$Z\mathbb{C}G$ is a commutative alg over $\mathbb{C}$ with also an herm inner product. $(a,b)^{\text{herm}} = \frac{1}{|G|} \sum_{g \in G} \overline{a(g)}\, b(g)$

There is a related complex sym bilinear, non-deg form defined as follows:

On characters $\overline{\chi_V(g)} = \chi_V(g^{-1})$ since eigenvals are roots of unity. so ~~[illegible]~~ the bilinear form

$$(a,b) = \frac{1}{|G|} \sum_{g} a(g^{-1})\, b(g)$$

agrees with $(\ ,\ )^{\text{herm}}$ on chars which span so it is non-degenerate (but obviously, it is different from $(\ ,\ )^{\text{herm}}$).

<u>Def'n</u> A (symmetric, commutative) Frobenius alg over $\mathbb{C}$ is a commutative algebra A over $\mathbb{C}$ equipped with a symmetric, non-deg pairing $\langle\ ,\ \rangle$ satisfying $\langle ab, c\rangle = \langle a, bc\rangle$.

So $R(G) \otimes \mathbb{C} \cong Z\mathbb{C}G$ is an example.

Frobenius algebras have a beautiful interpretation in terms of topology which is important in physics

We will explain $\left\{ \begin{matrix} \text{Frobenius} \\ \text{algs over } \mathbb{C} \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} 1+1 \text{ dim'l} \\ \text{TQFTs} \end{matrix} \right\}$

Categorical description of Frobenius algebras:

If A is the underlying $\mathbb{C}$-vector space of an algebra, we can regard the algebra structure as defined by two ~~elements~~ maps:

$$m : A \otimes A \longrightarrow A \qquad \text{(multiplication)}$$

$$1_A : \mathbb{C} \longrightarrow A \qquad \text{(Unit)}$$

satisfying various axioms: all the usual properties of multiplication and unit can be formulated in terms of maps: e.g.:

Associativity:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "Id \otimes m"] \arrow[d, "m \otimes Id"] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
A \arrow[r, "\cong"] \arrow[rrr, bend right, "Id_A"'] & A \otimes \mathbb{C} \arrow[r, "Id \otimes 1_A"'] & A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

etc...

The non-deg. pairing gives an isomorphism $A \to A^*$ so dualizing m and 1_A gives co-unit and co-multiplication

$$\mu : A \to \mathbb{C} \qquad \Delta : A \to A \otimes A$$

example in $Z\mathbb{C}G$ $\quad \mu(a) = (a, 1) = \frac{1}{|G|} a(Id)$

$$\langle a, b \rangle = \mu(ab) = \mu\Big(\sum_{g,g'} a(g) b(g') gg' \Big)$$

$$= \frac{1}{|G|} \sum_{g} a(g^{-1}) b(g)$$

Jan 29th

(Symmetric, commutative) Frobenius algebras have an entirely topological formulation which is easy to understand, even if you don't know much topology.

First the abstract definition:

Def'n A $(1+1)$-dim'l topological quantum field theory (TQFT) is functor Z of symmetric, monoidal, tensor categories:

$$Z : 2\underline{Cob} \longrightarrow Vect_{\mathbb{C}}$$

from the category of oriented 2dim cobordisms to the category of $\mathbb{C}$-vector spaces.

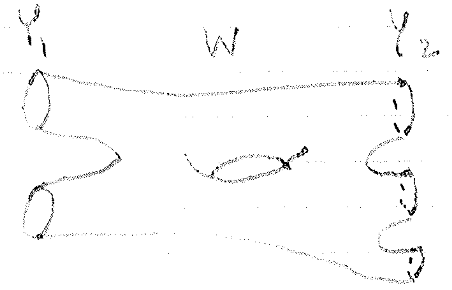
This amounts to the following data

(1) $Z(-)$ assigns to each compact, oriented 1-manifold a cx-vector space and disjoint unions should correspond to tensor products. i.e

$$Z(S^1) = A \quad , \quad Z(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{r}) = Z(S^1) \otimes \cdots \otimes Z(S^1)$$
$$= A^{\otimes r}$$
$$Z(\phi) = \mathbb{C}$$

