Elliptic Integrals and Functions

Devised and coded by **John Trenholme**
Maple worksheet "EllipticStuff.mw"
Started 2014-07-04 Changes to 2014-07-16

```
> restart; version = kernelopts( version);
  StringTools:-FormatTime( "It is now %Y-%m-%d %T");
  startTime := time():
```

version = *Maple 17.02, X86 64 WINDOWS, Sep 5 2013, Build ID 872941*

"It is now 2014-07-16 23:10:49" **(1)**

Set numeric precision.

```
> WD := 20: `Working Digits = WD` = WD;
  Digits := WD;
```

Working Digits = *WD* = 20

Digits := 20 **(2)**

▶ *Set Default Plot Options*

Elliptic Integral of the First Kind

Show some example values, including negative parameter and close-to-unity cases. Negative parameters have a value with a zero imaginary part, not that there's anything wrong with that.

```
> `---- parameter, complete elliptic integral of first kind ----`;
  for x_ from -1 to 0.9 by 0.1 do
    x_, Re(EllipticK(1.0 * sqrt(x_)))
  end do;
  # approach unity by inverse powers of 10
  for p_ from -1 to -8 by -1 do
    x_ := evalf(1 - 10^p_):
    print(evalf(x_, 9), EllipticK(sqrt(x_)))
  end do:
  `---- check the accuracy of the last value ----`;
  Digits := 50; print(evalf(x_, 9), EllipticK(sqrt(x_)));
  Digits := WD;
```

---- parameter, complete elliptic integral of first kind ----

−1, 1.3110287771460599052

−0.9, 1.3293621856564093625

−0.8, 1.3488465121932685780

−0.7, 1.3696211944090494098

−0.6, 1.3918518556639100293

−0.5, 1.4157372084259561989

−0.4, 1.4415183761084805043

-0.3, 1.4694917220921212563

-0.2, 1.5000268912867475206

-0.1, 1.5335928197134568815

0., 1.5707963267948966192

0.1, 1.6124413487202193982

0.2, 1.6596235986105280009

0.3, 1.7138894481787910620

0.4, 1.7775193714912533235

0.5, 1.8540746773013719184

0.6, 1.9495677498060258827

0.7, 2.0753631352924691439

0.8, 2.2572053268208536551

0.9, 2.5780921133481731882

0.900000000, 2.5780921133481731882

0.990000000, 3.6956373629898746774

0.999000000, 4.8411325605502970346

0.999900000, 5.9915893405069963676

0.999990000, 7.1427724505817782292

0.999999000, 8.2940514636154424853

0.999999900, 9.4453423977326174448

0.999999990, 10.596634757087660327

---- check the accuracy of the last value ----

$Digits := 50$

0.999999990, 10.596634757087660320255540297468325968698268052002

$Digits := 20$ **(3)**

Jacobi Elliptic Functions

Let's try to get an elliptic function period (period = 4 * EllipticK) close to 8. The parameter will be around 0.65.

```
> EllipticK(sqrt(0.65));
```

2.0075983984243763017 **(4)**

Ask Maple for an exact answer. Somehow, we get two copies of the same number. C'est la vie.

```
> `period-8 m` = solve(EllipticK(sqrt(x)) = 2.0, x);
```

period-8 m = (0.64385621914775464687, 0.64385621914775464687) **(5)**

Check the result with a 15-digit parameter.

```
> period = 4 * EllipticK(sqrt(0.643856219147755));
```

period = 8.0000000000000017308 **(6)**

Plot the elliptic functions of period 8.

