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STABILITY THEOREMS FOR SPACES OF RATIONAL CURVES

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1. Introduction

Let Σ be a compact Riemann surface, and X, a Riemannian manifold. In the spirit of Morse theory, one would hope that the energy functional

$$E(f) = \int_{\Sigma} |df|^2 \tag{1.1}$$

on the space of smooth based maps $\mathrm{Map}(\Sigma, X)$, would encode a lot of the topology of the space in terms of the critical points of the functional. These critical points are harmonic maps; the absolute minima, in particular, are often of special interest. It has been known, however, at least since the work of Sachs–Uhlenbeck that the Palais–Smale condition C fails for this functional, making a good Morse theory impossible. Nevertheless, in some cases, there is a sense in which the conclusions of Morse theory seem to hold, at least asymptotically. For simplicity, let us suppose that $\pi_1(X) = 0$, and that $\pi_2(X)$ is a free Abelian group of rank r so that the homotopy class of any map $f \in \mathrm{Map}(\Sigma, X)$ is given by a multi-degree $\mathbf{k}$. We can try to compare the homology (homotopy) groups of the space of absolute minima $\mathrm{Min}_{\mathbf{k}}(\Sigma, X)$ with the homology (homotopy) groups of the full mapping space $\mathrm{Map}_{\mathbf{k}}(\Sigma, X)$. If X is Kähler, then by a theorem of Eells and Wood [10] the space of absolute minima $\mathrm{Min}_{\mathbf{k}}(\Sigma, X)$ is just the space $\mathrm{Hol}_{\mathbf{k}}(\Sigma, X)$ of based holomorphic maps (or anti-holomorphic maps, depending on orientation) from Σ to X, as long as $\mathrm{Hol}_{\mathbf{k}}(\Sigma, X)$ is non-empty. We are then led to consider the inclusion

$$\mathrm{Hol}_{\mathbf{k}}(\Sigma, X) \to \mathrm{Map}_{\mathbf{k}}(\Sigma, X)\,.$$

In several important cases, one can prove stability theorems, which, loosely speaking, say that the homology (homotopy) groups of the space of holomorphic maps is isomorphic to the homology (homotopy) groups of the entire mapping space through a range that grows with $\mathbf{k}$, as $\mathbf{k}$ moves to infinity in a suitable positive cone. These results are compatible with the Morse theoretic picture, in that in known cases, the indices of the higher critical points also tend to infinity with $\mathbf{k}$, so that if Morse theory were to hold, the homotopy type of the space of minima would tend to that of the whole space.

Topological stability theorems of the type described had their origins in questions coming from control theory on the one hand and gauge theory on the other. More specifically Atiyah and Jones [2] asked whether such a theorem existed when Σ is replaced by S^4 and the functional instead of (1.1) is the Yang–Mills functional of instanton gauge theory. Later it was realized by Atiyah that this was equivalent to considering a mapping space $\mathrm{Map}(S^2, \mathcal{L}G)$, where $\mathcal{L}G$ is the loop group corresponding to G, the compact Lie group associated to the gauge theory. A positive answer to the Atiyah–Jones conjecture was given a few years ago by the authors and B. Mann [6] for the group $G = SU(2)$, following on a "stable" result of Taubes [36] and then more generally for the various classical compact Lie groups by Tian [37, 38] and Kirwan [28]. The conjecture should hold for more general four-manifolds than S^4; see [18] for the case of a ruled complex surface. However the first proof of any such stability theorem was given in the mapping space ("sigma model") case by Segal [35] for $X = \mathbb{P}^n$. Segal's theorem was then generalized to the case X a complex Grassmannian by Kirwan [27], and certain $SL(n, \mathbb{C})$ flags by Guest [16]. Later, Mann and Milgram [32] increased the range of the isomorphisms obtained by Kirwan for Grassmannians and treated [33] all $SL(n, \mathbb{C})$ flag manifolds. Moreover, the essential technique of L-stratifications used by the authors in the present and previous papers [5, 6] was introduced in [32, 33]. More recently, topological stability theorems were proven for any generalized flag manifold G/P by the authors [5, 21], following on a stable result of Gravesen [14] and for toric varieties by Guest [17]. We refer to [22] for a survey.

In all of the above cases an essential ingredient is representing the minima $\mathrm{Min}_{\mathbf{k}}(\Sigma, X)$ by a labelled configuration space. This description is essentially confined to spaces which admit actions of groups of dimension equal to that of the space. In what follows we prove the stability theorem for what is more or less the largest class of manifolds to which this particle description applies. It includes, for example, all the preceeding (non-singular) cases, as well as many smooth compact spherical varieties (for definitions, see Sec. 2 below). It is interesting to note that, with the exception of the Atiyah-Jones conjecture for which the target space X is infinite-dimensional, all other varieties for which the theorem was known to hold were spherical; the present paper covers many X for which this is not the case, sim-

ple examples being provided by blow-ups of $\mathbb{P}^n$ along varieties in the hyperplane at infinity. One is still left, however, with the following important question:
QUESTION: What is the most general complex target space X which admits a topological stability theorem of the type described above?

For such a theorem to hold, the space of based holomorphic maps should build up in a regular way the entire space of based maps. The space $\mathrm{Map}(\Sigma, X)$ is often quite complicated topologically; thus, the spaces $\mathrm{Hol}_{\mathbf{k}}(\Sigma, X)$ of parameterized based holomorphic curves of a particular genus on X, must have, at least stably, an equivalent structure. In particular, for $\Sigma = \mathbb{P}^1 = S^2$ the Riemann sphere, if the loop space $\Omega^2(X)$ is non-trivial, there should be a plethora of rational curves on X, passing through, for example, any finite set of points. Such varieties are called rationally connected [29]. The role of rational curves in varieties is appearing to be more and more essential in their study, whether it is to define quantum cohomology, or to study higher dimensional varieties in Mori's program (in which the rationally connected varieties play an important role) (See e.g. [29, 31]). Our theorem can be seen as another manifestation of this, in a more homotopy theoretic setting; it is interesting that the topological property of stability should appear to be linked to some sort of rationality property of the manifold.

We thus prove the stability results for a large subclass of rationally connected varieties, in fact rational varieties on which a complex solvable linear group acts with a free dense open orbit, which we call *principal almost solv-varieties*. While many of the techniques in treating the geometry of the maps are taken directly from [5], the greater generality of the situation means that many of the proofs must be modified in a non-trivial way. In these cases we refer to [5] for much of the detail, only indicating the changes needed. However, the topological questions involved in passing from homology groups to homotopy groups (analyzing covering spaces) require a much finer analysis than that of [5], and the last two sections of the paper are devoted to it.

Actually, we can do a little better than principal almost solv-varieties, and remove the condition that the group be linear, by noticing that stability holds trivially for Abelian varieties, and making use of the Albanese map $\alpha : X \to \mathrm{Alb}(X)$. It turns out that in our case α is a locally trivial fibration whose fibres F are principal almost solv-varieties, and that the long exact sequence in homotopy implies that $\pi_2(X) \simeq \pi_2(F)$. We will show that $\pi_2(X)$ is free, and that it embeds into a fixed lattice $\mathbb{Z}^\kappa$. The multi-degree of an element of a homotopy class is then given in terms of this embedding. Our first main result is:

Theorem 1.1 (Homology Stability Theorem). *Let X be a smooth compact Kähler manifold which has a holomorphic action of a connected complex solvable Lie group S with an open orbit N on which S acts freely. Then for every multi-degree $\mathbf{k} = (k_1, \dots, k_\kappa)$ the inclusion $\iota(\mathbf{k}(X))$ induces an isomorphism in homology with $\mathbb{Z}$ coefficients through dimension $q(\mathbf{k})$ between the space of holomorphic maps* *of degree* $\mathbf{k}$ *and the space of continuous maps of degree* $\mathbf{k}$:

$$(\iota(\mathbf{k}(X)))_t : H_t(\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*; \mathbb{Z}) \cong H_t(\Omega^2_{\mathbf{k}} X; \mathbb{Z})$$

for

$$t \leq q(\mathbf{k}) = c_0 l(\mathbf{k}) - 1 .$$

Here $c_0(X) > 0$ *is a constant depending only on the space* X, *and* $l(\mathbf{k}) = \min(k_i)$.

To prove this theorem, we define stabilization maps from the spaces of holomorphic maps of degree $\mathbf{k}$ to spaces of maps of higher degree $\mathbf{k}'$. The proof of Theorem 1.1 uses a result of Gravesen [14], which gives a homology isomorphism between the loop space and the limit of the spaces of holomorphic maps. Theorem 1.1 tells us that the stabilization of the homology occurs in the nicest possible way, so that one does not, for instance, have for a fixed homology group, an infinite sequence of classes being created in moduli spaces of maps of ever higher degrees, only to disappear as one increases the degree a bit further.

In particular cases, the constant $c_0(X)$ can be estimated; some comments will be given after the proof of Theorem 1.1 in Sec. 4 below. The controlling factor is the structure of the singular set of the complement X_∞ of the free dense orbit of the solvable group. If, for example, X_∞ has smooth components with normal crossings (e.g. $\mathbb{P}^n$, toric varieties, Bott–Samelson varieties), one can obtain $c_0(X) = 1$; for cases such as $\mathbb{P}^n$, this is of course still far from the $c_0(X) = 2n - 1$ obtained by Segal [35]; it is however the known range for toric varieties [17].

Since $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ and $\Omega^2 X$ are not always simply connected, our second main result, Theorem 1.2 below, does not trivially follow from Theorem 1.1. Indeed, in general, the gap between homology isomorphisms and homotopy isomorphisms can be quite large, for example if one considers homology spheres versus homotopy spheres.

Theorem 1.2 (Homotopy Stability Theorem). *Let* X *be as in Theorem* 1.1. *Then for all* $\mathbf{k}$ *the inclusion* $\iota(\mathbf{k}(X))$ *is a homotopy equivalence through dimension* $c_0 l(\mathbf{k}) - s - 2$, *where* c_0 *and* $l(\mathbf{k})$ *are as in Theorem* 1.1, *and* s *is the rank of* $\pi_3(X) = \pi_1(\Omega^2 X)$.

The proof of Theorem 1.2, which is regrettably but necessarily quite technical since it must give information on the structure of the homology of cyclic covers of quite complicated spaces, the strata at infinity described above, is given in Secs. 5 and 6.

2. Principal Almost Solv-Varieties

2.1. *Almost homogeneous spaces*

Let $(X, \mathcal{O}_X)$ be an irreducible complex space, and let G be a connected complex Lie group acting as biholomorphic maps of $(X, \mathcal{O}_X)$. The definition given below is due to Remmert and Van de Ven [cf. 1, 19, 24].

Definition 2.1. Let X be an irreducible complex space and let G be a complex Lie group acting as biholomorphic transformations on X. We say that X is *almost homogeneous* if G has an open orbit N in X. If G acts freely on N so that N can be identified with G itself, we call X a *principal almost homogeneous space.*

Notice that a given complex space X can be almost homogeneous with respect to different Lie groups or with respect to different actions of the same Lie group. When possible confusion can arise we will specify the Lie group G and/or the open orbit N.

The complement $X\backslash N$ of N in X is an analytic subspace of X [1], and X can be viewed as an equivariant compactification of N by adding $X\backslash N$. We shall refer to $X\backslash N$ as "infinity" in X, and denote it by X_∞. (When $X = \mathbb{P}^n$, with the action of $\mathbb{C}^n$, X_∞ is indeed the hyperplane at infinity; when one switches to the group $(\mathbb{C}^*)^n$, X_∞ is the union of the coordinate hyperplanes). The Lie group G acts biholomorphically on the irreducible components of X_∞. As a variety X_∞ may have many irreducible components. For example for $X = G/P$, a generalized flag manifold, with the action of the unipotent group "opposite" to P, one finds that X_∞ is the union of the closures of all codimension one Bruhat cells in the Bruhat decomposition of G/P. The number of irreducible components of X_∞ then equals the number of codimension one Bruhat cells. When P is a Borel subgroup, this is the rank r of the group; more generally, one has $s \leq r$ components, with s equal to the dimension of the center of the Levi factor of the parabolic subgroup P.

Our first result provides a wealth of examples of almost homogeneous spaces. The proof of the following proposition is straightforward.

Proposition 2.1. *Let X be an almost homogeneous space with respect to the Lie group G, and let V be a complex subvariety of X_∞ of codimension greater than* 1 *in X that is stable under the action of G. Let $\pi : \tilde{X} \to X$ denote the blow-up of X along V. Then $\tilde{X}$ is almost homogeneous with respect to G, with open orbit $\tilde{N}$ given by the inverse image of N. In particular, if N is a principal orbit in X so is $\tilde{N}$ in $\tilde{X}$.*

We shall be especially interested in certain types of almost homogeneous spaces, namely those where the Lie group in question is a solvable complex Lie group S.

Definition 2.2. An irreducible complex space X that is almost homogeneous with respect to a connected solvable complex Lie group S is called an *almost solv-space.* Furthermore, if the action of S is free on the open orbit, X is called a *principal almost solv-space.* If X is a complex manifold, we use the terminology *almost solv-manifold.* If X is an algebraic variety and S is a connected solvable linear algebraic group acting algebraically on X, then X is called an *almost solv-variety.*

Examples are given below.

Proposition 2.2. *Let X be a normal almost solv-variety. Then infinity X_∞ has pure codimension one in X; hence, X_∞ is a Weil divisor on X.*

Proof. By a theorem of Snow [24] the open orbit $N \simeq \mathbb{C}^k \times (\mathbb{C}^*)^l$ as a complex manifold. Hence, N is holomorphically convex. Thus, for each component of X_∞, there are holomorphic functions on N which do not extend holomorphically across it. If this component had codimension ≥ 2, Hartog's theorem for normal varieties [25, p. 124] would imply that every holomorphic function would extend across it. Thus, every component of X_∞ has codimension 1 in X. $\square$

Proposition 2.3. *Any principal almost solv-variety is rational.*

Proof. As above let $n = k+l$ be the complex dimension of X. Since $N \simeq \mathbb{C}^k \times (\mathbb{C}^*)^l$, we have embeddings as dense open sets $X \hookleftarrow N \hookrightarrow \mathbb{P}^n$ which defines a birational map. $\square$

2.2. *Spherical varieties*

Definition 2.3. Let G be a connected complex reductive algebraic group. A normal algebraic variety X on which G acts as a group of algebraic transformations is called a *spherical variety* if some Borel subgroup B of G has a dense orbit in X. Let H be the stabilizer in G of a point in the dense orbit; the homogeneous space G/H is an open dense subvariety of X. The subgroup $H \subset G$ is called *spherical*.

It is clear that a spherical variety is an almost solv-variety with respect to the connected solvable linear complex group B. One can, however, often find a smaller solvable subgroup which has a *free* dense orbit. To see this we need to use the structure of spherical homogeneous spaces due to Brion, Luna, and Vust [8]. Let G, H, B be as in Definition 2.3. Then BH is open in G. Let P denote the subgroup of G defined by $P = \{p \in G | pBH = BH\}$. Since P contains B it is parabolic, and thus has a Levi decomposition $P \simeq L \rtimes U$ where L is reductive and $U = R_u(P)$ is the unipotent radical. Let $C = R(L)$ be the radical of L. It is an algebraic torus and the connected component of the center of L. Let N denote the dense open B-orbit in $G/H \subset X$. A theorem of [8] (see also [7]) says that as varieties $N \simeq U \times (C/C \cap H)$. Consider the exact sequence of algebraic groups

$$\langle e \rangle \to C \cap H \to C \to C/C \cap H \to \langle e \rangle .$$

If $C \cap H$ is connected then this sequence splits [3]. Let $T \subset C$, $T \simeq C/C \cap H$ denote the image of $C/C \cap H$ under such a splitting. For any such choice of T we denote by S the subgroup of B which is the semi-direct product $T \rtimes U$. Then by construction S acts freely on $N \simeq U \times (C/C \cap H)$. We have arrived at:

Proposition 2.4. *Let G be a reductive algebraic group, and let H, B, C, T be subgroups of G as described above. Suppose further that $C \cap H$ is connected. Then $S = T \rtimes U$ acts freely on the dense open B-orbit N. Hence, every spherical variety X with $C \cap H$ connected is a principal almost solv-variety with respect to the connected solvable linear algebraic group S.*

Remark 2.1. The following simple examples due to Akhiezer shows that the connectivity of $C \cap H$ in Proposition 2.4 is indeed necessary. First take $G = SL(2, \mathbb{C})$ and $H = \mathbb{C}^*$ any torus in G. Then any solvable subgroup S of G with a dense orbit must be a Borel subgroup for dimensional reasons. Moreover, the intersection of S with any torus contains the center $= \mathbb{Z}_2$, so S never acts freely. In this case $S = B$ and C is its reductive factor, and one easily sees that $C \cap H = \mathbb{Z}_2$. For another example let G be any complex semi-simple Lie group, and take H to be an extension of a maximal unipotent subgroup by a finite group. Then the intersection $C \cap H$ will be a non-trivial finite group.

