NAIL062 P&P Logic: Worksheet 7 – Properties of structures and theories

Teaching goals: After completing, the student

- understands the notion of substructure, generated substructure, can find them
- understands the notion of expansion and reduct of a structure, can define them formally and give examples
- understands the notions of [simple, conservative] extension, can formulate the definitions and the corresponding semantic criterion (for both expansions and reducts), and apply it to an example
- understands the notion of extension by definition, can define it formally and give examples
- can decide whether a given theory is a extension by definition, construct an extension by a given definition
- understands the notion of definability in a structure, can find definable subsets/relations

In-class problems

Problem 1. Consider $\underline{\mathbb{Z}_4} = \langle\{0, 1, 2, 3\}; +, -, 0\rangle$ where $+$ is binary addition modulo 4 and $-$ is the unary function returning the *inverse* for $+$ with respect to the *neutral* element 0.

(a) Is $\underline{\mathbb{Z}_4}$ a model of the theory of groups (i.e. is it a *group*)?
(b) Determine all substructures $\underline{\mathbb{Z}_4}\langle a\rangle$ generated by some $a \in \mathbb{Z}_4$.
(c) Does $\underline{\mathbb{Z}_4}$ contain any other substructures?
(d) Is every substructure of $\underline{\mathbb{Z}_4}$ a model of the theory of groups?
(e) Is every substructure of $\underline{\mathbb{Z}_4}$ elementarily equivalent to $\underline{\mathbb{Z}_4}$?

Solution. *(a) Yes, one can check that $\underline{\mathbb{Z}_4}$ satisfies all axioms of the theory of groups ($+$ is associative, 0 is neutral for $+$, $-x$ is the inverse of x w.r.t. $+$ and 0).*
(b) $\underline{\mathbb{Z}_4}\langle 0\rangle = \underline{\mathbb{Z}_4} \restriction \{0\}$ *(*the trivial group*),* $\underline{\mathbb{Z}_4}\langle 1\rangle = \underline{\mathbb{Z}_4}\langle 3\rangle = \underline{\mathbb{Z}_4}$, $\underline{\mathbb{Z}_4}\langle 2\rangle = \underline{\mathbb{Z}_4} \restriction \{0, 2\}$ *(a two-element group* isomorphic *to* $\underline{\mathbb{Z}_2}$*).*
(c) No, as soon as we have the element 1 or 3, the generated substructure is the whole $\underline{\mathbb{Z}_4}$.
(d) Yes, the theory of groups is universal *(closed under substructures), hence substructures of models (groups) are also models (*subgroups*).*
(e) No, the language of group theory is with equality, and any finite model size can be expressed by a sentence, so finite models of different sizes cannot be elementarily equivalent. However, we do not even need to express model size directly. It suffices to use "group properties" to distinguish them: e.g. the sentence $(\forall x)x = 0$ distinguishes the trivial group $\underline{\mathbb{Z}_4} \restriction \{0\}$ from the two-element group $\underline{\mathbb{Z}_4} \restriction \{0, 2\}$ and from $\underline{\mathbb{Z}_4}$; and e.g. $(\forall x)x + x = 0$ is valid in $\underline{\mathbb{Z}_4} \restriction \{0, 2\}$ but not in $\underline{\mathbb{Z}_4}$.

Problem 2. Let $\underline{\mathbb{Q}} = \langle\mathbb{Q}; +, -, \cdot, 0, 1\rangle$ be the field of rationals with the standard operations.

(a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of the theory of groups?
(b) Can the reduct $\langle\mathbb{Q}, \cdot, 1\rangle$ be extended to a model of the theory of groups?
(c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementarily equivalent to $\underline{\mathbb{Q}}$?
(d) Let $\mathrm{Th}(\underline{\mathbb{Q}})$ denote the set of all sentences true in $\underline{\mathbb{Q}}$. Is $\mathrm{Th}(\underline{\mathbb{Q}})$ a complete theory?

Solution. *(a) Yes, $\underline{\mathbb{Q}} = \langle\mathbb{Q}; +, -, 0\rangle$ (the additive group reduct).*
(b) No, no matter how we interpret the function symbol $-$ as a function $f : \mathbb{Q} \to \mathbb{Q}$, the axiom $x + (-x) = 0$ would require that for every $q \in \mathbb{Q}$ we have $q \cdot f(q) = 1$, which is impossible due to $q = 0$.

