

## NAIL062 P&P LOGIC: WORKSHEET 10 – RESOLUTION IN PREDICATE LOGIC

**Teaching goals:** After completing, the student

- understands the notion of unification and can perform the Unification Algorithm
- knows the necessary notions from the resolution method in predicate logic (resolution rule, resolvent, resolution proof/refutation, resolution tree), can formally define them, give examples, and explain the differences compared to propositional logic
- can apply the resolution method to solve a given problem (word problem, etc.), performing all necessary steps (conversion to PNF, Skolemization, conversion to CNF)
- can construct a resolution refutation of a given (possibly infinite) CNF formula (if it exists), can draw the resolution tree including the unifications used
- can extract an unsatisfiable conjunction of ground instances of axioms from a res. tree
- knows the notion of LI-resolution, can find an LI-refutation of a given theory (if exists)
- has become familiar with selected notions from model theory

### IN-CLASS PROBLEMS

**Problem 1.** Every barber shaves all those who do not shave themselves. No barber shaves anyone who shaves themselves. Formalize and prove by resolution that: There are no barbers.

**Solution.** First, we choose a suitable language. In the text we identify a property of objects “*x is a barber*” and a relation between two objects “*x shaves y*”. We use the language  $L = \langle B, S \rangle$  without equality, where  $B$  is a unary relational symbol,  $B(x)$  means “*x is a barber*”,  $S$  is a binary relational symbol,  $S(x, y)$  means “*x shaves y*”.

In this language we formalize the statements from the problem:

- Every barber shaves all those who do not shave themselves:

$$\varphi_1 = (\forall x)(B(x) \rightarrow (\forall y)(\neg S(y, y) \rightarrow S(x, y)))$$

- No barber shaves anyone who shaves themselves:

$$\varphi_2 = \neg(\exists x)(B(x) \wedge (\exists y)(S(x, y) \wedge S(y, y)))$$

- There are no barbers:

$$\psi = \neg(\exists x)B(x)$$

Our goal is to show that in the theory  $T = \{\varphi_1, \varphi_2\}$  the sentence  $\psi$  is valid. We prove this by contradiction, starting with the theory  $T \cup \{\neg\psi\} = \{\varphi_1, \varphi_2, \neg\psi\}$ . Using Skolemization we obtain an equisatisfiable CNF formula  $S$ , then find its resolution refutation, showing that  $S$  and hence  $T \cup \{\neg\psi\}$  is unsatisfiable.

Convert to PNF, Skolemize, remove universal quantifiers, and convert to CNF:

- $\varphi_1 \rightsquigarrow B(x) \rightarrow (\neg S(y, y) \rightarrow S(x, y)) \sim \neg B(x) \vee S(y, y) \vee S(x, y)$
- $\varphi_1 \rightsquigarrow \neg(B(x) \wedge S(x, y) \wedge S(y, y)) \sim \neg B(x) \vee \neg S(x, y) \vee \neg S(y, y)$
- $\neg\psi \rightsquigarrow B(c)$  (where  $c$  is a new constant symbol)

Before starting Skolemization make sure you have sentences. And remember that we must Skolemize the sentence  $\neg\psi$ , not  $\psi$ . (The negation of a Skolem variant is typically not equisatisfiable with the negation of the original formula! By Skolemizing  $\neg(\exists x)B(x)$  we would get  $\neg B(x)$ , whose negation is equivalent to  $B(x)$ , i.e. “everyone is a barber”, whereas the correct procedure yields “(a witness)  $c$  is a barber”.)

