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Mathematical aspects of variational grid generation II

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Abstract: Variational grid generation techniques are now used to produce grids suitable for solving numerical partial differential equations in irregular geometries. In this paper the existence and uniqueness of solutions of the volume and smoothness problems that are used in variational grid generation are studied. An analysis of the Euler–Lagrange (EL) equations near the identity shows that the volume problem is difficult. These variational problems use a reference grid to specify the properties of the desired grid. Replication of reference grid properties is analyzed. Examples are given that show the effectiveness of the reference grid concept.

Keywords: Variational grid generation, Euler–Lagrange equations, reference grid.

Introduction

In the variational methods introduced by Steinberg and Roache [7] which are based on those of Brackbill and Saltzman [1], two functionals are presented that provide (i) the measure of spacing between the grid lines (smoothness) and (ii) the measure of the area of the grid cells. The minimization problem is usually solved by calculating the Euler–Lagrange (EL) equation for the variational problem. The computer creates a grid by solving a central finite differences approximation of the EL equations.

The code used in this paper is based on the ideas presented in [7] on surface grid generation, where the code generates a grid on the boundary (curve) and a grid on the interior; the surface (or region) must be parameterized. We would like to exercise a more refined control over the grid. There are many ways to do this, but the following is convenient. We imagine that the physical region is rather complex. We also assume that we have another region, called the reference region, that is somewhat like the physical region but usually considerable simpler, and that we can make a grid on the reference region that contains the essential features of the grid that we want to put on the physical object. The reference grid is thought of as being in reference space. Note that the reference space must have the same dimension as the physical object. It is also important to notice that the reference grid not only specifies the grid properties in the

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interior of the geometric object, but also determines the grid properties on the boundary of the geometric object [7].

In m-dimensions the integrals to be minimized are: for smoothness

$$I_s = \int_B \sum_{j=1}^{m} \left[\left\| \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}_j} \right\|^2 \Big/ \left\| \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\nu}_j} \right\| \right] |\mathrm{d}\boldsymbol{\nu}| \tag{1}$$

with

$$C_s = \int_B \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial v_i}{\partial \nu_j} |\partial \boldsymbol{\nu}| = \text{constant}, \tag{2}$$

for volume (area)

$$I_v = \int_B J^2 \left[\begin{smallmatrix} \boldsymbol{v} \\ \boldsymbol{\nu} \end{smallmatrix} \right] / J \left[\begin{smallmatrix} \boldsymbol{\tau} \\ \boldsymbol{\nu} \end{smallmatrix} \right] |\mathrm{d}\boldsymbol{\nu}| \tag{3}$$

with

$$C_v = \int_B J \left[\begin{smallmatrix} \boldsymbol{v} \\ \boldsymbol{\nu} \end{smallmatrix} \right] |\mathrm{d}\boldsymbol{\nu}| = \text{constant}. \tag{4}$$

Here $\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$, $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_m)$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_m)$, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\nu})$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{\nu})$, B is a 'box' in $\boldsymbol{\nu}$ space. Also, $\boldsymbol{\tau}$ is a given map from B to the reference region and $\boldsymbol{u}$ maps B onto the physical region; it is given on the boundary and must be calculated in the interior of the region (see Fig. 1). In this case the constraints are automatically satisfied [7].

1. Euler–Lagrange equations in 2-D

In the case $m = 2$ (see Fig. 1), we set

$$\boldsymbol{u} = (x, y), \qquad \boldsymbol{\nu} = (\xi, n), \qquad \boldsymbol{\tau} = (\alpha, \beta).$$

Hence, the integral to be minimized for smoothness is:

$$I_s = \int_B \frac{x_\xi^2 + y_\xi^2 + x_\nu^2 + y_\nu^2}{\left(\alpha_\xi^2 + \beta_\eta^2 + \alpha_\eta^2 + \beta_\xi^2\right)^{1/2}} \mathrm{d}\xi \, \mathrm{d}\eta,$$

and for volume

$$I_v = \int_B \frac{\left(x_\xi y_\eta - x_\eta y_\xi\right)^2}{\left(\alpha_\xi \beta_\eta - \alpha_\eta \beta_\xi\right)} \mathrm{d}\xi \, \mathrm{d}\eta,$$