(c) *Yes, for example* $\underline{\mathbb{Q}} \restriction \mathbb{Z} = \langle \mathbb{Z}; +, -, \cdot, 0, 1\rangle$ *(the ring of integers): in it not every nonzero element has a multiplicative inverse, so the sentence* $(\forall x)(\neg x = 0 \to (\exists y) x \cdot y = 1)$ *fails (e.g.* 2 *has no inverse in* $\mathbb{Z}$*, while it does in* $\mathbb{Q}$*). (From this it follows that the theory of fields is not openly axiomatizable, otherwise substructures of fields would be fields.)*

(d) *Yes, the so-called* theory of a structure *is always complete: for every sentence* ψ *either* $\mathrm{Th}(\underline{\mathbb{Q}}) \models \psi$ *or* $\mathrm{Th}(\underline{\mathbb{Q}}) \models \neg\psi$*, because* $\underline{\mathbb{Q}} \models \psi$ *or* $\underline{\mathbb{Q}} \models \neg\psi$*.*

Problem 3. Consider the theory $T = \{x = c_1 \lor x = c_2 \lor x = c_3\}$ in the language $L = \langle c_1, c_2, c_3\rangle$ with equality.

(a) Is T complete?

(b) How many simple extensions of T are there, up to equivalence? How many are complete? Write down all complete ones and at least three incomplete ones.

(c) Is the theory $T' = T \cup \{x = c_1 \lor x = c_4\}$ in the language $L' = \langle c_1, c_2, c_3, c_4\rangle$ an extension of T? Is T' a simple extension of T? Is T' a conservative extension of T?

Solution. *The theory says that every element is one of the three constants. But these constants need not be distinct. First find all models up to isomorphism; there are five (draw them):*

- $\mathcal{A}_1 = \langle \{0\}; 0, 0, 0\rangle$ *(one-element model,* $c_1^{\mathcal{A}_1} = c_2^{\mathcal{A}_1} = c_3^{\mathcal{A}_1} = 0$*)*
- $\mathcal{A}_2 = \langle \{0, 1\}; 0, 0, 1\rangle$ *(two-element model,* $c_1^{\mathcal{A}_2} = c_2^{\mathcal{A}_2} \neq c_3^{\mathcal{A}_2}$*)*
- $\mathcal{A}_3 = \langle \{0, 1\}; 0, 1, 1\rangle$ *(two-element model,* $c_1^{\mathcal{A}_3} \neq c_2^{\mathcal{A}_3} = c_3^{\mathcal{A}_3}$*)*
- $\mathcal{A}_4 = \langle \{0, 1\}; 0, 1, 0\rangle$ *(two-element model,* $c_1^{\mathcal{A}_4} = c_3^{\mathcal{A}_4} \neq c_2^{\mathcal{A}_4}$*)*
- $\mathcal{A}_5 = \langle \{0, 1, 2\}; 0, 1, 2\rangle$ *(three-element model, constants are distinct)*

(a) *It is not complete; for example the sentence* $c_1 = c_2$ *is independent in* T*: it is valid in* $\mathcal{A}_1$ *but not in* $\mathcal{A}_3$*. (Equivalently, by the semantic criterion, models* $\mathcal{A}_1$ *and* $\mathcal{A}_3$ *are not elementarily equivalent.)*

(b) *Simple extensions correspond to subsets of* $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$*, there are* $2^5 = 32$ *of them; complete ones correspond to singletons (complete theories of individual models), so there are 5.*

Simple extensions that are not complete:

- T — *models* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$
- $T \cup \{x = y \lor x = z\}$ — *models* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$
- $T \cup \{(\exists x)(\exists y)\neg x = y\}$ — *models* $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$

  (Note: $(\exists x)(\exists y)\neg x = y \sim \neg(\forall x)(\forall y) x = y \not\sim \neg x = y \sim (\forall x)(\forall y)\neg x = y$*.)*

$\vdots$

- $\{x = x \land \neg x = x\}$ — *the inconsistent theory, has no model*

Complete simple extensions:

- $\mathrm{Th}(\mathcal{A}_1) \sim \{x = y\}$
- $\mathrm{Th}(\mathcal{A}_2) \sim \{(\exists x)(\exists y)\neg x = y, x = y \lor x = z, c_1 = c_2, \neg c_2 = c_3\}$
- $\mathrm{Th}(\mathcal{A}_3) \sim \{(\exists x)(\exists y)\neg x = y, x = y \lor x = z, \neg c_1 = c_2, c_2 = c_3\}$
- $\mathrm{Th}(\mathcal{A}_4) \sim \{(\exists x)(\exists y)\neg x = y, x = y \lor x = z, c_1 = c_3, \neg c_1 = c_2\}$
- $\mathrm{Th}(\mathcal{A}_5) \sim \{x = c_1 \lor x = c_2 \lor x = c_3, \neg(c_1 = c_2 \lor c_1 = c_3 \lor c_2 = c_3)\}$

(c) *The theory additionally says that every element is either the interpretation of* c_1 *or* c_4*. Thus models have at most two elements; up to isomorphism they are:*

- $\mathcal{A}'_1 = \langle \{0\}; 0, 0, 0, 0\rangle$
- $\mathcal{A}'_2 = \langle \{0, 1\}; 0, 0, 1, 1\rangle$

- $\mathcal{A}'_3 = \langle\{0,1\}; 0,1,1,1\rangle$
- $\mathcal{A}'_4 = \langle\{0,1\}; 0,1,0,1\rangle$

The theory T' is an extension of T. All consequences of T are valid in T'; semantically: the reducts of models of T' to the original language L are models of T (e.g. the reduct of $\mathcal{A}'_1$ to L is $\mathcal{A}_1$). It is not a simple extension, since we enlarged the language.

It is also not a conservative extension: for example the sentence $(\forall x)(\forall y)(\forall z)(x = y \vee x = z)$ is a sentence in the original language L, it is valid in T' but it was not valid in T. Semantically: the three-element model $\mathcal{A}_5$ of T cannot be expanded to an L'-structure that models T', i.e. the reducts of models of T' to L do not yield all models of T.

Problem 4. Let T' be an extension of $T = \{(\exists y)(x+y=0), (x+y=0) \wedge (x+z=0) \rightarrow y = z\}$ in the language $L = \langle +, 0, \leq\rangle$ with equality by definitions of $<$ and unary $-$ with axioms

$$-x = y \quad\leftrightarrow\quad x + y = 0$$
$$x < y \quad\leftrightarrow\quad x \leq y \;\wedge\; \neg(x = y)$$

Find formulas in the language L that are equivalent in T' to the following formulas.
(a) $(-x) + x = 0$ (b) $x + (-y) < x$ (c) $-(x+y) < -x$

Solution. *Note that the axioms express existence and uniqueness for the definition of the function symbol $-$, so this is a proper extension by definition. We proceed according to the (proof of the) claim from the lecture:*

(a) $(\exists z)(x + z = 0 \wedge z + x = 0)$ *(The subformula $x + z = 0$ says that 'z is $-x$' and the second subformula says that '$(-x) + x = 0$'.)*

(b) First replace the term $-y$ by its definition:

$$(\exists z)(y + z = 0 \wedge x + z < x)$$

Now replace the relation symbol $<$:

$$(\exists z)(y + z = 0 \wedge x + z \leq x \wedge \neg(x + z = x))$$

(c) $(\exists u)(\exists v)((x + y) + u = 0 \wedge x + v = 0 \wedge u \leq v \wedge \neg u = v)$ *(Where 'u is $-(x+y)$' and 'v is $-x$'.)*

Problem 5. Consider the language $L = \langle F\rangle$ with equality, where F is a binary function symbol. Find formulas defining the following sets (without parameters):

(a) the interval $(0, \infty)$ in $\mathcal{A} = \langle\mathbb{R}, \cdot\rangle$ where $\cdot$ is multiplication of real numbers
(b) the set $\{(x, 1/x) \mid x \neq 0\}$ in the same structure $\mathcal{A}$
(c) the set of all singleton subsets of $\mathbb{N}$ in $\mathcal{B} = \langle\mathcal{P}(\mathbb{N}), \cup\rangle$
(d) the set of all prime numbers in $\mathcal{C} = \langle\mathbb{N} \cup \{0\}, \cdot\rangle$

