NAIL062 P&P Logika: Worksheet 3 – Algebra of propositions, SAT

Teaching goals: After completing, the student

- understands the relationship between propositions/theories up to [T-]equivalence and sets of models (the so-called algebra of propositions), can apply in concrete examples
- can encode a given problem as an instance of SAT
- has gained practical experience with using a SAT solver
- understands the algorithm for solving 2-SAT using the implication graph (including finding all models), and can apply it to an example
- understands the algorithm for solving Horn-SAT using unit propagation, and can apply it to an example
- understands the DPLL algorithm and can apply it to an example

In-class problems

Problem 1. Let $|\mathbb{P}| = n$ and let $\varphi \in \mathrm{VF}_{\mathbb{P}}$ be a proposition such that $|M(\varphi)| = k$. Determine (up to equivalence):

(a) the number of propositions ψ such that $\varphi \models \psi$ or $\psi \models \varphi$,
(b) the number of theories over $\mathbb{P}$ in which φ is valid,
(c) the number of complete theories over $\mathbb{P}$ in which φ is valid,
(d) the number of theories T over $\mathbb{P}$ such that $T \cup \{\varphi\}$ is consistent.

Now, consider a contradictory theory $\{\varphi, \psi\}$ where $|M(\psi)| = p$. Compute (up to equivalence):

(e) the number of propositions χ such that $\varphi \vee \psi \models \chi$,
(f) the number of theories in which $\varphi \vee \psi$ is valid.

Problem 2. Build the implication graph of the given 2-CNF formula. Is it satisfiable? If yes, find some solution: (a) the proposition φ below, (b) $\varphi \wedge \neg p_1$, (c) $\varphi \wedge \neg p_1 \wedge (p_1 \vee p_2)$.

$$\varphi = (p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee \neg p_1)$$

Problem 3. Use unit propagation to decide whether the following Horn formula is satisfiable. If yes, find a satisfying assignment.

$$(\neg p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2) \wedge p_1 \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge$$
$$(p_1 \vee \neg p_2 \vee \neg p_4) \wedge (\neg p_2 \vee \neg p_3 \vee \neg p_4) \wedge (p_4 \vee \neg p_5 \vee \neg p_6)$$

Problem 4. Use the DPLL algorithm to decide if the following CNF formula is satisfiable:

$$(\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2) \wedge (p_2 \vee \neg p_3) \wedge (p_1 \vee p_3)$$

Problem 5. Given a directed graph, we want to determine whether it is acyclic and, if so, find a topological ordering. Encode this problem as SAT.

Extra practice

Problem 6. Consider the following propositions φ and ψ over $\mathbb{P} = \{p, q, r, s\}$:

$$\varphi = (\neg p \vee q) \rightarrow (p \wedge r)$$
$$\psi = s \rightarrow q$$

(a) Determine the number (up to equivalence) of propositions χ over $\mathbb{P}$ such that $\varphi \wedge \psi \models \chi$.
(b) Determine the number (up to equivalence) of complete theories T over $\mathbb{P}$ such that $T \models \varphi \wedge \psi$.

(c) Find an axiomatization for each (up to equivalence) complete theory T over $\mathbb{P}$ such that $T \models \varphi \wedge \psi$.

Problem 7. Using the unit propagation algorithm, find all models of:

$$(\neg a \vee \neg b \vee c \vee \neg d) \wedge (\neg b \vee c) \wedge d \wedge (\neg a \vee \neg c \vee e) \wedge$$
$$(\neg c \vee \neg d) \wedge (\neg a \vee \neg d \vee \neg e) \wedge (a \vee \neg b \vee \neg e)$$

Problem 8. Solve using the implication graph as in Example 2, and also using the DPLL algorithm as in Example 4:

(a) $(p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee p_1)$

(b) $(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1 \wedge \neg p_7$

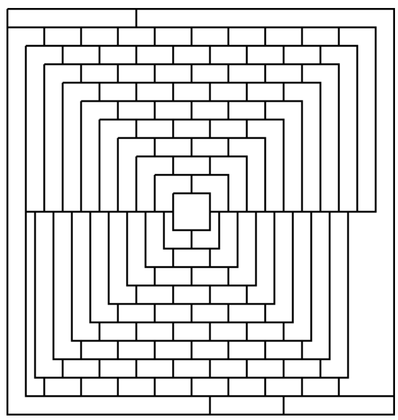
Problem 9. Can the numbers 1 to n be colored with two colors so that there is no monochromatic solution of the equation $a+b=c$ for any $1 \leq a < b < c \leq n$? Construct a propositional CNF formula φ_n that is satisfiable iff such a coloring exists. Try $n = 8$ first.

Try at home: Write a script that generates φ_n in DIMACS CNF format. Use a SAT solver to find the smallest n for which such a coloring does not exist (i.e., every 2-coloring contains a monochromatic triple $a < b < c$ with $a + b = c$).

Problem 10. The four-color theorem implies that the following maps can be colored with four colors so that no two adjacent regions share the same color. Find such a coloring using a SAT solver.

(a) Map of regions of Czechia

(b) A harder instance



For further thought

Problem 11. For a given proposition φ in CNF, find a 3-CNF formula φ' such that φ' is satisfiable if and only if φ is satisfiable. Describe an efficient algorithm for constructing φ' given φ (i.e., a *reduction* from the SAT problem to the 3-SAT problem).

Problem 12. Encode the problem of sorting a given n-tuple of integers into SAT.

Problem 13. Encode into SAT the well-known riddle about a farmer who needs to transport a wolf, a goat, and a cabbage across a river.