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ON THE FOLDING OF NUMERICALLY GENERATED GRIDS: USE OF A REFERENCE GRID

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SUMMARY

When variational grid generation techniques are used to produce grids suitable for solving numerical partial differential equations in irregular geometries, a reference grid can be used to determine the properties of the desired grid. Grid folding is a major problem in all methods of numerical grid generation; this paper studies the use of a reference grid, in variational smoothing methods, to prevent such foldings. The effects of various types of reference grid are compared and natural reference grids are shown to prevent foldings in the examples studied. The analysis is performed for a model problem with only one free point, and is then applied for finer grids. In addition, a short description of a discrete variational method is given. The reference grid concept applies to this discrete formulation.

INTRODUCTION

Grid folding is one of the main concerns in numerical grid generation. In this paper we analyse a variational grid generation method that uses a reference grid as a tool in preventing such folding. A reference grid is a region somewhat like the physical region where the grid is to be placed but which can be much simpler (see Figure 1). The idea is to define, on the reference grid, the properties of the grid which we want to transfer to the given physical object.[1] (It is worth noting that the reference grid is not limited to the determination of the grid properties in the interior of the geometric object, but also can be used to determine the grid properties on the boundary.[1])

In this paper we are considering two of the integrals from the variational methods introduced by Steinberg and Roache,[1] which are similar to the methods of Brackbill and Saltzman;[2] the two functionals presented provide (1) the measure of spacing between the grid lines (smoothness) and (2) the measure of the area of the grid cells. The minimization problem is usually solved by calculating the Euler–Lagrange (E–L) equations for the variational problem; the computer creates a grid by solving a centred finite difference approximation for the Euler–Lagrange equations.

Our study is motivated by the fact that the Winslow[3] grid generator (homogeneous TTM[4]) produces folded grids in certain electrode configurations.[5, 6] In these configurations, the regions studied were roughly like the region between two concentric ellipses; the grid was 100×15. It was demonstrated that the folding was not due to nonlinear effects or coding error. The folding is intrinsic to the Winslow[3] (or homogeneous TTM[4]) method when applied to certain geometries, for finite grid sizes. Although the theorem of Mastin proves that the grids will not fold in two dimensions[8] or three dimensions[9] in the continuum limit, the results of Roache and Steinberg[5] conclusively and succinctly demonstrated that the theorem does not apply for finite mesh increments. The difficulty manifested itself even in the simplest possible grids and the results for the simple grids proved relevant to more complex and finer resolution cases (see References 5, 9 and 10). When only 3×3 grids are considered, as the case for the present analyses, the region degenerates to the area between two triangles, as in Figure 3.

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A straightforward discretization and minimization of the two integrals provides an alternative to solving the E–L equations. In theory, direct discretization and minimization should provide the same solution; there are, instead, several differences in these methods.[11] In the straightforward discretization approach, the derivatives are replaced by centred finite differences and the integrals are replaced by summations over the grid points. However, only first derivatives of the co-ordinates x and y appear on the direct minimization problem. A centred finite difference discretization at the grid point (x_{ij},y_{ij}) does not involve the values of x_{ij} and y_{ij}. This produces strong decoupling problems for the direct approach that are not seen in the E–L approach.[12]

In the E–L formulation,[1] there are certain integral constraints that are satisfied. In the straightforward direct formulation, the analogues of these constraints are not automatically satisfied.[11] The straightforward discretization approach transforms the smoothness integral into a linear minimization problem with a linear constraint, while the area integral is transformed into a nonlinear minimization problem with a nonlinear constraint.[9, 11] Such problems are much harder to solve than the unconstrained problems that occur in the continuous cases. A better analogue formulation of the variational grid generation method is obtained when the properties to be controlled are considered direclty.[9, 11] There have been efforts in generating grids by the optimization of direct properties (see Reference 12); however, the characterization of the method presented here, as well as its properties and the minimization procedure used, differs considerably from theirs.

VARIATIONAL METHODS

In m-dimensions, from the Steinberg and Roache formulations,[1] the two integrals chosen to be minimized are:

for smoothness

$$I_s = \int_B \sum_{j=1}^{m} \left[\frac{\left\| \frac{\partial \mathbf{v}}{\partial \boldsymbol{\nu}_j} \right\|}{\left\| \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\nu}_j} \right\|} \right]^2 |\mathrm{d}\boldsymbol{\nu}| \tag{1}$$

with the constraint

$$C_s = \int_B \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial v_i}{\partial \nu_j} |\partial \boldsymbol{\nu}| = \text{constant}, \tag{2}$$

for area (volume)

$$I_v = \int_B \frac{J^2 \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\nu} \end{bmatrix}}{J \begin{bmatrix} \boldsymbol{\tau} \\ \boldsymbol{\nu} \end{bmatrix}} |\mathrm{d}\boldsymbol{\nu}| \tag{3}$$

with the constraint

$$C_v = \int_B J \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\nu} \end{bmatrix} |\mathrm{d}\boldsymbol{\nu}| = \text{constant}. \tag{4}$$

