LOGIC TUTORIAL: WORKSHEET 2

Topics: Theories. Normal forms, CNF and DNF, SAT and 3-SAT. 3-SAT. 2-SAT and implication graph, Horn-SAT and unit propagation. Encoding problems in SAT.

Problem 1. Consider the propositional theory $T = \{\neg q \to (\neg p \vee q),\ \neg p \to q,\ r \to q\}$ over the language $\{p, q, r, s\}$. Which of the following propositions are valid, contradictory, independent, satisfiable, equivalent in T?

(a) p, q, r, s, $\neg p$, $\neg s$
(b) $p \vee q$, $p \vee r$, $p \vee s$, $q \vee s$
(c) $p \wedge q$, $q \wedge s$, $p \to q$, $s \to q$

Problem 2. Given a formula φ in CNF or DNF, A) count the number of models and B) describe all its models.

a) $(p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) \vee (p_2 \wedge p_3 \wedge \neg p_4) \vee (\neg p_3) \vee (p_2 \wedge p_4) \vee (p_1 \wedge p_3 \wedge p_5) \vee (p_3 \wedge \neg p_4 \wedge p_2)$
b) $(p_1 \vee \neg p_2 \vee p_3 \vee \neg p_4) \wedge (p_2 \vee p_3 \vee \neg p_4) \wedge (\neg p_3) \wedge (p_2 \vee p_4) \wedge (p_1 \vee p_3 \vee p_5) \wedge (p_3 \vee \neg p_4 \vee p_2)$

Problem 3. Transform the following propositional formulas into CNF and DNF A) using truth tables (determining models), B) using syntactic rules:

a) $(\neg p \vee q) \to (\neg q \wedge r)$,
b) $(\neg p \to (\neg q \to r)) \to p$,
c) $((p \to \neg q) \to \neg r) \to \neg p$.

Problem 4. Given a formula φ in CNF, find a 3-CNF formula φ' such that φ' is satisfiable iff φ is satisfiable. Describe an efficient algorithm to construct φ' given φ (a reduction from SAT to 3-SAT).

Problem 5. Find the (shortest possible) CNF and DNF representations of the Boolean majority function $\mathrm{maj} : {}^{3}2 \to 2$ which outputs the majority vote of the input values.

Problem 6. Can you find CNF and DNF representations of the Boolean n-ary parity function $\mathrm{par} : {}^{n}2 \to 2$ defined by $\mathrm{par}(x_1, \dots, x_n) = (x_1 + \cdots + x_n) \bmod 2$ which outputs the XOR of all input values? Try it for small values of n.

Problem 7. Let $\mathbb{P}$ be a countably infinite set of propostional letters. Show that it is no longer true that every $K \subseteq {}^{\mathbb{P}}2$ can be modelled by both a CNF and a DNF formula. Find such a set of models K which cannot be modelled by either.

Problem 8. Construct the implication graph of the following 2-CNF formula. Is the formula satisfiable? If it is, find a solution.

a) $(p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee \neg p_1)$,
b) $(p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee p_1)$,
c) $(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1 \wedge \neg p_7$.

Problem 9. Apply the unit propagation algorithm to determine whether the following Horn formula is satisfiable. It it is, find a satisfying assignment.

$$(\neg p_1 \vee \neg p_3 \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge p_1 \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge$$
$$(\neg p_2 \vee \neg p_4 \vee p_1) \wedge (\neg p_4 \vee \neg p_3 \vee \neg p_2) \wedge (p_4 \vee \neg p_5 \vee \neg p_6)$$

Problem 10. Can a 4×4 chessboard with two opposite corners removed be perfectly covered by domino tiles? Encode the problem as a SAT formula. Generalize to all even n.

Problem 11. Can you color integers from 1 to n with two colors so that the equation $a + b = c$ has no monochromatic solutions with $1 \leq a < b < c \leq n$? Construct a CNF formula φ_n which is satisfiable, if and only if such a coloring is possible. Try $n = 8$ first.

Try this at home: Write a script generating φ_n in DIMACS CNF format. Use a SAT solver to find the smallest n for which no such coloring exists (i.e., any 2-coloring has a monochromatic triple $a < b < c$ with $a + b = c$).

Problem 12. Encode the problem of sorting three integer numbers in SAT.

Problem 13. The famous Four Colour Theorem says that the following map can be colored by 4 colors so that no two adjacent regions share the same colour. Find such a coloring (using a SAT solver).