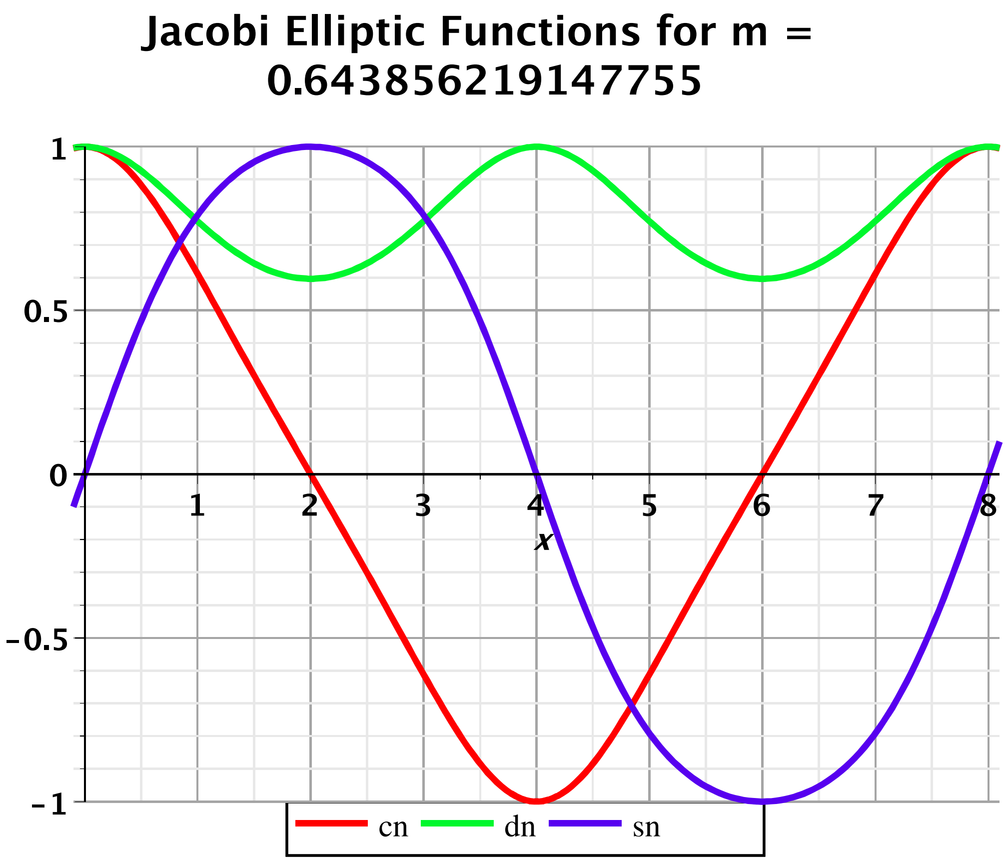
```
> m_ := 0.643856219147755;
```

```
k_ := sqrt(m_);
period = 4.0 * EllipticK(k_);
plot([JacobiCN(x, k_), JacobiDN(x, k_), JacobiSN(x, k_)],
  x = -0.1 .. 4 * EllipticK(k_) + 0.1, legend = ["cn", "dn", "sn"],
  titlefont = ['HELVETICA', 13, 'BOLD'],
  title = typeset("Jacobi Elliptic Functions for m = ", m_));
```