Here, $\mathbf{v} = (v_1, v_2, \ldots, v_m)$, $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_m)$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_m)$, $\mathbf{v} = \mathbf{v}(\boldsymbol{\nu})$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{\nu})$, B is a 'box' in $\boldsymbol{\nu}$ space. Also, $\boldsymbol{\tau}$ is a given map from B to the reference region. $\mathbf{u}$ maps B on to the physical region; it is given on the boundary and must be calculated in the interior of the region (see Figure 1). It has been shown that for this method the constraints are automatically satisfied.[1]

The notation used follows Reference 11, (see Figure 1). In the case $m = 2$ (see Figure 1), we set

$$\mathbf{v} = (x,y), \quad \boldsymbol{\nu} = (\xi, \eta), \quad \boldsymbol{\tau} = (\alpha,\beta).$$

The grid is generated in physical space which is labelled with the variable (x,y). The computations are done in logical space which is labelled with the variables (ξ,η). The grid properties are defined in the reference space which is labelled with the variables (α,β). The region in logical space is

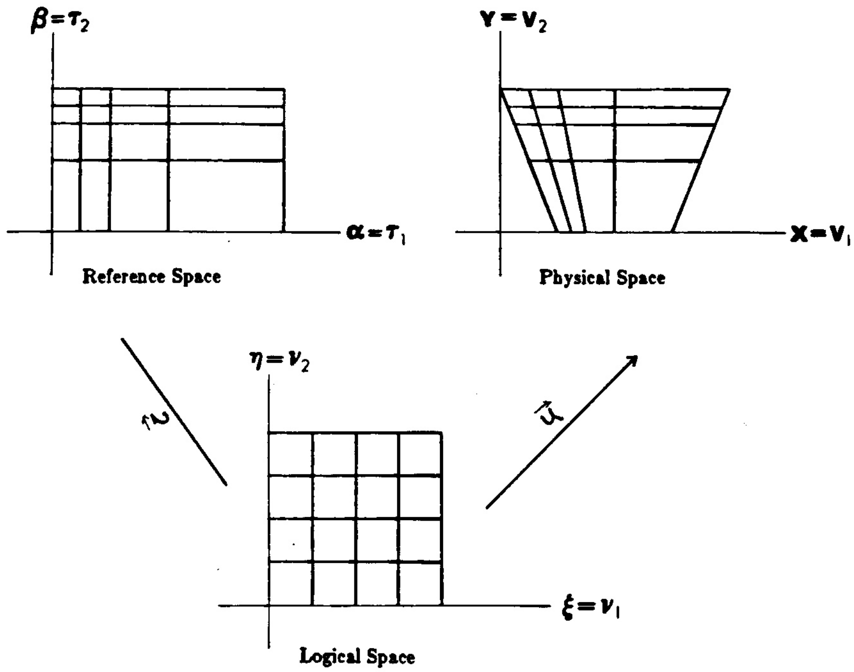

Figure 1

chosen as a unit square B. The minimization problem will define a mapping from $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ from logical space to physical space.

Then[9, 13] the Euler–Lagrange equations for the smoothness are

$$\begin{aligned} \frac{1}{A}x_{\xi\xi} + \frac{1}{B}x_{\eta\eta} &= \frac{x_\xi A_\xi}{A^2} + \frac{x_\eta B_\eta}{B^2}, \\ \frac{1}{A}y_{\xi\xi} + \frac{1}{B}y_{\eta\eta} &= \frac{y_\xi A_\xi}{A^2} + \frac{y_\eta B_\eta}{B^2}, \end{aligned} \tag{5}$$

where

$$A = (\alpha_\xi^2 + \beta_\xi^2)^{1/2}, \quad B = (\alpha_\eta^2 + \beta_\eta^2)^{1/2},$$

and

$$A_\xi = \frac{\alpha_\xi \alpha_{\xi\xi} + \beta_\xi \beta_{\xi\xi}}{A}, \quad B_\eta = \frac{\alpha_\eta \alpha_{\eta\eta} + \beta_\eta \beta_{\eta\eta}}{B}$$