Solution. *(a)* $(\exists y)F(y, y) = x \wedge \neg(\forall y)F(x, y) = x$ *(The number x is a square, and it is not zero.)*

(b) $(\exists z)(F(x, y) = z \wedge (\forall u)F(z, u) = u)$ *(The product equals one.)*

(c) $(\forall y)(\forall z)(F(y, z) = x \rightarrow y = x \vee z = x) \wedge \neg(\forall y)F(x, y) = y$ *(Whenever the set is the union of two sets, it equals one of them. And it is not empty.)*

(d) Same as (c), $(\forall y)(\forall z)(F(y, z) = x \rightarrow y = x \vee z = x) \wedge \neg(\forall y)F(x, y) = x$ *(Whenever the product of two numbers equals a prime, one of them equals the prime, and a prime is not zero.)*

Extra practice

Problem 6. Let $T = \{\neg E(x,x), E(x,y) \to E(y,x), (\exists x)(\exists y)(\exists z)(E(x,y) \wedge E(y,z) \wedge E(x,z) \wedge \neg(x = y \vee y = z \vee x = z)), \varphi\}$ be a theory in the language $L = \langle E \rangle$ with equality, where E is a binary relation symbol and φ expresses that "there are exactly four elements."

(a) Consider the expansion $L' = \langle E, c \rangle$ of the language by a new constant symbol c. Determine the number (up to equivalence) of theories T' in L' that are extensions of T.

(b) Does T have any *conservative* extension in the language L'? Justify your answer.

Problem 7. Let $T = \{x = f(f(x)), \varphi, \neg c_1 = c_2\}$ be a theory in the language $L = \langle f, c_1, c_2 \rangle$ with equality, where f is a unary function symbol, c_1, c_2 are constant symbols, and the axiom φ expresses that "there are exactly three elements."

(a) Determine how many pairwise nonequivalent complete simple extensions the theory T has. Write down two of them.

(b) Let $T' = \{x = f(f(x)), \varphi, \neg f(c_1) = f(c_2)\}$ be a theory in the same language, with φ same as above. Is T' an extension of T? Is T an extension of T'? If so, is it a conservative extension? Provide justification.

Problem 8. Consider $L = \langle P, R, f, c, d \rangle$ with equality and the following two formulas:

$$\begin{aligned} \varphi: &\quad P(x,y) \leftrightarrow R(x,y) \wedge \neg x = y \\ \psi: &\quad P(x,y) \to P(x, f(x,y)) \wedge P(f(x,y), y) \end{aligned}$$

Consider the following L-theory:

$$\begin{aligned} T = \{&\varphi,\ \psi,\ \neg c = d, \\ &R(x,x), \\ &R(x,y) \wedge R(y,x) \to x = y, \\ &R(x,y) \wedge R(y,z) \to R(x,z), \\ &R(x,y) \vee R(y,x)\} \end{aligned}$$

(a) Find an expansion of the structure $\langle \mathbb{Q}, \leq \rangle$ to the language L that is a model of T.

(b) Is the sentence $(\forall x)R(c,x)$ valid/contradictory/independent in T? Justify all 3 answers.

(c) Find two nonequivalent complete simple extensions of T, or justify why they do not exist.

(d) Let $T' = T \setminus \{\varphi, \psi\}$ be a theory in the language $L' = \langle R, f, c, d \rangle$. Is the theory T a conservative extension of the theory T'? Provide justification.

For further thought

Problem 9. Let $T_n = \{\neg c_i = c_j \mid 1 \leq i < j \leq n\}$ be a theory in the language $L_n = \langle c_1, \ldots, c_n \rangle$ with equality, where $c_1, \ldots, c_n$ are constant symbols.

(a) For a given finite $k \geq 1$, count k-element models of the theory T_n up to isomorphism.

(b) Determine the number of countable models of the theory T_n up to isomorphism.

(c) For which pairs of values n and m is T_n an extension of T_m? For which pairs is it a conservative extension? Justify your answer.