# NAIL062 P&P Logika: Worksheet 3 – Algebra of propositions, SAT

**Teaching goals:** After completing, the student

- understands the relationship between propositions/theories up to [$T$-]equivalence and sets of models (the so-called algebra of propositions), can apply in concrete examples
- can encode a given problem as an instance of SAT
- has gained practical experience with using a SAT solver
- understands the algorithm for solving 2-SAT using the implication graph (including finding all models), and can apply it to an example
- understands the algorithm for solving Horn-SAT using unit propagation, and can apply it to an example
- understands the DPLL algorithm and can apply it to an example

## In-class problems

**Problem 1.** Let $|\mathbb{P}| = n$ and let $\varphi \in \mathrm{VF}_{\mathbb{P}}$ be a proposition such that $|M(\varphi)| = k$. Determine (up to equivalence):

(a) the number of propositions $\psi$ such that $\varphi \models \psi$ or $\psi \models \varphi$,
(b) the number of theories over $\mathbb{P}$ in which $\varphi$ is valid,
(c) the number of complete theories over $\mathbb{P}$ in which $\varphi$ is valid,
(d) the number of theories $T$ over $\mathbb{P}$ such that $T \cup \{\varphi\}$ is consistent.

Now, consider a contradictory theory $\{\varphi, \psi\}$ where $|M(\psi)| = p$. Compute (up to equivalence):

(e) the number of propositions $\chi$ such that $\varphi \vee \psi \models \chi$,
(f) the number of theories in which $\varphi \vee \psi$ is valid.

**Solution.** *(a) The condition can be expressed in terms of sets of models:* $\mathrm{M}(\varphi) \subseteq \mathrm{M}(\psi)$ *or* $\mathrm{M}(\psi) \subseteq \mathrm{M}(\varphi)$*. There are* $2^n$ *total models, and* $|\mathrm{M}(\varphi)| = k$*. We want to count possible sets* $\mathrm{M}(\psi)$*. The condition* $\mathrm{M}(\varphi) \subseteq \mathrm{M}(\psi)$ *holds for* $2^{2^n-k}$ *sets (i.e. the number of supersets of the given* $k$*-element set inside a* $2^n$*-element universe), and the condition* $\mathrm{M}(\psi) \subseteq \mathrm{M}(\varphi)$ *holds for* $2^k$ *sets. We must subtract one to avoid double-counting the case* $\mathrm{M}(\psi) = \mathrm{M}(\varphi)$*. Altogether there are* $2^{2^n-k} + 2^k - 1$ *possible model sets, hence that many propositions* $\psi$ *up to equivalence.*

*(b)* $T \models \varphi$ *iff* $\mathrm{M}(T) \subseteq \mathrm{M}(\varphi)$*; the number of such model sets* $\mathrm{M}(T)$ *is* $2^k$*.*

*(c) Additionally we require* $|\mathrm{M}(T)| = 1$*; the number of* 1*-element subsets of the* $k$*-element set is* $k$*.*

*(d) In terms of models the condition says* $\mathrm{M}(T \cup \{\varphi\}) \neq \emptyset$*. Since* $\mathrm{M}(T \cup \{\varphi\}) = \mathrm{M}(T) \cap \mathrm{M}(\varphi)$*, we count how many possible sets* $\mathrm{M}(T)$ *have a nonempty intersection with the* $k$*-element set* $\mathrm{M}(\varphi)$*. One way to express this is* $(2^k - 1) \cdot 2^{2^n-k}$*, where* $2^k - 1$ *counts the nonempty possible intersections* $\mathrm{M}(T) \cap \mathrm{M}(\varphi)$ *and* $2^{2^n-k}$ *counts arbitrary choices for membership of models outside* $\mathrm{M}(\varphi)$*.*

*(e) Since* $\{\varphi, \psi\}$ *is contradictory we have* $\emptyset = \mathrm{M}(\varphi, \psi) = \mathrm{M}(\varphi) \cap \mathrm{M}(\psi)$*. We count sets* $\mathrm{M}(\chi)$ *with* $\mathrm{M}(\varphi \vee \psi) \subseteq \mathrm{M}(\chi)$*. By the Lindenbaum–Tarski algebra* $\mathrm{M}(\varphi \vee \psi) = \mathrm{M}(\varphi) \cup \mathrm{M}(\psi)$*. Disjointness gives* $|\mathrm{M}(\varphi) \cup \mathrm{M}(\psi)| = k + p$*, so the number of choices for* $\mathrm{M}(\chi)$ *is* $2^{2^n-(k+p)}$*.*

