

Inference for Numerical Data

Computational Mathematics and Statistics

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October 21, 2025

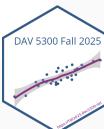
One Minute Paper Results

What was the most important thing you learned during this class?

conclusions
hypothesis
sample
limit
central
testing square
chisquared
grows error
large different means
chi size
theorem
normal distribution
dice

What important question remains unanswered for you?

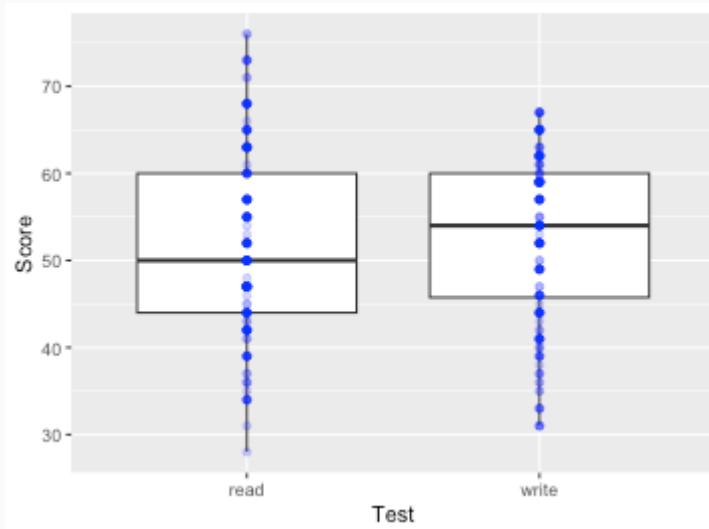
bootstrap
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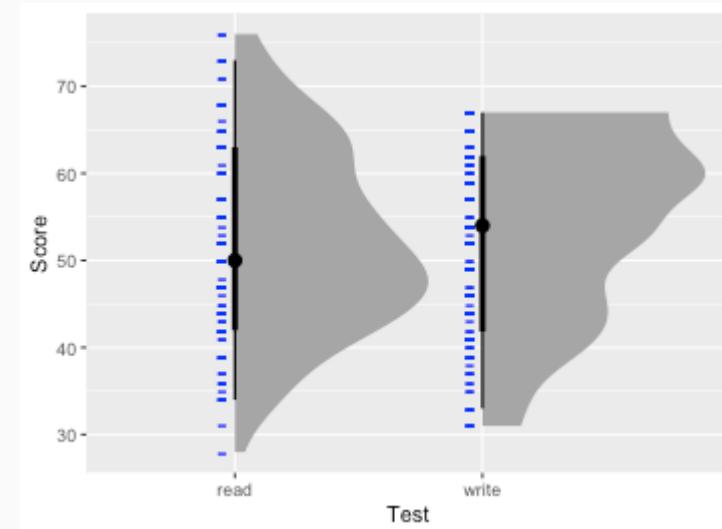
High School & Beyond Survey

200 randomly selected students completed the reading and writing test of the High School and Beyond survey. The results appear to the right. Does there appear to be a difference?

```
data(hsb2) # in openintro package  
hsb2.melt <- melt(hsb2[,c('id','read', 'write')], id='-'  
ggplot(hsb2.melt, aes(x=variable, y=value)) +      geom_  
  geom_point(alpha=0.2, color='blue') + xlab('Test')
```



```
ggplot(hsb2.melt, aes(x=variable, y=value)) +  
  ggdist::stat_halfeye() +  
  geom_point(color='blue', position = position_nudge(0,  
xlab('Test') + ylab('Score')
```

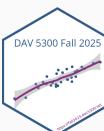


High School & Beyond Survey

```
head(hsb2)
```

```
## # A tibble: 6 × 11
##   id gender race  ses    schtyp prog      read  write  math science socst
##   <int> <chr>  <chr> <fct> <fct>      <int> <int> <int> <int> <int>
## 1    70 male    white low   public general     57    52    41    47    57
## 2   121 female  white middle public vocational  68    59    53    63    61
## 3    86 male    white high  public general     44    33    54    58    31
## 4   141 male    white high  public vocational  63    44    47    53    56
## 5   172 male    white middle public academic    47    52    57    53    61
## 6   113 male    white middle public academic    44    52    51    63    61
```

Are the reading and writing scores of each student independent of each other?



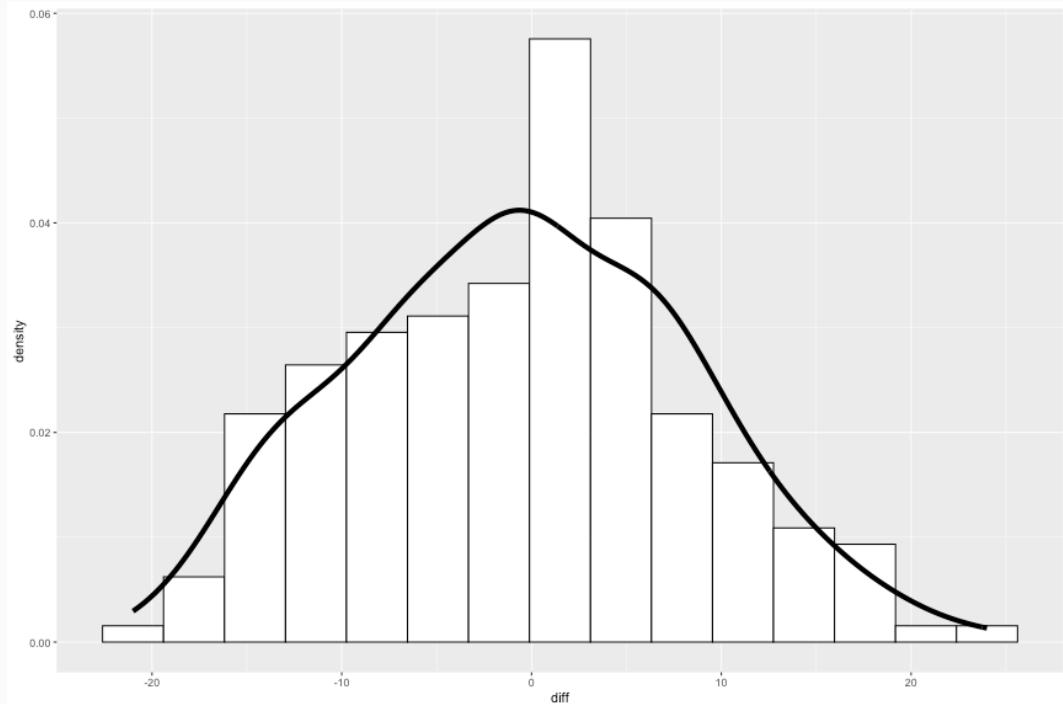
Analyzing Paired Data

- When two sets of observations are not independent, they are said to be paired.
- To analyze these type of data, we often look at the difference.

```
hsb2$diff <- hsb2$read - hsb2$write  
head(hsb2$diff)
```

```
## [1] 5 9 11 19 -5 -8
```

```
ggplot(hsb2, aes(x = diff)) +  
  geom_histogram(aes(y = ..density..), bins = 15, col  
  geom_density(size = 2)
```



Setting the Hypothesis

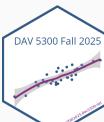
What are the hypothesis for testing if there is a difference between the average reading and writing scores?

H_0 : There is no difference between the average reading and writing scores.

$$\mu_{diff} = 0$$

H_A : There is a difference between the average reading and writing score.

$$\mu_{diff} \neq 0$$



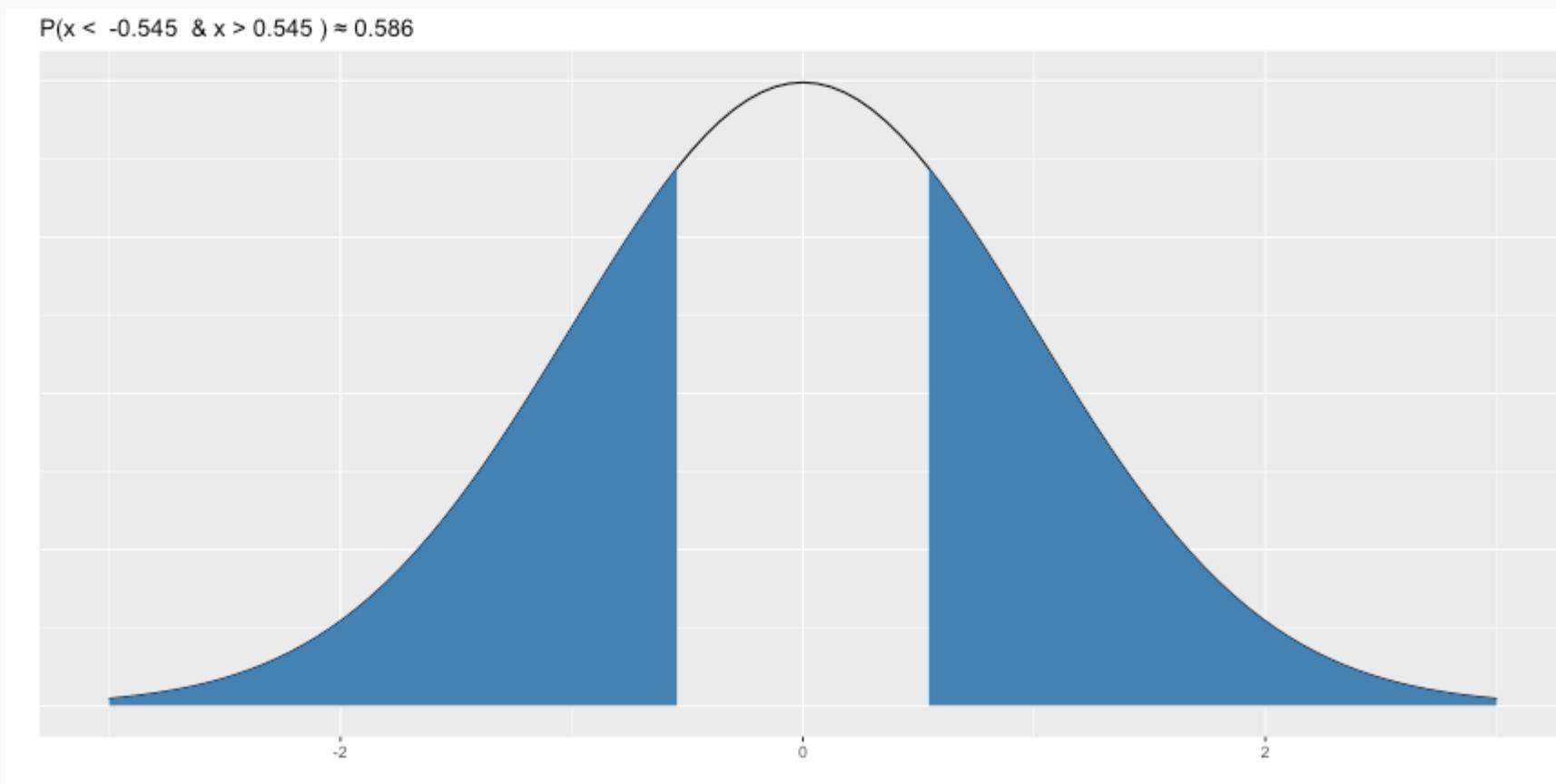
Nothing new here...

- The analysis is no different than what we have done before.
- We have data from one sample: differences.
- We are testing to see if the average difference is different than 0.



Calculating the test-statistic and the p-value

The observed average difference between the two scores is -0.545 points and the standard deviation of the difference is 8.887 points. Do these data provide convincing evidence of a difference between the average scores on the two exams (use $\alpha = 0.05$)?



Calculating the test-statistic and the p-value

$$Z = \frac{-0.545 - 0}{\frac{8.887}{\sqrt{200}}} = \frac{-0.545}{0.628} = -0.87$$

$$p-value = 0.1949 \times 2 = 0.3898$$

Since p-value > 0.05, we **fail to reject the null hypothesis**. That is, the data do not provide evidence that there is a statistically significant difference between the average reading and writing scores.

```
2 * pnorm(mean(hsb2$diff), mean=0, sd=sd(hsb2$diff)/sqrt(nrow(hsb2)))
```

```
## [1] 0.3857741
```



Evaluating the null hypothesis

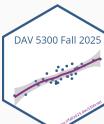
Interpretation of the p-value

The probability of obtaining a random sample of 200 students where the average difference between the reading and writing scores is at least 0.545 (in either direction), if in fact the true average difference between the score is 0, is 38%.

