$\Omega^2(BSP)$ **NOTES**

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CONTENTS

These are Ravi's notes, for Jim, but also to set some notation.

$\boxed{x} = x_0/x_1$. $\boxed{p_\infty} = \infty = [1,0] = V(x_1)$.

I think $y = x_1/x_0$.

0.1. *The space* V.

We start with a vector space $\boxed{U = V}$ with an alternating form. (I expect everything I type here to work fine for O in place of Sp with change of signs, but I stick to the symplectic case for concreteness and to avoid confusion.) Choosing a splitting $V = L \oplus L^*$ is called a polarization or Lagrangian splitting. Jim says it is a "Weinstein normal form".

More generally, we can work over an arbitrary base, and all statements will behave well with respect to base change. So most generally, we work over $\operatorname{Spec} \mathbb{Z}$, and V is a free sheaf on $\operatorname{Spec} \mathbb{Z}$ of rank $2n$, etc. I will continue to use vector space language for simplicity.

1. BACKGROUND ON THE SYMPLECTIC AFFINE GRASSMANNIAN

1.1. *The loop space of* V.

The loop space of V is

$$\boxed{H} = \boxed{V((z))} := V[[z]] \oplus z^{-1}V[z^{-1}].$$

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We define $\boxed{H^+} = V[[z]]$ and $\boxed{H^-} = z^{-1}V[z^{-1}]$. H has a residue pairing:

$$\boxed{\langle f(z), g(z)\rangle} := \mathrm{Res}_{z=0}\, \omega(f(-z), g(z))\ dz.$$

This residue pairing is skew-symmetric and nondegenerate. H is thought of as some sort of infinite-dimensional symplectic Hilbert space.

Question: what's the reason for the sign in pairing? I think in our pairing (see later on), there is no sign.

1.2. *The Lagrangian Sato Grassmannian.*

The Lagrangian Sato Grassmannian parametrizes "Lagrangians" of the loop space that are close to H^+. It is contained in the index zero component of the (usual — not symplectic) Sato Grassmannian (not defined here). The Lagrangian Sato Grassmannian is often denoted LGr_∞ (or $LGr(H)$). I will use temporary notation $\boxed{\text{LgStGr}}$ to remind us of its meaning. (This is a macro which can be easily changed.)

The Nth truncation is denoted $\boxed{\text{LgStGr}^{(N)}} = \text{LgStGr}^{(N)}(H)$, and is isomorphic to

$$LG(N \dim V, 2N \dim V).$$

We can show that LgStGr_∞ is the limit of $LGr(nN, 2nN)$ in our naive homotopy category. Thus its homotopy type is $LGr(\infty, 2\infty) = U/O = \Omega(Sp)$.

Remark: This will likely be relevant for us, because we will be "increasing V" which will be changing the symplectic affine Grassmannian, but the ambient Lagrangian Sato Grassmannian (even for finite V) will be what we want.

1.3. *The symplectic affine Grassmannian.*

The symplectic affine Grassmannian is the z-stable locus inside the Lagrangian Sato Grassmannian. The symplectic affine Grassmannian is unfortunately often denoted Gr or $Gr_{Sp_{2n}}$ or Gr_{Sp}. I will use temporary notation $\boxed{\text{SpAfGr}}$ to remind us. This is a temporary macro which can be easily changed.

Fact: $\text{SpAfGr}_{Sp(V)}$ is homotopic to $\Omega(Sp(V))$.

Vague argument (that I've not thought through): $LSp(V)$ parametrizes maps from loops to $Sp(V)$ — smooth maps from S^1. $\boxed{L^+} = L^+(V)$ is the subset where the map extends holomorphically to the disk. Then show that $L/L^+ \cong LGr$, by describing a transitive action of L, and identifying the stabilizer as L^+. But L^+ is contractible. (Presumably I should look at [PS] for more. It might be in a later paper of Nadler, perhaps [Na], see one of the references of [Z].) (There is an argument of this sort in the algebraic category, involving algebraic loops; and also in the continuous category.)

