Confidence Intervals for Proportions

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Standard Error for a Proportion

What is the sampling distribution?

- Usually we do not know the population proportion p.
- We cannot find the standard deviation of the sampling distribution:

$$\sqrt{\frac{pq}{n}}$$

- After taking a sample, we only know the sample proportion, which we use as an approximation.
- The standard error is given by

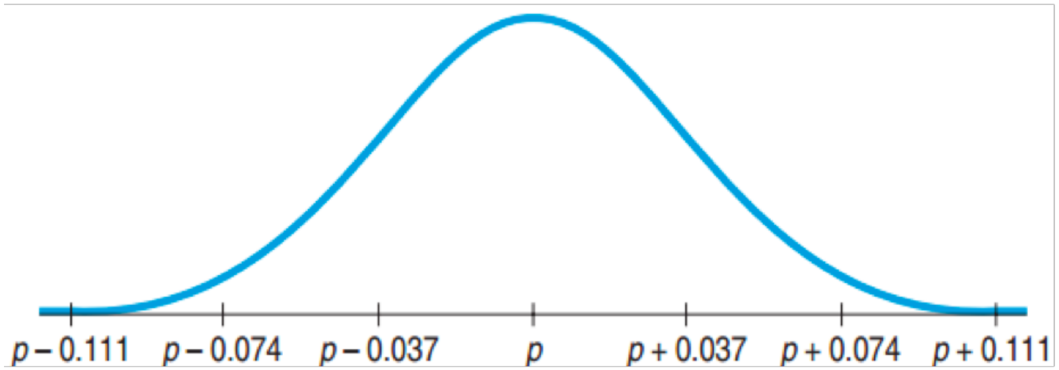
$$\sqrt{\frac{\hat{p}\hat{q}}{n}}$$

Facebook Daily Status Updates

A recent survey found that 48 of 156 or 30.8% update their Facebook status daily.

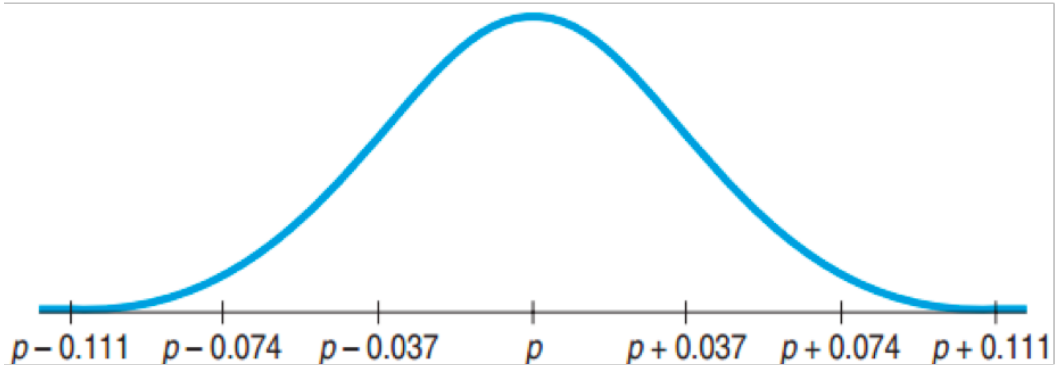
$$SE(\hat{p}) = \sqrt{\frac{\hat{p}\hat{q}}{n}} = \sqrt{\frac{(0.308)(0.692)}{156}} \approx 0.037$$

The sampling distribution is approximately normal



Interpreting this Normal Curve

- By normality, about 95% ofall possible samples of 156 young Facebook users will have $\hat{p}$'s within 2 SE's of p.
- If $\hat{p}$ is close to p, then p is close to $\hat{p}$.
- If you stand at $\hat{p}$, then you can be 95% sure that p is within 2SE's from where you are standing.



What You Cannot Say About p if You Know $\hat{p}$

30.8% of all Facebook users update their status daily.

- We can't make such absolute statements about p.

It is probably true that 30.8% of all Facebook users update their status daily.

- We still cannot commit to a specific value for p, only a range.

We don't know exactly what percent of all Facebook users update their status daily, but we know it is within the interval 30.8% ± 2 X 3.7%.

- We cannot be certain it is in this interval.

What You Can Say About p if You Know $\hat{p}$

We don’t know exactly what percent of all Facebook users update their status daily, but the interval from 23.4% and 38.2% probably contains the true proportion.