Calculating 95% Confidence Interval

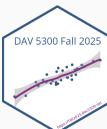
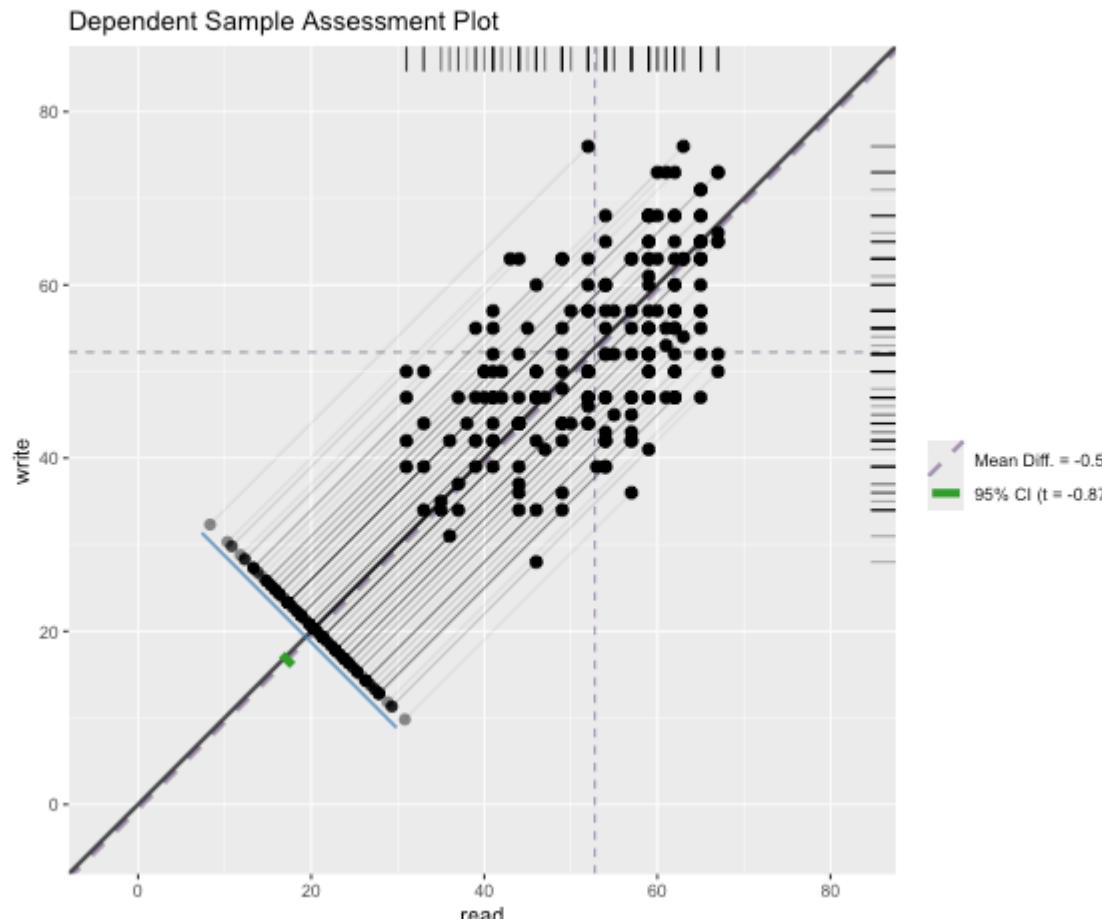
$$-0.545 \pm 1.96 \frac{8.887}{\sqrt{200}} = -0.545 \pm 1.96 \times 0.628 = (-1.775, 0.685)$$

Note that the confidence interval spans zero!



Visualizing Dependent Sample Tests

```
# remotes::install_github('briandk/granovaGG')
library(granovaGG)
granovagg.ds(as.data.frame(hsb2[,c('read', 'write')])))
```



SAT Scores by Sex

```
data(sat)  
head(sat)
```

```
##   Verbal.SAT Math.SAT Sex  
## 1      450     450   F  
## 2      640     540   F  
## 3      590     570   M  
## 4      400     400   M  
## 5      600     590   M  
## 6      610     610   M
```

Is there a difference in math scores between males and females?

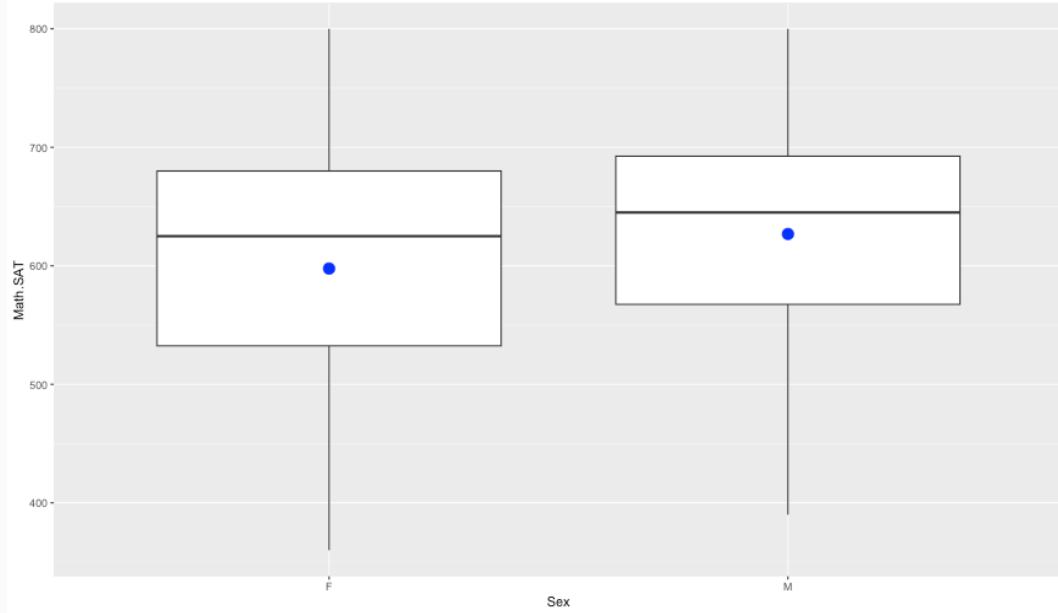


SAT Scores by Sex

```
tab <- describeBy(sat$Math.SAT,  
                   group=sat$Sex,  
                   mat=TRUE, skew=FALSE)  
tab[,c(2,4:7)]
```

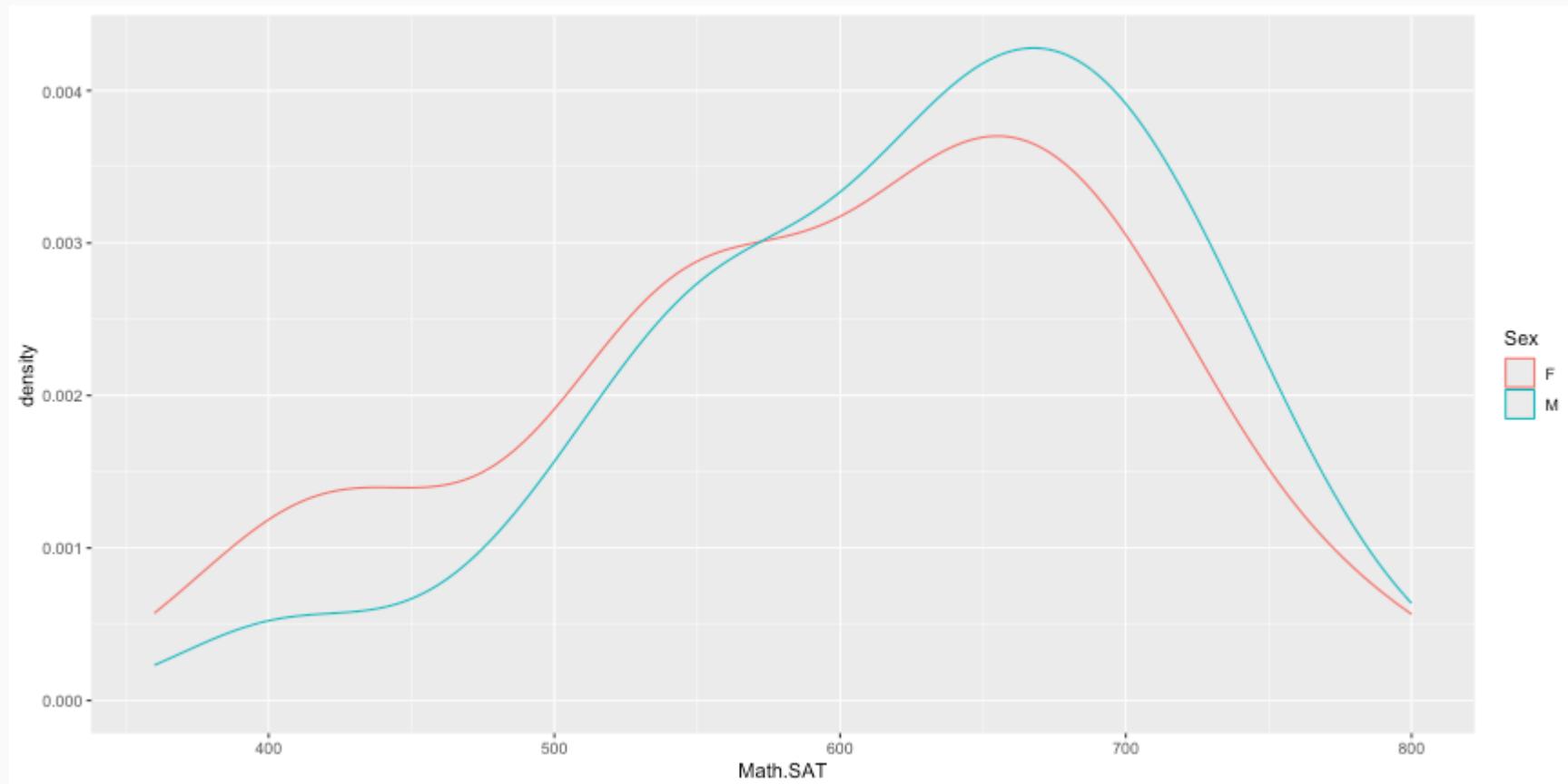
```
##      group1 n     mean       sd median  
## X11      F 82 597.6829 103.70065    625  
## X12      M 80 626.8750  90.35225    645
```

```
ggplot(sat, aes(x=Sex, y=Math.SAT)) +  
  geom_boxplot() +  
  geom_point(data = tab, aes(x=group1, y=mean),  
             color='blue', size=4)
```



Distributions

```
ggplot(sat, aes(x=Math.SAT, color = Sex)) + geom_density()
```



95% Confidence Interval

We wish to calculate a 95% confidence interval for the average difference between SAT scores for males and females.

Assumptions:

1. Independence within groups.
2. Independence between groups.
3. Sample size/skew



Confidence Interval for Difference Between Two Means

- All confidence intervals have the same form: point estimate \pm ME
- And all ME = critical value * SE of point estimate
- In this case the point estimate is $\bar{x}_1 - \bar{x}_2$ Since the sample sizes are large enough, the critical value is z^* So the only new concept is the standard error of the difference between two means...

Standard error for difference in SAT scores

$$SE_{(\bar{x}_M - \bar{x}_F)} = \sqrt{\frac{s_M^2}{n_M} + \frac{s_F^2}{n_F}}$$

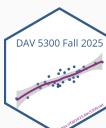
$$SE_{(\bar{x}_M - \bar{x}_F)} = \sqrt{\frac{90.4}{80} + \frac{103.7}{82}} = 1.55$$

Calculate the 95% confidence interval:

