NAIL062 P&P Logic: Worksheet 8 – The tableau method in predicate logic

Teaching goals: After completing, the student

- understands how the tableau method in predicate logic differs from propositional logic, can formally define all necessary concepts
- knows atomic tableaux for quantifiers, understands their use
- can construct a finished tableau for a given formula from a given (even infinite) theory
- can describe the canonical model for a given finished noncontradictory branch
- understands the axioms of equality, their relation to congruences, quotient structures
- can apply the tableau method to solve a given problem (word problem, etc.)
- understands tableau method for languages with equality, can apply to simple examples
- knows the compactness theorem of predicate logic, can apply it

In-class problems

Problem 1. Assume that:

- *All guilty people are liars.*
- *At least one of the accused is also a witness.*
- *No witness lies.*

Prove by the tableau method that: *Not all the accused are guilty.* Specifically:

(a) Choose a suitable language $\mathcal{L}$. Will it be with equality, or without equality?
(b) Formalize our knowledge and the statement to be proved as sentences $\alpha_1, \alpha_2, \alpha_3, \varphi$ in $\mathcal{L}$.
(c) Construct a tableau proof of the sentence φ from the theory $T = \{\alpha_1, \alpha_2, \alpha_3\}$.

Solution. *(a) Let us choose the language $\mathcal{L} = \langle V, L, O, S\rangle$ without equality, where V, L, O and S are unary relation symbols meaning "to be a guilty person / a liar / an accused / a witness".*
(b)

$$\begin{aligned}
\alpha_1 &= (\forall x)(V(x) \rightarrow L(x))\\
\alpha_2 &= (\exists x)(O(x) \wedge S(x))\\
\alpha_3 &= \neg(\exists x)(S(x) \wedge L(x))\\
\varphi &= \neg(\forall x)(O(x) \rightarrow V(x))
\end{aligned}$$

(c) We construct a finished tableau from the theory $T = \{\alpha_1, \alpha_2, \alpha_3\}$ with the item $F\varphi$ at the root. We will see that all branches are closed, so this is a tableau proof. (In blue the attachment of axioms is marked, in red are the roots of atomic tableau entries of the "for all" type, which we could avoid drawing if our conventions allowed it.)

$$\mathrm{F}\neg(\forall x)(O(x) \to V(x))$$

$$\mathrm{T}(\forall x)(O(x) \to V(x))$$

$$\mathrm{T}(\exists x)(O(x) \wedge S(x))$$

$$\mathrm{T}O(c_0) \wedge S(c_0)$$

$$\mathrm{T}O(c_0)$$

$$\mathrm{T}S(c_0)$$

$$\mathrm{T}(\forall x)(O(x) \to V(x))$$

$$\mathrm{T}O(c_0) \to V(c_0)$$

$$\mathrm{F}O(c_0) \qquad \mathrm{T}V(c_0)$$

$$\otimes \qquad \mathrm{T}(\forall x)(V(x) \to L(x))$$

$$\mathrm{T}(\forall x)(V(x) \to L(x))$$

$$\mathrm{T}V(c_0) \to L(c_0)$$

$$\mathrm{F}V(c_0) \qquad \mathrm{T}L(c_0)$$

$$\otimes \qquad \mathrm{T}\neg(\exists x)(S(x) \wedge L(x))$$

$$\mathrm{F}(\exists x)(S(x) \wedge L(x))$$

$$\mathrm{F}(\exists x)(S(x) \wedge L(x))$$

$$\mathrm{F}S(c_0) \wedge L(c_0)$$

$$\mathrm{F}S(c_0) \qquad \mathrm{F}L(c_0)$$

$$\otimes \qquad \otimes$$

Problem 2. Consider the following statements:

(i) Zero is a small number.

(ii) A number is small iff it is close to zero.
(iii) The sum of two small numbers is a small number.
(iv) If x is close to y, then $f(x)$ is close to $f(y)$.

