

Teaching goals: After completing, the student

- understands the notions of propositional logic semantics (truth value, truth function, model, validity, tautology, inconsistency, independence, satisfiability, equivalence), can formally define them and provide examples
- can decide whether a set of logical connectives is universal
- knows the terminology for propositions in CNF and DNF
- can convert a given proposition or a finite theory into CNF and DNF, both using the set of models and using equivalent transformations
- understands the terminology of properties of theories (inconsistent, consistent/satisfiable, complete, consequences, T -equivalence), can define them formally and give examples
- understands the notion of [simple, conservative] extension, can formally define them and provide examples
- in a concrete case, can decide whether the case is a [simple, conservative] extension, and justify it both from the definition and using the semantic criterion

IN-CLASS PROBLEMS

Problem 1. Give an example of a proposition in the language $\mathbb{P} = \{p, q, r\}$ that is (a) valid (b) contradictory, (c) independent, (d) equivalent to $(p \wedge q) \rightarrow \neg r$, (e) has exactly the models $\{(1, 0, 0), (1, 0, 1), (0, 0, 1)\}$.

Problem 2. Are these sets of logical connectives universal? (a) $\{\vee, \rightarrow, \leftrightarrow\}$, (b) $\{\downarrow\}$ where \downarrow is the Peirce arrow (NOR).

Problem 3. Convert the following proposition to CNF and to DNF. Do this (a) semantically (using a truth table), (b) via equivalent transformations:

$$(\neg p \vee q) \rightarrow (\neg q \wedge r)$$

Problem 4. Let $T = \{p \leftrightarrow q, \neg p \rightarrow \neg q, q \vee r\}$ be a theory in the language $\mathbb{P} = \{p, q, r\}$.

- Decide whether the theory T is [inconsistent/satisfiable/complete].
- Give an example of a proposition φ that is [true/false/independent] in T .
- Give an example of an extension T' of the theory T (if it exists, and if possible not equivalent to T) that is [simple / conservative / complete / conservative simple / complete simple / complete conservative]. Also give an example of an extension T' of the theory T that is neither conservative nor simple.
- Using your example extensions, show that the semantic criterion holds (i.e., the statement defining the notion of a [conservative] extension using expansions/reducts of models).

Problem 5. Prove or refute (or state the correct relationship) that for every theory T and propositions φ, ψ in the language \mathbb{P} the following hold:

- $T \models \varphi$ iff $T \not\models \neg\varphi$
- $T \models \varphi$ and $T \models \psi$ iff $T \models \varphi \wedge \psi$
- $T \models \varphi$ or $T \models \psi$ iff $T \models \varphi \vee \psi$
- $T \models \varphi \rightarrow \psi$ and $T \models \psi \rightarrow \chi$ iff $T \models \varphi \rightarrow \chi$

EXTRA PRACTICE

Problem 6. Let $T = \{\neg q \rightarrow (\neg p \vee q), \neg p \rightarrow q, r \rightarrow q\}$ be a theory in the language $\{p, q, r\}$.

- (a) Give an example of the following: a proposition valid in T , inconsistent with T , independent in T , satisfiable in T , and a pair of T -equivalent propositions.
- (b) Which of these are valid, contradictory, independent, satisfiable in T ? T -equivalent?

$$p, \neg q, \neg p \vee q, p \rightarrow r, \neg q \rightarrow r, p \vee q \vee r$$

Problem 7. Are the following sets of logical connectives universal? Justify your answer.

- (a) $\{\vee, \wedge, \rightarrow\}$
- (b) $\{\uparrow\}$ where \uparrow is the Sheffer stroke (NAND)

Problem 8. Find the set of models of the given proposition. Use that it is in DNF or CNF.

- (a) $(\neg p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \vee (p_2 \wedge \neg p_3)$
- (b) $(\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2) \wedge (p_2 \vee \neg p_3)$

Problem 9. Convert to CNF and DNF by both methods: $(\neg p \rightarrow (\neg q \rightarrow r)) \rightarrow p$

Problem 10. Find the (shortest possible) CNF and DNF representations of the Boolean function $\text{maj}: \{0, 1\}^3 \rightarrow \{0, 1\}$ that returns the majority value among the 3 inputs.

Problem 11. The same assignment as in Example 4, but for the theory $T = \{(p \wedge q) \rightarrow r, \neg r \vee (p \wedge q)\}$ in the language $\mathbb{P} = \{p, q, r\}$.

Problem 12. Prove or refute (or state the correct relationship) that for arbitrary theories T, S over \mathbb{P} the following hold:

- (a) $S \subseteq T \Rightarrow \text{Csq}(T) \subseteq \text{Csq}(S)$
- (b) $\text{Csq}(S \cup T) = \text{Csq}(S) \cup \text{Csq}(T)$
- (c) $\text{Csq}(S \cap T) = \text{Csq}(S) \cap \text{Csq}(T)$

FOR FURTHER THOUGHT

Problem 13. Show that \wedge and \vee are not sufficient to define all Boolean operators, i.e., that $\{\wedge, \vee\}$ is not a *universal* set of logical connectives.

Problem 14. Consider the Boolean operator $\text{IFTE}(p, q, r)$ defined as “if p then q else r ”.

- (a) Construct the truth table.
- (b) Show that all basic Boolean operators $(\neg, \rightarrow, \wedge, \vee, \dots)$ can be expressed using IFTE and the constants TRUE and FALSE.

Problem 15. Let \mathbb{P} be a countably infinite set of propositional variables.

- (a) Show that it is no longer true that every $K \subseteq \mathbb{M}_{\mathbb{P}}$ can be axiomatized by a proposition in CNF and also by a proposition in DNF.
- (b) Give an example of a set of models K that cannot be axiomatized by a proposition in CNF nor by a proposition in DNF.

Problem 16. Find CNF and DNF representations of n -ary parity, i.e., the Boolean function $\text{par}: \{0, 1\}^n \rightarrow \{0, 1\}$, which returns the XOR of all input values:

$$\text{par}(x_1, \dots, x_n) = (x_1 + \dots + x_n) \bmod 2$$

Try it for small values of n .

Problem 17. Consider the infinite propositional theory $T = \{p_i \rightarrow p_{i+1} \mid i \in \mathbb{N}\}$ over $\text{var}(T)$.

- (a) Find all models of T .
- (b) Which propositions of the form $p_i \rightarrow p_j$ are consequences of T ?