NAIL062 P&P Logic: Worksheet 9 – Prep for resolution in predicate logic

Teaching goals: After completing, the student

- can convert formulas into prenex normal form (PNF)
- understands the notion of a Skolem variant, can Skolemize a given theory
- can transform a given open theory into CNF, write it in set representation
- knows Herbrand's theorem, can demonstrate it on an example, describe a Herbrand model

In-class problems

Problem 1. Convert the following formulas into PNF. Then find their Skolem variants.

(a) $(\forall y)((\exists x)P(x,y) \to Q(y,z)) \wedge (\exists y)((\forall x)R(x,y) \vee Q(x,y))$
(b) $(\exists x)R(x,y) \leftrightarrow (\forall y)P(x,y)$
(c) $\neg((\forall x)(\exists y)P(x,y) \to (\exists x)(\exists y)R(x,y)) \wedge (\forall x)\neg(\exists y)Q(x,y)$

Problem 2. Convert into an equisatisfiable CNF formula, write in set representation.

(a) $(\forall y)(\exists x)P(x,y)$
(b) $\neg(\forall y)(\exists x)P(x,y)$
(c) $\neg(\exists x)((P(x) \to P(c)) \wedge (P(x) \to P(d)))$
(d) $(\exists x)(\forall y)(\exists z)(P(x,z) \wedge P(z,y) \to R(x,y))$

Problem 3. Let $T = \{(\exists x)R(x), (\exists y)\neg P(x,y), (\exists y)(\forall z)(\neg R(x) \vee P(y,z))\}$ be a theory in the language $L = \langle P, R\rangle$ without equality. Find an open theory T' equisatisfiable with T. Convert T' into CNF and write the resulting formula S in set representation.

Problem 4. Let $T = \{\varphi_1, \varphi_2\}$ be a theory in the language $L = \langle R\rangle$ with equality, where:

$$\varphi_1 = \quad (\exists y)R(y,x)$$
$$\varphi_2 = \quad (\exists z)(R(z,x) \wedge R(z,y) \wedge (\forall w)(R(w,x) \wedge R(w,y) \to R(w,z)))$$

(a) Using Skolemization, construct an openly axiomatized theory T' (possibly in the extended language L') equisatisfiable with T.
(b) Let $\mathcal{A} = \langle \mathbb{N}, R^{\mathcal{A}}\rangle$, where $(n,m) \in R^{\mathcal{A}}$ iff n divides m. Find an expansion $\mathcal{A}'$ of the L-structure $\mathcal{A}$ to the language L' such that $\mathcal{A}' \models T'$. (The set of natural numbers $\mathbb{N}$ includes zero, see ISO 80000-2:2019.)

Problem 5. Construct a Herbrand model of the given theory, or find an unsatisfiable conjunction of ground instances of its axioms (c, d are constant symbols in the language).

(a) $T = \{\neg P(x) \vee Q(f(x),y), \neg Q(x,d), P(c)\}$
(b) $T = \{\neg P(x) \vee Q(f(x),y), Q(x,d), P(c)\}$
(c) $T = \{P(x,f(x)), \neg P(x,g(x))\}$
(d) $T = \{P(x,f(x)), \neg P(x,g(x)), P(g(x),f(y)) \to P(x,y)\}$

Extra practice

Problem 6. The theory of fields T in the language $L = \langle +, -, \cdot, 0, 1\rangle$ contains only one axiom φ that is not open: $x \neq 0 \;\to\; (\exists y)(x \cdot y = 1)$. We know that $T \models 0 \cdot y = 0$ and $T \models\; (x \neq 0 \;\wedge\; x \cdot y = 1 \;\wedge\; x \cdot z = 1) \;\to\; y = z$.

(a) Find the Skolem form φ_S of the formula φ with a new function symbol f.
(b) Consider the theory T' obtained from T by replacing φ with φ_S. Is φ valid in T'?

(c) Can every model of T be *uniquely* expanded to a model of T'?

Now consider the formula $\psi = x \cdot y = 1 \lor (x = 0 \land y = 0)$.

(d) Are the axioms of existence and uniqueness for $\psi(x, y)$ and the variable y valid in T?
(e) Construct an extension T'' of the theory T by definition of f using the formula ψ.
(f) Is T'' equivalent to the theory T'?
(g) Find an L-formula that is T''-equivalent to the formula: $f(x \cdot y) = f(x) \cdot f(y)$

Problem 7. We know the following holds:

- *If a brick is on (another) brick, then it is not on the ground.*
- *Every brick is on (another) brick or on the ground.*
- *No brick is on a brick that is itself on (another) brick.*

We want to prove by resolution the following statement: *"If a brick is on (another) brick, the lower brick is on the ground."*. Construct the corresponding CNF formula S, and try to find its resolution refutation.

For further thought

Problem 8. The Skolem form does not have to be equivalent to the original formula; verify that the following holds:

(a) $\models (\forall x)P(x, f(x)) \to (\forall x)(\exists y)P(x, y)$
(b) $\not\models (\forall x)(\exists y)P(x, y) \to (\forall x)P(x, f(x))$