Powell’s Conjugate Direction Method

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Necessary preliminaries

Define “norm” to measure length of vectors. Also set up routine to print give number of decimal places.

In[1]:=
```
Clear[norm, nn, dp, x];
norm[x_] = Sqrt[x.x];
dp = 9;
nn[x_] := NumberForm[N[x], {20, dp}];
```

Line search routines. (The following newton method and line search comes from Professor Ellingham)

In[5]:=
```
Clear[newton1, f1, x1, x, linesearch, ff, xx, p];
newton1[f1_, x1_] = N[x1 - f1'[x1] / f1''[x1]];
linesearch[ff_, xx_, p_, t0_] := Module[{phi, t},
  Clear[phi, t];
  phi[t_] = ff[xx + t * p];
  t = t0;
  t = newton1[phi, t];
  t = newton1[phi, t];
  t = newton1[phi, t];
  t = newton1[phi, t]
 ]
```

Initialize Powell’s Conjugate Direction Method

Initialize the starting point of the method x[0], the iteration counter k, and the dimensions of the function n. Also initialize u[i], the direction vectors, as the column vectors of the identity matrix.

In[8]:=
```
pcdminit[x0_, n0_] := Module[{},
  Clear[k, x, u, n, i, p];
  x[0] = x0; k = 0; n = n0; p[0] = x0;
  For[i = 1, i <= n, i++,
   u[i] = IdentityMatrix[n][[i]];
  ];
  pcgdprint;
 ]
```

One Iteration of Powell’s Conjugate Direction Method

Calculates a new approximation to the minimizer of f and updates the direction vectors

In[9]:=
```
pcdm := Module[{t},
  Clear[t];
  For[i = 1, i ≤ n, i++,
   t = linesearch[f, x[i - 1], u[i], 0];
   x[i] = N[x[i - 1] + t * u[i]];
  ];
  For[i = 1, i < n, i++,
   u[i] = u[i + 1];
  ];
  u[n] = N[x[n] - x[0]];
  t = linesearch[f, x[0], u[n], 0];
  x[0] = x[0] + t * u[n];
  k = k + 1;
  p[k] = x[0];
  pcdmprint;
 ]
```

Print Method

Prints the iteration number, the new starting point (approximation of the minimizer of f after k iterations), and the direction vectors

In[10]:=
```
pcdmprint := Module[{},
  Print[k, " ", x[0] // nn];
  For[i = 1, i <= n, i++,
   Print["u", i, ": ", u[i]];
  ];
 ]
```

Define Function 1

We are going to define our function f and its first and second derivatives g and h as a function of a vector rather than as a function of three individual variables. The function f has a minimizer somewhere near {1,0,2}.

In[11]:=
```
Clear[f, g, h, x, x1, x2, x3, xi];
f[{x1_, x2_, x3_}] =
  (x1 + x2 - 1)^2 + x2^2 - (4 / (5 + (x2 + x3 - 2)^2)) + 0.4 * ArcTan[x1 + x2 + x3] + 2
x = {x1, x2, x3};
g[{x1_, x2_, x3_}] = Map[Function[xi, D[f[x], xi]], x];
g[x] // MatrixForm
h[{x1_, x2_, x3_}] = Map[Function[xi, D[g[x], xi]], x];
h[x] // MatrixForm
```

Out[12]=
$$2 + x2^2 + (-1 + x1 + x2)^2 - \frac{4}{5 + (-2 + x2 + x3)^2} + 0.4\,\mathrm{ArcTan}[x1 + x2 + x3]$$

Out[15]//MatrixForm=
$$\begin{pmatrix} 2\,(-1 + x1 + x2) + \frac{0.4}{1+(x1+x2+x3)^2} \\ 2\,x2 + 2\,(-1 + x1 + x2) + \frac{8\,(-2+x2+x3)}{\left(5+(-2+x2+x3)^2\right)^2} + \frac{0.4}{1+(x1+x2+x3)^2} \\ \frac{8\,(-2+x2+x3)}{\left(5+(-2+x2+x3)^2\right)^2} + \frac{0.4}{1+(x1+x2+x3)^2} \end{pmatrix}$$

