Numerical Methods for Stochastic Ordinary Differential Equations (SODEs)

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Introduction

- Deterministic ODEs vs. Stochastic Differential Equations

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- Brownian Motion and Wiener Process
  1. Definitions, Properties, Examples
  2. Sample Paths in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$

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  1. Itô and Stratonovich Calculus

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- Geometric Brownian Motion
- Euler-Maruyama Method
- Milstein Method
- Monte Carlo Method
  1. What is a Monte Carlo Simulation?
  2. Approximation of Logistic Equation
  3. Approximation of Geometric Brownian Motion

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- Higher Order Taylor and Runge Kutta Methods

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- Applications include population dynamics, neuron activity, option pricing, radio-astronomy, satellite orbit stability, blood clotting, turbulent diffusion, Josephson tunneling in semiconductors, stochastic differential geometry, and many more.
- Filtering problems - algorithms that use measurements over time that contain "noise", and give estimates for unknown quantities.

Deterministic ODEs

Consider the ordinary differential equation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)) & \text{for } t > 0 \\ \mathbf{x}(0) = x_0 & x_0 \in \mathbb{R}^n \end{cases}$$

where $\mathbf{f}$ is a given smooth vector field, and the solution $\mathbf{x}(t) : [0, \infty) \to \mathbb{R}^n$ is the *trajectory*.

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$$\begin{cases} \dot{x}(t) = x(t)(1 - x(t)) & \text{for } t > 0 \\ x(0) = x_0 & x_0 \in \mathbb{R} \end{cases}$$

which has the exact solution $x(t) = \frac{1}{1+\left(\frac{1}{x_0}-1\right)e^{-t}}$

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Let Ω be a non-empty set, $\mathcal{U}$ be a σ-algebra of subsets of Ω, and $\mathbb{P}$ be the probability measure on $\mathcal{U}$. We define a ***probability space*** to be the triple $(\Omega, \mathcal{U}, \mathbb{P})$.

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Definition

Let $(\Omega, \mathcal{U}, \mathbb{P})$ be a probability space and $\mathcal{B}$ be the Borel subsets of $\mathbb{R}$. Then the mapping

$$X : \Omega \to \mathbb{R} \tag{1}$$

is a ***random variable*** if for each $B \in \mathcal{B}$, then $X^{-1}(B) \in \mathcal{U}$.

Stochastic ODEs (cont.)

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Now we can modify the general deterministic ODE that we have seen. Mimicking what we saw for ODEs, we write

$$\begin{cases} \dot{X}_t = f(t, X_t) + F(t, X_t)\xi_t & \text{for } t > 0 \\ X_0 = x_0 & x_0 \in \mathbb{R} \end{cases}$$

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where F and f are sufficiently smooth functions, and X_t is a stochastic process. But what is ξ_t?

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$$\frac{dX_t}{dt} = f(t, X_t) + F(t, X_t)\frac{dW_t}{dt} \tag{2}$$

where W_t turns out to be *Brownian motion*, or a *Wiener process*. Symbolically (being careful about what $\frac{d}{dt}$ means!) we write

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This seems to say that the time derivative of a Brownian motion is white noise. We will see this is not quite correct (in the usual sense), once we define what Brownian motion is.

White Noise (A more formal definition)

Definition

Let $\mathcal{T}$ be an indexing set, and $X := \{X_t\}_{t \in \mathcal{T}}$ be a stochastic process. Then X is a ***Gaussian random field*** (or *Gaussian process* if $\mathcal{T} \subset \mathbb{R}$) if $(X_{t_1}, \dots, X_{t_n})$ is a Gaussian random vector for all $t_1, \dots, t_n \in \mathcal{T}$.

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Definition

Let $\mathcal{A} := \mathcal{A}(\mathbb{R}^n)$ denote the collection of all Borel-measurable subsets of $\mathbb{R}^n$ that have finite Lebesgue measure. Then ***white noise*** on $\mathbb{R}^n$ is a mean-zero, set indexed, Gaussian random field $\{\xi(A)\}_{A\in\mathcal{A}}$, with covariance function

$$E\left[\xi(A_1)\xi(A_2)\right] := m(A_1 \cap A_2) \quad \text{for all } A_1, A_2 \in \mathcal{A} \tag{4}$$

where m denotes Lebesgue measure.

