if x is a point in the stratum. We thus have

$$\dim \pi^{-1}(\mathscr{Q}_0^{\mathrm{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) \times S_{\Gamma}^{\mathbf{v}'}\mathbb{C}) \leq \dim \mathscr{Q}_0^{\mathrm{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) + \frac{1}{2}\dim \mathscr{Q}(\mathbf{v}', \mathbf{w}).$$

This is strictly smaller than dim $\mathcal{Q}(\mathbf{v}, \mathbf{w})$ unless $\mathbf{v}' = 0$. It means that $\mathcal{Q}_0^{\text{reg}}(\mathbf{v}, \mathbf{w})$ cannot be empty. Hence the dimension of $\mathcal{Q}_0(\mathbf{v}, \mathbf{w})$ is equal to dim $\mathcal{Q}(\mathbf{v}, \mathbf{w})$, and the codimension of the stratum $\mathcal{Q}_0^{\text{reg}}(\mathbf{v} - \mathbf{v}', \mathbf{w}) \times S_{\mathbf{v}}^{\text{reg}}(\mathbb{C})$ is equal to

$$\dim \mathcal{Q}(\mathbf{v}', \mathbf{w}) - m.$$

Therefore the dimension of the fiber $\pi^{-1}(x)$ is smaller than the half of codimension of the stratum unless m=0, i.e., $\mathbf{v}'=0$.

4(v). **Freeness.** We will consider equivariant homology groups of $\mathcal{Q}(\mathbf{v}, \mathbf{w})$ and $\mathcal{O}(\mathbf{v}, \mathbf{w})$ with respect to $\mathbb{G}_{\mathbf{w}}$ -actions later. We state the following:

Theorem 4.10. $H_*^{\mathbb{G}_{\mathbf{w}}}(\mathcal{Q}(\mathbf{v}, \mathbf{w}))$ and $H_*^{\mathbb{G}_{\mathbf{w}}}(\mathcal{O}(\mathbf{v}, \mathbf{w}))$ are free of finite rank as $H_{\mathbb{G}_{\mathbf{w}}}^*(\mathrm{pt}) = S(\mathbb{G}_{\mathbf{w}})$ -modules.

This is a general property of a variety with torus action such that Bialynicki-Birula decomposition gives an α -partition (see e.g., [22, §7.1]).

5. Hecke correspondences

We introduce Hecke correspondences in products of handsaw quiver varieties. They will give generators $E_i^{(r)}$, $F_i^{(r)}$ of the finite W-algebra in §6. These are given essentially in [6], but we reformulate them because

- (1) their analogy to Hecke correspondences [20, §4] for ordinary quiver varieties will become clear;
- (2) a modification from [6] seems more natural.
- 5(i). **Definition.** Fix $i \in I$. Take dimension vectors \mathbf{w} , \mathbf{v}^1 , \mathbf{v}^2 such that $\mathbf{v}^2 = \mathbf{v}^1 + \alpha_i$, and consider the product $\mathcal{Q}(\mathbf{v}^1, \mathbf{w}) \times \mathcal{Q}(\mathbf{v}^2, \mathbf{w})$. We denote by V_i^1 (resp. V_i^2) the vector bundle $V_i \boxtimes \mathcal{O}_{\mathcal{Q}(\mathbf{v}^1, \mathbf{w})}$ (resp. $\mathcal{O}_{\mathcal{Q}(\mathbf{v}^1, \mathbf{w})} \boxtimes V_i$). We regard B_1^p , B_2^p , a^p , b^p (p = 1, 2) as vector bundle homomorphisms between V_i^p , W_i^p 's.

We introduce the following variant of the complex (2.7):

(5.1)
$$\mathbf{L}(V^2, V^1) \xrightarrow{\sigma} \mathbf{E}(V^2, V^1) \oplus \mathbf{L}(V^2, V^1) \\ \oplus \\ \mathbf{L}(W, V^1) \oplus \mathbf{E}(V^2, W) \xrightarrow{\tau} \mathbf{E}(V^2, V^1),$$

where σ and τ are defined by

$$\sigma(\xi) = \begin{pmatrix} \xi B_1^2 - B_1^1 \xi \\ \xi B_2^2 - B_2^1 \xi \\ \xi a^2 \\ -b^1 \xi \end{pmatrix}, \quad \tau \begin{pmatrix} C_1 \\ C_2 \\ A \\ B \end{pmatrix} = B_1^1 C_2 - C_2 B_1^2 + C_1 B_2^2 - B_2^1 C_1 + a^1 B + A b^2.$$

This is a complex thanks to $\mu = 0$, and by the same argument as in [20, 5.2], σ is injective and τ is surjective, both pointwise on $\mathcal{Q}(\mathbf{v}^1, \mathbf{w}) \times \mathcal{Q}(\mathbf{v}^2, \mathbf{w})$. Therefore $\operatorname{Ker} \tau / \operatorname{Im} \sigma$ is a vector bundle of rank

$$\dim \mathbf{L}(W, V^1) + \dim \mathbf{E}(V^2, W) = \sum_{j} \left(\dim V_j^1 \dim W_j + \dim V_j^2 \dim W_{j+1} \right).$$

We define its section s by

$$s = (0 \oplus 0 \oplus a^1 \oplus -b^2) \mod \operatorname{Im} \sigma.$$

Let Z(s) denote the zero set of s. Then there exists $\xi \in \mathbf{L}(V^2, V^1)$ such that

$$\xi B_1^2 = B_1^1 \xi, \quad \xi B_2^2 = B_2^1 \xi, \quad \xi a^2 = a^1, \quad b^2 = b^1 \xi.$$

The stability condition implies that ξ is surjective. Then (B_1^1, B_2^1, a^1, b^1) is isomorphic to data on the quotient $V^2/\operatorname{Ker}\xi$ induced from (B_1^2, B_2^2, a^2, b^2) .

Since $\mathbf{v}^2 = \mathbf{v}^1 + \alpha_i$, Ker ξ must be 1-dimensional on i and 0 otherwise. It is preserved by B_2^2 by one of the above equations. Thus $B_2^2 | \operatorname{Ker} \xi$ is a scalar. The decomposition $\mathcal{Q} = \mathcal{Q}^0 \times \mathbb{C}$ induces a decomposition $Z(s) = \mathcal{P}_i(\mathbf{v}^2, \mathbf{w}) \times \mathbb{C}$, where $\mathcal{P}_i(\mathbf{v}^2, \mathbf{w})$ is defined by $B_2^2 | \operatorname{Ker} \xi = 0$.

Again by the same argument as in [20, 5.7], the differential ∇s is surjective on Z(s). Therefore Z(s) and hence $\mathscr{P}_i(\mathbf{v}^2, \mathbf{w})$ is a nonsingular subvariety.

Note that the construction is $\mathbb{G}_{\mathbf{w}}$ -equivariant. In particular, $\mathscr{P}_i(\mathbf{v}^2, \mathbf{w})$ has an induced $\mathbb{G}_{\mathbf{w}}$ -equivariant structure.

Remark only Z(s) was considered in [6].

