More Homework problems Math 502

1. **Symmetries of the icosohedron.** The icosohedron is the regular polyhedron having 20 sides, 30 edges, and 12 vertices. Partition the edges into 5 equivalence classes where two edges are equivalent if and only if they are parallel or perpendicular.

   (a) Show that the symmetry group of the icosohedron is isomorphic to the alternating group A_5 by studying the action on the 5 equivalence classes of edges.

   (b) Using the character table of A_5 from the previous homework, decompose the permutation representation of A_5 acting on the set of faces into irreducible representations.

   (c) Decompose the permutation representation of A_5 acting on the set of vertices into irreducible representations.

   (d) Decompose the permutation representation of A_5 acting on the set of edges into irreducible representations.

2. **Character table of $SL_2(\mathbb{F}_3)$ and the binary tetrahedral group.** Let $G = SL_2(\mathbb{F}_3)$ be the group of 2×2 matrices whose entries are in $\mathbb{F}_3 = \{0, 1, -1\}$, the field with three elements.

   (a) Determine the set of conjugacy classes of G. (Hint: this is easily done by hand; it is helpful to note that transposition and multiplication by the central element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts on the set of conjugacy classes).

   (b) Find a normal subgroup $N \subset G$ such that G/N has order three and thus use the three irreducible representations of G/N to provide the first three irreducible representations of G. Fill in their values in the character table.

   (c) To determine the remaining irreducible representations, consider the action of G on $\mathbb{P}^1(\mathbb{F}_3) = (\mathbb{F}_3 \times \mathbb{F}_3 - \{(0,0)\}) / \mathbb{F}_3^\times$ where $\mathbb{F}_3^\times = \{1, -1\}$ is the group of units. The set $\mathbb{P}^1(\mathbb{F}_3)$ has four elements and so we may define a three dimensional representation V so that the permutation representation of G on $\mathbb{P}^1(\mathbb{F}_3)$ is isomorphic to $V \oplus \mathbb{C}$ where $\mathbb{C}$ is the trivial representation. Compute the character of V and show it is an irreducible representation.

   (d) Show that the representation V is a real representation and that $\Lambda^3 V \cong \mathbb{C}$ so that V gives a map $\rho_V : G \to SO(3)$. Show that this map factors through $G/\{\pm 1\} = PSL_2(\mathbb{F}_3)$ which is a group of order 12.

   (e) Show that the action of $PSL_2(\mathbb{F}_3)$ on the set $\mathbb{P}^1(\mathbb{F}_3)$ identifies $PSL_2(\mathbb{F}_3)$ as the symmetry group of the tetrahedron which is A_4 (V is the standard representation of A_4 under this identification).

   (f) From the previous parts we see that $G/\{\pm 1\} \cong A_4$. Show that however, G is not isomorphic to S_4.

   (g) Instead we wish to show that G is the subgroup of $SU(2)$ under the preimage of symmetries of the tetrahedron $A_4 \subset SO(3)$ under the double cover $SU(2) \to SO(3)$. Namely, we wish to show that G is isomorphic to the *binary tetrahedral group*. To do this, find a two dimensional quaternionic representation W such that $\mathrm{Sym}^2(W) \cong V$ (we discussed this phenomenon in class).

   (h) Determine the character of W and use W and its tensor products with the non-trivial one dimensional G-representations to complete the character table of G.

   (i) **(Bonus question)** Find an explicit embedding $G \subset SU(2)$ viewing $SU(2)$ as the set of quaternions of unit length. Namely, find a set of 24 unit quaternions whose multiplication table is the same as $SL_2(\mathbb{F}_3)$.

3. **Orthogonality of the columns of the character table.** If $g, h \in G$ are conjugate we denote it by $g \sim h$. Let $z(g)$ be the order of the centralizer of g. Show that

$$\sum_R \chi_R(h) \cdot \overline{\chi}_R(g) = \begin{cases} z(g) & \text{if } h \sim g \\ 0 & \text{if } h \not\sim g \end{cases}$$

where the sum is over all irreducible representations. Hint: consider the class function $1_{(g)}$ which takes the value 1 on all elements of G which are conjugate to g and takes the value 0 on all other elements. Write this class function as a linear combination of characters of irreducible representations.