For spherical varieties, there is a notion of rank, defined (see Brion [7]) to be the dimension of $C/C \cap H$ in the proof of Proposition 2.4. More generally, we have

Definition 2.4. We define the *rank* of any principal almost solv-variety X, denoted by rk (X), to be the dimension of the torus S/S_u, where S is the connected solvable algebraic group of Definition 2.2 and S_u is the subgroup of unipotent elements of S.

The rank is in a sense the crudest invariant of a principal almost solv-variety. Even in the spherical case a given spherical variety can be given the structure of a principal almost solv-variety in many inequivalent ways. This will be illustrated below.

2.3. *Examples of principal almost solv-varieties*

Example 2.1 Generalized flag manifolds. Let G be a semi-simple complex Lie group, and let P be a parabolic subgroup of G. The quotient G/P is a generalized flag manifold, and is obviously spherical. The unipotent radical U of P acts freely on an open orbit of G/P, as do all the conjugates of U. One can choose this conjugate to be "opposite" to P, so that its Lie algebra is a sum of those root spaces not lying in the Lie algebra of P. A variety X is a complete spherical variety of $\mathrm{rk}(X) = 0$ if and only if the spherical subgroup is parabolic and $X = G/P$ [7].

Example 2.2 Toric varieties. By definition a toric variety is a variety together with an action of $(\mathbb{C}^*)^n$ which is free on an open dense orbit. These varieties are spherical, with $G = (\mathbb{C}^*)^n$, and $H = \langle e \rangle$, and principal almost solv-varieties. Moreover, X is a toric variety if and only if $\mathrm{rk} X = \dim X$.

Example 2.3 $\mathbb{P}^n$ as a principal almost solv-variety. Some varieties can be principal solv-varieties in a variety of distinct ways. This is the case for projective spaces $\mathbb{P}^n$. Indeed, one can obtain them as flag manifolds (rank 0), toric varieties (rank n), or indeed, any rank between zero and n. For $\mathbb{P}^2$, one has for example a whole infinite sequence of rank one solvable groups acting in affine coordinates by

$$T_{\lambda,\zeta}(x, y) = (\lambda^k x, \lambda^l y + \zeta),$$

where $k \neq 0$ and k, l are coprime or $l = 0$. These are, in fact, all the ways to obtain $\mathbb{P}^2$ as a rank 1 almost solv-variety, up to isomorphism.

Example 2.4 Some examples of complete symmetric varieties. Let σ be the involution of $PGL(n,\mathbb{C})$ induced by conjugation by the matrix

$$\begin{pmatrix} 0 & I_r & 0 \\ I_r & 0 & 0 \\ 0 & 0 & I_s \end{pmatrix},$$

$n = 2r + s$, and let H be the fixed point subgroup of σ. The variety $SL(n,\mathbb{C})/H$ is spherical, and De Concini and Procesi [9] construct a spherical compactification X of this space which is smooth. One can show that the subgroup P is the parabolic subgroup whose Lie algebra is the sum of root spaces with roots α satisfying

$$\langle (r, r-1, \ldots, 1, -r, -r+1, \ldots, -1, 0, 0, \ldots, 0), \alpha \rangle \geq 0,$$

and that $C \cap H$ is connected.

Next we have examples of principal almost solv-varieties which are not spherical:

Example 2.5 Equivariant blow-UPS. As noted in Proposition 2.1, one can take any S-invariant subvariety of codimension greater than one in a principal almost-solv variety X and blow it up to obtain another principal almost-solv variety $\tilde{X}$. The invariant subvariety must then lie in X_∞. Taking X to be $\mathbb{P}^n$, for example, one can blow up any subvariety of the plane at infinity and obtain a principal almost-solv variety for the group $\mathbb{C}^n$. We note that this blowing up can reduce the automorphism group; for example, blowing up a set of points in $\mathbb{P}^n$ reduces the automorphism group from $PGL(n,\mathbb{C})$ to the subgroup which fixes that set. Blowing up a suitably generic set of points along infinity in $\mathbb{P}^n$ thus reduces the automorphism group to the $\mathbb{C}^n$ of translations, which acts trivially along infinity. Such a variety is definitely not spherical. More generally, one obtains a whole set of examples from the standard ones by blowing up subvarieties which are invariant under the action of the solvable group.

Example 2.6 Bott–Samelson varieties. Let us consider a given reductive group G, a fixed maximal torus T, and a Borel subgroup B, with negative simple roots $s_1, \ldots, s_r$ in the Lie algebra of G corresponding to generators $w_1, w_2, \ldots, w_r$ of the Weyl group. For each i, we have the parabolic subgroups P_i whose Lie algebra is obtained by adjoining the root space of s_i to the Lie algebra of B, so that $P_i/B = \mathbb{P}^1$. For each word $\tilde{w} = w_{i_1} w_{i_2} \cdots w_{i_k}$ in the generators of the Weyl group, we consider the quotient

$$P(\tilde{w}) = P_{i_1} \times P_{i_2} \times \cdots \times P_{i_k} / B^k,$$

where B^k acts by $(b_{i_1}, b_{i_2}, \ldots, b_{i_k})(p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = (p_{i_1} b_{i_1}^{-1}, b_{i_1} p_{i_2} b_{i_2}^{-1}, \ldots, b_{i_{k-1}} p_{i_k} b_{i_k}^{-1})$. This is smooth, and when the word is a reduced expression for an element

w for the Weyl group, it is a desingularization of the Schubert variety S_w given as the closure of the orbit BwB/B in G/B. Over the open cell, the map to BwB/B is an isomorphism. The cell BwB/B is a free orbit of a unipotent subgroup U_w of B, and this action lifts to $P(\tilde{w})$, giving it the structure of a principal almost-solv variety. One can show that the divisor at infinity has smooth components with normal crossings. These varieties are frequent objects of study in representation theory [26].

Example 2.7 Products. One can of course, take products of almost-solv-varieties to obtain new ones. For example, one obtains the compact complex symmetric spaces (in the sense of Borel [4]) associated to a reductive group G by taking products of flag manifolds and certain toric varieties ([30]).

2.4. *The topology of principal almost solv-varieties*

For any rational variety X, in fact for unirational varieties, there is a vanishing theorem $H^q(X, \mathcal{O}) = 0$ for $q > 0$. In addition, if X is smooth, the exponential exact sequence gives for the line bundles on X:

$$\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})\,. \tag{2.1}$$

Next, returning to our decomposition $X = N \cup X_\infty$, we write X_∞ as the union of irreducible algebraic hypersurfaces $X_\infty = \cup_{\alpha=1}^{\kappa} X_\alpha$, and have:

Proposition 2.5. *Let X be a connected smooth principal almost solv-variety, of complex dimension n. Then X is simply connected, $H_2(X, \mathbb{Z})$ is torsion free and* $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z}) \simeq H_2(X, \mathbb{Z}) \simeq \mathbb{Z}^{\kappa - l}$ *where κ is the number of irreducible components of X_∞ and $l = \mathrm{rk}(X)$.*

Proof. By Proposition 2.3, a principal almost solv-variety is rational, and the fundamental group is a birational invariant for algebraic varieties [13, p. 494]. Thus, X is simply connected.

The action of the solvable linear algebraic group S on the pair (X, N) gives the commuting diagram, with exact rows, in homology with integer coefficients:

$$\begin{array}{ccccccc}
\cdots & H_*(S) \otimes H_*(N) & \xrightarrow{id \times \alpha_*} & H_*(S) \otimes H_*(X) & \xrightarrow{id \times j} & H_*(S) \otimes H_*(X, N) & \\
 & \Big\downarrow \beta_* & & \Big\downarrow \beta_* & & \Big\downarrow \beta_* & \\
\cdots & H_*(N) & \xrightarrow{\alpha_*} & H_*(X) & \xrightarrow{j} & H_*(X, N) & \\
 & \xrightarrow{id \times \partial} & H_*(S) \otimes H_{*-1}(N) & \xrightarrow{id \times \alpha_{*-1}} & \cdots & & \\
 & & \Big\downarrow \beta_* & & & & \\
 & \xrightarrow{\partial} & H_{*-1}(N) & \xrightarrow{\alpha_{*-1}} & \cdots & &
\end{array} \tag{2.2}$$

where β_* is the induced map in homology. Under our assumptions, $H_*(N) = H_*(S)$, and $H_*(S)$ is generated by $H_1(S)$.

In dimension one the simple connectivity of X implies that $\partial : H_2(X,N) \to H_1(N) = \mathbb{Z}^l$ is onto. Since a set of generators for $H_i(N)$ has the form $\{\beta_*(e_j \otimes v_j)\}$ where the e_j run over $H_{i-1}(S)$ and the v_j generate $H_1(N)$, the formula

$$\partial\beta_*(e_j \otimes w_j) = -\beta_*(e_j \otimes \partial(w_j)) \tag{2.3}$$

shows that $\partial : H_i(X,N) \to H_{i-1}(N)$ is onto for each i. This implies that in each dimension greater than or equal to two we have the short exact sequence

$$0 \to H_i(X) \to H_i(X,N) \to \mathbb{Z}^{l_{i-1}} \to 0 \tag{2.4}$$

where $H_{i-1}(N) = \mathbb{Z}^{l_{i-1}}$, and consequently, splittings

$$H_i(X,N) = H_i(X) \oplus \mathbb{Z}^{l_{i-1}}\,.$$

By Alexander-Poincare duality

$$H_2(X,N) \simeq H^{2n-2}(X_\infty) \simeq \mathbb{Z}^\kappa\,.$$

So the exact sequence (2.4) shows that $H_2(X) \simeq \mathbb{Z}^{\kappa-l}$. In particular $H_2(X)$ is torsion free, and so by the universal coefficients theorem, $H_2(X) \simeq H^2(X)$. $\square$

Let r denote the rank of $\mathrm{Pic}(X) \simeq H_2(X) \simeq \pi_2(X)$. The components of $\Omega^2 X$ are labeled by a multi-degree $\mathbf{k} = (k_1, \ldots, k_r)$ and $\Omega^2_{\mathbf{k}} X$ denotes such a component. There is a basis $\{L_i\}_{i=1}^r$ for $\mathrm{Pic}(X)$ such that for $f \in \Omega^2_{\mathbf{k}} X$, $c_1(f^* L_i) = k_i$ for all $i = 1, \ldots, r$. It is also convenient to give a description in terms of the line bundles $[X_\alpha]$ associated to the divisor X_α with certain relations. Dualizing the exact sequence (2.4) gives

$$\begin{tikzcd}
0 \arrow[r] & H^1(N) \arrow[r] \arrow[d, "\simeq"] & H^2(X,N) \arrow[r] \arrow[d, "\simeq"] & H^2(X) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathbb{Z}^l \arrow[r] & H_{2n-2}(X_\infty) \arrow[r, "\psi"] & \mathrm{Pic}(X) \arrow[r] & 0.
\end{tikzcd} \tag{2.5}$$

The map ψ sends the class $\mathrm{cl}(X_\alpha)$ in $H_{2n-2}(X_\infty)$ representing the irreducible component X_α to the line bundle $[X_\alpha]$. As this is an epimorphism of free Abelian groups, for each L_i there is a sum V_i of components X_α such that $\psi(\mathrm{cl}(V_i)) = L_i$. Thus, L_i and $[V_i]$ define the same element of $\mathrm{Pic}(X)$, so $c_1(f^*[V]) = k_i$. Alternatively, k_i is the intersection number of $f(S^2)$ with the variety V_i at infinity.

3. The Poles and Principal Parts Description

3.1. *The sheaf of principal parts*

For any complex space X we let $\mathcal{O}(X)$ denote the sheaf of germs of holomorphic maps from $\mathbb{P}^1$ into X, and $\mathcal{O}(U,X)$ denote the holomorphic sections of $\mathcal{O}(X)$ over the open set $U \subset \mathbb{P}^1$. If the space X has a base point, $\mathcal{O}(X)$ will be a sheaf of pointed sets. We denote the global holomorphic maps by $\mathrm{Hol}(\mathbb{P}^1, X)$, and the global based holomorphic maps by $\mathrm{Hol}(\mathbb{P}^1, X)^*$. There is natural inclusion

$$\mathrm{Hol}(\mathbb{P}^1, X)^* \hookrightarrow \Omega^2 X \tag{3.1}$$

into the space of all based continuous maps $\Omega^2 X$.

Let X be an almost solv-variety, with $N = X \backslash X_\infty$ the dense open orbit in X. Following Gravesen [14], we define the presheaf of meromorphic maps from $\mathbb{P}^1$ to N by setting for each open U in $\mathbb{P}^1$:

$$\mathcal{M}(U, N) = \mathcal{O}(U, X) \backslash \mathcal{O}(U, X_\infty) \tag{3.2}$$

and we let $\mathcal{M}(N)$ denote its associated sheaf. Let S be the connected solvable linear algebraic group that acts biholomorphically on X and transitively on N. There is a natural action of the sheaf $\mathcal{O}(S)$ on $\mathcal{M}(N)$ given on the presheaf level by pointwise multiplication: for each open set $U \subset \mathbb{P}^1$, the action of $\mathcal{O}(U, S)$ on $\mathcal{M}(U, N)$ sends a local section $f_U \in \mathcal{M}(U, N)$ to the local section $g_U \cdot f_U$ defined by $g_U \cdot f_U(z) = g_U(z) \cdot f_U(z)$. This allows us to define the quotient sheaf

$$\mathcal{PP} \simeq \mathcal{M}(N)/\mathcal{O}(S) \tag{3.3}$$

to be the sheaf of germs of equivalence classes of meromorphic maps where f, $f' \in \mathcal{M}(U, N)$ are equivalent if there is $g \in \mathcal{O}(U, S)$ such that $f' = g \cdot f$. The sheaf $\mathcal{PP}$ is called the sheaf of *principal parts* in X. Notice that the stalk $\mathcal{PP}_z$ is actually independent of the location $z \in \mathbb{P}^1$ of the pole: one can simply translate the maps from point to point, by an automorphism of $\mathbb{P}_1$; over $\mathbb{C} \subset \mathbb{P}^1$, this can be done unambiguaously by translations. We thus denote the space $\mathcal{PP}_z$ of "local principal parts" simply by $\mathcal{LPP}$.

To give an idea of what the stalk of this sheaf looks like, we give a few examples. We consider the stalk at $z = 0$.

(1) When $X = \mathbb{P}^n$, $N = \mathbb{C}^n$ and $S = \mathbb{C}^n$, the principal parts construction yields exactly the classical principal parts of a meromorphic map. Indeed, one has that the local form of a map into $\mathbb{P}^n$ near $z = 0$ is given by an n-tuple of Laurent series. Let us write the ith entry of this n-tuple as $a_k z^k + a_{k+1} z^{k+1} + \cdots$. Normalizing these series under the additive action of maps into $\mathbb{C}^n$ just allows us to kill off all the positive order terms, leaving just the negative order terms, i.e. the principal part. We note that in our sense a principal part encodes both some discrete information (the order of the pole) as well as some continuous information (the principal part or Laurent tail).

(2) When $X = \mathbb{P}^n$, $N = (\mathbb{C}^*)^n$ and $S = (\mathbb{C}^*)^n$, one obtains the other classical description of a map in terms of zeroes in homogeneous coordinates. Indeed, one can normalize the series $a_k z^k + a_{k+1} z^{k+1} + \cdots$ under the multiplicative action of maps into $(\mathbb{C}^*)$ to the monomial z^k, so that the information contained in the principal part is simply the order of each component, or, equivalently, the multiplicity

of intersection with each component of X_∞, which in this case are the coordinate hyperplanes. The information contained in the principal part is then discrete in nature.

(3) The same calculation of a normal form holds for the maps into a general toric variety: indeed, one has the same local coordinates as in (2) above. Thus the information contained in a principal part in the toric case is simply the multiplicity of intersection with the components of the divisor at infinity, which in this case are simply the closures of the codimension one orbits. This ties in with the description of the maps given in [17].

(4) A computation of some spaces of principal parts for the space of full flags in $\mathbb{C}^3$ is given in [5].

The sheaf $\mathcal{PP}$ in many ways resembles the sheaf of divisors on a complex manifold, in that when one considers a fixed section and restricts it to stalks, one obtains the trivial element (corresponding to maps into N) generically, and, over exceptional points (the *support* of the section), some non-trivial element. We will, in all cases, refer to the points in the support as *poles*, even though in some examples, such as $\mathbb{P}_1$ in its toric description, some of the poles are actually zeroes!

3.2. *Configurations of principal parts*

Now assume that X is a smooth almost solv-variety. An element $P \in H^0(\mathbb{P}^1, \mathcal{PP})$ can be represented by a sequence of pairs (U_i, f_i) where $\{U_i, i = 0, \ldots, n\}$ is a finite cover of $\mathbb{P}^1$ and $f_i \in \mathcal{M}(U_i, N)$. The pull-back $f_i^* X_\infty$ is a divisor on U_i, i.e. a finite sum of points. Furthermore, we can choose the cover $\{U_i\}$ so that $U_i \cap U_j = \emptyset$ for $i,\ j > 1$, $f_0(U_0) \subset N$, and $f_i(U_i), i = 1, \ldots, n$ intersects infinity X_∞ at a single point $f_i(z_i)$. We shall refer to such covers as *P-good* covers. A global section in $H^0(\mathbb{P}^1, \mathcal{PP})$ is called a *configuration of principal parts*. It consists of a finite number of points $z_i \in \mathbb{P}^1$ (location of the poles), together with the *local* principal parts data, a non-trivial element in the stalk $\mathcal{LPP} = \mathcal{PP}_{z_i} \simeq \mathcal{M}_{z_i}/\mathcal{O}(S)_{z_i}$ at each point z_i. We view $H^0(\mathbb{P}^1, \mathcal{PP})$ as a labelled configuration space with $\mathcal{LPP}$ as the space of labels. The topology on $H^0(\mathbb{P}^1, \mathcal{PP})$ is the quotient topology from $\mathcal{M}(U, N)$ with the later given the compact-open topology. See [14] for details.