SODE in standard form

Returning back to SODEs, we can write a general SODE in the general differential form

$$\begin{cases} dX_t = f(t, X_t)\ dt + F(t, X_t)\ dW_t & \text{for } t > 0 \\ X_0 = x_0 & x_0 \in \mathbb{R} \end{cases}$$

where the terms dX_t and FdW_t are called *stochastic differentials*.

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where the terms dX_t and FdW_t are called *stochastic differentials*. We say the stochastic process X_t "solves" the SODE provided

$$X_t = x_0 + \int_0^t f(t, X_s)\ ds + \int_0^t F(t, X_s)\ dW_s \quad \text{for all } t > 0 \tag{5}$$

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For those who have taken 207A, this is similar to the integral form of the deterministic problem we saw earlier

$$x(t) = x_0 + \int_0^t f(s, x(s))ds \qquad (6)$$

Solution

We stated previously that the stochastic process X_t “solves” the SODE provided

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Problems

- What is Brownian motion W_t?
- How do we integrate with respect to a Brownian motion?
- Does (7) make sense, and if so, show a solution exists.

Brownian Motion and Wiener Process

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- In a stochastic process there is randomness, even if the initial condition is known. There are infinitely many directions in which the process may evolve.
- Brownian motion was first observed in 1826 by R. Brown, as the result of pollen particles being moved by water molecules in a container.

Definition

A ***Wiener process***, also called *standard Brownian motion* is a continuous-time stochastic process with certain criteria. Specifically, $W_0 = 0$, $W_t - W_s \sim N(0, t-s)$ for $t \geq s \geq 0$, and W_t has independent increments.

Brownian motion Properties

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- Construction of Brownian motion due to Lèvy, gives Brownian motion as

$$W_t(\omega) = \sum_{k=0}^{\infty} A_k(\omega) s_k(t) \tag{8}$$

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  where $\{A_k(\omega)\}_{k=0}^{\infty}$ are a sequence of independent $N(0,1)$ random variables from the same probability space.
- Brownian motion sample paths, $t \mapsto W_t(\omega)$ are uniformly Hölder continuous for each exponent $0 < \gamma < \frac{1}{2}$.
- Brownian motion paths are almost surely nowhere differentiable.

Brownian motion Properties (cont.)

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Definition

Let $\{X_t | t \geq 0\}$ be a stochastic process such that $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$. If

$$\mathbb{E}(X_t | \mathcal{U}_s) = X_s \quad \text{a.s. for all } t \geq s \geq 0, \tag{9}$$

where $\mathcal{U}_s$ is the σ-algebra generated by random variables up to and including X_s, then $\{X_t | t \geq 0\}$ is a ***martingale***.

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where $\mathcal{U}_s$ is the σ-algebra generated by random variables up to and including X_s, then $\{X_t | t \geq 0\}$ is a ***martingale***. If

$$\mathbb{P}(X_t \in B | \mathcal{U}_s) = \mathbb{P}(X_t \in B | X_s) \quad \text{a.s. for all } 0 \leq s \leq t \tag{10}$$

and Borel sets B of $\mathbb{R}$, then $\{X_t | t \geq 0\}$ is a ***Markov process***.

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2) W_t is neighborhood recurrent, but not point recurrent in dimension $d = 2$
*A Markov process, $\{X_t | t \geq 0\}$, is **neighborhood recurrent** if for every $x \in \mathbb{R}^d$ and $\epsilon > 0$, there is a random sequence $t_n \nearrow \infty$ such that $X_{t_n} = B(x, \epsilon)$ for all $n \in \mathbb{N}$ almost surely.*