Out[17]//MatrixForm=
$$\begin{pmatrix} 2 - \frac{0.8\,(x1+x2+x3)}{\left(1+(x1+x2+x3)^2\right)^2} & 2 - \frac{0.8\,(x1+x2+x3)}{\left(1+(x1+x2+x3)^2\right)^2} & -\frac{0.8\,(x1+x2+x3)}{\left(1+(x1+x2+x3)^2\right)^2} \\ 2 - \frac{0.8\,(x1+x2+x3)}{\left(1+(x1+x2+x3)^2\right)^2} & 4 - \frac{32\,(-2+x2+x3)^2}{\left(5+(-2+x2+x3)^2\right)^3} + \frac{8}{\left(5+(-2+x2+x3)^2\right)^2} - \frac{0.8\,(x1+x2+x3)}{\left(1+(x1+x2+x3)^2\right)^2} & -\frac{32\,(-2+x2+x3)^2}{\left(5+(-2+x2+x3)^2\right)^3} + \frac{8}{\left(5+(-2+x2+x3)^2\right)^2} - \\ -\frac{0.8\,(x1+x2+x3)}{\left(1+(x1+x2+x3)^2\right)^2} & -\frac{32\,(-2+x2+x3)^2}{\left(5+(-2+x2+x3)^2\right)^3} + \frac{8}{\left(5+(-2+x2+x3)^2\right)^2} - \frac{0.8\,(x1+x2+x3)}{\left(1+(x1+x2+x3)^2\right)^2} & -\frac{32\,(-2+x2+x3)^2}{\left(5+(-2+x2+x3)^2\right)^3} + \frac{8}{\left(5+(-2+x2+x3)^2\right)^2} - \end{pmatrix}$$