An element in $\mathrm{Hol}(\mathbb{P}^1, X)^*$ will naturally determine a section in $H^0(\mathbb{P}^1, \mathcal{PP})$, whose poles are the points mapped to X_∞. We now fix the basing condition. We choose the base point in $\mathbb{P}^1 \simeq S^2 \simeq \mathbb{C} \cup \{\infty\}$ to be the north pole $\{\infty\}$ and the base point of X to be a fixed point $* \in N \subset X$. This precludes a map in $\mathrm{Hol}(\mathbb{P}^1, X)^*$ having a pole at $\infty \in \mathbb{P}^1$. Let $H^0(\mathbb{C}, \mathcal{PP})$ denote the subspace of $H^0(\mathbb{P}^1, \mathcal{PP})$ whose poles are all located in $\mathbb{C} \simeq \mathbb{P}^1 - \{\infty\}$, so that our holomorphic map determines an element of $H^0(\mathbb{C}, \mathcal{PP})$. There is then a map

$$\mathrm{Hol}(\mathbb{P}^1, X)^* \to H^0(\mathbb{C}, \mathcal{PP}), \tag{3.4}$$

which we shall see is an inclusion, so that the holomorphic maps are determined by their principal parts. We are interested in when a configuration of principal parts $P \in H^0(\mathbb{C}, \mathcal{PP})$ comes from a holomorphic map $f \in \mathrm{Hol}(\mathbb{P}^1, X)^*$. In general there is an obstruction cocycle coming from the action of $(\mathbb{C}^*)^l \subset S$ which we shall describe shortly.

3.3. *Virtual multi-degrees and label spaces*

We have just seen that a based holomorphic map $f \in \mathrm{Hol}(\mathbb{P}^1, G/P)^*$ corresponds to a finite set of points $f^*X_\infty \subset \mathbb{P}^1$ together with labels in a space $\mathcal{LPP}$. These labels carry information about the map. One piece of information the labels always carry is that of a virtual multi-degree.

Indeed, any element LP of $\mathcal{LPP}$ is represented by a map f of a disk in $\mathbb{C}$ into X, with the center of the disk being mapped to X_∞ and the rest of the disk being sent to $N = X \backslash X_\infty$. The maps of the disk into S acts on this by homotopies, and so one has a well-defined multiplicity or *virtual degree* of LP as an element of $H_2(X, N; \mathbb{Z})$ which from the proof of Proposition 2.5 is $\mathbb{Z}^\kappa$. There exist natural generators for $H_2(X, N; \mathbb{Z})$, given by the Alexander-Poincaré duals to the components X_α of the divisor X_∞. In this basis, we write the multiplicity as a vector

$$(m_1, \ldots, m_\kappa)\,. \tag{3.5}$$

Analytically, we can obtain the m_α by considering the holomorphic sections s_α of the line bundles $[X_\alpha]$ vanishing at the hypersurface X_α. These are invariant up to a factor by the action of S. The multiplicity m_α is the order of the vanishing of the pull-back f^*s_α.

Let P belong to $H^0(\mathbb{C}, \mathcal{PP})$, with poles z_i, $i = 1, \ldots, r$, and local representatives $f_i : U_i \to X$ for the principal parts. We have multiplicities associated to each f_i, and so, summing, a multiplicity or virtual degree for P:

$$\mathbf{k}(P) = \sum_{i=1}^{r} \mathbf{m}^i = (k_1, \ldots, k_\kappa) \qquad k_\alpha = \sum_{i=1}^{r} m_\alpha^i\,. \tag{3.6}$$

We define some notation. If the point z_i is mapped by f_i to the divisor X_α, we will refer to z_i as an α-pole. We note that z_i can be both an α-pole and an β-pole, by being mapped to $X_\alpha \cap X_\beta$. An α-pole is called *simple* if $m_\beta = 1$ for $\beta = \alpha$ and $m_\beta = 0$ if $\beta \neq \alpha$. The label α will sometimes be referred to as the *color* of the pole.

We can also define the *scalar multiplicity* of the ith pole by

$$|\mathbf{m}^i| = \sum_{\alpha=1}^{\kappa} m_\alpha^i\,. \tag{3.7}$$

Similarily, the scalar virtual degree is $|\mathbf{k}| = \sum_\alpha k_\alpha$.

We can decompose $H^0(\mathbb{C}, \mathcal{PP})$ as the disjoint union

$$H^0(\mathbb{C}, \mathcal{PP}) = \bigsqcup H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})\,, \tag{3.8}$$

where $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ is the subspace of those principal parts with virtual multi-degree equal to $\mathbf{k}$. In the same way, the local principal parts space decomposes according to multi-degree:

$$\mathcal{LPP} = \bigsqcup_{\mathbf{m}} \mathcal{LPP}_{\mathbf{m}}\,. \tag{3.9}$$

Example 3.1. To fix our ideas, we will consider a simple example, that of degree one based maps from $\mathbb{P}^1$ to $\mathbb{P}^2$, to see what configurations we can obtain. The based holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^2$ are given in homogeneous coordinates by $f(z) = [z - z_0, z - z_1, z - z_2]$. The only constraint for this to be a well defined map is that all homogeneous coordinates cannot vanish simultaneously. Thus the space of degree one maps in $\mathrm{Hol}(\mathbb{P}^1, \mathbb{P}^2)^*$ is $\mathbb{C}^3 - \{z_0 = z_1 = z_2\}$. The poles and principal parts description of this space varies according to the choice of group.

(1) When one considers $\mathbb{P}^2$ with the action of translation of $\mathbb{C}^2$, infinity is a single hyperplane divisor, say X_0, defined by setting the first homogeneous coordinate equal to zero. There is one pole, with position z_0. Fixing the pole, the label space $\mathcal{LPP}_1$ corresponds to the residues $z_0 - z_1$ and $z_0 - z_2$, and so $\mathcal{LPP}_1 \simeq \mathbb{C}^2 - \{(0, 0)\}$. The space of maps is then an $\mathcal{LPP}_1$ fibration over $\mathbb{C}$.
(2) At the opposite extreme is the toric description of $\mathbb{P}^2$. Here infinity is the union $\cup_{\alpha=0}^2 X_\alpha$ of the coordinate hyperplane divisors $X_\alpha = \{[x_0, x_1, x_2] | x_\alpha = 0\}$, $\alpha = 0, 1, 2$. There are three "poles" of different colors, with position z_0, z_1, z_2 with the constraint that all three z_i are not equal, corresponding to the fact that the three coordinate lines do not intersect. The spaces of degree one maps splits into strata: the generic stratum corresponds to the z_i being distinct, so that there is one pole of multiplicity (1, 0, 0), one of multiplicity (0, 1, 0), and one of multiplicity (0, 0, 1). The other 3 strata correspond to two of the poles coinciding, so that there is one stratum of maps with one pole of multiplicity (1, 1, 0) and one of multiplicity (0, 0, 1), one stratum with multiplicities (1, 0, 1), (0, 1, 0) and one stratum with multiplicities (0, 1, 1), (1, 0, 0). In all cases, the label spaces $\mathcal{LPP}_{(i,j,k)}$ are single points.
(3) The intermediate case corresponds to the rank one examples of Example 2.3. In this case infinity X_∞ is the singular variety $\{[x_0, x_1, x_2] | x_0 x_1 = 0\}$. There are two poles z_0, z_1, and the mapping space divides into two strata, according to whether z_0, z_1 coincide or not. When the poles are different, one has one pole of multiplicity (1, 0), with label in $\mathcal{LPP}_{(1,0)} = pt$ and one pole of multiplicity (0, 1), with label in $\mathcal{LPP}_{(0,1)} = \mathbb{C}$. When the poles coincide, one then has one pole of multiplicity (1, 1), and the label space $\mathcal{LPP}_{(1,1)}$ is $\mathbb{C}^*$.

The "additivity" of principal parts plays a crucial role in stability theorems. Let P, $P' \in H^0(\mathbb{C}, \mathcal{PP})$ be such that they have no poles in common, then their union $P \cup P'$ is a configuration of principal parts in an obvious way. Let us take an isotopy of $\mathbb{C}$ into a disk D; this defines a diffeomorphism $H^0(\mathbb{C}, \mathcal{PP}) \simeq H^0(D, \mathcal{PP})$, (which is not holomorphic). This allows us to define on $H^0(\mathbb{C}, \mathcal{PP})$ the structure of a homotopy-associative monoid: one first maps the configurations of principal parts over $\mathbb{C}$ into configurations of principal parts over disjoint disks D and D', then adds them. This gives

Proposition 3.1. *There is a continuous map*

$$A : H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}) \times H^0_{\mathbf{l}}(\mathbb{C}, \mathcal{PP}) \to H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP})\,,$$

where $\mathbf{k}+\mathbf{l}$ *has components* $k_\alpha + l_\alpha$.

3.4. *Configurations and holomorphic maps*

It is quite straightforward to see that the map $\mathrm{Hol}(\mathbb{P}^1, X)^* \to H^0(\mathbb{C}, \mathcal{PP})$ is injective. Indeed, one has that any two based maps with the same principal parts differ by the action of a global map $\mathbb{P}^1 \to S$; these maps are constant, and the basing condition guarantees that the constant is the identity, so that the maps coincide.

We now consider the question of which configurations of principal parts correspond to holomorphic maps. When a configuration of principal parts P is obtained from a based holomorphic map $f \in \mathrm{Hol}(\mathbb{P}^1, X)^*$, the corresponding multi-degree $\mathbf{k}$ is constrained to lie in the image of $H_2(X;\mathbb{Z}) \subset H_2(X, N;\mathbb{Z})$. We will show that this is the only obstruction to a configuration of principal parts being derived from a map.

As with principal parts we denote the subspace of $\mathrm{Hol}(\mathbb{P}^1, X)^*$ of maps having multi-degree $\mathbf{k}$ by $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$. For any complex space M with a sheaf of groups $\mathcal{O}(S)$ we recall the definition of $H^1(M, \mathcal{O}(S))$. Given a cover $\{U_i\}$ of M by open sets we consider holomorphic maps $g_{ij} : U_i \cap U_j \to \mathcal{O}(S)$ which satisfy the cocycle condition $g_{ij}g_{jk}g_{ki} = e$ in $U_i \cap U_j \cap U_k$. Two such maps g_{ij} and g'_{ij} are equivalent if for each i there is a holomorphic map $\phi_i : U_i \to \mathcal{O}(S)$ such that for all i, j we have $g'_{ij} = \phi_i^{-1} g_{ij} \phi_j$. Then $H^1(M, \mathcal{O}(S))$ is defined to be the limit, under refinement of the cover, of the set of equivalence classes of such equivalence maps and it classifies equivalence classes of S-bundles over M. If the cover is acyclic (Leray), one can compute $H^1(M, \mathcal{O}(S))$ directly from the cover, without taking limits, as in the Abelian case.

Let us now consider our case, where X is a principal almost homogeneous space with respect to the complex Lie group S. We recall that S acts freely on the open orbit N, so that N can be identified with S itself. Given a configuration of principal parts P represented by local holomorphic maps $f_i \in \mathcal{M}(U_i, N)$ with respect to the P-good cover $\{U_i\}$, we consider the restrictions of f_i and f_j to the intersection $U_i \cap U_j$. Since S acts freely on N, there is a unique holomorphic map $g_{ij} : U_i \cap U_j \to \mathcal{O}(S)$ such that $f_i = g_{ij} \cdot f_j$ in $U_i \cap U_j$. Moreover, by refining the cover if necessary we can check that a different representative $f'_i \in \mathcal{M}(U_i, N)$ of P gives rise to a cocycle g'_{ij} that is related to g_{ij} by $g'_{ij} = g_i g_{ij} g_j^{-1}$ where $g_i : U_i \to \mathcal{O}(S)$ are holomorphic. So we get a well-defined element $[g_{ij}] \in H^1(\mathbb{P}^1, \mathcal{O}(S))$. Principal parts arising from a holomorphic map give the trivial cocycle, and so we have a sequence of maps of pointed sets

$$\langle e \rangle \longrightarrow \mathrm{Hol}(\mathbb{P}^1, X)^* \longrightarrow H^0(\mathbb{C}, \mathcal{PP}) \xrightarrow{\delta} H^1(\mathbb{P}^1, \mathcal{O}(S))\,, \tag{3.10}$$

and we would like to think of this as being exact.

Definition 3.1 [15]. Let A, B, C be sets with C a set with a neutral element e. Then the sequence

$$A \longrightarrow B \xrightarrow{\nu} C$$

is said to be *exact* (at B) if the kernel $\nu^{-1}(e)$ equals image(A).

Warning: If $A = 0$ (a one point set), then exactness at B does not imply the injectivity of ν. We refer the reader to the lecture notes of Grothendieck [15] for further detail.

In our case, the set $H^1(\mathbb{P}^1, \mathcal{O}(S))$ has a preferred or "neutral" element, namely the class of the identity element in $C^1(U_i \cap U_j, \mathcal{O}(S))$. To obtain exactness, note that an element of $H^0(\mathbb{C}, \mathcal{PP})$ that maps to a trivial cocycle in $H^1(\mathbb{P}^1, \mathcal{O}(S))$ can be split as $\phi_i\phi_j^{-1}$. As this then guarantees that the local maps f_i patch together to give a global section in $H^0(\mathbb{P}^1, \mathcal{M}(N))^*$, we have exactness, that is the following Mittag–Leffler type result:

Theorem 3.1. *Let X be a smooth complex principal almost homogeneous space. Then the configurations of principal parts* (*i.e. elements of* $H^0(\mathbb{C}, \mathcal{PP})$) *that can be represented by based holomorphic maps* (*i.e. elements of* $\mathrm{Hol}(\mathbb{P}^1, X)^*$) *are precisely those lying in the kernel of the map δ of* (3.10).

Let us return to the case at hand when S is a connected solvable algebraic group. There is a short exact sequence of sheaves of groups on $\mathbb{P}^1$,

$$\langle e\rangle \longrightarrow \mathcal{O}(R) \longrightarrow \mathcal{O}(S) \longrightarrow \mathcal{O}(T) \longrightarrow \langle e\rangle\,, \tag{3.11}$$

where T is an algebraic torus of rank l and R is the normal subgroup of unipotent elements of S. Since R is unipotent we have a vanishing theorem for $H^1(\mathbb{P}^1, \mathcal{O}(R))$. When R is Abelian and unipotent this is quite classical, since $\mathcal{O}(R)$ is isomorphic to a sum of copies of the structure sheaf $\mathcal{O}$. However, when R is not Abelian this is not so well known, so we give the proof; see also [14].

Lemma 3.1. $H^1(\mathbb{P}^1, \mathcal{O}(R)) = 0$.

