

## NAIL062 P&P LOGIC: WORKSHEET 6 – BASICS OF PREDICATE LOGIC

**Teaching goals:** After completing, the student

- understands the notions of structure and signature, can formally define them and provide examples
- understands the notions of the syntax of predicate logic (language, term, atomic formula, formula, theory, free variable, open formula, sentence, instance, variant), can formally define them and provide examples
- understands semantic notions of predicate logic (value of a term, truth value, validity [under assignment], model, being valid/contradictory in a model/theory, independence [in a theory], consequence of a theory), can formally define, give examples
- understands the notion of a complete theory and its relation to elementary equivalence of structures, can define both and apply them to examples
- knows basic examples of theories (graph theories, orders, algebraic theories)
- can describe models of a given theory

### IN-CLASS PROBLEMS

**Problem 1.** Are the following formulas variants of  $(\forall x)(x < y \vee (\exists z)(z = y \wedge z \neq x))$ ?

- $(\forall z)(z < y \vee (\exists z)(z = y \wedge z \neq z))$
- $(\forall y)(y < y \vee (\exists z)(z = y \wedge z \neq y))$
- $(\forall u)(u < y \vee (\exists z)(z = y \wedge z \neq u))$

**Solution.** Let  $\psi = (x < y \vee (\exists z)(z = y \wedge z \neq x))$ , so the formula is  $(\forall x)\psi$ .

- No,  $z$  is not substitutable for  $x$  into  $\psi$ : a new bound occurrence would be created.
- No,  $y$  has a free occurrence in  $\psi$ .
- Yes,  $u$  is a new variable: in that case one can always form a variant.

**Problem 2.** Let  $\mathcal{A} = (\{a, b, c, d\}; \triangleright^A)$  be a structure in the language with a single binary relation symbol  $\triangleright$ , where  $\triangleright^A = \{(a, c), (b, c), (c, c), (c, d)\}$ .

- I. Which of the following formulas are true in  $\mathcal{A}$ ?
- II. For each of them find a structure  $\mathcal{B}$  (if one exists) such that  $\mathcal{B} \models \varphi$  iff  $\mathcal{A} \not\models \varphi$ .

- $x \triangleright y$
- $(\exists x)(\forall y)(y \triangleright x)$
- $(\exists x)(\forall y)((y \triangleright x) \rightarrow (x \triangleright x))$
- $(\forall x)(\forall y)(\exists z)((x \triangleright z) \wedge (z \triangleright y))$
- $(\forall x)(\exists y)((x \triangleright z) \vee (z \triangleright y))$

**Solution.** We can visualize the structures as directed graphs.

- I. No — intuitively the formula would state that the relation  $\triangleright^A$  contains all pairs (edges); from the definition  $\text{PH}^{\mathcal{A}}(x \triangleright y)[e] = 0$  for example for  $e(x) = a, e(y) = a$ .  
II. For example  $\mathcal{B}_0 = (\{0\}; \triangleright^{\mathcal{B}_0})$  with  $\triangleright^{\mathcal{B}_0} = \{(0, 0)\}$ .
- I. No — intuitively the graph has no vertex that would have arcs from all vertices; from the definition:  $\text{PH}^{\mathcal{A}}(\varphi)[e] = \max_{u \in A} \text{PH}^{\mathcal{A}}((\forall y)(y \triangleright x))[e(x/u)] = \max_{u \in A} \min_{v \in A} \text{PH}^{\mathcal{A}}(y \triangleright x)[e(x/u, y/v)] = 0$ , e.g. for  $u = a$  we may take  $v = a$ .  
II. For example  $\mathcal{B}_0$  as above.
- I. Yes (assign to  $x$  for instance the element  $a$ ), the antecedent is not satisfied for any assignment to  $y$ , hence the implication is always true. (Intuitively, the formula states

that there exists a vertex that either has a loop or no edge leads to it from any vertex.)

II. For example  $\mathcal{B}_1 = (\{0, 1\}; \triangleright^{\mathcal{B}_1})$  with  $\triangleright^{\mathcal{B}_1} = \{(0, 1), (1, 0)\}$ .

- (d) I. No. II: For example  $\mathcal{B}_0$ .
- (e) I. No. II: For example  $\mathcal{B}_0$ .

**Problem 3.** Prove (semantically) or find a counterexample: For every structure  $\mathcal{A}$ , formula  $\varphi$ , and sentence  $\psi$ ,

- (a)  $\mathcal{A} \models (\psi \rightarrow (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \rightarrow \varphi)$
- (b)  $\mathcal{A} \models (\psi \rightarrow (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \rightarrow \varphi)$
- (c)  $\mathcal{A} \models ((\exists x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \rightarrow \psi)$
- (d)  $\mathcal{A} \models ((\forall x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\exists x)(\varphi \rightarrow \psi)$

Does it hold for every formula  $\psi$  with a free variable  $x$ ? And for every formula  $\psi$  in which  $x$  is not free?