and then [7] the EL equations for the smoothness are:

$$\frac{1}{A} x_{\xi\xi} + \frac{1}{B} x_{\eta\eta} = \frac{x_\xi A_\xi}{A^2} + \frac{x_\eta B_\eta}{B^2}, \qquad \frac{1}{A} y_{\xi\xi} + \frac{1}{B} y_{\eta\eta} = \frac{y_\xi A_\xi}{A^2} + \frac{y_\eta B_\eta}{B^2}, \tag{5}$$

where

$$A = \left(\alpha_\xi^2 + \beta_\xi^2\right)^{1/2}, \qquad B = \left(\alpha_\eta^2 + \beta_\eta^2\right)^{1/2},$$

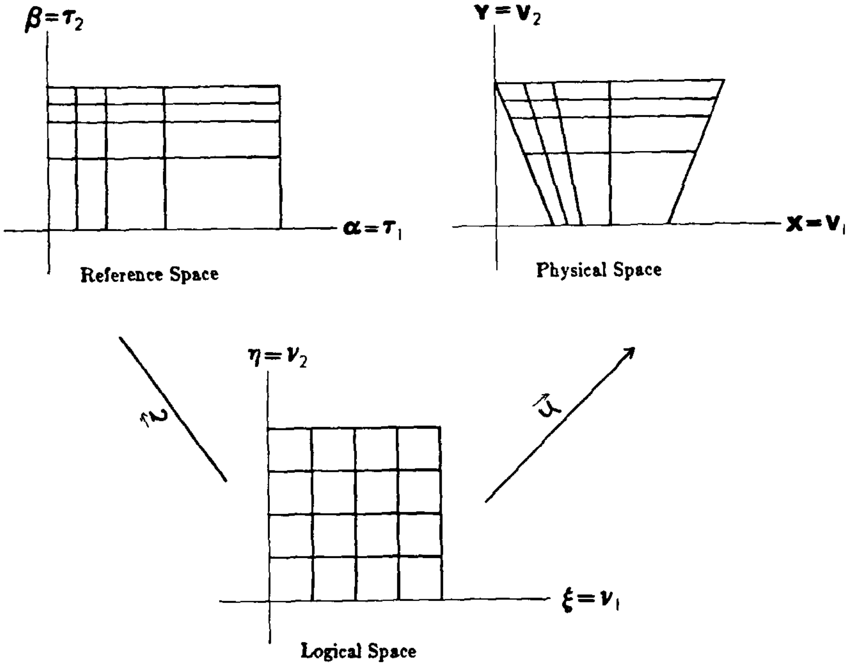


Fig. 1.

and

$$A_\xi = \frac{\alpha_\xi \alpha_{\xi\xi} + \beta_\xi \beta_{\xi\xi}}{A}, \qquad B_\eta = \frac{\alpha_\eta \alpha_{\eta\eta} + \beta_\eta \beta_{\eta\eta}}{B}.$$

The EL equations for the volume are [7]

$$\begin{aligned} &J\left[(J_{\mathrm{R}})_\xi y_\eta - (J_{\mathrm{R}})_\eta y_\xi\right] + (J_{\mathrm{R}})\left[y_\xi J_\eta - y_\eta J_\xi\right] = 0, \\ &J\left[-(J_{\mathrm{R}})_\xi x_\eta + (J_{\mathrm{R}})_\eta x_\xi\right] + (J_{\mathrm{R}})\left[x_\eta J_\xi - x_\xi J_\eta\right] = 0, \end{aligned} \tag{6}$$

where J_{R} is the Jacobian of the reference mapping τ and J is the Jacobian of the mapping we want to construct (see Fig. 1). For the smoothness problem, A and B are fixed and positive if the reference map is proper. The EL equations are linear, elliptic and uncoupled. Note also that this is true for m dimensions as well as it is for 2 dimensions.