Proof. We have the composition series $\langle e\rangle = R_n < R_{n-1} < \cdots < R_0 = R$ for some positive integer n, with each factor R_i/R_{i+1} Abelian. So we have the exact sequence

$$\langle e\rangle \longrightarrow \mathcal{O}(R_{i+1}) \longrightarrow \mathcal{O}(R_i) \longrightarrow \mathcal{O}(R_i/R_{i+1}) \longrightarrow \langle e\rangle$$

of sheaves of groups on $\mathbb{P}^1$. According to Sec. 5.3 of Grothendieck [14], this gives an exact sequence (as in Definition 3.1)

$$H^1(\mathbb{P}^1, \mathcal{O}(R_{i+1})) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(R_i)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(R_i/R_{i+1}))\,.$$

But since $R_n = \langle e\rangle$, R_{n-1} is Abelian, and each factor group R_i/R_{i+1} is Abelian, the first and last terms of this sequence vanish. But this implies the vanishing of the middle term. An easy induction finishes the proof. $\square$

We are now ready for:

Proposition 3.1. *Let X be a principal almost solv-variety. Then there is a map $H^1(\mathbb{P}^1, \mathcal{O}(S)) \xrightarrow{\nu} H^1(\mathbb{P}^1, \mathcal{O}(T)) \simeq \mathbb{Z}^l$. Furthermore, $\gamma \in H^1(\mathbb{P}^1, \mathcal{O}(S))$ maps to zero in $\mathbb{Z}^l \simeq H^1(\mathbb{P}^1, \mathcal{O}(T))$ if and only if it represents the trivial bundle.*

Proof. The exact sequence (3.11) implies the existence of the map ν (See [14, Sec. 5.3]). If γ represents the trivial bundle in $H^1(\mathbb{P}^1, \mathcal{O}(S))$, one easily sees that $\nu(\gamma) = 0$. Conversely, suppose that γ maps to 0 in $H^1(\mathbb{P}^1, \mathcal{O}(T))$, and choose a good cover U_i, so that the only non-empty double intersections are $U_0 \cap U_i$, $i = 1, \ldots, n$. Let $s_{0i} \in C^1(U_0 \cap U_i, \mathcal{O}(S))$ be a cocycle representing γ. We can use the fact that (3.11) splits to write $s_{0i} = u_{0i}t_{0i}$ with $u_{0i} \in C^1(U_0 \cap U_i, \mathcal{O}(R))$, and $t_{0i} \in C^1(U_0 \cap U_i, \mathcal{O}(T))$. By hypothesis, one can write $t_{0i} = t_0^{-1}t_i$, so that s_{0i} can be replaced by the equivalent cocycle $\tilde{s}_{0i} = t_0 s_{0i} t_i^{-1} = t_0(u_{0i}t_0^{-1}t_i)t_i^{-1} = t_0 u_{0i} t_0^{-1}$. Since the subgroup R is normal, this last cocycle lies in $\mathcal{O}(R)$, and therefore, by Lemma 3.1, can be written as $u_0 u_i$, expressing $\tilde{s}_{0i}$, and therefore s_{0i}, as a coboundary. □

We recall that a principal part gives a well defined element of the relative homology $H_2(X, N)$. Also, there is a well defined map from $H^1(\mathbb{P}^1, \mathcal{O}(S))$ to $H_1(N; \mathbb{Z})$, given with respect to any good cover of $\mathbb{P}^1$ by $U_0 = \mathbb{P}^1 - \{p_1, \ldots, p_j\}$ and disjoint disks U_i centred at p_i, as follows: a cocycle g_{0i} for this covering can be restricted to circles around the punctures, giving a well defined element of $H_1(N; \mathbb{Z})$, under the identification of N with S. The map $H^1(\mathbb{P}^1, \mathcal{O}(S)) \to H_1(N; \mathbb{Z})$ factors through $H^1(\mathbb{P}^1, \mathcal{O}(T))$, and in fact gives an isomorphism $H^1(\mathbb{P}^1, \mathcal{O}(T)) \simeq H_1(N; \mathbb{Z})$. With this identification, we have an exact sequence of commutative diagrams:

$$\begin{array}{ccccccccc} \langle e \rangle & \longrightarrow & \mathrm{Hol}\,(\mathbb{P}^1, X)^* & \longrightarrow & H^0(\mathbb{C}, \mathcal{PP}) & \xrightarrow{\delta} & H^1(\mathbb{P}^1, \mathcal{O}(S)) & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow \psi & & \downarrow \rho & & \\ 0 & \longrightarrow & H_2(X) & \longrightarrow & H_2(X, N) & \longrightarrow & H_1(N) & \longrightarrow & 0\,. \end{array} \tag{3.12}$$

The map ψ associates to each configuration of principal parts a virtual multi-degree $(k_1, \ldots, k_\kappa)$ in $H_2(X, N)$. The "real" or "good" multi-degrees $(k_1, \ldots, k_\kappa)$ are those in the subgroup $H_2(X) \simeq \mathbb{Z}^{\kappa - l} \subset H_2(X, N)$. From the diagram, one has:

Proposition 3.2. *A configuration of principal parts $P \in H^0(\mathbb{P}^1, \mathcal{PP})$ represents a holomorphic map $f \in \mathrm{Hol}(\mathbb{P}^1, X)^*$ if and only if the virtual multi-degree $\mathbf{k} = (k_1, \ldots, k_\kappa)$ maps to zero in $H_1(N)$. For multi-degrees which do map to zero, one has a homeomorphism*

$$H^{\mathbf{0}}_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}) \simeq \mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*\,.$$

In particular, this gives the principal part spaces the structure of algebraic varieties.

4. Homology Stabilization of Principal Parts

4.1. *A stratification of* $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$

In this section unless otherwise stated X will denote a smooth principal almost solv-variety with respect to a connected solvable linear algebraic group S. We will exploit the isomorphism given by Proposition 3.2 to give a proof of homology stability theorems. The proof relies on two key phenomena. The first is that spaces of labelled particles, such as configurations of principal parts, exhibit a natural stability property in homology as one increases the number of particles. Thus one must check that the space behaves sufficiently like a space of many particles. This is ensured by the second key point, proving that the space of maps with poles of high multiplicity (and therefore fewer poles) has a high homological codimension. This in turn is done by showing that the space is smooth, and that the geometric codimension of the high multiplicity set is high.

Proposition 4.1. *Let X be a smooth complex principal almost solv-variety, and* $\mathbf{k}$ *a good virtual multi-degree. Then the space* $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}) = \mathrm{Hol}_{\mathbf{k}}(\mathbb{P}, X)^*$ *is a smooth finite dimensional complex manifold.*

Proof. The deformation theory proof of smoothness of $\mathrm{Hol}(\mathbb{P}^1, X)^*$ in ([5, Theorem 4.1]) carries over without any essential changes to our case. This is due to the fact that the transitivity of the action of G on the dense affine open set N guarantees that at a map f the pull-back f_i^*TX splits as the sum of positive line bundles, i.e.

$$f_i^*TX \simeq \oplus_i(\mathcal{O}(\lambda_i)) \tag{4.1}$$

with $\lambda_i \geq 0$. The obstruction to smoothness $H^1(\mathbb{P}^1, f^*TX)$ then vanishes. □

Let us fix an irreducible component X_α of X_∞. We let $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})$ denote the subspace of $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ of those principal parts with only one α-pole, which is then of multiplicity k_α; the other poles can be of arbitrary multiplicity. More generally, for any S-stable subvariety $V \subset X_\alpha$, we define $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})(X, V)$ to be the subset of $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})$ consisting of those principal parts P whose α-pole have their image in V. In particular, we are interested in the cases when $V = X^*_\alpha$, the smooth locus at infinity, or $V = X^s_\alpha = X_\alpha - X^*_\alpha$, the singular locus at infinity. Also, if $\mathbf{k}$ is a multi-index, we let $H^0_{\mathbf{k},\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ be the subvariety of $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ with only one pole of multiplicity $\mathbf{k}$, i.e. α-multiplicity k_α for each $\alpha = 1, \ldots, \kappa$. We remark that $H^0_{\mathbf{k},\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ could be empty.

Proposition 4.2. *Taking "codimension" to mean complex codimension in* $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$,

(i) *The codimension of* $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})(X, X^*_\alpha)$ *is* $k_\alpha - 1$.
(ii) *The codimension of* $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})(X, X^s_\alpha)$ *is bounded below by* $\max\{C_\alpha k_\alpha - C'_\alpha, 1\}$.

(iii) *There are constants* $0 < C_\alpha \leq 1$ *and* $C'_\alpha \geq 0$ *depending only on* α *and* X *such that the codimension of* $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})$ *is bounded below by* $C_\alpha k_\alpha - C'_\alpha$, *and by* 1 *if* $k_\alpha > 1$.

(iv) *There are constants* $0 < \hat{C} \leq 1$ *and* $\hat{C}' \geq 0$ *depending only on* X *such that the codimension of* $H^0_{\mathbf{k},\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ *is bounded below by* $\hat{C}|\mathbf{k}| - \hat{C}'$.

(v) *Unless* $k_\alpha = \delta_{\alpha,\beta}$ *for some* β, *the codimension of* $H^0_{\mathbf{k},\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ *is at least one.*

Proof. The proof differs only slightly from the proof of ([5, Theorem 5.20]), given there for the generalized flag manifolds G/P. We summarize here the salient points. We begin with the proof of (i). Let us place ourselves at a point P of $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})(X, X^*_\alpha)$, with α-pole at z. We stabilize $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})(X, X^*_\alpha)$ by adding in a large number of principal parts located away from the support of P, in such a way that the final multidegree $\mathbf{k} + \mathbf{l}$ corresponds to a space of holomorphic maps. We then have a map $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}) \to H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP})$, mapping P to a configuration P' corresponding to a map f'. One has a local product structure $H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP}) \simeq H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}) \times H^0_{\mathbf{l}}(\mathbb{C}, \mathcal{PP})$. The codimension of $H^0_{\mathbf{k},\alpha}(\mathbb{C}, \mathcal{PP})(X, X^*_\alpha)$ at P is the same as the codimension at P' of the subvariety of $H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP})(X, X^*_\alpha)$ consisting of maps with one α-pole of multiplicity k_α and all its other α-poles simple.

One then can use the fact that the pull-back $f'^*(TX)$ is sufficiently positive to show that the evaluation map taking elements of $H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP})$ to their k_α-jets at z, is submersive at P'. Again, details are given in ([5, Theorem 5.20]).

The problem is reduced to understanding tangencies in the space of jets. One shows that the jets tangent to X_α to order k_α have codimension k_α; this tells us that the maps having at least one point with a jet tangent to X_α to order k_α have codimension $k_\alpha - 1$. As we are on the smooth locus, we can choose local coordinates $x_1, \ldots, x_n$, with X_α cut out by $x_1 = 0$. The desired result is then a simple observation.

For (ii), one must also analyse jets at singular points of the hypersurface X_α. It then appears that the codimension of the subspace of jets vanishing to order k_α is not necessarily k_α, but can be lower. By blowing up, however, one can obtain the estimate $C_\alpha k_\alpha - C'_\alpha$. Again, we refer to [5]. A general position argument to note that the generic holomorphic map does not meet the singular locus of X_α, showing that the codimension is at least one. Part (iii) follows from (i) and (ii).

Parts (iv) and (v) are obtained by repeating the arguments of (i), but with the divisor X_∞ instead of its components X_α. □

Referring to (3.9), we can let $\mathcal{LPP}_{\mathbf{k}}$ be the subspace of $H^0_{\mathbf{k},\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ with the pole at 0, so that $H^0_{\mathbf{k},\mathbf{k}}(\mathbb{C}, \mathcal{PP}) = \mathcal{LPP}_{\mathbf{k}} \times \mathbb{C}$.

As in [5], we consider a collection of multi-indices

$$\mathcal{M} = \{\mathbf{m}^1, \ldots, \mathbf{m}^r\} \tag{4.2}$$

satisfying the constraint (3.6).

Definition 4.1. Let $\mathcal{V}_{\mathcal{M}}$ be the subset of all elements of $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ that have r poles at distinct points $z_1, \ldots, z_r$ of $\mathbb{C}$ such that the multiplicity of the pole at z_i is $\mathbf{m}^i$.

Now we have the "pole location map"

$$\Pi : H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}) \longrightarrow SP^{|\mathbf{k}|}(\mathbb{C}), \tag{4.3}$$

defined by sending a principal part P to the unordered set of points of $\mathbb{C}$ defining the configuration of poles counting multiplicity. That is, if $z_i \in \mathbb{C}$ is a pole of multiplicity $\mathbf{m}^i$ then z_i occurs $|\mathbf{m}^i|$ times. The map Π is not necessarily surjective, as for example, there can be a sequence of divisors at infinity with empty intersection. Thus, not all poles can necessarily coalesce. Nevertheless, the pole location map (4.3) restricted to $\mathcal{V}_{\mathcal{M}}$ gives the locally trivial fibration

$$\Pi_{\mathcal{M}} : \mathcal{V}_{\mathcal{M}} \longrightarrow \mathbb{DP}^r(\mathbb{C}) \tag{4.4}$$

where $\mathbb{DP}^r(\mathbb{C})$ denotes the deleted r-fold product; that is, the space of r distinct unordered points in $\mathbb{C}$. The fiber of Π in (4.4) is $\prod_{i=1}^r \mathcal{LPP}_{\mathbf{m}^i}$ which is not necessarily smooth. Thus, the strata $\mathcal{V}_{\mathcal{M}}$ are not necessarily smooth; however, they are varieties, as they are cut out of $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ by the vanishing of derivatives of locally defined analytic functions. Furthermore, one can subdivide the spaces $\mathcal{LPP}_{\mathbf{m}^i}$ into smooth strata; this induces a subdivision $\mathcal{V}_{\mathcal{M},i}$ of the $\mathcal{V}_{\mathcal{M}}$. One can order the $\mathcal{M}$ in such a way that together with Proposition 4.2 one has

Theorem 4.1. *There is a stratification of* $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ *by smooth complex varieties* $\mathcal{V}_{\mathcal{M},j}$ *which satisfies*:

(i) $\dim \mathcal{V}_{\mathcal{M},j} < \dim \mathcal{V}_{\mathcal{M}',i}$ *only if* $\mathcal{M}' < \mathcal{M}$, *or, if* $\mathcal{M}' = \mathcal{M}$, *then* $i < j$.
(ii) $\mathcal{V}_{\mathcal{M},j} \subset \overline{\mathcal{V}_{\mathcal{M}',i}}$ *only if* $\mathcal{M}' \leq \mathcal{M}$, *or, if* $\mathcal{M}' = \mathcal{M}$, *then* $i \leq j$.
(iii) *The set* $\{\mathcal{M}, j\}$ *is finite.*

Furthermore, there is a positive constant $c(X)$, *which is independent of the stratum indices* $\mathcal{M}$, j *and the multi-degree* $\mathbf{k}$, *so that the complex codimension of* $\mathcal{V}_{\mathcal{M},j}$ *in* $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$,

$$\operatorname{codim}(\mathcal{M}, j; \mathbf{k}) = \operatorname{codim}(\mathcal{V}_{\mathcal{M},j} \subset H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}))$$

is bounded below by

$$c(X) \sum_i (|\mathbf{m}^i| - 1) = c(X)(|\mathbf{k}| - r)$$

where r *is the number of poles. Here, referring to Proposition* 4.2, $c(X)$ *is a constant such that* $c(X)n - 1 \leq \max(1, \hat{C}n - \hat{C}')$ *for all* n.

This last fact follows from Proposition 4.2 and the local decomposition of $H^0_{\mathbf{k}}$ $(\mathbb{C}, \mathcal{PP})$ near $\mathcal{V}_{\mathcal{M}}$ as a product $\Pi_{i=1}^r H^0_{\mathbf{m}^i}(\mathbb{C}, \mathcal{PP})$. Thus the big strata are the ones with many poles of low multiplicity. As in [5], we construct a stabilization map as follows: for each irreducible component X_α fix a local principal part of multiplicity one. We have

Proposition 4.3. *There is a rational map of multi-degree* $(k_1, \dots, k_\kappa)$ *with all* $k_i > 0$.

Proof. By Proposition 2.3, the variety X is birational to $\mathbb{P}^n$, and maps $f : \mathbb{P}^1 \to X$ get transformed into maps $\tilde{f} : \mathbb{P}^1 \to \mathbb{P}^n$. The constraint that, say, $0 \in \mathbb{P}^1$ gets mapped into X_α gets transformed into a constraint on $\tilde{f}(0)$, $\tilde{f}'(0)$, $\tilde{f}''(0), \dots,$ $\tilde{f}^{(s)}(0)$, for some s. However, maps into $\mathbb{P}^n$ are given by polynomials, and one can arrange by interpolation for a map to satisfy any finite set of constraints such as those given above, at a set of points $z_1, z_2, \dots, z_k$ of $\mathbb{P}^1$. In other words, one can ensure that the map f meets each X_α. $\square$

4.2. *Stabilization of principal parts*

Proposition 4.3 tells us that the plane of good virtual degrees $(k_1, \dots, k_\kappa)$ intersects the interior of the positive quadrant $k_i > 0$. Let us choose some small good multi-degree $\mathbf{l}$ in this quadrant, and build a configuration $P_\mathbf{l}$ of multi-degree $\mathbf{l}$ with simple poles from our local principal parts. Using (3.8), we add $P_\mathbf{l}$ to an arbitrary $P \in H^0_\mathbf{k}(\mathbb{C}, \mathcal{PP})$. This gives a continuous map

$$\iota(\mathbf{k}, \mathbf{k}+\mathbf{l}) : H^0_\mathbf{k}(\mathbb{C}, \mathcal{PP}) \longrightarrow H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP}) . \tag{4.5}$$

This map preserves the stratification, so that it can be studied stratum by stratum. The arguments in [5] then carry over word for word to give:

Theorem 4.2. *For all* $\mathbf{k}$ *the inclusion* (4.5) *induces an isomorphism in homology*

$$(\iota(\mathbf{k}, \mathbf{k}+\mathbf{l}))_t : H_t(H^0_\mathbf{k}(\mathbb{C}, \mathcal{PP}); \mathbb{Z}) \cong H_t(H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP}); \mathbb{Z})$$

for

$$t \le q = q(\mathbf{k}) = c_0(X) l(\mathbf{k}) - 1 ,$$

where $c_0(X) > 0$ *is a constant which depends only on* X *and where* $l(\mathbf{k}) = \min (k_1, \dots, k_\kappa)$.

