Teaching goals: After completing, the student
- understands the notions of propositional syntax (language, [atomic] proposition, tree of a proposition, subproposition, theory), can formally define them and give examples
- understands the notions of model, consequence of a theory, can formally define them and give examples
- can formalize given system (word/computational problem, etc.) in propositional logic
- can find models of a given theory
- can decide whether a given proposition is a consequence of a given theory
- has experience applying (with instructor assistance) the tableau method and resolution method to prove properties of a given system (e.g., to solve a word problem)

In-class problems

Problem 1. Lost in a labyrinth, we face three doors—red, blue, and green. Exactly one leads out; the other two hide dragons. On the doors we read the following inscriptions:
- Red door: *"The way out is behind this door."*
- Blue door: *"The way out is not behind this door."*
- Green door: *"The way out is not behind the blue door."*

We know that at least one inscription is true and at least one is false. Which way leads out?

(a) Choose a suitable language (a set of propositional variables) $\mathbb{P}$.
(b) Formalize all knowledge as a theory T in the language $\mathbb{P}$. (Note: The axioms are not the inscriptions on the doors; those need not be true.)
(c) Find all models of the theory T.
(d) Formalize the statements "The way out is behind the red/blue/green door" as formulas $\varphi_1, \varphi_2, \varphi_3$ over $\mathbb{P}$. Is any of these formulas a consequence of T?
(e) Try using the tableau method: Construct a tableau from the theory T with the item $F\varphi_i$ at the root. Will all branches be contradictory? (Try to come up with the correct steps for the construction of the tableau, take inspiration from the example in the lecture.)
(f) Try using the resolution method: Convert the axioms of T, as well as $\neg\varphi_i$, into conjunctive normal form (CNF). Try to construct a resolution refutation and draw it in the form of a resolution tree. (Note: Don't forget to negate the formula φ_i you are proving.)

Solution. *(a) A natural choice is the language $\mathbb{P} = \{p_1, p_2, p_3\}$, where p_i means 'the way out is behind the i-th door', with the doors ordered red, blue, green as in the assignment. (We could also have used 'there is a dragon behind the i-th door' or 'the inscription on the i-th door is true'. The important thing is that the chosen language is as small as possible. If we have p_i, we can for example express 'there is a dragon behind the i-th door' as '$\neg p_i$', so we don't need another propositional variable. Furthermore, we want the properties from the assignment to be as easy to formalize as possible.)*

(b) From the assignment we understand that 'there is a dragon' means the same as 'there is no way out'. The fact that the way out is behind exactly one of the doors can be expressed by saying that it is behind at least one door, and for each pair of doors it is not behind both:

$$\alpha_1 = (p_1 \lor p_2 \lor p_3) \land (\neg p_1 \lor \neg p_2) \land (\neg p_1 \lor \neg p_3) \land (\neg p_2 \lor \neg p_3)$$

Now for the inscriptions on the doors, first let's formalize their meaning:

- *Red door: "p_1"*
- *Blue door: "$\neg p_2$"*
- *Green door: "$\neg p_2$"*

At least one of these inscriptions is true: $p_1 \vee \neg p_2 \vee \neg p_2$, which simplifies to

$$\alpha_2 = p_1 \vee \neg p_2$$

Similarly, the fact that at least one of the inscriptions is false can be formalized as $\neg p_1 \vee \neg\neg p_2 \vee \neg\neg p_2$, which after simplification (think about why this is equivalent) gives:

$$\alpha_3 = \neg p_1 \vee p_2$$

The resulting theory is thus:

$$T = \{\alpha_1, \alpha_2, \alpha_3\}$$
$$= \{(p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3), p_1 \vee \neg p_2, \neg p_1 \vee p_2\}$$

(c) *Later we will learn to find models using the tableau method, but for now an 'inefficient' procedure: First we find the models of one axiom. Since the first axiom is rather complicated, it may be better to start with axiom α_2. (In principle, we could try all 8 models of the language $\mathbb{P}$, and for each of them compute the truth value of α_2.) We obtain:*

$$M_{\mathbb{P}}(\alpha_2) = \{(0,0,0), (0,0,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

