NAIL062 P&P Logic: Worksheet 5 – The resolution method

Teaching goals: After completing, the student

- knows the necessary concepts of the resolution method (resolution rule, resolvent, resolution proof/refutation, resolution tree), can formally define them and provide examples
- can work with formulas in CNF and their models in set representation
- can construct a resolution refutation of a given (even infinite) CNF formula (if it exists), and also draw the corresponding resolution tree
- knows the notion of a substitution tree, can formally define it and construct it for a concrete CNF formula
- can apply the resolution method to solve a given problem (word problem, etc.)

In-class problems

Problem 1. Let φ denote the formula $\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$. Show that φ is a tautology:

(a) Convert $\neg\varphi$ to CNF and write the resulting formula as S in set representation.
(b) Find a resolution refutation of S.

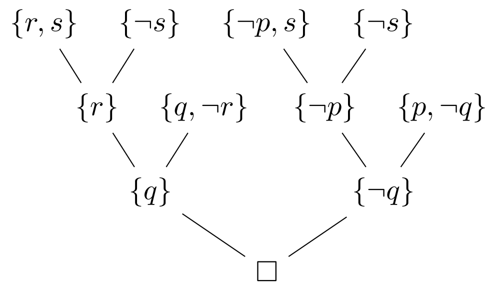
Solution. *(a) Using equivalent transformations:* $\neg\varphi = \neg(\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)) \sim \neg(\neg\neg(p \vee q) \vee (\neg p \wedge \neg q)) \sim \neg(p \vee q \vee (\neg p \wedge \neg q)) \sim \neg p \wedge \neg q \wedge \neg(\neg p \wedge \neg q) \sim \neg p \wedge \neg q \wedge (p \vee q)$

$$S = \{\{\neg p\}, \{\neg q\}, \{p, q\}\}$$

(b) Resolution refutation: $\{\neg p\}, \{p, q\}, \{q\}, \{\neg q\}, \square$ *(draw the corresponding resolution tree).*

Problem 2. Prove by resolution that in $T = \{\neg p \rightarrow \neg q, \neg q \rightarrow \neg r, (r \rightarrow p) \rightarrow s\}$ the statement s holds.

Solution. *Convert the theory* $T \cup \{\neg s\}$ *to CNF and write it in set representation. We have* $(r \rightarrow p) \rightarrow s \sim \neg(\neg r \vee p) \vee s \sim (r \wedge \neg p) \vee s \sim (r \vee s) \wedge (\neg p \vee s)$*; the other axioms convert easily. We obtain* $S = \{\{p, \neg q\}, \{q, \neg r\}, \{r, s\}, \{\neg p, s\}, \{\neg s\}\}$*. We depict the resolution refutation by a resolution tree:*



Problem 3. Let the propositional variables r, s, t represent (respectively) that *"Radka / Sára / Tom is at school"* and write $\mathbb{P} = \{r, s, t\}$. We know that:

- *If Tom is not at school, then Sára is not there either.*
- *Radka does not go to school without Sára.*
- *If Radka is not at school, then Tom is there.*

(a) Formalize our knowledge as a theory T in the language $\mathbb{P}$.

(b) Using the resolution method, prove that T entails that *Tom is at school*: Write a formula S in set representation that is unsatisfiable exactly when this holds, and find a resolution refutation of S. Draw the resolution tree.
(c) Determine the set of models of the theory T.

Solution. *(a)* $T = \{\neg t \to \neg s, \neg(r \land \neg s), \neg r \to t\}$
(b) We obtain S *from the theory* $T \cup \{\neg t\}$ *by converting to CNF:* $S = \{\{t, \neg s\}, \{\neg r, s\}, \{r, t\}, \{\neg t\}\}$

$$
\begin{array}{ccccccc}
\{r,t\} & & \{\neg t\} & & & & \\
 & \diagdown\ \diagup & & & & & \\
 & \{r\} & & \{\neg r, s\} & \{t,\neg s\} & & \{\neg t\} \\
 & & \diagdown\ \diagup & & & \diagdown\ \diagup & \\
 & & \{s\} & & & \{\neg s\} & \\
 & & & \diagdown\quad\diagup & & & \\
 & & & \square & & &
\end{array}
$$

(c) We use that $T \models t$ *(proved in (b)). Because of this, the first and third axioms are satisfied,* $T \sim \{t, \neg(r \land \neg s)\}$*. From this it follows easily that* $\mathrm{M}(T) = \{(0,0,1), (0,1,1), (1,1,1)\}$*.*

Problem 4. Construct a *substitution tree* for the following formula. Based on this tree, construct a resolution refutation according to the procedure from the proof of the Completeness Theorem for resolution.