We recall that the proof proceeded by considering the stabilization map stratum by stratum; the homology of the whole is then obtained from a Leray type spectral sequence, whose E^1-term is the homology of the strata suspended by the real codimension of the stratum. Ordering the strata as in Theorem 4.1, labelling them as $\mathcal{V}_{\mathcal{M}_1} \ge \mathcal{V}_{\mathcal{M}_2} \ge \dots$, one has $E^1_{ij} = H_{i+j-\mathrm{codim}}(\mathcal{V}_{\mathcal{M}_i})$ The fact that the stabilization maps preserves the spectral sequence gives a map between the E^1 terms:

$$E^1(H^0_\mathbf{k}(\mathbb{C}, \mathcal{PP}))_{ij} \longrightarrow E^1(H^0_{\mathbf{k}+\mathbf{l}}(\mathbb{C}, \mathcal{PP}))_{ij}$$

which commutes with differentials. The isomorphism on the homology of the whole spaces for a given range follows from the isomorphism on the constituent pieces, through the same range, plus one.

On the stratum $\mathcal{V}_{\mathcal{M}_i}$, the homology stabilizes in a range which is half the minimum (over the set of colors) of the numbers of simple poles in $\mathbf{m}$. (The map in homology is injective on the strata; one then simply has to look for the smallest class one can build in the charge $\mathbf{m}+\mathbf{l}$ stratum which one cannot build in the charge $\mathbf{m}$ stratum.) To the stability range of the homology map on the stratum one must then add the real codimension of the stratum, which is bounded from below, using Theorem 4.1, by $2(c(X)(|\mathbf{k}|-r))$. The range for the stability of the whole space is then $c_0(X)l(\mathbf{k})-1$, with $c_0=\min(2c(X),1/2)$.

Using the maps $\iota(\mathbf{k},\mathbf{k}+\mathbf{l})$ of (4.5) one can form the direct limit $\varinjlim H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})$ and obtain natural inclusions

$$\iota(\mathbf{k}): H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP}) \to \varinjlim H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})\,. \tag{4.6}$$

Theorem 4.2 tells us then that the map $\iota(\mathbf{k})$ induces isomorphisms in homology in the range $t\le c_0|\mathbf{k}|-1$. We now use the argument of Gravesen [14], showing how the limit space is homologically equivalent to $\Omega^2(X)$. This argument, given there for maps to the $SL(n,\mathbb{C})$ flag manifolds only, depends on having a pole and principal parts description, so it holds equally well in the more general case of a principal almost solv-variety. Indeed, the argument of Gravesen, in essence, works inductively, building up on one side the continuous maps of the 2-sphere (or, more generally, any surface) into X from maps of disks into X, and using natural fibration properties of spaces of maps under restriction. On the particle space side, the same procedure is used, building up the space of particles over the sphere from particles over disks, with quasi-fibrations replacing fibrations in the restriction maps. At the last step in the induction (adding the last disk and closing off the surface), one has to take the mapping telescope $\widetilde{H^0}(\mathbb{C},\mathcal{PP})$ of the stabilized principal part space $\varinjlim H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})$, and similarly the mapping telescope $\widetilde{\Omega^2 X}$ of the stabilized mapping space; some of the restriction maps are only homology fibrations, and the final result is that one has isomorphisms

$$j_*: H_*(\widetilde{H^0}(\mathbb{C},\mathcal{PP})) \longrightarrow H_*(\widetilde{\Omega^2 X})\,. \tag{4.7}$$

Using then the fact that the components of $\Omega^2 X$ are homotopy equivalent to each other we have

Theorem 4.3. *Let X be a smooth compact principal almost solv-variety, and let $\mathbf{k}$ be a good multi-degree. Then there exists a constant $c_0>0$ such that for all $t<c_0 l(\mathbf{k})|-1$, there are homology isomorphisms*:

$$j_*: H_t(H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})) \longrightarrow H_t(\Omega^2_{\mathbf{k}}X)\,.$$

We are now ready to extend the theorem to the more general case of Theorem 1.1; this extension, in essence, adds to the action of our linear algebraic group that of a compact complex Lie group (a torus). In terms of maps, this makes essentially no difference, as the two-fold loop space of a torus is contractible; in terms of holomorphic maps, it also makes no difference, since the holomorphic maps of $\mathbb{P}^1$ into the torus are constants:

Proof of Theorem 1.1. By hypothesis the Kähler manifold X is almost homogeneous with respect to the connected complex solvable Lie group S. Then by a theorem of Remmert and Van de Ven [1], the Albanese map $\alpha : X \longrightarrow \mathrm{Alb}(X)$ is a locally trivial holomorphic fibre bundle, and by a theorem of Oeljeklaus [1, 34], the fibre F is a smooth connected, simply connected projective algebraic variety. Moreover, the fibration is S-equivariant [19, 24], so that F is almost homogeneous with respect to a subgroup S_0 of S, and has an S_0-equivariant embedding of F in some projective space. It follows that S_0 is a solvable linear algebraic group and that F is a smooth projective algebraic variety with an open orbit biholomorphic to S_0 itself. In other words F is a smooth projective principal almost solv-variety. This allows us to describe the space $\pi_2(X)$ of components of the mapping space $\Omega_2(X)$ in the same way as before. Since the long exact sequence in homotopy gives us an isomorphism $\pi_2(F) \simeq \pi_2(X)$, we can then think of $\pi_2(X)$ as being embedded in $H_2(F, N(F)) = \mathbb{Z}^r$, as before. The theorem follows by noticing that on the one hand, every map from $\mathbb{P}^1$ to a torus is null-homotopic, and on the other, that every such holomorphic map is constant. Everything then lives in the fiber, and so it suffices to prove the theorem for maps into F. But this is the content of Theorem 4.3. □

We now examine how one can evaluate the constant c_0. As we saw, the stability range of each stratum's contribution to the spectral sequence was governed by two numbers.

(a) The stability range for the homology in each stratum. This is 1/2 of the minimum over the different components X_α of the number of simple α-poles of elements of the stratum.
(b) The codimension of the stratum within the space of holomorphic maps. We saw that this was governed in turn by the codimension ℓ in the space of k-jets of maps from $\mathbb{C}$ to X of the subvariety of jets intersecting X_∞ with multiplicity k.

Let us suppose that X_∞ has smooth components with normal crossings. Then it is easy to see from expressions in local coordinates that the complex codimension ℓ is just k. From this, one then has that the strata with ρ simple poles in $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$ has complex codimension $|\mathbf{k}| - \rho$, hence real codimension $2(|\mathbf{k}| - s)$. Combining this with the stability range for the homology of the stratum gives $c_0 = 1/2$.

To improve on this one must, in essence, compute some differentials in the spectral sequence. It turns out that just computing one differential improves things considerably. Indeed, the cycles of dimension between $\rho/2$ and ρ which can cause the stability theorem to fail all contain factors (under loop sum and bracket operations) of the form $[e, e']$, the one-dimensional cycle obtained by moving an α-pole labelled by the base point e in $\mathcal{LPP}_\alpha$ in a circle around a fixed β-pole labelled by the base point e' in $\mathcal{LPP}_\beta$. One simply has to show that this cycle is killed when one adds in the higher co-dimensional strata. To do this, one adds in the stratum with a double pole instead of the two simple poles. Geometrically, the cycle, which is a circle, is filled in to a disk in which the two poles are allowed to coalesce; this is quite easy to do in local coordinates, using the normal crossing condition, and moving an embedded disk. The fact that these cycles then do not contribute to the homology of the total space improves the stability range to $c_0 = 1$, again when X_∞ has smooth components with normal crossings. A similar calculation, showing that a similar cycle is trivial in the case of $SU(2)$ instanton moduli, is given in more detail in [23]. It also increases the stability range there to the optimal $c_0 = 1$.

5. The Fundamental Group and Covering Spaces

5.1. *The fundamental group*

A crucial ingredient in the proof of the homotopy stability of our mapping spaces is that $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ has Abelian fundamental group when the components of $\mathbf{k}$ are large enough. We have:

Theorem 5.1. *If the good multi-degree $\mathbf{k} = (k_1, \ldots, k_\kappa)$ satisfies $k_i \geq 2$ for all i, then the fundamental group $\pi_1(\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*)$ is Abelian.*

Remark 5.1. In the stable range, one then has that the fundamental group of $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ is simply the fundamental group of the mapping space, that is $\pi_3(X)$.

Proof. We begin by remarking that for dimensional reasons (Theorem 4.2) any loop in $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}) = \mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ can be deformed to the generic locus $\mathcal{S}$ consisting of configurations of points with simple poles; this means that $\pi_1(\mathcal{S})$ surjects onto $\pi_1(H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP}))$. The locus $\mathcal{S}$ can be described as follows. One has κ different colors, corresponding to the different types of poles, that is the different divisors at infinity, and associated to each color, a label space $\mathcal{LPP}_i$ of principal parts of multiplicity 1 in the ith color, 0 in the others. The space $\mathcal{LPP}_i$ is connected, as it is a quotient of the space of germs of maps into X meeting the ith component of infinity transversally, in its smooth locus, and this latter space is connected. The generic stratum $\mathcal{S}$ then consists of configurations of $|\mathbf{k}| = \sum_i k_i$ distinct unordered points in the plane, with, for each i, k_i points labelled by elements of $\mathcal{LPP}_i$. In other words, consider the product $\Pi_{i=1}^{\kappa} \mathbb{DP}^{k_i}$ of the deleted symmetric products of k_i points in $\mathbb{C}$ (i.e. unordered k_i-tuples of distinct points in $\mathbb{C}$). Let $\mathcal{D}$ be the subspace of $\Pi_{i=1}^{\kappa} \mathbb{DP}^{k_i}$ consisting of those elements $(\alpha_1, \ldots, \alpha_\kappa)$ which considered as $|\mathbf{k}|$ points of $\mathbb{C}$ have no points in common. One has a fibration

$$\Pi_i(\mathcal{LPP}_i)^{k_i} \to \mathcal{S} \longrightarrow \mathcal{D}\,. \tag{5.1}$$

Let G be the subgroup of the braid group B_k on $|\mathbf{k}| = \sum_i k_i$ letters consisting of those elements which map to $S_{k_1} \times \cdots \times S_{k_\kappa}$ under the standard homomorphism

$B_{|\mathbf{k}|} \to S_{|\mathbf{k}|}$ of the braid group onto the symmetric group $S_{|\mathbf{k}|}$ on $|\mathbf{k}|$ letters. $\mathcal{D}$ is a $K(G,1)$, and the $\mathcal{LPP}_i$ are connected, so one has the exact sequence of groups

$$0 \to \pi_1(\Pi_i(\mathcal{LPP}_i)^{k_i}) \longrightarrow \pi_1(\mathcal{S}) \to \pi_1(\mathcal{D}) \longrightarrow 0\,. \tag{5.2}$$

Using the translation action on principal parts, one can lift paths in the base $\mathcal{D}$ to $\mathcal{S}$. The monodromy of this lift along a loop ℓ in $\mathcal{D}$ is simply given by the action of the image of ℓ in $S_{k_1} \times \cdots \times S_{k_\kappa}$ on $\Pi_i(\mathcal{LPP}_i)^{k_i}$. The fibration thus admits sections: one simply chooses the same label in $\mathcal{LPP}_i$ for each point of color i. This allows us to split the sequence (5.2) and so decompose any element of $\pi_1(\mathcal{S})$ into a product of a loop in the fibers and a loop in $\pi_1(\mathcal{D})$.

The fundamental group of the full $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ is a quotient of that of $\mathcal{S}$ and one now would like to see what additional relations on $\pi_1(\mathcal{S})$ are given by the inclusion of $\mathcal{S}$ into $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$. These relations are given by glueing in the complex codimension one strata. The key relation turns out to be that particles of the same color are allowed to coalesce: as in [5], Sec. 9, one can bring two principal parts of multiplicity one and of the same color together into a double principal part. This kills the braiding between particles of the same color. To realize this coalescence for a given color ρ in terms of a family of germs, one can choose a coordinate x on X such that X_ρ is cut out locally by $x = 0$. Now consider a family of curves $f_a : \mathbb{C} \to X$ whose x-component is given by $x(t,a) = t^2 - a$; for $a \neq 0$, one has two simple poles at $x = \pm\sqrt{a}$, and for $a = 0$ one has a double pole. Glueing in these extra strata includes $\mathcal{S}$ into a bigger space $\mathcal{S}'$ for which double poles are allowed. Similarly, $\mathcal{D}$ is glued into a bigger space $\mathcal{D}'$ for which double points are allowed. Let i denote this inclusion map.

The fundamental group of $\mathcal{D}'$ is Abelian, if all the k_i are at least two. Indeed, let α be a standard braid group generator in $\pi_1(\mathcal{D})$, interchanging two poles of the same color, while leaving the others fixed; the preceeding paragraph tells us that $i_*(\alpha) = e$, the identity element; more generally, $i_*(\alpha\beta\alpha^{-1}) = i_*(\beta)$. This means that one can use *any* of the points of a given color to represent in $\mathcal{D}$ an element of $\pi_1(\mathcal{D}')$. The natural thing is then to use only one of our points: thus, if $\mathcal{D}_0$ is the space of κ-tuples of distinct points in the unit disk in $\mathbb{C}$, one for each color, and if $j : \mathcal{D}_0 \to \mathcal{D}$ is the stabilization map of adding in fixed extra points outside the disk, then the composition $i \circ j$ induces a surjection on π_1.

Now let us suppose that there are two points p_ℓ and q_ℓ of each colour ℓ. One can then represent an element γ of $\pi_1(\mathcal{D}')$ as a braid in the p_ℓ and another element δ as a disjoint (i.e. lying in another disk) braid in the q_ℓ. In other words, we can represent γ, δ as elements $(\hat\gamma, 0)$, $(0, \hat\delta)$ in $\pi_1(\mathcal{D}_0 \times \mathcal{D}_0)$. Such elements commute, and so $\pi_1(\mathcal{D}')$ is commutative.

The same holds true when one considers the space of labelled configurations: the key point is that the loop α' which interchanges two points with the same labels is contractible. This again allows us to "localize" loops onto configurations involving only one pole of each colour; if we have at our disposal at least two points of each colour, we can localize any two loops into loops of configurations living in disjoint disks, which must commute. In short, $\pi_1(\mathcal{S}')$ is Abelian.

Thus, the relations put in by allowing two points of the same color to coincide make the group Abelian, and as $\pi_1(\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*)$ is obtained by putting in possibly more relations (adding more strata and so going to an even smaller quotient), it too must be Abelian. □

Example 5.1. The classical case of holomorphic maps of the Riemann sphere to itself is a good example here. The solvable group that we select is $\mathbb{C}^*$ acting on $\mathbb{P}^1$, fixing zero and infinity. In this case the open orbit is just a copy of $\mathbb{C}^*$, and "∞" is $0 \sqcup \infty$. Thus the data describing a holomorphic map are its zeros and poles. Any zeros can come together as can any poles, but roots and poles cannot come together. Thus, each stratum will consist of pairs $\langle x_1, \ldots, x_n\rangle$, $\langle y_1, \ldots, y_m\rangle$ (the x_i's are zeros and the y_j poles). Attached to these points are integers which are the multiplicities of the zeros and poles, with the sole requirement that the sums of the multiplicities of zeros and the sums of the multiplicities of poles be equal to a fixed degree n. The fundamental group is $\mathbb{Z}$, with generator induced by looping a zero around a fixed pole [35]. Note that the label spaces are discrete finite sets and do not introduce any generators into the fundamental group.

5.2. *Coverings of the strata*

We have just seen that the fundamental groups of our spaces of holomorphic maps are stably the first homology groups, and we know that these groups stabilize. The stabilization maps extend to the universal cover; we have already seen that they preserved our strata. Our next step is to see how the universal cover of $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ behaves when one restricts to the strata $\mathcal{V}_{\mathcal{M}}$. These strata correspond to multiplicity patterns for the poles; there will be a certain number of *simple* poles of each color, say n_i for the ith color, $i = 1, \ldots, k$ and then multiple poles, of which there are, say n_i, $i = k+1, \ldots, \ell$ of various types i, taking values in principal part spaces which we also label as $\mathcal{LPP}_i$. We restrict our attention to those strata with at least one simple pole of each color, and note that the codimension of those strata with no simple poles in a given color increases linearly with $|\mathbf{k}|$. Set $n = \sum_{i=1}^{\ell} n_i$. We can describe the stratum $\mathcal{V}_{\mathcal{M}}$ as

$$C_n(\mathbb{C}) \times_{\Pi_i S_{n_i}} \Pi_i(\mathcal{LPP}_i)^{n_i}\,.$$

Here $C_j(X)$ denotes the configuration space of ordered j-tuples of distinct points in X and the symmetric group S_j, and its subgroup $\Pi_i S_{n_i}$ act by changing the ordering.