We want to prove that: *(v) If x and y are small numbers, then $f(x+y)$ is close to $f(0)$.*

(a) Formalize the statements as sentences $\varphi_1, \dots, \varphi_5$ in $L = \langle M, B, f, +, 0\rangle$ without equality.
(b) Construct a finished tableau from the theory $T = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ with the item $F\varphi_5$ at the root. Decide whether $T \models \varphi_5$.
(c) If they exist, give at least two complete simple extensions of the theory T.

Solution. *(a)*

$$\begin{aligned}
\varphi_1 &= M(0)\\
\varphi_2 &= (\forall x)(M(x) \leftrightarrow B(x,0))\\
\varphi_3 &= (\forall x)(\forall y)(M(x) \wedge M(y) \rightarrow M(x+y))\\
\varphi_4 &= (\forall x)(\forall y)(B(x,y) \rightarrow B(f(x), f(y)))\\
\varphi_5 &= (\forall x)(\forall y)(M(x) \wedge M(y) \rightarrow B(f(x+y), f(0)))
\end{aligned}$$

(b) The tableau is closed, so we have $T \vdash \varphi_5$ and by completeness $T \models \varphi_5$. Note that the axiom $\varphi_1 = M(0)$ is not needed:

$$
\begin{array}{c}
\mathrm{F}(\forall x)(\forall y)(M(x) \wedge M(y) \to B(f(x+y), f(0))) \\
| \\
\mathrm{F}(\forall y)(M(c_0) \wedge M(y) \to B(f(c_0+y), f(0))) \\
| \\
\mathrm{F}M(c_0) \wedge M(c_1) \to B(f(c_0+c_1), f(0)) \\
| \\
\mathrm{T}M(c_0) \wedge M(c_1) \\
| \\
\mathrm{F}B(f(c_0+c_1), f(0)) \\
| \\
\mathrm{T}M(c_0) \\
| \\
\mathrm{T}M(c_1) \\
| \\
\mathrm{T}(\forall x)(\forall y)(M(x) \wedge M(y) \to M(x+y)) \\
| \\
\mathrm{T}(\forall x)(\forall y)(M(x) \wedge M(y) \to M(x+y)) \\
| \\
\mathrm{T}(\forall y)(M(c_0) \wedge M(y) \to M(c_0+y)) \\
| \\
\mathrm{T}(\forall y)(M(c_0) \wedge M(y) \to M(c_0+y)) \\
| \\
\mathrm{T}M(c_0) \wedge M(c_1) \to M(c_0+c_1) \\
\begin{array}{cc}
\begin{array}{c}
\mathrm{F}M(c_0) \wedge M(c_1) \\
\begin{array}{cc}
\mathrm{F}M(c_0) & \mathrm{F}M(c_1) \\
\otimes & \otimes
\end{array}
\end{array}
&
\begin{array}{c}
\mathrm{T}M(c_0+c_1) \\
| \\
\mathrm{T}(\forall x)(M(x) \leftrightarrow B(x,0)) \\
| \\
\mathrm{T}(\forall x)(M(x) \leftrightarrow B(x,0)) \\
| \\
\mathrm{T}M(c_0+c_1) \leftrightarrow B(c_0+c_1, 0) \\
\begin{array}{cc}
\begin{array}{c}
\mathrm{T}M(c_0+c_1) \\
| \\
\mathrm{T}B(c_0+c_1, 0) \\
| \\
\mathrm{T}(\forall x)(\forall y)(B(x,y) \to B(f(x), f(y))) \\
| \\
\mathrm{T}(\forall x)(\forall y)(B(x,y) \to B(f(x), f(y))) \\
| \\
\mathrm{T}(\forall y)(B(c_0+c_1, y) \to B(f(c_0+c_1), f(y))) \\
| \\
\mathrm{T}(\forall y)(B(c_0+c_1, y) \to B(f(c_0+c_1), f(y))) \\
| \\
\mathrm{T}B(c_0+c_1, 0) \to B(f(c_0+c_1), f(0)) \\
\begin{array}{cc}
\mathrm{F}B(c_0+c_1, 0) & \mathrm{T}B(f(c_0+c_1), f(0)) \\
\otimes & \otimes
\end{array}
\end{array}
&
\begin{array}{c}
\mathrm{F}M(c_0+c_1) \\
| \\
\mathrm{F}B(c_0+c_1, 0) \\
\otimes
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

