Homework problem set 5   Math 423/502

1. Let $x \in R$ be an element which is not a zero-divisor. Show that the first Tor group is the module of "x-torsion in M", namely:
$$\mathrm{Tor}_1(R/(x), M) = \{m \in M \mid xm = 0\}\,.$$

2. Let I and J be ideals of R. Let IJ be the ideal generated by elements ab where $a \in I$ and $b \in B$. Show that
$$\mathrm{Tor}_1(R/I, R/J) = (I \cap J)/(IJ)$$
and use this to show that $I \cap J = IJ$ if $I + J = R$.

3. Let $x \in R$ be an element which is not a zero-divisor. Compute
$$\mathrm{Ext}^i_R(R/(x), M)$$
and in particular compute $\mathrm{Ext}^i(\mathbf{Z}/n, \mathbf{Z}/m)$ for all $n, m \in \mathbf{Z}$.

4. Let I be any ideal in a ring R. Prove that
$$\mathrm{Ext}^1(R/I, R/I) = \mathrm{Hom}(I/I^2, R/I) = \mathrm{Hom}(\mathrm{Tor}_1(R/I, R/I), R/I).$$
(Use problem 2).