# 1 Introduction

In this report, we will detail the physics behind free-fall down a vertical mineshaft, using a variety of physical models. We will analyze models that account for variable gravitational acceleration, drag forces, the Coriolis force, as well as models that account for a variable density profile of the Earth. We will compare these models primarily by comparing the fall times of the mass down the mineshaft.

For the analyses done in this report, we suppose a 1 kg test mass falling down, first, a 4 km deep, 5 m wide vertical mineshaft. We then consider other scenarios involving an infinitely deep mineshaft, i.e. one that continues through the entire planet, and relate the crossing time to the density of the planet.

# 2 Fall Time Down a 4 km Mineshaft

These motion of a mass in freefall can represented by the differential equation

$$\frac{d^2y}{dt^2} = -g(y) + \alpha \left|\frac{dy}{dt}\right|^\gamma \tag{1}$$

where $g(y)$ is the acceleration due to gravity as a function of height $y$, $\alpha$ is the drag coefficient, and $\gamma$ is the speed dependence of drag. This differential equation was solved numerically using the solve_ivp() function from the scipy.integrate package. To find the fall time, the time $t$ at which $y = -4000\ m$ is saved and returned.

For the first model, we assume constant acceleration and no drag, taking $g(y) = 9.81\ \frac{m}{s^2}$ and $\alpha = 0$. The fall time using this model is $t = 28.6\ s$.

For the second model, we assume variable acceleration due to a constant density Earth, still with no drag, we model the gravitational acceleration to be

$$g(y) = g_0 \left(\frac{y + R_E}{R_E}\right) \tag{2}$$

where $g_0 = 9.81\ \frac{m}{s^2}$ and $R_E = 6378.1\ km$ is the radius of the Earth. The fall time obtained from this model is $t = 28.6$ s, equal to the first fall time, within 3 significant figures.

For the third model, we still consider variable acceleration, but now consider non-zero drag. We assume $\gamma = 2$ and calibrate the drag coefficient to $\alpha = 0.004$, where the terminal velocity of the mass settles to be around 50 $\frac{m}{s}$. The fall time obtained from this model is $t = 84.3\ s$.

Comparing the three models, the change in fall time when considering variable, linear acceleration is very small ($\delta t = 1.5\ ms$). This is because the depth of the mineshaft is negligible compared to the radius of the Earth, so the difference in acceleration has almost no effect. Considering drag, however, has a major effect on the fall time ($\delta t = 55.7\ s$). This is because the mass can no longer acceleration to a velocity greater than 50 $\frac{m}{s}$, drastically increasing the fall time.

# 3 Feasibility of Depth Measurement Approach

We now present a further amendment to our model by considering the Coriolis force, changing the model to a system of second-order ODEs with an additional Coriolis force term,

$$\frac{d^2x}{dt^2} = -2\Omega\frac{dy}{dt} \tag{3}$$

$$\frac{d^2y}{dt^2} = -g(y) + \alpha \left|\frac{dy}{dt}\right|^\gamma + 2\Omega\frac{dx}{dt} \tag{4}$$

where $\Omega = 7.272 \times 10^{-5}\ \frac{rad}{s}$ is the angular velocity of the Earth's rotation. We, again, use the scipy.integrate package to numerically solve this system of ODEs. Using the same initial conditions and drag coefficient as section 2, the mass is found to have a fall time of $t = 84.3\ s$, however, if the mineshaft is 5 m wide, the mass displace 2.5 m in the x-direction and hit the wall at a time $t = 29.7\ s$ and a depth of $y = -1296.6\ m$. Therefore, the mass does not fall uninterrupted, and will collide with and bounce off of the walls as it falls, unpredictably affecting the fall time, thus, this method of determining the fall time is not recommended, and more complex considerations should be made.

Even when neglecting drag, setting $\alpha = 0$, the mass still hits the wall at $t = 21.9\ s$, before it reaches the bottom at $t = 28.6\ s$. Drag does have an effect on the time for the mass to hit the wall, because the Coriolis terms depend on velocity, which is decreased by the presence of drag.

