Teaching goals: After completing, the student

- understands the notion of unification and can perform the Unification Algorithm
- knows the necessary notions from the resolution method in predicate logic (resolution rule, resolvent, resolution proof/refutation, resolution tree), can formally define them, give examples, and explain the differences compared to propositional logic
- can apply the resolution method to solve a given problem (word problem, etc.), performing all necessary steps (conversion to PNF, Skolemization, conversion to CNF)
- can construct a resolution refutation of a given (possibly infinite) CNF formula (if it exists), can draw the resolution tree including the unifications used
- can extract an unsatisfiable conjunction of ground instances of axioms from a res. tree
- knows the notion of LI-resolution, can find an LI-refutation of a given theory (if exists)
- has become familiar with selected notions from model theory

IN-CLASS PROBLEMS

Problem 1. Every barber shaves all those who do not shave themselves. No barber shaves anyone who shaves themselves. Formalize and prove by resolution that: There are no barbers.

Problem 2. The following statements describe a genetic experiment:

- (i) Every sheep was either born from another sheep or cloned (but not both).
- (ii) No cloned sheep gave birth.

We want to show by resolution that: (iii) If a sheep gave birth, it was itself born. Specifically:

- (a) Express as sentences φ_1 , φ_2 , φ_3 in $L = \langle B, C \rangle$ without equality (B is binary, C unary relation symbol, B(x,y) means 'sheep x gave birth to sheep y', C(x) 'sheep x was cloned').
- (b) Using Skolemization of these sentences or their negations, construct a set of clauses S (possibly in an extended language) that is unsatisfiable exactly when $\{\varphi_1, \varphi_2\} \models \varphi_3$.
- (c) Find a resolution refutation of S, draw the resolution tree with unifications used.
- (d) Does S have an LI-refutation?

Problem 3. Let $T = \{ \neg (\exists x) R(x), (\exists x) (\forall y) (P(x,y) \rightarrow P(y,x)), (\forall x) ((\exists y) (P(x,y) \land P(y,x)) \rightarrow R(x)), (\forall x) (\exists y) P(x,y) \}$ be a theory in the language $L = \langle P, R \rangle$ without equality.

- (a) Using Skolemization, find an open theory T' equisatisfiable with T.
- (b) Convert T' to an equivalent theory S in CNF. Write S in set representation.
- (c) Find a resolution refutation of S. Indicate the unification used at each step.
- (d) Find an unsatisfiable conjunction of ground instances of clauses from S. Hint: use the unifications from (c).

EXTRA PRACTICE

Problem 4. Find a resolution refutation:

$$S = \{ \{ P(a,x,f(y)), P(a,z,f(h(b))), \neg Q(y,z) \}, \ \{ \neg Q(h(b),w), H(w,a) \}, \ \{ \neg H(v,a) \}, \ \{ \neg P(a,w,f(h(b))), H(x,a) \}, \ \{ P(a,u,f(h(u))), H(u,a), Q(h(b),b) \} \}$$

Problem 5. Let $L = \langle <, a, b, c \rangle$ be without equality, where a, b, c are constant symbols ('apples/bananas/cherries') and x < y expresses "fruit y is better than fruit x". We know:

- (i) The relation "being better" is a strict partial order (irreflexive, asymmetric, transitive).
- (ii) Pears are better than apples.

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Prove by resolution: (iii) If cherries are better than bananas, then apples aren't better than cherries.

- (a) Express statements (i), (ii), (iii) as open formulas in the language L.
- (b) Using these formulas, find a CNF formula S that is unsatisfiable exactly when (i) and (ii) imply (iii). Write S in set representation.
- (c) Prove by resolution that S is unsatisfiable. Illustrate the refutation with a resolution tree, indicate the unification used at each step. *Hint: 4 resolution steps are enough.*
- (d) Find a conjunction of ground instances of axioms of S that is unsatisfiable.
- (e) Is S refutable by LI-resolution?

Problem 6. Let $T = \{\varphi\}$ be in $L = \langle U, c \rangle$ with equality, where U is unary relational and c is a constant symbol, and φ expresses "There are at least 5 elements for which U(x) holds."

- (a) Find two non-equivalent complete simple extensions of T.
- (b) Is the theory T openly axiomatizable? Give justification.

Problem 7. Let $T = \{U(x) \to U(f(x)), (\exists x)U(x), \neg(f(x) = x), \varphi\}$ be a theory in the language $L = \langle U, f \rangle$ with equality, where U is a unary relational symbol, f is a unary function symbol, and φ expresses that "there are at most 4 elements."

- (a) Is the theory T an extension of the theory $S = \{(\exists x)(\exists y)(\neg x = y \land U(x) \land U(y)), \varphi\}$ in the language $L' = \langle U \rangle$? Is it a conservative extension? Justify.
- (b) Is the theory T openly axiomatizable? Justify.

Problem 8. Let $T = \{(\forall x)(\exists y)S(y) = x, \ S(x) = S(y) \to x = y\}$ be a theory in the language $L = \langle S \rangle$ with equality, where S is a unary function symbol.

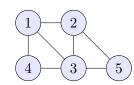
- (a) Find an extension T' of the theory T by definition of a new unary function symbol P such that $T' \models S(S(x)) = y \leftrightarrow P(P(y)) = x$.
- (b) Is the theory T' openly axiomatizable? Give justification.

Problem 9. Let T be an extension of the theory $DeLO^-$ (i.e., dense linear orders with a minimal element and without a maximal element) by a new axiom $c \leq d$ in the language $L = \langle \leq, c, d \rangle$ with equality, where c, d are new constant symbols.

- (a) Are $(\exists x)(x \leq d \land x \neq d)$ and $(\forall x)(x \leq d)$ valid / contradictory / independent in T?
- (b) Write two non-equivalent complete simple extensions of the theory T.

Problem 10. Consider the following graph.

- (a) Find all automorphisms.
- (b) Which subsets of the set of vertices V are definable? Give the defining formulas. (Hint: Use (a).)
- (c) Which binary relations on V are definable?



FOR FURTHER THOUGHT

Problem 11. Let $T = \{(\forall x)(\exists y)S(y) = x, S(x) = S(y) \to x = y\}$ be a theory in the language $L = \langle S \rangle$ with equality, where S is a unary function symbol.

- (a) Let $\mathcal{R} = \langle \mathbb{R}, S \rangle$, where S(r) = r + 1 for $r \in \mathbb{R}$. For which $r \in \mathbb{R}$ is the set $\{r\}$ definable in \mathcal{R} from the parameter 0?
- (b) Is the theory T openly axiomatizable? Give justification.
- (c) Is the extension T' of T by the axiom S(x) = x an ω -categorical theory? Is T' complete?
- (d) For which $0 < n \in \mathbb{N}$ does there exist an *L*-structure \mathcal{B} of size *n* elementarily equivalent to \mathcal{R} ? Does there exist a countable structure \mathcal{B} elementarily equivalent to \mathcal{R} ?