The Euler–Lagrange equations for the volume are[9, 13]

$$\begin{aligned} b_{v1}x_{\xi\xi} + b_{v2}x_{\xi\eta} + b_{v3}x_{\eta\eta} + a_{v1}y_{\xi\xi} + a_{v2}y_{\xi\eta} + a_{v3}y_{\eta\eta} &= \frac{((J_R)_\eta y_\xi - (J_R)_\xi y_\eta)J}{J_R}, \\ a_{v1}x_{\xi\xi} + a_{v2}x_{\xi\eta} + a_{v3}x_{\eta\eta} + c_{v1}y_{\xi\xi} + c_{v2}y_{\xi\eta} + c_{v3}y_{\eta\eta} &= \frac{((J_R)_\xi x_\eta - (J_R)_\eta x_\xi)J}{J_R} \end{aligned} \tag{6}$$

where

$$\begin{aligned} a_{v1} &= -x_\eta y_\eta, & b_{v1} &= y_\eta^2, & c_{v1} &= x_\eta^2, \\ a_{v2} &= x_\xi y_\eta + x_\eta y_\xi, & b_{v2} &= -2y_\xi y_\eta, & c_{v2} &= -2x_\xi x_\eta, \\ a_{v3} &= -x_\xi y_\xi, & b_{v3} &= y_\xi^2, & c_{v3} &= x_\xi^2; \end{aligned}$$

where J_R is the Jacobian of the reference mapping τ and J is the Jacobian of the mapping we want to construct (see Figure 1).

Note that when the reference grid is a unit square then E–L are simplified. In this case the smoothness equations correspond to those of Hirt[14] and not those of Winslow[3] (or homogeneous TTM[4]). As is well known, the Hirt smoother folds grids more easily than the Winslow (homogeneous TTM) smoother, so preventing folding for the Hirt smoother is more difficult than for the Winslow smoother. Here, for our model problem, we will solve the more difficult problem.

THE REAL PROBLEM

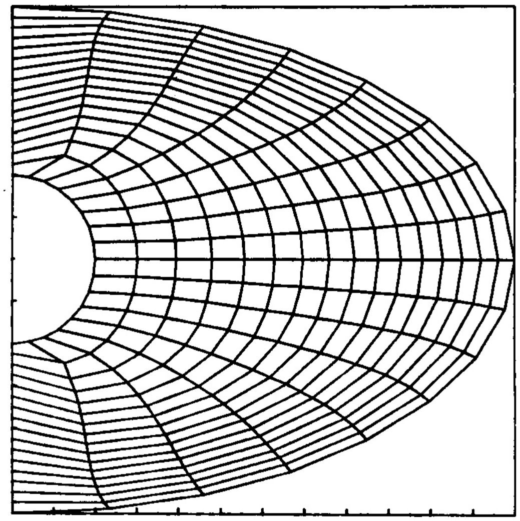

Figure 2

THE MODEL PROBLEM

The real problem to be studied is the generation of a grid between two ellipses (see Figure 2). In general, such problems are too complex to solve analytically. To gain some understanding, the simplest of these problems will be studied. A main point of this paper is that such simple problems are relevant to real problems. In the simplest problems there is only one free point in the grid, that is, the grid is 3×3 (when the boundary points are counted) (see Figure 3). In this situation the region between two ellipses reduces to the area between two triangles. Two parameters, d and e, are used to describe the physical region; this leads to some complexity but, as will be demonstrated below, the analysis shows that folding does not depend on a single parameter such as eccentricity of the ellipses. There are nine points in this grid; they are labelled by the numbers 0 through 8 (see Table I). These labels indicate which points are mapped into each other by the transformation from logical to physical space (again, see Figure 1).

Because this grid contains only one free point, the centred-finite-difference approximation to the Euler–Lagrange equations for any variational problems that determines such a grid will contain two unknowns, the co-ordinates (x_{11}, y_{11}) of this point. However, the physical region is symmetric about the line $y = 3$, so $y_{11} = 3$. We are left only with the determination of x_{11} for various problems.

FOLDING ANALYSIS

Previous studies in the variational methods introduced by Steinberg and Roache[1] showed that the smoothness integral folds the model problem and the volume integral always produces unfolded grids for the model problem.[10] This analysis was performed on the discretization (centred finite differences) of the Euler–Lagrange equations (5) and (6) for the two integrals (1) and (3), using the unit square as a reference grid; note that in this case the E–L equations are simplified. In the 3×3 case (see Figure 3) symmetry determines the y-co-ordinate of the free point, hence the x-co-ordinate is the only variable to be determined. For the smoothness integral the x-co-ordinate of the free point is

$$x_{11} = (d + e)/4.$$

Note that (see Figure 3)