Running pcdm on Function 1

In[18]:=
```
pcdminit[{1, 0, 2}, 3]
```

In[19]:=
```
pcdm
```

1 $\{0.979514363, 1.471097098\times10^{-17}, 1.859477104\}$

u1: {0, 1, 0}

u2: {0, 0, 1}

u3: $\{-0.0202451, 1.45382\times10^{-17}, -0.138873\}$

In[20]:=
```
pcdm
```

2 {0.977766995, 0.011254857, 1.838129648}

u1: {0, 0, 1}

u2: $\{-0.0202451, 1.45382\times10^{-17}, -0.138873\}$

u3: {-0.00150111, 0.00966873, -0.018339}

In[21]:=
```
pcdm
```

3 {0.955146269, 0.022425406, 1.836286683}

u1: $\{-0.0202451, 1.45382\times10^{-17}, -0.138873\}$

u2: {-0.00150111, 0.00966873, -0.018339}

u3: {-0.00160663, 0.000793384, -0.000130896}

In[22]:= **pcdm**

4 {0.955145783, 0.022427109, 1.836281939}

u1: {-0.00150111, 0.00966873, -0.018339}

u2: {-0.00160663, 0.000793384, -0.000130896}

u3: $\{-4.86487\times10^{-7}, 1.70235\times10^{-6}, -4.74329\times10^{-6}\}$

In[23]:= **pcdm**

5 {0.955145782, 0.022427109, 1.836281938}

u1: {-0.00160663, 0.000793384, -0.000130896}

u2: $\{-4.86487\times10^{-7}, 1.70235\times10^{-6}, -4.74329\times10^{-6}\}$

u3: $\{-1.15756\times10^{-11}, -6.6874\times10^{-12}, -7.36902\times10^{-11}\}$

In[24]:= **pcdm**

6 {0.955145782, 0.022427109, 1.836281938}

u1: $\{-4.86487\times10^{-7}, 1.70235\times10^{-6}, -4.74329\times10^{-6}\}$

u2: $\{-1.15756\times10^{-11}, -6.6874\times10^{-12}, -7.36902\times10^{-11}\}$

u3: $\{1.9984\times10^{-15}, -1.94289\times10^{-16}, -2.44249\times10^{-15}\}$

In[25]:= **pcdm**

7 {0.955145782, 0.022427109, 1.836281938}

u1: $\{-1.15756\times10^{-11}, -6.6874\times10^{-12}, -7.36902\times10^{-11}\}$

u2: $\{1.9984\times10^{-15}, -1.94289\times10^{-16}, -2.44249\times10^{-15}\}$

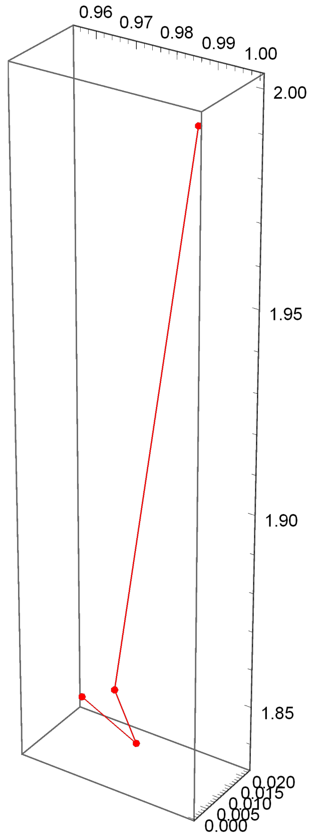
u3: $\{0., -1.73472\times10^{-17}, 0.\}$

Results for Function 1

We stop because we got the same point to 9 decimal places. Thus the approximation of the minimizer to 9 decimal places after 6 iterations of pcdm is {0.955145782,0.022427109,1.836281938}. If we would continue to run the method, the direction vectors would become 0 and we would get an invalid solution.
Below is the graph of the points found.

In[26]:=
```
Show[Graphics3D[{Red, Line[{p[0], p[1], p[2], p[3], p[4], p[5]}],
    PointSize[0.03], Point[p[0]], Point[p[1]], Point[p[2]],
    Point[p[3]], Point[p[4]], Point[p[4]]}, Axes → True, ViewPoint → {2, -3, 2}]]
```

Out[26]=



In[27]:=

Define Function 2

This function has a global minimum at (1,1,1)

In[28]:=
```
Clear[f, g, h, x, x1, x2, x3, xi];
f[{x1_, x2_, x3_}] = 10 (x2 - x1^2)^2 + (1 - x1)^2 + (E^x3 - E)^2;
x = {x1, x2, x3};
g[{x1_, x2_, x3_}] = Map[Function[xi, D[f[x], xi]], x];
(*g[x]//MatrixForm*)
h[{x1_, x2_, x3_}] = Map[Function[xi, D[g[x], xi]], x];
(*h[x]//MatrixForm*)
```

Running pcdm on Function 2

In[33]:=
```
pcdminit[{-1, 0, 0}, 3]
```

In[34]:=
```
pcdm
```

```
1 {-0.279107345, 0.136277717, -2.549700036}
u1: {0, 1, 0}
u2: {0, 0, 1}
u3: {1.53946, 0.29102, -5.44487}
```

In[35]:= **pcdm**

```
2 {-5.715778716, -0.774198676, 24.807258542}
u1: {0, 0, 1}
u2: {1.53946, 0.29102, -5.44487}
u3: {2.70629, 0.453221, -13.6179}
```

In[36]:= **pcdm**

```
3 {-5.394800649, -0.716378759, 22.807258543}
u1: {1.53946, 0.29102, -5.44487}
u2: {2.70629, 0.453221, -13.6179}
u3: {0.962934, 0.17346, -6.}
```

In[37]:= **pcdm**

```
4 {-4.966829891, -0.639285537, 20.807258543}
u1: {2.70629, 0.453221, -13.6179}
u2: {0.962934, 0.17346, -6.}
u3: {1.28391, 0.23128, -6.}
```

