

**Teaching goals:** After completing, the student

- understands the concept of unification and can perform the Unification Algorithm
- knows the necessary concepts from the resolution method in predicate logic (resolution rule, resolvent, resolution proof/refutation, resolution tree), can formally define them, give examples, and explain the differences compared to propositional logic
- can apply the resolution method to solve a given problem (word problems, etc.), performing all necessary steps (conversion to PNF, Skolemization, conversion to CNF)
- can construct a resolution refutation of a given (possibly infinite) CNF formula (if it exists), can draw the resolution tree including the unifications used
- can extract an unsatisfiable conjunction of ground instances of axioms from a res. tree
- knows the notion of LI-resolution, can find an LI-refutation of a given theory (if exists)
- has become familiar with selected concepts from model theory

#### IN-CLASS PROBLEMS

**Problem 1.** *Every barber shaves all those who do not shave themselves. No barber shaves anyone who shaves themselves. Formalize and prove by resolution that: There are no barbers.*

**Problem 2.** The following statements describe a genetic experiment:

- (i) *Every sheep was either born from another sheep or cloned (but not both).*
- (ii) *No cloned sheep gave birth.*

We want to show by resolution that: (iii) *If a sheep gave birth, it was itself born.* Specifically:

- (a) Express as sentences  $\varphi_1, \varphi_2, \varphi_3$  in  $L = \langle P, K \rangle$  without equality ( $P$  is binary,  $K$  unary rel. symbol,  $P(x, y)$  means ‘sheep  $x$  gave birth to sheep  $y$ ’,  $K(x)$  ‘sheep  $x$  was cloned’).
- (b) Using Skolemization of these sentences or their negations, construct a set of clauses  $S$  (possibly in an extended language) that is unsatisfiable exactly when  $\{\varphi_1, \varphi_2\} \models \varphi_3$ .
- (c) Find a resolution refutation of  $S$ , draw the resolution tree with unifications used.
- (d) Does  $S$  have an LI-refutation?

**Problem 3.** Let  $T = \{\neg(\exists x)R(x), (\exists x)(\forall y)(P(x, y) \rightarrow P(y, x)), (\forall x)((\exists y)(P(x, y) \wedge P(y, x)) \rightarrow R(x)), (\forall x)(\exists y)P(x, y)\}$  be a theory in the language  $L = \langle P, R \rangle$  without equality.

- (a) Find an open equisatisfiable theory  $T'$  for  $T$  by Skolemization.
- (b) Convert  $T'$  to an equivalent theory  $S$  in CNF. Represent  $S$  as sets.
- (c) Find a resolution refutation of  $S$ . Indicate the unification used at each step.
- (d) Find an unsatisfiable conjunction of the ground instances of clauses from  $S$ . *Hint: use the unifications from (c).*

#### EXTRA PRACTICE

**Problem 4.** Find a resolution refutation:

$$S = \{\{P(a, x, f(y)), P(a, z, f(h(b))), \neg Q(y, z)\}, \{\neg Q(h(b), w), H(w, a)\}, \{\neg H(v, a)\}, \{\neg P(a, w, f(h(b))), H(x, a)\}, \{P(a, u, f(h(u))), H(u, a), Q(h(b), b)\}\}$$

**Problem 5.** Let  $L = \langle <, j, h, s \rangle$  be without equality, where  $j, h, q$  are constant symbols (‘apples/pears/plums’) and  $x < y$  expresses “fruit  $y$  is better than fruit  $x$ ”. We know that:

- (i) *The relation “being better” is a strict partial order (irreflexive, asymmetric, transitive).*
- (ii) *Pears are better than apples.*

Prove by resolution: (iii) *If plums are better than pears, then apples aren't better than plums.*

- Express statements (i), (ii), (iii) as open formulas in the language  $L$ .
- Using these formulas, find a CNF formula  $S$  that is unsatisfiable exactly when (i), (ii) imply (iii). Write  $S$  in set representation.
- Prove by resolution that  $S$  is unsatisfiable. Illustrate the refutation with a resolution tree, indicate the unification used at each step. *Hint: 4 resolution steps are enough.*
- Find the conjunction of the basic instances of the axioms of  $S$  that is unsatisfiable.
- Is  $S$  refutable by LI-resolution?