# 4 Crossing Times for Homogeneous & Non-homogeneous Earths

In the previous sections, our consideration of the variable gravitational acceleration assumed a constant Earth density, yielding a linear $g(y)$. In this section, we will consider different density profiles for the Earth as described by

$$\rho_n(r) = \rho_n \left(1 - \frac{r}{R_E}\right)^n \tag{5}$$

giving a function of mass

$$M_n(r) = \int_V \rho(r) dV = 4\pi \int_0^r \rho(r) r^2 dr \tag{6}$$

where $r$ is the distance from the center of the Earth, and $\rho_n$ is a normalization constant that ensures the total mass is constant for all $n$.

Following Newton's Law of Gravitation, the ODE describing the motion of mass is, now neglecting drag and Coriolis forces, is

$$\frac{d^2y}{dt^2} = -\frac{GM_n(y)}{y^2} \tag{7}$$

To analyze the effect of different density profiles, we will consider an infinitely deep mine, and determine the "crossing timeöf the mass i.e. the time for the mass to reach the center of the Earth. In particular, we will compare the $n = 0$ and $n = 9$ cases. Figure 1 shows the position and velocity for various n, including $n = 0$ and $n = 9$.

For $n = 0$, i.e. constant density, the crossing time was found to be $t = 1267.3\ s$. For $n = 9$, the crossing time was found to be $t = 943.9\ s$. Plots of position and velocity of the mass vs time are shown on Fig. 2, including $n = 0$ and $n = 9$, as well as some other density profiles. The density profile of the Earth is significant to the particle's motion, because the gravitational force depends on the proportion of Earth's mass that is interior to the height of the particle.

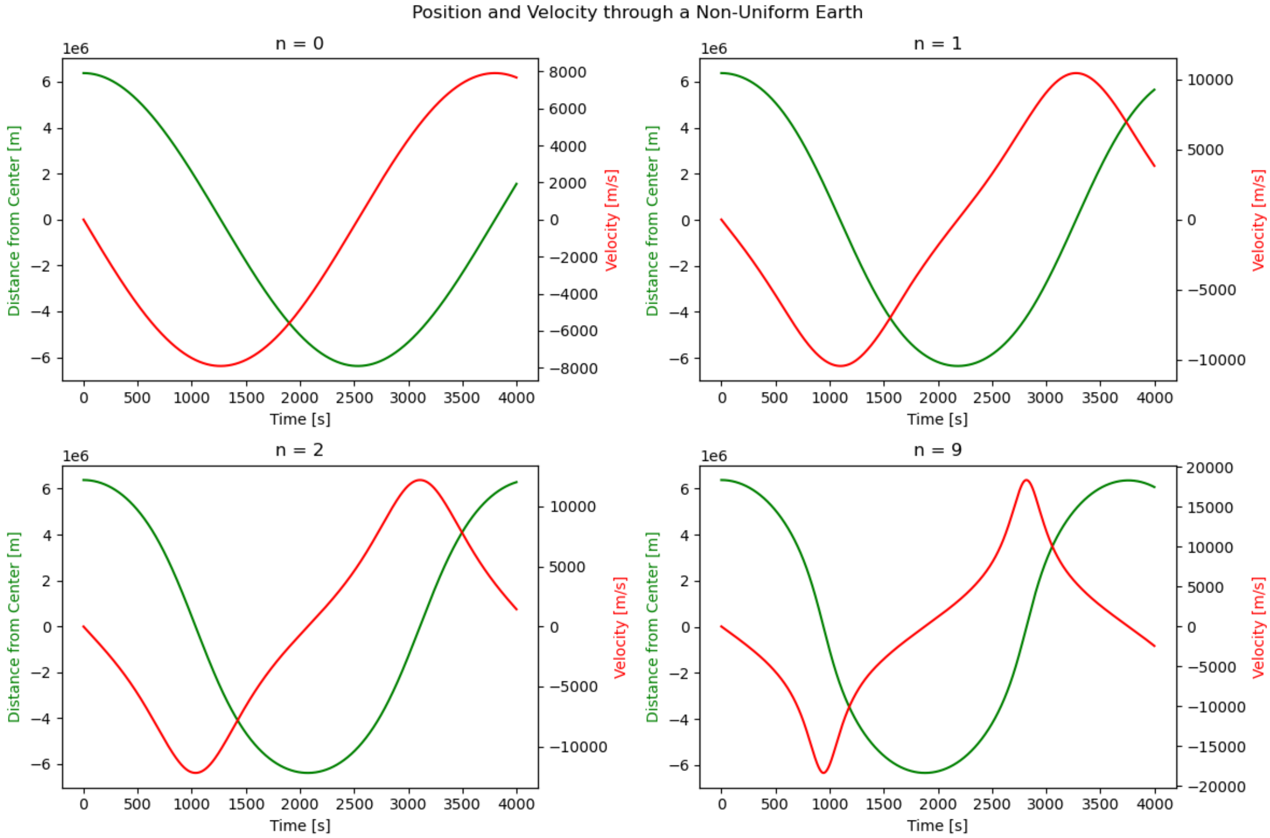
The same process was done to find the crossing time of a mineshaft dug through the Moon, assuming constant density ($n = 0$), yielding a time $t = 1624.9\ s$. The crossing time is proportional to $\rho^{-1/2}$, which can be seen when comparing the density of the Earth $\rho_E = 5494.9\ \frac{kg}{m^3}$ to the density of the Moon, $\rho_M = 3341.8\ \frac{kg}{m^3}$. Fig. 2 shows this relationship more explicitly, displaying a $f(\rho) = A\rho^{-1/2}$ fit over 3 computed fall times.