In[38]:= **pcdm**

```
5 {-4.584693059, -0.572126928, 18.807258550}
u1: {0.962934, 0.17346, -6.}
u2: {1.28391, 0.23128, -6.}
u3: {1.14641, 0.201476, -6.}
```

In[39]:= **pcdm**

```
6 {-4.207664519, -0.504769680, 16.807258595}
u1: {1.28391, 0.23128, -6.}
u2: {1.14641, 0.201476, -6.}
u3: {1.13109, 0.202072, -6.}
```

In[40]:= **pcdm**

```
7 {-3.811952478, -0.434233321, 14.807258931}
u1: {1.14641, 0.201476, -6.}
u2: {1.13109, 0.202072, -6.}
u3: {1.18713, 0.211608, -5.99998}
```

In[41]:= **pcdm**

```
8 {-3.426994009, -0.365882708, 12.807261413}
u1: {1.13109, 0.202072, -6.}
u2: {1.18713, 0.211608, -5.99998}
u3: {1.15485, 0.205047, -5.99984}
```

In[42]:= **pcdm**

```
9 {-3.041097718, -0.297135203, 10.807279743}
u1: {1.18713, 0.211608, -5.99998}
u2: {1.15485, 0.205047, -5.99984}
u3: {1.15748, 0.206205, -5.99886}
```

In[43]:= **pcdm**

```
10 {-2.652263149, -0.227927501, 8.807414763}
u1: {1.15485, 0.205047, -5.99984}
u2: {1.15748, 0.206205, -5.99886}
u3: {1.16504, 0.207362, -5.99204}
```

In[44]:= **pcdm**

```
11 {-2.265893989, -0.159193715, 6.808398341}
u1: {1.15748, 0.206205, -5.99886}
u2: {1.16504, 0.207362, -5.99204}
u3: {1.15064, 0.204694, -5.95322}
```

In[45]:= **pcdm**

```
12 {-1.880115462, -0.090520392, 4.815231273}
u1: {1.16504, 0.207362, -5.99204}
u2: {1.15064, 0.204694, -5.95322}
u3: {1.14283, 0.203437, -5.90455}
```

In[46]:= **pcdm**

```
13 {-1.500272819, -0.022917174, 2.854301783}
u1: {1.15064, 0.204694, -5.95322}
u2: {1.14283, 0.203437, -5.90455}
u3: {1.49755, 0.266529, -7.73106}
```

In[47]:= **pcdm**

```
14 {-1.122852520, 0.044253677, 0.903590734}
u1: {1.14283, 0.203437, -5.90455}
u2: {1.49755, 0.266529, -7.73106}
u3: {1.21322, 0.215922, -6.2706}
```

In[48]:= **pcdm**

```
15 {-0.381903753, 0.176148718, -2.924256219}
u1: {1.49755, 0.266529, -7.73106}
u2: {1.21322, 0.215922, -6.2706}
u3: {0.835028, 0.148642, -4.31387}
```

In[49]:= **pcdm**

```
16 {-0.287637563, 0.192925926, -3.410902182}
u1: {1.21322, 0.215922, -6.2706}
u2: {0.835028, 0.148642, -4.31387}
u3: {0.0940978, 0.0167472, -0.485777}
```

In[50]:= **pcdm**

17 {-0.222974529, 0.200012377, -5.335192400}

u1: {0.835028, 0.148642, -4.31387}

u2: {0.0940978, 0.0167472, -0.485777}

u3: $\{-4.95724\times10^{-8}, -5.43266\times10^{-9}, 1.47521\times10^{-6}\}$

In[51]:= **pcdm**

18 {-0.219440790, 0.200654899, -5.348954692}

u1: {0.0940978, 0.0167472, -0.485777}

u2: $\{-4.95724\times10^{-8}, -5.43266\times10^{-9}, 1.47521\times10^{-6}\}$

u3: {0.00758104, 0.00137842, -0.0295246}

In[52]:= **pcdm**

19 {-0.178779004, 0.208057577, -5.503826279}

u1: $\{-4.95724\times10^{-8}, -5.43266\times10^{-9}, 1.47521\times10^{-6}\}$

u2: {0.00758104, 0.00137842, -0.0295246}

u3: {2.77825, 0.505794, -10.5817}

In[53]:= **pcdm**

```
20 {0.623936660, 0.354198530, -8.562983591}
u1: {0.00758104, 0.00137842, -0.0295246}
u2: {2.77825, 0.505794, -10.5817}
u3: {0.802716, 0.146141, -3.05916}
```