5(ii). **Line bundle.** From the construction in the previous section, we have a natural line bundle \mathcal{L}_i over $\mathscr{P}_i(\mathbf{v}^2, \mathbf{w})$ given by Ker ξ . This is a $\mathbb{G}_{\mathbf{w}}$ -equivariant line bundle. Let us denote the pull-back of the character q under the projection $\mathbb{G}_{\mathbf{w}} \to \mathbb{C}^*$ also by q for brevity. We twist \mathcal{L}_i by as

$$\mathcal{L}_i' \stackrel{\text{def.}}{=} q^{1-i} \mathcal{L}_i.$$

6. Convolution algebra

We fix a \widetilde{I} -graded vector space W with dimension vector \mathbf{w} throughout this section. We also choose a pyramid π so that its row lengths p_1, \ldots, p_n are given by dim $W_1, \ldots, \dim W_n$. Hence we assume the condition (4.6).

In [6] a $W^{\hbar}(\pi)$ -module structure was constructed on

$$\bigoplus_{\mathbf{v}} H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{Q}(\mathbf{v}, \mathbf{w})) \otimes_{S(\mathbb{G}_{\mathbf{w}})} \mathbb{S}(\mathbb{G}_{\mathbf{w}}),$$

the direct sum of the localized equivariant cohomology group of Laumon spaces. Here the variable \hbar for the Rees algebra is identified with the generator \hbar of $\mathbb{C}[\hbar] = H^*_{\mathbb{C}^*}(\mathrm{pt})$. More precisely the Rees algebra with respect to the shifted Kazhdan filtration.

In this section we slightly modify this result working on non-localized equivariant homology group and using the framework in [20, §8], [22, §9]. The author believes that this version is more natural, as

- (1) it fits with the theory of a convolution algebra,
- (2) it gives the Rees algebra for the Kazhdan filtration, instead of the shifted one.
- 6(i). Closed embedding. Suppose that V^1 is an I-graded vector space and V^2 is its I-graded subspace. We consider $\mathbf{M}(\mathbf{v}^1, \mathbf{w})$, $\mathbf{M}(\mathbf{v}^2, \mathbf{w})$ as in §2(i), where \mathbf{v}^1 , \mathbf{v}^2 , \mathbf{w} are dimension vectors of V^1 , V^2 , W as before. We embed $\mathbf{M}(\mathbf{v}^2, \mathbf{w})$ into $\mathbf{M}(\mathbf{v}^1, \mathbf{w})$ by choosing a complementary subspace of V^2 in V^1 . It induces a closed embedding

$$\mathcal{Q}_0(\mathbf{v}^2, \mathbf{w}) \to \mathcal{Q}_0(\mathbf{v}^1, \mathbf{w}),$$

as in [22, Lemma 2.5.3]. It is independent of the choice of the complementary subspace.

6(ii). A Steinberg type variety. Suppose that two dimension vectors \mathbf{v}^1 , \mathbf{v}^2 are given. We have closed embeddings $\mathcal{Q}_0(\mathbf{v}^1, \mathbf{w})$, $\mathcal{Q}_0(\mathbf{v}^2, \mathbf{w}) \hookrightarrow \mathcal{Q}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w})$. Then we introduce the following variety, as analog of Steinberg variety defined as in [20, §7]:

Definition 6.1. Let us consider π as a morphism from $\mathcal{Q}(\mathbf{v}^a, \mathbf{w})$ to $\mathcal{Q}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w})$ and introduce the fiber product

$$Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w}) \stackrel{\text{def.}}{=} \{ (x^1, x^2) \in \mathcal{Q}(\mathbf{v}^1, \mathbf{w}) \times \mathcal{Q}(\mathbf{v}^2, \mathbf{w}) \mid \pi(x^1) = \pi(x^2) \}.$$

This is called a Steinberg type variety.

Take $([B_1^1, B_2^1, a^1, b^1], [B_1^2, B_2^2, a^2, b^2]) \in \mathscr{P}_i(\mathbf{v}^2, \mathbf{w})$. Then $\pi([B_1^1, B_2^1, a^1, b^1]), \pi([B_1^2, B_2^2, a^2, b^2])$ are the same point. However if we only suppose $([B_1^1, B_2^1, a^1, b^1], [B_1^2, B_2^2, a^2, b^2]) \in Z(s)$, then traces of powers of B_2^2 may differ from those of B_2^1 as $B_2^2|\ker \xi$ possibly is nonzero. This gives a reason why $\mathscr{P}_i(\mathbf{v}^2, \mathbf{w})$ is more natural.

6(iii). Convolution algebra. We now consider three dimension vectors \mathbf{v}^1 , \mathbf{v}^2 , \mathbf{v}^3 and the triple product $\mathcal{Q}(\mathbf{v}^1, \mathbf{w}) \times \mathcal{Q}(\mathbf{v}^2, \mathbf{w}) \times \mathcal{Q}(\mathbf{v}^3, \mathbf{w})$. The projection to the product $\mathcal{Q}(\mathbf{v}^a, \mathbf{w}) \times \mathcal{Q}(\mathbf{v}^b, \mathbf{w})$ of the a^{th} and b^{th} factors is denoted by p_{ab} .

The map

$$p_{13} \colon p_{12}^{-1} Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w}) \cap p_{23}^{-1} Z(\mathbf{v}^2, \mathbf{v}^3, \mathbf{w}) \to \mathcal{Q}(\mathbf{v}^1, \mathbf{w}) \times \mathcal{Q}(\mathbf{v}^3, \mathbf{w})$$

is proper, and its image is contained in $Z(\mathbf{v}^1, \mathbf{v}^3, \mathbf{w})$. Hence we can define the convolution product

$$H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w})) \otimes H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^2, \mathbf{v}^3, \mathbf{w})) \to H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^1, \mathbf{v}^3, \mathbf{w})).$$

Taking direct sum over all \mathbf{v}^1 , \mathbf{v}^2 , we have an algebra structure on $\bigoplus H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w}))$. However, this does not contain the unit, which is the direct $\operatorname{product} \prod_{\mathbf{v}} [\Delta \mathcal{Q}(\mathbf{v}, \mathbf{w})]$ of the classes of diagonals $\Delta \mathcal{Q}(\mathbf{v}, \mathbf{w}) \subset Z(\mathbf{v}, \mathbf{v}, \mathbf{w})$. It is practically no harm at all, as we can consider $[\Delta \mathcal{Q}(\mathbf{v}, \mathbf{w})]$ as a collection of idempotents. But it does not fit with the usual convention for the W-algebra, which has unit. Therefore as in $[22, \S 9.1]$, we introduce a slightly larger space $\prod' H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w}))$, consisting of elements $(F_{\mathbf{v}^1, \mathbf{v}^2})$ of $\prod H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w}))$ such that $F_{\mathbf{v}^1, \mathbf{v}^2} = 0$ for all but finitely many choices of \mathbf{v}^2 for fixed \mathbf{v}^1 and the same is true for 1 and 2 exchanged. Then $\prod' H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w}))$ is an algebra with unit.

Let $Z(\mathbf{w})$ denote the disjoint union $\coprod Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w})$ and define $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$ as the above space $\coprod' H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w}))$.

