Sample tutorial test: Propositional logic

Time limit: 45 minutes. Total points: 100.

1. Consider three brothers, each of whom either always tells the truth or always lies.

   (i) The eldest says: *"Both my brothers are liars."*

   (ii) The middle one says: *"The youngest is a liar."*

   (iii) The youngest says: *"The eldest is a liar."*

   Let the propositional variables p_1, p_2, p_3 represent (in order) that *"the eldest / middle / youngest brother is truthful"* and denote $\mathbb{P} = \{p_1, p_2, p_3\}$.

   (a) Write propositions (in the form of equivalences) φ_1, φ_2, φ_3 over $\mathbb{P}$ representing the knowledge derived (in order) from (i), (ii), (iii). *(15 points)*

   (b) Write a theory S in set notation obtained by converting φ_1, φ_2, φ_3, p_3 or their negations into CNF, which is unsatisfiable if and only if it follows from statements (i), (ii), (iii) that *"the youngest is truthful"*. *(15 points)*

   (c) Prove by resolution that S is unsatisfiable. Represent the resolution refutation with a resolution tree. *(20 points)*

2. Let $T = \{(\neg p \wedge q) \to r,\ (q \to r) \leftrightarrow p\}$ be a theory over $\mathbb{P} = \{p, q, r\}$.

   (a) Using the tableaux method, determine all models of theory T. *(20 points)*

   (b) Is T an extension of the theory $S = \{q \to p\}$ over $\{p, q\}$? Is T a conservative extension of S? Justify. *(15 points)*

   (c) Determine how many pairwise inequivalent propositions over $\mathbb{P}$ are there that are independent in both S and T. Justify. *(15 points)*