(2) To each oriented cobordism W from Y_1 to Y_2

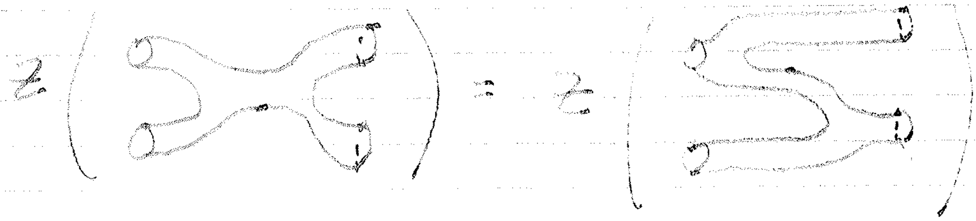
$Z(W) : Z(Y_1) \to Z(Y_2)$ linear map



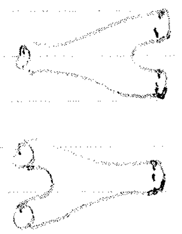
"Incoming" versus "Outgoing" boundaries distinguished by whether or not orientation agrees with induced orientations

$$A \otimes A \xrightarrow{Z(W)} A \otimes A \otimes A$$

(3) if $W \cong W'$ diffeomorphic by a boundary preserving oriented diffeomorphism, then $Z(W) = Z(W')$



④ $Z(W_1 \sqcup W_2) = Z(W_1) \otimes Z(W_2)$

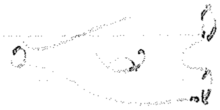
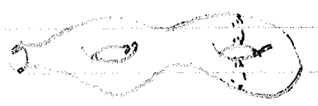
$Z\Big($  $\Big)$ $=$ $Z\big($  $\big)$ $A \to A\otimes A$

$\otimes$ $\qquad Z \otimes$

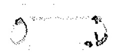
$Z\big($  $\big)$ $A\otimes A \to A$

$A\otimes A\otimes A \longrightarrow A\otimes A\otimes A$

⑤ concatenation of cobords ⟷ composition of maps.

$Z\big($  $\big) \circ Z\big($  $\big) = Z\big($  $\big)$

$A \to A\otimes A \to \mathbb{C}$ $\qquad\qquad A \longrightarrow \mathbb{C}$

⑥ $Z\big($  $\big) = Id_A$

Rk. $(n+1)$-dim'l TQFT. same idea using n manifolds and $n+1$ dim'l boundaries

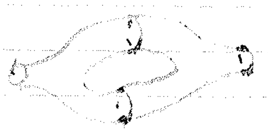
Rk. If W is a closed 2 manifold then it can be regarded as a cobordism from ϕ to ϕ

$Z\big($  $\big) : \mathbb{C} \to \mathbb{C}$ determined by number $\lambda \in \mathbb{C}$

by axiom ③ $Z(\text{closed mfd})$ is a topo invariant of manifold.

string theory : $Z(S^1) = A$ is the "Hilbert space of 1-particle states"

$Z(W)$ gives evolution of states through a world sheet.



strings divide and recombine

$A \longrightarrow A \otimes A \longrightarrow A$

(Theorem, Dijkgraaf) There is a bijective correspondence:

$$\left\{ \begin{matrix} \text{1+1 dim'l TQFT} \\ \text{over } \mathbb{C} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{symmetric, commutative,} \\ \text{Frobenius alg's over } \mathbb{C} \end{matrix} \right\}$$

pf. Given a TQFT Z define a Frobenius algebra as follows. Let $A = Z(S^1)$

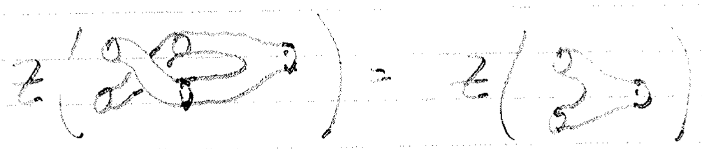
$Z($  $) : A \otimes A \longrightarrow A$ multiplication

$Z($  $) : \mathbb{C} \longrightarrow A$ unit

$Z($  $) : A \otimes A \longrightarrow \mathbb{C}$ pairing.

check axioms of Frob. algebra:

commutativity:

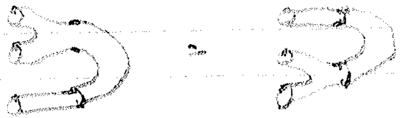


Associativity:

-Unit axiom:

$$A \to A\otimes A \to A$$
$$x \mapsto x\otimes 1_A \mapsto x\cdot 1_A$$

$\langle ab,c\rangle = \langle a,bc\rangle$



Pairing is non-deg. $\Leftrightarrow$ $A \to A^*$ is an isomorphism
$a \mapsto \langle a,-\rangle$

i.e. it has an inverse $A^* \to A$ $\Leftrightarrow$ a copairing $1 \xrightarrow{\tilde{g}} A\otimes A$