The EL equations for the volume problem can also be written in the following way:

$$\begin{aligned} &b_{v1} x_{\xi\xi} + b_{v2} x_{\xi\eta} + b_{v3} x_{\eta\eta} + a_{v1} y_{\xi\xi} + a_{v2} y_{\xi\eta} + a_{v3} y_{\eta\eta} = \left((J_{\mathrm{R}})_\eta y_\xi - (J_{\mathrm{R}})_\xi y_\eta\right) J/J_{\mathrm{R}}, \\ &a_{v1} x_{\xi\xi} + a_{v2} x_{\xi\eta} + a_{v3} x_{\eta\eta} + c_{v1} y_{\xi\xi} + c_{v2} y_{\xi\eta} + c_{v3} y_{\eta\eta} = \left((J_{\mathrm{R}})_\xi x_\eta - (J_{\mathrm{R}})_\eta x_\xi\right) J/J_{\mathrm{R}}, \end{aligned}$$

where

$$\begin{aligned} &a_{v1} = -x_\eta y_\eta, && b_{v1} = y_\eta^2, && c_{v1} = x_\eta^2, \\ &a_{v2} = x_\xi y_\eta + x_\eta y_\xi, && b_{v2} = -2 y_\xi y_\eta, && c_{v2} = -2 x_\xi x_\eta, \\ &a_{v3} = -x_\xi y_\xi, && b_{v3} = y_\xi^2, && c_{v3} = x_\xi^2. \end{aligned}$$

For the case when there is no reference grid, the righthand side of the EL equations is zero; in this case the equations are the same as the ones presented by Brackbill and Saltzman [1]. It will be shown that these equations are nonlinear, not elliptic, and coupled. To see this situation more clearly, we now do an analysis for the simplest form of these equations.

1.1. Near identity analysis in 2-D

In order to study solutions of the EL equations that are nearly identity maps, $x=\xi$, $y=\eta$, the reference map is chosen to be the identity. To do a near identity analysis we view the EL equations as quasi-linear partial differential equations of the form

$$Af_{\xi\xi}+Bf_{\xi\eta}+Cf_{\eta\eta}=D,$$

where A, B, C, D depend on f, f_ξ, f_η. Here A, B, C, D are made constant by choosing fixed f, f_ξ, f_η.

To study near identity maps, set $x=\xi$, $y=\eta$; so $x_\xi=1$, $x_\eta=0$, $y_\xi=0$ and $y_\eta=1$. The EL equations for the smoothness integral become

$$x_{\xi\xi}+x_{\eta\eta}=0, \qquad y_{\xi\xi}+y_{\eta\eta}=0, \tag{7}$$

which is an uncoupled elliptic system of PDEs. However, in the case of the volume integral we get a degenerate system,

$$x_{\xi\xi}+y_{\xi\eta}=0, \qquad x_{\xi\eta}+y_{\eta\eta}=0, \tag{8}$$

which can be easily checked to be not elliptic.

Based on this, we should expect difficulties with the codes that are used for solving these problems, since they are elliptic solvers. However, experience shows the opposite: [4], [5] and [6]. This may seen surprising; however, the programs that are used solve an approximation of the EL equations so they do not deal with the degenerate problem.

1.2. Replication of reference grid properties in 2-D

One of the questions to be asked with respect to the usefulness of the reference grid concept is the possibility of being able to reproduce any reference grid on the physical object. We do not expect an arbitrary reference grid to be replicated; the simplest test of the replication property is given is the reference region is chosen to be the same as the physical region and checking to see if the reference mapping satisfy the Euler–Lagrange equations. Surprisingly, this does not always happen. When we choose the mapping, $x=\alpha$, and $y=\beta$, the smoothness equations for (x, y) become

$$C_1\alpha_{\xi\xi}+C_2\alpha_{\eta\eta}+C_3\beta_{\xi\xi}+C_4\beta_{\eta\eta}=0, \qquad -C_3\alpha_{\xi\xi}+C_4\alpha_{\eta\eta}+D_3\beta_{\xi\xi}+D_4\beta_{\eta\eta}=0,$$

where

$$C_1=B^3\beta_\xi^2, \qquad C_2=A^3\beta_\eta^2, \qquad C_3=-B^3\alpha_\xi\beta_\xi,$$
$$C_4=-A^3\alpha_\eta\beta_\eta, \qquad D_3=-B^3\alpha_\xi^2, \qquad D_4=-A^3\alpha_\eta^2,$$

with

$$A=\left(\alpha_\xi^2+\beta_\xi^2\right)^{1/2}, \qquad B=\left(\alpha_\eta^2+\beta_\eta^2\right)^{1/2}.$$