We also denote by $\mathscr{Q}(\mathbf{w})$ (resp. $\mathscr{O}(\mathbf{w})$) the disjoint union $\bigsqcup \mathscr{Q}(\mathbf{v}, \mathbf{w})$ (resp. $\mathscr{O}(\mathbf{v}, \mathbf{w})$). For these we set $H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{Q}(\mathbf{w})) = \bigoplus H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{Q}(\mathbf{v}, \mathbf{w}))$ and $H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{O}(\mathbf{w})) = \bigoplus H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{O}(\mathbf{v}, \mathbf{w}))$. These are modules of $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$.

6(iv). Image of generators. The rest of this section is devoted to a construction of a homomorphism ρ from the finite W-algebra $W^{\hbar}(\pi)$ to the convolution algebra $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$. We first define the assignment on generators $E_i^{(r)}$, $F_i^{(r)}$, $D_i^{(r)}$ in this subsection.

We set

$$\rho(E_i^{(r)}) \stackrel{\text{def.}}{=} \prod_{\mathbf{v}} c_1(\mathcal{L}_i')^{r-s_{i,i+1}-1} \cap [{}^{\omega}\mathscr{P}_i(\mathbf{v}, \mathbf{w})],$$
$$\rho(F_i^{(r)}) \stackrel{\text{def.}}{=} - \prod_{\mathbf{v}} c_1(\mathcal{L}_i')^{r-s_{i+1,i}-1} \cap [\mathscr{P}_i(\mathbf{v}, \mathbf{w})].$$

Note that these are different from ones in [6, §4], where Z(s) in §5 was used instead of $\mathscr{P}_i(\mathbf{v}, \mathbf{w})$. More precisely, the above are multiplied by \hbar from the original ones, which are *not* contained in $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$.

To define the image of $D_i^{(r)}$, we recall generating functions $D_i(u)$, $C_i(u)$ and $Z_N(u)$ in (1.15), and $A_i(u)$ in (1.16).

For $i \in \widetilde{I}$ let $\underline{\mathcal{E}}_i$ be the universal bundle over $\mathcal{Q} \times \mathbb{P}^1$ given by considering \mathcal{Q} as the moduli space of flags $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = W \otimes \mathcal{O}_{\mathbb{P}^1}$ of locally free sheaves on \mathbb{P}^1 as in §3. (This is denoted by $\underline{\mathcal{W}}_i$ in [6].) Recall that it is given by $\operatorname{Ker} \alpha_i$ in the quiver description, where α_i was defined in (3.1).

We modify α_i to a homomorphism on \mathcal{Q} as

$$\bar{\alpha}_i \colon W_1 \oplus \cdots \oplus W_i \oplus q^{-1}V_i \to V_i$$

 $\bar{\alpha}_i \stackrel{\text{def.}}{=} (B_1^{i-1}a, \dots, a, -B_2),$

and define \mathcal{E}_i as Ker $\bar{\alpha}_i$. Here q^{-1} is introduced so that $\bar{\alpha}_i$ is $\mathbb{G}_{\mathbf{w}}$ -equivariant. Let

$$c_{1/u}(\mathcal{E}_i) = \sum_{r=0}^{\infty} u^{-r} c_r(\mathcal{E}_i)$$

be the generating function of Chern classes of \mathcal{E}_i . We define

$$\rho(\mathsf{A}_i(u)) \stackrel{\text{def.}}{=} u^{\operatorname{rank} \mathcal{E}_i} c_{1/u}(\mathcal{E}_i),$$

where $A_i(u)$ is the Rees algebra version of the original $A_i(u)$, which means that we replace various u - k by $u - k\hbar$.

This is again slightly different from one in [6, §4]. Our definition gives $\prod_{j=1}^{i} \prod_{a=1}^{p_j} (u - \mathbf{p}_{ij}^{(a)})$ instead of $\prod_{j=1}^{i} \prod_{a=1}^{p_j} (u - \hbar^{-1} \mathbf{p}_{ij}^{(a)})$ as the eigenvalue on an element corresponding to the $\mathbb{T}_{\mathbf{w}}$ -fixed point, where $\mathbf{p}_{ij}^{(a)}$ is defined in [6, (3.15)].

Consider the last operator $Z_N(u) = A_n(u)$ in the above definition. Since $\mathcal{E}_n = W_1 \oplus \cdots \oplus W_n$ is a trivial vector bundle of rank N over \mathcal{Q} (with a nontrivial $\mathbb{G}_{\mathbf{w}}$ -equivariant structure), we have $\rho(Z_N(u)) \in H^*_{\mathbb{G}_{\mathbf{w}}}(\mathrm{pt})[u] = S(\mathbb{G}_{\mathbf{w}})[u]$. (More precisely, it is a polynomial of degree N whose leading term is u^N .) This will be compatible with the fact that the coefficients of $Z_N(u)$ generate the center $Z(W(\pi))$ of $W(\pi)$.

6(v). A homomorphism from the W-algebra to the convolution algebra. The convolution algebra $H^{\mathbb{G}_{\mathbf{w}}}_*(Z(\mathbf{w}))$ is an algebra over $H^*_{\mathbb{G}_{\mathbf{w}}}(\operatorname{pt}) = S(\mathbb{G}_{\mathbf{w}})$. Let $H^{\mathbb{G}_{\mathbf{w}}}_*(Z(\mathbf{w}))$ /tor be the image of $H^{\mathbb{G}_{\mathbf{w}}}_*(Z(\mathbf{w}))$ in $H^{\mathbb{G}_{\mathbf{w}}}_*(Z(\mathbf{w})) \otimes_{S(\mathbb{G}_{\mathbf{w}})} S(\mathbb{G}_{\mathbf{w}})$.

Theorem 6.2. The assignment ρ induces an algebra homomorphism

$$\rho \colon W^{\hbar}(\pi) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar, \hbar^{-1}] \to \left(H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w})) / \mathrm{tor} \right) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar, \hbar^{-1}].$$

Moreover the center $Z(W^{\hbar}(\pi))$ of $W^{\hbar}(\pi)$ is mapped isomorphically to $S(\mathbb{G}_{\mathbf{w}}) = H_{\mathbb{G}_{\mathbf{w}}}^*(\mathrm{pt})$ under ρ .

The last assertion has been explained just before. The first statement follows from the proof of the main result in [6]. Let us explain how one should modify it.

First note that the pyramid is *left-justified* (or equivalently $s_{i+1,i} = 0$ for all i) in [6], while ours is arbitrary. This is not essential as there is an explicit algebra isomorphism $W(\pi) \cong W(\dot{\pi})$ for two pyramids with the same row length [8, §3.5]. Thus we may assume π is left-justified.