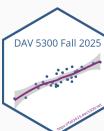
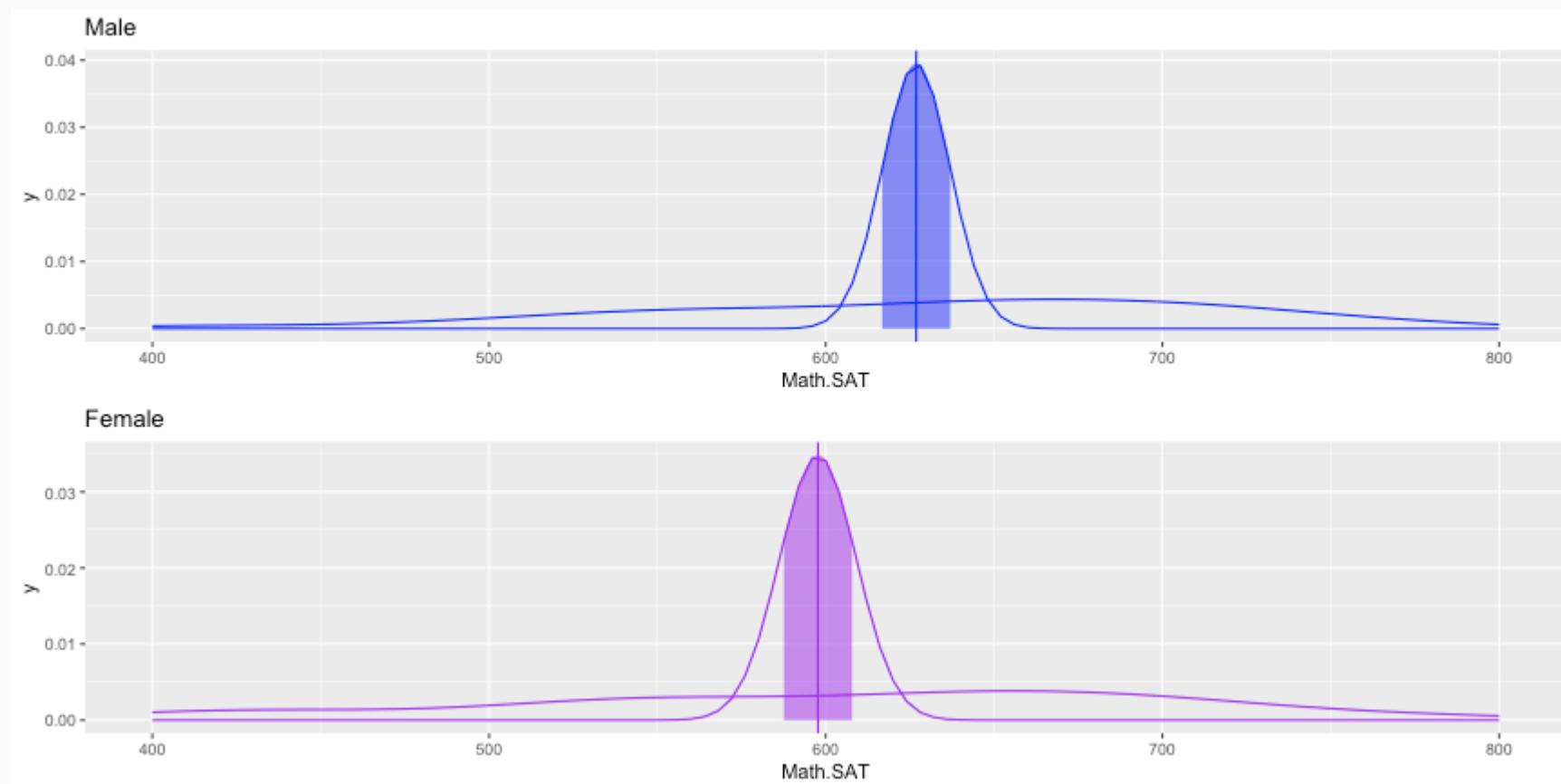
$$(\bar{x}_M - \bar{x}_F) \pm 1.96SE_{(\bar{x}_M - \bar{x}_F)}$$

$$(626.9 - 597.7) \pm 1.96 \times 1.55$$

$$29.2 \pm 3.038 = (26.162, 32.238)$$



Visualizing independent sample tests



What about smaller sample sizes?

What if you want to compare the quality of one batch of Guinness beer to the next?

- Sample sizes necessarily need to be small.
- The CLT states that the sampling distribution approximates normal as $n \rightarrow \text{Infinity}$
- Need an alternative to the normal distribution.
- The t distribution was developed by William Gosset (under the pseudonym *student*) to estimate means when the sample size is small.

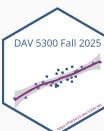
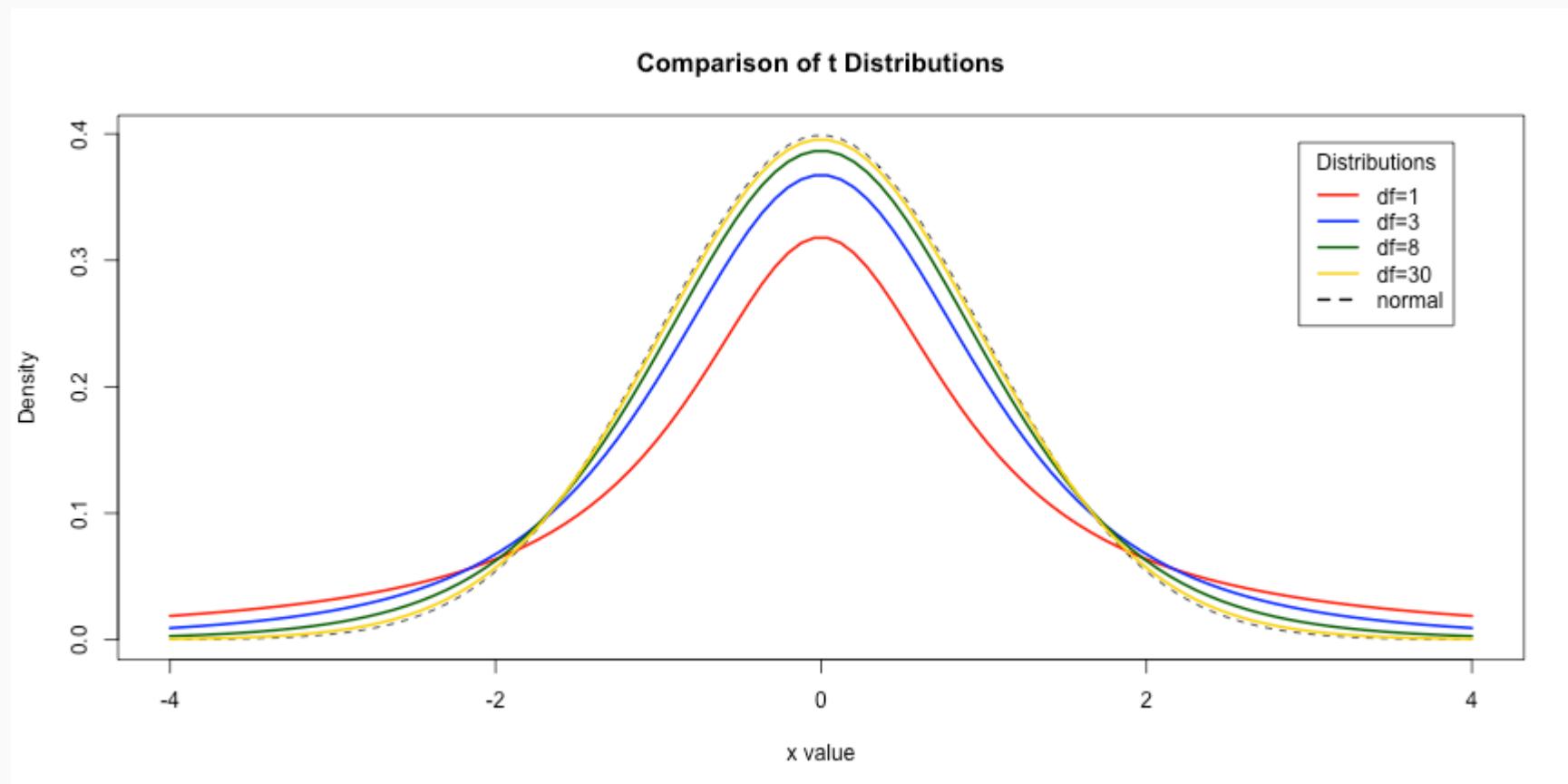
Confidence interval is estimated using

$$\bar{x} \pm t_{df}^* SE$$

Where df is the degrees of freedom ($df = n - 1$)



t -Distributions



t-test in R

The `pt` and `qt` will give you the *p*-value and critical value from the *t*-distribution, respectively.

Critical value for $p = 0.05$, degrees of freedom
= 10

```
qt(0.025, df = 10)
```

```
## [1] -2.228139
```

p-value for a critical value of 2, degrees of
freedom = 10

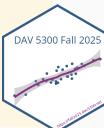
```
pt(2, df=10)
```

```
## [1] 0.963306
```

The `t.test` function will calculate a null hypothesis test using the *t*-distribution.

```
t.test(Math.SAT ~ Sex, data = sat)
```

```
##  
##      Welch Two Sample t-test  
##  
## data: Math.SAT by Sex  
## t = -1.9117, df = 158.01, p-value = 0.05773  
## alternative hypothesis: true difference in means bet  
## 95 percent confidence interval:  
## -59.3527145  0.9685682  
## sample estimates:  
## mean in group F mean in group M  
##                 597.6829                626.8750
```



Analysis of Variance (ANOVA)

Analysis of Variance (ANOVA)

The goal of ANOVA is to test whether there is a discernible difference between the means of several groups.

Hand Washing Example

Is there a difference between washing hands with: water only, regular soap, antibacterial soap (ABS), and antibacterial spray (AS)?

- Each tested with 8 replications
- Treatments randomly assigned

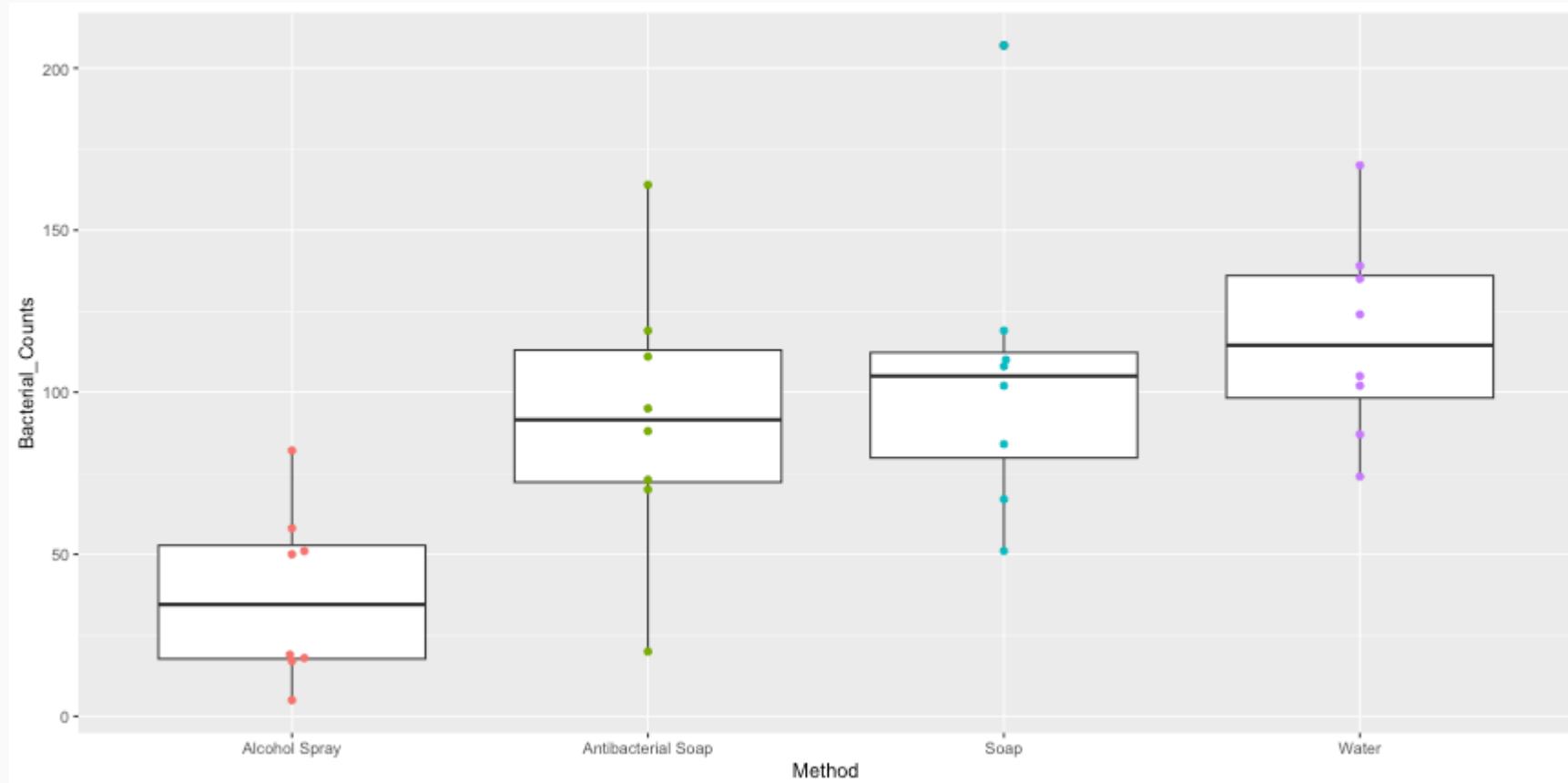
For ANOVA:

- The means all differ.
- Is this just natural variability?
- Null hypothesis: All the means are the same.
- Alternative hypothesis: The means are not all the same.

Source: De Veaux, R.D., Velleman, P.F., & Bock, D.E. (2014). *Intro Stats, 4th Ed.* Pearson.

