

Teaching goals: After completing, the student

- understands the notion of unification and can perform the Unification Algorithm
- knows the necessary notions from the resolution method in predicate logic (resolution rule, resolvent, resolution proof/refutation, resolution tree), can formally define them, give examples, and explain the differences compared to propositional logic
- can apply the resolution method to solve a given problem (word problem, etc.), performing all necessary steps (conversion to PNF, Skolemization, conversion to CNF)
- can construct a resolution refutation of a given (possibly infinite) CNF formula (if it exists), can draw the resolution tree including the unifications used
- can extract an unsatisfiable conjunction of ground instances of axioms from a res. tree
- knows the notion of LI-resolution, can find an LI-refutation of a given theory (if exists)
- has become familiar with selected notions from model theory

IN-CLASS PROBLEMS

Problem 1. *Every barber shaves all those who do not shave themselves. No barber shaves anyone who shaves themselves. Formalize and prove by resolution that: There are no barbers.*

Solution. *First, we choose a suitable language. In the text we identify a property of objects “ x is a barber” and a relation between two objects “ x shaves y ”. We use the language $L = \langle B, S \rangle$ without equality, where B is a unary relational symbol, $B(x)$ means “ x is a barber”, S is a binary relational symbol, $S(x, y)$ means “ x shaves y ”.*

In this language we formalize the statements from the problem:

- *Every barber shaves all those who do not shave themselves:*

$$\varphi_1 = (\forall x)(B(x) \rightarrow (\forall y)(\neg S(y, y) \rightarrow S(x, y)))$$

- *No barber shaves anyone who shaves themselves:*

$$\varphi_2 = \neg(\exists x)(B(x) \wedge (\exists y)(S(x, y) \wedge S(y, y)))$$

- *There are no barbers:*

$$\psi = \neg(\exists x)B(x)$$

Our goal is to show that in the theory $T = \{\varphi_1, \varphi_2\}$ the sentence ψ is valid. We prove this by contradiction, starting with the theory $T \cup \{\neg\psi\} = \{\varphi_1, \varphi_2, \neg\psi\}$. Using Skolemization we obtain an equisatisfiable CNF formula S , then find its resolution refutation, showing that S and hence $T \cup \{\neg\psi\}$ is unsatisfiable.

Convert to PNF, Skolemize, remove universal quantifiers, and convert to CNF:

- $\varphi_1 \rightsquigarrow B(x) \rightarrow (\neg S(y, y) \rightarrow S(x, y)) \sim \neg B(x) \vee S(y, y) \vee S(x, y)$
- $\varphi_2 \rightsquigarrow \neg(B(x) \wedge S(x, y) \wedge S(y, y)) \sim \neg B(x) \vee \neg S(x, y) \vee \neg S(y, y)$
- $\neg\psi \rightsquigarrow B(c)$ (where c is a new constant symbol)

Before starting Skolemization make sure you have sentences. And remember that we must Skolemize the sentence $\neg\psi$, not ψ . (The negation of a Skolem variant is typically not equisatisfiable with the negation of the original formula! By Skolemizing $\neg(\exists x)B(x)$ we would get $\neg B(x)$, whose negation is equivalent to $B(x)$, i.e. “everyone is a barber”, whereas the correct procedure yields “(a witness) c is a barber”.)

In set notation we have:

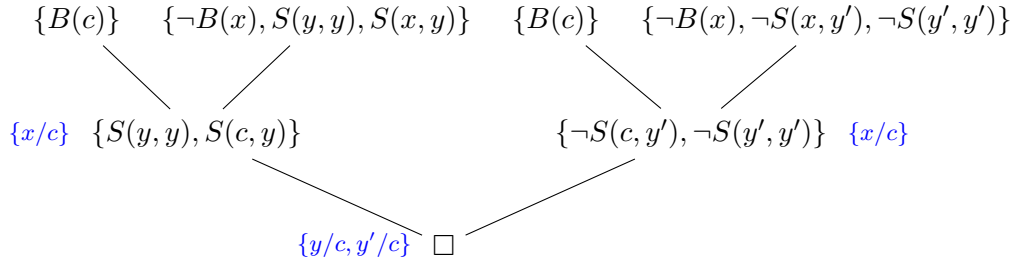
$$S = \{\{\neg B(x), S(y, y), S(x, y)\}, \{\neg B(x), \neg S(x, y), \neg S(y, y)\}, \{B(c)\}\}$$

Resolution refutation:

$$\{B(c)\}, \{\neg B(x), S(y, y), S(x, y)\}, \{S(y, y), S(c, y)\}, \{\neg B(x), \neg S(x, y'), \neg S(y', y')\}, \\ \{\neg S(c, y'), \neg S(y', y')\}, \square$$

The first two clauses are from S , the third is their resolvent using the unification $\{x/c\}$. The fourth clause is a variant of a clause from S , variable y renamed to y' to satisfy the technical condition of disjoint variable sets in resolved clauses. The fifth clause is the resolvent of the first and fourth clauses using unification $\{x/c\}$. The last, empty clause \square is the resolvent of clauses 3 and 5 with unification $\{y/c, y'/c\}$.

