NAIL062 P&P Logic: Worksheet 10 – Resolution in predicate logic

Teaching goals: After completing, the student

- understands the notion of unification and can perform the Unification Algorithm
- knows the necessary notions from the resolution method in predicate logic (resolution rule, resolvent, resolution proof/refutation, resolution tree), can formally define them, give examples, and explain the differences compared to propositional logic
- can apply the resolution method to solve a given problem (word problem, etc.), performing all necessary steps (conversion to PNF, Skolemization, conversion to CNF)
- can construct a resolution refutation of a given (possibly infinite) CNF formula (if it exists), can draw the resolution tree including the unifications used
- can extract an unsatisfiable conjunction of ground instances of axioms from a res. tree
- knows the notion of LI-resolution, can find an LI-refutation of a given theory (if exists)
- has become familiar with selected notions from model theory

In-class problems

Problem 1. *Every barber shaves all those who do not shave themselves. No barber shaves anyone who shaves themselves.* Formalize and prove by resolution that: *There are no barbers.*

Problem 2. The following statements describe a genetic experiment:

(i) Every sheep was either born from another sheep or cloned (but not both).
(ii) No cloned sheep gave birth.

We want to show by resolution that: *(iii) If a sheep gave birth, it was itself born.* Specifically:

(a) Express as sentences φ_1, φ_2, φ_3 in $L = \langle B, C \rangle$ without equality (B is binary, C unary relation symbol, $B(x, y)$ means 'sheep x gave birth to sheep y', $C(x)$ 'sheep x was cloned').
(b) Using Skolemization of these sentences or their negations, construct a set of clauses S (possibly in an extended language) that is unsatisfiable exactly when $\{\varphi_1, \varphi_2\} \models \varphi_3$.
(c) Find a resolution refutation of S, draw the resolution tree with unifications used.
(d) Does S have an LI-refutation?

Problem 3. Let $T = \{\neg(\exists x)R(x),\ (\exists x)(\forall y)(P(x, y) \to P(y, x)),\ (\forall x)((\exists y)(P(x, y) \wedge P(y, x)) \to R(x)),\ (\forall x)(\exists y)P(x, y)\}$ be a theory in the language $L = \langle P, R \rangle$ without equality.

(a) Using Skolemization, find an open theory T' equisatisfiable with T.
(b) Convert T' to an equivalent theory S in CNF. Write S in set representation.
(c) Find a resolution refutation of S. Indicate the unification used at each step.
(d) Find an unsatisfiable conjunction of ground instances of clauses from S. *Hint: use the unifications from (c).*

Extra practice

Problem 4. Find a resolution refutation:

$$S = \{\{P(a, x, f(y)), P(a, z, f(h(b))), \neg Q(y, z)\},\ \{\neg Q(h(b), w), H(w, a)\},\ \{\neg H(v, a)\},$$
$$\{\neg P(a, w, f(h(b))), H(x, a)\},\ \{P(a, u, f(h(u))), H(u, a), Q(h(b), b)\}\}$$

Problem 5. Let $L = \langle <, a, b, c \rangle$ be without equality, where a, b, c are constant symbols ('apples/bananas/cherries') and $x < y$ expresses *"fruit y is better than fruit x"*. We know:

(i) The relation "being better" is a strict partial order (irreflexive, asymmetric, transitive).
(ii) Pears are better than apples.

Prove by resolution: *(iii) If cherries are better than bananas, then apples aren't better than cherries.*

(a) Express statements (i), (ii), (iii) as open formulas in the language L.
(b) Using these formulas, find a CNF formula S that is unsatisfiable exactly when (i) and (ii) imply (iii). Write S in set representation.
(c) Prove by resolution that S is unsatisfiable. Illustrate the refutation with a resolution tree, indicate the unification used at each step. *Hint: 4 resolution steps are enough.*
(d) Find a conjunction of ground instances of axioms of S that is unsatisfiable.
(e) Is S refutable by LI-resolution?

Problem 6. Let $T = \{\varphi\}$ be in $L = \langle U, c\rangle$ with equality, where U is unary relational and c is a constant symbol, and φ expresses *"There are at least 5 elements for which $U(x)$ holds."*

(a) Find two non-equivalent complete simple extensions of T.
(b) Is the theory T openly axiomatizable? Give justification.

Problem 7. Let $T = \{U(x) \rightarrow U(f(x)), (\exists x)U(x), \neg(f(x) = x), \varphi\}$ be a theory in the language $L = \langle U, f\rangle$ with equality, where U is a unary relational symbol, f is a unary function symbol, and φ expresses that "there are at most 4 elements."

(a) Is the theory T an extension of the theory $S = \{(\exists x)(\exists y)(\neg x = y \wedge U(x) \wedge U(y)), \varphi\}$ in the language $L' = \langle U\rangle$? Is it a conservative extension? Justify.
(b) Is the theory T openly axiomatizable? Justify.

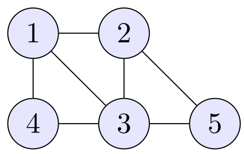
Problem 8. Let $T = \{(\forall x)(\exists y)S(y) = x,\ S(x) = S(y) \rightarrow x = y\}$ be a theory in the language $L = \langle S\rangle$ with equality, where S is a unary function symbol.

(a) Find an extension T' of the theory T by definition of a new unary function symbol P such that $T' \models S(S(x)) = y \leftrightarrow P(P(y)) = x$.
(b) Is the theory T' openly axiomatizable? Give justification.

Problem 9. Let T be an extension of the theory $DeLO^-$ (i.e., dense linear orders with a minimal element and without a maximal element) by a new axiom $c \leq d$ in the language $L = \langle \leq, c, d\rangle$ with equality, where c, d are new constant symbols.

(a) Are $(\exists x)(x \leq d \wedge x \neq d)$ and $(\forall x)(x \leq d)$ valid / contradictory / independent in T?
(b) Write two non-equivalent complete simple extensions of the theory T.

Problem 10. Consider the following graph.



(a) Find all automorphisms.
(b) Which subsets of the set of vertices V are definable? Give the defining formulas. *(Hint: Use (a).)*
(c) Which binary relations on V are definable?

For further thought

Problem 11. Let $T = \{(\forall x)(\exists y)S(y) = x,\ S(x) = S(y) \rightarrow x = y\}$ be a theory in the language $L = \langle S\rangle$ with equality, where S is a unary function symbol.

(a) Let $\mathcal{R} = \langle \mathbb{R}, S\rangle$, where $S(r) = r + 1$ for $r \in \mathbb{R}$. For which $r \in \mathbb{R}$ is the set $\{r\}$ definable in $\mathcal{R}$ from the parameter 0?
(b) Is the theory T openly axiomatizable? Give justification.
(c) Is the extension T' of T by the axiom $S(x) = x$ an ω-categorical theory? Is T' complete?
(d) For which $0 < n \in \mathbb{N}$ does there exist an L-structure $\mathcal{B}$ of size n elementarily equivalent to $\mathcal{R}$? Does there exist a countable structure $\mathcal{B}$ elementarily equivalent to $\mathcal{R}$?