Boxplot

```
ggplot(hand_washing, aes(x = Method, y = Bacterial_Counts)) + geom_boxplot() +  
  geom_beeswarm(aes(color = Method)) + theme(legend.position = 'none')
```



Descriptive Statistics

```
desc <- psych::describeBy(hand_washing$Bacterial_Counts, group = hand_washing$Method, mat = TRUE, skew = FALSE)
names(desc)[2] <- 'Method' # Rename the grouping column
desc$Var <- desc$sd^2 # We will need the variance latter, so calculate it here
desc
```

##	item	Method	vars	n	mean	sd	median	min	max	range	se	Var
## X11	1	Alcohol Spray	1	8	37.5	26.55991	34.5	5	82	77	9.390345	705.4286
## X12	2	Antibacterial Soap	1	8	92.5	41.96257	91.5	20	164	144	14.836008	1760.8571
## X13	3	Soap	1	8	106.0	46.95895	105.0	51	207	156	16.602496	2205.1429
## X14	4	Water	1	8	117.0	31.13106	114.5	74	170	96	11.006492	969.1429

```
( k <- length(unique(hand_washing$Method)) )
```

```
## [1] 4
```

```
( n <- nrow(hand_washing) )
```

```
## [1] 32
```

```
( grand_mean <- mean(hand_washing$Bacterial_Counts) )
```

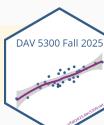
```
## [1] 88.25
```

```
( grand_var <- var(hand_washing$Bacterial_Counts) )
```

```
## [1] 2237.613
```

```
( pooled_var <- mean(desc$Var) )
```

```
## [1] 1410.143
```



Contrasts

A contrast is a linear combination of two or more factor level means with coefficients that sum to zero.

```
desc$contrast <- (desc$mean - mean(desc$mean))  
mean(desc$contrast) # Should be 0!
```

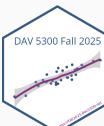
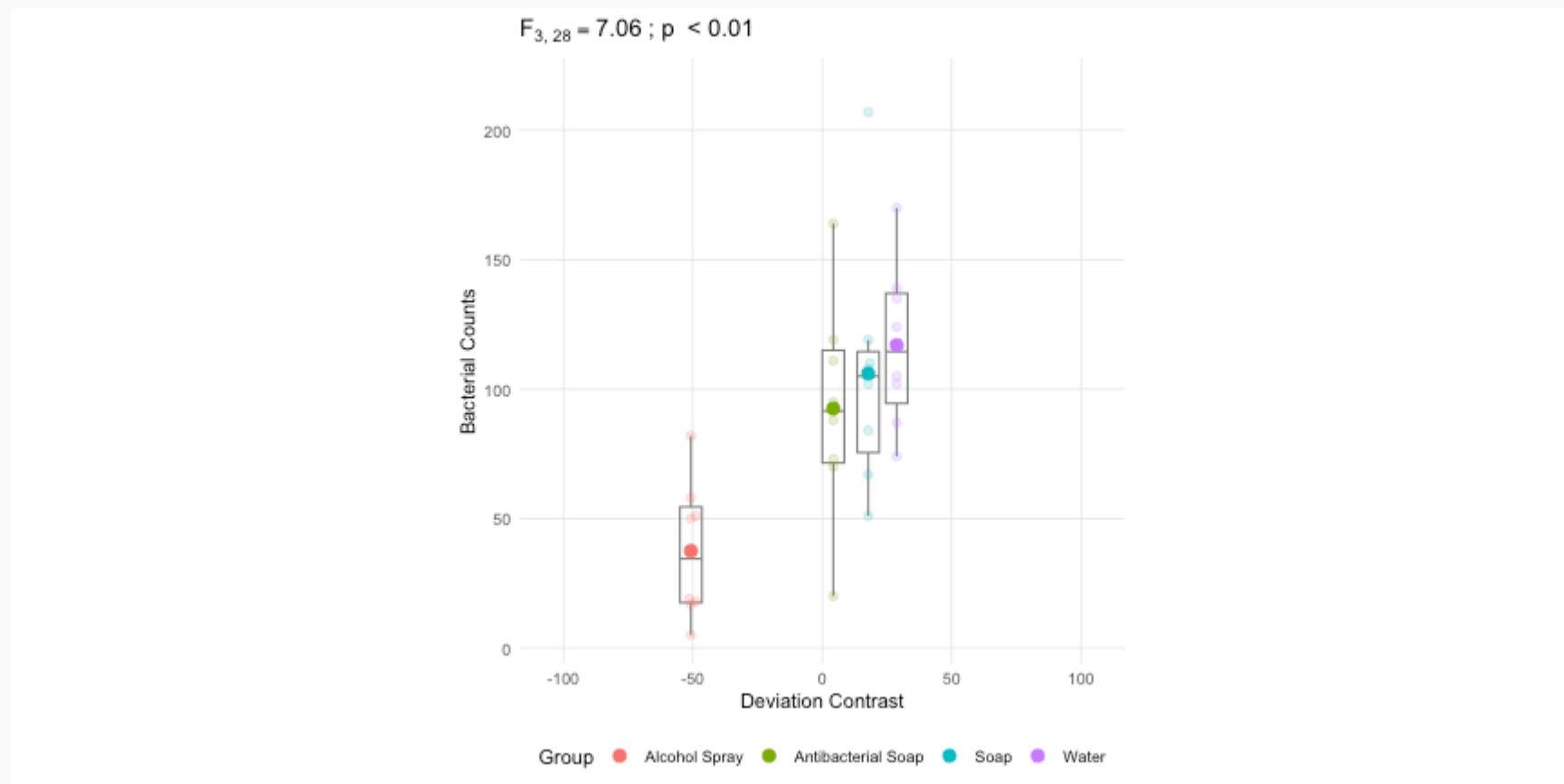
```
## [1] 0
```

```
desc
```

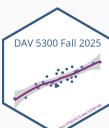
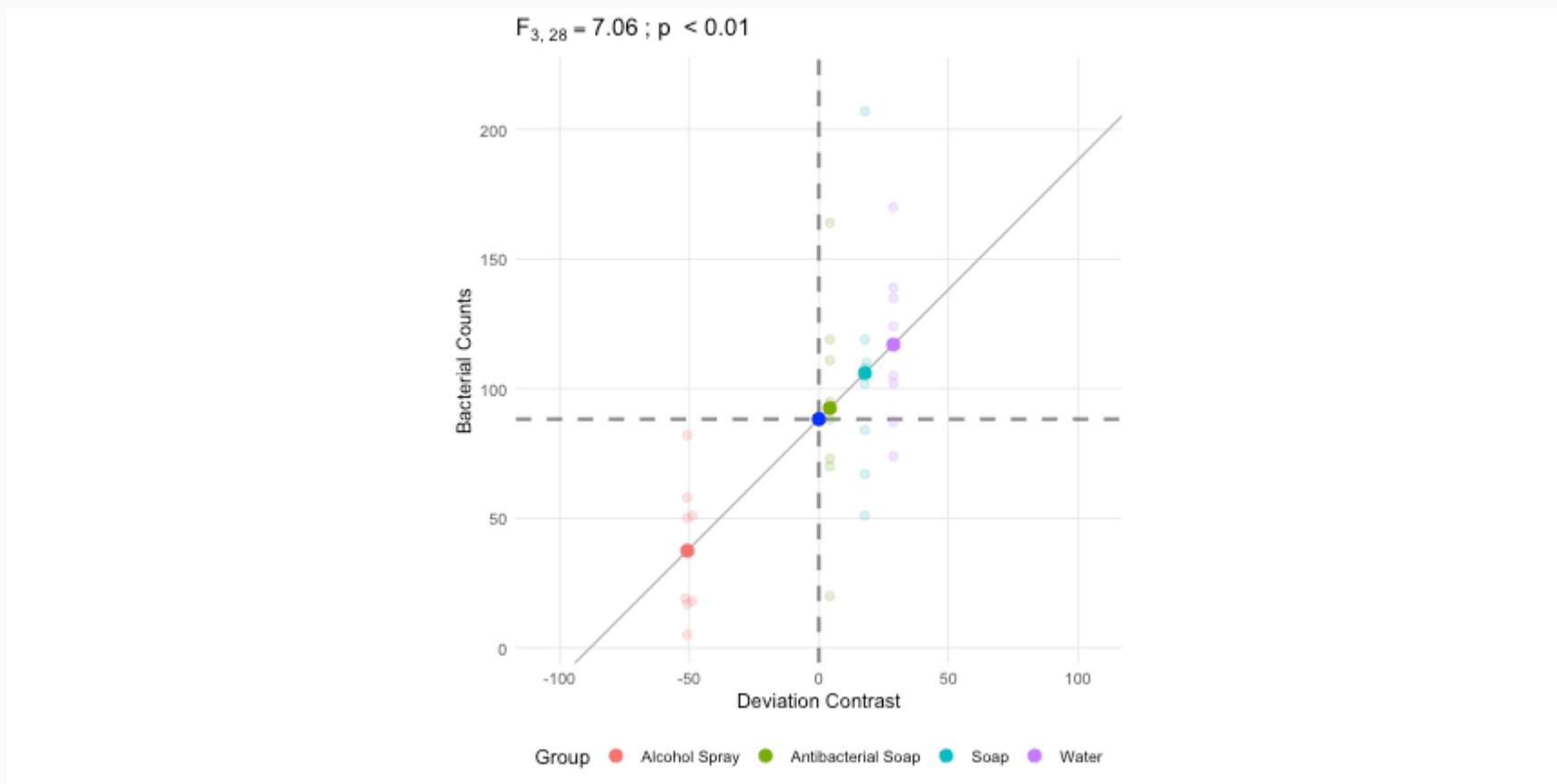
##	item	Method	vars	n	mean	sd	median	min	max	range	se	Var	contrast
## X11	1	Alcohol Spray	1	8	37.5	26.55991	34.5	5	82	77	9.390345	705.4286	-50.75
## X12	2	Antibacterial Soap	1	8	92.5	41.96257	91.5	20	164	144	14.836008	1760.8571	4.25
## X13	3	Soap	1	8	106.0	46.95895	105.0	51	207	156	16.602496	2205.1429	17.75
## X14	4	Water	1	8	117.0	31.13106	114.5	74	170	96	11.006492	969.1429	28.75



Plotting using contrasts



Grade Mean and Unit Line (slope = 1, intercept = \bar{x})

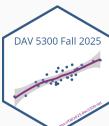
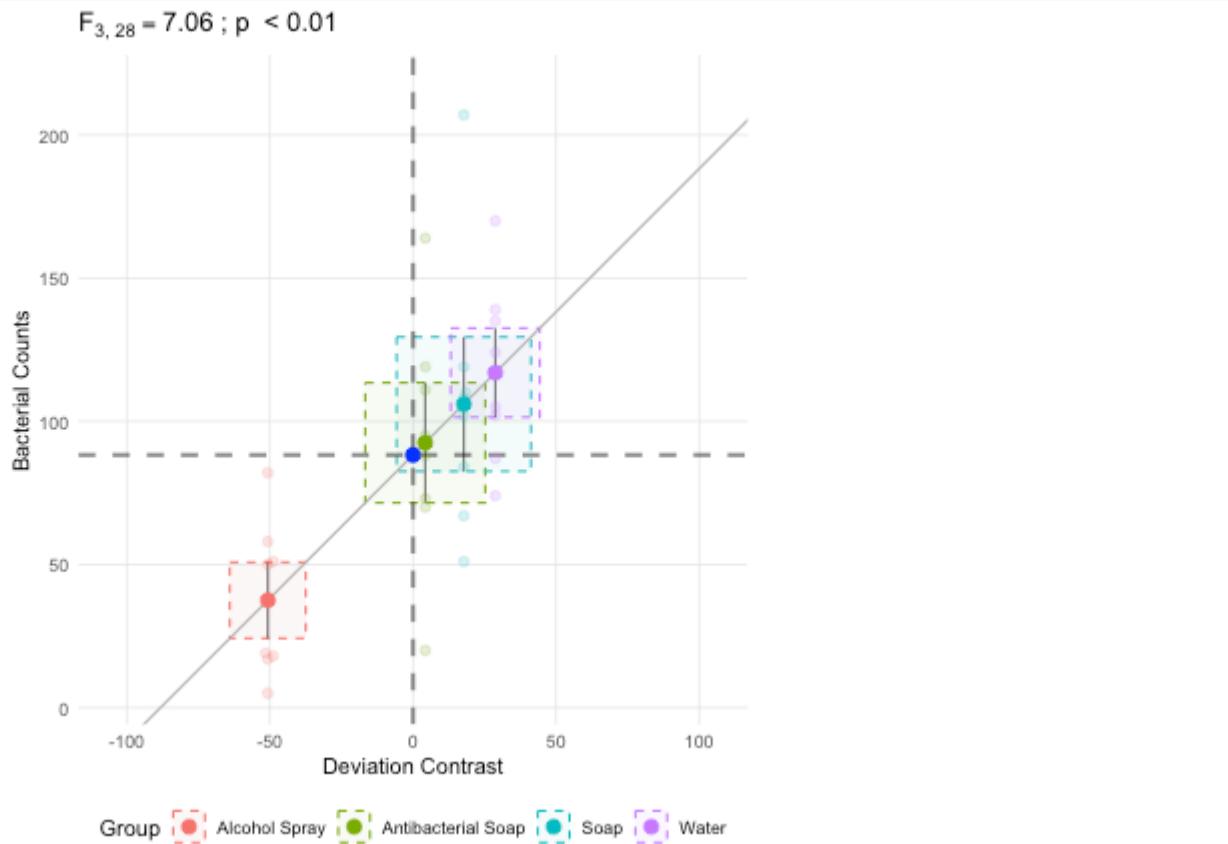