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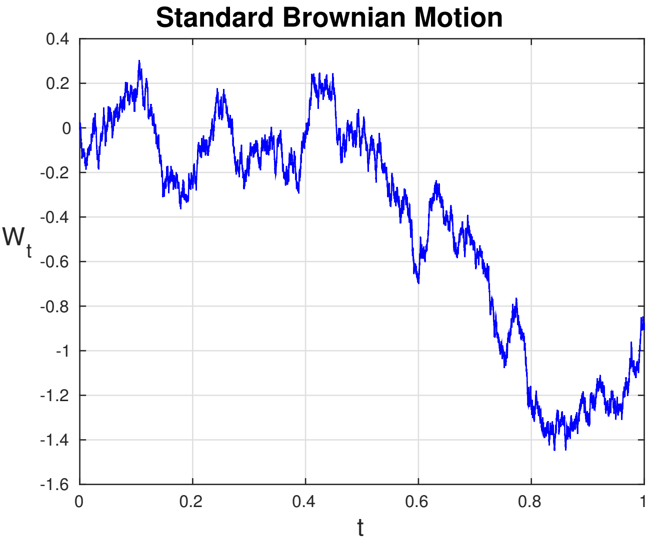
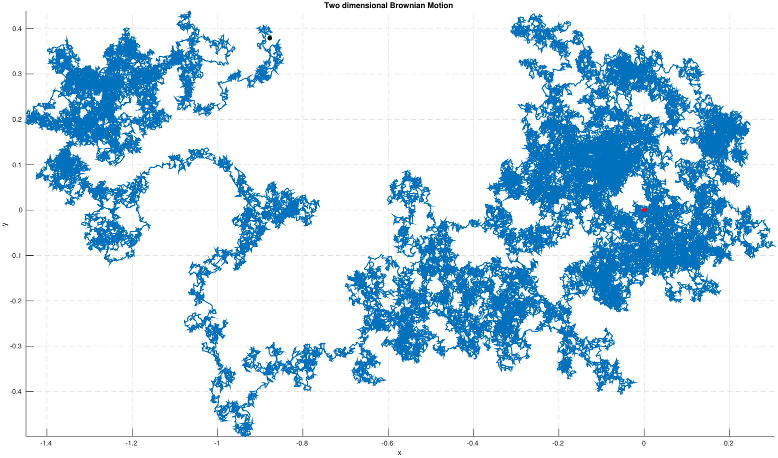
3) W_t is transient in dimension $d \geq 3$
*A Markov process, $\{X_t | t \geq 0\}$, is **transient** if it converges to infinity almost surely.*

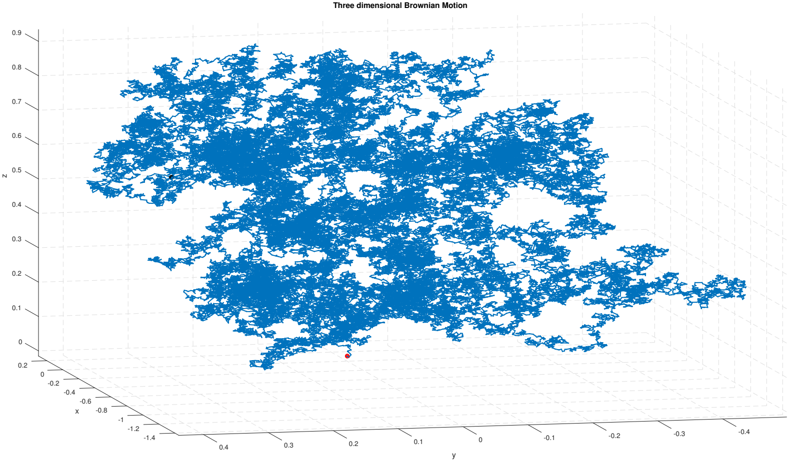
Brownian Motion 1D Sample Path



Brownian Motion in $\mathbb{R}^2$ Sample Path

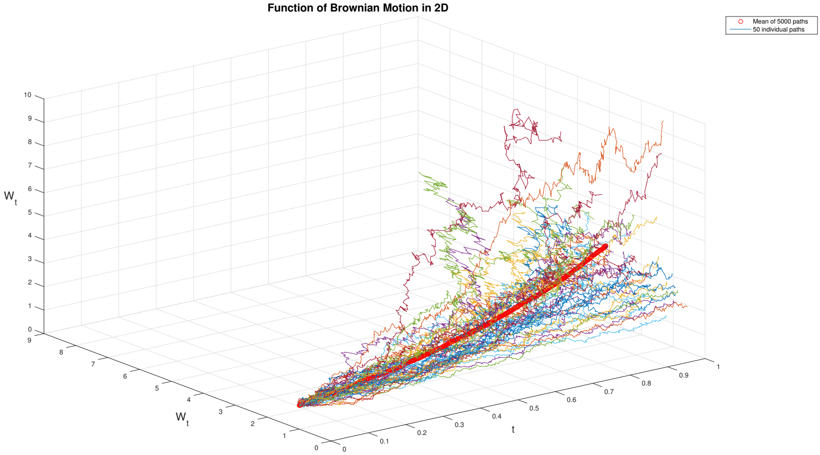


Brownian Motion in $\mathbb{R}^3$ Sample Path



Function of a Brownian Motion in 2D

Sample paths of $f(t, W_t) = e^{t+\frac{1}{2}W_t}$ in 2D.