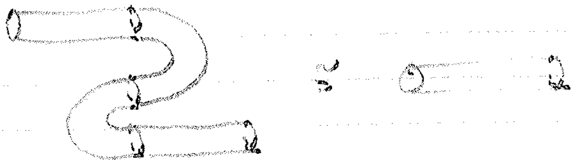
$A\otimes A \xrightarrow{g} A$

such that

$$\begin{tikzcd} A \arrow[r, "1\otimes\tilde{g}"] \arrow[rr, bend right, "id"'] & A\otimes A\otimes A \arrow[r, "g\otimes 1"] & A \end{tikzcd}$$

[in a basis g_{ij}
has inverse $\tilde{g}^{nm}$

$\sum_m \tilde{g}^{nm} g_{mk} = \delta^n_k$



Converse: a TQFT is determined by its values on pair of pants, cap, tube (morse theory, every Riemann surf has a pair of

Goal: find a geometric interpretation of $Z_G(-)$ the TQFT corresponding to $Z\mathbb{C}G$, use our theory of characters to deduce nice geometric results.

We have an obvious basis for $Z\mathbb{C}G$ labelled by conjugacy classes $\{e_\alpha\}$ $\quad e_\alpha = \sum_{g \in \alpha} g$

Tensor calculus: Given a basis for A we can express a linear map $f : A^{\otimes r} \to A^{\otimes s}$ via tensor coefficients:

$$f(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r}) = \sum_{\beta_1 \cdots \beta_s} f^{\beta_1 \cdots \beta_s}_{\alpha_1 \cdots \alpha_r} e_{\beta_1} \otimes \cdots \otimes e_{\beta_s}$$

we use summation convention: repeated indices, one up one down, are summed over $\quad f(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r}) = f^{\beta_1 \cdots \beta_s}_{\alpha_1 \cdots \alpha_r} e_{\beta_1} \otimes \cdots \otimes e_{\beta_s}$

f is determined by tensor coefs $f^{\beta_1 \cdots \beta_s}_{\alpha_1 \cdots \alpha_r} \in \mathbb{C}$

e.g. $f : A \to A$ $\quad f^\beta_\alpha$ matrix entries.

Composition: $A \otimes A \xrightarrow{f} A \otimes A \xrightarrow{g} A$

$$e_\alpha \otimes e_\beta \longmapsto f^{\gamma\delta}_{\alpha\beta} e_\gamma \otimes e_\delta \longmapsto f^{\gamma\delta}_{\alpha\beta} g^{\varepsilon}_{\gamma\delta} e_\varepsilon$$

i.e. $(g \circ f)^{\varepsilon}_{\alpha\beta} = f^{\gamma\delta}_{\alpha\beta} g^{\varepsilon}_{\gamma\delta}$

Kronecker δ $\quad \delta^\beta_\alpha = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$

so $f : A \to A$ has inverse $g \iff f^\alpha_\beta g^\beta_\delta = \delta^\alpha_\delta$

Non-degenerate from $g : A\otimes A \to \mathbb{C}$ given by $g_{\alpha\beta}$

Non-deg means induced map $A \to A^*$ is an iso.

The inverse map $A^* \to A$ corresponds to copairing $\mathbb{C} \to A\otimes A$ given by $g^{\alpha\beta}$; the inverse condition is $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$.

The form g (or metric g) gives us a way of raising and lowering indices:

$$m : A\otimes A \to A \iff A\otimes A\otimes A^* \to \mathbb{C} \quad \text{then}$$

via $g : A^* \to A$ we get $A\otimes A\otimes A \to \mathbb{C}$

$$v\otimes w\otimes u \longmapsto g(m(v\otimes w), u)$$

in tensor language:

$m : A\otimes A \to A$ is given by $m^{\delta}_{\alpha\beta}$

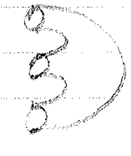
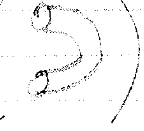
$A\otimes A\otimes A \to \mathbb{C}$ is given by $m_{\alpha\beta\sigma} := m^{\delta}_{\alpha\beta} g_{\sigma\delta}$

so for example co-multiplication $A \to A\otimes A$ is given by $m^{\varepsilon\eta}_{\gamma} := m^{\delta}_{\alpha\beta} g_{\delta\sigma} g^{\alpha\varepsilon} g^{\beta\eta}$

Any Frobenius alg. is determined by m, g, μ mult, pairing, counit.

$$A\otimes A\otimes A \xrightarrow{m} \mathbb{C} \qquad A\otimes A \xrightarrow{g} \mathbb{C} \qquad A \xrightarrow{\mu} \mathbb{C}$$

