NAIL062 P&P Logic: Worksheet 6 – Basics of predicate logic

Teaching goals: After completing, the student

- understands the notions of structure and signature, can formally define them and provide examples
- understands the notions of the syntax of predicate logic (language, term, atomic formula, formula, theory, free variable, open formula, sentence, instance, variant), can formally define them and provide examples
- understands semantic notions of predicate logic (value of a term, truth value, validity [under assignment], model, being valid/contradictory in a model/theory, independence [in a theory], consequence of a theory), can formally define, give examples
- understands the notion of a complete theory and its relation to elementary equivalence of structures, can define both and apply them to examples
- knows basic examples of theories (graph theories, orders, algebraic theories)
- can describe models of a given theory

In-class problems

Problem 1. Are the following formulas variants of $(\forall x)(x < y \lor (\exists z)(z = y \land z \neq x))$?

(a) $(\forall z)(z < y \lor (\exists z)(z = y \land z \neq z))$
(b) $(\forall y)(y < y \lor (\exists z)(z = y \land z \neq y))$
(c) $(\forall u)(u < y \lor (\exists z)(z = y \land z \neq u))$

Solution. *Let $\psi = (x < y \lor (\exists z)(z = y \land z \neq x))$, so the formula is $(\forall x)\psi$.*

(a) No, z is not substitutable for x into ψ: a new bound occurrence would be created.
(b) No, y has a free occurrence in ψ.
(c) Yes, u is a new variable: in that case one can always form a variant.

Problem 2. Let $\mathcal{A} = (\{a, b, c, d\}; \rhd^A)$ be a structure in the language with a single binary relation symbol $\rhd$, where $\rhd^A = \{(a, c), (b, c), (c, c), (c, d)\}$.

I. Which of the following formulas are true in $\mathcal{A}$?
II. For each of them find a structure $\mathcal{B}$ (if one exists) such that $\mathcal{B} \models \varphi$ iff $\mathcal{A} \not\models \varphi$.

(a) $x \rhd y$
(b) $(\exists x)(\forall y)(y \rhd x)$
(c) $(\exists x)(\forall y)((y \rhd x) \to (x \rhd x))$
(d) $(\forall x)(\forall y)(\exists z)((x \rhd z) \land (z \rhd y))$
(e) $(\forall x)(\exists y)((x \rhd z) \lor (z \rhd y))$

Solution. *We can visualize the structures as directed graphs.*

(a) I. No — intuitively the formula would state that the relation $\rhd^{\mathcal{A}}$ contains all pairs (edges); from the definition $\mathrm{PH}^{\mathcal{A}}(x \rhd y)[e] = 0$ for example for $e(x) = a$, $e(y) = a$.
II. For example $\mathcal{B}_0 = (\{0\}; \rhd^{\mathcal{B}_0})$ with $\rhd^{\mathcal{B}_0} = \{(0, 0)\}$.

(b) I. No — intuitively the graph has no sink; from the definition: $\mathrm{PH}^{\mathcal{A}}(\varphi) = \max_{u \in A} \mathrm{PH}^{\mathcal{A}}((\forall y)(y \rhd x))[e(x/u)] = \max_{u \in A} \min_{v \in A} \mathrm{PH}^{\mathcal{A}}(y \rhd x)[e(x/u, y/v)] = 0$, e.g. for $u = a$ we may take $v = a$.
II. For example $\mathcal{B}_0$ as above.

(c) I. Yes (assign to x for instance the element a), the antecedent is not satisfied for any assignment to y, hence the implication is always true.
II. For example $\mathcal{B}_1 = (\{0, 1\}; \rhd^{\mathcal{B}_1})$ with $\rhd^{\mathcal{B}_1} = \{(0, 1)\}$.

(d) I. No. II: For example $\mathcal{B}_0$.
(e) I. No. II: For example $\mathcal{B}_0$.

Problem 3. Prove (semantically) or find a counterexample: For every structure $\mathcal{A}$, formula φ, and sentence ψ,
(a) $\mathcal{A} \models (\psi \to (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \to \varphi)$
(b) $\mathcal{A} \models (\psi \to (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \to \varphi)$
(c) $\mathcal{A} \models ((\exists x)\varphi \to \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \to \psi)$
(d) $\mathcal{A} \models ((\forall x)\varphi \to \psi) \Leftrightarrow \mathcal{A} \models (\exists x)(\varphi \to \psi)$
Does it hold for every formula ψ with a free variable x? And for every formula ψ in which x is not free?