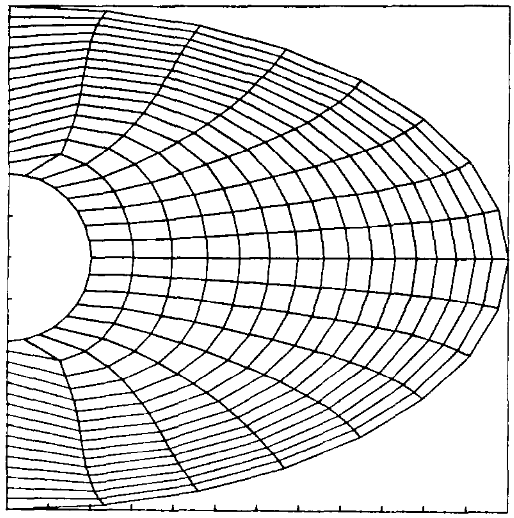


Fig. 2. The Real Problem.

In general, since these equations are nontrivial, it is not possible to replicate the reference grid but in simple geometries. In particular, if the reference grid is a quadrilateral, x and y are linear; hence the above equations are satisfied, so the smoothness integral replicates the reference grid. In the case of the volume control, a similar calculation can be done. After some algebra, the constraining equations become an identity. Thus, the reference grid always can be replicated.

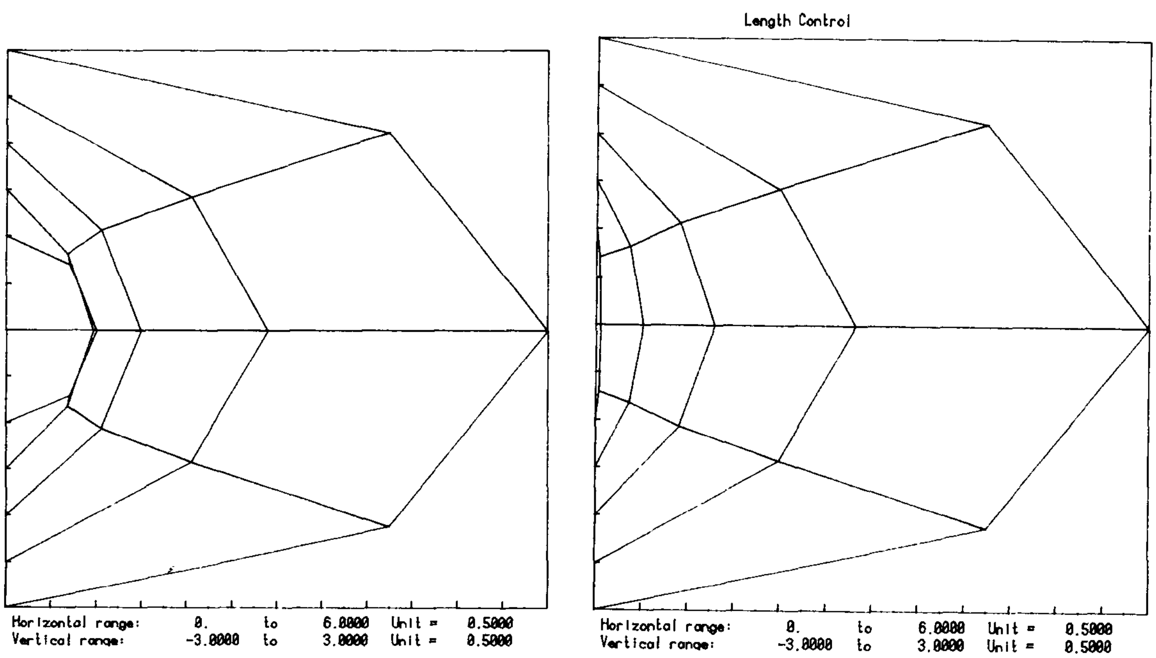


Fig. 3. Evenly spaced 5 by 5 grid (smoothness-noreference grid).