*(f)* $\mathrm{M}(T)$ *must be a subset of the* $(k+p)$*-element set* $\mathrm{M}(\varphi \vee \psi)$*, hence there are* $2^{k+p}$ *possibilities.*

**Problem 2.** Build the implication graph of the given 2-CNF formula. Is it satisfiable? If yes, find some solution: (a) the proposition $\varphi$ below, (b) $\varphi \wedge \neg p_1$, (c) $\varphi \wedge \neg p_1 \wedge (p_1 \vee p_2)$.

$$\varphi = (p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee \neg p_1)$$

**Solution.** *(a) Construct the implication graph. One finds two strongly connected components:* $C = \{p_1, p_2, \neg p_3, p_4, \neg p_5\}$ *and* $\overline{C} = \{\neg p_1, \neg p_2, p_3, \neg p_4, p_5\}$*, and there are no edges between them. After contracting the components we obtain a two-vertex graph* $\mathcal{G}^*$ *with no edges; it has two topological orders* $(C, \overline{C})$ *and* $(\overline{C}, C)$*, which correspond to the models* $(0,0,1,0,1)$ *and* $(1,1,0,1,0)$ *respectively.*

*(b) The components are the same, but adding* $\neg p_1$ *forces an edge* $C \to \overline{C}$ *in* $\mathcal{G}^*$*, so the only topological order is* $(C, \overline{C})$*, yielding the model* $(0,0,1,0,1)$*.*

*(c) With the extra clause the implication graph becomes strongly connected; its single component contains complementary literals, so the formula is unsatisfiable.*

**Problem 3.** Use unit propagation to decide whether the following Horn formula is satisfiable. If yes, find a satisfying assignment.

$$(\neg p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2) \wedge p_1 \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge$$
$$(p_1 \vee \neg p_2 \vee \neg p_4) \wedge (\neg p_2 \vee \neg p_3 \vee \neg p_4) \wedge (p_4 \vee \neg p_5 \vee \neg p_6)$$

**Solution.** *Perform unit propagation step by step starting over the literals* $p_1, p_2, p_3, \neg p_4$ *(in order); the remaining clause will be* $\neg p_5 \vee \neg p_6$*. It suffices to assign* $p_5 = 0$ *or* $p_6 = 0$ *to make the whole formula true. The models are* $\{(1,1,1,0,0,1), (1,1,1,0,1,0), (1,1,1,0,1,1)\}$*.*

**Problem 4.** Use the DPLL algorithm to decide if the following CNF formula is satisfiable:

$$(\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2) \wedge (p_2 \vee \neg p_3) \wedge (p_1 \vee p_3)$$

**Solution.** *The formula has no unit clauses and no pure literals, so we must branch, say on* $p_1$*:*

- *For* $\varphi \wedge p_1$*: unit propagation yields* $\neg p_2 \wedge p_2 \wedge (p_2 \vee \neg p_3)$*. Propagating* $\neg p_2$ *produces* $\square \wedge \neg p_3$*, which containts the empty clause* $\square$*, so this branch is unsatisfiable.*
- *For* $\varphi \wedge \neg p_1$*: unit propagation yields* $\neg p_2 \wedge (p_2 \vee \neg p_3) \wedge p_3$*. Propagating* $\neg p_2$ *gives* $\neg p_3 \wedge p_3$*, after unit propagation over* $\neg p_3$ *we get the empty clause* $\square$*, so this branch is again unsatisfiable.*

*In both (all) branches we proved a contradiction, therefore the formula is unsatisfiable.*

**Problem 5.** Given a directed graph, we want to determine whether it is acyclic and, if so, find a topological ordering. Encode this problem as SAT.