Itô Calculus

Recall the general SODE

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Definition

For a smooth function, $f(t, X_t)$, the ***Itô integral*** of f with respect to standard Brownian motion W_t is given by

$$\int_0^t f_s(s, X_s) dW_s = \lim_{n \to \infty} \sum_{i=1}^{n-1} f(t_{i-1}, X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \quad (11)$$

where we evaluate the function f at the *left endpoints* of some partition $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = T\}$.

Itô Lemma (Chain Rule)

For a twice differentiable function $f(t,x)$, its Taylor series is

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \dots \tag{12}$$

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Assume that $dX_t = \mu_t\ dt + \sigma_t dW_t$. Let $x = X_t$ and $dx = \mu_t dt + \sigma_t dW_t$ for dX_t, we then get

$$\begin{aligned} df(t,X_t) = \frac{\partial f}{\partial t}dt + &\frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dW_t) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t^2 dt^2 + 2\mu_t\sigma_t dt dW_t + \sigma_t^2 dW_t^2) + \dots \end{aligned}$$

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$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \dots \tag{12}$$

Assume that $dX_t = \mu_t\ dt + \sigma_t dW_t$. Let $x = X_t$ and $dx = \mu_t dt + \sigma_t dW_t$ for dX_t, we then get

$$\begin{aligned} df(t,X_t) = &\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dW_t) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t^2 dt^2 + 2\mu_t\sigma_t dt dW_t + \sigma_t^2 dW_t^2) + \dots \end{aligned}$$

Neglecting "small" high order terms such as $dtdW_t$ and dW_t^2, we have

$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}dW_t$$

This is Itô's chain rule.

Stratonovich Integral

Definition

For a smooth function, $f(t, X_t)$, the ***Stratonovich integral*** of f with respect to standard Brownian motion W_t is given by

$$\int_0^T f(t, W_t) \circ dW_t = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\tau_k, W_{\tau_k})(W_{t+1} - W_t) \tag{13}$$

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where we evaluate the function f at the *midpoints* of some partition $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = T\}$.

Example

Using the Itô formulation, we can compute the following simple integral

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Notice that we get two different answers for the same integral. The Itô integral requires a "correction term", whereas the Stratonovich integral is what we would normally expect.

Ito-Stratonovich Conversion

The Ito and Stratonovich Conversion is by the following

$$\int_0^T f'(W_t) \circ dW_t = \frac{1}{2}\int_0^T \frac{\partial f}{\partial W}(t, W_t)dt + \int_0^T f'(W_t)dW_t \quad (16)$$

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We can convert from Ito to Stratonovich SODEs or vice versa whenever one is convenient.

Definition

$$\frac{dX_t}{dt} = f(t, X_t) + g(t, X_t)\ \xi \qquad (Ito) \quad (17)$$

$$\frac{dX_t}{dt} = (f(t, X_t) - \frac{1}{2}g(t, X_t)\frac{\partial}{\partial X_t}g(t, X_t)) + g(t, X_t) \circ \xi \quad (18)$$

$$(Stratonovich)$$

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We will run numerical experiments on a simple SODE that has an *exact* solution. We consider Geometric Brownian motion

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Geometric Brownian Motion (cont.)

