Inferences About Means

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The Central Limit Theorem (CLT)

When a random sample is drawn from a population with mean m and standard deviation s, the sampling distribution has:

- Mean: μ
- Standard deviation: $\frac{\sigma}{\sqrt{n}}$
- Approximately Normal distribution as long as the sample size is large.
- The larger the distribution, the closer to Normal.

Weight of Angus Cows

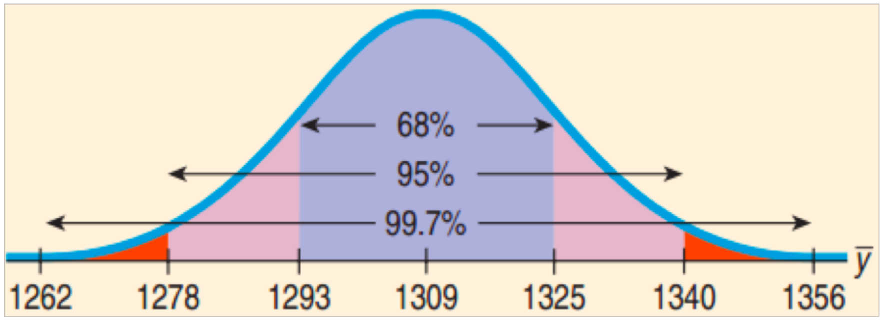
The weight of Angus cows is Normally distributed with $\mu = 1309$ pounds and $\sigma = 157 lbs$. What does the CLT say about the mean weight of a random sample of 100 Angus cows?

- Sample means $\bar{y}$ will average 1309 pounds.
- $SD(\bar{Y}) = \frac{\sigma}{\sqrt{n}} = \frac{157}{\sqrt{100}} = 15.7 lbs$

The CLT says the sampling distribution will be approximately Normal: N(1309, 15.7).

For the means of all random samples,

- 68% will be between 1293.3 and 1324.7 lb.
- 95% will be between 1277.6 and 1340.4 lb.
- 99.7% will be between 1261.9 and 1356.1 lb.

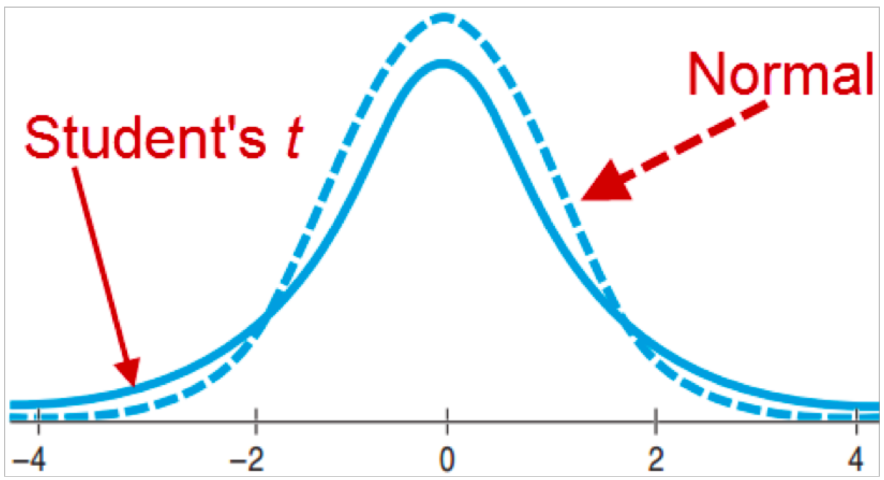


The Challenge of the CLT

- CLT tells us $SD(\bar{y}) = \frac{\sigma}{\sqrt{n}}$
- We would like to use this for Confidence Intervals and Hypothesis Testing.
- Unfortunately, we almost never know σ.
- Using s almost works: $SE(\bar{y}) = \frac{s}{\sqrt{n}}$, but not quite.
- When using s, the Normal model has some error.
- William Gosset came up with new models, one for each n that works better.