Within Group Variance (error)

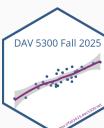
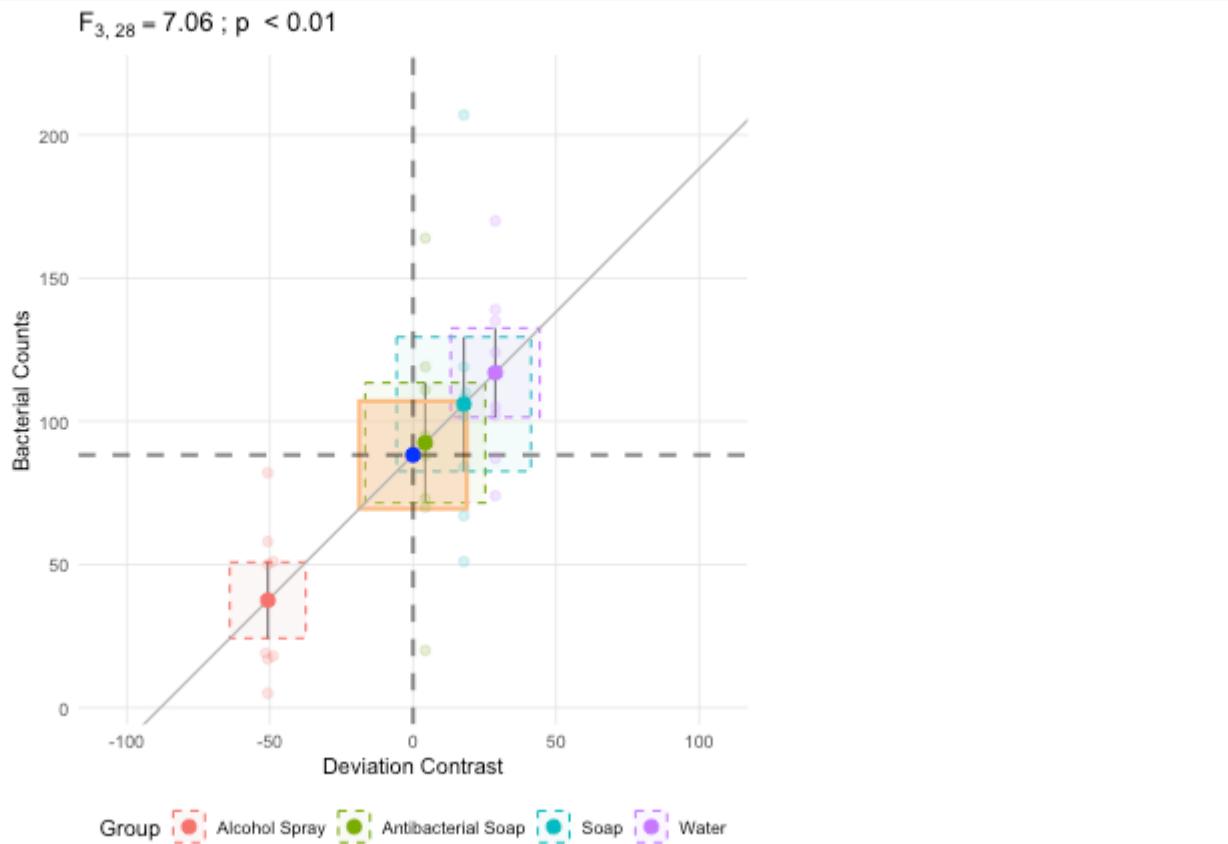
$$SS_{within} = \sum_k \sum_i (\bar{x}_{ik} - \bar{x}_k)^2$$



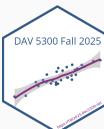
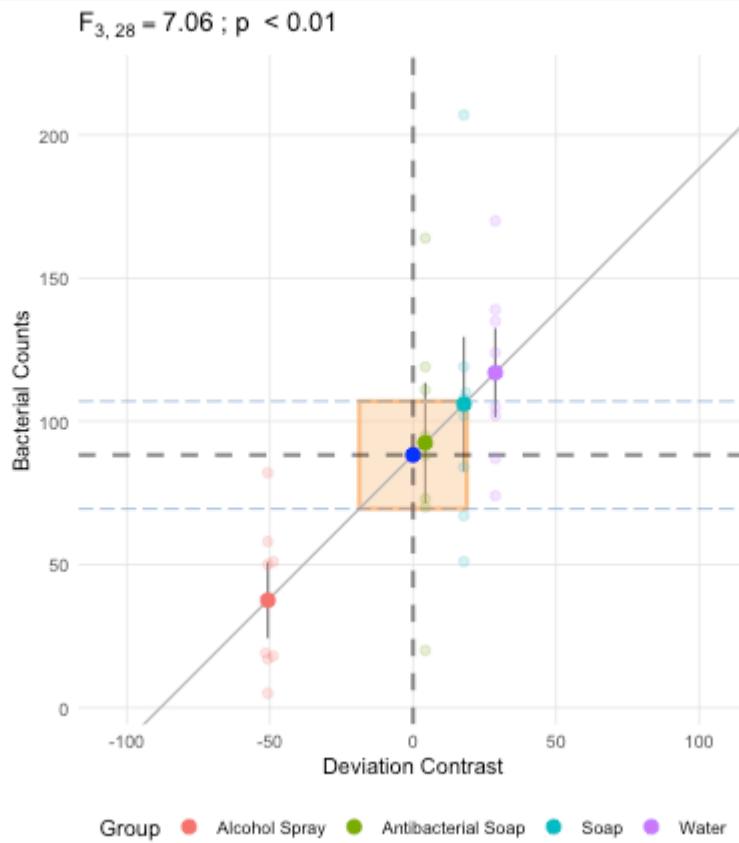
Within Group Variance (error)



Within Group Variance (error)



Within Group Variance (error)

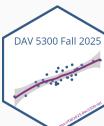
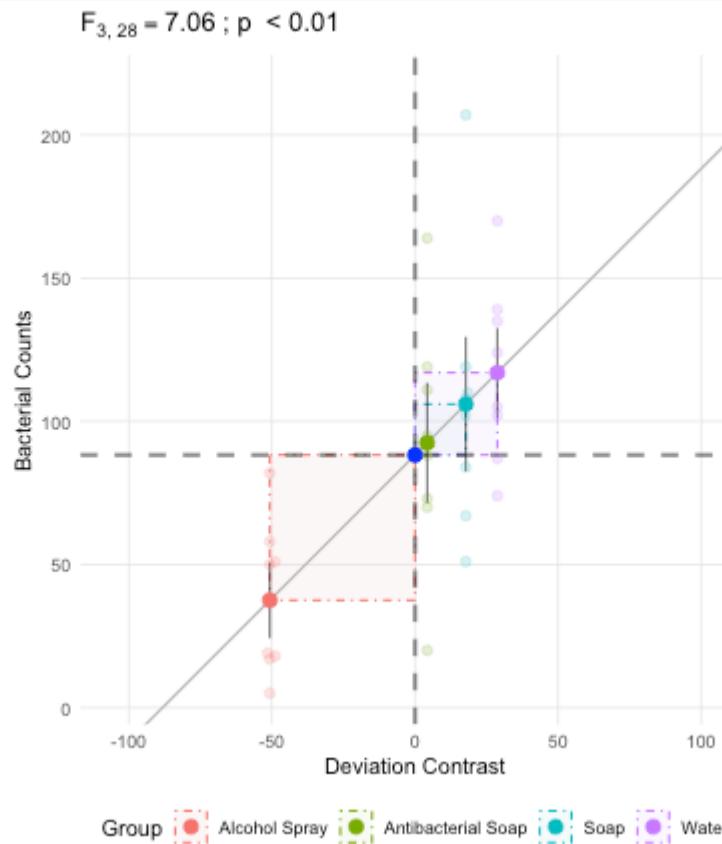


Between Group Variance

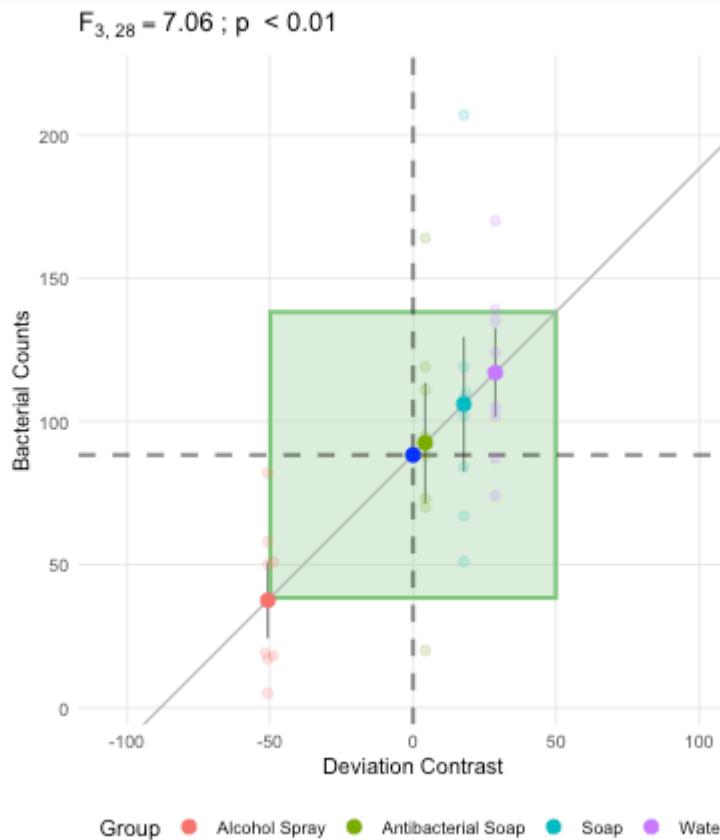
$$SS_{between} = \sum_k n_k (\bar{x}_k - \bar{x})^2$$