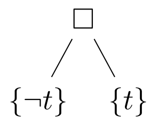
$$S = \{\{p, r\}, \{q, \neg r\}, \{\neg q\}, \{\neg p, t\}, \{\neg s\}, \{s, \neg t\}\}$$

Solution. *We branch preferentially on propositional variables in unit clauses. (Once we encounter the empty clause, we know the branch is contradictory; we do not need the rest of the formula; here, due to lack of space, we will not write it.)*

$$
\begin{array}{l}
S \\
\quad S^{q} = \{\{p,r\}, \square, \dots\} \quad \otimes \\
\quad S^{\bar q} = \{\{p,r\}, \{\neg r\}, \{\neg p, t\}, \{\neg s\}, \{s, \neg t\}\} \\
\qquad S^{\bar q r} = \{\square, \dots\} \quad \otimes \\
\qquad S^{\bar q \bar r} = \{\{p\}, \{\neg p, t\}, \{\neg s\}, \{s, \neg t\}\} \\
\qquad\quad S^{\bar q \bar r p} = \{\{t\}, \{\neg s\}, \{s, \neg t\}\} \\
\qquad\qquad S^{\bar q \bar r p s} = \{\{t\}, \square\} \quad \otimes \\
\qquad\qquad S^{\bar q \bar r p \bar s} = \{\{t\}, \{\neg t\}\} \\
\qquad\qquad\quad S^{\bar q \bar r p \bar s t} = \{\square\} \quad \otimes \\
\qquad\qquad\quad S^{\bar q \bar r p \bar s \bar t} = \{\square\} \quad \otimes \\
\qquad\quad S^{\bar q \bar r \bar p} = \{\square, \dots\} \quad \otimes
\end{array}
$$

The substitution tree provides a "recipe" for constructing a resolution refutation (which is key to the proof of the completeness theorem for resolution). We proceed from leaves to root, i.e., by the number of variables in the formulas. For the formulas at the leaves of the substitution tree we have one-element resolution refutations $\square$*.*

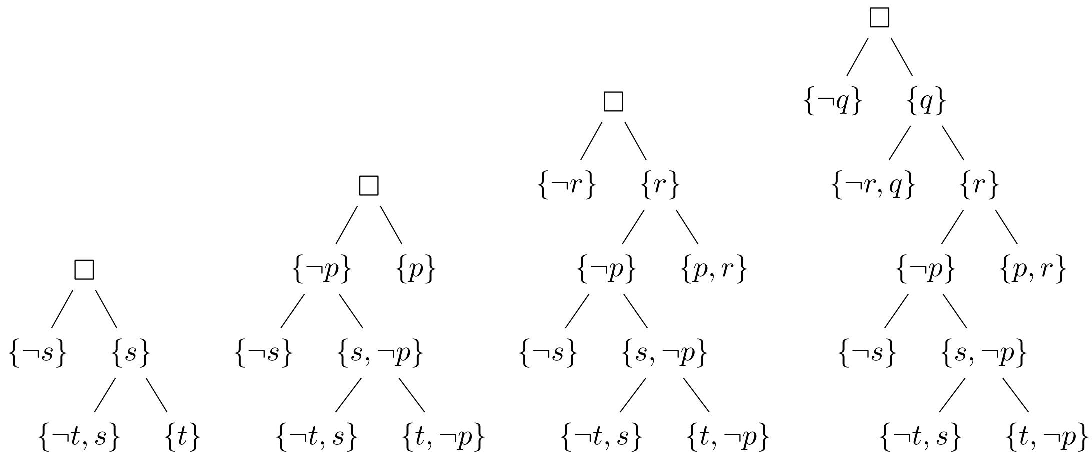
The formula $S^{\bar q \bar r p \bar s} = \{\{t\}, \{\neg t\}\}$ *has a one-step resolution refutation:*