Now we check in which of these models axiom α_3 holds:

$$M_{\mathbb{P}}(\alpha_2, \alpha_3) = \{(0,0,0), (0,0,1), (1,1,0), (1,1,1)\}$$
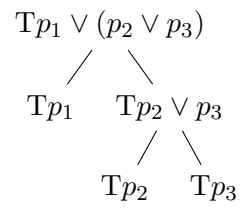
Finally, we check the validity of α_1 in each of these 4 models:

$$M_{\mathbb{P}}(T) = M_{\mathbb{P}}(\alpha_1, \alpha_2, \alpha_3) = \{(0,0,1)\}$$

(d) *With our choice of language $\mathbb{P}$ the formalization is simple: $\varphi_1 = p_1$, $\varphi_2 = p_2$, $\varphi_3 = p_3$. Being a consequence of theory T means holding in every model of T (note: in every model, not just 'in some model'—this is a common mistake). In our case, T has only one model, and we immediately see that φ_3 holds in it while φ_1 and φ_2 do not. Thus, the only consequence of T among these is φ_3.*

(e) *When using the tableau method, we proceed as in the introductory chapter of the lecture notes (Section 1.1.5). To prove that φ_3 holds in T, we construct a tableau from the theory T, where we put the assumption $\mathrm{F}p_3$ at the root, since we are proving by contradiction (F means 'False', T means 'True').*
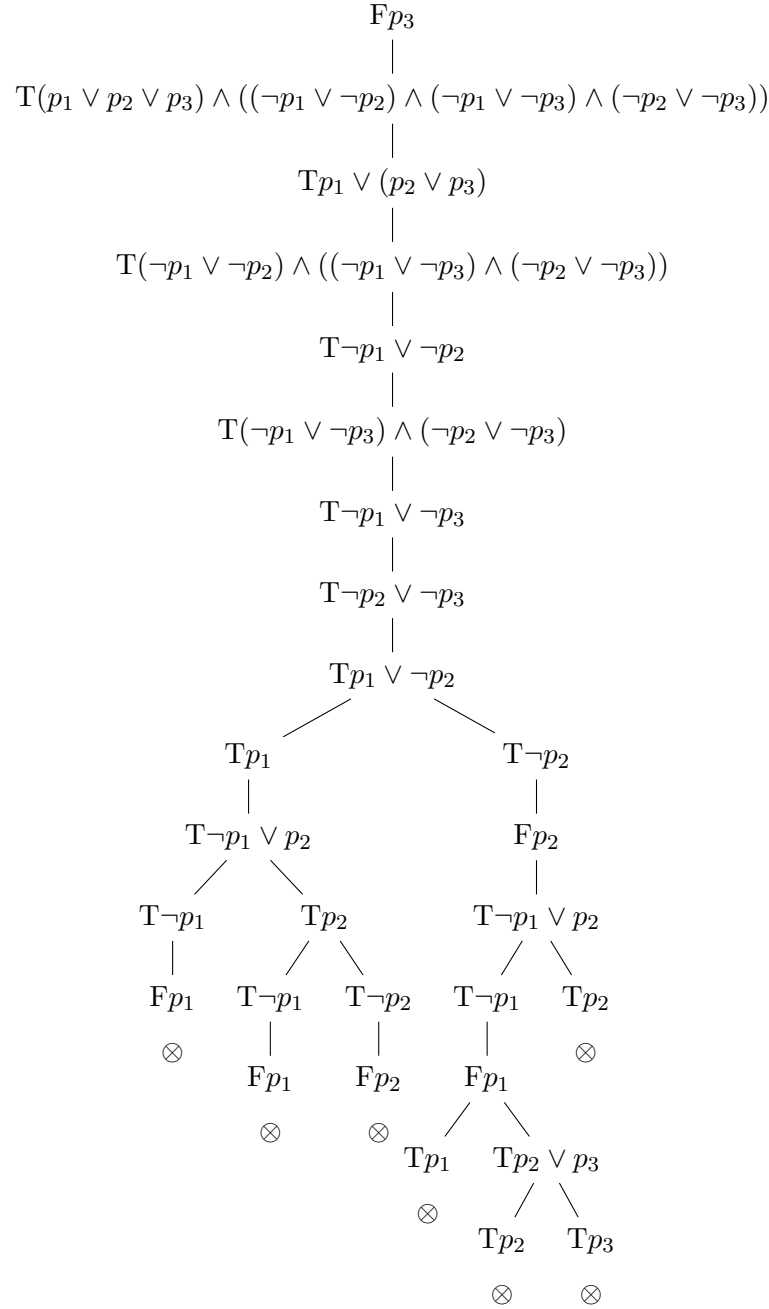
Recall that we expand a tableau by attaching assumptions about the validity of the axioms $\mathrm{T}\alpha_i$ ($i \in \{1, 2, 3\}$) and by reducing items (attaching the corresponding atomic tableaux). The order in which we do this can significantly affect the size of the resulting tableau proof. It is good first to reduce items whose atomic tableaux do not branch, or that do branch but one of the branches immediately becomes contradictory. We attach the axioms only when needed. It is often helpful to think through how we would proceed in the proof ourselves, and then build the tableau accordingly.