**Solution.** (a) It would be simpler to use the tableau method, but we want to practice a semantic proof. Intuitively, since  $\psi$  is a sentence, the variable assignment of  $x$  does not play a role in computing the truth value of  $\psi$ , so the equivalence holds. Compute from the definitions:  $\mathcal{A} \models (\psi \rightarrow (\exists x)\varphi)$  holds iff it holds under every variable assignment  $e : \text{Var} \rightarrow \mathcal{A}$ . Compute the truth value. Use the fact that  $f_{\rightarrow}(a, b) = \max(1 - a, b)$ :

$$\begin{aligned} & \text{PH}^{\mathcal{A}}(\psi \rightarrow (\exists x)\varphi)[e] \\ &= f_{\rightarrow}(\text{PH}^{\mathcal{A}}(\psi)[e], \text{PH}^{\mathcal{A}}((\exists x)\varphi)[e]) \\ &= \max(1 - \text{PH}^{\mathcal{A}}(\psi)[e], \text{PH}^{\mathcal{A}}((\exists x)\varphi)[e]) \\ &= \max(1 - \text{PH}^{\mathcal{A}}(\psi)[e], \max_{a \in A} \text{PH}^{\mathcal{A}}(\varphi)[e(x/a)]) \end{aligned}$$

Similarly for the formula on the right:

$$\begin{aligned} & \text{PH}^{\mathcal{A}}((\exists x)(\psi \rightarrow \varphi))[e] \\ &= \max_{a \in A} \text{PH}^{\mathcal{A}}(\psi \rightarrow \varphi)[e(x/a)] \\ &= \max_{a \in A} (\max(1 - \text{PH}^{\mathcal{A}}(\psi)[e(x/a)], \text{PH}^{\mathcal{A}}(\varphi)[e(x/a)])) \end{aligned}$$

Because  $\psi$  is a sentence, it does not contain a free occurrence of the variable  $x$ , hence  $\text{PH}^{\mathcal{A}}(\psi)[e(x/a)] = \text{PH}^{\mathcal{A}}(\psi)[e]$ . From this we see that:

$$\begin{aligned} &= \max_{a \in A} (\max(1 - \text{PH}^{\mathcal{A}}(\psi)[e], \text{PH}^{\mathcal{A}}(\varphi)[e(x/a)])) \\ &= \max(1 - \text{PH}^{\mathcal{A}}(\psi)[e], \max_{a \in A} (\text{PH}^{\mathcal{A}}(\varphi)[e(x/a)])) \end{aligned}$$

Both truth values are the same, so the equivalence holds. For this argument it is sufficient that  $x$  is not free in  $\psi$ .

If  $x$  is free in  $\psi$ , the equivalence does not hold. For example in language  $L = \langle c \rangle$  with equality, where  $c$  is a constant symbol:

- $\varphi$  is  $\neg x = x$ ,
- $\psi$  is  $x = c$ ,
- $\mathcal{A} = (\{0, 1\}; 0)$  (i.e.  $c^{\mathcal{A}} = 0$ ).

We have  $\mathcal{A} \not\models (x = c \rightarrow (\exists x)\neg x = x)$ , because it is not valid under the variable assignment  $e(x) = 0$ . But  $\mathcal{A} \models (\exists x)(x = c \rightarrow \neg x = x)$ , because  $x$  can be assigned the element 1, and then the antecedent is not satisfied.

(b), (c), (d) are solved similarly.

**Problem 4.** Decide whether  $T$  (in the language  $L = \langle U, f \rangle$  with equality) is complete. If they exist, give two elementarily non-equivalent models, and two non-equivalent complete simple extensions:

- (a)  $T = \{U(f(x)), \neg x = y, x = y \vee x = z \vee y = z\}$
- (b)  $T = \{U(f(x)), \neg(\forall x)(\forall y)x = y, x = y \vee x = z \vee y = z\}$
- (c)  $T = \{U(f(x)), \neg x = f(x), \neg(\forall x)(\forall y)x = y, x = y \vee x = z \vee y = z\}$
- (d)  $T = \{U(f(x)), \neg(\forall x)x = f(x), \neg(\forall x)(\forall y)x = y, x = y \vee x = z \vee y = z\}$

**Solution.** (a) Beware, this theory is inconsistent. Note that  $\neg x = y$  is inconsistent: it is not true in any model, because it fails under the variable assignment  $e(x) = a, e(y) = a$  for any element  $a \in A$ . (It is equivalent to its universal closure  $(\forall x)(\forall y)\neg x = y$ .) An inconsistent theory is not complete by definition, and all its extensions are also inconsistent, so it has no complete simple extension.