The restriction $\tilde{\mathcal{V}}_{\mathcal{M}} \to \mathcal{V}_{\mathcal{M}}$ of the covering of $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ is Abelian and is determined by a subgroup of the first homology group of $\mathcal{V}_{\mathcal{M}} = C_n(\mathbb{C}) \times_{\Pi_i \mathcal{S}_{n_i}} \Pi_i(\mathcal{LPP}_i)^{n_i}$. There is a fibration

$$\Pi_i(\mathcal{LPP}_i)^{n_i} \longrightarrow C_n(\mathbb{C}) \times_{\Pi_i \mathcal{S}_{n_i}} \Pi_i(\mathcal{LPP}_i)^{n_i} \longrightarrow C_n(\mathbb{C})/\Pi_i S_{n_i} \tag{5.3}$$

which, as above, splits. As the base is an Eilenberg–Maclane space, we have a semi-direct product

$$\pi_1(\mathcal{V}_{\mathcal{M}}) = [\Pi_i(\pi_1(\mathcal{LPP}_i))^{n_i}] \rtimes \pi_1(C_n(\mathbb{C})/\Pi_i S_{n_i})\,.$$

For the Abelianization of $\pi_1(C_n(\mathbb{C})/\Pi_i S_{n_i})$, one sees that there are various winding numbers one can define: we are dealing with n_i particles for each color i, which can move around each other. We therefore have the total winding numbers for particles of color i moving around particles of color i', where i can equal i', giving a factor of $\mathbb{Z}^{\ell(\ell+1)/2}$ in the Abelianization. On the other hand, the fundamental group is generated by elementary loops which send a particle of color i around a particle of color i'; the braiding relations tell us in essence that when we Abelianize, we cannot tell which particle of color i we are sending around, so that the Abelian quotient is at most $\mathbb{Z}^{\ell(\ell+1)/2}$. In turn , then we have that $H_1(\mathcal{V}_{\mathcal{M}})$ will be a quotient of $[\Pi_i(H_1(\mathcal{LPP}_i))^{n_i}] \times \mathbb{Z}^{\ell(\ell+1)/2}$, where we must still quotient out the action of the fundamental group of the base (by conjugation) on that of the fibers; as this action is by permutation of particles of the same color, one obtains for the first homology group:

$$H_1(\mathcal{V}_{\mathcal{M}}) = [\Pi_i(H_1(\mathcal{LPP}_i))] \times \mathbb{Z}^{\ell(\ell+1)/2}\,.$$

We note in particular that restricting any Abelian covering of $\mathcal{V}_{\mathcal{M}}$ to the fibers of the fibration (5.3) is induced by addition maps $H_1(\mathcal{LPP}_i)^{n_i} \longrightarrow H_1(\mathcal{LPP}_i)$. Also, the proof of Theorem 5.1 shows that our coverings quotient out the winding of particles labelled by simple poles of the same color.

We can further decompose the space $\mathcal{V}_{\mathcal{M}}$ as an iterate fibration, by successively projecting out the points labelled by $\mathcal{LPP}_\ell$, then those labelled by $\mathcal{LPP}_{\ell-1}$ and so on. In this way we obtain a sequence of fibrations:

$$\begin{array}{lllll}
\mathcal{V}_{\ell-1} & \longrightarrow & \mathcal{V}_{\mathcal{M}} & \longrightarrow & C_{n_\ell}(\mathbb{C}) \times_{S_{n_\ell}} \mathcal{LPP}_\ell^{n_\ell} \\
\mathcal{V}_{\ell-2} & \longrightarrow & \mathcal{V}_{\ell-1} & \longrightarrow & C_{n_{\ell-1}}(\mathbb{C} - \{y_{\ell,1}, \dots, y_{\ell,n_\ell}\}) \times_{S_{n_{\ell-1}}} \mathcal{LPP}_{\ell-1}^{n_{\ell-1}} \\
\mathcal{V}_{\ell-3} & \longrightarrow & \mathcal{V}_{\ell-2} & \longrightarrow & C_{n_{\ell-2}}(\mathbb{C} - \{y_{\ell,1}, \dots, y_{\ell,n_\ell}, y_{\ell-1,1}, \dots, y_{\ell-1,n_{\ell-1}}\}) \times_{S_{n_{\ell-2}}} \mathcal{LPP}_{\ell-2}^{n_{\ell-2}} \\
\vdots & & \vdots & & \vdots
\end{array} \tag{5.4}$$

$$C_{n_1}(\mathbb{C} - \{y_{\ell,1}, \dots, y_{2,n_2}\}) \times_{S_{n_1}} \mathcal{LPP}_1^{n_1} \longrightarrow \mathcal{V}_2 \longrightarrow C_{n_2}(\mathbb{C} - \{y_{\ell,1}, \dots, y_{3,n_3}\}) \times_{S_{n_2}} \mathcal{LPP}_2^{n_2}\,.$$

All of these fibrations have sections. There is then a corresponding diagram for the fundamental groups, which at each step is exact and splits as a semi-direct product, and which at the ith fibration projects out the loops involving the ith particle space and the paths of particles of type i around fixed punctures corresponding to particles of type $i+1, \dots, \ell$. Restricting the maps of fundamental groups to the subgroup defining the covers, we obtain a corresponding diagram of iterate fibrations for the covers $\tilde{\mathcal{V}}_{\mathcal{M}}$. Let $\tilde{C}_{n_i|n_{i+1},\dots,n_\ell}(\mathcal{LPP}_i)$ denote the cover of $C_{n_i}(\mathbb{C} - \{y_{\ell,1}, \dots, y_{i+1,n_{i+1}}\}) \times_{S_{n_i}} \mathcal{LPP}_i^{n_i}$. Then we have the sequence of fibrations

$$\begin{array}{ccccc} \tilde{\mathcal{V}}_{\ell-1} & \longrightarrow & \tilde{\mathcal{V}}_{\mathcal{M}} & \longrightarrow & \tilde{C}_{n_\ell|\cdot}(\mathcal{LPP}_\ell) \\ \tilde{\mathcal{V}}_{\ell-2} & \longrightarrow & \tilde{\mathcal{V}}_{\ell-1} & \longrightarrow & \tilde{C}_{n_{\ell-1}|n_\ell}(\mathcal{LPP}_{\ell-1}) \\ \tilde{\mathcal{V}}_{\ell-3} & \longrightarrow & \tilde{\mathcal{V}}_{\ell-2} & \longrightarrow & \tilde{C}_{n_{\ell-2}|n_{\ell-1},n_\ell}(\mathcal{LPP}_{\ell-2}) \\ & \vdots & \vdots & \vdots & \\ \multicolumn{5}{c}{\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1) \longrightarrow \mathcal{V}_2 \longrightarrow \tilde{C}_{n_2|n_3,\ldots,n_\ell}(\mathcal{LPP}_2)\,.} \end{array} \tag{5.5}$$

Proposition 5.1. *All the coverings in the fibrations* (5.4) *are Abelian.*

Proof. Let us write one of the fibrations in (5.4) and their coverings in (5.5) as

$$\begin{array}{ccccc} \tilde{\mathcal{V}}' & \xrightarrow{\tilde{a}} & \tilde{\mathcal{V}} & \xrightarrow{\tilde{b}} & \tilde{\mathcal{C}} \\ \big\downarrow \sigma' & & \big\downarrow \sigma & & \big\downarrow \rho \\ \mathcal{V}' & \xrightarrow{a} & \mathcal{V} & \xrightarrow{b} & \mathcal{C}\,\cdot \end{array}$$

The downward maps all induce injections on the level of fundamental groups, and the bottom row splits $\pi_1(\mathcal{V})$ as a semi-direct product of $\pi_1(\mathcal{V}')$ and $\pi_1(\mathcal{C})$. The covering $\tilde{C} \to C$ corresponds to setting $\rho_*(\pi_1(\tilde{\mathcal{C}})) = b_*\sigma_*(\pi_1(\tilde{\mathcal{V}}))$. One must check that the commutator of $\pi_1(\mathcal{C})$ lies in $\rho_*(\pi_1(\tilde{\mathcal{C}}))$, and similarily that the commutator of $\pi_1(\mathcal{V}')$ lies in $\sigma'_*(\pi_1(\tilde{\mathcal{V}}'))$. A small diagram chase gives this from the fact that both these commutators lie in the commutator of $\pi_1(\mathcal{V})$, which in turn lies in $\sigma_*(\pi_1(\tilde{\mathcal{V}}))$. We see then that all the covers are governed by subgroups of the first homology groups. □

We are interested in the effect on these spaces of stabilization by adding in simple poles; we will examine these one color at a time. To a configuration in the space $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$, with poles located at $z_1, \ldots, z_n$ in $\mathbb{C}$, we add a pole located at $\max(|z_i|) + 1$, labelled by the base point in $\mathcal{LPP}_1$. As noted in the previous section, this map preserves our stratification, and we are now interested in the effect of the map on the level of the covers. In the stable range, the covers are governed by the first homology of $H^0_{\mathbf{k}}(\mathbb{C}, \mathcal{PP})$, which is constant. The stabilization preserves all the spaces in the iterate fibrations (5.4), (5.5), and in fact only changes the last fibers of (5.4), (5.5), adding in an extra particle. For these last fibers, we have a commutative diagram:

$$\begin{array}{ccc} \tilde{C}_{n_1|n_2,\cdots,n_\ell}(\mathcal{LPP}_1) & \rightarrow & \tilde{C}_{n_1+1|n_2,\cdots,n_\ell}(\mathcal{LPP}_1) \\ \big\downarrow & & \big\downarrow \\ C_{n_1}(\mathbb{C} - \{y_{\ell,1},..,y_{2,n_2}\}) \times_{\mathcal{S}_{n_1}} \mathcal{LPP}_1^{n_1} & \rightarrow & C_{n_1+1}(\mathbb{C} - \{y_{\ell,1},..,y_{2,n_2}\}) \times_{\mathcal{S}_{n_1+1}} \mathcal{LPP}_1^{n_1+1} \end{array}$$

where n_1 denotes the number of simple poles of the given color.

To show that the homology of the universal cover $\widetilde{H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})}$ of $H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})$ stabilizes through an appropriate dimension q, we proceed as in Theorem 4.2, using the spectral sequence which assembles the homology of the different strata $\tilde{\mathcal{V}}_{\mathcal{M}}$, which reduces the problem to showing that the homology of the stratum $\tilde{\mathcal{V}}_{\mathcal{M}}$ stabilizes in dimensions $q - (\mathrm{codim}(\tilde{\mathcal{V}}_{\mathcal{M}}))$. In turn for these strata, one can use the spectral sequences for the fibrations (5.5), which, as the bases are not changed by the stabilization, reduces the stability question to showing that the coefficient systems $H_i(\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1))$ stabilize in an appropriate range. Our Theorem 1.2, with the appropriate range, then will follow from

Theorem 5.2. *Let s be the rank of $\pi_3(X)$. The stabilization map*

$$\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1) \longrightarrow \tilde{C}_{n_1+1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1)$$

induces isomorphisms in homology groups H_i for $i < n_1 - s - 1$.

The proof of this theorem occupies the next section of the paper.

Using Theorem 5.2, then, one obtains as in Sec. 4. that the homology of $\widetilde{H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})}$ stabilizes in the range $t \le c_0\ell(\mathbf{k}) - s - 2$. One then has to make sure that the space is simple in this range of dimensions, that is that the fundamental group acts trivially on the homology of the covering. This can be seen as follows. Let $\mathbf{k}_0 = (1, 1, \ldots, 1)$. We saw above that the fundamental group of $H^0_{\mathbf{k}_0}(\mathbb{C},\mathcal{PP})$ surjects under the stabilization map to that of $H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})$. Let an element τ in the fundamental group of $H^0_{\mathbf{k}_0}(\mathbb{C},\mathcal{PP})$ act by the deck transformation T on the universal cover, and let the corresponding element τ' of the fundamental group of $H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})$ act by T'; we have a commuting diagram on the universal covers:

$$\begin{array}{ccc} H^0_{\mathbf{k}_0}\widetilde{(\mathbb{C},\mathcal{PP})} \times H^0_{\mathbf{k}-\mathbf{k_0}}\widetilde{(\mathbb{C},\mathcal{PP})} & \xrightarrow{\tilde{I}} & H^0_{\mathbf{k}}\widetilde{(\mathbb{C},\mathcal{PP})} \\ (T,1)\Big\downarrow & & T'\Big\downarrow \\ H^0_{\mathbf{k}_0}\widetilde{(\mathbb{C},\mathcal{PP})} \times H^0_{\mathbf{k}-\mathbf{k_0}}\widetilde{(\mathbb{C},\mathcal{PP})} & \xrightarrow{\tilde{I}} & H^0_{\mathbf{k}}\widetilde{(\mathbb{C},\mathcal{PP})} \end{array}, \tag{5.6}$$

where $\tilde{I}$ represents the loop sum map. Now let us suppose that we have a homology cycle A in $\widetilde{H^0_{\mathbf{k}}(\mathbb{C},\mathcal{PP})}$ in the stable range for $\mathbf{k} - \mathbf{k}_0$; we can represent it on the left hand side of (5.6) as a class $\mathrm{pt} \times A$. For obvious geometrical reasons, $(T, 1)$ acts trivially on $\mathrm{pt} \times A$; but then, referring to the above diagram, $T'_*(A) = A$, and so our spaces are simple in the appropriate range.

Once one has this simplicity, one can then apply, e.g. the result of Hilton and Roitberg ([20, Corollary 3.4]) to show that the homology of the universal covers of $\mathrm{Hol}_{\mathbf{k}}(\mathbb{P}^1, X)^*$ stabilizes to that of the universal covers of the mapping spaces $\mathrm{Map}_{\mathbf{k}}(\mathbb{P}^1, X)$; this then proves Theorem 1.2.

6. Abelian Covers of Labelled Particle Spaces on Punctured Planes

6.1. *Abelian covers of the label spaces*

To prove Theorem 5.2, we want to understand the behaviour of the homology of the covers $\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1)$ of $C_{n_1}(\mathbb{C}-\{y_{\ell,1},\ldots,y_{2,n_2}\})\times_{S_{n_1}}\mathcal{LPP}_1^{n_1}$ as we stabilize. We first must understand what type of cover we are dealing with. From (5.6) and the discussion following it, we have that the subgroup of the first homology group of $C_{n_1}(\mathbb{C}-\{y_{\ell,1},\ldots,y_{2,n_2}\})\times_{S_{n_1}}\mathcal{LPP}_1^{n_1}$ which governs the cover is a subgroup of $H_1(\mathcal{LPP}_1;\mathbb{Z})\times\mathbb{Z}^{\ell-1}$. The $\mathbb{Z}^{\ell-1}$ factor corresponds to the total winding of particles of color 1 around the punctures corresponding to the other colors. We note that, as in Theorem 5.1, any two simple poles with labels in $\mathcal{LPP}_1$ are allowed to coalesce in $H^0_{\mathbf{k}}(\mathbb{C},X)$, and so for our covers there is no factor $\mathbb{Z}$ corresponding to the self-winding of particles of the same color.

We begin with the case of no punctures; one ends up with this case by restricting our covers to the subspace $C_{n_1}(\mathbb{C})\times_{S_{n_1}}\mathcal{LPP}_1^{n_1}$ corresponding to a (non-holomorphic!) embedding $\mathbb{C}\to\mathbb{C}-\{y_{\ell,1},\ldots,y_{2,n_2}\}$. Having done this, the connected components of the cover have as group of deck transformations a quotient π of $H_1(\mathcal{LPP}_1)$; restricting to a fiber $\mathcal{LPP}_1^{n_1}$ the covers correspond to the composition of the diagonal map $H_1(\mathcal{LPP}_1)^{n_1}\to H_1(\mathcal{LPP}_1)$ with projection to π.

In what follows, let X denote $\mathcal{LPP}_1$, k denote n_1 and $\tilde{X}$ the covering of X with π as deck transformations. Suppose that $\tilde{X}$ is given by a classifying map ρ:

$$\begin{array}{ccc}\tilde{X} & \xrightarrow{\tilde{\rho}} & E_\pi\\ \downarrow & & \downarrow\\ X & \xrightarrow{\rho} & B_\pi\end{array}\tag{6.1}$$

where π is an Abelian group. Then, the multiplication homomorphism $\pi\times\pi\to\pi$, $(a,b)\mapsto ab$ induces a map

$$\mu_k:\underbrace{B_\pi\times B_\pi\times\cdots\times B_\pi}_{k-times}\longrightarrow B_\pi$$

and consequently the composition $\mu_k\circ\rho^k:X^k\longrightarrow B_\pi$. Moreover, $\mu_k\circ\rho^k$ is invariant under the permutation action of the symmetric group, $\mathcal{S}_k$, on X^k. Consequently the induced π-covering map

$$\pi_k:\tilde{X}_k\longrightarrow X^k\tag{6.2}$$

is $\mathcal{S}_k$-equivariant, and the connected components of $\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1)$ over $C_k(\mathbb{C})\times_{S_k}X^k$ are given by $C_k(\mathbb{C})\times_{S_k}\tilde{X}_k$. We note that $\tilde{X}_k=((\tilde{X})^k)/\pi^{k-1}$ but $\pi^{k-1}\subset\pi^k$ is not the usual inclusion and, in the end, this is what will cause us to have to take extra care in our calculations.

6.2. *A spectral sequence for covers* $C_k(\mathbb{C})\times_{S_k}\tilde{X}_k$

Determining the cohomology of the resulting total space is naturally somewhat involved. In order to provide sufficient information to show that we get stabilization we will have to introduce somewhat novel spectral sequences converging to the cohomology of $\tilde{X}_k$ and that of the associated spaces $C_k(Y)\times_{\mathcal{S}_k}(\tilde{X}_k)$ which take as input the cohomology of X, and some data associated to the classifying map of $\tilde{X}$.

