Homework problems on Tensor, Symmetric, and Alternating products Math 423/502

Let V_1 and V_2 be complex vector spaces of dimension d_1 and d_2 respectively. The tensor product $V_1 \otimes V_2$ can be defined as the vector space linearly spanned by vectors given by $v_1 \otimes v_2$ where $v_i \in V_i$ subject to the relations

$$(av_1 + a'v_1') \otimes v_2 = a(v_1 \otimes v_2) + a'(v_1' \otimes v_2)$$
$$v_1 \otimes (av_2 + a'v_2') = a(v_1 \otimes v_2) + a'(v_1 \otimes v_2').$$

Here $v_i, v_i' \in V_i$ and $a, a' \in \mathbb{C}$.

1. Let $\mathrm{Hom}(V_1, V_2)$ be the vector space of linear maps $\phi : V_1 \to V_2$. Let $V_1^* = \mathrm{Hom}(V_1, \mathbb{C})$ be the dual space of linear functions on V_1.

   (a) Show that the map
   $$V_1^* \otimes V_2 \to \mathrm{Hom}(V_1, V_2)$$
   given by taking
   $$\phi \otimes v_2 \mapsto (v_1 \mapsto \phi(v_1)v_2)$$
   is an isomorphism of vector spaces.

   (b) Show that if V_1 and V_2 are G-representations, then the above isomorphism is an isomorphism of G-representations. (In your solution, you will want to carefully recall how to define the G-representation structure on tensor products, dual vector spaces, and Hom spaces).

2. The symmetric group S_n acts on $V^{\otimes n}$ by
   $$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$
   We may decompose $V^{\otimes n}$ into sums of irreducible representations of S_n and we define $\mathrm{Sym}^n(V) \subset V^{\otimes n}$, respectively $\Lambda^n V \subset V^{\otimes n}$, to be the summand of the trivial, respectively alternating, S_n representation.

   (a) Show that
   $$V \otimes V \cong \mathrm{Sym}^2 V \oplus \Lambda^2 V$$
   and show that
   $$V^{\otimes n} \ncong \mathrm{Sym}^n V \oplus \Lambda^n V$$
   for $n > 2$.

   (b) We could alternatively define $\mathrm{Sym}^n V$, respectively $\Lambda^n V$, to be the vector space linearly spanned by symbols $v_1 \cdots v_n$, respectively $v_1 \wedge \cdots \wedge v_n$ with $v_i \in V$ subject to the relations
   $$(av_1 + a'v_1') \cdot v_2 \cdots v_n = a(v_1 \cdots v_n) + a'(v_1' \cdots v_n) \text{ respectively,}$$
   $$(av_1 + a'v_1') \wedge v_2 \wedge \cdots \wedge v_n = a(v_1 \wedge \cdots \wedge v_n) + a'(v_1' \wedge \cdots \wedge v_n)$$
   and
   $$v_1 \cdots v_i \cdot v_{i+1} \cdots v_n = v_1 \cdots v_{i+1} \cdot v_i \cdots v_n \text{ respectively,}$$
   $$v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_n = -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_n$$
   for all i from 1 to $n - 1$. Show that these two definitions agree.

3. Compute the dimension of $\operatorname{Sym}^n V$ and $\Lambda^n V$ if the dimension of V is d.

4. Show that
$$\Lambda^n(V \oplus W) \cong \bigoplus_{k=0}^{n} \Lambda^k V \otimes \Lambda^{n-k} W.$$

5. Suppose that $f : V \to V$ is a linear map. Define linear maps
$$\wedge^k f : \Lambda^k V \to \Lambda^k V$$
by
$$v_1 \wedge \cdots \wedge v_k \mapsto f(v_1) \wedge \cdots \wedge f(v_k).$$
Let d be the dimension of V. Show that the map $\wedge^d f$ is multiplication by $\det(f)$.

6. Suppose that V is a G-representation. Then $\operatorname{Sym}^n(V)$ and $\Lambda^n(V)$ have the structure of G-representations inherited from the G-representation structure on $V^{\otimes n}$. Suppose that V has dimension d and that $\Lambda^d V \cong \mathbb{C}$, that is $\Lambda^d V$ is isomorphic to the trivial G representation. Show that
$$\Lambda^{d-1} V \cong V^*$$
as G-representations.