NAIL062 P&P Logic: Worksheet 7 – Properties of structures and theories

Teaching goals: After completing, the student

- understands the notion of substructure, generated substructure, expansion, reduct and can find them
- understands the notion of expansion and reduct of a structure, can define them formally and give examples
- understands the notions of [simple, conservative] extension, can formulate the definitions and the corresponding semantic criterion (for both expansions and reducts), and apply it to an example
- understands the notion of extension by definition, can define it formally and give examples
- can decide whether a given theory is a extension by definition, construct an extension by a given definition
- understands the notion of definability in a structure, can find definable subsets/relations

In-class problems

Problem 1. Consider $\underline{\mathbb{Z}_4} = \langle \{0,1,2,3\}; +, -, 0\rangle$ where $+$ is binary addition modulo 4 and $-$ is the unary function returning the *inverse* for $+$ with respect to the *neutral* element 0.

(a) Is $\underline{\mathbb{Z}_4}$ a model of the theory of groups (i.e. is it a *group*)?
(b) Determine all substructures $\underline{\mathbb{Z}_4}\langle a\rangle$ generated by some $a \in \mathbb{Z}_4$.
(c) Does $\underline{\mathbb{Z}_4}$ contain any other substructures?
(d) Is every substructure of $\underline{\mathbb{Z}_4}$ a model of the theory of groups?
(e) Is every substructure of $\underline{\mathbb{Z}_4}$ elementarily equivalent to $\underline{\mathbb{Z}_4}$?

Problem 2. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}; +, -, \cdot, 0, 1\rangle$ be the field of rationals with the standard operations.

(a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of the theory of groups?
(b) Can the reduct $\langle \mathbb{Q}, \cdot, 1\rangle$ be extended to a model of the theory of groups?
(c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementarily equivalent to $\underline{\mathbb{Q}}$?
(d) Let $\mathrm{Th}(\underline{\mathbb{Q}})$ denote the set of all sentences true in $\underline{\mathbb{Q}}$. Is $\mathrm{Th}(\underline{\mathbb{Q}})$ a complete theory?

Problem 3. Consider the theory $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ in the language $L = \langle c_1, c_2, c_3\rangle$ with equality.

(a) Is T complete?
(b) How many simple extensions of T are there, up to equivalence? How many are complete? Write down all complete ones and at least three incomplete ones.
(c) Is the theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ in the language $L' = \langle c_1, c_2, c_3, c_4\rangle$ an extension of T? Is T' a simple extension of T? Is T' a conservative extension of T?

Problem 4. Let T' be an extension of $T = \{(\exists y)(x+y = 0), (x+y = 0)\wedge(x+z = 0) \to y = z\}$ in the language $L = \langle +, 0, \leq\rangle$ with equality by definitions of $<$ and unary $-$ with axioms

$$-x = y \quad\leftrightarrow\quad x + y = 0$$
$$x < y \quad\leftrightarrow\quad x \leq y \;\wedge\; \neg(x = y)$$

Find formulas in the language L that are equivalent in T' to the following formulas.

(a) $(-x) + x = 0$ (b) $x + (-y) < x$ (c) $-(x + y) < -x$

Problem 5. Let the language $L = \langle F\rangle$ with equality, where F is a binary function symbol. Find formulas defining the following sets (without parameters):

(a) the interval $(0, \infty)$ in $\mathcal{A} = \langle \mathbb{R}, \cdot \rangle$ where $\cdot$ is multiplication of real numbers
(b) the set $\{(x, 1/x) \mid x \neq 0\}$ in the same structure $\mathcal{A}$
(c) the set of all at-most-singleton subsets of $\mathbb{N}$ in $\mathcal{B} = \langle \mathcal{P}(\mathbb{N}), \cup \rangle$
(d) the set of all prime numbers in $\mathcal{C} = \langle \mathbb{N} \cup \{0\}, \cdot \rangle$

Extra practice

Problem 6. Let $T = \{\neg E(x,x), E(x,y) \to E(y,x), (\exists x)(\exists y)(\exists z)(E(x,y) \wedge E(y,z) \wedge E(x,z) \wedge \neg(x = y \vee y = z \vee x = z)), \varphi\}$ be a theory in the language $L = \langle E \rangle$ with equality, where E is a binary relation symbol and φ expresses that "there are exactly four elements."

(a) Consider the expansion $L' = \langle E, c \rangle$ of the language by a new constant symbol c. Determine the number (up to equivalence) of theories T' in L' that are extensions of T.
(b) Does T have any *conservative* extension in the language L'? Justify your answer.

Problem 7. Let $T = \{x = f(f(x)), \varphi, \neg c_1 = c_2\}$ be a theory in the language $L = \langle f, c_1, c_2 \rangle$ with equality, where f is a unary function symbol, c_1, c_2 are constant symbols, and the axiom φ expresses that "there are exactly three elements."

(a) Determine how many pairwise nonequivalent simple complete extensions the theory T has. Write down two of them. *(3 points)*
(b) Let $T' = \{x = f(f(x)), \varphi, \neg f(c_1) = f(c_2)\}$ be a theory in the same language, with φ as above. Is T' an extension of T? Is T an extension of T'? If so, is it a conservative extension? Provide justification. *(2 points)*

Problem 8. Consider $L = \langle P, R, f, c, d \rangle$ with equality and the following two formulas:

$$\varphi: \quad P(x,y) \leftrightarrow R(x,y) \wedge \neg x = y$$
$$\psi: \quad P(x,y) \to P(x, f(x,y)) \wedge P(f(x,y), y)$$

Consider the following L-theory:

$$\begin{aligned} T = \{&\varphi,\ \psi,\ \neg c = d, \\ &R(x,x), \\ &R(x,y) \wedge R(y,x) \to x = y, \\ &R(x,y) \wedge R(y,z) \to R(x,z), \\ &R(x,y) \vee R(y,x)\} \end{aligned}$$

(a) Find an expansion of the structure $\langle \mathbb{Q}, \leq \rangle$ to the language L that is a model of T.
(b) Is the sentence $(\forall x) R(c,x)$ true/false/independent in T? Justify all three answers.
(c) Find two nonequivalent complete simple extensions of T, or justify why they do not exist.
(d) Let $T' = T \setminus \{\varphi, \psi\}$ be a theory in the language $L' = \langle R, f, c, d \rangle$. Is the theory T a conservative extension of the theory T'? Provide justification.

For further thought

Problem 9. Let $T_n = \{\neg c_i = c_j \mid 1 \leq i < j \leq n\}$ denote the theory of the language $L_n = \langle c_1, \ldots, c_n \rangle$ with equality, where $c_1, \ldots, c_n$ are constant symbols.

(a) For a given finite $k \geq 1$, count k-element models of the theory T_n up to isomorphism.
(b) Determine the number of countable models of the theory T_n up to isomorphism.
(c) For which pairs of values n and m is T_n an extension of T_m? For which pairs is it a conservative extension? Justify your answer.