$$m\_ := 0.643856219147755$$

$$k\_ := 0.80240651738863325362$$

$$period = 8.0000000000000017308$$



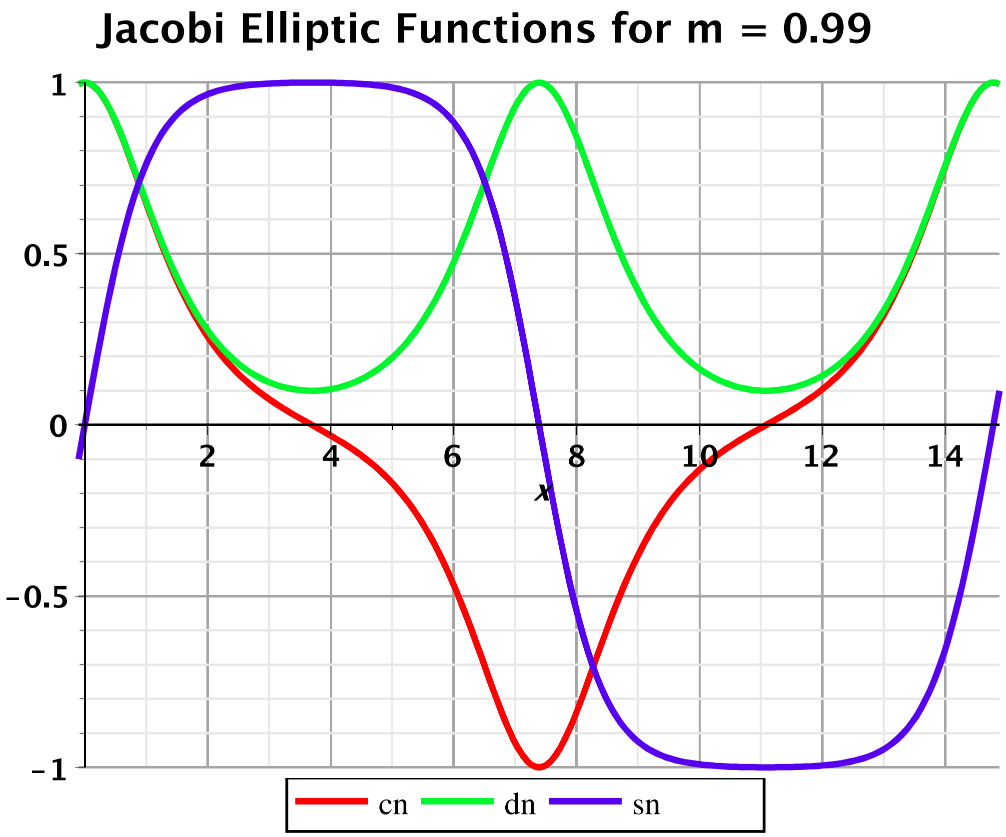
Plot a case closer to the singularity.

```
> m_ := 0.99;
  k_ := sqrt(m_);
  period = 4.0 * EllipticK(k_);
  plot([JacobiCN(x, k_), JacobiDN(x, k_), JacobiSN(x, k_)],
    x = -0.1 .. 4 * EllipticK(k_) + 0.1, legend = ["cn", "dn", "sn"],
    title = typeset("Jacobi Elliptic Functions for m = ", m_));
```

$$m\_ := 0.99$$

$$k\_ := 0.99498743710661995473$$

$period = 14.782549451959498710$



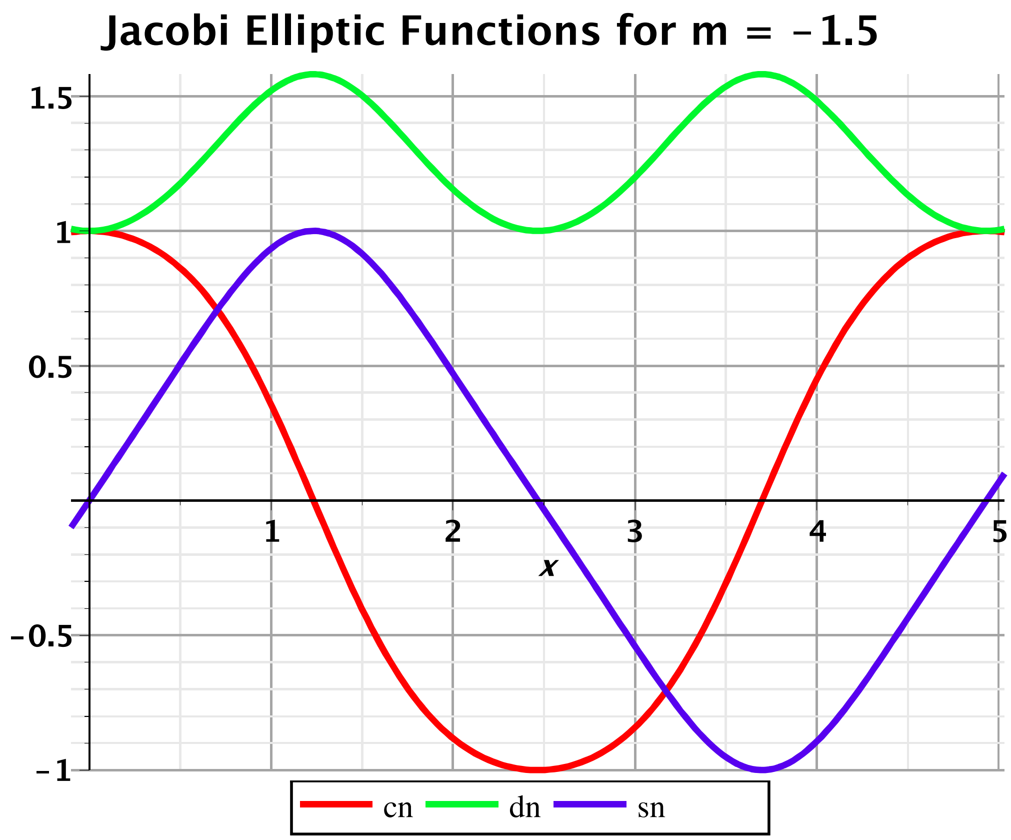
When the parameter is negative, the Dn(x|m) function becomes larger than unity. The period has a harmless zero-value imaginary part.

```
> m_ := -1.5;
  k_ := sqrt(m_);
  period = Re(4.0 * EllipticK(k_));
  plot([JacobiCN(x, k_), JacobiDN(x, k_), JacobiSN(x, k_)],
    x = -0.1 .. Re(4 * EllipticK(k_)) + 0.1,
    legend = ["cn", "dn", "sn"],
    title = typeset("Jacobi Elliptic Functions for m = ", m_));
```