Solution. *(a) It would be simpler to use the tableau method, but we want to practice a semantic proof. Intuitively, since ψ is a sentence, the variable assignment of x does not play a role in computing the truth value of ψ, so the equivalence holds. Compute from the definitions: $\mathcal{A} \models (\psi \to (\exists x)\varphi)$ holds iff it holds under every variable assignment* $e : \mathrm{Var} \to \mathcal{A}$. *Compute the truth value. Use the fact that $f_{\to}(a,b) = \max(1-a, b)$:*

$$\begin{aligned}
&\mathrm{PH}^{\mathcal{A}}(\psi \to (\exists x)\varphi)[e]\\
=&f_{\to}(\mathrm{PH}^{\mathcal{A}}(\psi)[e], \mathrm{PH}^{\mathcal{A}}((\exists x)\varphi)[e])\\
=&\max(1 - \mathrm{PH}^{\mathcal{A}}(\psi)[e], \mathrm{PH}^{\mathcal{A}}((\exists x)\varphi)[e])\\
=&\max(1 - \mathrm{PH}^{\mathcal{A}}(\psi)[e], \max_{a\in A} \mathrm{PH}^{\mathcal{A}}(\varphi)[e(x/a)])
\end{aligned}$$

Similarly for the formula on the right:

$$\begin{aligned}
&\mathrm{PH}^{\mathcal{A}}((\exists x)(\psi \to \varphi))[e]\\
=&\max_{a\in A} \mathrm{PH}^{\mathcal{A}}(\psi \to \varphi)[e(x/a)]\\
=&\max_{a\in A}(\max(1 - \mathrm{PH}^{\mathcal{A}}(\psi)[e(x/a)], \mathrm{PH}^{\mathcal{A}}(\varphi)[e(x/a)]))
\end{aligned}$$

Because ψ is a sentence, it does not contain a free occurrence of the variable x, hence $\mathrm{PH}^{\mathcal{A}}(\psi)[e(x/a)] = \mathrm{PH}^{\mathcal{A}}(\psi)[e]$. *From this we see that:*

$$\begin{aligned}
=&\max_{a\in A}(\max(1 - \mathrm{PH}^{\mathcal{A}}(\psi)[e], \mathrm{PH}^{\mathcal{A}}(\varphi)[e(x/a)]))\\
=&\max(1 - \mathrm{PH}^{\mathcal{A}}(\psi)[e], \max_{a\in A}(\mathrm{PH}^{\mathcal{A}}(\varphi)[e(x/a)]))
\end{aligned}$$

Both truth values are the same, so the equivalence holds. For this argument it is sufficient that x is not free in ψ.

If x is free in ψ, the equivalence does not hold. For example in language $L = \langle c \rangle$ with equality, where c is a constant symbol:

- *φ is $\neg x = x$,*
- *ψ is $x = c$,*
- $\mathcal{A} = (\{0, 1\}; 0)$ *(i.e. $c^{\mathcal{A}} = 0$).*

We have $\mathcal{A} \not\models (x = c \to (\exists x)\neg x = x)$, because it is not valid under the variable assignment $e(x) = 0$. But $\mathcal{A} \models (\exists x)(x = c \to \neg x = x)$, because x can be assigned the element 1, and then the antecedent is not satisfied.

(b), (c), (d) are solved similarly.

Problem 4. Decide whether T (in the language $L = \langle U, f\rangle$ with equality) is complete. If they exist, give two elementarily non-equivalent models, and two non-equivalent complete simple extensions:

(a) $T = \{U(f(x)),\ \neg x = y,\ x = y \vee x = z \vee y = z\}$
(b) $T = \{U(f(x)),\ \neg(\forall x)(\forall y)x = y,\ x = y \vee x = z \vee y = z\}$
(c) $T = \{U(f(x)),\ \neg x = f(x),\ \neg(\forall x)(\forall y)x = y,\ x = y \vee x = z \vee y = z\}$
(d) $T = \{U(f(x)),\ \neg(\forall x)x = f(x),\ \neg(\forall x)(\forall y)x = y,\ x = y \vee x = z \vee y = z\}$

Solution. *(a) Beware, this theory is inconsistent. Note that $\neg x = y$ is inconsistent: it is not true in any model, because it fails under the variable assignment $e(x) = a, e(y) = a$ for any element $a \in A$. (It is equivalent to its universal closure $(\forall x)(\forall y)\neg x = y$.) An inconsistent theory is not complete by definition, and all its extensions are also inconsistent, so it has no complete simple extension.*