**Solution.** *Sketch of a solution. Use the language* $\mathbb{P} = \{p_{uv} \mid u, v \in V\}$ *where* $p_{uv}$ *means "vertex* $u$ *appears strictly before* $v$ *in the ordering". Enforce that this is a strict (irreflexive, antisymmetric, transitive) order by axioms:*

- $\neg p_{vv}$ *for all* $v \in V$*,*
- $p_{uv} \to \neg p_{vu}$ *for all* $u, v \in V$*,*
- $(p_{uv} \wedge p_{vw}) \to p_{uw}$ *for all* $u, v, w \in V$*.*

*It remains to enforce that all graph edges go forward in the topological order:*

- $p_{uv}$ *for each edge* $(u, v) \in E$*.*

*Finally, convert the above axioms to CNF. In set notation we get:*

$$S = \{\{\neg p_{vv}\}, \{\neg p_{uv}, \neg p_{vu}\}, \{\neg p_{uv}, \neg p_{vw}, p_{uw}\} \mid u, v, w \in V\} \cup \{\{p_{uv}\} \mid (u, v) \in E\}.$$

## Extra practice

**Problem 6.** Consider the following propositions $\varphi$ and $\psi$ over $\mathbb{P} = \{p, q, r, s\}$:

$$\varphi = (\neg p \vee q) \to (p \wedge r)$$
$$\psi = s \to q$$

(a) Determine the number (up to equivalence) of propositions $\chi$ over $\mathbb{P}$ such that $\varphi \wedge \psi \models \chi$.
(b) Determine the number (up to equivalence) of complete theories $T$ over $\mathbb{P}$ such that $T \models \varphi \wedge \psi$.
(c) Find an axiomatization for each (up to equivalence) complete theory $T$ over $\mathbb{P}$ such that $T \models \varphi \wedge \psi$.

**Problem 7.** Using the unit propagation algorithm, find all models of:

$$(\neg a \vee \neg b \vee c \vee \neg d) \wedge (\neg b \vee c) \wedge d \wedge (\neg a \vee \neg c \vee e) \wedge$$
$$(\neg c \vee \neg d) \wedge (\neg a \vee \neg d \vee \neg e) \wedge (a \vee \neg b \vee \neg e)$$

**Problem 8.** Solve using the implication graph as in Example **??**, and also using the DPLL algorithm as in Example **??**:

(a) $(p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee p_1)$
(b) $(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1 \wedge \neg p_7$

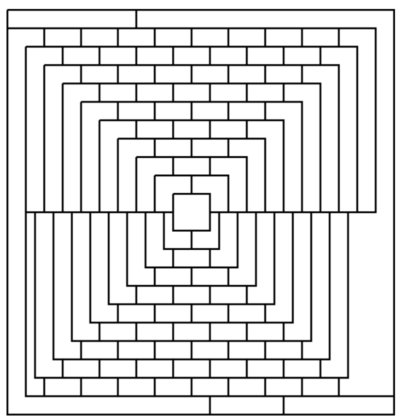
**Problem 9.** Can the numbers 1 to $n$ be colored with two colors so that there is no monochromatic solution of the equation $a+b=c$ for any $1 \leq a < b < c \leq n$? Construct a propositional CNF formula $\varphi_n$ that is satisfiable iff such a coloring exists. Try $n = 8$ first.

Try at home: Write a script that generates $\varphi_n$ in DIMACS CNF format. Use a SAT solver to find the smallest $n$ for which such a coloring does not exist (i.e., every 2-coloring contains a monochromatic triple $a < b < c$ with $a + b = c$).

**Problem 10.** The four-color theorem implies that the following maps can be colored with four colors so that no two adjacent regions share the same color. Find such a coloring using a SAT solver.

(a) Map of regions of Czechia

(b) A harder instance



## For further thought

**Problem 11.** For a given proposition $\varphi$ in CNF, find a 3-CNF formula $\varphi'$ such that $\varphi'$ is satisfiable if and only if $\varphi$ is satisfiable. Describe an efficient algorithm for constructing $\varphi'$ given $\varphi$ (i.e., a *reduction* from the SAT problem to the 3-SAT problem).

**Problem 12.** Encode the problem of sorting a given $n$-tuple of integers into SAT.

**Problem 13.** Encode into SAT the well-known riddle about a farmer who needs to transport a wolf, a goat, and a cabbage across a river.