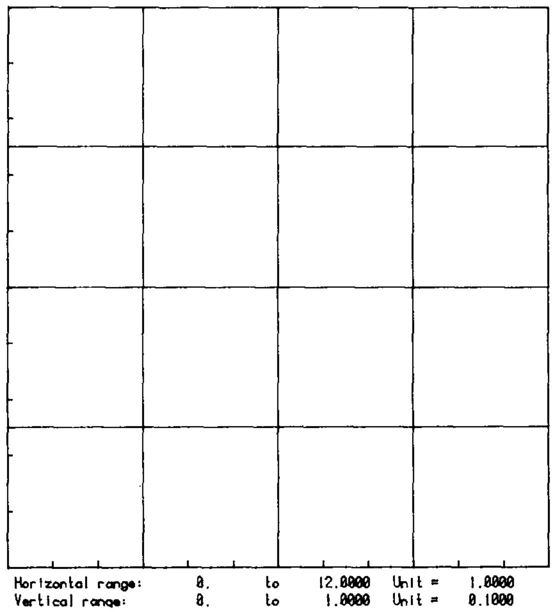


Fig. 4(a). Evenly spaced reference grid.

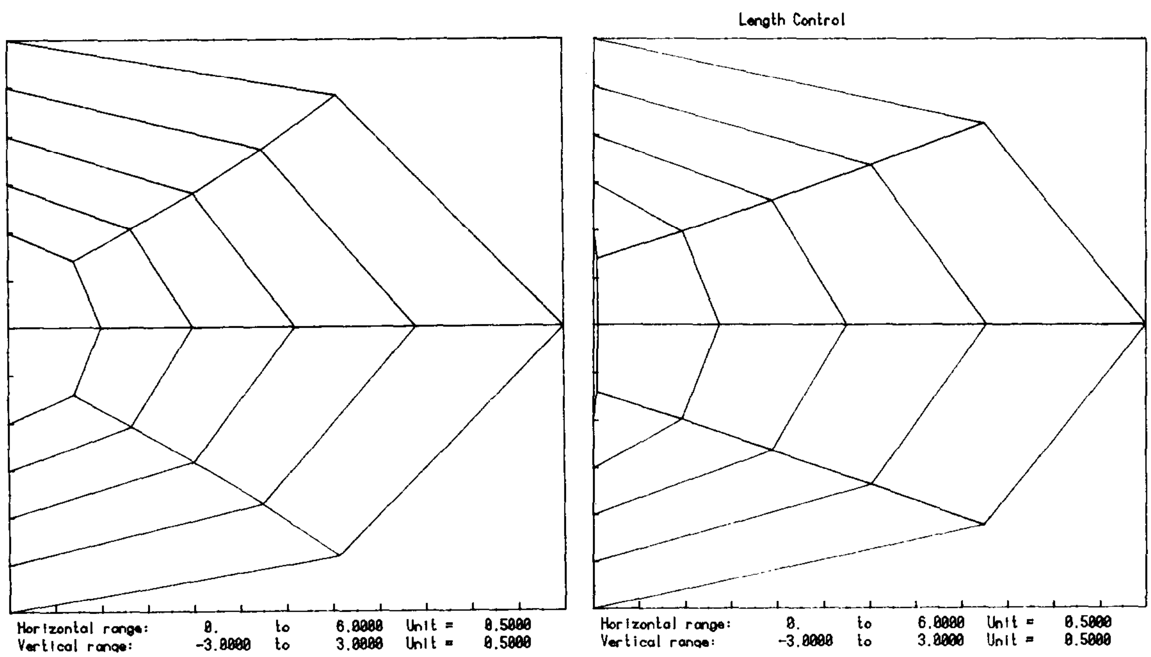


Fig. 4(b). Evenly spaced 5 by 5 grid (smoothness).