Next we check that the image of generators $E_i^{(s)}$, $F_i^{(s)}$, $D_i^{(s)}$ satisfy the defining relation in §1(ii). In $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$, we replace the group $\mathbb{G}_{\mathbf{w}}$ by its maximal torus $\mathbb{T}_{\mathbf{w}}$ and take the tensor product with $\mathbb{S}(\mathbb{T}_{\mathbf{w}})$. Then the target of ρ is replaced by $H_*^{\mathbb{T}_{\mathbf{w}}}(Z(\mathbf{w})) \otimes_{S(\mathbb{T}_{\mathbf{w}})} \mathbb{S}(\mathbb{T}_{\mathbf{w}})$. Note that this process kills the torsion part of $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$, but is injective on $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$ /tor. By the localization theorem in the equivariant homology it is isomorphic to $H_*(Z(\mathbf{w}))^{\mathbb{T}_{\mathbf{w}}} \otimes_{\mathbb{C}} \mathbb{S}(\mathbb{T}_{\mathbf{w}})$. From the analysis in §4, we have $Z(\mathbf{w})^{\mathbb{T}_{\mathbf{w}}} = \mathcal{Q}(\mathbf{w})^{\mathbb{T}_{\mathbf{w}}} \times \mathcal{Q}(\mathbf{w})^{\mathbb{T}_{\mathbf{w}}}$. Now it is enough to check the relation in the faithful representation $H_*(\mathcal{Q}(\mathbf{w}))^{\mathbb{T}_{\mathbf{w}}} \otimes_{\mathbb{C}} \mathbb{S}(\mathbb{T}_{\mathbf{w}})$. This is exactly the calculation done in [6] except that we need to check that our modification of generators gives the Rees algebra with respect to the Kazhdan's filtration, not the shifted one.

Finally higher root vectors $E_{i,j}^{(r)}$, $F_{i,j}^{(r)}$ are generated by generators if we invert \hbar (see (1.13)). Therefore we got the assertion.

The author conjectures that $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))$ is torsion free and ρ gives an isomorphism

$$W^{\hbar}(\pi) \to H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w})).$$

As a first step towards this conjecture, it should be possible to show that ρ gives a homomorphism $W^{\hbar}(\pi) \to H^{\mathbb{G}_{\mathbf{w}}}_*(Z(\mathbf{w}))/\text{tor}$, i.e., images of higher root vectors are in $H^{\mathbb{G}_{\mathbf{w}}}_*(Z(\mathbf{w}))/\text{tor}$. This is related to the commutativity of products on $H^{\mathbb{G}_{\mathbf{w}}}_*(Z(\mathbf{w}))$ at $\hbar=0$, as $W^{\hbar}(\pi)$ is specialized to the coordinate ring of the Slodowy slice. The author plans to come back to this problem in future.

7. Standard modules = Verma modules

In this section, we identify Verma modules with natural modules of the convolution algebra $H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w}))/\text{tor}$, called *standard modules*. Remark that the same terminology was used for a different class of modules in [8].

7(i). Standard modules. Let M(P) be a Verma module of $W(\pi)$ with the Drinfeld polynomial P. The center $Z(W(\pi))$ acts on M(P) via the character $\chi \colon Z(W(\pi)) \to \mathbb{C}$ given by

$$\chi(Z_N(u)) = P_1(u)P_2(u)\cdots P_n(u).$$

According to the isomorphism $S(\mathbb{G}_{\mathbf{w}}) = H^*_{\mathbb{G}_{\mathbf{w}}}(\mathrm{pt}) \cong Z(W^{\hbar}(\pi))$, χ corresponds to a conjugacy class of a semisimple element s in $\mathrm{Lie}\,G_{\mathbf{w}}$. We consider s as an element in $\mathrm{Lie}\,\mathbb{G}_{\mathbf{w}}$, putting $\hbar = 1$ in the \mathbb{C}^* -component. In concrete terms, the characteristic polynomial of the $\mathrm{GL}(W_i)$ -component of s is $P_i(u)$. Let \mathbb{C}_s be the $S(\mathbb{G}_{\mathbf{w}})$ -module corresponding to s.

Let $\mathcal{Q}(\mathbf{w})^s$, $Z(\mathbf{w})^s$ denote the s-fixed point subvariety in $\mathcal{Q}(\mathbf{w})$ and $Z(\mathbf{w})$ respectively. Here the s-fixed point subvariety is the zero locus of the vector field corresponding to s, or equivalently the subvariety fixed by $\exp(ts)$ for all $t \in \mathbb{R}$. We use the corresponding notation for other handsaw quiver varieties also. The localization theorem of equivariant homology groups (see e.g., [4]) implies the restriction homomorphism with support in $\mathcal{Q}(\mathbf{w})^s \times \mathcal{Q}(\mathbf{w})^s$

$$H_*^{\mathbb{G}_{\mathbf{w}}}(Z(\mathbf{w})) \otimes_{S(\mathbb{G}_{\mathbf{w}})} \mathbb{C}_s \to H_*(Z(\mathbf{w})^s)$$

is an isomorphism. Moreover if we correct the homomorphism by the Euler class of the normal bundle of the second factor of $\mathcal{Q}(\mathbf{w})^s \times \mathcal{Q}(\mathbf{w})^s$, it becomes an algebra isomorphism, as in [9, 5.11.10].

Similarly we have an isomorphism

$$H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{O}(\mathbf{w})) \otimes_{S(\mathbb{G}_{\mathbf{w}})} \mathbb{C}_s \to H_*(\mathscr{O}(\mathbf{w})^s),$$

which is compatible with the convolution product. Here $\mathscr{O}(\mathbf{w})^s$ is the s-fixed point subvariety in $\mathscr{O}(\mathbf{w})$. More generally if we take $x \in \mathscr{Q}_0(\mathbf{w})^s$ and its inverse image $\mathscr{Q}(\mathbf{w})_x^s \stackrel{\text{def.}}{=} \pi^{-1}(x) \cap \mathscr{Q}(\mathbf{w})^s$,

$$H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{Q}(\mathbf{w})_r^s) \otimes_{S(\mathbb{G}_{\mathbf{w}})} \mathbb{C}_s \to H_*(\mathscr{Q}(\mathbf{w})_r^s)$$

is also compatible with the convolution product. The previous $\mathcal{O}(\mathbf{w})^s$ is the special case x = 0. Following [22, §13] we call these *standard modules*.

Remark 7.1. As we mentioned in the introduction, the finite AGT conjecture is formulated for arbitary compact Lie group K. However, the only space corresponding to $\mathcal{Q}_0(\mathbf{w})$ can be defined, and its resolution $\mathcal{Q}(\mathbf{w})$ cannot be generalized. The convolution algebra can be defined as an Ext-algebra of the sum of the $\mathbb{G}_{\mathbf{w}}$ -equivariant intersection cohomology complexes of $\mathcal{Q}_0^{\text{reg}}(\mathbf{v},\mathbf{w})$ with various \mathbf{v} . Similarly the convolution algebra of the fixed point subvariety $\mathcal{Q}_0^{\bullet}(\mathbf{w}^{\bullet})$ can be defined. However the 'correction term' given by the Euler class of the normal bundle does not make sense. The author does *not* know how to define a homomorphism from the specialization of the convolution algebra to the convolution algebra of the fixed point set.

7(ii). Graded handsaw quiver varieties. We study the s-fixed subvariety $\mathcal{Q}(\mathbf{w})^s$ in this subsection. We call $\mathcal{Q}(\mathbf{w})^s$ the graded handsaw quiver variety in analogy with graded quiver varieties studied in [22, 25].

Let $x \in \mathcal{Q}(\mathbf{w})^s$ and take its representative (B_1, B_2, a, b) . As in the discussion in §2(ii) the point x is fixed by s if and only if there exists a semisimple element $\xi \in \text{Lie } G = \bigoplus_{i \in I} \mathfrak{gl}(V_i)$ such that

$$[B_1, \xi] = 0$$
, $[B_2, \xi] = B_2$, $-\xi a = -as$, $b\xi = (s+1)b$.