$m_{\alpha\beta\sigma}$, $g_{\alpha\beta}$, μ_{α}

$Z\Big($  $\Big)$ $\quad Z\Big($  $\Big)$ $\quad Z\Big($  $\Big)$

In the case of $\mathbb{Z}G$ $\{\alpha\}$ conj. classes of G

$$e_\alpha = \sum_{g \in \alpha} g$$

7

recall $\mu\left(\sum a(g) g \right) = \frac{1}{|G|} a(id)$

i.e. $\mu_\alpha = \mu(e_\alpha) = \begin{cases} \frac{1}{|G|} & \alpha = \{id\} \\ 0 & \alpha \neq \{id\} \end{cases}$

$$g\left(\sum a(g) g, \sum b(g) g \right) = \frac{1}{|G|} \sum a(g) b(g^{-1})$$

$g_{\alpha\beta} = g(e_\alpha, e_\beta) = \begin{cases} 0 & \alpha \neq \bar{\beta} \\ \frac{|\alpha|}{|G|} & \alpha = \bar{\beta} \end{cases}$ $\bar{\beta}$ conj class of inverse elts of β

let $z(\alpha) = |\text{Centralizer}|$ $g_{\alpha\beta} = \frac{1}{z(\alpha)} \delta_{\alpha\bar{\beta}}$

$$g^{\alpha\beta} = z(\alpha) \delta^{\alpha\bar{\beta}}$$

$$M_{\alpha\beta\gamma} = g(m(e_\alpha \otimes e_\beta) \otimes e_\gamma) = \langle e_\alpha \cdot e_\beta, e_\gamma \rangle = \langle e_\alpha \cdot e_\beta \cdot e_\gamma, 1 \rangle$$
$$= \mu(e_\alpha \cdot e_\beta \cdot e_\gamma) = \frac{1}{|G|} \cdot \left\{ \text{id. term of } \sum_{g \in \alpha} \sum_{h \in \beta} \sum_{k \in \gamma} ghk \right\}$$

$$M_{\alpha\beta\gamma} = \frac{1}{|G|} \# \left\{ g \in \alpha, h \in \beta, k \in \gamma : ghk = 1 \right\}$$

Idea: make these numbers geometric - they should count something.

Jan 31st.

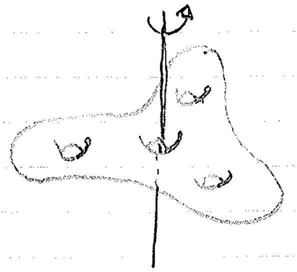
Def'n A free G-space is a topological space P on which G acts freely and continuously, i.e. for all $g \in G$

$$\phi_g : P \to P \quad \text{continuous map.}$$

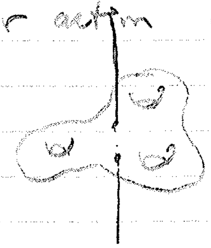
such that $\phi_{gh}(x) = \phi_g(\phi_h(x))$, $\phi_{id} = Id_X$

(free $x \neq \phi_g(x)$ unless $g = id$).

e.g. $\mathbb{Z}/_3$ acts freely on:



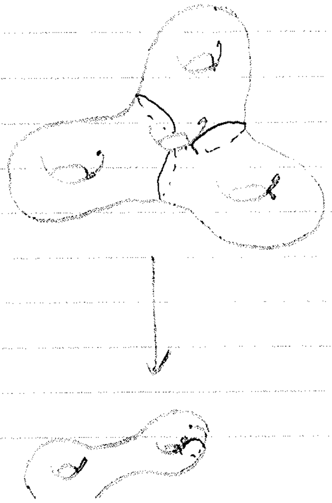
but similar action on

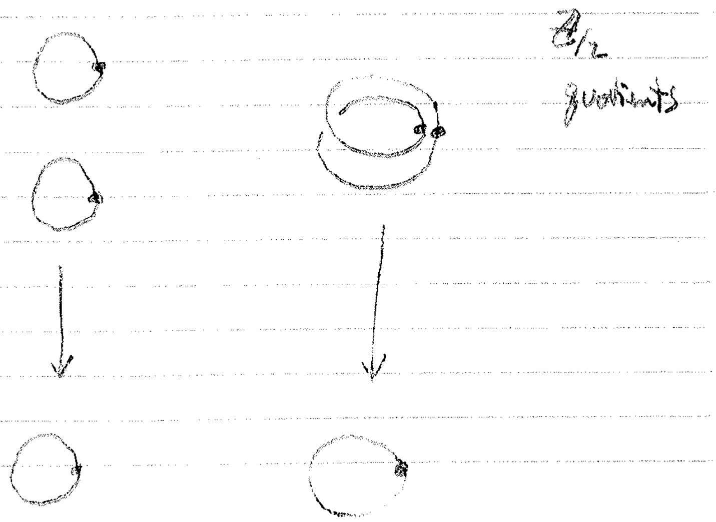


not free (2 fixed pts)

each orbit is a copy of G. Let $X = P/_G$ be the orbit space: we identify $x \sim \phi_g(x)$

