

Homework problems on Tensor, Symmetric, and Alternating products Math 423/502

Let V_1 and V_2 be complex vector spaces of dimension d_1 and d_2 respectively. The tensor product $V_1 \otimes V_2$ can be defined as the vector space linearly spanned by vectors given by $v_1 \otimes v_2$ where $v_i \in V_i$ subject to the relations

$$\begin{aligned}(av_1 + a'v'_1) \otimes v_2 &= a(v_1 \otimes v_2) + a'(v'_1 \otimes v_2) \\ v_1 \otimes (av_2 + a'v'_2) &= a(v_1 \otimes v_2) + a'(v_1 \otimes v'_2).\end{aligned}$$

Here $v_i, v'_i \in V_i$ and $a, a' \in \mathbb{C}$.

- Let $\text{Hom}(V_1, V_2)$ be the vector space of linear maps $\phi : V_1 \rightarrow V_2$. Let $V_1^* = \text{Hom}(V_1, \mathbb{C})$ be the dual space of linear functions on V_1 .

- Show that the map

$$V_1^* \otimes V_2 \rightarrow \text{Hom}(V_1, V_2)$$

given by taking

$$\phi \otimes v_2 \mapsto (v_1 \mapsto \phi(v_1)v_2)$$

is an isomorphism of vector spaces.

- Show that if V_1 and V_2 are G -representations, then the above isomorphism is an isomorphism of G -representations. (In your solution, you will want to carefully recall how to define the G -representation structure on tensor products, dual vector spaces, and Hom spaces).

- The symmetric group S_n acts on $V^{\otimes n}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

We may decompose $V^{\otimes n}$ into sums of irreducible representations of S_n and we define $\text{Sym}^n(V) \subset V^{\otimes n}$, respectively $\Lambda^n V \subset V^{\otimes n}$, to be the summand of the trivial, respectively alternating, S_n representation.

- Show that

$$V \otimes V \cong \text{Sym}^2 V \oplus \Lambda^2 V$$

and show that

$$V^{\otimes n} \not\cong \text{Sym}^n V \oplus \Lambda^n V$$

for $n > 2$.

- We could alternatively define $\text{Sym}^n V$, respectively $\Lambda^n V$, to be the vector space linearly spanned by symbols $v_1 \cdots v_n$, respectively $v_1 \wedge \cdots \wedge v_n$ with $v_i \in V$ subject to the relations

$$\begin{aligned}(av_1 + a'v'_1) \cdot v_2 \cdots v_n &= a(v_1 \cdots v_n) + a'(v'_1 \cdots v_n) \text{ respectively,} \\ (av_1 + a'v'_1) \wedge v_2 \wedge \cdots \wedge v_n &= a(v_1 \wedge \cdots \wedge v_n) + a'(v'_1 \wedge \cdots \wedge v_n)\end{aligned}$$

and

$$\begin{aligned}v_1 \cdots v_i \cdot v_{i+1} \cdots v_n &= v_1 \cdots v_{i+1} \cdot v_i \cdots v_n \text{ respectively,} \\ v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_n &= -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_n\end{aligned}$$

for all i from 1 to $n - 1$. Show that these two definitions agree.

3. Compute the dimension of $\text{Sym}^n V$ and $\Lambda^n V$ if the dimension of V is d .
4. Suppose that $f : V \rightarrow V$ is a linear map. Define linear maps

$$\wedge^k f : \Lambda^k V \rightarrow \Lambda^k V$$

by

$$v_1 \wedge \cdots \wedge v_k \mapsto f(v_1) \wedge \cdots \wedge f(v_k).$$

Let d be the dimension of V . Show that the map $\wedge^d f$ is multiplication by $\det(f)$.

5. Suppose that V is a G -representation. Then $\text{Sym}^n(V)$ and $\Lambda^n(V)$ have the structure of G -representations inherited from the G -representation structure on $V^{\otimes n}$. Suppose that V has dimension d and that $\Lambda^d V \cong \mathbb{C}$, that is $\Lambda^d V$ is isomorphic to the trivial G representation. Show that

$$\Lambda^{d-1} V \cong V^*$$

as G -representations.