Homework problems Math 502

1. Show that

$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right),$$
$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right).$$

2. **The Character table of $\mathbf{Z}/4\mathbf{Z}$.** Write down the character table of $\mathbf{Z}/4\mathbf{Z}$. Demonstrate the orthogonality of the characters of irreducible representations with three examples.

3. **The Character table of D_8.** Let S be the square in $\mathbf{R}^2$ with verticies $x_1 = (-1, 1)$, $x_2 = (1, 1)$, $x_3 = (1, -1)$, and $x_4 = (-1, -1)$. The symmetry group of S, is the group with eight elements

$$D_8 = \{1, \tau, \tau^2, \tau^3, \sigma, \sigma\tau, \sigma\tau^2, \sigma\tau^3\},$$

where τ is given by counterclockwise rotation through 90 degrees and σ is reflection about the y-axis.

- Using the relations $\sigma\tau\sigma = \tau^3$, $\sigma^2 = 1$, and $\tau^4 = 1$, show that the five conjugacy classes of D_8 are $\{1\}$, $\{\tau, \tau^3\}$, $\{\tau^2\}$, $\{\sigma, \sigma\tau^2\}$, and $\{\sigma\tau, \sigma\tau^3\}$.
- Let U be the trivial representation and let V be the complexification of the real two dimensional representation obtained from the above action on $\mathbf{R}^2$. Find the characters χ_U, χ_V, and $\chi_{\Lambda^2 V}$.
- Show that U, V, and $\Lambda^2 V$ are irreducible and distinct.
- Use the properties of the character table to deduce the characters of the remaining two irreducible representations U' and U''.
- Find the decomposition into irreducible representations of the permutation representation of D_8 acting on the vertices $\{x_1, x_2, x_3, x_4\}$.

4. **Symmetries of the Cube.** Show that the permutation group S_4 is isomorphic to the group of rotational symmetries of the cube by considering the action on the four diagonals. Using the character table of S_4 from class, decompose the permutation representations given by the action of S_4 on

- The six faces,
- The eight vertices,
- The twelve edges.

Express in terms of the irreducible representations, U, V, U', $V' = V \otimes U'$, and W.

5. **The Character table of A_5.** Consider $A_5 \subset S_5$ the alternating group.

- Show that A_5 has 5 conjugacy classes and give the number of elements in each.

- Show that the restrictions of the irreducible S_5 representations U, V, W are irreducible and compute their character.
- Use the orthogonality of the character table to deduce the characters of the remaining two irreducible representations Y and Z.
- Decompose $\Lambda^2 W$ into irreducible representations.