In[54]:= **pcdm**

21 {0.683079253, 0.442789821, 20.218214460}

u1: {2.77825, 0.505794, -10.5817}

u2: {0.802716, 0.146141, -3.05916}

u3: $\{1.67769\times10^{-9}, 2.51306\times10^{-9}, 8.16433\times10^{-7}\}$

In[55]:= **pcdm**

22 {1.031675521, 0.504451399, 18.218214471}

u1: {0.802716, 0.146141, -3.05916}

u2: $\{1.67769\times10^{-9}, 2.51306\times10^{-9}, 8.16433\times10^{-7}\}$

u3: {1.04579, 0.184985, -6.}

In[56]:= **pcdm**

23 {1.321436074, 0.554800948, 16.218214553}

u1: $\{1.67769\times10^{-9}, 2.51306\times10^{-9}, 8.16433\times10^{-7}\}$

u2: {1.04579, 0.184985, -6.}

u3: {0.869281, 0.151049, -5.99999}

In[57]:= **pcdm**

24 {1.532851173, 0.590085818, 14.218215159}

u1: {1.04579, 0.184985, -6.}

u2: {0.869281, 0.151049, -5.99999}

u3: {0.634241, 0.105854, -5.99996}

In[58]:= **pcdm**

25 {1.816110718, 0.639184925, 12.218219633}

u1: {0.869281, 0.151049, -5.99999}

u2: {0.634241, 0.105854, -5.99996}

u3: {0.849741, 0.147291, -5.99972}

In[59]:= **pcdm**

26 {2.077580192, 0.684094468, 10.218252701}

u1: {0.634241, 0.105854, -5.99996}

u2: {0.849741, 0.147291, -5.99972}

u3: {0.784143, 0.134683, -5.99787}

In[60]:= **pcdm**

27 {2.329565174, 0.727180807, 8.218497526}

u1: {0.849741, 0.147291, -5.99972}

u2: {0.784143, 0.134683, -5.99787}

u3: {0.753713, 0.128876, -5.98147}

In[61]:= **pcdm**

28 {2.595580045, 0.772969128, 6.220346544}

u1: {0.784143, 0.134683, -5.99787}

u2: {0.753713, 0.128876, -5.98147}

u3: {0.758772, 0.130605, -5.69946}

In[62]:= **pcdm**

```
29 {2.851069529, 0.816772856, 4.236803074}
u1: {0.753713, 0.128876, -5.98147}
u2: {0.758772, 0.130605, -5.69946}
u3: {0.506274, 0.0868008, -3.93056}
```

In[63]:= **pcdm**

```
30 {3.069254556, 0.854110726, 2.516804273}
u1: {0.758772, 0.130605, -5.69946}
u2: {0.506274, 0.0868008, -3.93056}
u3: {0.24884, 0.0425838, -1.96166}
```

In[64]:= **pcdm**

```
31 {3.099674305, 0.859345016, 2.287649759}
u1: {0.506274, 0.0868008, -3.93056}
u2: {0.24884, 0.0425838, -1.96166}
u3: {0.0328038, 0.00564452, -0.247114}
```

In[65]:= **pcdm**

```
32 {0.770385502, 0.408807260, 1.293560557}
u1: {0.24884, 0.0425838, -1.96166}
u2: {0.0328038, 0.00564452, -0.247114}
u3: {-0.000078317, -0.0000151483, -0.000033424}
```