1.4. *Truncations of the symplectic affine Grassmannian.* The Nth truncation of the symplectic affine Grassmannian is called $\boxed{\mathrm{SpAfGr}_{\mathrm{Sp}}^{(N)}} = \mathrm{Gr}^{(N)}$. (Caution: I am worried that the terminology Gr gets used both for the affine and the Sato case.)

1.5. *Singularities of the truncation.* Apparently $\mathrm{SpAfGr}^{(N)}$ is singular in codimension $\dim V$, with possible reference [MOV].

2. MODULI OF VECTOR BUNDLES WITH SYMPLECTIC STRUCTURE

Now let $\boxed{\Omega^2_{alg}(BSp(2n))}$ be the moduli space of vector bundles on $\mathbb{P}^1$, framed at p_∞ by V, with its form.

Let $\boxed{\Omega^2_{alg}(BSp(2n))^{[N]}}$ parametrize those bundles $\mathcal{F}$ such that $\mathcal{F}(N)$ is globally generated. For example, $\Omega^2_{alg}(BSp(2n))^{[N]}$ is empty if $N < 0$.

2.1. *Basic facts about this space.* The space $\Omega^2_{alg}(BSp(2n))$ is an Artin stack.

The space $\Omega^2_{alg}(BSp(2n))$ is an $Sp(2n)$ bundle over the "unframed" moduli space, which is also thus an Artin stack. This latter space is smooth, because the automorphisms/deformation/obstructions of a principle bundle E is given by the cohomology of the adjoint bundle $ad(E)$, so automorphisms are $H^0(ad(E))$, deformations are $H^1(ad(E))$, and obstructions are $H^2(ad(E))$, which in this case are zero. (The adjoint bundle is the "twisted Lie algebra bundle", as I think Jim was telling me.)

The space $\Omega^2_{alg}(BSp(2n))$ is the union of an increasing sequence of open substacks $\Omega^2_{alg}(BSp(2n))^{[N]}$.

Each $\Omega^2_{alg}(BSp(2n))^{[N]}$ is quasicompact and finite type.

$\Omega^2_{alg}(BSp(2n))^{[0]}$ is a reduced point.

The space of bundles trivialized in a formalized neighborhood of p_∞ is apparently, as an ind-scheme, the affine symplectic Grassmannian $\mathrm{SpAfGrSp}(2n)$.

2.2. *Sketch of why these things are true.* These are all well-known facts, but we will also end reproving them, so we can have confidence in these statements. Here is the architecture of our argument.

First: $\Omega^2_{\mathrm{alg}}(BSp(2n))$ and $\Omega^2_{\mathrm{alg}}(BSp(2n))^{[N]}$ are stacks in the usual (smooth etc.) topology (not yet obviously algebraic stacks).

(ii) We construct a finite type affine scheme (explicitly, with generators and relations), that will parametrize the same things as $\Omega^2_{\mathrm{alg}}(BSp(2n))^{[N]}$, with in addition a Zariski-splitting. This is an affine bundle over $\Omega^2_{\mathrm{alg}}(BSp(2n))^{[N]}$. Thus $\Omega^2_{\mathrm{alg}}(BSp(2n))^{[N]}$ is an algebraic stack (i.e., an Artin stack).

We have open embeddings of these truncations, and their union is the entire space, so $\Omega^2_{\mathrm{alg}}(BSp(2n))$ is an algebraic stack.

$\Omega^2_{\mathrm{alg}}(BSp(V))$ parametrizes the following.

- $\boxed{\mathcal{F}}$ is a rank $2n$ vector bundle on $\mathbb{P}^1$. (Caution: earlier the bundle on $\mathbb{P}^1$ was considered to be rank n.)
- We have an identification $\mathcal{F}|_{p_\infty} \xrightarrow{\sim} V$
- $\boxed{\phi_{\mathcal{F}}} : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee$ satisfying $\phi_{\mathcal{F}}^\vee = -\phi_{\mathcal{F}}$, and $\phi_{\mathcal{F}}|_{p_\infty} = \phi_V$ (where ϕ_V comes from the alternating form). Or equivalently: $\boxed{\psi_{\mathcal{F}}} : \mathcal{F} \otimes \mathcal{F} \to \mathcal{O}$, with appropriate hypotheses.