(c) *We find two elementarily non-equivalent models of T:*
- $\mathcal{A} = \langle \{0\}; M^{\mathcal{A}}, B^{\mathcal{A}}, f^{\mathcal{A}}, +^{\mathcal{A}}, 0^{\mathcal{A}} \rangle$ *where* $M^{\mathcal{A}} = \{0\}$, $B^{\mathcal{A}} = \{(0,0)\}$, $f^{\mathcal{A}} = \{(0,0)\}$, $+^{\mathcal{A}} = \{((0,0),0)\}$, *and* $0^{\mathcal{A}} = 0$
- $\mathcal{B} = \langle \{0,1\}; M^{\mathcal{B}}, B^{\mathcal{B}}, f^{\mathcal{B}}, +^{\mathcal{B}}, 0^{\mathcal{B}} \rangle$ *where* $M^{\mathcal{B}} = \{0\}$, $B^{\mathcal{B}} = \{(0,0),(1,1)\}$, $f^{\mathcal{B}} = \{(0,0),(1,1)\}$, $+^{\mathcal{B}} = \{((0,0),0),((0,1),1),((1,0),1),((1,1),0)\}$, *and* $0^{\mathcal{B}} = 0$

The complete simple extensions are then $\mathrm{Th}(\mathcal{A})$ *and* $\mathrm{Th}(\mathcal{B})$ *(i.e. all L-sentences true in $\mathcal{A}$ respectively $\mathcal{B}$). The theory of a structure is always a complete theory. They are not equivalent for example because $(\forall x)M(x)$ holds in $\mathcal{A}$ but not in $\mathcal{B}$. (Keep in mind that the language is without equality, so we need a sentence without equality.)*

Problem 3. Consider the language $L = \langle c \rangle$ with equality, where c is a constant symbol. Using the tableau method prove that the formula $x = c$ is valid in $T = \{(\exists x)(\forall y)x = y\}$.

Solution. *We construct a finished tableau from the theory T with the item $\mathrm{F}(\forall x)x = c$ at the root (formulae in tableau items must be sentences). Since the language has equality, we may also use the equality axioms for the language L, or rather their universal closures: $(\forall x)x = x$ and $(\forall x_1)(\forall x_2)(\forall y_1)(\forall y_2)(x_1 = y_1 \wedge x_2 = y_2 \rightarrow (x_1 = x_2 \rightarrow y_1 = y_2))$.*

$$\mathrm{F}(\forall x)x = c$$

$$\mathrm{F}c_0 = c$$

$$\mathrm{T}(\exists x)(\forall y)x = y$$

$$\mathrm{T}(\forall y)c_1 = y$$

$$\mathrm{T}(\forall y)c_1 = y$$

$$\mathrm{T}c_1 = c$$

$$\mathrm{T}(\forall y)c_1 = y$$

$$\mathrm{T}c_1 = c_0$$

$$\mathrm{T}(\forall x_1)(\forall x_2)(\forall y_1)(\forall y_2)(x_1 = y_1 \wedge x_2 = y_2 \to (x_1 = x_2 \to y_1 = y_2))$$

$$\mathrm{T}(\forall x_1)(\forall x_2)(\forall y_1)(\forall y_2)(x_1 = y_1 \wedge x_2 = y_2 \to (x_1 = x_2 \to y_1 = y_2))$$

$$\mathrm{T}(\forall x_2)(\forall y_1)(\forall y_2)(c_1 = y_1 \wedge x_2 = y_2 \to (c_1 = x_2 \to y_1 = y_2))$$

$$\mathrm{T}(\forall x_2)(\forall y_1)(\forall y_2)(c_1 = y_1 \wedge x_2 = y_2 \to (c_1 = x_2 \to y_1 = y_2))$$