If P is a manifold then $X = P/_G$ is a manifold





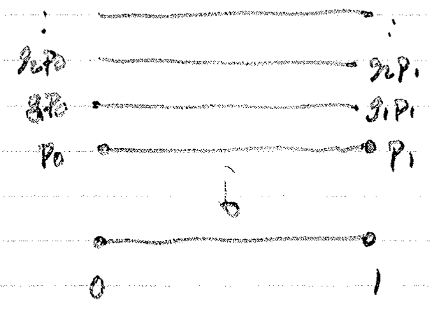
Def'n A principal G-bundle over X is a free G-space P such that $X = P/G$. $P = X \times G$ is called the trivial bundle.

Question: given a Riemann surface [illegible] X how many principal G bundles over X are there?

(Related to Galois theory: If $P \to X$ is a pricipal G bundle then the field of meromophic fncs on P is a Galois extension of the field of meromorphic fnc on X with galois gp G).

We begin by studying Principal G bundles over S^1.

We use the fact that all principal G bundles over $[0,1]$ are trivial. $S^1 = [0,1]/_{0 \sim 1}$ so we can get $P \to S^1$ by gluing $G \times [0,1]$ to itself:



we can make a principal bundle on S^1 by gluing p_0 to gp_1 some $g \in G$

once this choice is made all other gluings are determined.

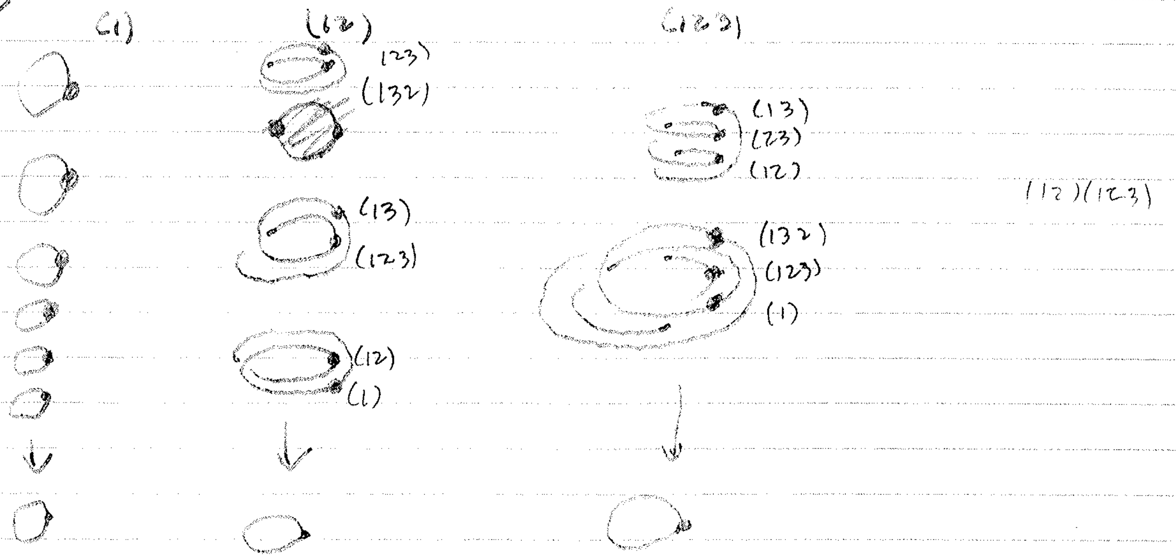
s'pose we had choosen a different labelling of the fiber over 0, i.e. some other $p_0' = hp_0$

then p_0' is glued to $hgh^{-1}p_1'$:

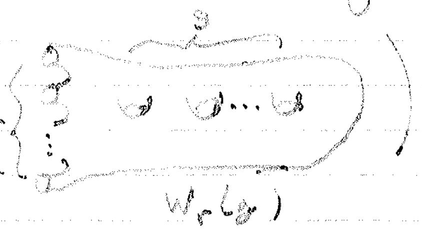
$$p_0' = hp_0 \sim hgp_1 = hgh^{-1}hp_1 = hgh^{-1}p_1'$$

so the bundle is determined by a conjugacy class called monodromy (around the circle).
of the bundle