We decompose V, W according to eigenvalues of ξ and s respectively:

$$V = \bigoplus V(m), \quad W = \bigoplus W(m).$$

Then the above equations mean that B_1 , a preserve eigenvalues while B_2 , b decrease them by 1. Combining the condition with the I-grading, we thus have

$$B_1(V_i(m)) \subset V_{i+1}(m), \quad B_2(V_i(m)) \subset V_i(m-1),$$

 $a(W_i(m)) \subset V_i(m), \quad b(V_i(m)) \subset W_{i+1}(m-1).$

We define graded dimension vectors

$$\mathbf{v}^{\bullet} = (\dim V_i(m))_{i \in I, m \in \mathbb{C}} \in \mathbb{Z}_{\geq 0}^{I \times \mathbb{C}}, \quad \mathbf{w}^{\bullet} = (\dim W_i(m))_{i \in \widetilde{I}, m \in \mathbb{C}} \in \mathbb{Z}_{\geq 0}^{\widetilde{I} \times \mathbb{C}}.$$

The conjugacy class of s is determined by \mathbf{w}^{\bullet} , and vice versa. The value \mathbf{v}^{\bullet} is constant on each connected component of $\mathcal{Q}(\mathbf{w})^s$. We denote by $\mathcal{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$ the union of components of $\mathcal{Q}(\mathbf{w})^s$ with given \mathbf{v}^{\bullet} . We denote by \mathbf{v} , \mathbf{w} the corresponding vectors

$$(\sum_{m} \dim V_i(m))_{i \in I} \in \mathbb{Z}^I_{\geq 0}, \quad (\sum_{m} \dim W_i(m))_{i \in \widetilde{I}} \in \mathbb{Z}^{\widetilde{I}}_{\geq 0}$$

respectively. We change the notation $\mathcal{Q}(\mathbf{w})^s$ to $\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})$ accordingly hereafter.

Let us observe that we can group V(m) according to $m \mod \mathbb{Z}$, and components of (B_1, B_2, a, b) between different groups vanish. This means $\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet})$ decomposes into a product of smaller varieties $\mathscr{Q}^{\bullet}(\mathbf{w}'^{\bullet})$, $\mathscr{Q}^{\bullet}(\mathbf{w}''^{\bullet})$, Thus we may assume all eigenvalues of s are integers. Then eigenvalues of s are also integers. This corresponds to the linkage principle [8, §6.3].

Let us rename the graded pieces as

(7.2)
$$V'_{j}(k) \stackrel{\text{def.}}{=} V_{\frac{j-k}{2}}(\frac{j+k}{2}), \quad W'_{j}(k) \stackrel{\text{def.}}{=} W_{\frac{j-k+1}{2}}(\frac{j+k-1}{2}).$$

Then the above implies

$$B_1(V'_j(k)) \subset V'_{j+1}(k-1), \quad B_2(V'_j(k)) \subset V'_{j-1}(k-1),$$

 $a(W'_j(k)) \subset V'_j(k-1), \quad b(V'_j(k)) \subset W'_j(k-1).$

Thus (B_1, B_2, a, b) defines a point in the quiver variety of type A for $V'_j = \bigoplus_k V'_j(k)$, $W'_j = \bigoplus_k W'_j(k)$ as in Remark 2.5. Moreover if we also regard the grading given by k, the above condition is nothing but that of the *graded quiver variety* introduced in [22, §4.1] and [25, §4]. We denote it by $\mathfrak{M}^{\bullet}(\mathbf{v}'^{\bullet}, \mathbf{w}'^{\bullet})$, where the dimension vectors are transformed by the rule (7.2).

A point in a graded quiver variety gives a point in a graded handsaw quiver variety conversely. Thus we get the following trivial, but main result in this paper:

Theorem 7.3. The graded handsaw quiver variety $\mathscr{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$ is isomorphic to the graded quiver variety $\mathfrak{M}^{\bullet}(\mathbf{v}'^{\bullet}, \mathbf{w}'^{\bullet})$.

This result allows us to apply various results for graded quiver varieties to handsaw quiver varieties $\mathcal{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$. For example, $\mathcal{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$ is connected, provided it is nonempty ([22, Th. 5.5.6]). It has no odd cohomology groups [22, Prop. 7.3.4]. We continue study in subsequent subsections.

Remark 7.4. The graded component $V'_j(k)$ was denoted by $V'_j(\varepsilon^k)$ in [25], where ε is a nonzero complex number. This is not a difference is as ε is not a root of unity in the current setting.

Note also that V was already an $I \times \mathbb{C}^*$ -graded module, and we did not put \bullet in the script in [25].

7(iii). Gelfand-Tsetlin character of a standard module. We consider $\mathscr{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})_x \stackrel{\text{def.}}{=} \mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet})_x \cap \mathscr{Q}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$, the fiber of the morphism π for graded handsaw quiver varieties. We have a decomposition

(7.5)
$$H_*(\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet})_x) = \bigoplus_{\mathbf{v}^{\bullet}} H_*(\mathscr{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})_x).$$

Proposition 7.6. The above (7.5) is the ℓ -weight space decomposition of the standard module $H_*(\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})_x)$, considered as a $W(\pi)$ -module.

The corresponding statement for graded quiver varieties were shown in [22, Prop. 13.4.5]. The proof works in our current setting. Let us sketch it.

Proof. Recall that we have a $\mathbb{G}_{\mathbf{w}}$ -equivariant vector bundle \mathcal{E}_i over $\mathcal{Q}(\mathbf{w})$. The operator $\mathsf{A}_i(u)$ is given by the multiplication of $u^{\mathrm{rank}\,\mathcal{E}_i}c_{1/u}(\mathcal{E}_i)$. On $H_*(\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet}))$ it acts through the restriction

$$H^*_{\mathbb{G}_{\mathbf{w}}}(\mathscr{Q}(\mathbf{w})) \to H^*_{\mathbb{G}_{\mathbf{w}}}(\mathscr{Q}(\mathbf{w})) \otimes_{H^*_{\mathbb{G}_{\mathbf{w}}}(\mathrm{pt})} \mathbb{C} \to H^*(\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet})),$$

where the first homomorphism is the specialization at χ , and the second is the pull-back homomorphism with respect to the inclusion $\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet}) \subset \mathscr{Q}(\mathbf{w})$. Then the $H^{>0}(\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet}))$ -part of $u^{\operatorname{rank} \mathcal{E}_i} c_{1/u}(\mathcal{E}_i)$ acts as a nilpotent operator since the ordinary cohomology group vanishes in sufficiently high degrees. The H^0 -part acts as scalar multiplication on each connected component of $\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet})$, i.e., $\mathscr{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$. And the scalar is given by characteristic polynomials of ξ and s, in other words \mathbf{v}^{\bullet} and \mathbf{w}^{\bullet} from the definition of \mathcal{E}_i . Therefore $H_*(\mathscr{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})_x)$ is an ℓ -weight space.