$$x_{11} = (d + e)/2$$

THE MODEL PROBLEM

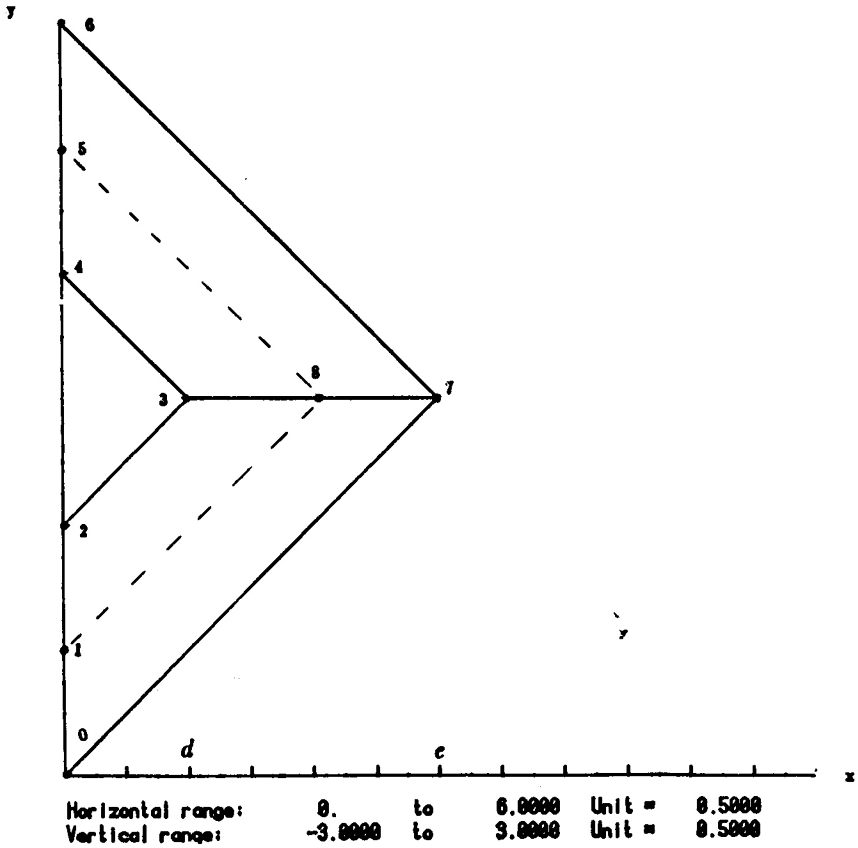

Figure 3

Table I

Number	Logical space	Physical space
	(ξ,η)	(x,y)
0	(0,0)	(0,0)
1	(0,1)	(0,1)
2	(0,2)	(0,2)
3	(1,2)	$(\alpha,3)$
4	(2,2)	(0,4)
5	(2,1)	(0,5)
6	(2,0)	(0,6)
7	(1,0)	$(\beta,3)$
8	(1,1)	(x_{11},y_{11})

places the free point at the midpoint of the segment joining point 3 to point 7, so that the smoothness always places the free point to the left of the midpoint. In order to get an unfolded grid, the following condition on the free point must be satisfied:

$$d < x_{11} < e.$$

This gives a bound on d with respect to e for the grid being unfolded,

$$d < e/3.$$

If a reference grid that is a general rectangle of width p and height q (see Figure 1) is used then, using centre finite differences on the E–L equations (5), the x_{11} co-ordinate of the free point is

$$x_{11} = \frac{p}{p+q}\left(\frac{d+e}{2}\right)$$

which clearly indicates the effect of the reference grid. This value depends on p and q, the width and height of the reference grid. We will see that an appropriate rectangle can be chosen in order to prevent the folding of the grid. In order to keep the free point between d and e, we need

$$d < \frac{p}{p+q}\left(\frac{d+e}{2}\right) < e.$$

Because p and q can be chosen so that $p/(p + 2q)$ is close to 1 and we know that $d < e$, it is possible, for q fixed, to choose $p > (2qd)/(e - d)$ and thus obtain an unfolded grid. It is important to notice that the free point cannot be placed at the centre of the segment determined by d and e since the co-ordinate of the free point is always to the left of the midpoint of this segment.

We now illustrate the effects of the reference grid in some simple problems. When the reference grid is very simple (a unit square) the grids can be folded, but in these examples, when the reference region is more like the physical region, then the grid is unfolded. The reference grids have height $q = 1$. A sequence of pictures (Figures 4(a) to 4(c) shows the effect of the reference grid in a 3 × 3 case. In Figure 4(a) the reference grid is the unit square, $p = 1$, in Figure 4(b) $p = 2$, and in Figure 4(c) $p = 4$. Note that the region in the model problem is approximately 1 × 4 units (roughly the distance between the midpoints of the sides). As the reference grid goes from a unit square to a long rectangle that is 1 × 4 units, the grid goes from folded to a more reasonable grid. The qualitative behaviour of the 3 × 3 grid applies to finer grid-resolution; a 5 × 5 grid is shown (Figures 5(a) to 5(d) to illustrate this. The physical region in the problem is approximately 2 × 12 units (using the difference of the minor axis and a distance slightly shorter that the circumference of the outer ellipse). In Figures 5(a) to 5(d), p take the values 1, 3, 6, 12.