Note also that we do not define atomic tableaux for conjunctions or disjunctions of three or more formulas. (We want the algorithm steps to be as simple as possible.) For example, in $\mathrm{T}p_1 \vee p_2 \vee p_3$, we first imagine the omitted parentheses, $\mathrm{T}p_1 \vee (p_2 \vee p_3)$, and then reduce in two steps by attaching $\mathrm{T}p_1$ and $\mathrm{T}p_2 \vee p_3$:

$$\text{T}p_1 \lor (p_2 \lor p_3)$$

$$\diagup \qquad \diagdown$$

$$\text{T}p_1 \qquad \text{T}p_2 \lor p_3$$

$$\diagup \quad \diagdown$$

$$\text{T}p_2 \qquad \text{T}p_3$$

(We also note that from the perspective of the tableau method it would be a bit better to have, instead of axiom α_1, four separate formulas whose conjunction it is. This would shorten the tableau construction. But the algorithm can handle axioms of any complexity.)
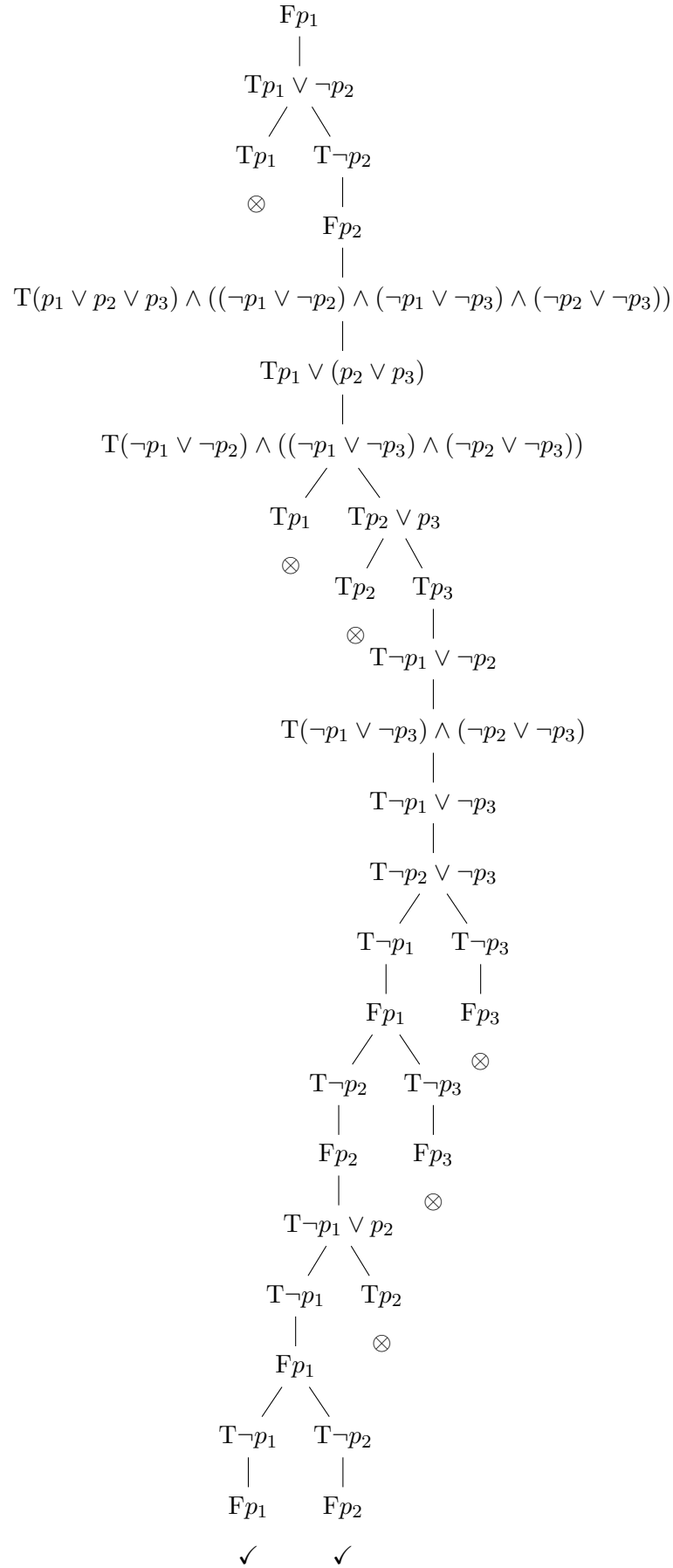
Here is one possible tableau proof ($\oplus$ marks a contradictory branch, $\checkmark$ a completed consistent one). At first glance it may look complicated, but in fact we are just running a simple algorithm. Work out where each item came from and where we see atomic tableaux:

$\mathrm{F}p_3$

$\mathrm{T}(p_1 \vee p_2 \vee p_3) \wedge ((\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3))$

$\mathrm{T}p_1 \vee (p_2 \vee p_3)$

$\mathrm{T}(\neg p_1 \vee \neg p_2) \wedge ((\neg p_1 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3))$

$\mathrm{T}\neg p_1 \vee \neg p_2$

$\mathrm{T}(\neg p_1 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3)$

$\mathrm{T}\neg p_1 \vee \neg p_3$

$\mathrm{T}\neg p_2 \vee \neg p_3$

$\mathrm{T}p_1 \vee \neg p_2$

$\mathrm{T}p_1 \qquad\qquad \mathrm{T}\neg p_2$

$\mathrm{T}\neg p_1 \vee p_2 \qquad\qquad \mathrm{F}p_2$

$\mathrm{T}\neg p_1 \quad \mathrm{T}p_2 \qquad\qquad \mathrm{T}\neg p_1 \vee p_2$

$\mathrm{F}p_1 \quad \mathrm{T}\neg p_1 \quad \mathrm{T}\neg p_2 \qquad \mathrm{T}\neg p_1 \quad \mathrm{T}p_2$

$\otimes \qquad\qquad \qquad\qquad\qquad \otimes$

$\mathrm{F}p_1 \qquad \mathrm{F}p_2 \qquad \mathrm{F}p_1$

$\otimes \qquad\qquad \otimes$

$\mathrm{T}p_1 \quad \mathrm{T}p_2 \vee p_3$

$\otimes$

$\mathrm{T}p_2 \quad \mathrm{T}p_3$

$\otimes \qquad\quad \otimes$

What if we tried to prove p_1 or p_2 from theory T? Let's show it for p_1 (try p_2 yourself). We put $\mathrm{F}p_1$ at the root. We proceed analogously, but at least one branch will remain consistent even after completion (adding all axioms and reducing all items). From the consistent branches we can then read off a counterexample (a model of T in which p_1 does not hold).

Below is a completed tableau. Verify that all axioms are used and all items reduced. We get two completed consistent branches. Looking at the assumptions about propositional
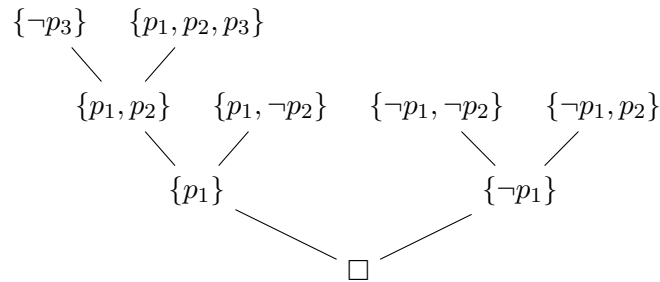
variables on them: in both we have $\mathrm{F}p_1$, $\mathrm{F}p_2$, $\mathrm{T}p_3$. This corresponds to the model $(0, 0, 1)$, which is indeed a counterexample: a model of T in which p_1 does not hold.