(b) Not complete. Informally, T says that the model has exactly two elements, and the outputs of $f^{\mathcal{A}}$ must lie inside $U^{\mathcal{A}}$. From this we know $U^{\mathcal{A}} \neq \emptyset$. If it is a one-element set, we have a single model (up to isomorphism); if it is two-element, we have in total three pairwise non-isomorphic (and thus elementarily non-equivalent) models (where $f^{\mathcal{A}}$ has no fixed point, has one fixed point, or has two fixed points, i.e. is the identity):

- $\mathcal{A}_1 = (\{0,1\}; U_1^{\mathcal{A}}, f_1^{\mathcal{A}})$ *where* $U_1^{\mathcal{A}} = \{0\}$ *and* $f_1^{\mathcal{A}} = \{(0,0),(1,0)\}$, *i.e.* $f_1^{\mathcal{A}}(0) = 0$, $f_1^{\mathcal{A}}(1) = 0$
- $\mathcal{A}_2 = (\{0,1\}; \{0,1\}, \{(0,1),(1,0)\})$,
- $\mathcal{A}_3 = (\{0,1\}; \{0,1\}, \{(0,0),(1,0)\})$,
- $\mathcal{A}_4 = (\{0,1\}; \{0,1\}, \{(0,0),(1,1)\})$.

(Draw the pictures!) The corresponding complete simple extensions can be written as $\mathrm{Th}(\mathcal{A}_i)$ *for* $i = 1,2,3,4$. *Or:*

- $T_1 = T \cup \{\neg(\forall x)U(x)\}$,
- $T_2 = T \cup \{U(x), \neg f(x) = x\}$,
- $T_3 = T \cup \{U(x), (\exists x)f(x) = x, (\exists x)\neg f(x) = x\}$,
- $T_4 = T \cup \{U(x), f(x) = x\}$.

(c) Similarly, it expresses that the model has exactly two elements and f has no fixed point. It is complete: there is a single model up to isomorphism, namely $\mathcal{A}_2$.

(d) The model has exactly two elements and f has at least one fixed point. It is not complete; its models up to isomorphism are $\mathcal{A}_3$ and $\mathcal{A}_4$.

Extra practice

Problem 5. Determine the free and bound occurrences of variables in the following formulas. Then convert them to variants in which no variable will have both free and bound occurence.

(a) $(\exists x)(\forall y)P(y,z) \vee (y = 0)$
(b) $(\exists x)(P(x) \wedge (\forall x)Q(x)) \vee (x = 0)$
(c) $(\exists x)(x > y) \wedge (\exists y)(y > x)$

Problem 6. Let φ denote the formula $(\forall x)((x = z) \vee (\exists y)(f(x) = y) \vee (\forall z)(y = f(z)))$. Which of the following terms are substitutable into φ?

(a) the term z for the variable x, the term y for the variable x,
(b) the term z for the variable y, the term $g(f(y), w)$ for the variable y,

(c) the term x for the variable z, the term y for the variable z,

Problem 7. Are the following sentences valid / contradictory / independent (in logic)?
(a) $(\exists x)(\forall y)(P(x) \vee \neg P(y))$
(b) $(\forall x)(P(x) \to Q(f(x))) \wedge (\forall x)P(x) \wedge (\exists x)\neg Q(x)$
(c) $(\forall x)(P(x) \vee Q(x)) \to ((\forall x)P(x) \vee (\forall x)Q(x))$
(d) $(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$
(e) $(\exists x)(\forall y)P(x, y) \to (\forall y)(\exists x)P(x, y)$

Problem 8. Decide whether the following hold for every formula φ. Prove (semantically, from the definitions) or provide a counterexample.
(a) $\varphi \models (\forall x)\varphi$
(b) $\models \varphi \to (\forall x)\varphi$
(c) $\varphi \models (\exists x)\varphi$
(d) $\models \varphi \to (\exists x)\varphi$

For further thought

Problem 9. Let $L = \langle +, -, 0\rangle$ be the language of group theory (with equality). The theory of groups T consists of the following axioms:

$$x + (y + z) = (x + y) + z$$
$$0 + x = x = x + 0$$
$$x + (-x) = 0 = (-x) + x$$

Decide whether the following formulas are true / false / independent in T. Justify.
(a) $x + y = y + x$
(b) $x + y = x \;\to\; y = 0$
(c) $x + y = 0 \;\to\; y = -x$
(d) $-(x + y) = (-y) + (-x)$