We denote the corresponding ℓ -weight by $e^{\mathbf{v}^{\bullet}}e^{\mathbf{w}^{\bullet}}$. In concrete terms, $e^{\mathbf{w}^{\bullet}}$ is an \widetilde{I} -tuple of characteristic polynomials $P_1(u), \ldots, P_n(u)$ of the $\mathrm{GL}(W_i)$ -components of s, and $e^{\mathbf{v}^{\bullet}}$ is given

by $(Q_1(u), \ldots, Q_n(u))$ where $Q_i(u)$ is the characteristic polynomial of ξ on the virtual vector space

$$q^{-1}V_i - V_i - q^{-1}V_{i-1} + V_{i-1}$$

where ξ acts on $q^{-1}V_i$ as $\xi - \mathrm{id}_{V_i}$. Then $e^{\mathbf{v}^{\bullet}}e^{\mathbf{w}^{\bullet}}$ is defined as $(P_1(u)Q_1(u), \ldots, P_n(u)Q_n(u))$. More concretely it is given by the formula

$$P_i(u)Q_i(u) = \prod_m (u-m)^{\dim W_i(m) + \dim V_i(m+1) - \dim V_i(m) - \dim V_{i-1}(m+1) + \dim V_{i-1}(m)}.$$

In terms of $\mathbf{v}^{\prime \bullet}, \mathbf{w}^{\prime \bullet}$, it is equal to the ℓ -weight defined in [25, (4.4)].

As a corollary, we have

Corollary 7.7.

$$\operatorname{ch} H_*(\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})_x) = \sum_{\mathbf{v}^{\bullet}} \operatorname{dim} H_*(\mathcal{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})_x) e^{\mathbf{v}^{\bullet}} e^{\mathbf{w}^{\bullet}}$$

Let us mention that the above Gelfand-Tsetlin character of a standard module is given in terms of Young tableaux [24] since the graded quiver varieties are of type A. (The result is originally due to Kuniba-Suzuki [15].)

It also follows from the definition of $A_i(u)$ that the summand

$$H_*(\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})_x \cap \mathcal{Q}(\mathbf{v}, \mathbf{w}))$$

is a generalized weight space decomposition with respect to $D_i^{(1)}$ considered in §1(vi). In particular, we see that the standard module is admissible, as each $H_*(\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})_x \cap \mathcal{Q}(\mathbf{v}, \mathbf{w}))$ is finite dimensional, and \mathbf{w} is a highest weight as we will see in Lemma 7.8 below.

7(iv). An ℓ -highest weight vector in a standard module. Let us continue the study of a standard module.

If we restrict the stratification of $\mathcal{Q}_0(\mathbf{w})$ in §2(iv) to the fixed point locus $\mathcal{Q}_0^{\bullet}(\mathbf{w}^{\bullet})$, the symmetric product parts are all concentrated at the origin, and hence $x \in \mathcal{Q}_0^{\bullet}(\mathbf{w}^{\bullet})$ is in $\mathcal{Q}_0^{\bullet \text{reg}}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$ for some \mathbf{v}^{\bullet} , extended by 0. We denote this \mathbf{v}^{\bullet} by \mathbf{v}_x^{\bullet} . Let us consider $\mathcal{Q}^{\bullet}(\mathbf{v}_x^{\bullet}, \mathbf{w}^{\bullet})_x$. Since $\pi : \mathcal{Q}(\mathbf{v}, \mathbf{w}) \to \mathcal{Q}_0(\mathbf{v}, \mathbf{w})$ is an isomorphism on $\mathcal{Q}_0^{\text{reg}}(\mathbf{v}, \mathbf{w})$, it consists of a single point, which we also denote by x.

Lemma 7.8. The fundamental class [x] is an ℓ -highest weight vector in the standard module $H_*(\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet})_x)$ with ℓ -weight $e^{\mathbf{v}_x^{\bullet}}e^{\mathbf{w}^{\bullet}}$.

Proof. Since $H_*(\mathcal{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})_x)$ is 1-dimensional, it is clear that [x] is an eigenvector for any $D_i^{(r)}$. Its ℓ -weight is $e^{\mathbf{v}_x^{\bullet}}e^{\mathbf{w}^{\bullet}}$ as calculated above.

It is killed by all $E_i^{(r)}$, as $\mathscr{Q}^{\bullet}(\mathbf{v'^{\bullet}}, \mathbf{w^{\bullet}}) \neq \emptyset$ implies each component of $\mathbf{v'^{\bullet}}$ is greater than or equal to the corresponding component of $\mathbf{v^{\bullet}}$ by [22, Prop. 13.3.1].

Note that $\mathcal{E}_{i-1} \subset \mathcal{E}_i$ is a vector subbundle over $\mathscr{Q}^{\bullet \operatorname{reg}}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$. This implies that the ℓ -weight $e^{\mathbf{v}^{\bullet}}e^{\mathbf{w}^{\bullet}}$ is ℓ -dominant if $\mathscr{Q}^{\bullet \operatorname{reg}}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet}) \neq \emptyset$, since

$$\frac{u^{\operatorname{rank}\mathcal{E}_{i}}c_{1/u}(\mathcal{E}_{i})}{u^{\operatorname{rank}\mathcal{E}_{i-1}}c_{1/u}(\mathcal{E}_{i-1})} = u^{\operatorname{rank}\mathcal{E}_{i}/\mathcal{E}_{i-1}}c_{1/u}(\mathcal{E}_{i}/\mathcal{E}_{i-1})$$

is a polynomial in u. In particular, the ℓ -weight in the previous lemma is ℓ -dominant.

We deduce the following criterion from [22, Th. 14.3.2(2)], since ℓ -dominance in this paper is equivalent to one in [22].

Proposition 7.9. $\mathscr{Q}^{\bullet \operatorname{reg}}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet}) \neq \emptyset$ if and only if $\mathscr{Q}^{\bullet}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet}) \neq \emptyset$ and $e^{\mathbf{v}^{\bullet}}e^{\mathbf{w}^{\bullet}}$ is ℓ -dominant.

7(v). Cyclicity. We now give a main result in this section.

Theorem 7.10. (1) The standard module $H_*(\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet})_x)$ is an ℓ -highest weight module with an ℓ -highest weight vector [x].

(2) The standard module $H_*(\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})_x)$ is the Verma module with ℓ -highest weight $e^{\mathbf{v}_x^{\bullet}}e^{\mathbf{w}^{\bullet}}$.

Proof. (1) Let us prove the assertion as a consequence of the corresponding statement for graded quiver varieties proved in [22, Prop. 13.3.1], after the equivariant K-theory is replaced by the equivariant homology group as in [30].

Consider the restriction of $F_i^{(r)}$ to

$$H_*(\mathcal{Q}^{\bullet}(\mathbf{v}^{2\bullet}, \mathbf{w}^{\bullet})) \to H_*(\mathcal{Q}^{\bullet}(\mathbf{v}^{1\bullet}, \mathbf{w}^{\bullet})).$$

It can be computed in the following way, as explained in $\S7(i)$:

- (1) replace $\mathscr{P}_i(\mathbf{v}, \mathbf{w})$ by the Koszul complex for the section s of Ker $\tau / \text{Im } \sigma$ in (5.1),
- (2) restrict the complex to $\mathcal{Q}^{\bullet}(\mathbf{v}^{1\bullet}, \mathbf{w}^{\bullet}) \times \mathcal{Q}^{\bullet}(\mathbf{v}^{2\bullet}, \mathbf{w}^{\bullet}),$
- (3) divide by the Euler class (evaluated at s) of the normal bundle of the second factor $\mathcal{Q}^{\bullet}(\mathbf{v}^{2\bullet}, \mathbf{w}^{\bullet})$ in $\mathcal{Q}(\mathbf{v}^2, \mathbf{w})$.