$$\mathrm{T}(\forall y_1)(\forall y_2)(c_1 = y_1 \wedge c_1 = y_2 \to (c_1 = c_1 \to y_1 = y_2))$$

$$\mathrm{T}(\forall y_1)(\forall y_2)(c_1 = y_1 \wedge c_1 = y_2 \to (c_1 = c_1 \to y_1 = y_2))$$

$$\mathrm{T}(\forall y_2)(c_1 = c_0 \wedge c_1 = y_2 \to (c_1 = c_1 \to c_0 = y_2))$$

$$\mathrm{T}(\forall y_2)(c_1 = c_0 \wedge c_1 = y_2 \to (c_1 = c_1 \to c_0 = y_2))$$

$$\mathrm{T}(c_1 = c_0 \wedge c_1 = c \to (c_1 = c_1 \to c_0 = c))$$

$$\begin{array}{cc|cc}
\multicolumn{2}{c|}{\mathrm{F}c_1 = c_0 \wedge c_1 = c} & \multicolumn{2}{c}{\mathrm{T}c_1 = c_1 \to c_0 = c} \\
\mathrm{F}c_1 = c_0 & \mathrm{F}c_1 = c & \mathrm{F}c_1 = c_1 & \mathrm{T}c_0 = c \\
\otimes & \otimes & \mathrm{T}(\forall x)x = x & \otimes \\
 & & \mathrm{T}(\forall x)x = x & \\
 & & \mathrm{T}c_1 = c_1 & \\
 & & \otimes &
\end{array}$$

Problem 4. Let L be a language with equality containing a binary relational symbol $\leq$ and let T be an L-theory such that T has an infinite model and the axioms of a linear order hold in T. Using the compactness theorem show that T has a model $\mathcal{A}$ with an *infinite descending chain*; that is, in A there exist elements c_i for every $i \in \mathbb{N}$ such that: $\cdots < c_{n+1} < c_n < \cdots < c_0$. (This implies that the notion of a *well-ordering* is not definable in first-order logic.)

Solution. *From the assumption we know that T has an infinite model $\mathcal{B}$, i.e. an infinite linear order. This could, however, be for example $\langle \mathbb{N}; \leq^{\mathbb{N}} \rangle$, which has no infinite descending chain. We need a model with an infinite descending chain; we obtain it from the Compactness Theorem (version for predicate logic):*

Expand the language L by adding countably many new constant symbols c_i ($i \in \mathbb{N}$). Denote the expanded language by L'. Consider the following L'-theory T':

$$T' = T \cup \{c_{i+1} \leq c_i \wedge \neg c_{i+1} = c_i \mid i \in \mathbb{N}\}$$

It is enough to show that T' has a model. Such a model must obviously be infinite and its reduct to the language L is the desired model $\mathcal{A}$ of the theory T which has an infinite descending chain $\cdots < c^{\mathcal{A}}_{n+1} < c^{\mathcal{A}}_n < \cdots < c^{\mathcal{A}}_0$.

By the compactness theorem we know that T' has a model iff every finite subset of T' has a model. If we take a finite subtheory $S \subseteq T'$, it contains only finitely many formulas $c_{i+1} \leq c_i \wedge \neg c_{i+1} = c_i$, for some finite set of indices $I \subseteq \mathbb{N}$. Let $\mathcal{B}$ be the infinite model of T from the assumption. (This model need not have an infinite descending chain! It might be for example $\langle \mathbb{N}; \leq^{\mathbb{N}} \rangle$.) In it we can choose any finite descending chain of length $|I|$ to interpret the constant symbols c_i for $i \in I$ (interpret the symbols $c_j \notin I$ arbitrarily), and thus obtain a model of S.

Extra practice

Problem 5. Consider the following statements:

(i) Every professor has written at least one textbook.
(ii) Every textbook was written by some professor.
(iii) For every professor, someone is studying with them.
(iv) Everyone who studies with some professor has read all textbooks by that professor.
(v) Every textbook has been read by someone.

