NAIL062 P&P Logic: Worksheet 2 – Semantics, properties of theories

Teaching goals: After completing, the student

- understands the notions of propositional logic semantics (truth value, truth function, model, validity, tautology, inconsistency, independence, satisfiability, equivalence), can formally define them and provide examples
- can decide whether a set of logical connectives is universal
- knows the terminology for propositions in CNF and DNF
- can convert a given proposition or a finite theory into CNF and DNF, both using the set of models and using equivalent transformations
- understands the terminology of properties of theories (inconsistent, consistent/satisfiable, complete, consequences, T-equivalence), can define them formally and give examples
- understands the notion of [simple, conservative] extension, can formally define them and provide examples
- in a concrete case, can decide whether the case is a [simple, conservative] extension, and justify it both from the definition and using the semantic criterion

In-class problems

Problem 1. Give an example of a proposition in the language $\mathbb{P} = \{p, q, r\}$ that is (a) valid (b) contradictory, (c) independent, (d) equivalent to $(p \wedge q) \rightarrow \neg r$, (e) has exactly the models $\{(1,0,0), (1,0,1), (0,0,1)\}$.

Solution. *For example: (a) $p \vee \neg p$, (b) $p \wedge \neg p$, (c) p, (d) $\neg p \vee \neg q \vee \neg r$, (e) $(p \vee r) \wedge \neg q$*

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Problem 2. Are these sets of logical connectives universal? (a) $\{\vee, \rightarrow, \leftrightarrow\}$, (b) $\{\downarrow\}$ where $\downarrow$ is the Peirce arrow (NOR).

Solution. *(a) No; prove by structural induction that every proposition has $(1, \ldots, 1)$ as a model.*

(b) Yes; we use the fact that $\{\neg, \vee, \wedge\}$ is universal, and express:

- $\neg x \sim x \downarrow x$
- $x \vee y \sim \neg(x \downarrow y) \sim (x \downarrow y) \downarrow (x \downarrow y)$
- $x \wedge y \sim \neg(\neg x \vee \neg y) \sim \neg x \downarrow \neg y \sim (x \downarrow x) \downarrow (y \downarrow y)$

Problem 3. Convert the following proposition to CNF and to DNF. Do this (a) semantically (using a truth table), (b) via equivalent transformations:

$$(\neg p \vee q) \rightarrow (\neg q \wedge r)$$

Solution. *(a) First find the models of the proposition: $\{(0,0,1), (1,0,0), (1,0,1)\}$. Describe each model by one elementary conjunction:*

$$(\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$$

CNF is obtained from the set of non-models, where each clause forbids one non-model:

$$\{(0,0,0), (0,1,0), (0,1,1), (1,1,0), (1,1,1)\}$$

$$(p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

(b) $(\neg p \vee q) \rightarrow (\neg q \wedge r) \sim \neg(\neg p \vee q) \vee (\neg q \wedge r) \sim (p \wedge \neg q) \vee (\neg q \wedge r)$ is a DNF; CNF is obtained by distribution, and then simplified: $(p \vee \neg q) \wedge (p \vee r) \wedge (\neg q \vee \neg q) \wedge (\neg q \vee r) \sim (p \vee r) \wedge \neg q$

Problem 4. Let $T = \{p \leftrightarrow q, \neg p \to \neg q, q \vee r\}$ be a theory in the language $\mathbb{P} = \{p, q, r\}$.

(a) Decide whether the theory T is [inconsistent/satisfiable/complete].
(b) Give an example of a proposition φ that is [true/false/independent] in T.
(c) Give an example of an extension T' of the theory T (if it exists, and if possible not equivalent to T) that is [simple / conservative / complete / conservative simple / complete simple / complete conservative]. Also give an example of an extension T' of the theory T that is neither conservative nor simple.
(d) Using your example extensions, show that the semantic criterion holds (i.e., the statement defining the notion of a [conservative] extension using expansions/reducts of models).

Solution. *We will need to know the models:* $\mathrm{M}(T) = \{(0,0,1),(1,1,0),(1,1,1)\}$

(a) It is not inconsistent, it is satisfiable, it is not complete.
(b) In the theory T, for example, $p \vee r$ is true, $\neg q \wedge \neg r$ is false, $p \vee q$ is independent.
(c) Let us give examples or justify non-existence:
1. *Simple:* $\{p \wedge q\}$
2. *Conservative:* $T_2 = \{(p \wedge q) \vee (\neg p \wedge \neg q), p \vee q \vee r, p \vee s\}$ *in the language* $\mathbb{P}' = \{p, q, r, s\}$
3. *Complete:* $\{\neg p, \neg q, r, \neg s\}$ *in the language* $\mathbb{P}' = \{p, q, r, s\}$
4. *Conservative simple: must be equivalent to T, e.g.,* $\{(p \wedge q) \vee (\neg p \wedge \neg q), p \vee q \vee r\}$
5. *Complete simple:* $\{p, q, \neg r\}$
6. *Complete conservative: does not exist; a non-complete theory cannot have a complete conservative extension (prove this).*
7. *Neither conservative nor simple:* $\{p \wedge q, r \vee s\}$ *in the language* $\mathbb{P}' = \{p, q, r, s\}$.