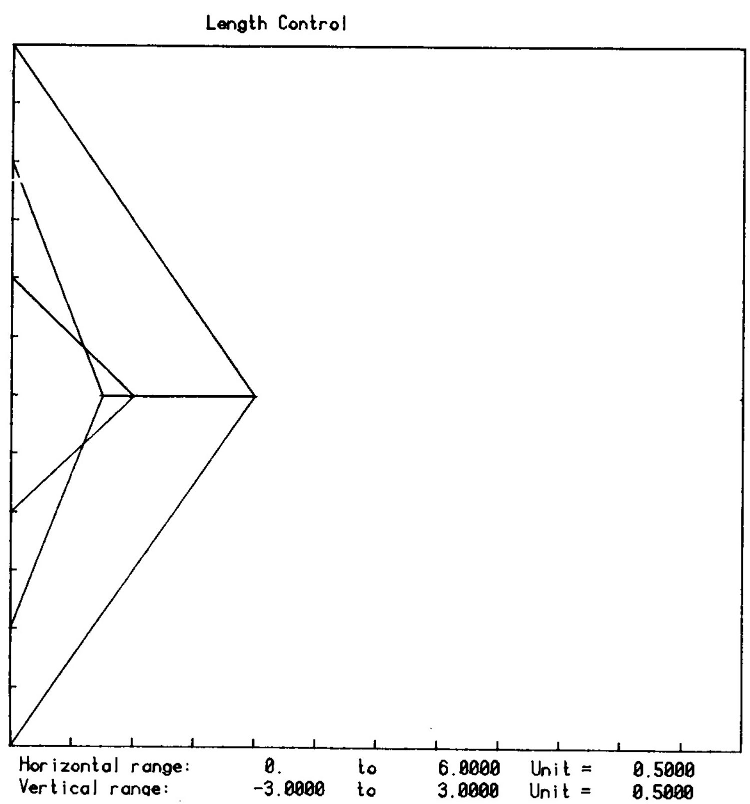

Figure 4(a). Reference grid (unit square)

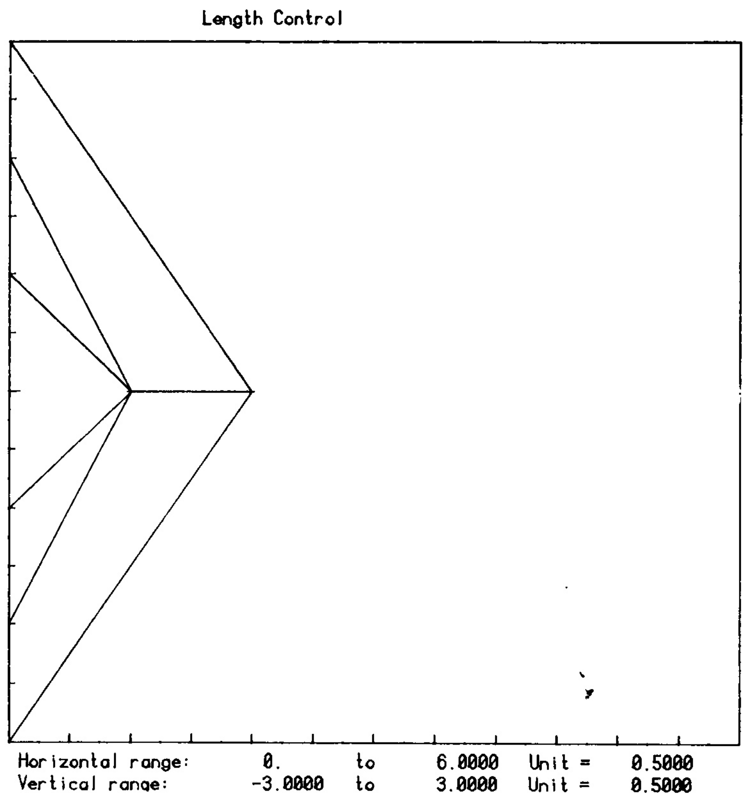

Figure 4(b). Reference grid (rectangle with height 1 and width 2)