In[66]:= **pcdm**

```
33 {0.652046089, 0.385252159, 0.994969097}
u1: {0.0328038, 0.00564452, -0.247114}
u2: {-0.000078317, -0.0000151483, -0.000033424}
u3: {-0.122011, -0.0242859, -0.307855}
```

In[67]:= **pcdm**

```
34 {0.645183563, 0.383946474, 1.000123606}
u1: {-0.000078317, -0.0000151483, -0.000033424}
u2: {-0.122011, -0.0242859, -0.307855}
u3: {-0.00640688, -0.00121899, 0.00481227}
```

In[68]:= **pcdm**

```
35 {0.645180997, 0.383945964, 1.000117296}
u1: {-0.122011, -0.0242859, -0.307855}
u2: {-0.00640688, -0.00121899, 0.00481227}
```

u3: $\{-2.57972\times10^{-6}, -5.13043\times10^{-7}, -6.34419\times10^{-6}\}$

In[69]:= **pcdm**

```
36 {0.645160033, 0.383942000, 1.000133019}
u1: {-0.00640688, -0.00121899, 0.00481227}
```

u2: $\{-2.57972\times10^{-6}, -5.13043\times10^{-7}, -6.34419\times10^{-6}\}$

u3: $\{-3.4639\times10^{-14}, -6.55032\times10^{-15}, 2.59792\times10^{-14}\}$

In[70]:= **pcdm**

37 {0.863830166, 0.698848989, 0.999990238}

u1: $\{-2.57972\times10^{-6}, -5.13043\times10^{-7}, -6.34419\times10^{-6}\}$

u2: $\{-3.4639\times10^{-14}, -6.55032\times10^{-15}, 2.59792\times10^{-14}\}$

u3: $\{1.95449\times10^{-8}, 2.81466\times10^{-8}, -1.27618\times10^{-11}\}$

In[71]:= **pcdm**

38 {0.964955652, 0.931627239, 0.978320227}

u1: $\{-3.4639\times10^{-14}, -6.55032\times10^{-15}, 2.59792\times10^{-14}\}$

u2: $\{1.95449\times10^{-8}, 2.81466\times10^{-8}, -1.27618\times10^{-11}\}$

u3: {0.037807, 0.0870271, -0.0081016}

In[72]:= **pcdm**

39 {0.991554529, 0.976555999, 0.980407023}

u1: $\{1.95449\times10^{-8}, 2.81466\times10^{-8}, -1.27618\times10^{-11}\}$

u2: {0.037807, 0.0870271, -0.0081016}

u3: {0.0156003, 0.0263508, 0.00122391}

In[73]:= **pcdm**

40 {0.998277795, 0.996158671, 1.000247442}

u1: {0.037807, 0.0870271, -0.0081016}

u2: {0.0156003, 0.0263508, 0.00122391}

u3: {0.000398546, 0.00116202, 0.00117611}

In[74]:= **pcdm**

41 {1.000000298, 1.000000476, 1.000000021}

u1: {0.0156003, 0.0263508, 0.00122391}

u2: {0.000398546, 0.00116202, 0.00117611}

u3: {0.00170694, 0.0038071, -0.000245187}

In[75]:= **pcdm**

42 {1.000000000, 1.000000000, 1.000000000}

u1: {0.000398546, 0.00116202, 0.00117611}

u2: {0.00170694, 0.0038071, -0.000245187}

u3: $\{-2.94712\times10^{-7}, -4.70802\times10^{-7}, -2.02098\times10^{-8}\}$

In[76]:= **pcdm**

```
43 {1.000000000, 1.000000000, 1.000000000}
```

u1: {0.00170694, 0.0038071, -0.000245187}

u2: $\{-2.94712\times10^{-7}, -4.70802\times10^{-7}, -2.02098\times10^{-8}\}$

u3: $\{-5.91931\times10^{-11}, -1.48281\times10^{-10}, -1.2065\times10^{-10}\}$

Results for Function 2

Appeared to be diverging but eventually converged after 43 iterations. The approximation of the minimizer of f is {1.000000000,1.000000000,1.000000000}