In set notation we have:

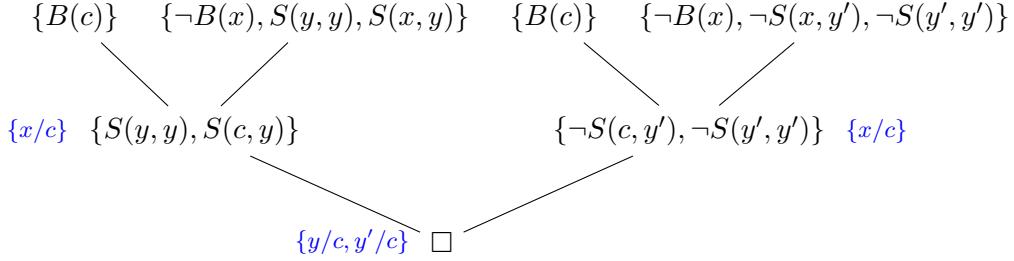
$$S = \{\{\neg B(x), S(y, y), S(x, y)\}, \{\neg B(x), \neg S(x, y), \neg S(y, y)\}, \{B(c)\}\}$$

*Resolution refutation:*

$$\begin{aligned} & \{B(c)\}, \{\neg B(x), S(y, y), S(x, y)\}, \{S(y, y), S(c, y)\}, \{\neg B(x), \neg S(x, y'), \neg S(y', y')\}, \\ & \{\neg S(c, y'), \neg S(y', y')\}, \square \end{aligned}$$

The first two clauses are from  $S$ , the third is their resolvent using the unification  $\{x/c\}$ . The fourth clause is a variant of a clause from  $S$ , variable  $y$  renamed to  $y'$  to satisfy the technical condition of disjoint variable sets in resolved clauses. The fifth clause is the resolvent of the first and fourth clauses using unification  $\{x/c\}$ . The last, empty clause  $\square$  is the resolvent of clauses 3 and 5 with unification  $\{y/c, y'/c\}$ .

Typically, we represent the refutation as a resolution tree, indicating the unifications used:



**Problem 2.** The following statements describe a genetic experiment:

- (i) Every sheep was either born from another sheep or cloned (but not both).
- (ii) No cloned sheep gave birth.

We want to show by resolution that: (iii) If a sheep gave birth, it was itself born. Specifically:

- (a) Express as sentences  $\varphi_1, \varphi_2, \varphi_3$  in  $L = \langle B, C \rangle$  without equality ( $B$  is binary,  $C$  unary relation symbol,  $B(x, y)$  means ‘sheep  $x$  gave birth to sheep  $y$ ’,  $C(x)$  ‘sheep  $x$  was cloned’).
- (b) Using Skolemization of these sentences or their negations, construct a set of clauses  $S$  (possibly in an extended language) that is unsatisfiable exactly when  $\{\varphi_1, \varphi_2\} \models \varphi_3$ .
- (c) Find a resolution refutation of  $S$ , draw the resolution tree with unifications used.
- (d) Does  $S$  have an LI-refutation?

**Solution.** Note that all objects are sheep, so no predicate for ‘being a sheep’ is needed. The procedure is similar to the previous example:

- (a) There are several ways to formulate the formulas; if we try to follow the text as closely as possible we get, for example:

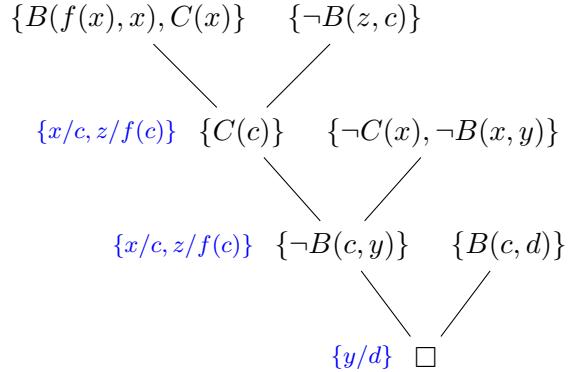
$$\begin{aligned} \varphi_1 &= (\forall x)((\exists y)B(y, x) \vee C(x)) \wedge \neg((\exists z)B(z, x) \wedge C(x)) \\ \varphi_2 &= \neg(\exists x)(C(x) \wedge (\exists y)B(x, y)) \\ \varphi_3 &= (\forall x)((\exists y)B(x, y) \rightarrow (\exists z)B(z, x)) \end{aligned}$$