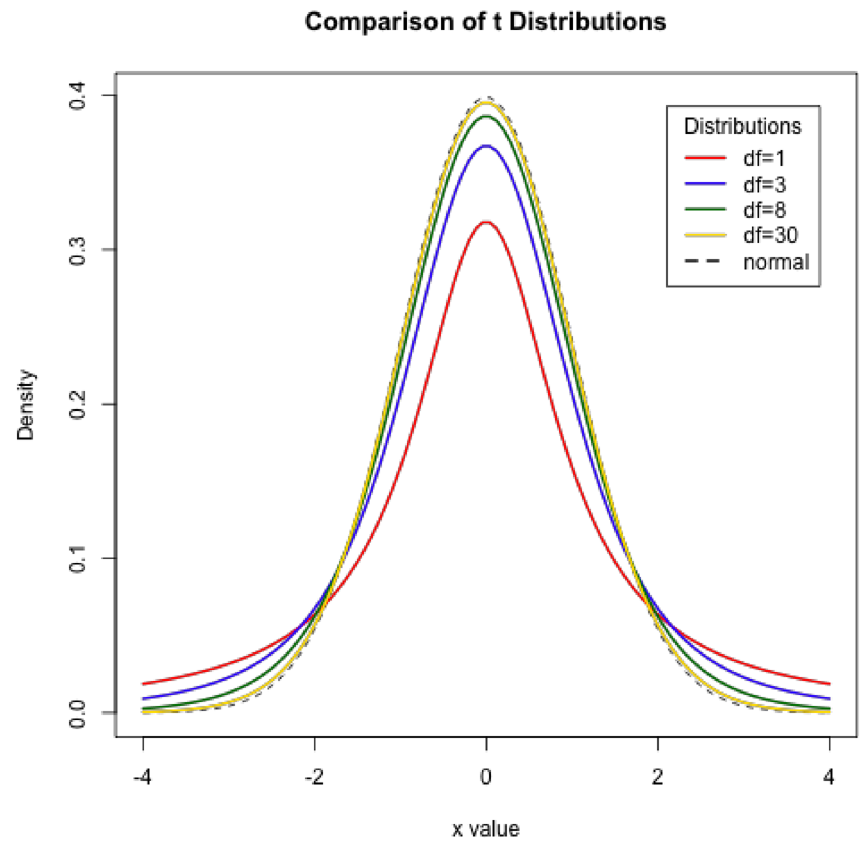
Gosset the Brewer

- At Guinness, Gosset experimented with beer.
- The Normal Model was not right, especially for small samples.
- Still bell shaped, but details differed, depending on n
- Came up with the “Student’s t Distribution” as the correct model



Degrees of Freedom

- For every sample size n there is a different Student's t distribution.
- Degrees of freedom: df = n – 1.
- Similar to the "n – 1" in the formula for sample standard deviation
- It is the number of independent quantities left after we've estimated the parameters.



Confidence Interval for Means

Sampling Distribution Model for Means

- With certain conditions (seen later), the standardized sample mean follows the Student's t model with n – 1 degrees of freedom.

$$t = \frac{\bar{y} - \mu}{SE(\bar{y})}$$

- We estimate the standard deviation with

$$SE(\bar{y}) = \frac{s}{\sqrt{n}}$$

One Sample t-Interval for the Mean

When the assumptions are met (seen later), the confidence interval for the mean is

$$\bar{y} = t^*_{n-1} \times SE(\bar{y})$$

The critical value t^*_{n-1} depends on the confidence level, C, and the degrees of freedom n – 1.

Contaminated Salmon

A study of mirex concentrations in salmon found

- n = 150, $\bar{y} = 0.0913$ ppm, s = 0.0495 ppm
- Find a 95% confidence interval for mirex concentrations in salmon.
- df = 150 – 1 = 149
- $SE(\bar{y}) = \frac{0.0485}{\sqrt{150}} \approx 0.0040$
- $t^*_{149} = 1.976$

```
qt(0.975, 149)
```

```
[1] 1.976
```

$$\bar{y} \pm t^*_{149} \times SE(\bar{y}) = 0.0913 \pm 1.976(0.0040)$$
$$\bar{y} \pm t^*_{149} \times SE(\bar{y}) = (0.0834, 0.0992)$$

- I'm 95% confident that the mean level of mirex concentration in farm-raised salmon is between 0.0834 and 0.0992 parts per million.

Notes about z and t

The Student's t distribution:

- Is unimodal.
- Is symmetric about its mean.
- Has higher tails than Normal.
- Is very close to Normal for large df.
- Is needed because we are using s as an estimate for s.

If you happen to know σ, which almost never happens, use the Normal model and not Student's t.

Assumptions and Conditions

Independence Condition

- Randomization Condition: The data should arise from a suitably randomized experiment.
- Sample size $<$ 10% of the population size.

