NAIL062 P&P Logic: Worksheet 8 – The tableau method in predicate logic

Teaching goals: After completing, the student

- understands how the tableau method in predicate logic differs from propositional logic, can formally define all necessary concepts
- knows atomic tableaux for quantifiers, understands their use
- can construct a finished tableau for a given formula from a given (even infinite) theory
- can describe the canonical model for a given finished noncontradictory branch
- understands the axioms of equality, their relation to congruences, quotient structures
- can apply the tableau method to solve a given problem (word problem, etc.)
- understands tableau method for languages with equality, can apply to simple examples
- knows the compactness theorem of predicate logic, can apply it

In-class problems

Problem 1. Assume that:

- *All guilty people are liars.*
- *At least one of the accused is also a witness.*
- *No witness lies.*

Prove by the tableau method that: *Not all of the accused are guilty.* Specifically:

(a) Choose a suitable language $\mathcal{L}$. Will it be with equality, or without equality?
(b) Formalize our knowledge and the statement to be proved as sentences $\alpha_1, \alpha_2, \alpha_3, \varphi$ in $\mathcal{L}$.
(c) Construct a tableau proof of the sentence φ from the theory $T = \{\alpha_1, \alpha_2, \alpha_3\}$.

Problem 2. Consider the following statements:

(i) Zero is a small number.
(ii) A number is small iff it is close to zero.
(iii) The sum of two small numbers is small.
(iv) If x is close to y, so is $f(x)$ to $f(y)$.

We want to prove that: *(v) If x and y are small numbers, then $f(x + y)$ is close to $f(0)$.*

(a) Formalize the statements as sentences $\varphi_1, \ldots, \varphi_5$ in $L = \langle S, C, f, +, 0\rangle$ without equality.
(b) Construct a finished tableau from the theory $T = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ with the item $F\varphi_5$ at the root. Decide whether $T \models \varphi_5$.
(c) If they exist, give at least two complete simple extensions of the theory T.

Problem 3. Consider the language $L = \langle c\rangle$ with equality, where c is a constant symbol. Using the tableau method prove that the formula $x = c$ is valid in $T = \{(\exists x)(\forall y)x = y\}$.

Problem 4. Let L be a language with equality containing a binary relational symbol $\leq$ and let T be an L-theory such that T has an infinite model and the axioms of linear order are valid in T. Using the compactness theorem show that T has a model $\mathcal{A}$ with an *infinite descending chain*; that is, in A there exist elements c_i for every $i \in \mathbb{N}$ such that: $\cdots < c_{n+1} < c_n < \cdots < c_0$. (This implies that the notion of a *well-ordering* is not definable in first-order logic.)

Extra practice

Problem 5. Consider the following statements:

(i) Every professor has written at least one textbook.
(ii) Every textbook was written by some professor.
(iii) Every professor has someone studying with them.
(iv) Everyone who studies with some professor has read all textbooks by that professor.

(v) Every textbook has been read by someone.

(a) Formalize *(i)–(v)* as sentences $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ in $L = \langle W, S, R, P, T\rangle$ without equality, where W, S, R are binary relation symbols ("x wrote y","x studies with y", "x read y") and P, T are unary relation symbols ("being a professor", "being a textbook").
(b) Construct a finished tableau from $T = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ with entry $F\varphi_5$ at the root.
(c) Is the sentence φ_5 valid in T? Is it contradictory in T? Is it independent in T? Justify.
(d) Does the theory T have a complete conservative extension? Justify.

Problem 6. Using the tableau method, prove the following rules for 'pulling out' quantifiers, where $\varphi(x)$ is a formula with a single free variable x, and ψ is a sentence.

(a) $\neg(\exists x)\varphi(x) \to (\forall x)\neg\varphi(x)$
(b) $(\forall x)\neg\varphi(x) \to \neg(\exists x)\varphi(x)$
(c) $((\exists x)\varphi(x) \to \psi) \to (\forall x)(\varphi(x) \to \psi)$
(d) $(\forall x)(\varphi(x) \to \psi) \to ((\exists x)\varphi(x) \to \psi)$

Problem 7. Let $F(x, y)$ represent *"there is a flight from x to y"* and $C(x, y)$ represent *"there is a connection from x to y"*. Assume that from Prague one can fly to Bratislava, London, and New York, and from New York to Paris, and that

- $(\forall x)(\forall y)(F(x, y) \to F(y, x))$,
- $(\forall x)(\forall y)(F(x, y) \to C(x, y))$,
- $(\forall x)(\forall y)(\forall z)(C(x, y) \wedge F(y, z) \to C(x, z))$.

Prove using the tableau method that there is a connection from Bratislava to Paris.

Problem 8. Let T be the following theory in the language $L = \langle R, f, c, d\rangle$ with equality, where R is a binary relation symbol, f a unary function symbol, and c, d constant symbols:

$$T = \{R(x, x), R(x, y) \wedge R(y, z) \to R(x, z), R(x, y) \wedge R(y, x) \to x = y, R(f(x), x)\}$$

Denote by T' the general closure of T. Let φ and ψ be the following formulas:

$$\varphi = R(c, d) \wedge (\forall x)(x = c \vee x = d) \qquad \psi = (\exists x)R(x, f(x))$$

(a) Construct a tableau proof of ψ from $T' \cup \{\varphi\}$. (For simplicity, in the tableau you may directly use the axiom $(\forall x)(\forall y)(x = y \to y = x)$, a consequence of the axioms of equality.)
(b) Show that ψ is not a consequence of T by finding a model of T in which ψ is not valid.
(c) How many complete simple extensions (up to $\sim$) does $T \cup \{\varphi\}$ have? Provide two examples.
(d) Is the following theory S in $L' = \langle R\rangle$ with equality a conservative extension of T?

$$S = \{R(x, x), R(x, y) \wedge R(y, z) \to R(x, z), R(x, y) \wedge R(y, x) \to x = y\}$$

For further thought

Problem 9. Prove syntactically, by transforming tableaux:

(a) *Theorem on Constants*: Let φ be a formula in the language L with free variables $x_1, \ldots, x_n$ and T a theory in L. Let L' be the extension of L with new constant symbols $c_1, \ldots, c_n$ and T' the theory T in L'. Then: $T \vdash (\forall x_1) \ldots (\forall x_n)\varphi$ if and only if $T' \vdash \varphi(x_1/c_1, \ldots, x_n/c_n)$
(b) *Deduction Theorem*: For any theory T (in closed form) and sentences φ, ψ, we have: $T \vdash \varphi \to \psi$ if and only if $T, \varphi \vdash \psi$

Problem 10. Let T^* be a theory with axioms of equality. Show using the tableau method:

(a) $T^* \models x = y \;\to\; y = x$ (symmetry)
(b) $T^* \models (x = y \;\wedge\; y = z) \;\to\; x = z$ (transitivity)

Hint: For (a) use the axiom of equality *(iii)* for $x_1 = x$, $x_2 = x$, $y_1 = y$ and $y_2 = x$, for (b) use *(iii)* for $x_1 = x$, $x_2 = y$, $y_1 = x$ and $y_2 = z$.