$$\text{F}p_1$$
$$|$$
$$\text{T}p_1 \vee \neg p_2$$

$$\text{T}p_1 \qquad \text{T}\neg p_2$$
$$\otimes$$
$$\text{F}p_2$$
$$|$$
$$\text{T}(p_1 \vee p_2 \vee p_3) \wedge ((\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3))$$
$$|$$
$$\text{T}p_1 \vee (p_2 \vee p_3)$$
$$|$$
$$\text{T}(\neg p_1 \vee \neg p_2) \wedge ((\neg p_1 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3))$$

$$\text{T}p_1 \qquad \text{T}p_2 \vee p_3$$
$$\otimes$$
$$\text{T}p_2 \qquad \text{T}p_3$$
$$\otimes$$
$$\text{T}\neg p_1 \vee \neg p_2$$
$$|$$
$$\text{T}(\neg p_1 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3)$$
$$|$$
$$\text{T}\neg p_1 \vee \neg p_3$$
$$|$$
$$\text{T}\neg p_2 \vee \neg p_3$$

$$\text{T}\neg p_1 \qquad \text{T}\neg p_3$$
$$|\qquad\qquad |$$
$$\text{F}p_1 \qquad \text{F}p_3$$
$$\otimes$$

$$\text{T}\neg p_2 \qquad \text{T}\neg p_3$$
$$|\qquad\qquad |$$
$$\text{F}p_2 \qquad \text{F}p_3$$
$$\otimes$$
$$\text{T}\neg p_1 \vee p_2$$

$$\text{T}\neg p_1 \qquad \text{T}p_2$$
$$|\qquad \otimes$$
$$\text{F}p_1$$

$$\text{T}\neg p_1 \qquad \text{T}\neg p_2$$
$$|\qquad\qquad |$$
$$\text{F}p_1 \qquad \text{F}p_2$$
$$\checkmark \qquad\quad \checkmark$$

(f) *When using the resolution method, we proceed as in the introductory chapter of the lecture notes, in Section 1.1.6. To prove that p_3 holds in T, we add to the theory T the formula $\neg p_3$ as an axiom. We thus have $T' = \{\alpha_1, \alpha_2, \alpha_3, \neg p_3\}$. Using equivalence transformations we convert into CNF (express as a conjunction of clauses, i.e., disjunctions of literals). In our case all axioms are already in CNF (α_1 is a conjunction of three clauses, the others are clauses themselves). The resulting CNF formula, written in set notation (for clarity it is good to write the literals in a fixed order), is:*

$$S = \{\{p_1, p_2, p_3\}, \{\neg p_1, \neg p_2\}, \{\neg p_1, \neg p_3\}, \{\neg p_2, \neg p_3\}, \{p_1, \neg p_2\}, \{\neg p_1, p_2\}, \{\neg p_3\}\}$$

Now we construct a resolution refutation. How do we choose the resolution steps? A good heuristic is that a shorter clause contains more information. So we try to start from the shortest clauses, here $\{\neg p_3\}$, and see where we can use newly derived ('learned') clauses. A clause often needs to be used repeatedly. Here is one possible resolution refutation, drawn as a resolution tree:

$$\{\neg p_3\} \quad \{p_1, p_2, p_3\}$$

$$\{p_1, p_2\} \quad \{p_1, \neg p_2\} \quad \{\neg p_1, \neg p_2\} \quad \{\neg p_1, p_2\}$$

$$\{p_1\} \qquad\qquad \{\neg p_1\}$$

$$\square$$

Verify that all leaves are clauses from S and all internal nodes are resolvents of their children. We define a resolution proof as a sequence of clauses (where clauses are either axioms or resolvents of previous ones). There are several such sequences corresponding to our tree; here is one possible:

$$\{\neg p_3\}, \{p_1, p_2, p_3\}, \{p_1, p_2\}, \{p_1, \neg p_2\}, \{p_1\}, \{\neg p_1, \neg p_2\}, \{\neg p_1, p_2\}, \{\neg p_1\}, \square$$

Problem 2. Consider the *vertex cover* of the following graph:
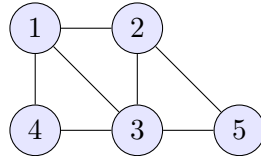
For a given $k > 0$, we want to determine if this graph has a vertex cover of size at most k.