(a) Formalize *(i)–(v)* as sentences $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ in $L = \langle N, S, P, D, U \rangle$ without equality.
(b) Construct a finished tableau from $T = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ with item $F\varphi_5$ at the root.
(c) Is the sentence φ_5 true in the theory T? Is it false in T? Is it independent in T? Justify.
(d) Does the theory T have a complete conservative extension? Justify.

Problem 6. Using the tableau method, prove the following rules for 'pulling out' quantifiers, where $\varphi(x)$ is a formula with a single free variable x, and ψ is a sentence.

(a) $\neg(\exists x)\varphi(x) \to (\forall x)\neg\varphi(x)$
(b) $(\forall x)\neg\varphi(x) \to \neg(\exists x)\varphi(x)$
(c) $((\exists x)\varphi(x) \to \psi) \to (\forall x)(\varphi(x) \to \psi)$
(d) $(\forall x)(\varphi(x) \to \psi) \to ((\exists x)\varphi(x) \to \psi)$

Problem 7. Let $L(x, y)$ represent *"there is a flight from x to y"* and $S(x, y)$ represent *"there is a connection from x to y"*. Assume that from Prague one can fly to Bratislava, London, and New York, and from New York to Paris, and that

- $(\forall x)(\forall y)(L(x, y) \to L(y, x))$,
- $(\forall x)(\forall y)(L(x, y) \to S(x, y))$,

- $(\forall x)(\forall y)(\forall z)(S(x,y) \wedge L(y,z) \to S(x,z))$.

Prove using the tableau method that there is a connection from Bratislava to Paris.

Problem 8. Let T be the following theory in the language $L = \langle R, f, c, d\rangle$ with equality, where R is a binary relation symbol, f a unary function symbol, and c, d constant symbols:

$$T = \{R(x,x), R(x,y) \wedge R(y,z) \to R(x,z), R(x,y) \wedge R(y,x) \to x = y, R(f(x),x)\}$$

Denote by T' the general closure of T. Let φ and ψ be the following formulas:

$$\varphi = R(c,d) \wedge (\forall x)(x = c \vee x = d) \qquad\qquad \psi = (\exists x)R(x, f(x))$$

(a) Construct a tableau proof of ψ from $T' \cup \{\varphi\}$. (For simplicity, in the tableau you may directly use the axiom $(\forall x)(\forall y)(x = y \to y = x)$, a consequence of the equality axioms.)
(b) Show that ψ is not a consequence of T by finding a model of T in which ψ is not valid.
(c) How many complete simple extensions (up to $\sim$) does $T \cup \{\varphi\}$ have? Provide two examples.
(d) Is the following theory S in $L' = \langle R\rangle$ with equality a conservative extension of T?

$$S = \{R(x,x), R(x,y) \wedge R(y,z) \to R(x,z), R(x,y) \wedge R(y,x) \to x = y\}$$

For further thought

Problem 9. Prove syntactically, using tableau transformations:

(a) *Theorem on Constants*: Let φ be a formula in the language L with free variables $x_1, \ldots, x_n$ and T a theory in L. Let L' be the extension of L with new constant symbols $c_1, \ldots, c_n$ and T' the theory T in L'. Then: $T \vdash (\forall x_1) \ldots (\forall x_n)\varphi$ if and only if $T' \vdash \varphi(x_1/c_1, \ldots, x_n/c_n)$
(b) *Deduction Theorem*: For any theory T (in closed form) and sentences φ, ψ, we have: $T \vdash \varphi \to \psi$ if and only if $T, \varphi \vdash \psi$

Problem 10. Let T^* be a theory with equality axioms. Show using the tableau method that:

(a) $T^* \models x = y \ \to \ y = x$ (symmetry)
(b) $T^* \models (x = y \ \wedge \ y = z) \ \to \ x = z$ (transitivity)

Hint: For (a) use the equality axiom (iii) for $x_1 = x$, $x_2 = x$, $y_1 = y$ and $y_2 = x$,
for (b) use (iii) for $x_1 = x$, $x_2 = y$, $y_1 = x$ and $y_2 = z$.