- Note, we admit we are unsure about both the exact proportion and whether it is in the interval.

We are 95% confident that between 23.4% and 38.2% of all Facebook users update their status daily

- Notice “% confident” and an interval rather than an exact value are stated.

Naming the Confidence Interval

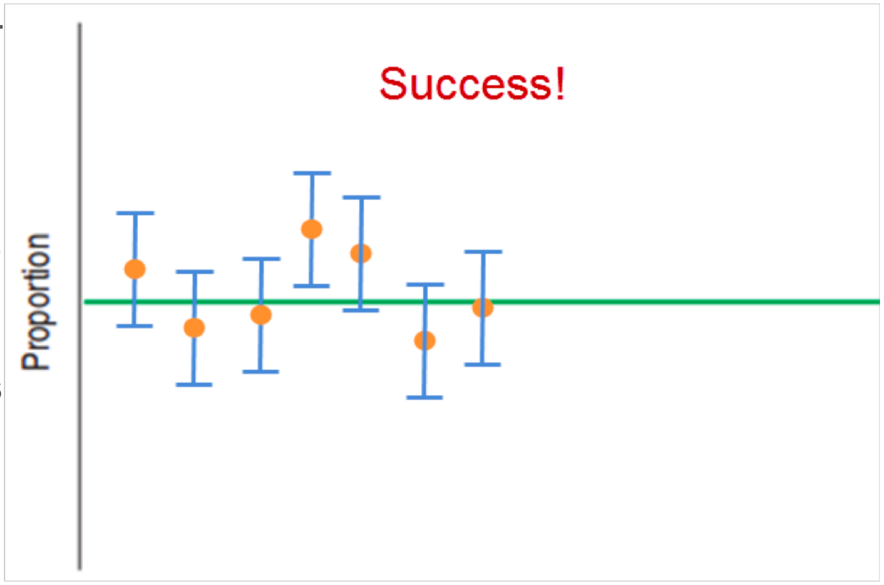
This confidence interval is a one-proportion z-interval.

- “One” since there is a single survey question.
- “Proportion” since we are interested in the proportion of Facebook users who update their status daily.
- “z-interval” since the distribution is approximately normal.

Capturing a Proportion

- The confidence interval may or may not contain the true population proportion.

- Consider repeating the study over an over again, each time with the same sample size.
  - Each time we would get a different $\hat{p}$
  - From each $\hat{p}$, a different confidence interval could be computed.
  - About 95% of these confidence intervals will capture the true proportion.
  - 5% will be duds.



Confidence Intervals

There are a huge number of confidence intervals that could be drawn.

- In theory, all the confidence intervals could be listed.
- 95% will “work” (capture the true proportion).
- 5% will be “duds” (not capture the true proportion).

What about our confidence interval (0.234, 0.382)?

- We will never know whether it captures the population proportion.

Confidence Interval on Global Warming

Yale and George Mason University interviewed 1010 US adults about beliefs and attitudes on global warming. They presented a 95% confidence interval on the proportion who think there is disagreement among scientists.

- Had the polling been done repeatedly, 95% of all random samples would yield confidence intervals that contain the true population proportion of all US adults who believe there is disagreement among scientists.

Facebook Status Updates

Technically Correct

- I am 95% confident that the interval from 23.4% to 38.2% captures the true proportion of Facebook users who update daily.

More Casual But Fine

- I am 95% confident that between 23.4% and 38.2% of Facebook users update daily.

Margin of Error

- Confidence interval for a population proportion:

$$\hat{p} = 2SE(\hat{p})$$

- The distance, $2SE(\hat{p})$, from $\hat{p}$ is called the margin of error.

- Confidence intervals also work for means, regression slopes, and others. In general, the confidence interval has the form:

$$Estimate \pm ME$$

Certainty vs. Precision

- Instead of a 95% confidence interval, any percent can be used.
- Increasing the confidence (e.g. 99%) increases the margin of error.
- Decreasing the confidence (e.g. 90%) decreases the margin of error.

Yale/George Mason Study

The poll of 1010 adults reported a margin of error of 3%. By convention, 95% with p = 0.5.

- How was the 3% computed?
  $SD(\hat{p}) = \sqrt{\frac{(0.5)(0.5)}{1010}} \approx 0.0157$
- For 95% confidence
  ME = 2(0.0157) = 0.031
- The margin of error is close to 3%.