(a) Choose a suitable language (a set of propositional variables) $\mathbb{P}$.

(b) Formalize in propositional logic the problem of whether the graph in the picture has a vertex cover of size at most k, for a fixed k. Denote the resulting theory by VC_k.

(c) Show that VC_2 has no models, i.e., the graph does not have a 2-element vertex cover.

(d) Could you use the tableau method for this? Think through the procedure.

(e) Could you use the resolution method for this? Think through the procedure.

(f) Find all 3-element vertex covers.

Solution. *A vertex cover is a set of vertices C containing at least one endpoint of every edge.*

(a) *A natural choice is one propositional variable p_v for each vertex $v \in V$, which says whether $v \in C$. Thus $\mathbb{P} = \{p_v \mid v \in V\} = \{p_1, p_2, p_3, p_4, p_5\}$.*

(b) *First, formalize that we have a vertex cover (of arbitrary size). For each edge $\{u,v\} \in E$ we express that u or v must be in C. We thus have a theory describing all vertex covers:*

$$VC = \{p_u \vee p_v \mid \{u,v\} \in E, u < v\}$$

It remains to express that at most k propositional variables are true, which we write as a disjunction of negations over all $(k+1)$-element subsets of vertices:

$$S_{\leq k} = \{\bigvee_{v \in I} \neg p_v \mid I \subseteq V, |I| = k+1\}$$

The resulting theory will therefore be $VC_k = VC \cup S_{\leq k}$.

(c) *For clarity, we list all axioms of the theory VC_2 here (though we don't have to):*

$$VC_2 = \{p_1 \vee p_2, p_1 \vee p_3, p_1 \vee p_4, p_2 \vee p_3, p_2 \vee p_5, p_3 \vee p_4, p_3 \vee p_5,$$
$$\neg p_1 \vee \neg p_2 \vee \neg p_3, \neg p_1 \vee \neg p_2 \vee \neg p_4, \neg p_1 \vee \neg p_2 \vee \neg p_5, \neg p_1 \vee \neg p_3 \vee \neg p_4, \neg p_1 \vee \neg p_3 \vee \neg p_5,$$
$$\neg p_1 \vee \neg p_4 \vee \neg p_5, \neg p_2 \vee \neg p_3 \vee \neg p_4, \neg p_2 \vee \neg p_3 \vee \neg p_5, \neg p_2 \vee \neg p_4 \vee \neg p_5, \neg p_3 \vee \neg p_4 \vee \neg p_5\}$$

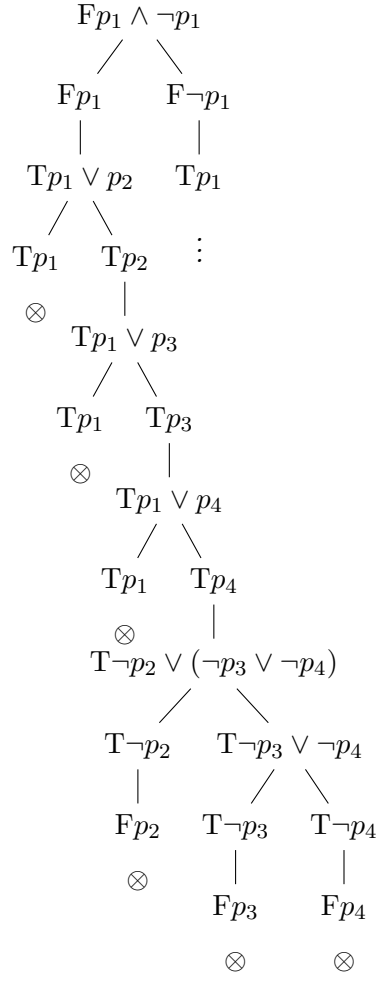*Using the 'inefficient' approach, we can, for example, notice that:*

$$M(p_1 \vee p_3, p_1 \vee p_4, p_3 \vee p_4, \neg p_1 \vee \neg p_3 \vee \neg p_4) = \{(0,a,1,1,b),(1,a,0,1,b),(1,a,1,0,b) \mid a,b \in \{0,1\}\}$$
$$M(p_2 \vee p_3, p_2 \vee p_5, p_3 \vee p_5, \neg p_2 \vee \neg p_3 \vee \neg p_5) = \{(a,0,1,b,1),(a,1,0,b,1),(a,1,1,b,0) \mid a,b \in \{0,1\}\}$$