Typically, we represent the refutation as a resolution tree, indicating the unifications used:



Problem 2. The following statements describe a genetic experiment:

- (i) Every sheep was either born from another sheep or cloned (but not both).
- (ii) No cloned sheep gave birth.

We want to show by resolution that: (iii) If a sheep gave birth, it was itself born. Specifically:

- (a) Express as sentences $\varphi_1, \varphi_2, \varphi_3$ in $L = \langle B, C \rangle$ without equality (B is binary, C unary relation symbol, $B(x, y)$ means ‘sheep x gave birth to sheep y ’, $C(x)$ ‘sheep x was cloned’).
- (b) Using Skolemization of these sentences or their negations, construct a set of clauses S (possibly in an extended language) that is unsatisfiable exactly when $\{\varphi_1, \varphi_2\} \models \varphi_3$.
- (c) Find a resolution refutation of S , draw the resolution tree with unifications used.
- (d) Does S have an LI-refutation?

Solution. Note that all objects are sheep, so no predicate for ‘being a sheep’ is needed. The procedure is similar to the previous example:

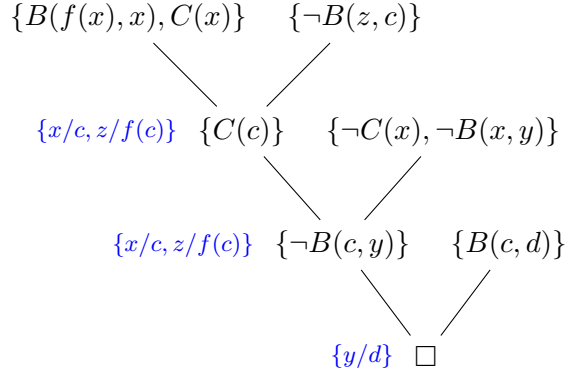
- (a) There are several ways to formulate the formulas; if we try to follow the text as closely as possible we get, for example:

$$\begin{aligned} \varphi_1 &= (\forall x)((\exists y)B(y, x) \vee C(x)) \wedge \neg((\exists z)B(z, x) \wedge C(x)) \\ \varphi_2 &= \neg(\exists x)(C(x) \wedge (\exists y)B(x, y)) \\ \varphi_3 &= (\forall x)((\exists y)B(x, y) \rightarrow (\exists z)B(z, x)) \end{aligned}$$

- (b) Start from the theory $\{\varphi_1, \varphi_2, \neg\varphi_3\}$ (proof by contradiction). Convert to PNF, Skolemize, remove universal quantifiers, convert to CNF, and write in set notation:

$$\begin{aligned} \bullet \varphi_1 &\sim (\forall x)(\exists y)(\forall z)((B(y, x) \vee C(x)) \wedge \neg(B(z, x) \wedge C(x))) \rightsquigarrow (B(f(x), x) \vee C(x)) \wedge \neg(B(z, x) \wedge C(x)) \sim \{\{B(f(x), x), C(x)\}, \{\neg B(z, x), \neg C(x)\}\} \\ \bullet \varphi_2 &\sim (\forall x)(\forall y)\neg(C(x) \wedge B(x, y)) \sim \{\{\neg C(x), \neg B(x, y)\}\} \\ \bullet \neg\varphi_3 &\sim (\exists x)(\exists y)(\forall z)\neg(B(x, y) \rightarrow B(z, x)) \rightsquigarrow \neg(B(c, d) \rightarrow B(z, c)) \sim \{\{B(c, d)\}, \{\neg B(z, c)\}\} \\ S &= \{\{B(f(x), x), C(x)\}, \{\neg B(z, x), \neg C(x)\}, \{\neg C(x), \neg B(x, y)\}, \{B(c, d)\}, \{\neg B(z, c)\}\} \end{aligned}$$

- (c) Resolution tree for $S \vdash_R \square$:



- (d) Yes, in (c) we managed to construct an LI-refutation. Even if we had not, existence of an LI-refutation follows from the completeness theorem of LI-resolution for Horn formulas; our CNF formula S is Horn.