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where we have used a direct application of Itô's chain rule:

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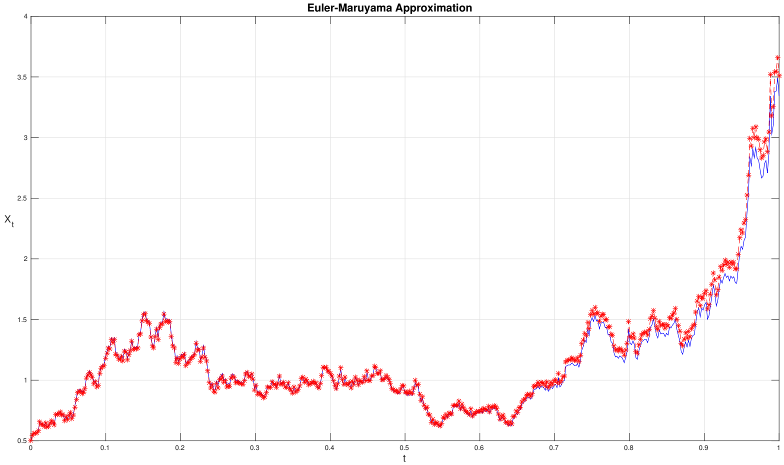
$$|\mathbb{E}p(X_n) - \mathbb{E}p(X_T)| \leq C \Delta t^{\gamma} \tag{22}$$

Euler-Maruyama Results



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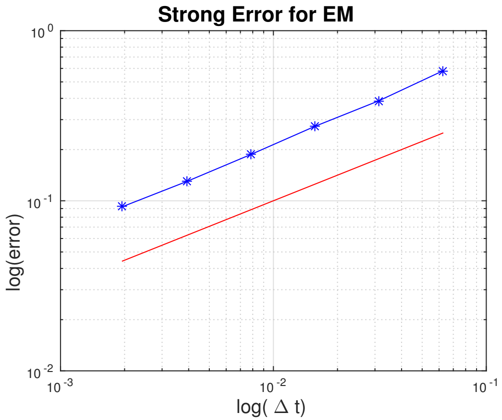

Figure : Log-log plot of error for various time steps. Red reference line is slope 1/2.

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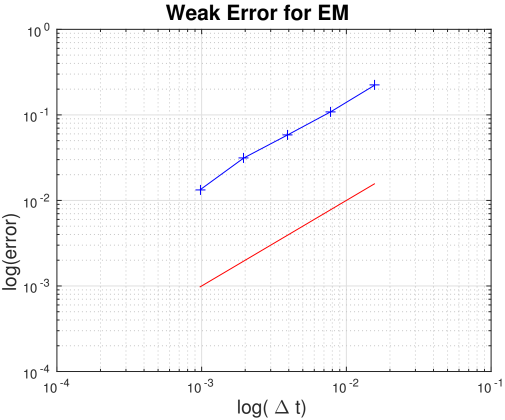

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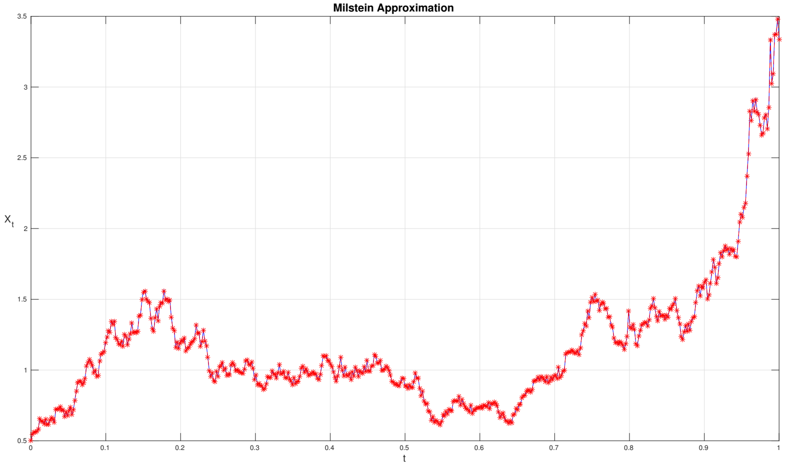
Milstein Method Results



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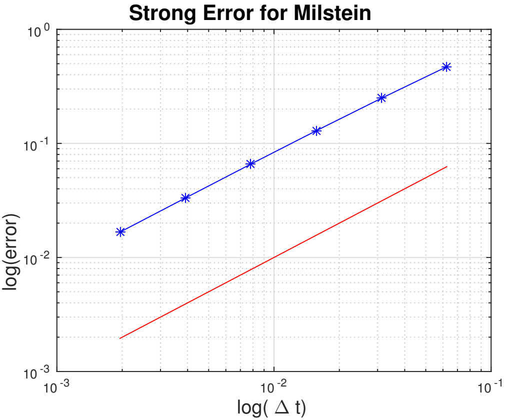