$m\_ := -1.5$

$k\_ := 1.2247448713915890491\,I$

$period = 4.9320596337570655180$



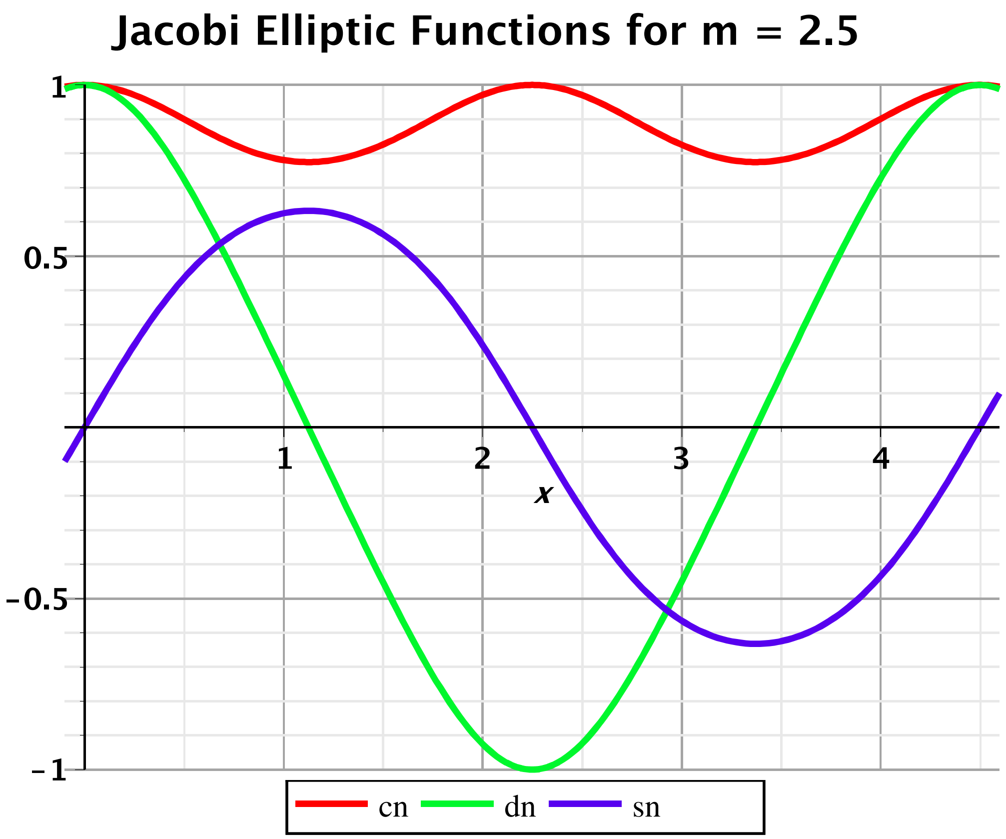
When the parameter is greater than unity, Cn and Dn switch roles, and Sn shrinks. The period is given by the real part of the complete elliptic integral of the first kind.

```
> m_ := 2.5;
  k_ := sqrt(m_);
  period = Re(4.0 * EllipticK(k_));
  plot([JacobiCN(x, k_), JacobiDN(x, k_), JacobiSN(x, k_)],
    x = -0.1 .. Re(4 * EllipticK(k_)) + 0.1,
    legend = ["cn", "dn", "sn"],
    title = typeset("Jacobi Elliptic Functions for m = ", m_));
```

