Sample tutorial test: Propositional logic

Time limit: 45 minutes. Total points: 100.

1. Consider three brothers, each of whom either always tells the truth or always lies.

    (*i*) The eldest says: *"Both my brothers are liars."*

    (*ii*) The middle one says: *"The youngest is a liar."*

    (*iii*) The youngest says: *"The eldest is a liar."*

    Let the propositional variables p_1, p_2, p_3 represent (in order) that *"the eldest / middle / youngest brother is truthful"* and denote $\mathbb{P} = \{p_1, p_2, p_3\}$.

    (a) Write formulas (in the form of equivalences) φ_1, φ_2, φ_3 over $\mathbb{P}$ representing the knowledge derived (in order) from (*i*), (*ii*), (*iii*). *(15 points)*

    (b) Write a theory S in set notation obtained by converting φ_1, φ_2, φ_3, p_3 or their negations into CNF, which is unsatisfiable if and only if it follows from statements (*i*), (*ii*), (*iii*) that *"the youngest is truthful"*. *(15 points)*

    (c) Prove by resolution that S is unsatisfiable. Represent the resolution refutation with a resolution tree. *(20 points)*

2. Let $T = \{(\neg p \wedge q) \to r, \ (q \to r) \leftrightarrow p\}$ be a theory over $\mathbb{P} = \{p, q, r\}$.

    (a) Using the tableaux method, determine all models of theory T. *(20 points)*

    (b) Is T an extension of the theory $S = \{q \to p\}$ over $\{p, q\}$? Is T a conservative extension of S? Justify. *(15 points)*

    (c) Determine how many pairwise inequivalent propositions over $\mathbb{P}$ are there that are independent in both S and T. Justify. *(15 points)*