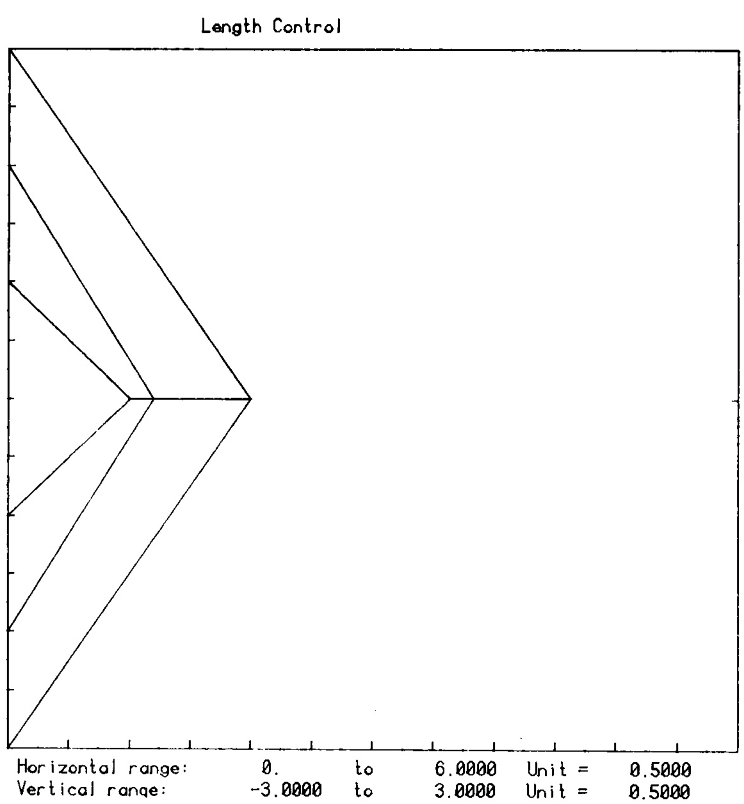

Figure 4(c). Reference grid (rectangle with height 1 and width 4)

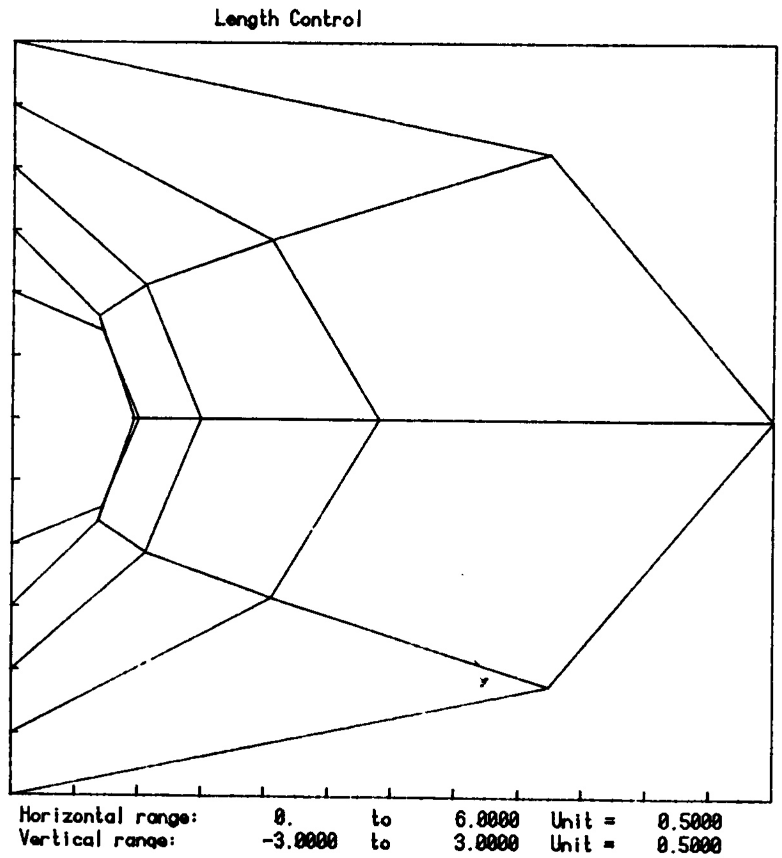

Figure 5(a). Reference grid (unit square)

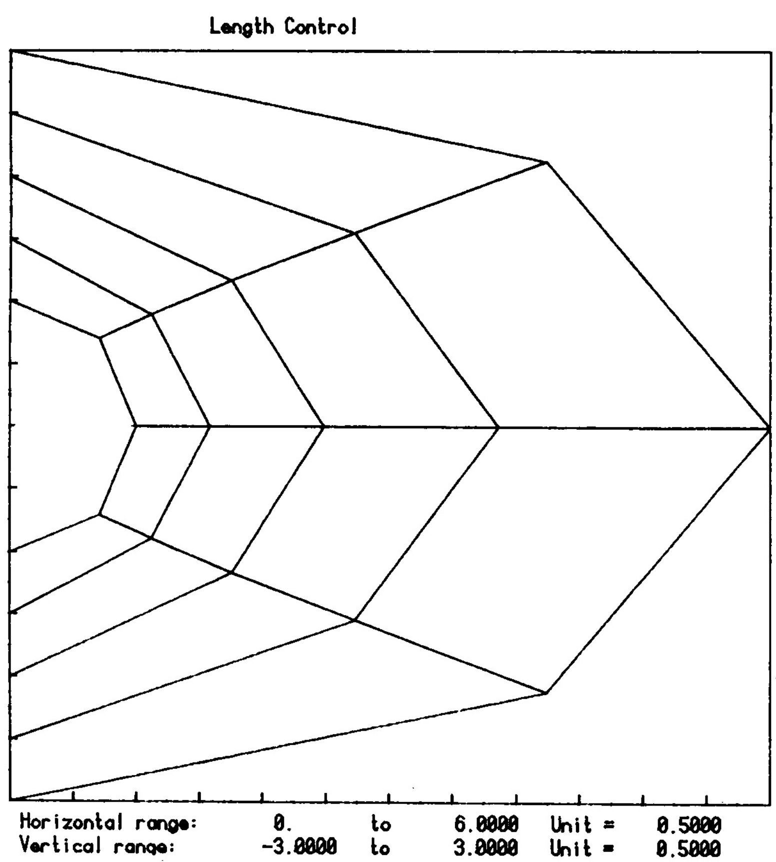

Figure 5(b). Reference grid (rectangle with height 1 and width 3)