$$m\_ := 2.5$$

$$k\_ := 1.58113883008418966660$$

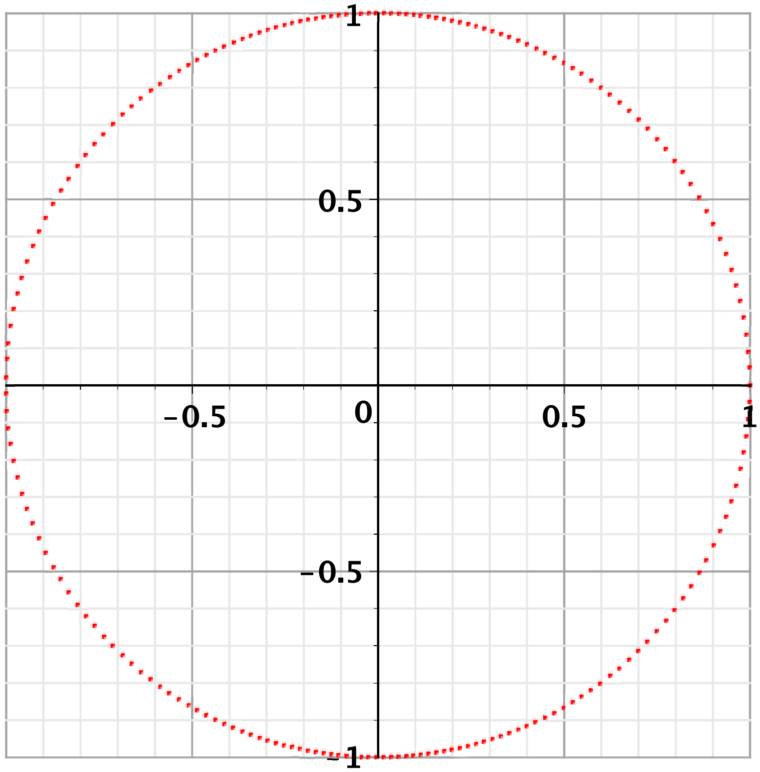
$$period = 4.49680783918666303356$$



Recall that (cn, sn) travels in a circle, since $cn^2 + sn^2 = 1$, but at a variable rate.

```
> k_ := 0.9;
  plot([JacobiCN(x, k_), JacobiSN(x, k_), x = 0 .. 4 * EllipticK(k_)],
    style = point, symbolsize = 4, scaling = constrained);
```

$$k\_ := 0.9$$



Get values at center of octants of period-8 elliptic functions. Include one negative-argument case as well.

```
> k_ := sqrt(0.643856219147755);
  period = 4.0 * EllipticK(k_);
  for x_ from -0.5 to 8 by 1 do
    x_, JacobiCN(x_, k_), JacobiDN(x_, k_), JacobiSN(x_, k_)
  end do;
```

$$k\_ := 0.80240651738863325362$$

$$period = 8.0000000000000017308$$

$$-0.5, 0.88358957167486910265, 0.92672649084087369761, -0.46826217958257247744$$

$$0.5, 0.88358957167486910265, 0.92672649084087369761, 0.46826217958257247744$$

$$1.5, 0.30154364975464892914, 0.64396328147504181994, 0.95345237284965923714$$

$$2.5, -0.30154364975464839783, 0.64396328147504165976, 0.95345237284965940518$$

$$3.5, -0.88358957167486872714, 0.92672649084087346709, 0.46826217958257318602$$

$$4.5, -0.88358957167486947816, 0.92672649084087392813, -0.46826217958257176886$$

$$5.5, -0.30154364975464946045, 0.64396328147504198013, -0.95345237284965906911$$

$$6.5, 0.30154364975464786653, 0.64396328147504149957, -0.95345237284965957321$$

$$7.5, 0.88358957167486835162, 0.92672649084087323657, -0.46826217958257389460 \quad \textbf{(7)}$$