Problem 3. Let $T = \{\neg(\exists x)R(x), (\exists x)(\forall y)(P(x, y) \rightarrow P(y, x)), (\forall x)((\exists y)(P(x, y) \wedge P(y, x)) \rightarrow R(x)), (\forall x)(\exists y)P(x, y)\}$ be a theory in the language $L = \langle P, R \rangle$ without equality.

- Using Skolemization, find an open theory T' equisatisfiable with T .
- Convert T' to an equivalent theory S in CNF. Write S in set representation.
- Find a resolution refutation of S . Indicate the unification used at each step.
- Find an unsatisfiable conjunction of ground instances of clauses from S . *Hint: use the unifications from (c).*

Solution. (a) By Skolemization we get:

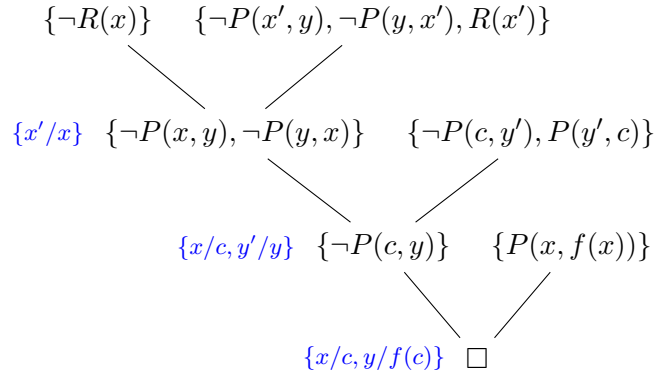
$$T' = \{\neg R(x), P(c, y) \rightarrow P(y, c), P(x, y) \wedge P(y, x) \rightarrow R(x), P(x, f(x))\}$$

(Be careful with the third axiom: $(\exists y)$ in the antecedent of the implication changes to $(\forall y)$.)

- (b) Easily convert to CNF:

$$S = \{\{\neg R(x)\}, \{\neg P(c, y), P(y, c)\}, \{\neg P(x, y), \neg P(y, x), R(x)\}, \{P(x, f(x))\}\}$$

- (c) Resolution tree for $S \vdash_R \square$:



(Note where we need to rename variables to ensure disjoint variable sets in resolved clauses.)

- (d) To obtain the conjunction of ground instances of axioms we can use the constructed resolution refutation. For each leaf of the resolution tree that is labelled by a clause C (up to

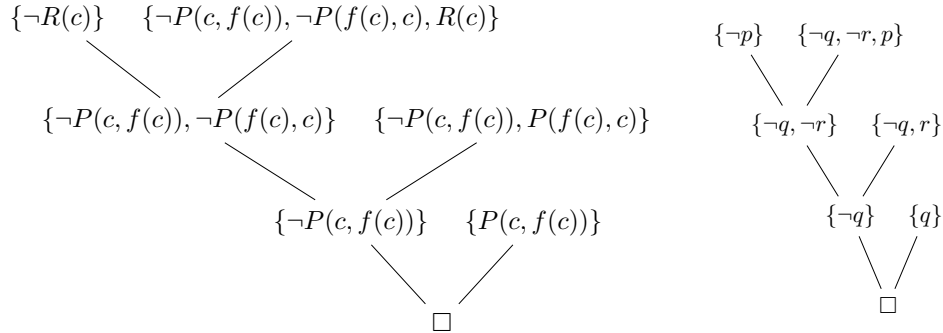
renaming it is a clause from S), we apply to C , in order, all the unifications that occur on the path from that leaf to the root:

- $\neg R(x) \cdot \{x'/x\} \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg R(c)$
- $\neg P(x', y) \vee \neg P(y, x') \vee R(x') \cdot \{x'/x\} \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg P(c, f(c)) \vee \neg P(f(c), c) \vee R(c)$
- $\neg P(c, y') \vee P(y', c) \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg P(c, f(c)) \vee P(f(c), c)$
- $P(x, f(x)) \cdot \{x/c, y/f(c)\} = P(c, f(c))$

If variables remain in some clauses, we substitute any constant term for them. Thus we obtain the unsatisfiable conjunction of ground instances of clauses from S :

$$\neg R(c) \wedge (\neg P(c, f(c)) \vee \neg P(f(c), c) \vee R(c)) \wedge (\neg P(c, f(c)) \vee P(f(c), c)) \wedge P(c, f(c))$$

Its resolution refutation “at the propositional level” has the same structure as the resolution refutation of S :



If we wanted to have ground instances of axioms of the original theory T , we would have to check which axiom was used to obtain each clause and apply the same unifications to it:

$$\neg R(c) \wedge (P(c, f(c)) \rightarrow P(f(c), c)) \wedge (P(c, f(c)) \wedge P(f(c), c) \rightarrow R(c)) \wedge P(c, f(c))$$