Between Group Variance

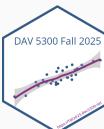


Between Group Variance



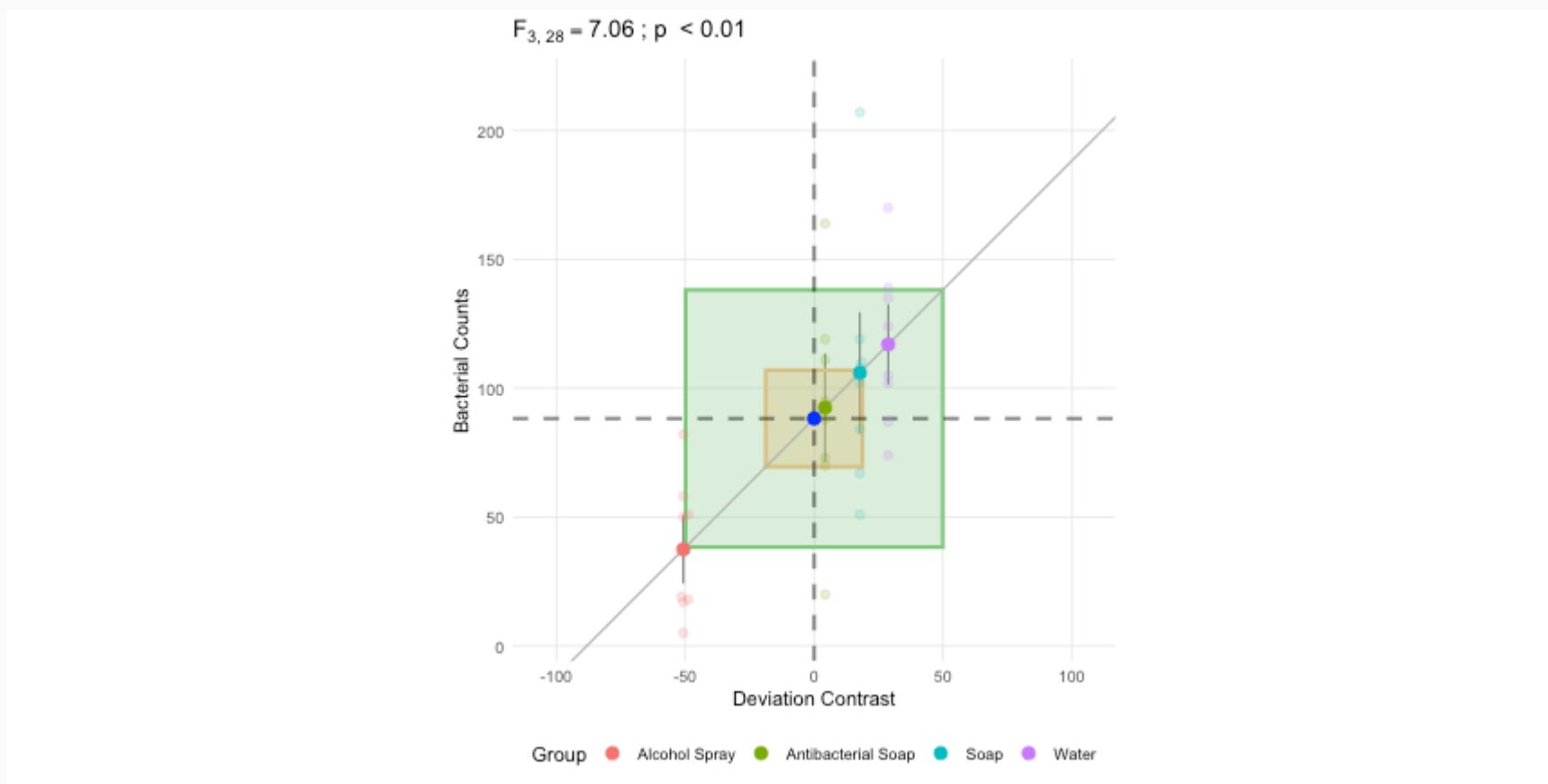
Mean Square

Source	Sum of Squares	df	MS
Between Group (Treatment)	$\sum_k n_k (\bar{x}_k - \bar{x})^2$	k - 1	$\frac{SS_{between}}{df_{between}}$
Within Group (Error)	$\sum_k \sum_i (\bar{x}_{ik} - \bar{x}_k)^2$	n - k	$\frac{SS_{within}}{df_{within}}$
Total	$\sum_n (x_n - \bar{x})^2$	n - 1	



$$MS_{Between}/MS_{Within} = F\text{-Statistic}$$

Mean squares can be represented as squares, hence the ratio of area of the two rectangles is equal to $\frac{MS_{Between}}{MS_{Within}}$ which is the F-statistic.



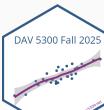
Washing type all the same?

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

H_A : at least one mean is different from the others.

Variance components we need to evaluate the null hypothesis:

- Between Sum of Squares: $SS_{between} = \sum_k n_k (\bar{x}_k - \bar{x})^2$
- Within Sum of Squares: $SS_{within} = \sum_k \sum_i (\bar{x}_{ik} - \bar{x}_k)^2$
- Between degrees of freedom: $df_{between} = k - 1$ (k = number of groups)
- Within degrees of freedom: $df_{within} = k(n - 1)$
- Mean square between (aka treatment): $MS_T = \frac{SS_{between}}{df_{between}}$
- Mean square within (aka error): $MS_E = \frac{SS_{within}}{df_{within}}$



Comparing MS_T (between) and MS_E (within)

Assume each washing method has the same variance.

Then we can pool them all together to get the pooled variance s_p^2

Since the sample sizes are all equal, we can average the four variances: $s_p^2 = 1410.14$

```
mean(desc$Var)
```

```
## [1] 1410.143
```

MS_T

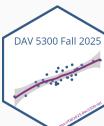
- Estimates s_p^2 if H_0 is true
- Should be larger than s_p^2 if H_0 is false

MS_E

- Estimates s_p^2 whether H_0 is true or not
- If H_0 is true, both close to s_p^2 , so MS_T is close to MS_E

Comparing

- If H_0 is true, $\frac{MS_T}{MS_E}$ should be close to 1
- If H_0 is false, $\frac{MS_T}{MS_E}$ tends to be > 1



The F-Distribution

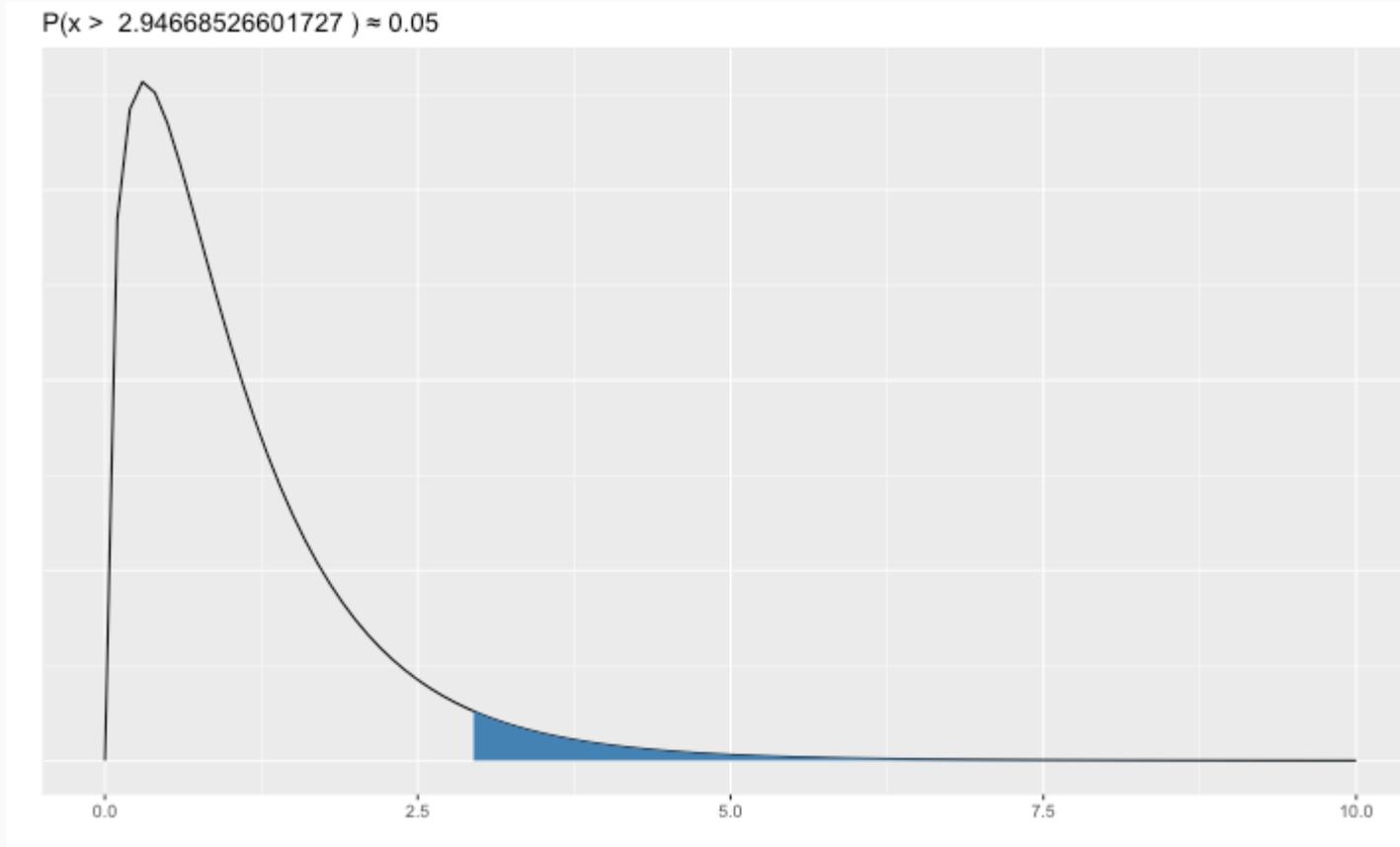
- How do we tell whether $\frac{MS_T}{MS_E}$ is larger enough to not be due just to random chance?
- $\frac{MS_T}{MS_E}$ follows the F-Distribution
 - Numerator df: $k - 1$ (k = number of groups)
 - Denominator df: $k(n - 1)$
 - n = # observations in each group
- $F = \frac{MS_T}{MS_E}$ is called the F-Statistic.

A Shiny App by Dr. Dudek to explore the F-Distribution:
<https://shiny.rit.albany.edu/stat/fdist/>



The F-Distribution (cont.)

```
df.numerator <- 4 - 1  
df.denominator <- 4 * (8 - 1)  
DATA606::F_plot(df.numerator, df.denominator, cv = qf(0.95, df.numerator, df.denominator))
```



ANOVA Table

Source	Sum of Squares	df	MS	F	p
Between Group (Treatment)	$\sum_k n_k (\bar{x}_k - \bar{x})^2$	k - 1	$\frac{SS_{between}}{df_{between}}$	$\frac{MS_{between}}{MS_{within}}$	area to right of $F_{k-1, n-k}$
Within Group (Error)	$\sum_k \sum_i (\bar{x}_{ik} - \bar{x}_k)^2$	n - k	$\frac{SS_{within}}{df_{within}}$		
Total	$\sum_n (x_n - \bar{x})^2$	n - 1			

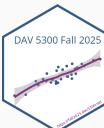
```
aov(Bacterial_Counts ~ Method, data = hand_washing) |> summary()
```

```
##          Df Sum Sq Mean Sq F value    Pr(>F)
## Method      3 29882   9961   7.064 0.00111 ***
## Residuals  28 39484   1410
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```



Assumptions and Conditions

- To check the assumptions and conditions for ANOVA, always look at the side-by-side boxplots.
 - Check for outliers within any group.
 - Check for similar spreads.
 - Look for skewness.
 - Consider re-expressing.
- Independence Assumption
 - Groups must be independent of each other.
 - Data within each group must be independent.
 - Randomization Condition
- Equal Variance Assumption
 - In ANOVA, we pool the variances. This requires equal variances from each group: Similar Spread Condition.



Your turn: M&Ms

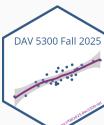
You each received a small pack of M&Ms. We want to know if each color of M&M is equally likely to be available. With your pack:

1. Count each color.
2. Scan the QR code and enter your results



$$H_0 : \mu_{orange} = \mu_{red} = \mu_{green} = \mu_{yellow} = \mu_{blue} = \mu_{brown}$$

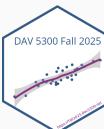
H_A : at least one mean is different from the others.



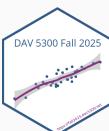
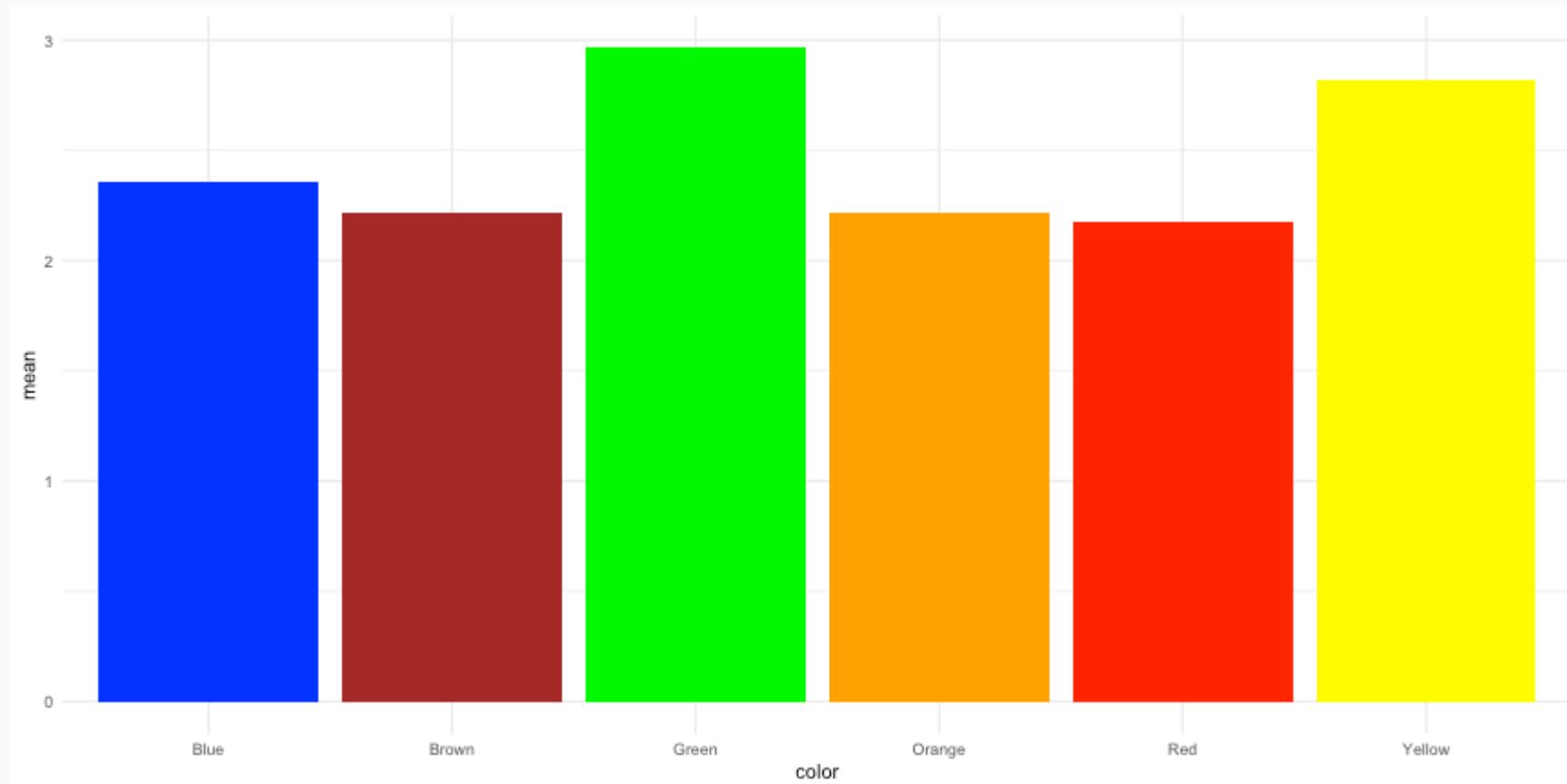
M&M Results