Move out many cycles to test whether values repeat. Should be same as 0.5 case above, but it changes at the

15th digit.

```
> k_ := sqrt(0.643856219147755); period = 4.0 * EllipticK(k_);
  x_ := 100000.0 * 4 * EllipticK(k_) + 0.5:
  x_, JacobiCN(x_, k_), JacobiDN(x_, k_), JacobiSN(x_, k_);
  Digits := 75;
  JacobiCN(x_, k_), JacobiDN(x_, k_), JacobiSN(x_, k_);
  Digits := WD;
```

$$k\_ := 0.80240651738863325362$$

$$period = 8.0000000000000017308$$

$8.0000050000000017308\ 10^{5}$, 0.88358957167486365542, 0.92672649084087035363, 0.46826217958258275612

$$Digits := 75$$

0.883589571674863655417600617942204190259186885086678124943111911770689092017, 0.926726490840870353630046414517424936201265084102096303261672593563751042359, 0.468262179582582756120153435618452578720580277610139204888010271967569295936

$$Digits := 20 \tag{8}$$

Do a very-small-parameter case, near the center of the first octant (i. e., at 1/16 of the period).

```
> k_ := sqrt(0.0001); period = 4.0 * EllipticK(k_);
  x_ := 0.4:
  x_, JacobiCN(x_, k_), JacobiDN(x_, k_), JacobiSN(x_, k_);
```

$$k\_ := 0.01$$

$$period = 6.2833423956486089440$$

$$0.4, 0.92106139629029055238, 0.99999241767604099267, 0.38941739080808954169 \tag{9}$$

Do a close-to-unity-parameter case, near the center of the first octant (i. e., at 1/16 of the period).

```
> k_ := sqrt(0.99999999); period = 4.0 * EllipticK(k_);
  x_ := 2.6:
  x_, JacobiCN(x_, k_), JacobiDN(x_, k_), JacobiSN(x_, k_);
```

$$k\_ := 0.99999999499999998750$$

$$period = 42.386539028350641308$$

$$2.6, 0.14773216672437526364, 0.14773219983074332824, 0.98902740453180638350 \tag{10}$$

Do a first-octant case with negative parameter.

```
> k_ := sqrt(-1.5); period = Re(4.0 * EllipticK(k_));
  x_ := 0.3:
  x_, Re(JacobiCN(x_, k_)), Re(JacobiDN(x_, k_)), Re(JacobiSN(x_, k_));
```

$$k\_ := 1.2247448713915890491\ I$$

$$period = 4.9320596337570655180$$

$$0.3, 0.95334383473080432094, 1.0661628858533518121, 0.30188662239450899028 \tag{11}$$

Do a first-octant case with greater-than-unity parameter.

```
> k_ := sqrt(2.5); period = Re(4.0 * EllipticK(k_));
  x_ := 0.3:
  x_, Re(JacobiCN(x_, k_)), Re(JacobiDN(x_, k_)), Re(JacobiSN(x_, k_));
```

$$k\_ := 1.5811388300841896660$$

$$period = 4.49680783918666630356$$

$$0.3, 0.95851072192295526069, 0.89266847715328346666, 0.28505647854194637371 \quad \textbf{(12)}$$

Worksheet timing:

```
> StringTools:-FormatTime( "Done %Y-%m-%d %T");
  `Elapsed time (seconds)` = time() - startTime;
```

"Done 2014-07-16 23:11:09"

$$Elapsed\ time\ (seconds) = 15.226 \quad \textbf{(13)}$$

```
> #~~~~~~~~~~~~~~~~~~~~~~~~~ end of file ~~~~~~~~~~~~~~~~~~~~~~~~~;
```