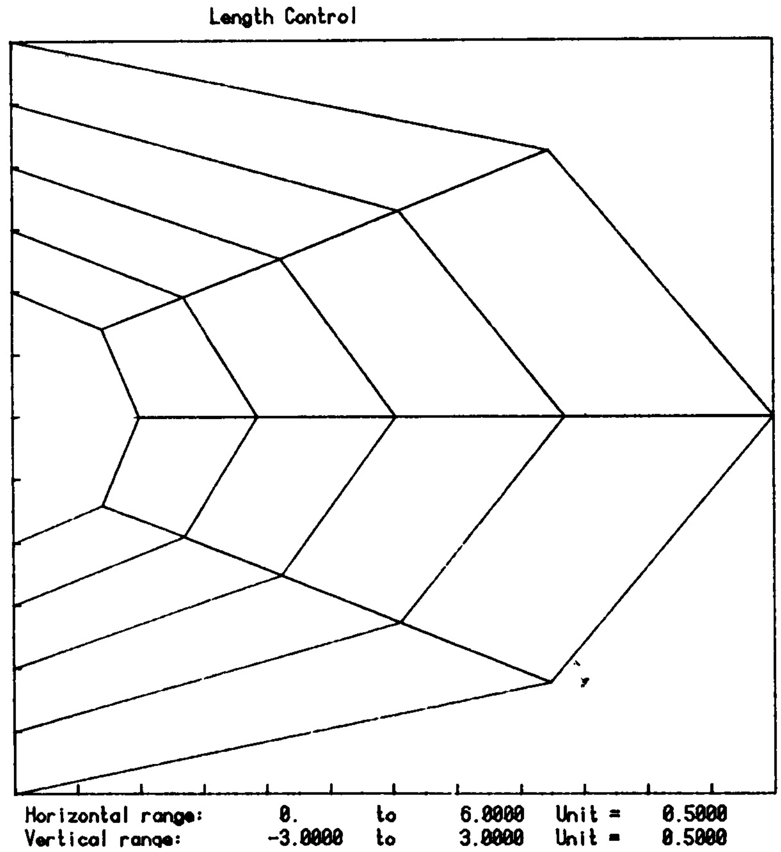

Figure 5(c). Reference grid (rectangle with height 1 and width 6)

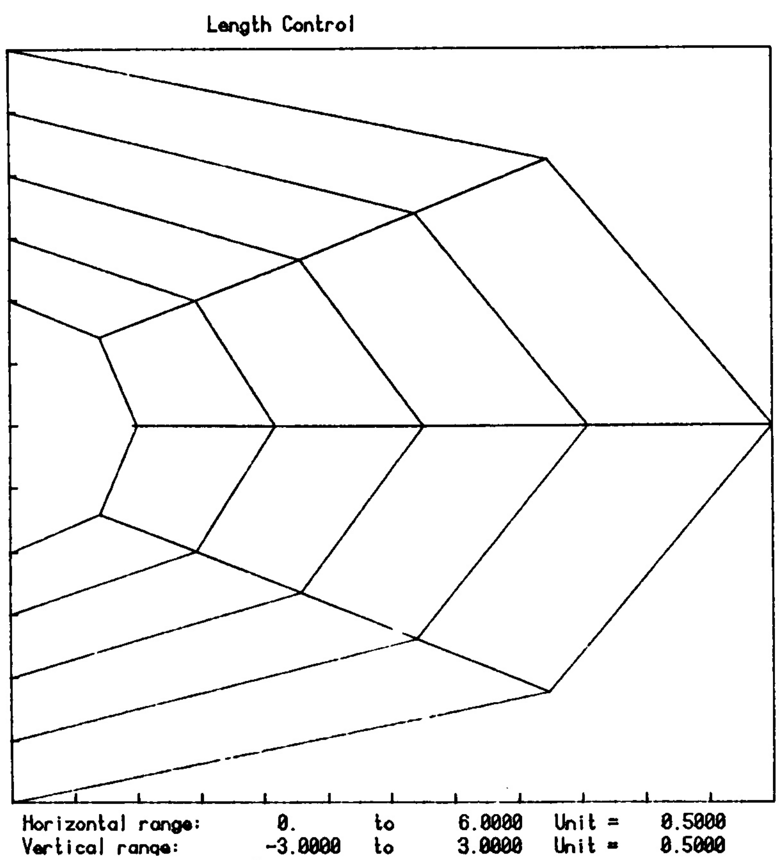

Figure 5(d). Reference grid (rectangle with height 1 and width 12)