How was this obtained? From the resolution refutation $\square$ *of the formula* $S^{\bar{q}\bar{r}p\bar{s}t}$ *we produce a resolution proof of the clause* $\{\neg t\}$ *from* $S^{\bar{q}\bar{r}p\bar{s}}$ *by, for each leaf that arose by deleting the literal* $\neg t$*, putting* $\neg t$ *back into it and into all clauses above it. (Here we have only one leaf, which is also the root* $\square$*.)*

Analogously, we produce a resolution proof of $\{t\}$ *from* $S^{\bar{q}\bar{r}p\bar{s}\bar{t}}$ *(we add the literal* t *to the nodes). And finally we add one resolution step that derives* $\square$ *from* $\{\neg t\}$ *and* $\{t\}$*. (If no leaf arose by deleting a literal from a clause of* $S^{\bar{q}\bar{r}p\bar{s}}$*, it means that the resolution refutation we have is already a resolution refutation of* $S^{\bar{q}\bar{r}p\bar{s}}$ *as well.)*

We proceed in the same way higher in the tree, for $S^{\bar{q}\bar{r}p}$*,* $S^{\bar{q}\bar{r}}$*,* $S^{\bar{q}}$*, and finally for* S*:*



Verify that the resulting tree indeed represents a resolution refutation of S*. Notice how its shape mirrors the shape of the substitution tree. (In our case the tree is a "hairy path", which need not be true in general, but the construction works the same way.)*

Extra practice

Problem 5. Find a resolution refutation of the following formulas:

(a) $\neg(((p \to q) \to \neg q) \to \neg q)$
(b) $(p \leftrightarrow (q \to r)) \wedge ((p \leftrightarrow q) \wedge (p \leftrightarrow \neg r))$

Problem 6. Tonia and Fabio describe to us their latest recipe for the best pizza in the world.

- Tonia said: "The recipe includes anchovies or basil or garlic."
- Tonia also said: "If cooked ham is not included, then basil is not included either."
- Fabio said: "Cooked ham is included in the recipe."
- Fabio further said: "Neither anchovies nor basil are included, but garlic is included."

We know that Tonia always tells the truth, while Fabio always lies.

(a) Express our knowledge as a propositional theory T in the language $\mathbb{P} = \{a, b, c, d\}$, where the propositional variables mean (in order) "anchovies / basil / garlic / cooked ham are included in the recipe".

(b) Using the resolution method, prove that T entails that "anchovies are included in the recipe". Draw the resolution tree.

Problem 7. The integers are afflicted by a mysterious disease spreading (in discrete steps) according to the following rules (valid for all integers at all time steps).

(i) A healthy number becomes ill exactly when precisely one neighboring number was ill (in the previous time).
(ii) An ill number recovers exactly when the previous number was ill (in the previous time).
(iii) At time 0 *the number* 0 *was ill, all other numbers were healthy.*

(a) Write theories T_1, T_2, T_3 expressing (respectively) statements (i), (ii), (iii) over the set of propositional atoms $\mathbb{P} = \{p_i^t \mid i \in \mathbb{Z}, t \in \mathbb{N}_0\}$, where the atom p_i^t expresses that "*number i is ill at time t.*"
(b) Convert the axioms from T_1, T_2, T_3 to CNF and write a theory S in set representation that is unsatisfiable exactly when $T_1 \cup T_2 \cup T_3 \models \neg p_1^2$, i.e. when "*Number* 1 *is healthy at time* 2." (It is enough to convert only the specific axioms from T_1, T_2, T_3 that entail $\neg p_1^2$, and include only the corresponding clauses in S.)
(c) Prove by resolution that S is unsatisfiable. Show the refutation as a resolution tree.

For further thought

Problem 8. Prove in detail that if $S = \{C_1, C_2\}$ is satisfiable and C is the resolvent of C_1 and C_2, then C is satisfiable as well.