Nearly Normal

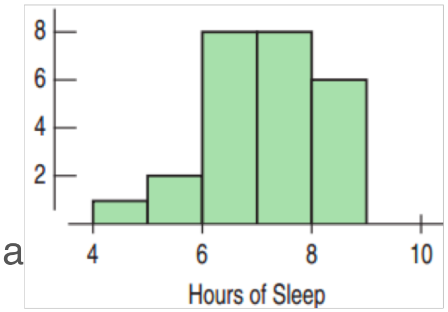
- For large sample sizes ($n > 40$), not severely skewed.
- ($15 \leq n \leq 40$): Need unimodal and symmetric.
- ($n < 15$): Need almost perfectly normal.
- Check with a histogram.

How Much Sleep do College Students Get?

Build a 90% Confidence Interval for the Mean.

Plan: Data on 25 Students

Model



- Randomization Condition
  The data are from a random survey.
- Nearly Normal Condition
  Unimodal and slightly skewed, so OK
- Use Student's t-Model with df = 25 – 1 = 24.
- One-sample t-interval for the mean

Mechanics

- n = 25; $\bar{y} = 6.64$; s = 1.075
- $SE(\bar{y}) = \frac{s}{\sqrt{n}} = \frac{1.075}{\sqrt{25}} = 0.215 hours$

```
qt(0.95, 24)
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```
[1] 1.711
```

- $t^*_{24} = 1.711$
- $ME = t^*_{24} \times SE\bar{y} = 1.711 \times 0.215$
  $= 0.368 hours$

90% $CI = 6.64 \pm 0.368$

How Much Sleep do College Students Get?

Conclusion: I'm 90 percent confident that the interval from 6.272 and 7.008 hours contains the true population mean number of hours that college students sleep.

What Not to Say

"90% of all students sleep between 6.272 and 7.008 hours each night."

- The CI is for the mean sleep, not individual students.

"We are 90% confident that a randomly selected student will sleep between 6.272 and 7.008 hours per night."

- We are 90% confident about the mean sleep, not an individual's sleep.

"The mean amount of sleep is 6.64 hours 90% of the time."

- The population mean never changes. Only sample means vary from sample to sample.

"90% of all samples will have a mean sleep between 6.272 and 7.008 hours per night."

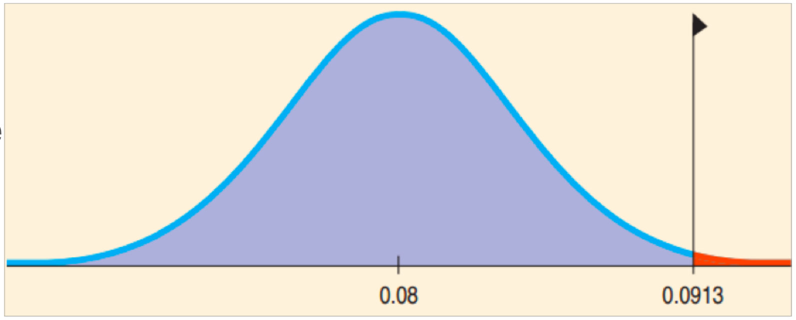
- This interval does not set the standard for all other intervals. This interval is no more likely to be correct than any other.

One-Sample t-Test for the Mean

- Assumptions are the same.
- $H_0 : \mu = \mu_0$
- $t^*_{n-1} = \frac{\bar{y} - \mu_0}{SE(\bar{y})}$
- Standard Error of $\bar{y}$: $SE(\bar{y}) = \frac{\sigma}{\sqrt{n}}$
- When the conditions are met and H_0 is true, the statistic follows the Student's t Model.
- Use this model to find the P-value.

Are the Salmon Unsafe?

EPA recommended mirex screening is 0.08 ppm.



- Are farmed salmon contaminated beyond the permitted EPA level?
- Recap: Sampled 150 salmon. Mean 0.0913 ppm, Standard Deviation 0.0495 ppm.
- $H_0 : \mu = 0.08$
- $H_A : \mu > 0.08$

Are the Salmon Unsafe?