# 5 Discussion and Future Work

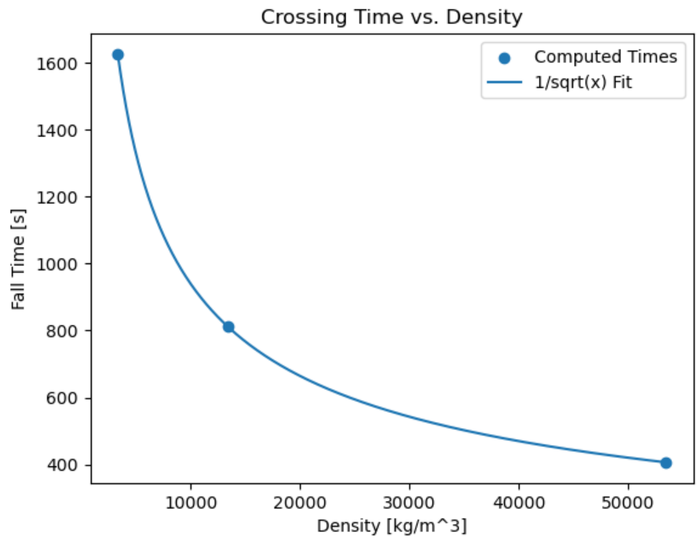
In this report, we describe various models for the free fall of a 1 kg mass down a vertical mineshaft. We compare models that consider the effects of drag, and the Coriolis force. We also consider the effect that the density profile of the planet has on the fall time.

We find that the dropping the mass down the mine would not be a successful method of measuring the mines depth, as the effect of the Coriolis force will push the mass into the sides of the mineshaft, unpredictably effecting its motion. We also find that, for a sufficiently deep mineshaft, the density profile of planet (or moon) plays a significant role in the fall time, and that, for constant density, the crossing time decreases as $\frac{1}{\sqrt{\rho}}$.

To better model this physical situation, one should consider the effect of a non-spherical Earth on the fall time, as that would change the mass distribution. A more rigorous model of the drag force would also be needed, as the properties of the air, including temperature and density, would change as the mass falls deeper into the mineshaft, affecting drag force.




Figur 1: Time evolution of position and velocity of the 1 kg mass for $n = 0, 1, 2, 9$ density profiles.




Figur 2: Crossing time plotted against density. The plot shows a $\rho^{-1/2}$ relation.