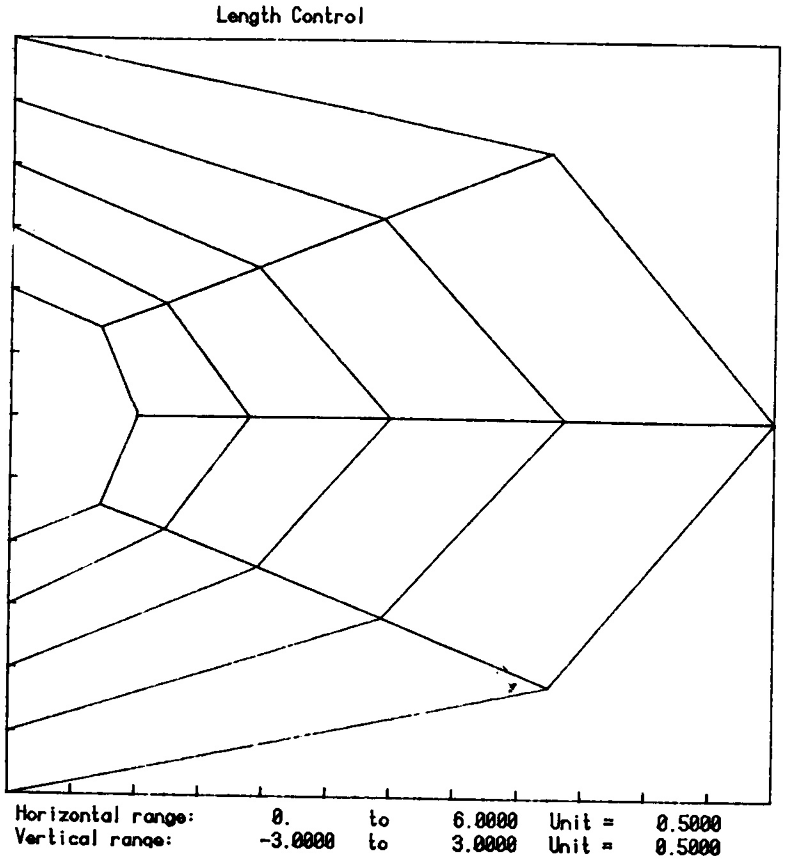

Figure 6. Reference grid (trapezoid with height 1 and width 12)

Again, as p increases from 1 to 6 the grid goes from a folded to a usable grid. Interestingly, $p = 12$ gives some improvement and this is the limit of what can be done using only length control.

A reference grid that is a trapezoid can also be used. It is a better match for the model problem and it is still an easy reference grid to work with. An example in Figure 6 shows the effects of a trapezoidal reference grid.

In the case of the area control using reference grids whose cells have constant area, the reference grid does not play any role so it will produce the same grids as in [4]. When a reference grid with different cell areas is used, its properties can be transferred to the physical object;[13] this can be accomplished by a simple stretching applied on the reference grid.

DISCRETE VARIATIONAL METHOD

A discrete variational formulation of the grid generation problem is obtained if the properties to be controlled are considered directly.[9, 11] To have the grid points evenly spaced, the sum of the squares of the segments between the grid lines should be minimized:

$$\text{minimize } F_S = \sum \delta_{ij}^2 \tag{7}$$

with the constraint

$$C_S = \sum \delta_{ij} = \text{constant}. \tag{8}$$

For controlling the area, the sum of the squares of the true discrete area of the quadrilateral cell should be minimized:

$$\text{minimize } F_A = \sum A_{ij}^2 \tag{9}$$

with the constraint

$$C_A = \sum A_{ij} = \text{constant}. \tag{10}$$

It is worth noting that in this direct variational approach for spacing between the grid lines, the constraint (8) is automatically satisfied since the sum of all the segments is a telescopic sum which depends only on the values of the boundary. For the area sum, the constraint (10) is the sum of the areas of the true quadrilateral cells which is shown to be the total area of the region, and this solely depends on the values of the boundary. Therefore, this discrete direct variational formulation is an analogue of the continuous variational formulation introduced by Steinberg and Roache.[1]

To control the two properties, a weighted combination of both sums is to be minimized:

$$F = \sigma F_S + (1 - \sigma) F_A, \quad 0 \leq \sigma \leq 1. \tag{11}$$

The full analysis,[9] the details of which may be omitted here, produce identical results as in the previous section for the same 3×3 problem, although the behaviour for finer resolution differs.

CONCLUSIONS

Variational grid generation methods that use a reference grid concept produce grids suitable for solving numerical partial differential equations. The reference grid is an effective tool for controlling grid properties; it plays an important role in obtaining a more refined control over the grid properties. The discrete method provides an alternative to solving the E–L equations. The reference grid concept is naturally applicable to the discrete formulation.

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