We shall work on the level of chain complexes. For M a space, let $C_\#(M)$ be the chain complex of M with coefficients in a field $\mathbb{F}$. Set $E = E_\pi$. We note that $C_\#(\tilde{X})$, $C_\#(E)$ can be chosen so that the group ring $\mathbb{F}(\pi)$ acts freely on them.

Proposition 6.1. *Up to (free) equivariant chain homotopy equivalance we can replace the (equivariant) chain map $\rho_\# : C_\#(\tilde{X}) \to C_\#(E)$ associated to the classifying map ρ by a surjection to $C_\#(E)$ of a free complex equivariantly homotopy equivalent to $C_\#(\tilde{X})$.*

(As this is standard we defer the proof to the end of this section.)

Then, if necessary replacing $C_\#(\tilde{X})$ by the construction of the proposition, we assume $C_\#(\tilde{X})$ is given together with a surjection $\rho_\#$ to $C_\#(E)$. Consequently we have the short exact sequence of ($\mathbb{F}(\pi)$-free) chain complexes

$$0 \longrightarrow L_\# \longrightarrow C_\#(\tilde{X}) \xrightarrow{\tilde{\rho}_\#} C_\#(E) \longrightarrow 0\,. \tag{6.3}$$

Remark 6.1. Tensoring with $\mathbb{F}$ over the group-ring $\mathbb{F}(\pi)$ and passing to cohomology gives $H^*(B_\pi,\mathbb{F})$ from the complex $C_\#(E)$, and $H^*(X,\mathbb{F})$ from the complex $C_\#(\tilde{X})$. The exact sequence above yields a long exact sequence, and the five-lemma then tells us that up to a shift,

$$H^*(B_\pi, X;\mathbb{F}) = H^*(L\otimes_{\mathbb{F}(\pi)}\mathbb{F})\,. \tag{6.4}$$

We then have

$$\begin{array}{ccccccc}
\cdots\to & H^i(C_\#(\tilde{X})\otimes_{\mathbb{F}(\pi)}\mathbb{F}) & \to & H^i(L\otimes_{\mathbb{F}(\pi)}\mathbb{F}) & \to & H^{i+1}(C_\#(E)\otimes_{\mathbb{F}(\pi)}\mathbb{F}) & \cdots \\
 & \| & & \| & & \| & \\
\cdots\xrightarrow{\rho^*} & H^i(X;\mathbb{F}) & \xrightarrow{j} & H^{i+1}(B_\pi,X;\mathbb{F}) & \to & H^{i+1}(B_\pi;\mathbb{F}) & \cdots\,.
\end{array} \tag{6.5}$$

From the $\mathbb{F}(\pi)$ chain subcomplex $L = L_\#$ of $C_\#(\tilde{X})$, we can build a spectral sequence converging to $H^*(\tilde{X}_k,\mathbb{F}) = H^*(C_\#(\tilde{X})^{\otimes k}\otimes_{\mathbb{F}(\pi^{k-1})}\mathbb{F})$. Let $A\subset H^*(X;\mathbb{F})$ be the image of $\rho^* : H^*(B_\pi;\mathbb{F})\to H^*(X;\mathbb{F})$. (For example, if $\tilde{X}\to X$ is a $\mathbb{Z}$-cover then it is classified by a map $\rho : X\to B_{\mathbb{Z}}\simeq S^1$, and $A = \langle\rho^*(\iota)\rangle$ with ι the generator of $H^1(S^1)$.) The exact sequence above relates A to the relative groups $H^*(B_\pi, X;\mathbb{F})$ and will give rise shortly to the appearance of A in the E_2-term of the spectral sequence.

Proposition 6.2. *There is a spectral sequence converging to $H^*(\tilde{X}_k;\mathbb{F})$ with E_1-term*

$$\left[\bigoplus_{r=0}^{k-1}\left\{H^*(B_\pi,X;\mathbb{F})^{\otimes r}\otimes H^*(B_\pi;\mathbb{F})^{\otimes k-r-1}\right\}\otimes_{\mathbb{F}(\mathcal{S}_r\times\mathcal{S}_{k-r})}\mathbb{F}(\mathcal{S}_k))\right]$$

$$\oplus H^*(B_\pi,X;\mathbb{F})^{\otimes k}$$

and E_2-term

$$\left[\bigoplus_{r=1}^{k-1}\left\{\left[\tilde{H}^*(X;\mathbb{F})/A\right]^{\otimes r}\otimes A^{\otimes k-r-1}\right\}\otimes_{\mathbb{F}(\mathcal{S}_r\times\mathcal{S}_{k-r})}\mathbb{F}(\mathcal{S}_k)\right]\oplus L_k\,,$$

where both $H^*(B_\pi,X;\mathbb{F})^{\otimes k}$ and L_k are at least k-connected.

Remark 6.2. The action of $\mathcal{S}_i$ on $H^*(B_\pi,X;\mathbb{F})^{\otimes i}$ is just permutation of coordinates while that of $\mathcal{S}_{k-i}$ on $H^*(B_\pi;\mathbb{F})^{\otimes k-i-1}$ is obtained from an $\mathcal{S}_{k-i}$-equivariant embedding of B_π^{k-i-1} into B_π^{k-i} derived from the embedding $\pi^{k-i-1}\subset\pi^{k-i}$ as the subgroup of all elements $(g_1,\ldots,g_{k-i})$ with product $g_1g_2\cdots g_{k-i}=1$. The fact that we are not dealing with a standard permutation action here will be the major difficulty with proving stability.

Proof. The inclusion $L_\#\subset C_\#(\tilde{X})$ gives rise to an $\mathcal{S}_k$-equivariant filtration of the k-fold tensor product, $(C_\#(\tilde{X}))^k$:

$$\begin{aligned}L^k&\subset(L^{k-1}\otimes C+L^{k-2}\otimes C\otimes L+\cdots+C\otimes L^{k-1})\\&\subset(L^{k-2}\otimes C^2+\cdots+C^2\otimes L^{k-2})\subset\cdots\subset C^k\end{aligned}$$

where $L=L_\#$, $C=C_\#(\tilde{X})$, and the exponents denote tensor product.

The relative quotients are of the form

$$\begin{aligned}L^k,(L^{k-1}\otimes C(E)\oplus\cdots\oplus C(E)\otimes L^{k-1}),(L^{k-2}\otimes C(E)^2\\ \oplus\cdots\oplus C(E)^2\otimes L^{k-2}),\ldots\end{aligned}$$

and taking $\otimes_{\mathbb{F}(\pi^{k-1})}\mathbb{F}$, we have an associated spectral sequence for $\tilde{X}_k=(\tilde{X}^k)/\pi^{k-1}$.

Now we show that the E_1-term of this spectral sequence is of the form desired. As the tensoring with $\mathbb{F}(\mathcal{S}_k)$ in the statement of the proposition simply accounts for the different possible orderings of the factors, it will suffice to show that

$$\begin{aligned}H^*((L^{\otimes r}\otimes C_\#(E)^{\otimes s})\otimes_{\mathbb{F}(\pi^{r+s-1})}\mathbb{F})&=H^*(L\otimes_{\mathbb{F}(\pi)}\mathbb{F})^{\otimes r}\otimes H^*(C_\#(E)\otimes_{\mathbb{F}(\pi)}\mathbb{F})^{\otimes s-1}\\&=H^*(B_\pi,X;\mathbb{F})^{\otimes r}\otimes H^*(B_\pi;\mathbb{F})^{\otimes s-1}\,,\end{aligned}\tag{6.6}$$

where $r+s=k$, and the action of $(g_1,\ldots,g_{r+s-1})\in\pi^{r+s-1}$ is via the rule

$$g_i(x_1\otimes\cdots\otimes x_{r+s})=(x_1\otimes\cdots\otimes g_i(x_i)\otimes\cdots\otimes g_i^{-1}(x_{r+s}))\,.\tag{6.7}$$

By this we mean that g_i only acts non-trivially on the ith and kth coordinates and there acts as specified. Taking the natural filtration by dimension on the first $k-1$ factors gives (yet another) spectral sequence whose E_2-term is the cohomology of the complex $\mathrm{Hom}_{\mathbb{F}(\pi^{r+s-1})}(L^{\otimes r}\otimes C_\#(E)^{\otimes s-1},\,H^*(C_\#(E);\mathbb{F}))$, namely, recalling Proposition 6.1,

$$\begin{aligned}&H^*_{\pi^{r+s-1}}(L^{\otimes r}\otimes C_\#(E)^{\otimes s-1};H^*(C_\#(E);\mathbb{F}))\\&\quad=H^*((L\otimes_{\mathbb{F}(\pi)}\mathbb{F})^{\otimes r}\otimes(C_\#(E)\otimes_{\mathbb{F}(\pi)}\mathbb{F})^{\otimes s-1})\end{aligned}$$

concentrated along the line $E_{*,0}$ since the action of π on

$$H^*(C_\#(E);\mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } * = 0\,, \\ 0 & \text{otherwise} \end{cases}$$

is trivial. It follows that $E_2 = E_\infty$ for this sequence, giving the equality (6.6).

Now that we have evaluated the E_1-term of our spectral sequence, note that there is a non-trivial d_1-differential associated to the differential in the long exact sequence in cohomology of (6.3) above. The effect of this differential, in view of the results above is to replace $H^*(L \otimes_{\mathbb{F}(\pi)} \mathbb{F})$ by $H^*(X)/A$ and $H^*(B_\pi;\mathbb{F})$ by A (see Remark 6.1). Thus we have determined the E_2-term of the spectral sequence, except for the part L_k corresponding to $H^*(L^{\otimes k} \otimes_{\mathbb{F}(\pi^{k-1})} \mathbb{F})$, which starts in dimension at least k. $\square$

For later use we record the following consequence of (6.6):

Lemma 6.1. *Recalling the identifications* (6.5) *and* (6.6), *the natural map*

$$H^*(B_\pi, X;\mathbb{F})^{\otimes r} \otimes H^*(\pi;\mathbb{F})^{\otimes s} \to H^*((L^{\otimes r} \otimes C_\#(E)^{\otimes s}) \otimes_{\mathbb{F}(\pi^{r+s-1})} \mathbb{F})$$

induced by the inclusion $\pi^{r+s-1} \hookrightarrow \pi^{r+s}$ of (6.7) *is a surjection and is equivariant with respect to the action of $\mathcal{S}_r \times \mathcal{S}_s$ which permutes coordinates.*

Next, we have

Lemma 6.2. *Let $\pi = \prod_1^m \mathbb{Z}/m_j \oplus \mathbb{Z}^t$ with generators $\alpha_1, \ldots, \alpha_m, \chi_1, \ldots, \chi_t$. Let $a_i \in H^1(B_\pi^s;\mathbb{F})$, $i = 1, \ldots, m$ be $\sum_{j=1}^s \alpha_{i,j}^*$, c_i in $H^1(B_\pi^s;\mathbb{F})$ be the sum of elements dual to the integral generators $c_i = \sum_{j=1}^s \chi_{i,j}^*$, $i = 1, \ldots, t$, and $b_i = \beta(a_i)$, the appropriate Bochstein of a_i. Then, thinking of $H^*(B_\pi, X;\mathbb{F})^{\otimes r} \otimes H^*(B_\pi;\mathbb{F})^{\otimes s}$ as being embedded in the tensor algebra of $H^*(B_\pi, X;\mathbb{F}) \oplus H^*(B_\pi;\mathbb{F})$, the kernel of the surjection above is the intersection of $H^*(B_\pi, X;\mathbb{F})^{\otimes r} \otimes H^*(B_\pi;\mathbb{F})^{\otimes s}$ with the ideal generated by the elements a_i, b_i, and c_i.*

Proof. The point is to understand the inclusion of $\pi^{s-1} \subset \pi^s$. Note that by definition π^{s-1} is the kernel of the *sum* map: $\mu_s : \pi^s \to \pi$, so we have an exact sequence

$$\pi^{s-1} \overset{i}{\hookrightarrow} \pi^s \xrightarrow{\mu_s} \pi$$

and the kernel of i^* is evidently the ideal generated by the image of $(\mu_s)^*$, but the elements above are precisely those images. $\square$

Proposition 6.3. *There is a spectral sequence converging to $H^*(C_k(Y) \times_{\mathcal{S}_k} \tilde{X}_k)$ with the π^{k-1}-action as above, that has E_2-term*

$$E_2^i = H^*_{\mathcal{S}_k}(C_k(Y); H^*(B_\pi, X;\mathbb{F})^{\otimes i} \otimes H^*(B_\pi;\mathbb{F})^{\otimes k-i-1} \otimes_{\mathbb{F}(\mathcal{S}_i \times \mathcal{S}_{k-i})} \mathbb{F}(\mathcal{S}_k))$$

for $i < k$, while E_2^k starts in dimension at least k and $E_2^j = 0$ for $j > k$.

Remark 6.3. The actions of the symmetric groups on $H^*(B_\pi, X; \mathbb{F})^{\otimes i}$ and $H^*(B_\pi; \mathbb{F})^{\otimes k-i-1}$ are the same as above, Also, since the $\mathcal{S}_k$-module above is induced up from an $\mathcal{S}_i \times \mathcal{S}_{k-i}$-module, we see that

$$\begin{aligned} &H^*_{\mathcal{S}_k}(C_k(Y); H^*(B_\pi, X; \mathbb{F})^{\otimes i} \otimes H^*(B_\pi; \mathbb{F})^{\otimes k-i-1} \otimes_{\mathbb{F}(\mathcal{S}_i \times \mathcal{S}_{k-i})} \mathbb{F}(\mathcal{S}_k)) \\ &\quad \cong H^*_{\mathcal{S}_i \otimes \mathcal{S}_{k-i}}(C_k(Y); H^*(B_\pi, X; \mathbb{F})^{\otimes i} \otimes H^*(B_\pi; \mathbb{F})^{\otimes k-i-1}) . \end{aligned}$$

Proof. The filtration of $C_\#(\tilde{X}^k) \otimes_{\mathbb{F}(\pi^{k-1})} \mathbb{F}$ constructed above is invariant under the action of $\mathcal{S}_k$. Consequently, it gives rise to a filtration of the chain complex for $C_k(Y) \times_{\mathcal{S}_k} \tilde{X}^k/(\pi^{k-1}) = C_k(Y) \times_{\mathcal{S}_k} \tilde{X}_k$, and the E_0-level term for this spectral sequence is $\mathrm{Hom}_{\mathbb{F}(\mathcal{S}_k)}(C_\#(C_k(Y), E_0(\text{fiber}))$ with differential dual to $\partial \otimes 1 + \epsilon(1 \otimes d_0)$ where d_0 is the first differential in the spectral sequence of the fiber. However, $C_k(Y)$ is a free $\mathbb{F}(\mathcal{S}_k)$-module so the usual spectral sequence for determining the cohomology of this complex reduces to the sequence of its edge terms which gives E_1 as a chain complex with (local) coefficients with values in the E_1 term of the fiber, and so $E_2 = H^*(C_k(Y);\ E_1(\text{fiber}))$ as asserted. □

Proof of Proposition 6.1. Let $f : V_\# \to W_\#$ be a chain map, then the *mapping cylinder* of f is the complex

$$MCyl(f)_\# = V_\# \oplus V_{\#-1} \oplus W_\#$$

with boundary defined by

$$\partial(v|v'|w) = (\partial v + v'| - \partial v'| - f(v') + \partial w) .$$

It is chain equivalent to $W_\#$ via the map ψ:

$$\psi : (v|v'|w) = w + f(v) ,$$

and converts the map f into the inclusion $I : V_\# \hookrightarrow MCyl(f)_\#$ with $I(v) = (v, 0, 0)$. Similarly, the *mapping cone* of f is the chain complex $MC(f) = V_{\#-1} \oplus W_\#$ with boundary given as $\partial(v|w) = (-\partial(v)|\partial(w) + f_*(v))$.