S_3 bundles on S^1 :



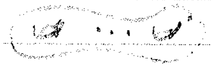
Theorem Let $Z_G(-)$ be the TQFT associated to the Frobenius alg. $Z\mathbb{C}G$. Let $Z_G(g)_{\alpha_1,\dots,\alpha_r}$ be the tensor coefs of



$$Z_G\left(W_r(g) \right) : A^{\otimes r} \longrightarrow \mathbb{C}$$

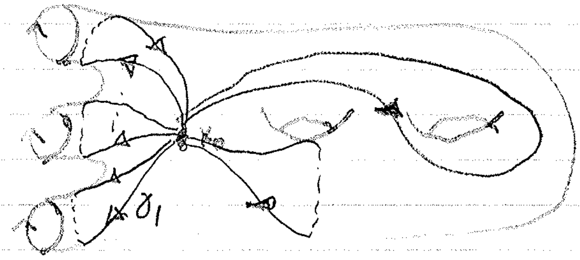
with respect to the basis $\{e_\alpha\}_{\alpha \text{ conj. class of } G}$ $\quad e_\alpha = \sum_{g \in \alpha} g$.

Then $Z(g)_{\alpha_1 \cdots \alpha_r}$ = # of principal bundles ~~of~~ on $W_r(g)$ having monodromy α_i about the ith boundary component. We count each bundle by $\frac{1}{\# \mathrm{Aut}\, P} = \frac{1}{|G|}$.

In particular $Z_G(g)$ = # princ. G-bundles on 

We sketch the proof. We show this forms a TQFT and we show it agrees with ZG on pants, tube, and cap.

Pick a base point $x_0 \in W_r(g)$ and choose a point $p_0 \in \pi^{-1}(x_0)$. Then each loop $\gamma : [0,1] \to W_r(g)$ with $\gamma(0) = \gamma(1) = x_0$ determines an element of G by monodromy.



$\gamma_1, \ldots, \gamma_r$ are loops homotopic to the loop round boundary components

This defines a homomorphism $\phi \in \mathrm{Hom}(\pi_1(W_r(g)), G)$ such that $\phi(\gamma_i) \in \alpha_i$ this uniquely determines the principal bundle. So

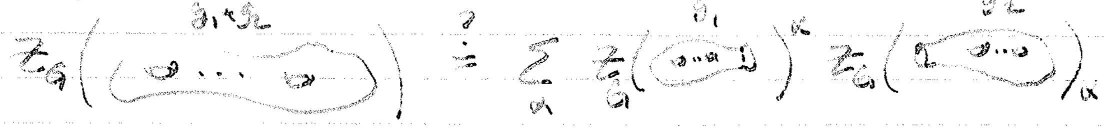
$$Z_G(W_r(g))_{\alpha_1 \cdots \alpha_r} = \frac{1}{|G|} \# \{ \phi \in \mathrm{Hom}(\pi_1(W_r(g)), G) : \phi(\gamma_i) \in \alpha_i \}$$

$$z(\alpha) = |\text{centralizer}(\alpha)| = \frac{|G|}{|\alpha|}$$

$$:= z(\beta_1)\cdots z(\beta_s)\, Z_G(W_{g+s}(g))_{\alpha_1 \cdots \alpha_r \bar{\beta}_1 \cdots \bar{\beta}_s}$$

Prop: The above defines a TQFT, i.e. composition law holds

e.g.



$$= \frac{1}{|G|} \sum_\alpha |\alpha|\, Z_G(W_1(g_1))_{\bar{\alpha}}\, Z_G(W_1(g_2))_\alpha$$

can glue bundles with opposite monodromy $\frac{|\alpha|}{|G|}$ number of ways to glue.

Since the principal bundle TQFT agrees with the one from ZG on cap, pants, & tube they agree in general.

Our original topological problem:

$$Z_G(g) = \#\ \text{Princ. } G\text{-bundles over closed genus } g \text{ surface}$$

$$= \frac{1}{|G|} \#\, \text{Hom}(\pi_1(\Sigma_g), G)$$

$$= \frac{1}{|G|} \# \left\{ (a_1, b_1, \dots, a_g, b_g) \in G^{2g} : \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \right\}$$

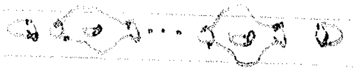
looks like a very difficult combinatorial group theory problem.