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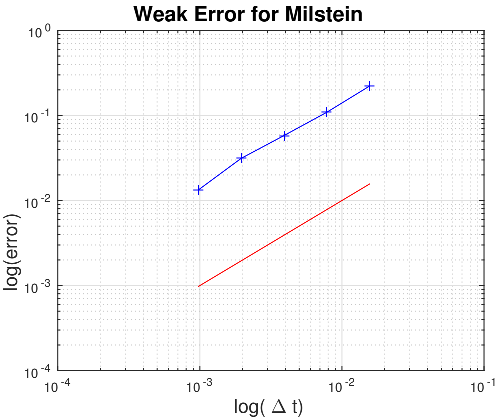

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Monte Carlo Methods

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$$\sigma_M^2 = \frac{1}{M-1}\sum_{j=1}^{M}\left(X_j - \bar{X}_M\right)^2 \tag{24}$$

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where we take $T = 10$ and $\epsilon = 0.1$. The term $U(D)$ denotes that we will choose y_0 independently from a uniform distribution on the set D, as defined above.

Approximation of the Logistic Equation (cont.)

The value of ϵ is chosen to such that some "error" is incurred when measuring the initial data, say for some population.

Example

The mean of the random initial data can be calculated as follows:

$$\bar{y}_0 = \frac{1}{2}(b-a) = \frac{1}{2}\left(\frac{1}{2} - \epsilon + \frac{1}{2} + \epsilon\right) = \frac{1}{2}$$

The true solution of the IVP is known for the mean, $\bar{y}_0$

$$\bar{y}(t) = \frac{e^t}{e^t + 1}$$

So we take

$$\mathbb{E}\left[y(T)\right] = \frac{e^T}{e^T + 1}$$

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- The error of the explicit Euler method is first order in Δt and the order of the error of the Monte Carlo sampling is $\mathcal{O}(M^{-1/2})$.
- To maintain a fixed level of accuracy, $\delta = C\Delta t$, we balance the two errors by holding the quantity $(\Delta t)^2 M$ fixed. This follows from the fact that, in general, the Var$(y_N(t)) = \sigma^2$ and $K = ||y_N(t)||_{C^2}$ are unknown.

Approximation of the Logistic Equation (cont.)

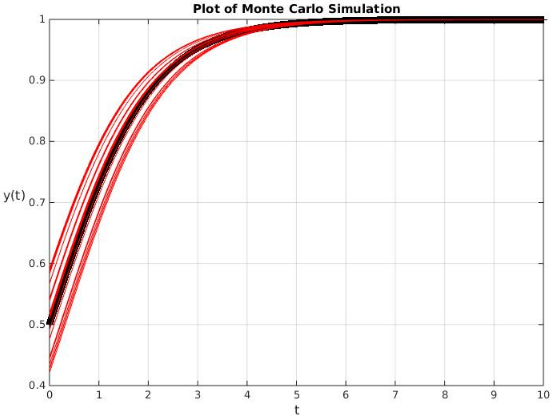

Figure : The function $\bar{y}(t)$ is given by the black '$+$' signs, which appear as a bold dark line. The plot also includes 20 uniform random initial conditions and corresponding solutions.

Approximation of the Logistic Equation (cont.)

Figure : Forward Euler is used with $(\Delta t)^2 M$ = constant. The blue line is the reference for slope of -1. The red is the approximation of the error which is first order in Δt.

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For simplicity, we take $T = 1$. The parameters will be taken as $\alpha = 2$, $\beta = 0.1$.

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- If we want the *weak approximation* of the solution to the SODE, we must use multiple paths, since the quantity of interest will be $\mathbb{E}[X_T]$.

Definition

A method has ***weak order convergence*** of γ if there exists a constant C such that for all functions p in some class

$$|\mathbb{E}[p(X_n)] - \mathbb{E}[p(X_\tau)]| \leq C\Delta t^\gamma \tag{26}$$

at any fixed $\tau = n\Delta t \in [0, T]$ and Δt sufficiently small.

Approximation of Geometric Brownian Motion (cont.)