These are the 2-element vertex covers of the subgraphs $\{1,3,4\}$ and $\{2,3,5\}$. Their intersection is

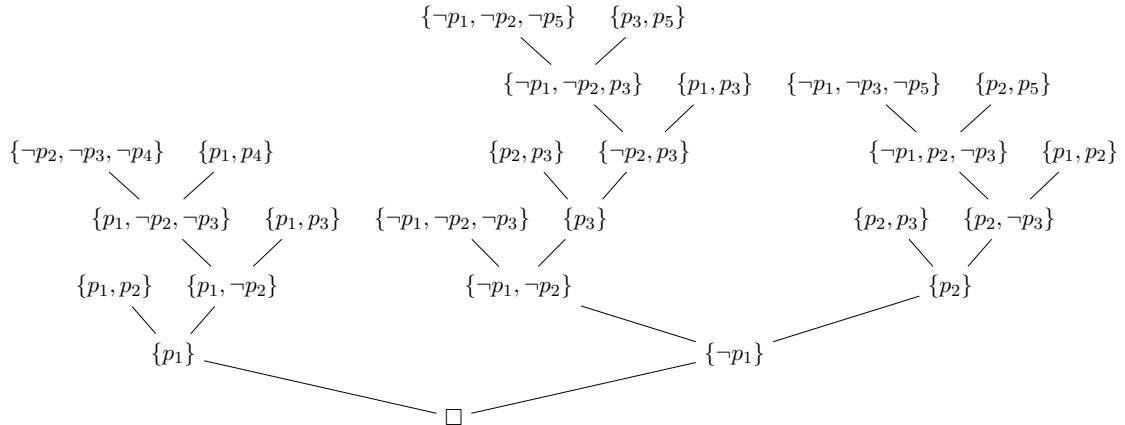$$\{(0,0,1,1,1),(0,1,1,1,0),(1,1,0,1,1),(1,0,1,0,1),(1,1,1,0,0)\}$$

But none of these models satisfies the remaining axioms; for example, in the first one $\neg p_3 \vee \neg p_4 \vee \neg p_5$ does not hold.

(d) *We use the tableau method to prove that VC_2 entails a contradiction, i.e., the formula $p_1 \wedge \neg p_1$. We put this into the root with the marker F ('False'), i.e., we prove inconsistency by contradiction. We show only part of the construction (the left branch).*

$$\mathrm{F}p_1 \wedge \neg p_1$$

$$\mathrm{F}p_1 \qquad \mathrm{F}\neg p_1$$

$$\mathrm{T}p_1 \vee p_2 \qquad \mathrm{T}p_1$$

$$\mathrm{T}p_1 \qquad \mathrm{T}p_2 \qquad \vdots$$

$$\otimes$$

$$\mathrm{T}p_1 \vee p_3$$

$$\mathrm{T}p_1 \qquad \mathrm{T}p_3$$

$$\otimes$$

$$\mathrm{T}p_1 \vee p_4$$

$$\mathrm{T}p_1 \qquad \mathrm{T}p_4$$

$$\otimes$$

$$\mathrm{T}\neg p_2 \vee (\neg p_3 \vee \neg p_4)$$

$$\mathrm{T}\neg p_2 \qquad \mathrm{T}\neg p_3 \vee \neg p_4$$

$$\mathrm{F}p_2 \qquad \mathrm{T}\neg p_3 \qquad \mathrm{T}\neg p_4$$

$$\otimes$$

$$\mathrm{F}p_3 \qquad \mathrm{F}p_4$$

$$\otimes \qquad \otimes$$

(Alternatively, we could use a tableau method adapted directly to searching for models: put the validity of the first axiom at the root and expand the tableau while adding assumptions of the validity of the other axioms; every model must agree with some branch, but all of them will turn out contradictory.)