EXTRA PRACTICE

Problem 4. Find a resolution refutation:

$$S = \{\{P(a, x, f(y)), P(a, z, f(h(b))), \neg Q(y, z)\}, \{ \neg Q(h(b), w), H(w, a) \}, \{ \neg H(v, a) \}, \\ \{ \neg P(a, w, f(h(b))), H(x, a) \}, \{ P(a, u, f(h(u))), H(u, a), Q(h(b), b) \}\}$$

Problem 5. Let $L = \langle <, a, b, c \rangle$ be without equality, where a, b, c are constant symbols (‘apples/bananas/cherries’) and $x < y$ expresses “fruit y is better than fruit x ”. We know:

- The relation “being better” is a strict partial order (irreflexive, asymmetric, transitive).
- Pears are better than apples.

Prove by resolution: (iii) If cherries are better than bananas, then apples aren’t better than cherries.

- Express statements (i), (ii), (iii) as open formulas in the language L .
- Using these formulas, find a CNF formula S that is unsatisfiable exactly when (i) and (ii) imply (iii). Write S in set representation.
- Prove by resolution that S is unsatisfiable. Illustrate the refutation with a resolution tree, indicate the unification used at each step. *Hint: 4 resolution steps are enough.*
- Find a conjunction of ground instances of axioms of S that is unsatisfiable.
- Is S refutable by LI-resolution?

Problem 6. Let $T = \{\varphi\}$ be in $L = \langle U, c \rangle$ with equality, where U is unary relational and c is a constant symbol, and φ expresses “There are at least 5 elements for which $U(x)$ holds.”

- (a) Find two non-equivalent complete simple extensions of T .
- (b) Is the theory T openly axiomatizable? Give justification.

Problem 7. Let $T = \{U(x) \rightarrow U(f(x)), (\exists x)U(x), \neg(f(x) = x), \varphi\}$ be a theory in the language $L = \langle U, f \rangle$ with equality, where U is a unary relational symbol, f is a unary function symbol, and φ expresses that “there are at most 4 elements.”

- (a) Is the theory T an extension of the theory $S = \{(\exists x)(\exists y)(\neg x = y \wedge U(x) \wedge U(y)), \varphi\}$ in the language $L' = \langle U \rangle$? Is it a conservative extension? Justify.
- (b) Is the theory T openly axiomatizable? Justify.

Problem 8. Let $T = \{(\forall x)(\exists y)S(y) = x, S(x) = S(y) \rightarrow x = y\}$ be a theory in the language $L = \langle S \rangle$ with equality, where S is a unary function symbol.

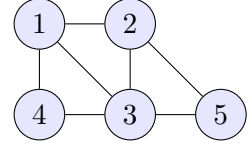
- (a) Find an extension T' of the theory T by definition of a new unary function symbol P such that $T' \models S(S(x)) = y \leftrightarrow P(P(y)) = x$.
- (b) Is the theory T' openly axiomatizable? Give justification.

Problem 9. Let T be an extension of the theory $DeLO^-$ (i.e., dense linear orders with a minimal element and without a maximal element) by a new axiom $c \leq d$ in the language $L = \langle \leq, c, d \rangle$ with equality, where c, d are new constant symbols.

- (a) Are $(\exists x)(x \leq d \wedge x \neq d)$ and $(\forall x)(x \leq d)$ valid / contradictory / independent in T ?
- (b) Write two non-equivalent complete simple extensions of the theory T .

Problem 10. Consider the following graph.

- (a) Find all automorphisms.
- (b) Which subsets of the set of vertices V are definable? Give the defining formulas. (*Hint: Use (a).*)
- (c) Which binary relations on V are definable?



FOR FURTHER THOUGHT

Problem 11. Let $T = \{(\forall x)(\exists y)S(y) = x, S(x) = S(y) \rightarrow x = y\}$ be a theory in the language $L = \langle S \rangle$ with equality, where S is a unary function symbol.

- (a) Let $\mathcal{R} = \langle \mathbb{R}, S \rangle$, where $S(r) = r + 1$ for $r \in \mathbb{R}$. For which $r \in \mathbb{R}$ is the set $\{r\}$ definable in \mathcal{R} from the parameter 0?
- (b) Is the theory T openly axiomatizable? Give justification.
- (c) Is the extension T' of T by the axiom $S(x) = x$ an ω -categorical theory? Is T' complete?
- (d) For which $0 < n \in \mathbb{N}$ does there exist an L -structure \mathcal{B} of size n elementarily equivalent to \mathcal{R} ? Does there exist a countable structure \mathcal{B} elementarily equivalent to \mathcal{R} ?