Critical Values

- For a 95% confidence interval, the margin of error was 2SE.
  - The 2 comes from the normal curve.
  - 95% of the area is within about 2SE from the mean.
- In general the number of SE is called the critical value. Since we use the normal distribution here we denote it z^*.

Finding the Margin of Error (Take 2)

Yale/George Mason Poll: 1010 US adults, 40% think scientists disagree about global warming. At 95% confidence ME = 3%

- Find the margin of error at 90% confidence.

$$SD(\hat{p}) = \sqrt{\frac{(0.4)(0.6)}{1010}} \approx 0.0154$$

- For 90%, $z^* \approx 1.645$: ME = (1.645)(0.0154) = 0.025.
- This gives a smaller margin of error which is good.
- Drawback: lower level of confidence which is bad

Assumptions and Conditions

Independence and Sample Size

Independence Condition

- If data is collected using SRS or a randomized experiment → Randomization Condition
- Some data values do not influence others.
- Check for the 10% Condition: The sample size is less than 10% of the population size.

Success/Failure Condition

- There must be at least 10 successes.
- There must be at least 10 failures.

One-Proportion z-Interval

- First check for randomization, independence, 10%, and conditions on sample size.
- Confidence level C, sample size n, proportion $\hat{p}$.
- Confidence interval: $\hat{p} \pm z^* SE(\hat{p})$
- $SE(\hat{p}) = \sqrt{\frac{\hat{p}\hat{q}}{n}}$
- z*: the critical value that specifies the number of SE's needed for C% of random samples to yield confidence intervals that capture the population proportion.

Do You Believe the Death Penalty is Applied Fairly?

- Sample size: 510
- Answers:
  - 58% “Fairly”
  - 36% “Unfairly”
  - 7% “Don’t Know”
- Construct a confidence interval for the population proportion that would reply “Fairly.”

Do You Believe the Death Penalty is Applied Fairly?

Plan:

- Find a 95% confidence interval for the population proportion.

Model:

- Randomization: Randomly selected by Gallup Poll
- 10% Condition: Population is all Americans
- Success/Failure Condition
  $(510)(0.58) = 296 > 10$, $(510)(0.42) = 214 > 10$
- Use the Normal Model to find a one-proportion z-interval.

Do You Believe the Death Penalty is Applied Fairly?

Mechanics:

- n = 510
- $\hat{p} = 0.58$
- $SE(\hat{p}) = \sqrt{\frac{(0.58)(0.42)}{510}} \approx 0.022$
- z* ≈ 1.96
- ME ≈ (1.96)(0.022) ≈ 0.043
- The 95% Confidence Interval is: 0.58 ± 0.043 or (0.537, 0.623)

Conclusion:

- I am 95% confident that between 57.3% and 62.3% of all US adults think that the death penalty is applied fairly.

What Sample Size?

$$ME = z^* \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

- For 95%, $z^* = 1.96$
- Values that make ME largest are: $\hat{p} = 0.5, \hat{q} = 0.5$
- For example, to ensure a ME < 3%:

$$0.03 = 1.96\sqrt{\frac{(0.5)(0.5)}{n}}$$

- Solving for n, gives $n \approx 1067.1$.
- We need to survey at least 1068 to ensure a ME less than 0.03 for the 95% confidence interval.

The Yale/George Mason Survey and Sample Size

Poll: 40% believe scientists disagree on global warming.

- For a follow-up survey, what sample size is needed to obtain a 95% confidence interval with $ME \leq 2$?

$$ME = z^* \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

$$0.02 = 1.96 \sqrt{\frac{(0.4)(0.6)}{n}}$$

- n ≈ 2304.96
- The group will need at least 2305 respondents.

Credit Cards and Sample Size

A pilot study showed that 0.5% of credit card offers in the mail end up with the person signing up.

- To be within 0.1% of the true rate with 95% confidence, how big does the test mailing have to be?

$$ME = z^* \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

$$0.001 = 1.96\sqrt{\frac{(0.005)(0.995)}{n}}$$

- n ≈ 19,111.96
- The test mailing should include at least 19,112 offers.

Thoughts on Sample Size and ME

- Obtaining a large sample size can be expensive and/or take a long time.
- For a pilot study, ME = 10% can be acceptable.
- For full studies, ME < 5% is better.
- Public opinion polls typically use ME = 3%, $n = 1000$.
- If p is expected to be very small such as 0.005, then much smaller ME such as 0.1% is required.