Theorem: $Z_G(\Sigma_g) = \sum_{\substack{\text{irr. repr} \\ R}} \left(\frac{|G|}{\dim R} \right)^{2g-2}$

Def'n. A TQFT / Frob. alg is semi-simple if the alg. is semi-simple: i.e. $\exists$ a basis v_α s.t. $v_\alpha \cdot v_\beta = \delta_{\alpha\beta} v_\alpha$.

Exercise Suppose A is a semi-simple Frob. alg with idempotent basis $e_1, \dots, e_n$ (so $e_i \cdot e_j = \delta_{ij} e_i$) let $\mu_i = \mu(e_i)$. Show that $\mu_i \neq 0$ and letting $\lambda_i = \mu_i^{-1}$, show that

$Z\Bigg($  $\Bigg) = \sum_{i=1}^{n} \lambda_i^{g-1}$

Hint: compute  , then 

Theorem $\mathbb{Z}G$ is semi-simple with idempotent basis

$$v_R = \dim R \frac{1}{|G|} \sum_{g \in G} \chi_R(g)\, g$$

Pf: For any repr. W and any irreducible R consider $\phi_R : W \longrightarrow W$

$$\phi_R(w) = \dim R \frac{1}{|G|} \sum_{g \in G} \chi_R(g)\, g \cdot w$$

ϕ_R is $\hat{G}$-linear:

$$\phi_R(hw) = \frac{\dim R}{|G|} \sum_{g \in G} \overline{\chi_R(g)}\,(h h^{-1}) g h w \qquad g' = h^{-1} g h$$

$$= \frac{\dim R}{|G|} \sum_{g' \in G} \overline{\chi_R(g')}\, h g' w = h \overset{*}{\phi}_R(w)$$

By Schur's lemma $\phi_R : R' \to R'$ is equal to $\lambda \, Id_{R'}$ if R' is irr. and in this case

$$tr(\phi_R) = \lambda \dim R' = \dim R \frac{1}{|G|} \sum_{g \in G} \overline{\chi_R(g)}\, \chi_{R'}(g) = \delta_{RR'} \dim R$$

so $\lambda = \delta_{RR'}$ thus ϕ_R is projection onto R summands of W.

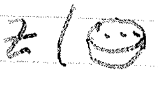
Let $W = R_{reg} = \mathbb{C}[G]$ then $\phi_R : \mathbb{C}[G] \to \mathbb{C}[G]$ is just multiplication by V_R and since $\phi_R \circ \phi_{R'} = \delta_{RR'} \phi_R$ we get $V_R \circ V_{R'} = \delta_{RR'} V_R$ in $\mathbb{C}[G]$ since V_R are central and span $Z\mathbb{C}[G]$, the theorem follows.

$$\mu(V_R) = \mu\left(\dim R \frac{1}{|G|} \sum_{g \in G} \overline{\chi_R(g)}\, g\right) = \dim R \frac{1}{|G|} \cdot \frac{1}{|G|} \overline{\chi_R(id)}$$

$$= \frac{\dim^2 R}{|G|^2} \quad \text{so} \quad \lambda_R = \frac{|G|^2}{(\dim R)^2}$$

Exercise $\Rightarrow$ $Z_G(g) = \sum_R \left(\frac{|G|}{\dim R}\right)^{2g-2}$

verify formula for $g=0$ and $g=1$ "by hand"

$Z($  $) = \frac{1}{|G|} \# \mathrm{Hom}(\overset{\{1\}}{\overset{\shortparallel}{\pi_1(S^2)}}, G) = \frac{1}{|G|}$ (the only bundle is the trivial bundle)

$$\overset{?}{=} \sum_R \frac{(\dim R)^2}{|G|^2} = \frac{1}{|G|^2}|G| = \frac{1}{|G|} \quad ✓$$

$Z($  $) = \frac{1}{|G|} \# \mathrm{Hom}(\overset{\mathbb{Z}\times\mathbb{Z}}{\overset{\shortparallel}{\pi_1(\text{torus})}}, G))$

$$= \frac{1}{|G|} \# \{ a,b \in G \quad \underbrace{ab=ba}_{a=bab^{-1}} \}$$

$$= \frac{1}{|G|} \sum_{a\in G} |C(a)| = \sum_{a \in G} \frac{1}{|\mathrm{conj}(a)|}$$

$$= \sum_{\substack{\text{conj}\\ \text{classes}}} 1 = \# \text{ conj classes.}$$

$$\overset{?}{=} \sum_R \left(\frac{\cancel{\dim}|G|}{\dim R} \right)^0 = \# \text{ irr repr} = \# \text{ conj. classes} \quad ✓$$