2.3. *Comparison to topology.* I think we quote Cohen-Lupercio-Segal or someone else to show that smooth maps to $BSp(V)$ are homotopic to holomorphic maps to $BSp(V)$. Then by GAGA (some justification needed) this is the same as algebraic maps to $BSp(V)$.

2.4. $\boxed{\Omega^2_{d,\mathrm{alg}}(BSp(V))}$ parametrizes the same, with the additional requirement that $\mathcal{F}(dp_\infty)$ is generated by global sections. (Equivalently, when you write $\mathcal{F}$ as a direct sum of line bundles, the summands are all of degree between $-d$ and d inclusive. This interpretation is *not helpful*.)

We define $\boxed{\mathcal{E}} = \mathcal{F}(dp_\infty)$ for convenience. This bundle is rank $2n$ and

We then have $\boxed{\phi_{\mathcal{E}}} : \mathcal{E} \to \mathcal{E}^\vee(2d)$, and

$$\boxed{\psi_{\mathcal{E}}} = \psi_{\mathcal{F}}(2d) : \mathcal{E} \otimes \mathcal{E} \to \mathcal{O}(2dp_\infty). \tag{1}$$

2.5. **Claim. —** *We have an induced isomorphism* $\mathcal{E}|_{p_\infty} \xrightarrow{\sim} V$.

(Proof omitted.)

2.6. *Zariski-framing.*

Define $\boxed{A} := H^0(\mathcal{E}(-p_\infty))$. $\mathcal{E}$ has rank $2n$ and degree $2nd$. Thus $\dim A = 2n(d-1) + 2n = 2nd$.

2.7. We now consider the Zariski-framed moduli space, which doesn't yet have a name. I temporarily dub it $\boxed{X(d)}$. $\dim X(d) = (\dim A)(\dim U) = (2dn)(2n) = 4dn^2$.

2.8. The data of the Zariski-framed bundle $\mathcal{E}$ is the data of A, plus $\alpha : A \to A$, and $j : A \to U$, along with an *open condition* on j and α.

The following matrix (where we are treating elements of A and U as column vectors) is required to be full rank for all $x \in \mathbb{C}$.

$$\begin{array}{|c|c}
\hline
x\mathrm{Id} - \alpha & A \\
\hline
j & U \\
\hline
A^\vee &
\end{array}$$

For each $x \in \mathbb{C}$, the locus where the matrix is not full rank is codimension $2n+1$. Thus as x varies, the locus where the matrix is not full rank is codimension $2n$. (That's not quite rigorous.)

2.9. *Relationships between* (α, j) *and the pairing* $(\psi_\mathcal{E}/\phi_\mathcal{E})$.

We unpack (1). We have

$$\boxed{\psi_{A\oplus U}} : \underbrace{(A \oplus U)}_{\dim=(2d+1)(2n)} \otimes (A \oplus U) \to \mathbb{C}[x_0, x_1]_{2d}$$

where the subscript means "homogeneous of degree 2d", or equivalently,

$$\boxed{\phi_{A\oplus U}} : A \oplus U \to (A^\vee \oplus U^\vee)[x_0, x_1]_{2d}$$

satisfying $\phi_{A\oplus U} = -\phi^\vee_{A\oplus U}$, satisfying three conditions:

(V) "vanishes on $(x-\alpha, -j)$ (on either factor)" (closed condition) and
(ND) "nondegenerate on the quotient for all $x \in \mathbb{C}$" (open condition)
(BP) $[x_0^{2d}]\phi_{A\oplus U}((a_1, u_1), (a_2, u_2)) = \psi_U(u_1, u_2)$ (base point condition, hence nondegeneracy at $x = \infty$)

The following is then tautological.

2.10. **Claim. —** *This precisely parametrizes our space. Hence our space is a quasiaffine variety.*

We unpack this. We write the components of ϕ out explicitly as follows.