**Problem 6.** Let  $T = \{\varphi\}$  be in  $L = \langle U, c \rangle$  with equality, where  $U$  is unary relational and  $c$  is a constant symbol, and  $\varphi$  expresses “There are at least 5 elements for which  $U(x)$  holds.”

- Find two non-equivalent simple complete extensions of  $T$ .
- Is the theory  $T$  openly axiomatizable? Give justification.

**Problem 7.** Let  $T = \{U(x) \rightarrow U(f(x)), (\exists x)U(x), \neg(f(x) = x), \varphi\}$  be a theory in the language  $L = \langle U, f \rangle$  with equality, where  $U$  is a unary relational symbol,  $f$  is a unary function symbol, and  $\varphi$  expresses that “there are at most 4 elements.”

- Is the theory  $T$  an extension of the theory  $S = \{(\exists x)(\exists y)(\neg x = y \wedge U(x) \wedge U(y)), \varphi\}$  in the language  $L' = \langle U \rangle$ ? Is it a conservative extension? Justify.
- Is the theory  $T$  openly axiomatizable? Justify.

**Problem 8.** Let  $T = \{(\forall x)(\exists y)S(y) = x, S(x) = S(y) \rightarrow x = y\}$  be a theory in the language  $L = \langle S \rangle$  with equality, where  $S$  is a unary function symbol.

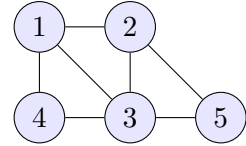
- Find an extension  $T'$  of the theory  $T$  by defining a new unary function symbol  $P$  such that  $T' \models S(S(x)) = y \leftrightarrow P(P(y)) = x$ .
- Is the theory  $T'$  openly axiomatizable? Give justification.

**Problem 9.** Let  $T$  be an extension of the theory  $DeLO^-$  (i.e., dense linear orders with a minimal element and without a maximal element) by a new axiom  $c \leq d$  in the language  $L = \langle \leq, c, d \rangle$  with equality, where  $c, d$  are new constant symbols.

- Are the sentences  $(\exists x)(x \leq d \wedge x \neq d)$  and  $(\forall x)(x \leq d)$  true / false / independent in  $T$ ?
- Write two non-equivalent simple complete extensions of the theory  $T$ .

**Problem 10.** Consider the following graph.

- Find all automorphisms.
- Which subsets of the set of vertices  $V$  are definable? Give the defining formulas. (*Hint: Use (a).*)
- Which binary relations on  $V$  are definable?



#### FOR FURTHER THOUGHT

**Problem 11.** Let  $T = \{(\forall x)(\exists y)S(y) = x, S(x) = S(y) \rightarrow x = y\}$  be a theory in the language  $L = \langle S \rangle$  with equality, where  $S$  is a unary function symbol.

- Let  $\mathcal{R} = \langle \mathbb{R}, S \rangle$ , where  $S(r) = r + 1$  for  $r \in \mathbb{R}$ . For which  $r \in \mathbb{R}$  is the set  $\{r\}$  definable in  $\mathcal{R}$  from the parameter 0?
- Is the theory  $T$  openly axiomatizable? Give justification.
- Is the extension  $T'$  of  $T$  by the axiom  $S(x) = x$  an  $\omega$ -categorical theory? Is  $T'$  complete?
- For which  $0 < n \in \mathbb{N}$  does there exist an  $L$ -structure  $\mathcal{B}$  of size  $n$  elementarily equivalent to  $\mathcal{R}$ ? Does there exist a countable structure  $\mathcal{B}$  elementarily equivalent to  $\mathcal{R}$ ?