The restriction of the vector bundle $\operatorname{Ker} \tau / \operatorname{Im} \sigma$ decomposes into sum of the s-fixed part and the other part. The s-fixed part gives a resolution of the intersection

$$\mathscr{P}_i(\mathbf{v}, \mathbf{w}) \cap \mathscr{Q}^{\bullet}(\mathbf{v}^{1\bullet}, \mathbf{w}^{\bullet}) \times \mathscr{Q}^{\bullet}(\mathbf{v}^{2\bullet}, \mathbf{w}^{\bullet}).$$

The other part is close to the normal bundle of the second factor: if we replace V^1 by V^2 , it precisely is the normal bundle, as can be seen from (2.7). The difference V^1/V^2 is nothing but the definition of the line bundle \mathcal{L}'_i . Therefore the restriction is given by a linear combination of correspondences

$$c_1(\mathcal{L}_i')^{r-s_{i+1,i}} \cap [\mathscr{P}_i(\mathbf{v}, \mathbf{w}) \cap \mathscr{Q}^{\bullet}(\mathbf{v}^{1\bullet}, \mathbf{w}^{\bullet}) \times \mathscr{Q}^{\bullet}(\mathbf{v}^{2\bullet}, \mathbf{w}^{\bullet})],$$

multiplied with various Chern classes of \mathcal{L}'_i and components of V^1 , V^2 . Chern classes of components of V^1 , V^2 are given by products of various operators $D_j^{(s)}$. Multiplication of a power of $c_1(\mathcal{L}'_i)$ is a restriction of $E_i^{(s)}$ with $s \geq r$ plus correction terms. The same is true for negative generators of the Yangian. Since the complex (5.1) is different,

The same is true for negative generators of the Yangian. Since the complex (5.1) is different, the restriction of $F_i^{(r)}$ is not the same, but the linear span of products of $F_i^{(r)}$ and various products of $D_j^{(s)}$ is the same for the finite W-algebra and Yangian. This statement is enough to conclude $H_*(\mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})_x)$ is ℓ -highest weight.

(2) Let us consider the dimension of the weight space

$$H_*(\mathscr{O}^{\bullet}(\mathbf{w}^{\bullet}) \cap \mathscr{Q}(\mathbf{v}, \mathbf{w})) = H_*^{\mathbb{G}_{\mathbf{w}}}(\mathscr{O}(\mathbf{v}, \mathbf{w})) \otimes_{S(\mathbb{G}_{\mathbf{w}})} \mathbb{C}_s$$

of the standard module $H_*(\mathscr{O}^{\bullet}(\mathbf{w}^{\bullet}))$. By Theorem 4.10 this is equal to the Euler number of $\mathscr{O}(\mathbf{v}, \mathbf{w})$, which is given by the formula in Theorem 4.4. This coincides with one given by [8, Th. 6.1]. Therefore the assertion for the special case x = 0 is proved.

A general case follows from the special case x = 0 from [22, Th. 3.3.2 and Rem. 3.3.3].

8. Simple modules

We give a character formula of a simple module in terms of dimensions of intersection cohomology groups of graded quiver varieties in this section.

8(i). IC sheaves. Combining Proposition 7.9 with Theorem 7.10, we find that the stratification

$$\mathscr{Q}^{\bullet}(\mathbf{w}^{\bullet}) = \bigsqcup_{\mathbf{v}^{\bullet}} \mathscr{Q}^{\bullet reg}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet}),$$

introduced at the beginning of $\S7(iv)$ is indexed by the set of vectors such that $e^{\mathbf{v}^{\bullet}}e^{\mathbf{w}^{\bullet}}$ is an ℓ -dominant ℓ -weight of the Verma module with ℓ -highest weight vector $e^{\mathbf{w}^{\bullet}}$. From Theorem 7.3 (or probably from somewhere in [8]) and the corresponding result for graded quiver varieties [22, Th. 14.3.2], we know that the set is finite.

We denote by $IC(\mathcal{Q}^{\bullet reg}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet}))$ the intersection cohomology complex associated with the constant sheaf on $\mathcal{Q}^{\bullet reg}(\mathbf{v}^{\bullet}, \mathbf{w}^{\bullet})$.

Take a point $x_{\mathbf{v}^{\bullet},\mathbf{w}^{\bullet}}$ from the stratum $\mathcal{Q}^{\bullet \operatorname{reg}}(\mathbf{v}^{\bullet},\mathbf{w}^{\bullet})$, and denote by $i_{\mathbf{v}^{\bullet},\mathbf{w}^{\bullet}}$ the inclusion $\{x\} \to \mathcal{Q}^{\bullet}(\mathbf{w}^{\bullet})$. The statement of the following theorem holds for any choice of $x_{\mathbf{v}^{\bullet},\mathbf{w}^{\bullet}}$, hence our notation does not cause any trouble.

Now we can state our main result in this paper.

Theorem 8.1. The composition multiplicity of L(Q) in M(P) is given by

(8.2)
$$[M(P):L(Q)] = \dim H^*(i^!_{\mathbf{v}_P^{\bullet},\mathbf{w}^{\bullet}}IC(\mathscr{Q}^{\bullet reg}(\mathbf{v}_Q^{\bullet},\mathbf{w}^{\bullet}))),$$

where \mathbf{v}_{P}^{\bullet} , \mathbf{v}_{Q}^{\bullet} are determined by $e^{\mathbf{v}_{P}^{\bullet}}e^{\mathbf{w}^{\bullet}} = P$, $e^{\mathbf{v}_{Q}^{\bullet}}e^{\mathbf{w}^{\bullet}} = Q$.

This is proved exactly as in [22, Th. 14.3.10]. We use Ginzburg's theory [9, §8.6] of simple modules of a convolution algebra to analyze $H_*(Z(\mathbf{w}^{\bullet}))$. Since the algebra homomorphism ρ in Theorem 6.2 may not be an isomorphism, it is not a priori clear whether a simple $H_*(Z(\mathbf{w}^{\bullet}))$ -module remains simple under the pull-back by ρ . But this difficulty can be overcome by the use of the cyclicity proved in Theorem 7.10, as in the proof of [22, Th. 14.3.2(3)].

8(ii). **KL polynomials.** As we noticed in §7(ii) graded handsaw quiver varieties are isomorphic to graded quiver varieties of type A. Using results known on graded quiver varieties, we identify the right hand side of (8.2) with the evaluation of a Kazhdan-Luszitg polynomial at 1 in this subsection. We give two proofs.