```
results_url <- 'https://docs.google.com/spreadsheets/d/1lAOM-wRsbD1z38jcc2yN6T4p1Qo13dCQlJcLGMM-yJk/edit?resourcekey='
results <- read_sheet(results_url)
results_melt <- results |> reshape2::melt(id.vars = 'Timestamp') |> rename(color = variable)
tab <- psych::describeBy(results_melt$value, group = results_melt$color, mat = TRUE, skew = FALSE) |>
  rename(color = group1)
tab
```

##	item	color	vars	n	mean	sd	median	min	max	range	se
## X11	1	Orange	1	28	2.214286	1.100745	2	0	5	5	0.2080213
## X12	2	Red	1	28	2.178571	1.334821	2	0	5	5	0.2522574
## X13	3	Green	1	28	2.964286	1.373887	3	1	6	5	0.2596403
## X14	4	Yellow	1	28	2.821429	1.188013	3	0	5	5	0.2245134
## X15	5	Blue	1	28	2.357143	1.339272	2	0	5	5	0.2530987
## X16	6	Brown	1	28	2.214286	1.315355	2	0	5	5	0.2485787



M&M Results (cont.)



M&M Results (cont.)

```
aov(value ~ color, data = results_melt) |> summary()
```

```
##          Df Sum Sq Mean Sq F value Pr(>F)
## color      5 16.67   3.335   2.038  0.076 .
## Residuals 162 265.04   1.636
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```



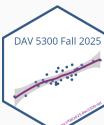
More Information

ANOVA Vignette in the `visualStats` package: <https://visualstats.bryer.org/anova.html>

The plots were created using the `visualStats::anova_vis()` function.

Shiny app:

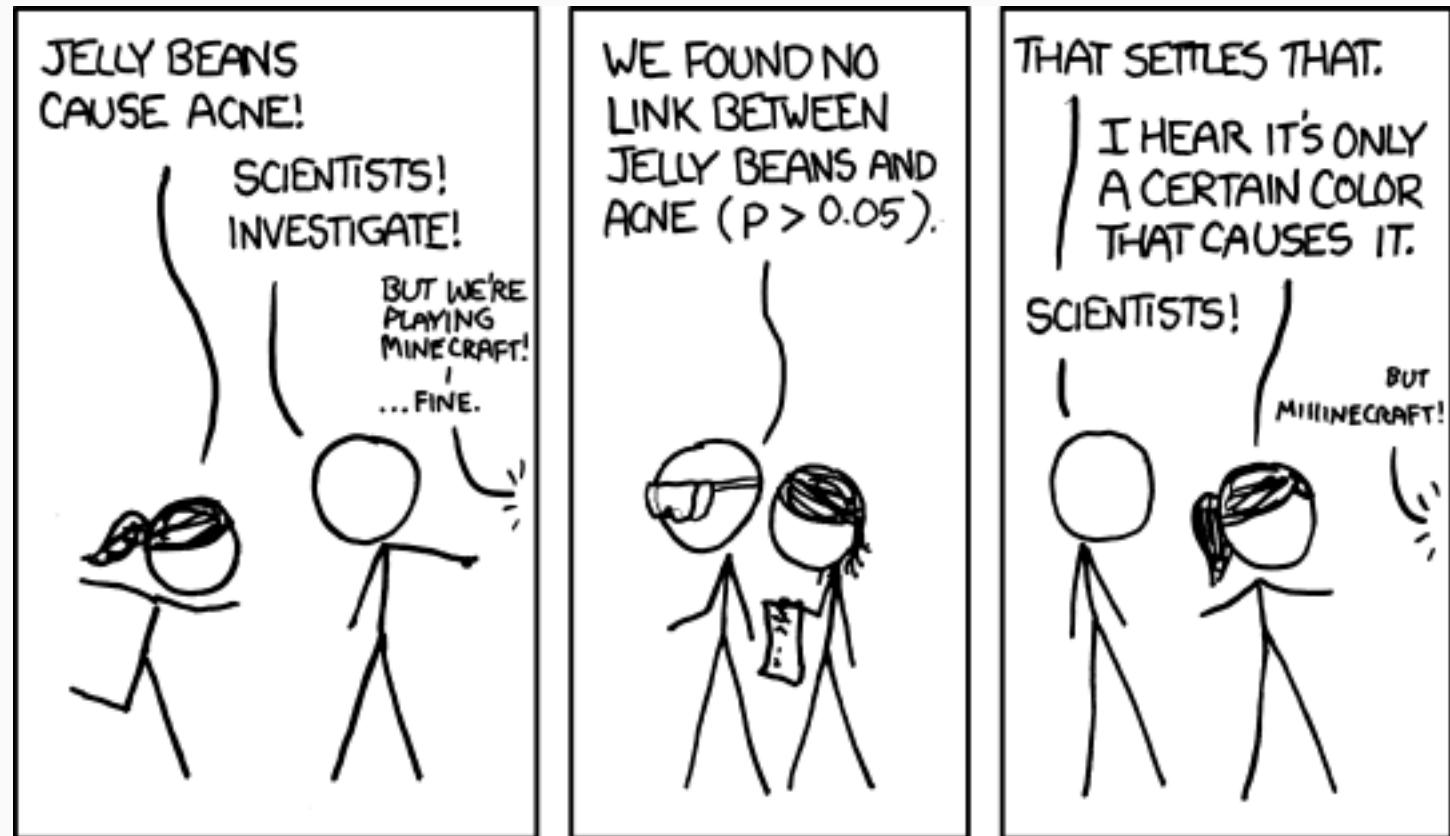
```
remotes::install_github('jbryer/VisualStats')
library(VisualStats)
anova_shiny()
```



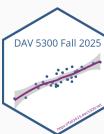
What Next?

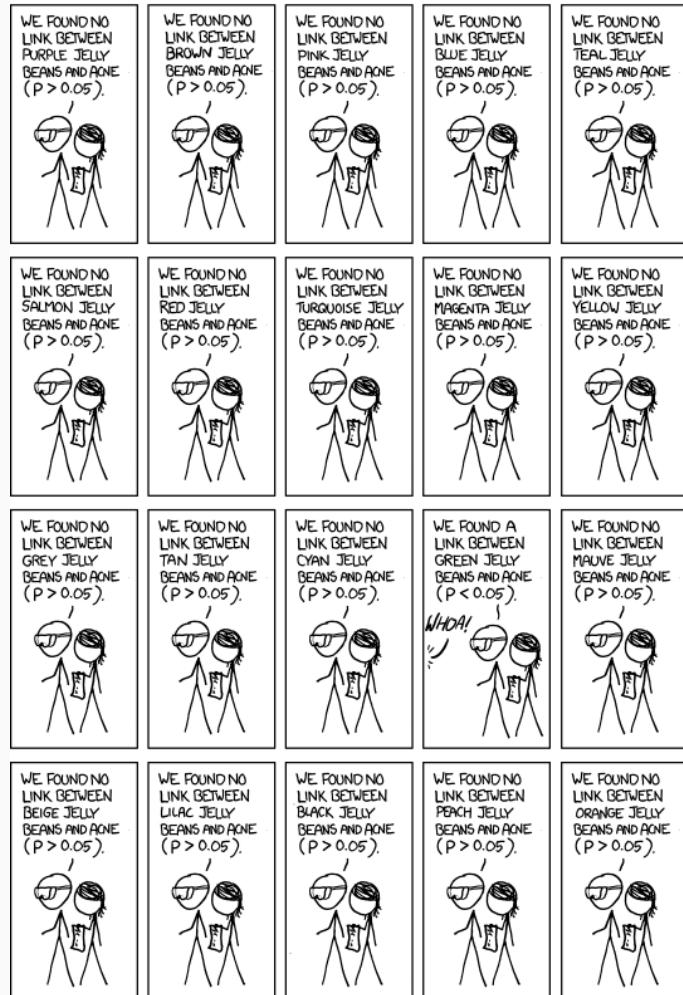
- P-value large \rightarrow Nothing left to say
- P-value small \rightarrow Which means are large and which means are small?
- We can perform a t-test to compare two of them.
- We assumed the standard deviations are all equal.
- Use s_p , for pooled standard deviations.
- Use the Students t-model, $df = N - k$.
- If we wanted to do a t-test for each pair:
 - $P(\text{Type I Error}) = 0.05$ for each test.
 - Good chance at least one will have a Type I error.
- **Bonferroni to the rescue!**
 - Adjust α to α/J where J is the number of comparisons.
 - 95% confidence $(1 - 0.05)$ with 3 comparisons adjusts to $(1 - 0.05/3) \approx 0.98333$.
 - Use this adjusted value to find t^{**} .

