TQFT problems Math 502

1. **Semi-simple TQFTs** A TQFT/Frobenius algebra $Z(-)/A$ is called *semi-simple* if A is semi-simple: i.e. there exists a basis $e_1, \ldots, e_n$ of A such that $e_i \cdot e_j = \delta_{ij} e_i$. Let $\mu_i = \mu(e_i)$ where μ is the co-unit.

   (a) Show that $\mu_i \neq 0$.

   (b) Define $\lambda_i = \mu_i^{-1}$. Show that the value of the TQFT on the closed, genus g surface is given by

   $$Z(g) = \sum_{i=1}^{n} \lambda_i^{g-1}.$$

   Hint: compute the "genus adding operator" $G : A \to A$ given by evaluating Z on the genus 1 cobordism from the circle to the circle by observing it is the composition of the coproduct with the product.

2. **Genus adding operator in various cases.** Given a TQFT/Frobenius algebra $Z(-)/A$, let $G : A \to A$ be the genus adding operator defined in the hint above. Let $Z(g)$ be the value of the TQFT on the closed genus g surface.

   (a) Let $A = \mathbb{C}[x]/(x^2 + 1)$, having comultiplication $\mu(a + bx) = a$. Compute $G : A \to A$ (as a matrix with respect to the basis $\{1, x\}$) and $Z(g)$.

   (b) Let $A = \mathbb{C}[x]/(x^2 - 1)$, having comultiplication $\mu(a + bx) = a$. Compute $G : A \to A$ (as a matrix with respect to the basis $\{1, x\}$) and $Z(g)$.

   (c) Let $A = \mathbb{C}[x]/(x^2)$, having comultiplication $\mu(a + bx) = b$. Compute $G : A \to A$ (as a matrix with respect to the basis $\{1, x\}$) and $Z(g)$.

   (d) (Some knowledge of topology required). Let $A = H^{ev}(M, \mathbb{C})$ be the even degreed part of the cohomology of a compact oriented even dimensional manifold M. A is a commutative algebra via cup product and has co-unit given by $\mu : A \to \mathbb{C}$ where $\mu(\omega) = \int_M \omega$ (in particular $\mu(\omega) = 0$ unless $\deg(\omega) = \dim(M)$). Note that Poincaré duality implies that the associated bilinear form is non-degenerate. Compute $G : A \to A$ and $Z(g)$ in this case.

   (e) Which of the above cases are semi-simple?

3. **The Fibonacci TQFT**. Let $A = \mathbb{C}[x]/(x^2 - x - 1)$ with $\mu(1) = -1$ and $\mu(x) = 2$. Compute the G, the genus adding operator, as a matrix with respect to the basis $\{1, x\}$. Let $W^1(g)$ be the genus g cobordism from the empty set to the circle. Show that $Z(W^1(g))$ viewed as an element of A is given by

   $$f_{g-1} + f_g x$$

   where f_n is the nth Fibonacci number.