

NAIL062 P&P LOGIC: WORKSHEET 7 – PROPERTIES OF STRUCTURES AND THEORIES

Teaching goals: After completing, the student

- understands the notion of substructure, generated substructure, can find them
- understands the notion of expansion and reduct of a structure, can define them formally and give examples
- understands the notions of [simple, conservative] extension, can formulate the definitions and the corresponding semantic criterion (for both expansions and reducts), and apply it to an example
- understands the notion of extension by definition, can define it formally and give examples
- can decide whether a given theory is an extension by definition, construct an extension by a given definition
- understands the notion of definability in a structure, can find definable subsets/relations

IN-CLASS PROBLEMS

Problem 1. Consider $\underline{\mathbb{Z}}_4 = \langle \{0, 1, 2, 3\}; +, -, 0 \rangle$ where $+$ is binary addition modulo 4 and $-$ is the unary function returning the *inverse* for $+$ with respect to the *neutral* element 0.

- (a) Is $\underline{\mathbb{Z}}_4$ a model of the theory of groups (i.e. is it a *group*)?
- (b) Determine all substructures $\underline{\mathbb{Z}}_4(a)$ generated by some $a \in \underline{\mathbb{Z}}_4$.
- (c) Does $\underline{\mathbb{Z}}_4$ contain any other substructures?
- (d) Is every substructure of $\underline{\mathbb{Z}}_4$ a model of the theory of groups?
- (e) Is every substructure of $\underline{\mathbb{Z}}_4$ elementarily equivalent to $\underline{\mathbb{Z}}_4$?

Problem 2. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}; +, -, \cdot, 0, 1 \rangle$ be the field of rationals with the standard operations.

- (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of the theory of groups?
- (b) Can the reduct $\langle \mathbb{Q}, \cdot, 1 \rangle$ be extended to a model of the theory of groups?
- (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementarily equivalent to $\underline{\mathbb{Q}}$?
- (d) Let $\text{Th}(\underline{\mathbb{Q}})$ denote the set of all sentences true in $\underline{\mathbb{Q}}$. Is $\text{Th}(\underline{\mathbb{Q}})$ a complete theory?

Problem 3. Consider the theory $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ in the language $L = \langle c_1, c_2, c_3 \rangle$ with equality.

- (a) Is T complete?
- (b) How many simple extensions of T are there, up to equivalence? How many are complete? Write down all complete ones and at least three incomplete ones.
- (c) Is the theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ in the language $L' = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T ? Is T' a simple extension of T ? Is T' a conservative extension of T ?

Problem 4. Let T' be an extension of $T = \{(\exists y)(x+y=0), (x+y=0) \wedge (x+z=0) \rightarrow y=z\}$ in the language $L = \langle +, 0, \leq \rangle$ with equality by definitions of $<$ and unary $-$ with axioms

$$\begin{aligned} -x = y &\leftrightarrow x + y = 0 \\ x < y &\leftrightarrow x \leq y \wedge \neg(x = y) \end{aligned}$$

Find formulas in the language L that are equivalent in T' to the following formulas.

- (a) $(-x) + x = 0$
- (b) $x + (-y) < x$
- (c) $-(x + y) < -x$

Problem 5. Consider the language $L = \langle F \rangle$ with equality, where F is a binary function symbol. Find formulas defining the following sets (without parameters):

- (a) the interval $(0, \infty)$ in $\mathcal{A} = \langle \mathbb{R}, \cdot \rangle$ where \cdot is multiplication of real numbers
- (b) the set $\{(x, 1/x) \mid x \neq 0\}$ in the same structure \mathcal{A}
- (c) the set of all singleton subsets of \mathbb{N} in $\mathcal{B} = \langle \mathcal{P}(\mathbb{N}), \cup \rangle$
- (d) the set of all prime numbers in $\mathcal{C} = \langle \mathbb{N} \cup \{0\}, \cdot \rangle$

EXTRA PRACTICE

Problem 6. Let $T = \{\neg E(x, x), E(x, y) \rightarrow E(y, x), (\exists x)(\exists y)(\exists z)(E(x, y) \wedge E(y, z) \wedge E(x, z) \wedge \neg(x = y \vee y = z \vee x = z)), \varphi\}$ be a theory in the language $L = \langle E \rangle$ with equality, where E is a binary relation symbol and φ expresses that “there are exactly four elements.”

- (a) Consider the expansion $L' = \langle E, c \rangle$ of the language by a new constant symbol c . Determine the number (up to equivalence) of theories T' in L' that are extensions of T .
- (b) Does T have any *conservative* extension in the language L' ? Justify your answer.

Problem 7. Let $T = \{x = f(f(x)), \varphi, \neg c_1 = c_2\}$ be a theory in the language $L = \langle f, c_1, c_2 \rangle$ with equality, where f is a unary function symbol, c_1, c_2 are constant symbols, and the axiom φ expresses that “there are exactly three elements.”

- (a) Determine how many pairwise nonequivalent complete simple extensions the theory T has. Write down two of them.
- (b) Let $T' = \{x = f(f(x)), \varphi, \neg f(c_1) = f(c_2)\}$ be a theory in the same language, with φ same as above. Is T' an extension of T ? Is T an extension of T' ? If so, is it a conservative extension? Provide justification.

Problem 8. Consider $L = \langle P, R, f, c, d \rangle$ with equality and the following two formulas:

$$\begin{aligned}\varphi : \quad & P(x, y) \leftrightarrow R(x, y) \wedge \neg x = y \\ \psi : \quad & P(x, y) \rightarrow P(x, f(x, y)) \wedge P(f(x, y), y)\end{aligned}$$

Consider the following L -theory:

$$\begin{aligned}T = \{ & \varphi, \psi, \neg c = d, \\ & R(x, x), \\ & R(x, y) \wedge R(y, x) \rightarrow x = y, \\ & R(x, y) \wedge R(y, z) \rightarrow R(x, z), \\ & R(x, y) \vee R(y, x)\}\end{aligned}$$

- (a) Find an expansion of the structure $\langle \mathbb{Q}, \leq \rangle$ to the language L that is a model of T .
- (b) Is the sentence $(\forall x)R(c, x)$ valid/contradictory/independent in T ? Justify all 3 answers.
- (c) Find two nonequivalent complete simple extensions of T , or justify why they do not exist.
- (d) Let $T' = T \setminus \{\varphi, \psi\}$ be a theory in the language $L' = \langle R, f, c, d \rangle$. Is the theory T a conservative extension of the theory T' ? Provide justification.

FOR FURTHER THOUGHT

Problem 9. Let $T_n = \{\neg c_i = c_j \mid 1 \leq i < j \leq n\}$ be a theory in the language $L_n = \langle c_1, \dots, c_n \rangle$ with equality, where c_1, \dots, c_n are constant symbols.

- (a) For a given finite $k \geq 1$, count k -element models of the theory T_n up to isomorphism.
- (b) Determine the number of countable models of the theory T_n up to isomorphism.
- (c) For which pairs of values n and m is T_n an extension of T_m ? For which pairs is it a conservative extension? Justify your answer.