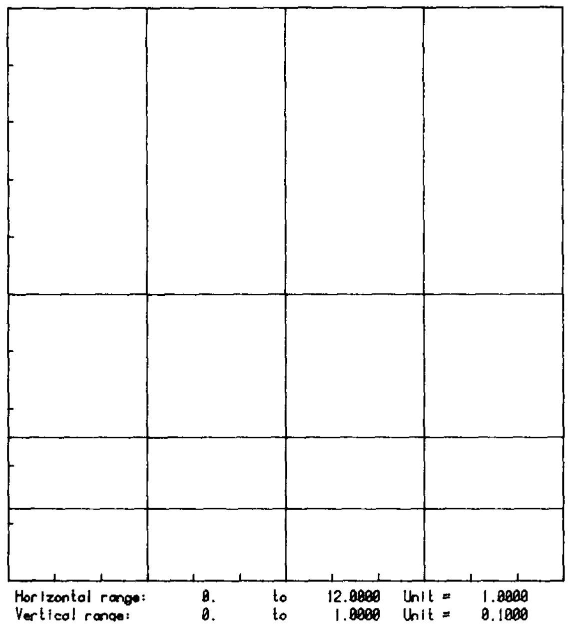


Fig. 5(a). Exponentially stretched 5 by 5 reference grid.

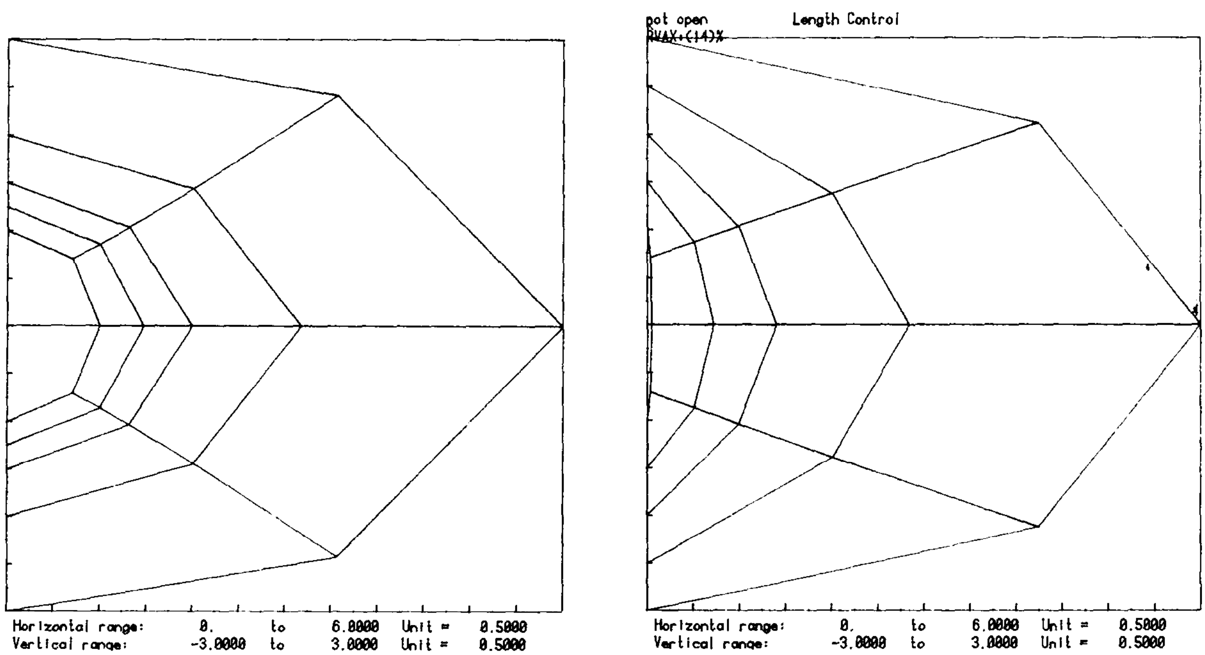


Fig. 5(b). Exponentially stretched 5 by 5 grid (smoothness).