(e) *All axioms are already in CNF. It suffices to find a resolution refutation of the theory VC_2 (that is, a resolution proof of the empty clause $\square$); it then follows that VC_2 cannot have a model. We draw only one possible resolution tree:*

$$\{\neg p_1, \neg p_2, \neg p_5\} \quad \{p_3, p_5\}$$

$$\{\neg p_1, \neg p_2, p_3\} \quad \{p_1, p_3\} \quad \{\neg p_1, \neg p_3, \neg p_5\} \quad \{p_2, p_5\}$$

$$\{\neg p_2, \neg p_3, \neg p_4\} \quad \{p_1, p_4\} \qquad \{p_2, p_3\} \quad \{\neg p_2, p_3\} \qquad \{\neg p_1, p_2, \neg p_3\} \quad \{p_1, p_2\}$$

$$\{p_1, \neg p_2, \neg p_3\} \quad \{p_1, p_3\} \quad \{\neg p_1, \neg p_2, \neg p_3\} \quad \{p_3\} \qquad \{p_2, p_3\} \quad \{p_2, \neg p_3\}$$

$$\{p_1, p_2\} \quad \{p_1, \neg p_2\} \qquad \{\neg p_1, \neg p_2\} \qquad \{p_2\}$$

$$\{p_1\} \qquad\qquad\qquad\qquad \{\neg p_1\}$$

$$\Box$$

(f) Here we state just the correct answer: $\mathrm{M}(VC_3) = \{(0,1,1,1,0),(1,0,1,0,1),(1,1,1,0,0)\}$, *which corresponds to the vertex sets* $\{2,3,4\}, \{1,3,5\}, \{1,2,3\}$. *Using the tableau method would be a bit more laborious, but you can try to prove (by the tableau method or by resolution) that vertex 3 must always be in the cover.*

EXTRA PRACTICE

Problem 3. Consider the following statements:

  (i) *Whoever is a good runner and is in good shape will finish a marathon.*
 (ii) *Whoever is unlucky and not in good shape will not finish a marathon.*
(iii) *Whoever finishes a marathon is a good runner.*
 (iv) *If I am lucky, I will finish a marathon.*
  (v) *I am in good shape.*

Similarly to the first example, describe the situation using propositional logic:

(a) Formalize these statements as a theory T over a suitable set of propositional variables.
(b) Find all models of the theory T.
(c) Try to use the *tableau method* to search for models.
(d) Write several different consequences of the theory T.
(e) Find a CNF theory equivalent to the theory T.

Problem 4. There are 3 brothers; each of them either always tells the truth or always lies.

  (i) The eldest says: *"Both my brothers are liars."*
 (ii) The middle one says: *"The youngest is a liar."*
(iii) The youngest says: *"The eldest is a liar."*

Using propositional logic, show that the youngest brother is truthful.

Problem 5. Consider a fixed Sudoku puzzle. Describe how to create a theory (in propositional logic) whose models correspond exactly to valid solutions.

Problem 6. Formalize the following statements in propositional logic:

(a) Rabbits in the area have not been observed and walking along the path is safe, but the blueberries along the path are ripe.
(b) If the blueberries along the path are ripe, then walking along the path is safe only if rabbits have not been observed in the area.

(c) It is not safe to walk along the path, but rabbits in the area have not been observed and the blueberries along the path are ripe.

(d) For walking along the path to be safe, it is necessary but not sufficient that the blueberries along the path are not ripe and rabbits have not been observed in the area.

(e) Walking along the path is not safe whenever the blueberries along the path are ripe and rabbits have been observed in the area.

Problem 7. Formalize the following properties of mathematical objects in propositional logic:

(a) For a fixed (finite) graph G, that it has a perfect matching.

(b) For a fixed partially ordered set, that it is totally (linearly) ordered.

(c) For a fixed partially ordered set, that it has a least element.

Problem 8. For the following formulas draw the formula tree, and find the set of models:

(a) $(p \to q) \leftrightarrow \neg(p \wedge \neg q)$ (b) $(p \leftrightarrow q) \leftrightarrow ((p \vee q) \to (p \wedge q))$

For further thought

Problem 9. Recall the definition of the *formula tree.*

(a) Prove in detail that every formula has a uniquely determined tree.

(b) Is it still true if we replace the symbols '(', ')' in the definition of a formula by '|'?

(c) What would happen if we omitted parentheses entirely?