NAIL062 P&P LOGIC: WORKSHEET 1 – INTRO TO PROPOSITIONAL LOGIC

Teaching goals: After completing, the student

- understands the notions of propositional syntax (language, [atomic] proposition, tree of a proposition, subproposition, theory), can formally define them and give examples
- understands the notions of model, consequence of a theory, can formally define them and give examples
- can formalize given system (word/computational problem, etc.) in propositional logic
- can find models of a given theory
- can decide whether a given proposition is a consequence of a given theory
- has experience applying (with instructor assistance) the tableau method and resolution method to prove properties of a given system (e.g., to solve a word problem)

IN-CLASS PROBLEMS

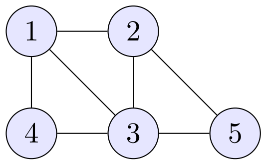
Problem 1. Lost in a labyrinth, we face three doors—red, blue, and green. Exactly one leads out; the other two hide dragons. On the doors we read the following inscriptions:

- Red door: *"The way out is behind this door."*
- Blue door: *"The way out is not behind this door."*
- Green door: *"The way out is not behind the blue door."*

We know that at least one inscription is true and at least one is false. Which way leads out?

(a) Choose a suitable language (a set of propositional variables) $\mathbb{P}$.
(b) Formalize all knowledge as a theory T in the language $\mathbb{P}$. (Note: The axioms are not the inscriptions on the doors; those need not be true.)
(c) Find all models of the theory T.
(d) Formalize the statements "The way out is behind the red/blue/green door" as propositions $\varphi_1, \varphi_2, \varphi_3$ over $\mathbb{P}$. Is any of these propositions a consequence of T?
(e) Try using the tableau method: Construct a tableau from the theory T with the item $F\varphi_i$ at the root. Will all branches be contradictory? (Try to come up with the correct steps for the construction of the tableau, take inspiration from the example in the lecture.)
(f) Try using the resolution method: Convert the axioms of T, as well as $\neg\varphi_i$, into conjunctive normal form (CNF). Try to construct a resolution refutation and draw it in the form of a resolution tree. (Note: Don't forget to negate the proposition φ_i you are proving.)

Problem 2. Consider the *vertex cover* of the following graph:



For a given $k > 0$, we want to determine if this graph has a vertex cover of size at most k.

(a) Choose a suitable language (a set of propositional variables) $\mathbb{P}$.
(b) Formalize in propositional logic the problem of whether the graph in the picture has a vertex cover of size at most k, for a fixed k. Denote the resulting theory by VC_k.
(c) Show that VC_2 has no models, i.e., the graph does not have a 2-element vertex cover.
(d) Would you be able to use the tableau method for this? Think through the procedure.
(e) Would you be able to use the resolution method for this? Think through the procedure.
(f) Find all 3-element vertex covers.

Extra practice

Problem 3. Consider the following statements:

(i) *Whoever is a good runner and is in good shape will finish a marathon.*
(ii) *Whoever is unlucky and not in good shape will not finish a marathon.*
(iii) *Whoever finishes a marathon is a good runner.*
(iv) *If I am lucky, I will finish a marathon.*
(v) *I am in good shape.*

Similarly to the first example, describe the situation using propositional logic:

(a) Formalize these statements as a theory T over a suitable set of propositional variables.
(b) Find all models of the theory T.
(c) Try to use the *tableau method* to search for models.
(d) Write several different consequences of the theory T.
(e) Find a CNF theory equivalent to the theory T.

Problem 4. There are 3 brothers; each of them either always tells the truth or always lies.

(i) The eldest says: *"Both my brothers are liars."*
(ii) The middle one says: *"The youngest is a liar."*
(iii) The youngest says: *"The eldest is a liar."*

Using propositional logic, show that the youngest brother is truthful.

Problem 5. Consider a fixed Sudoku puzzle. Describe how to create a theory (in propositional logic) whose models correspond exactly to valid solutions.

Problem 6. Formalize the following statements in propositional logic:

(a) Rabbits in the area have not been observed and walking along the path is safe, but the blueberries along the path are ripe.
(b) If the blueberries along the path are ripe, then walking along the path is safe only if rabbits have not been observed in the area.
(c) It is not safe to walk along the path, but rabbits in the area have not been observed and the blueberries along the path are ripe.
(d) For walking along the path to be safe, it is necessary but not sufficient that the blueberries along the path are not ripe and rabbits have not been observed in the area.
(e) Walking along the path is not safe whenever the blueberries along the path are ripe and rabbits have been observed in the area.

Problem 7. Formalize the following properties of mathematical objects in propositional logic:

(a) For a fixed (finite) graph G, that it has a perfect matching.
(b) For a fixed partially ordered set, that it is totally (linearly) ordered.
(c) For a fixed partially ordered set, that it has a least element.

Problem 8. For the following propositions draw their trees, and find the set of models:
(a) $(p \to q) \leftrightarrow \neg(p \wedge \neg q)$ (b) $(p \leftrightarrow q) \leftrightarrow ((p \vee q) \to (p \wedge q))$

For further thought

Problem 9. Recall the definition of the *tree of a proposition.*

(a) Prove in detail that every proposition has a uniquely determined tree.
(b) Is it still true if we replace the symbols '(', ')' in the definition of a proposition by '|'?
(c) What would happen if we omitted parentheses entirely?