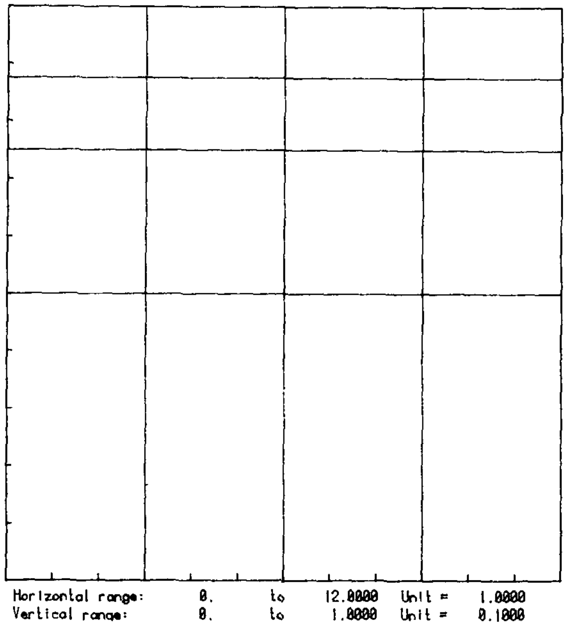
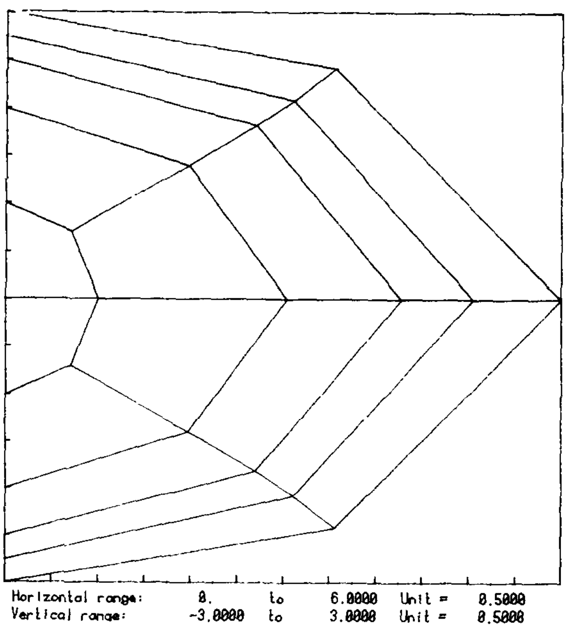


Fig. 6(a). Exponentially stretched 5 by 5 reference grid.



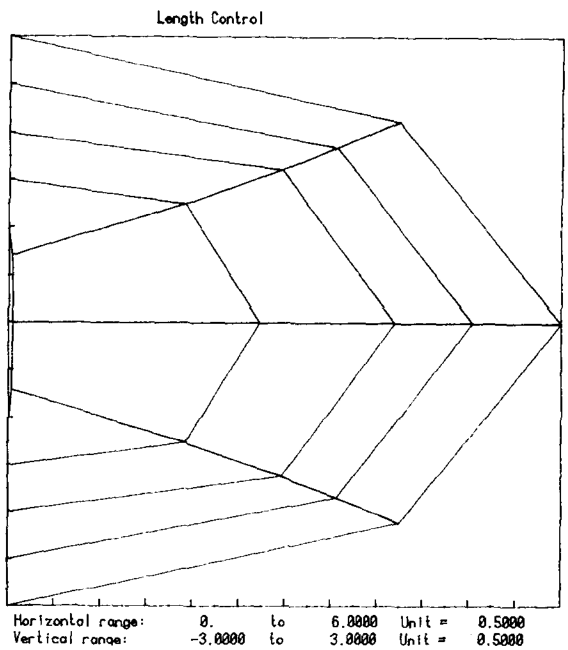


Fig. 6(b). Exponentially stretched 5 by 5 grid (smoothness).

2. Model problem

We use the region between two ellipses as a standard test problem [4] (see Fig. 2). This region is chosen because most grid generators have problems with this type of geometry [4]. The grid in this figure was generated using the ideas presented in [7] and was one of the best grids that we could generate without the reference grid concept. The grid in Fig. 3 was generated with no reference grid. The grid in Fig. 4(b) was generated using an evenly spaced grid in a rectangle as a reference grid (see Fig. 4(a)). The grids in Fig. 5(b) and Fig. 6(b) were generated using an exponentially spaced grid in a rectangle as a reference grid (see Fig. 5(a) and Fig. 6(a)). These grids clearly illustrate the effects of the reference grid. Note that the grids in Figs. 3, 4(b), 5(b) and 6(b) were generated using only smoothness control.

3. Conclusions

The near the identity analysis shows that the Euler–Lagrange equations appearing in the variational grid generation method need not be elliptic. Reference grids are not reproduced exactly in the physical region, but they play an important role in exercising a more refined control over the grid properties. Variational grid generators that use the reference grid concept produce grids suitable for solving numerical partial differential equations.

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