In our case, in order to convert the chain map $\rho_\#$ into a surjection we combine the two above constructions and replace $C_\#(\tilde{X})$ by the chain complex

$$C_\#(\tilde{X}) \oplus C_{\#+1}(E_\pi) \oplus C_{\#-1}(\tilde{X}) \oplus C_\#(E_\pi) \oplus C_\#(\tilde{X}) . \tag{6.8}$$

If we write an element as

$$\{c_n, e_{n+1}|c'_{n-1}, e'_n|c''_n\} ,$$

the boundary is defined by

$$\begin{aligned} &\partial\{(c_n, e_{n+1}|c'_{n-1}, e'_n|c''_n\} \\ &\quad = \{-\partial c_n + c'_{n-1}, \rho_\#(c_n) + \partial e_{n+1} + e'_n|\partial c'_{n-1}, -\rho_\#(c'_{n-1}) - \partial e'_n| - c'_{n-1} - \partial c''_n\} . \end{aligned}$$

Now, if $(V_\#, \partial)$ is a chain complex, then the *suspension* $(S(V)_\#, \partial)$ is defined as $(V_\#, -\partial)$. The map $\epsilon : V_\# \to S(V)_\#$ defined by $\epsilon(v) = (-1)^{|v|}v$ is easily seen to be a chain equivalence. (Normally, the suspension $SV_{\#}$ is regraded, so that $SV_n = V_{n-1}$, but this would introduce needless indices into our constructions, hence we leave the grading degree as it was. The important point for us is the fact that the resulting complex is equivariantly chain equivalent to the original one.) Here, the chain equivalence of (6.8) with $SC_{\#}(\tilde{X})$ is given by sending the 5-tuple above to $c_n'' + c_n$, while the surjection to $SC_{\#}(E)$ is given by sending the 5-tuple to $e_n' - \rho_{\#}\pi(c_n'')$. □

6.3. *The cohomology of $C_k(\mathbb{C})$ with coefficients in the E_1-term of the fiber*

We want the stabilization map on the last fiber of our iterate fibration,

$$\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1) \to \tilde{C}_{n_1+1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1)\,,$$

to induce isomorphisms in homology through a suitable range. We first examine the case $n_2 = n_3 = \cdots n_\ell = 0$, that is, when there are no punctures. (Alternatively, one can look at the subspace of particles over $\mathbb{C} \subset \mathbb{C} - \{y_{\ell,1}, \ldots, y_{2,n_2}\}$, and take a connected component of the cover.) Setting $n_1 = k$, $\mathcal{LPP}_1 = X$ as above, one then has a map

$$C_k(\mathbb{C}) \times_{\mathcal{S}_k} (\tilde{X}_k) \longrightarrow C_{k+1}(\mathbb{C}) \times_{\mathcal{S}_{k+1}} (\tilde{X}_{k+1})\,. \tag{6.9}$$

Proposition 6.4. *The map* (6.9) *induces isomorphisms in cohomology in dimensions less than $k - s - 1$, where s is the rank of π.*

Proof. In view of the above spectral sequences, we want to understand the stability properties of

$$\begin{aligned} &H^*_{\mathcal{S}_k}(C_k(\mathbb{C}); H^*(B_\pi, X; \mathbb{F})^{\otimes i} \otimes H^*(B_\pi; \mathbb{F})^{\otimes k-i-1} \otimes_{\mathbb{F}(\mathcal{S}_i \times \mathcal{S}_{k-i})} \mathbb{F}(\mathcal{S}_k)) \\ &\quad \longrightarrow H^*_{\mathcal{S}_{k+1}}(C_{k+1}(\mathbb{C}); H^*(B_\pi, X; \mathbb{F})^{\otimes i} \otimes H^*(B_\pi; \mathbb{F})^{\otimes k-i} \otimes_{\mathbb{F}(\mathcal{S}_i \times \mathcal{S}_{k-i+1})} \mathbb{F}(\mathcal{S}_{k+1}))\,. \end{aligned} \tag{6.10}$$

We note that the stabilization adds a point labelled by the base point in X, and so on the algebraic level is just tensoring by a 0-cycle. Since $H^0(B_\pi, X; \mathbb{F}) = 0$, the stabilization factors through an increase in the number of $H^*(B_\pi; \mathbb{F})$-factors. The space $C_k(\mathbb{C})/\mathcal{S}_k$ is the classifying space of the braid group Γ_k, which maps to the symmetric group $\mathcal{S}_k$. We are therefore determining the stability properties of the cohomology of the braid group with coefficients in the $\mathbb{F}(\mathcal{S}_k)$-modules above. Loop space theory tells us how to determine the cohomology of Γ_k with coefficients in modules of the form $A^{\otimes k}$, where $\mathcal{S}_k$ acts to permute coordinates. But the modules in question here are not of this form. So we will have to introduce further arguments. We begin by giving the stability results for $H^*(C_k(\mathbb{C}); A^{\otimes k})$ where A is a tensor product $E(e_1, \ldots, e_s) \otimes \mathbb{F}[b_1, \ldots, b_r]$ with $\dim(e_i) = 1$, $\dim(b_j) = 2$. □

Lemma 6.3 ([6, Lemma 6.8]). *Let A be as above, then the cohomology map induced by the natural map of chain complexes*

$$\begin{aligned}\mathcal{C}_{\#}(C_k(\mathbb{C}) \otimes_{\mathbb{F}(\mathcal{S}_k)} A^k) &\longrightarrow \mathcal{C}_{\#}(C_{k+1}(\mathbb{C}) \otimes_{\mathbb{F}(\mathcal{S}_k \times 1)} A^k \otimes 1)\\ &\longrightarrow \mathcal{C}_{\#}(C_{k+1}(\mathbb{C}) \otimes_{\mathbb{F}(\mathcal{S}_{k+1})} A^{k+1})\end{aligned}$$

is an isomorphism through dimensions less than $2k - s + 1$.

We next determine stability dimensions for the map (6.10) for the case where π is *cyclic*: in the cohomology of Γ_k, we replace the tensor products $A^{\otimes k}$ by the modules in Lemma 6.2. Assume that $\pi = \mathbb{Z}$ or $\mathbb{Z}_{p^i} = \mathbb{Z}/p^i\mathbb{Z}$ with p a prime, so $H^*(\pi; \mathbb{F}) = E(e_1)$ or $E(\alpha_1) \otimes \mathbb{F}[\beta_2]$ and

$$H^*(\pi^k; \mathbb{F}) = \begin{cases} E(\alpha_1, \ldots, \alpha_k) & \text{if } \pi = \mathbb{Z}\,, \\ E(\alpha_1, \ldots, \alpha_k) \otimes \mathbb{F}[\beta_1, \ldots, \beta_k] & \text{if } \pi = \mathbb{Z}_{p^i}\,. \end{cases}$$

We only do the case $\pi = \mathbb{Z}$, as the others are similar. We first determine the structure of the ideal described in Lemma 6.2 and the corresponding quotient. We define $V(m) = E(\alpha_1, \ldots, \alpha_m)/(\sum \alpha_i)$ in the notation of Lemma 6.2 and we have

Lemma 6.4. *There is a short exact sequence*

$$0 \longrightarrow V(m) \xrightarrow{\cup \sum \alpha_i} E(\alpha_1, \ldots, \alpha_m) \xrightarrow{\pi} V(m) \longrightarrow 0\,.$$

Proof. For convenience denote by $a(m)$ the sum $\Sigma_i \alpha_i \in E(\alpha_1, \ldots \alpha_m)$. The kernel of the projection is composed of elements of the form $\omega a(m)$, and there is a natural map from $V(m)$ to the kernel given by $\omega \mapsto \omega a(m)$. It is obviously surjective, and to see that it is injective, suppose that $\omega a(m) = 0$. Note that $a(m) = a(m-1) + \alpha_m$, and we can write

$$E(\alpha_1, \ldots, \alpha_m) = E(\alpha_1, \ldots, \alpha_{m-1}) \oplus E(\alpha_1, \ldots, \alpha_{m-1})\alpha_m\,.$$

Set $\omega = \theta + \nu\alpha_m$, then $0 = \omega a(m) = \theta a(m-1) + (-\nu a(m-1) + \theta)\alpha_m$, and so $\theta = \nu a(m-1)$ giving $\omega = \nu a(m)$.

Now expanding out the exact sequence of Lemma 6.4 with respect to the grading gives

$$\begin{array}{l} 0 \longrightarrow \mathbb{F} \longrightarrow E^1(\alpha_1, \ldots, \alpha_m) \longrightarrow V^1(m) \longrightarrow 0 \\ 0 \longrightarrow V^1(m) \longrightarrow E^2(\alpha_1, \ldots, \alpha_m) \longrightarrow V^2(m) \longrightarrow 0 \\ 0 \longrightarrow V^2(m) \longrightarrow E^3(\alpha_1, \ldots, \alpha_m) \longrightarrow V^3(m) \longrightarrow 0\,. \\ \qquad\;\; \vdots \qquad\quad \vdots \qquad\qquad \vdots \end{array} \tag{6.11}$$

From Lemmas 6.1 and 6.2, we then have an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^*(B_\pi, X; \mathbb{F})^{\otimes r} \otimes V^*(k-r) \longrightarrow H^*(B_\pi, X; \mathbb{F})^{\otimes r} \otimes H^*(\pi; \mathbb{F})^{\otimes k-r} \\ &\longrightarrow H^*((L^{\otimes r} \otimes C_{\#}(E)^{\otimes k-r}) \otimes_{\mathbb{F}(\pi^{k-1})} \mathbb{F}) \longrightarrow 0\,. \end{aligned}$$

The stability result for homology with coefficients in the last module will follow from that of the other two. The middle one has the stability property in dimensions $< 2k$, by Lemma 6.4. For the first, we can build up inductively using (6.11) and Lemma 6.3.

We note that stabilization gives maps

$$\begin{aligned} &H^*_{\mathcal{S}_k}(C_k(\mathbb{C}); H^*(B_\pi, X)^r \otimes V^*(k-r) \otimes_{\mathbb{F}(\mathcal{S}_r \times \mathcal{S}_{k-r})} \mathbb{F}(\mathcal{S}_k)) \\ &\qquad \longrightarrow H^*_{\mathcal{S}_{k+1}}(C_{k+1}(\mathbb{C}); H^*(B_\pi, X)^r \otimes V^*(k-r+1) \otimes_{\mathbb{F}(\mathcal{S}_r \times \mathcal{S}_{k-r+1})} \mathbb{F}(\mathcal{S}_{k+1}))\,, \end{aligned}$$

since $1 \notin H^*(B_\pi, X)$. This fact also shifts the stability up by r dimensions. On the other hand each of the short exact sequences above gives rise to a long exact sequence

$$\begin{aligned} \cdots \longrightarrow\; &H^*_{\mathcal{S}_k}(C_k(\mathbb{C}); H^*(B_\pi, X)^{\otimes r} \otimes V^i(k-r) \otimes_{\mathbb{F}(\mathcal{S}_r \times \mathcal{S}_{k-r})} \mathbb{F}(\mathcal{S}_k)) \\ \longrightarrow\; &H^*_{\mathcal{S}_k}(C_k(\mathbb{C}); H^*(B_\pi, X)^{\otimes r} \otimes E^{i+1}(\alpha_1, \ldots, \alpha_k) \otimes_{\mathbb{F}(\mathcal{S}_r \times \mathcal{S}_{k-r})} \mathbb{F}(\mathcal{S}_k)) \\ \longrightarrow\; &H^*_{\mathcal{S}_k}(C_k(\mathbb{C}); H^*(B_\pi, X)^{\otimes r} \otimes V^i(k-r) \otimes_{\mathbb{F}(\mathcal{S}_r \times \mathcal{S}_{k-r})} \mathbb{F}(\mathcal{S}_k)) \longrightarrow \cdots \end{aligned}$$

which, in turn gives an inductive method for determining

$$\begin{aligned} &H^*_{\mathcal{S}_k}(C_k(\mathbb{C}); H^*(B_\pi, X; \mathbb{F})^{\otimes r} \otimes V^i(k-r) \otimes_{\mathbb{F}(\mathcal{S}_r \times \mathcal{S}_{k-r})} \mathbb{F}(\mathcal{S}_k)) \\ &\qquad = H^*(\Gamma_{k,r}; H^*(B_\pi, X; \mathbb{F})^{\otimes r} \otimes V^i(k-r))\,, \end{aligned}$$

where $\Gamma_{k,r}$ is the subgroup of Γ_k given as the inverse image of $\mathcal{S}_{k-r} \times \mathcal{S}_r$ under the projection $\Gamma_k \to \mathcal{S}_k$. Of course, we do not need the entire calculation, but what is clear is that the five lemma shows that we lose at most one dimension of stabilization passing from $V^i(k-r)$ to $V^{i+1}(k-r)$. But since, at the next stage the dimension of $V^i(k-r)$ is augmented by one (since there it is $V^i(k-r) \cup a(k)$), we actually do not lose this dimension. We then find that homology stabilizes in the first term for dimensions less than $2(k-r)+r$ and so the homology of the third term stabilizes in dimensions $2(k-r)+r-1 \geq k-1$.

There remains the question of stability for arbitrary π. The resulting stability dimensions will then decrease as we replace the cyclic group by a general finitely generated Abelian group by a simple function of the number of generators of π since the construction preserves tensor product. That is to say, if $\pi = \pi_1 \times \pi_2$ then $H^*(\pi^k) = H^*(\pi_1^k) \otimes H^*(\pi_2^k)$; similarly the quotients $H^*((L^{\otimes r} \otimes C_\#(E)^{\otimes k-r}) \otimes_{\mathbb{F}(\pi^{k-1})} \mathbb{F})$ for π decompose into tensor products of the corresponding factors for π_1, π_2. One then obtains stability for π-covers from that of the π_i-covers, with the possible loss of one dimension due to "edge effects". One then has the stability for the terms of the spectral sequence (6.10), and so, with the loss of one dimension, for the cohomology of the spaces (6.9). This concludes the proof of Proposition (6.4). $\square$

6.4. *Putting in the punctures*

We exploit the result for particles over $\mathbb{C}$ to obtain the corresponding stability result for the strata $\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1)$. Keeping $n_i = k$, $X = \mathcal{LPP}_1$ as before, we renumber the punctures y_{ij} as y_s, $s = 1, \ldots, m$, where $m = \sum_{i=2}^{\ell} n_i$, and suppose that the points y_i are placed along the y axis. Let $\mathbb{R}_i^+$ denote the half-lines $\{(x, y_i)|x\rangle 0\}$, and denote by D the complement of the $\mathbb{R}_i^+$. We stratify $C_k(D - \{y_1, \ldots, y_m\}) \times_{\mathcal{S}_k} X^k$ according to the number of particles on the $\mathbb{R}_i^+$ with strata

$$\mathcal{D}_{j,r_1,\ldots,r_m} = (C_j(D) \times C_{r_1} \times \cdots \times C_{r_m}) \times_{\mathcal{S}_j \times \mathcal{S}_{r_1} \times \cdots \times \mathcal{S}_{r_m}} X^k\,.$$

As there is a natural ordering of points along lines, one can rewrite this as

$$\mathcal{D}_{j,r_1,\ldots,r_m} = (C_j(D) \times \Delta^{k-j}) \times_{\mathcal{S}_j} X^k$$

where Δ^{k-j} is an open $k-j$-cell, and $\mathcal{S}_j$ acts on $C_j(D)$ and the first j factors of X^k. Passing to the covers, we have strata

$$\tilde{\mathcal{D}}_{j,r_1,\ldots,r_m} = [(C_j(D) \times \Delta^{k-j}) \times_{\mathcal{S}_j} \tilde{X}_k] \times G\,, \tag{6.12}$$

where G is a discrete Abelian group, corresponding to pure winding around the punctures. This factor appears because staying on the stratum precludes winding around the punctures. We have the following easy lemma, giving the codimension of the strata:

Lemma 6.5. *The strata $\tilde{\mathcal{D}}_{j,r_1,\ldots,r_m}$ are smooth and have real codimension $k-j$.*

We can compute the cohomology of $\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1)$ from the spectral sequence associated to this stratification, in which the cohomology of the strata appears suspended by the codimension. The stabilization preserves the stratification and so the stability result for $\tilde{C}_{n_1|n_2,\ldots,n_\ell}(\mathcal{LPP}_1)$ follows from the following:

Proposition 6.5. *The stabilization map*

$$\tilde{\mathcal{D}}_{j,r_1,\ldots,r_m} \to \tilde{\mathcal{D}}_{j+1,r_1,\ldots,r_m}$$

induces isomorphisms in cohomology in dimensions $< j - s - 1$.

Proof. In analogy with Proposition 6.3, one has a spectral sequence for the strata whose E_2 term is the cohomology of $C_j(Y)$ with coefficients in the $\mathcal{S}_j$-module

$$H^*(B_\pi, X; \mathbb{F})^{\otimes i} \otimes H^*(B_\pi; \mathbb{F})^{\otimes k-i-1} \otimes_{\mathbb{F}(\mathcal{S}_i \times \mathcal{S}_{k-i})} \mathbb{F}(\mathcal{S}_k) + H_k\,, \tag{6.13}$$

where H_k is k-connected, and so does not enter into consideration. To see what this $\mathcal{S}_j$-module is, let us write

$$A = H^*(B_\pi, X; \mathbb{F})\,,$$
$$B = H^*(B_\pi; \mathbb{F})\,.$$

As a vector space, (6.13) is

$$\oplus_{(i,k-i)\text{shuffles}} A^{\otimes i} \otimes B^{\otimes k-i-1}\,.$$

Think of each of these terms corresponding to the choice of i "slots" amongst k boxes; the action of $\mathcal{S}_j$ on these depends on the number ℓ of these slots lying in the first j boxes. As an $\mathcal{S}_j$-module, one can identify it with

$$\oplus_{\ell=0}^{\min(j,i)}(A^{\otimes\ell}\otimes B^{\otimes j-\ell-1}\otimes_{\mathbb{F}(\mathcal{S}_\ell\times\mathcal{S}_{j-\ell})}\mathbb{F}(\mathcal{S}_j))\otimes((A^{\otimes i-l}\otimes B^{\otimes k-i-j+\ell})^{C(k-j,k-\ell)},$$

where $C(k-\ell,k-j)=(k-\ell)!/[(k-j)!(j-\ell)!]$. The action of the $\mathcal{S}_j$ on the last factors is trivial. The coefficient modules are then tensor products of the $\mathcal{S}_j$-modules for which the stability result was proven in Sec. 6.3, and so the stability result follows for these also. This completes the proof of Theorem 5.2, and hence, the proof of Theorem 1.2. □

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