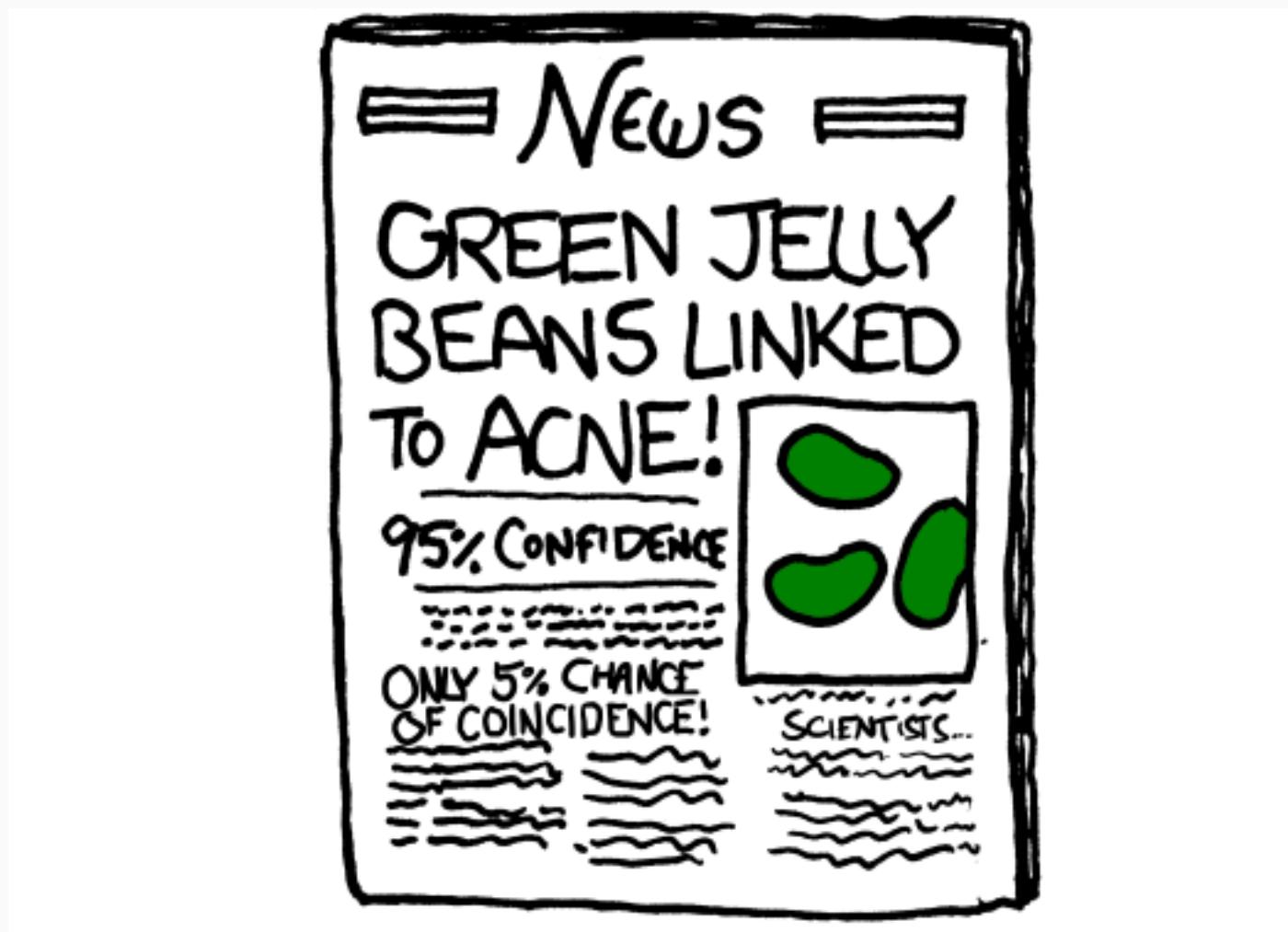




Source: <https://xkcd.com/882/>

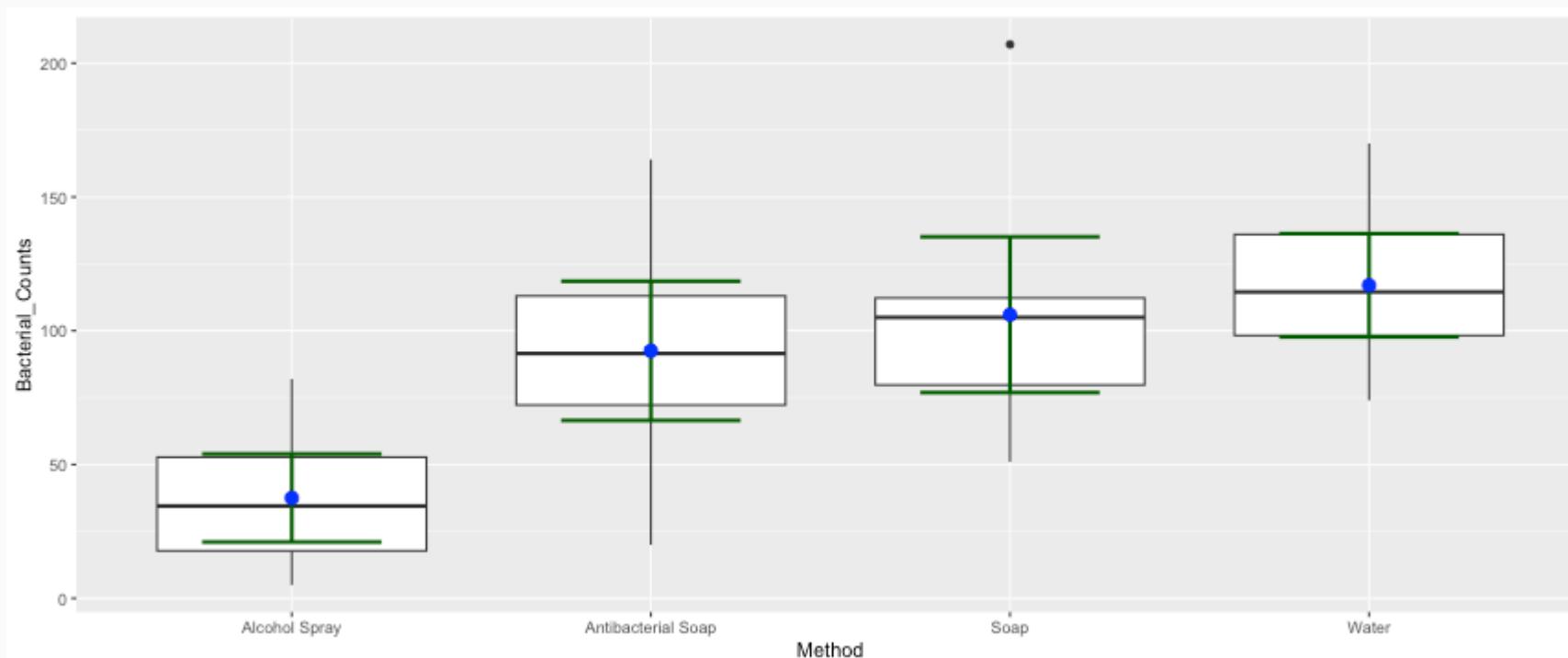






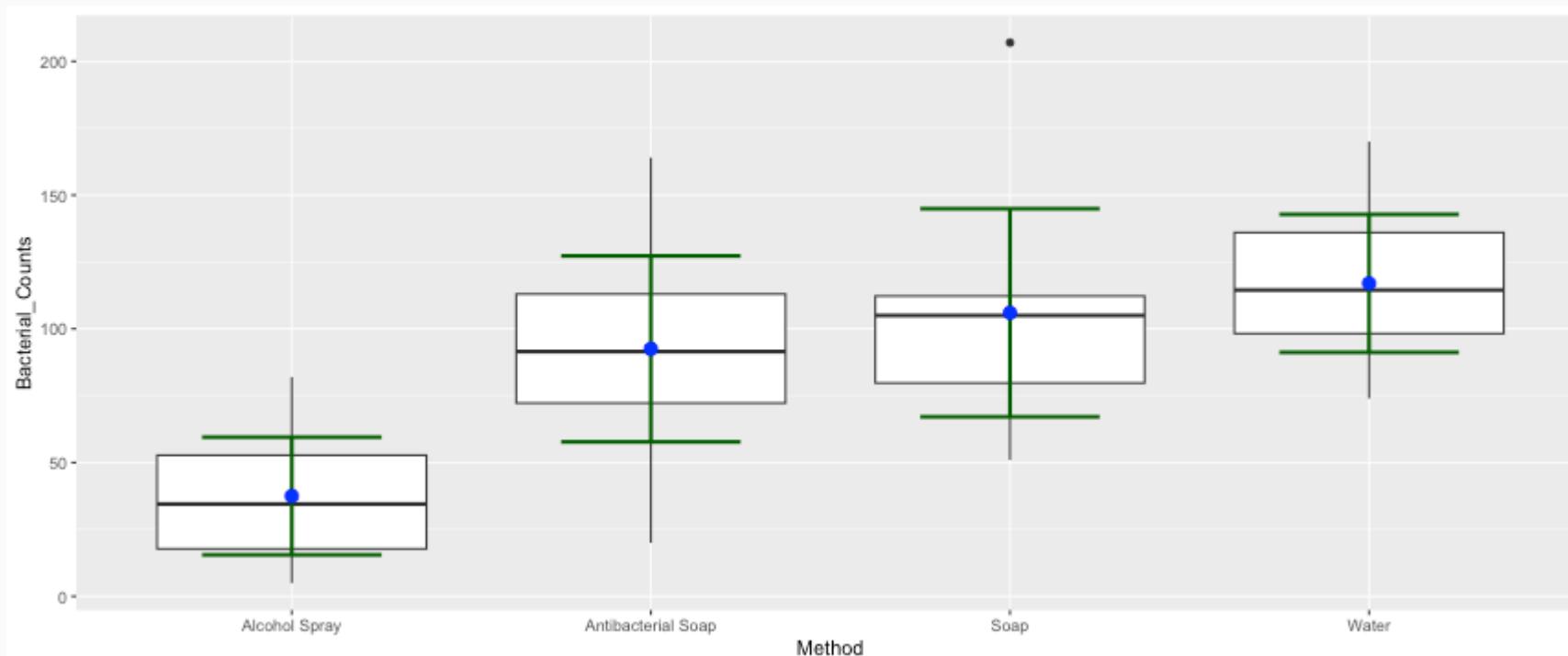
Multiple Comparisons (no Bonferroni adjustment)

```
cv <- qt(0.05, df = 15)
tab <- describeBy(hand_washing$Bacterial_Counts, group = hand_washing$Method, mat = TRUE)
ggplot(hand_washing, aes(x = Method, y = Bacterial_Counts)) + geom_boxplot() +
  geom_errorbar(data = tab, aes(x = group1, y = mean,
                                 ymin = mean - cv * se, ymax = mean + cv * se),
                color = 'darkgreen', width = 0.5, size = 1) +
  geom_point(data = tab, aes(x = group1, y = mean), color = 'blue', size = 3)
```



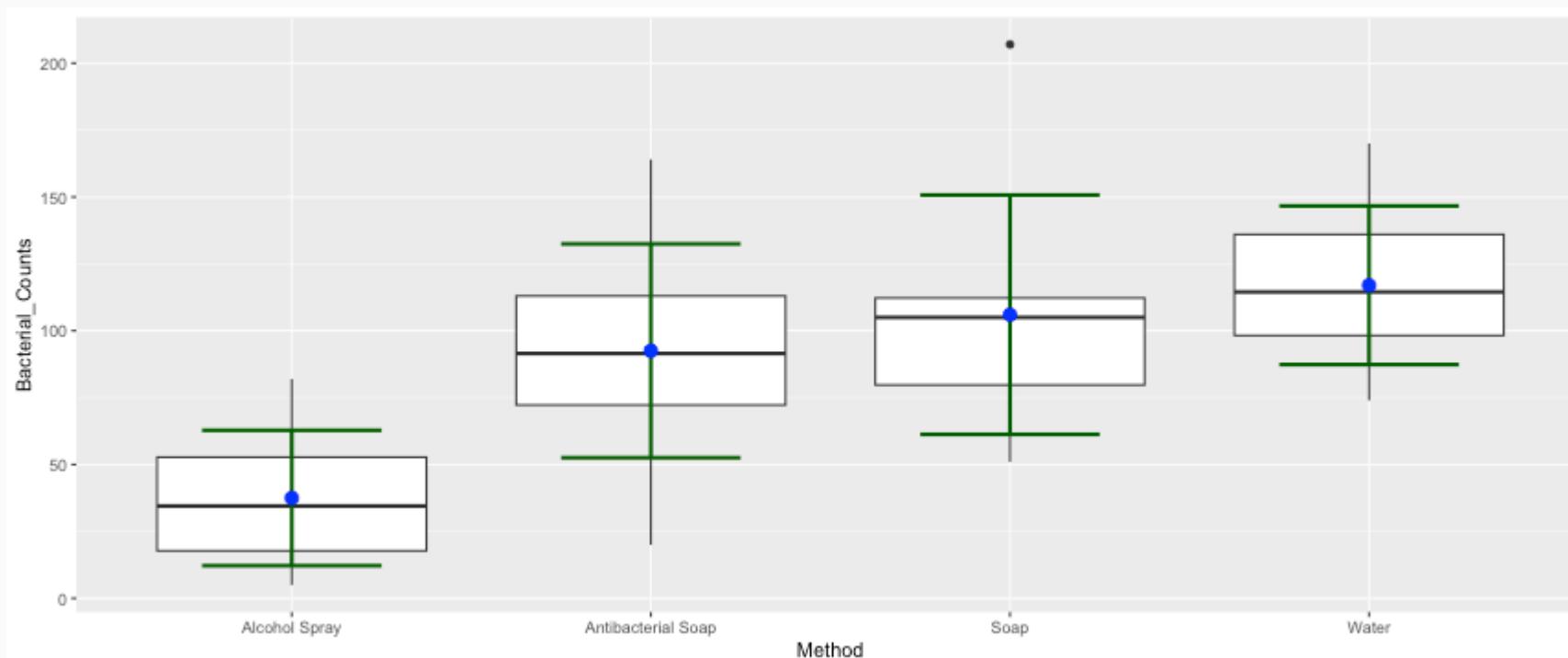
Multiple Comparisons (3 paired tests)

```
cv <- qt(0.05 / 3, df = 15)
tab <- describeBy(hand_washing$Bacterial_Counts, group = hand_washing$Method, mat = TRUE)
ggplot(hand_washing, aes(x = Method, y = Bacterial_Counts)) + geom_boxplot() +
  geom_errorbar(data = tab, aes(x = group1, y = mean,
                                 ymin = mean - cv * se, ymax = mean + cv * se),
                color = 'darkgreen', width = 0.5, size = 1) +
  geom_point(data = tab, aes(x = group1, y = mean), color = 'blue', size = 3)
```



Multiple Comparisons (6 paired tests)

```
cv <- qt(0.05 / choose(4, 2), df = 15)
tab <- describeBy(hand_washing$Bacterial_Counts, group = hand_washing$Method, mat = TRUE)
ggplot(hand_washing, aes(x = Method, y = Bacterial_Counts)) + geom_boxplot() +
  geom_errorbar(data = tab, aes(x = group1, y = mean,
                                 ymin = mean - cv * se, ymax = mean + cv * se ),
                color = 'darkgreen', width = 0.5, size = 1) +
  geom_point(data = tab, aes(x = group1, y = mean), color = 'blue', size = 3)
```



One Minute Paper

1. What was the most important thing you learned during this class?
2. What important question remains unanswered for you?



<https://forms.gle/bEk6KegZ1AMvQHNT5>