Relationship with covering spaces:

There is a correspondence:

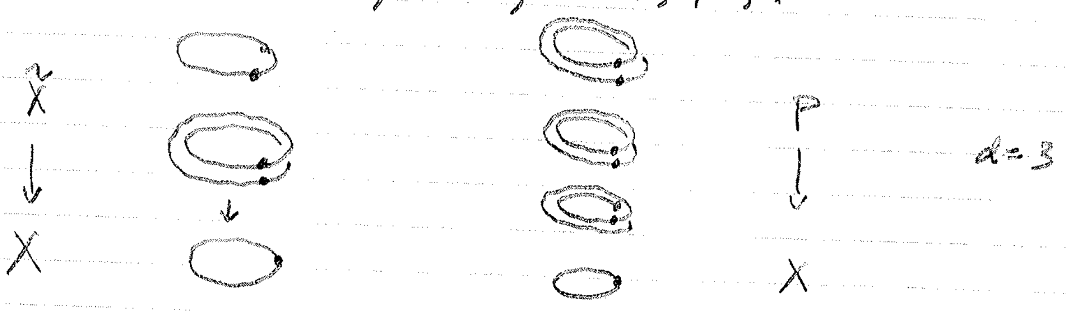
$\left\{ \begin{array}{c} \text{Principal } S_d\text{-bundles} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{degree } d, \text{ not nec. connected,} \\ \text{covering spaces of } X \end{array} \right\}$

given a principal S_d-bundle $P \to X$ we can make a degree d cover $\tilde{X} \to X$ by $\tilde{X} = (P \times \{1,\dots,d\})/S_d$

conversely, given a degree d cover $\tilde{X} \xrightarrow{\pi} X$ define $P = \{ \phi : \{1,\dots,d\} \hookrightarrow \tilde{X}$ s.t. $\pi \circ \phi$ maps to a point, i.e. labellings of the fibers $\tilde{X} \to x \}$

we must give P a topology so that the labellings vary continuously.

S_d acts on P by acting on $\{1,\dots,d\}$.



so

$$\#\text{ deg } d \text{ covering spaces of } \Sigma_g = \sum_{\substack{\text{irr}\\ \text{rep of } S_d}} \left(\frac{d!}{\dim R}\right)^{2g-2}$$

not nec. conn.

in particular $\#$ of $\overset{\deg d}{\text{covers}}$ of T^2 = $p(d)$ = # of partitions

very fun to see this directly from covering space theory:

# of connected covering spaces of degree d $= \frac{1}{d}$ # index d subgps of $\mathbb{Z} \times \mathbb{Z}$.

exercise: show RHS is $\frac{1}{d}\sigma(d)$ $\sigma(d)$ = # of divisors of d.

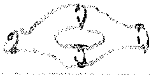
let N_d = # of possibly disco covers $= p(d)$

let n_d = # of connected covers $= \frac{1}{d}\sigma_1(d)$

let $F(g) = \sum_{d=1}^{\infty} n_d g^d$

let $Z(g) = \sum_{d=0}^{\infty} N_d g^d$

show $Z(g) = \prod_{n=1}^{\infty}(1-g^n)^{-1} =$ ~~log~~ $\exp(F(g))$

Handle operator $H : Z\big($  $\big) : A \longrightarrow A$

Let $\{\gamma_i\}$ be a basis for A

$H : \gamma_i \longmapsto m_i^{jk}\, \gamma_j \otimes \gamma_k \longmapsto m_i^{jk}\, m_{jk}^{n}\, \gamma_n$ *

Let $\{\gamma^j\}$ be the dual basis, i.e. $\langle \gamma_i , \gamma^j \rangle = \delta_i^j$ $\quad \gamma^j = g^{jk}\gamma_k$

Let $\Gamma = \gamma_i \circ \gamma^i \in A$

<u>Claim</u>: H is multiplication by Γ

Pf: $\gamma_i \longmapsto (\gamma_j \circ \gamma^j) \circ \gamma_i$

$$= \gamma_j\, g^{jk} \gamma_k \circ \gamma_i$$

$$= g^{jk} \gamma_j\, m_{ki}^{e}\, \gamma_e$$

$$= g^{jk}\, m_{ki}^{e}\, m_{je}^{n}\, \gamma_n$$

$$= g^{jk}\, g_{ka}\, m_i^{ea}\, m_{je}^{n}\, \gamma_n$$

$$= m_i^{ej}\, m_{je}^{n}\, \gamma_n \qquad \boxtimes.$$