Define

$$\boxed{T_A} : A \to A^\vee[x_0, x_1]_{2d} \qquad T_A = T_{A,0}x_0^{2d} + \cdots + T_{A,2d}x_1^{2d} \qquad \text{so } T_{A,i}^\vee = -T_{A,i} \qquad T_{A,0} = 0$$

$$\boxed{T_U} : U \to U^\vee[x_0, x_1]_{2d} \qquad T_U = T_{U,0}x_0^{2d} + \cdots + T_{U,2d}x_1^{2d} \qquad \text{so } T_{U,i}^\vee = -T_{U,i} \qquad T_{U,0} = \phi_U$$

$$\boxed{T_{AU}} : A \to U^\vee[x_0, x_1]_{2d} \qquad T_{AU} = T_{AU,0}x_0^{2d} + \cdots + T_{AU,2d}x_1^{2d} \qquad T_{AU,0} = 0$$

The final column gives the third condition (BP).

Condition (V) translates to

$$\phi_{A\oplus U}((x_0 - x_1\alpha)a, ja) = 0 \in A^\vee \oplus U^\vee$$

for any $x \in \mathbb{C}$, $a \in A$.

The $A^\vee$ component of this gives:

$$0 = \sum x_0^{2d-1-i}x_i \left(T_{A,i}(x_0 a - x_1\alpha(a)) + T_{AU,i}^\vee(-j(a)x_1)\right)$$

The $U^\vee$ component of this gives:

$$0 = \sum x_0^{2d-1-i}x_i \left(T_{AU,i}(x_0 a - x_1\alpha(a)) + T_{U,i}^\vee(-j(a)x_1)\right)$$

We thus have two sequences of conditions from these.

$$0 = T_{A,i+1} - T_{A,i} \circ \alpha - T_{AU,i}^\vee \circ j : A \to A^\vee$$

and

$$0 = T_{AU,i+1} - T_{AU,i} \circ \alpha - T_{U,i} \circ j : A \to U^\vee$$

They can be considered to include the case $i = -1$, which are the previously-stated relations $T_{A,0} = 0$ and $T_{AU,0} = 0$.

Thus the closed condition (V) means that if we choose $T_{U,i}$ freely for $i = 1, \dots, 2d$, then everything else is determined.

2.11. *Checking dimensions.* $T_{U,i} : U \to U^\vee$ must satisfy $T_{U,i} = -T_{U,i}^\vee$, so the dimension of choices of one of them is $1 + \cdots + (2n-1) = (2n-1)(2n)/2 = n(2n-1)$. Thus the total number of choices is for all of them is $2dn(2n-1) = 4dn^2 - 2dn$. This does not agree with §2.7, which gives $4dn^2$!

3. For future use

For future use, define $j\!\!\alpha : A \to tU[[t]]$ by

$$j\!\!\alpha(a) = jat + j\alpha a t^2 + j\alpha^2 a t^3 + \cdots$$

This is the generating function for $j\alpha^{k-1}a$.

3.1. **Claim. —** $j\!\!\alpha \pmod{t}^{2d+1}$ *is an injection* $A \hookrightarrow tU \oplus \cdots \oplus t^{2d}U$.

3.2. *Sketch of proof.*

Consider $0 \to \mathcal{E}(-2d-1) \to \mathcal{E}(-1) \to \mathcal{E}(-1)/\mathcal{E}(-2d-1) \to 0$. Now $\mathcal{E}$ is the direct sum of line bundles with degrees at most $2d$, so $H^0(\mathcal{E}(-2d-1) = 0$ from which

$$H^0(\mathcal{E}(-1)) \to H^0(\mathcal{E}(-1)/\mathcal{E}(-2d-1))$$

is an injection. □

Now A has rank $2dn$, and the right side has rank $4dn$.

3.3. *Recovering* (A, j, α) *from this subset of* $tU \oplus \cdots \oplus t^{2d}U \oplus t^{2d+1}U$.

A is just the subset.

j is just $[t]j\!\!\alpha$.

We can almost get α, but not quite. We can recover it from the image in $tU \oplus \cdots t^{2d+2}U$: simply truncate the first and slide left.

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