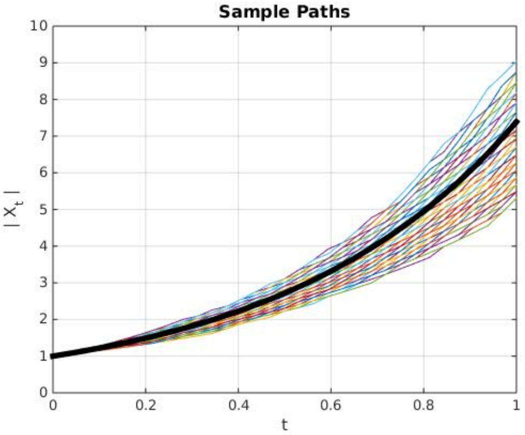

Figure : Sample paths for $M = 500$ for the solution to the Geometric Brownian motion. The value of $\mathbb{E}[X_T] = e^{\mu T} = e^2$. The dark line represents the function $y(t) = e^{2t}$.

Approximation of Geometric Brownian Motion (cont.)

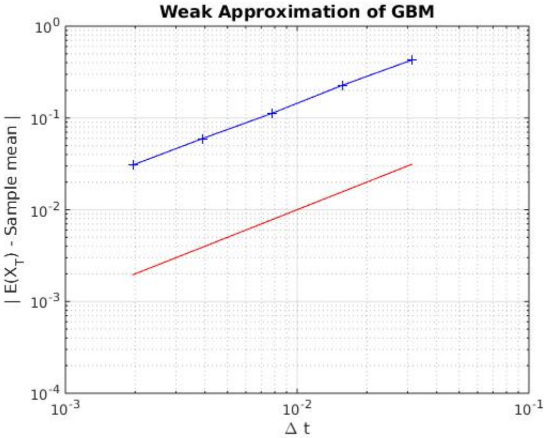

Figure : Weak Approximation error for GBM. The red reference line is that of slope 1.

Strong 1.0 Order Runge-Kutta Method

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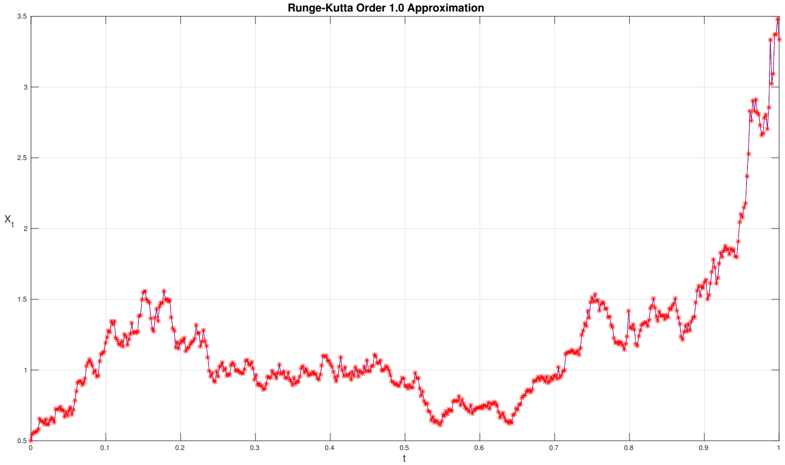
- Of course by the name, the method is of strong order 1.0.

Strong 1.0 Order Runge-Kutta Method Results



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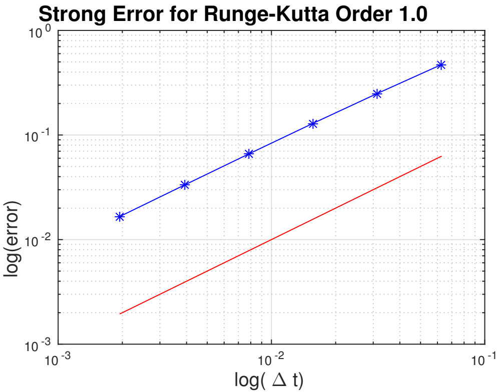

Figure : Log-log plot of error for various time steps. The red reference line is that of slope 1.