(b) Not complete. Informally,  $T$  says that the model has exactly two elements, and the outputs of  $f^A$  must lie inside  $U^A$ . From this we know  $U^A \neq \emptyset$ . If it is a one-element set, we have a single model (up to isomorphism); if it is two-element, we have in total three pairwise non-isomorphic (and thus elementarily non-equivalent) models (where  $f^A$  has no fixed point, has one fixed point, or has two fixed points, i.e. is the identity):

- $\mathcal{A}_1 = (\{0, 1\}; U_1^A, f_1^A)$  where  $U_1^A = \{0\}$  and  $f_1^A = \{(0, 0), (1, 0)\}$ , i.e.  $f_1^A(0) = 0, f_1^A(1) = 0$
- $\mathcal{A}_2 = (\{0, 1\}; \{0, 1\}, \{(0, 1), (1, 0)\})$ ,
- $\mathcal{A}_3 = (\{0, 1\}; \{0, 1\}, \{(0, 0), (1, 0)\})$ ,
- $\mathcal{A}_4 = (\{0, 1\}; \{0, 1\}, \{(0, 0), (1, 1)\})$ .

(Draw the pictures!) The corresponding complete simple extensions can be written as  $\text{Th}(\mathcal{A}_i)$  for  $i = 1, 2, 3, 4$ . Or:

- $T_1 = T \cup \{\neg(\forall x)U(x)\}$ ,
- $T_2 = T \cup \{U(x), \neg f(x) = x\}$ ,
- $T_3 = T \cup \{U(x), (\exists x)f(x) = x, (\exists x)\neg f(x) = x\}$ ,
- $T_4 = T \cup \{U(x), f(x) = x\}$ .

(c) Similarly, it expresses that the model has exactly two elements and  $f$  has no fixed point. It is complete: there is a single model up to isomorphism, namely  $\mathcal{A}_2$ .

(d) The model has exactly two elements and  $f$  has at least one fixed point. It is not complete; its models up to isomorphism are  $\mathcal{A}_3$  and  $\mathcal{A}_4$ .

#### EXTRA PRACTICE

**Problem 5.** Determine the free and bound occurrences of variables in the following formulas. Then convert them to variants in which no variable will have both free and bound occurrence.

- (a)  $(\exists x)(\forall y)P(y, z) \vee (y = 0)$
- (b)  $(\exists x)(P(x) \wedge (\forall x)Q(x)) \vee (x = 0)$
- (c)  $(\exists x)(x > y) \wedge (\exists y)(y > x)$

**Problem 6.** Let  $\varphi$  denote the formula  $(\forall x)((x = z) \vee (\exists y)(f(x) = y) \vee (\forall z)(y = f(z)))$ . Which of the following terms are substitutable into  $\varphi$ ?

- (a) the term  $z$  for the variable  $x$ , the term  $y$  for the variable  $x$ ,
- (b) the term  $z$  for the variable  $y$ , the term  $g(f(y), w)$  for the variable  $y$ ,
- (c) the term  $x$  for the variable  $z$ , the term  $y$  for the variable  $z$ ,

**Problem 7.** Are the following sentences valid / contradictory / independent (in logic)?

- (a)  $(\exists x)(\forall y)(P(x) \vee \neg P(y))$
- (b)  $(\forall x)(P(x) \rightarrow Q(f(x))) \wedge (\forall x)P(x) \wedge (\exists x)\neg Q(x)$
- (c)  $(\forall x)(P(x) \vee Q(x)) \rightarrow ((\forall x)P(x) \vee (\forall x)Q(x))$
- (d)  $(\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\exists x)P(x) \rightarrow (\exists x)Q(x))$
- (e)  $(\exists x)(\forall y)P(x, y) \rightarrow (\forall y)(\exists x)P(x, y)$

**Problem 8.** Decide whether the following hold for every formula  $\varphi$ . Prove (semantically, from the definitions) or provide a counterexample.

- (a)  $\varphi \models (\forall x)\varphi$
- (b)  $\models \varphi \rightarrow (\forall x)\varphi$
- (c)  $\varphi \models (\exists x)\varphi$
- (d)  $\models \varphi \rightarrow (\exists x)\varphi$

#### FOR FURTHER THOUGHT

**Problem 9.** Let  $L = \langle +, -, 0 \rangle$  be the language of group theory (with equality). The theory of groups  $T$  consists of the following axioms:

$$\begin{aligned} x + (y + z) &= (x + y) + z \\ 0 + x &= x = x + 0 \\ x + (-x) &= 0 = (-x) + x \end{aligned}$$

Decide whether the following formulas are true / false / independent in  $T$ . Justify.

- (a)  $x + y = y + x$
- (b)  $x + y = x \rightarrow y = 0$
- (c)  $x + y = 0 \rightarrow y = -x$
- (d)  $-(x + y) = (-y) + (-x)$