(d) Construct the corresponding sets of models and verify the condition; we show it only for 2.:

$$\mathrm{M}_{\mathbb{P}'}(T_2) = \{(0,0,1,1),(1,1,0,0),(1,1,0,1),(1,1,1,0),(1,1,1,1)\}$$

We see that restricting the models of T_2 to the language $\mathbb{P}$ yields only models of T, so it is an extension; and every model of T can be expanded to some model of T_2, hence the extension is conservative.

Problem 5. Prove or refute (or state the correct relationship) that for every theory T and propositions φ, ψ in the language $\mathbb{P}$ the following hold:

(a) $T \models \varphi$ iff $T \not\models \neg\varphi$
(b) $T \models \varphi$ and $T \models \psi$ iff $T \models \varphi \wedge \psi$
(c) $T \models \varphi$ or $T \models \psi$ iff $T \models \varphi \vee \psi$
(d) $T \models \varphi \to \psi$ and $T \models \psi \to \chi$ iff $T \models \varphi \to \chi$

Solution. *We give only the correct answers and counterexamples; prove them yourself (from the definitions).*

(a) False, e.g., for $T = p \vee q$, $\varphi = p$. (If T is consistent, the direction $\Rightarrow$ holds.)
(b) True.
(c) False, e.g., for $T = p \vee q$, $\varphi = p$, $\psi = q$. The direction $\Rightarrow$ holds.
(d) False, e.g., for $T = \{p \to r\}$, $\varphi = p$, $\psi = q$, $\chi = r$. The direction $\Rightarrow$ holds.

Extra practice

Problem 6. Let $T = \{\neg q \to (\neg p \vee q),\ \neg p \to q,\ r \to q\}$ be a theory in the language $\{p, q, r\}$.

(a) Give an example of the following: a proposition valid in T, inconsistent with T, independent in T, satisfiable in T, and a pair of T-equivalent propositions.

(b) Which of these are valid, contradictory, independent, satisfiable in T? T-equivalent?

$$p,\ \neg q,\ \neg p \vee q,\ p \to r,\ \neg q \to r,\ p \vee q \vee r$$

Problem 7. Are the following sets of logical connectives universal? Justify your answer.
(a) $\{\vee, \wedge, \to\}$ (b) $\{\uparrow\}$ where $\uparrow$ is the Sheffer stroke (NAND)

Problem 8. Find the set of models of the given proposition. Use that it is in DNF or CNF.
(a) $(\neg p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \vee (p_2 \wedge \neg p_3)$
(b) $(\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2) \wedge (p_2 \vee \neg p_3)$

Problem 9. Convert to CNF and DNF by both methods: $(\neg p \to (\neg q \to r)) \to p$

Problem 10. Find the (shortest possible) CNF and DNF representations of the Boolean function $\mathrm{maj}\colon \{0,1\}^3 \to \{0,1\}$ that returns the majority value among the 3 inputs.

Problem 11. The same assignment as in Example 4, but for the theory $T = \{(p \wedge q) \to r, \neg r \vee (p \wedge q)\}$ in the language $\mathbb{P} = \{p, q, r\}$.

Problem 12. Prove or refute (or state the correct relationship) that for arbitrary theories T, S over $\mathbb{P}$ the following hold:
(a) $S \subseteq T \Rightarrow \mathrm{Csq}(T) \subseteq \mathrm{Csq}(S)$
(b) $\mathrm{Csq}(S \cup T) = \mathrm{Csq}(S) \cup \mathrm{Csq}(T)$
(c) $\mathrm{Csq}(S \cap T) = \mathrm{Csq}(S) \cap \mathrm{Csq}(T)$

For further thought

Problem 13. Show that $\wedge$ and $\vee$ are not sufficient to define all Boolean operators, i.e., that $\{\wedge, \vee\}$ is not a *universal* set of logical connectives.

Problem 14. Consider the Boolean operator $\mathrm{IFTE}(p, q, r)$ defined as "if p then q else r".
(a) Construct the truth table.
(b) Show that all basic Boolean operators $(\neg, \to, \wedge, \vee, \dots)$ can be expressed using IFTE and the constants TRUE and FALSE.

Problem 15. Let $\mathbb{P}$ be a countably infinite set of propositional variables.
(a) Show that it is no longer true that every $K \subseteq \mathrm{M}_{\mathbb{P}}$ can be axiomatized by a proposition in CNF and also by a proposition in DNF.
(b) Give an example of a set of models K that cannot be axiomatized by a proposition in CNF nor by a proposition in DNF.

Problem 16. Find CNF and DNF representations of n-ary parity, i.e., the Boolean function $\mathrm{par}\colon \{0,1\}^n \to \{0,1\}$, which returns the XOR of all input values:

$$\mathrm{par}(x_1, \dots, x_n) = (x_1 + \cdots + x_n) \bmod 2$$

Try it for small values of n.

Problem 17. Consider the infinite propositional theory $T = \{p_i \to p_{i+1} \mid i \in \mathbb{N}\}$ over $\mathrm{var}(T)$.
(a) Find all models of T. (b) Which propositions of the form $p_i \to p_j$ are consequences of T?