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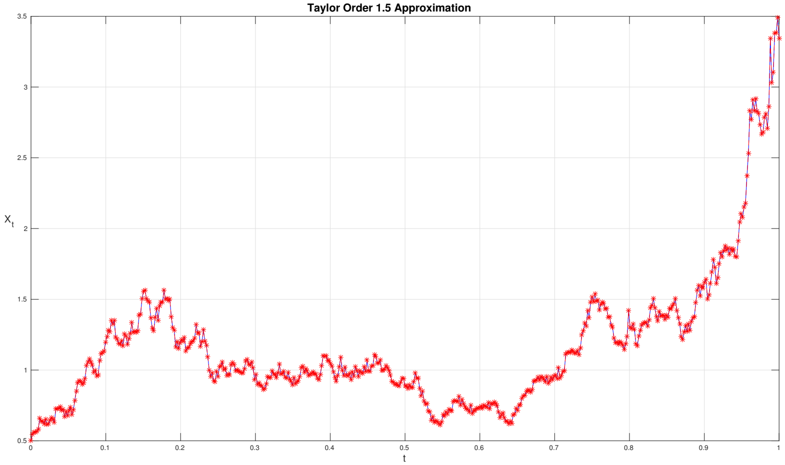
Strong 1.5 Order Taylor Method Results



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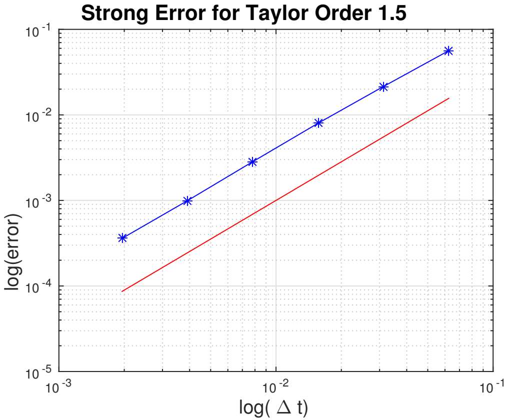

Figure : Log-log plot of error for various time steps. The red reference line is that of slope 1.5.

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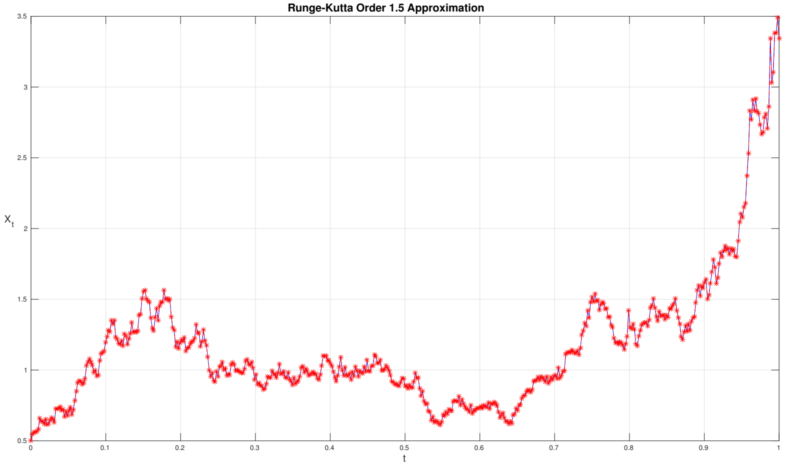

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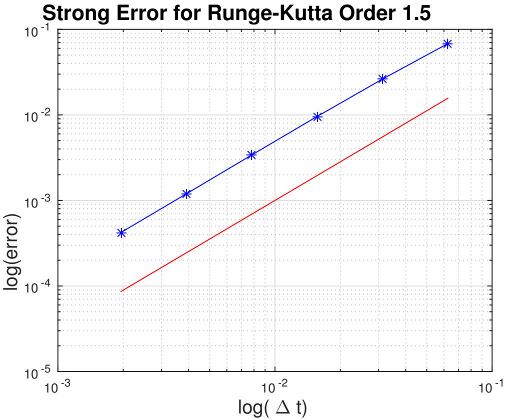

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Conclusion

Numerical Results for all Methods (Strong Order Error)

Error at t = T				
Δt	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Euler	7.8616e-01	2.1330e-01	7.1713e-02	2.0921e-02
Milstein	7.5554e-01	9.1122e-02	9.3412e-03	9.3298e-04
RK 1.0	7.5554e-01	9.1122e-02	9.3412e-03	9.3298e-04
RK 1.5	1.1295e-01	4.0662e-03	1.4553e-04	4.2337e-06
Taylor 1.5	1.2955e-01	5.0419e-03	1.6959e-04	5.1033e-06