By [22, Th. 14.3.10] the right hand side of (8.2) is equal to the composition multiplicity for finite dimensional representations of a quantum affine algebra $\hat{\mathbf{U}}_{\varepsilon}$ specialized at $q = \varepsilon$, which is not a root of unity:

$$[M^{\hat{\mathbf{U}}_{\varepsilon}}(P'):L^{\hat{\mathbf{U}}_{\varepsilon}}(Q')] = \dim H^*(i^!_{\mathbf{v}^{\bullet}_{\mathbf{p}},\mathbf{w}^{\bullet}}IC(\mathscr{Q}^{\bullet \mathrm{reg}}(\mathbf{v}^{\bullet}_{Q},\mathbf{w}^{\bullet}))),$$

where $M^{\hat{\mathbf{U}}_{\varepsilon}}(P')$ (resp. $L^{\hat{\mathbf{U}}_{\varepsilon}}(Q')$) is a standard (resp. simple) module of $\hat{\mathbf{U}}_{\varepsilon}$. Here the Drinfeld polynomial P' is given by the formula

$$P_j'(u) = \prod_{k} (1 - \varepsilon^{-k} u)^{\dim W_j'(k) + \dim V_{j+1}'(k) + \dim V_{j-1}'(k) - \dim V_j'(k+1) - \dim V_j'(k-1)},$$

when $\mathscr{Q}^{\bullet}(\mathbf{v}_{P}^{\bullet}, \mathbf{w}^{\bullet})$ is identified with $\mathfrak{M}^{\bullet}(\mathbf{v}'^{\bullet}, \mathbf{w}'^{\bullet})$ as in Theorem 7.3. And the same applies for Q'.

If we fix an ℓ -highest weight $e^{\mathbf{w}^{\bullet}}$, we only have a finitely many ℓ -dominant ℓ -weight in the Verma module, and hence we can work on a quantum affine algebra $\mathbf{U}_{\varepsilon}(\widehat{\mathfrak{sl}}_M)$ for some M. However, if we only fix the finite W-algebra $W(\pi)$, we cannot have a bound on M: if we have a bound on j-k, but not on j+k in (7.2), both j and k can arbitrary. Therefore the above $\hat{\mathbf{U}}_{\varepsilon}$ should be understood as $\mathbf{U}_{\varepsilon}(\widehat{\mathfrak{sl}}_{\infty})$, defined as the limit of $\mathbf{U}_{\varepsilon}(\widehat{\mathfrak{sl}}_M)$ as $M \to \infty$.

By [30] this composition multiplicity is the same as that for Yangian. And it is given by a Kazhdan-Lusztig polynomial thanks to Arakawa's result [2, Th. 16]. Let us state this result in terms of P, Q. We assume all zeros of Drinfeld polynomials of P are integers. A general case is reduced to this case as mentioned in §7(ii). We identify integral weights of \mathfrak{gl}_N with elements in \mathbb{Z}^N . We define integral weights λ , μ' of \mathfrak{gl}_N by

$$\lambda \stackrel{\text{def.}}{=} \underbrace{(n, \dots, n, \underbrace{n-1, \dots, n-1}_{p_n \text{ times}}, \dots, \underbrace{1, \dots, 1}_{p_1 \text{ times}})},$$

$$\mu' \stackrel{\text{def.}}{=} (-m_n^1, \dots - m_n^{p_n}, -m_{n-1}^1, \dots, -m_{n-1}^{p_{n-1}}, \dots, -m_1^1, \dots, -m_1^{p_1}),$$

where $m_i^1, \ldots, m_i^{p_i}$ are zeros of $P_i(u)$. We take a dominant weight μ so that $\mu' = w\mu$ for some $w \in S_N$. Note that w is well-defined in the double coset $S_\lambda \backslash S_N / S_\mu$, where S_λ , S_μ are stabilizers of λ , μ respectively. We define ν' from zeros of Q as above. Then Arakawa showed that $\nu' = x\mu$ for some $x \in S_N$ when the composition multiplicity is nonzero. We let x_{LR} , w_{LR} the longest length representatives of x, w in $S_\lambda \backslash S_N / S_\mu$.

Theorem 8.3. We have

$$[M(P):L(Q)] = P_{w_{LR},x_{LR}}(1)$$

where $P_{yz}(q)$ is the Kazhdan-Lusztig polynomial associated with S_N .

In Arakawa's result there are restrictions on x, w coming from the rank of the underlying Lie algebra \mathfrak{sl} . But as we take limit explained as above, we do not impose such a restriction.

This result gives a new proof of a conjecture by Brundan-Kleshchev [8, Conj. 7.17], which was proved by Losev [17, Th. 4.1, Th. 4.3], in a wider context, by a different method.

Arakawa's proof depends on a solution of Kazhdan-Lusztig conjecture for \mathfrak{sl}_N . Let us give another more direct proof. (It will give a new proof of Kazhdan-Lusztig conjecture, when $\mathbf{w} = (1, 1, \dots, 1)$ and $W(\mathfrak{gl}_N, e) = U(\mathfrak{gl}_N)$.)

Let $O(\rho)$ be the nilpotent orbit corresponding to an \mathfrak{sl}_2 -triple ρ . Let $S(\rho)$ denote Slodowy slice at $\rho \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. By [19, Th. 8.4], a quiver variety $\mathfrak{M}(\mathbf{v}', \mathbf{w}')$ of type A_{ℓ} is an intersection $\overline{O(\rho)} \cap S(\rho')$ where ρ and ρ' are given by dimension vectors by the formula

$$\mathbf{u} = (u_1, \dots, u_\ell) \stackrel{\text{def.}}{=} \mathbf{w}' - \mathbf{C}\mathbf{v}', \quad \rho \longleftrightarrow (1^{u_1} 2^{u_2} \cdots \ell^{u_\ell} (\ell+1)^{v'_\ell}),$$
$$\rho' \longleftrightarrow (1^{w'_1} 2^{w'_2} \cdots \ell^{w'_\ell}),$$

where C is the Cartan matrix of type A, and we write partitions corresponding to ρ , ρ' .

The \mathbb{C}^* -action on the quiver variety coincides with the standard \mathbb{C}^* -action on the Slodowy slice, defined using the \mathfrak{sl}_2 -triple ρ' :

$$t^2 \operatorname{Ad} \left(\rho \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix} \right).$$

(See e.g., [28, §7.4].) And $\prod_j \operatorname{GL}(W_j')$ is identified with the centralizer of ρ' . Then the graded quiver variety $\mathfrak{M}_0^{\bullet}(\mathbf{v}'^{\bullet}, \mathbf{w}'^{\bullet})$ is identified with the closure of an orbit $O(\rho^{\bullet})$ of the space of representations of a type A quiver, intersected with a slice $S(\rho'^{\bullet})$ to another orbit. Those orbits are parametrized by multisegments, collections of segments in \mathbb{Z} . A segment [a, b] corresponds to an indecomposable representation

$$\cdots 0 0 \mathbb{C} \to \mathbb{C} \to \cdots \to \mathbb{C} 0 0 \cdots$$

where \mathbb{C} starts from the vertex a, and ends at the vertex b. For example, ρ'^{\bullet} corresponds to the union of segments

$$[\frac{-j-k+3}{2},\frac{j-k+1}{2}]=[-m+1,i]$$

for each $W'_j(k) = W_i(m)$. Thus the multisegment is given by λ , μ' defined above. The multisegment corresponding to ρ^{\bullet} is similarly given by λ , ν' .

Now the dimension of the intersection cohomology group in question was given by a Kazhdan-Lusztig polynomial by [31]. It is known that the Kazhdan-Lusztig polynomial is equal to one given above, thanks to [13].

Another possible approach is to express the algorithm of [25] in terms of Young tableaux by a method in [24]. The author does not have ability to determine Weyl group elements obtained in this approach. So he leaves this problem as an exercise for the reader.

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