One-Sample t-Test for the Mean

- n=150; df=149; $\bar{y} = 0.0913$; s=0.0495
- $SE(\bar{y}) = \frac{0.0495}{\sqrt{150}} \approx 0.0040$
- $t^*_{149} = \frac{0.0913-0.08}{0.0040} = 2.825$
- $P(t_149 > 2.825) = 0.0027$

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1 - pt(2.825, 149)
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[1] 0.002688
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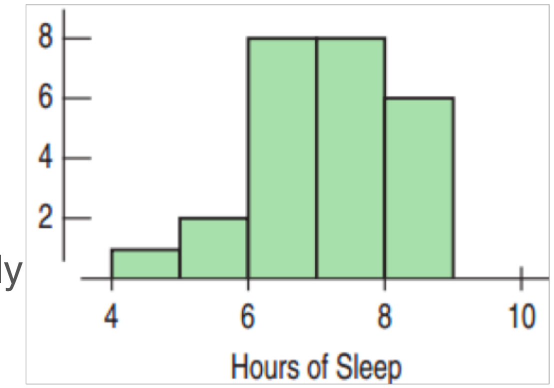
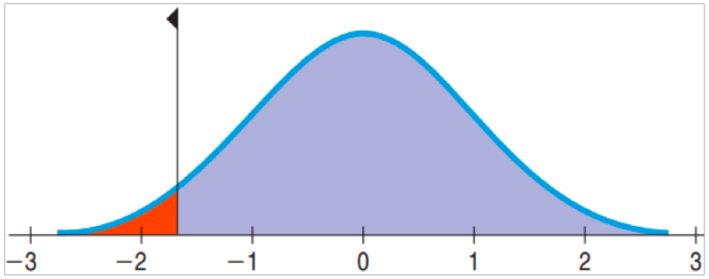
- Since the P-value is so low, reject H0 and conclude that the population mean mirex level does exceed the EPA screening value.

Do Students Get at Least Seven Hours of Sleep? 25 Surveyed.



- Plan: Does the mean amount of sleep exceed 7 hours?
- Hypotheses: $H_0 : \mu = 7; H_A : \mu > 7$
- Model
  - Randomization Condition: The students were randomly andindependently selected .
  - Nearly Normal Condition: unimodal and symmetric
- Use the Student's t-model, df = 24
- One-sample t-test for the mean

Do Students Get at Least Seven Hours of Sleep? 25 Surveyed.



Mechanics

- n = 25; $\bar{y} = 6.64$; s = 1.075
- $SE(\bar{y}) = \frac{s}{\sqrt{n}} = \frac{1.075}{\sqrt{25}} = 0.215 hours$
- $t = \frac{\bar{y}-\mu_0}{SE(\bar{y})} = \frac{6.64-7}{0.215} \approx -1.67$
- P-value= $P(t_{25} < -1.67) \approx 0.054$

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pt(-1.67, 24)
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[1] 0.05396
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Conclusion: P-value = 0.054 says thatif students do sleep an average of 7 hrs., samples of 25 students can be expected to have an observed mean of 6.64 hrs. or less about 54 times in 1000.

- With 0.05 cutoff, there is not quite enough evidence to conclude that the mean sleep is less than 7.
- The 90% CI: (6.272, 7.008) contains 7.
- Collecting a larger sample would reduce the ME and give us a better handle on the true mean hours of sleep.

Intervals and Tests

Confidence Intervals

- Start with data and find plausible values for the parameter.
- Always 2-sided

Hypothesis Tests

- Start with a proposed parameter value and then use the data to see if that value is not plausible.
- 2-sided test: Within the confidence interval means fail to reject H_0. P-value = 1 – C is the cutoff.
- 1-sided test: P-value = (1 – C)/2 is the cutoff.

Sleep, Confidence Intervals and Hypothesis Tests

90% Confidence interval: (6.272, 7.008)

- For a 2-tailed test with a 10% cutoff, any $6.272 \leq \mu_0 \leq 7.008$ would result in failing to reject H_0.
- For a 1-tailed test ("<") with a 5% cutoff, any $\mu_0 \leq 7.008$ would result in failing to reject H_0.
- For a 1-tailed test (">") with a 5% cutoff, any $6.272 \leq \mu_0$ would result in failing to reject H_0.

The Challenge of Finding the Sample Size

$$ME = t^*_{n-1} \times \frac{s}{\sqrt{n}}$$

To find the necessary sample size in order to have a small enough margin of error:

- Decide on acceptable ME.
- Determine s: Use a pilot to estimate s.
- Determine t^*_{n-1}: Use z* as an estimate. By the 68-95-99.7 Rule, use 2 for 95% confidence.