NAIL062 P&P Logic: Worksheet 6 – Basics of predicate logic

Teaching goals: After completing, the student

- understands the notion of a structure and a signature, can formally define them and provide examples
- understands the notions of the syntax of predicate logic (language, term, atomic formula, formula, theory, free variable, open formula, sentence, instance, variant), can formally define them and provide examples
- understands the notions of the semantics of predicate logic (value of a term, truth value, validity [under an assignment], model, truth/falsity in a model/in a theory, independence [in a theory], consequence of a theory), can formally define them and provide examples
- understands the notion of a complete theory and its relation to elementary equivalence of structures, can define both and apply them to examples
- knows basic examples of theories (graph theories, orders, algebraic theories)
- can describe models of a given theory

In-class problems

Problem 1. Are the following formulas variants of the formula $(\forall x)(x < y \vee (\exists z)(z = y \wedge z \neq x))$?

(a) $(\forall z)(z < y \vee (\exists z)(z = y \wedge z \neq z))$
(b) $(\forall y)(y < y \vee (\exists z)(z = y \wedge z \neq y))$
(c) $(\forall u)(u < y \vee (\exists z)(z = y \wedge z \neq u))$

Problem 2. Let $\mathcal{A} = (\{a, b, c, d\}; \rhd^A)$ be a structure in the language with a single binary relation symbol $\rhd$, where $\rhd^A = \{(a, c), (b, c), (c, c), (c, d)\}$.

I. Which of the following formulas are true in $\mathcal{A}$?
II. For each of them find a structure $\mathcal{B}$ (if one exists) such that $\mathcal{B} \models \varphi$ iff $\mathcal{A} \not\models \varphi$.

(a) $x \rhd y$
(b) $(\exists x)(\forall y)(y \rhd x)$
(c) $(\exists x)(\forall y)((y \rhd x) \rightarrow (x \rhd x))$
(d) $(\forall x)(\forall y)(\exists z)((x \rhd z) \wedge (z \rhd y))$
(e) $(\forall x)(\exists y)((x \rhd z) \vee (z \rhd y))$

Problem 3. Prove (semantically) or find a counterexample: For every structure $\mathcal{A}$, formula φ, and sentence ψ,

(a) $\mathcal{A} \models (\psi \rightarrow (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \rightarrow \varphi)$
(b) $\mathcal{A} \models (\psi \rightarrow (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \rightarrow \varphi)$
(c) $\mathcal{A} \models ((\exists x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \rightarrow \psi)$
(d) $\mathcal{A} \models ((\forall x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\exists x)(\varphi \rightarrow \psi)$

Does it hold for every formula ψ with a free variable x? And for every formula ψ in which x is not free?

Problem 4. Decide whether T (in the language $L = \langle U, f \rangle$ with equality) is complete. If they exist, give two elementarily non-equivalent models, and two non-equivalent complete simple extensions:

(a) $T = \{U(f(x)),\ \neg x = y,\ x = y \vee x = z \vee y = z\}$

(b) $T = \{U(f(x)),\ \neg(\forall x)(\forall y)x = y,\ x = y \lor x = z \lor y = z\}$
(c) $T = \{U(f(x)),\ \neg x = f(x),\ \neg(\forall x)(\forall y)x = y,\ x = y \lor x = z \lor y = z\}$
(d) $T = \{U(f(x)),\ \neg(\forall x)x = f(x),\ \neg(\forall x)(\forall y)x = y,\ x = y \lor x = z \lor y = z\}$

Extra practice

Problem 5. Determine the free and bound occurrences of variables in the following formulas. Then convert them to variants in which no variable occurs both free and bound at the same time.

(a) $(\exists x)(\forall y)P(y,z) \lor (y = 0)$
(b) $(\exists x)(P(x) \land (\forall x)Q(x)) \lor (x = 0)$
(c) $(\exists x)(x > y) \land (\exists y)(y > x)$

Problem 6. Let φ denote the formula $(\forall x)((x = z) \lor (\exists y)(f(x) = y) \lor (\forall z)(y = f(z)))$. Which of the following terms are substitutable into φ?

(a) the term z for the variable x, the term y for the variable x,
(b) the term z for the variable y, the term $g(f(y), w)$ for the variable y,
(c) the term x for the variable z, the term y for the variable z,

Problem 7. Are the following sentences true / false / independent (in logic)?

(a) $(\exists x)(\forall y)(P(x) \lor \neg P(y))$
(b) $(\forall x)(P(x) \to Q(f(x))) \land (\forall x)P(x) \land (\exists x)\neg Q(x)$
(c) $(\forall x)(P(x) \lor Q(x)) \to ((\forall x)P(x) \lor (\forall x)Q(x))$
(d) $(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$
(e) $(\exists x)(\forall y)P(x,y) \to (\forall y)(\exists x)P(x,y)$

Problem 8. Decide whether the following hold for every formula φ. Prove (semantically, from the definitions) or provide a counterexample.

(a) $\varphi \models (\forall x)\varphi$
(b) $\models \varphi \to (\forall x)\varphi$
(c) $\varphi \models (\exists x)\varphi$
(d) $\models \varphi \to (\exists x)\varphi$

For further thought

Problem 9. Let $L = \langle +, -, 0 \rangle$ be the language of group theory (with equality). The theory of groups T consists of the following axioms:

$$x + (y + z) = (x + y) + z$$
$$0 + x = x = x + 0$$
$$x + (-x) = 0 = (-x) + x$$

Decide whether the following formulas are true / false / independent in T. Provide justification.

(a) $x + y = y + x$
(b) $x + y = x \ \to\ y = 0$
(c) $x + y = 0 \ \to\ y = -x$
(d) $-(x + y) = (-y) + (-x)$