- (b) Start from the theory  $\{\varphi_1, \varphi_2, \neg\varphi_3\}$  (proof by contradiction). Convert to PNF, Skolemize, remove universal quantifiers, convert to CNF, and write in set notation:

- $\varphi_1 \sim (\forall x)(\exists y)(\forall z)((B(y, x) \vee C(x)) \wedge \neg(B(z, x) \wedge C(x))) \rightsquigarrow (B(f(x), x) \vee C(x)) \wedge \neg(B(z, x) \wedge C(x)) \sim \{\{B(f(x), x), C(x)\}, \{\neg B(z, x), \neg C(x)\}\}$
- $\varphi_2 \sim (\forall x)(\forall y)\neg(C(x) \wedge B(x, y)) \sim \{\{\neg C(x), \neg B(x, y)\}\}$
- $\neg\varphi_3 \sim (\exists x)(\exists y)(\forall z)\neg(B(x, y) \rightarrow B(z, x)) \rightsquigarrow \neg(B(c, d) \rightarrow B(z, c)) \sim \{\{B(c, d)\}, \{\neg B(z, c)\}\}$

$$S = \{\{B(f(x), x), C(x)\}, \{\neg B(z, x), \neg C(x)\}, \{\neg C(x), \neg B(x, y)\}, \{B(c, d)\}, \{\neg B(z, c)\}\}$$

- (c) Resolution tree for  $S \vdash_R \square$ :



- (d) Yes, in (c) we managed to construct an LI-refutation. Even if we had not, the existence of an LI-refutation can be seen from the completeness theorem of LI-resolution for Horn formulas; while our CNF formula  $S$  is not Horn, it would become Horn if we replaced  $C$  by its negation (i.e., formalized “sheep is uncloned”).

**Problem 3.** Let  $T = \{\neg(\exists x)R(x), (\exists x)(\forall y)(P(x, y) \rightarrow P(y, x)), (\forall x)((\exists y)(P(x, y) \wedge P(y, x)) \rightarrow R(x)), (\forall x)(\exists y)P(x, y)\}$  be a theory in the language  $L = \langle P, R \rangle$  without equality.

- (a) Using Skolemization, find an open theory  $T'$  equisatisfiable with  $T$ .
- (b) Convert  $T'$  to an equivalent theory  $S$  in CNF. Write  $S$  in set representation.
- (c) Find a resolution refutation of  $S$ . Indicate the unification used at each step.
- (d) Find an unsatisfiable conjunction of ground instances of clauses from  $S$ . Hint: use the unifications from (c).

**Solution.** (a) By Skolemization we get:

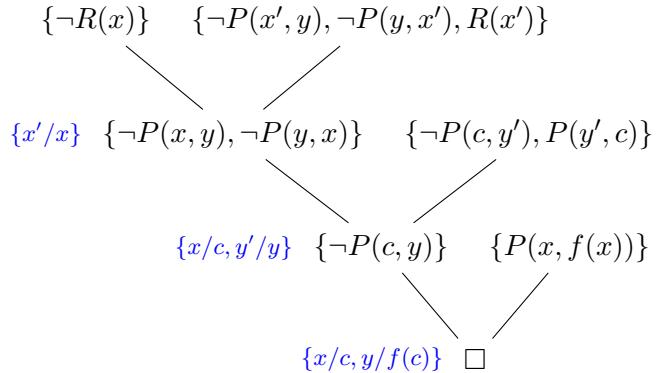
$$T' = \{\neg R(x), P(c, y) \rightarrow P(y, c), P(x, y) \wedge P(y, x) \rightarrow R(x), P(x, f(x))\}$$

(Be careful with the third axiom:  $(\exists y)$  in the antecedent of the implication changes to  $(\forall y)$ .)

- (b) Easily convert to CNF:

$$S = \{\{\neg R(x)\}, \{\neg P(c, y), P(y, c)\}, \{\neg P(x, y), \neg P(y, x), R(x)\}, \{P(x, f(x))\}\}$$

- (c) Resolution tree for  $S \vdash_R \square$ :



(Note where we need to rename variables to ensure disjoint variable sets in resolved clauses.)

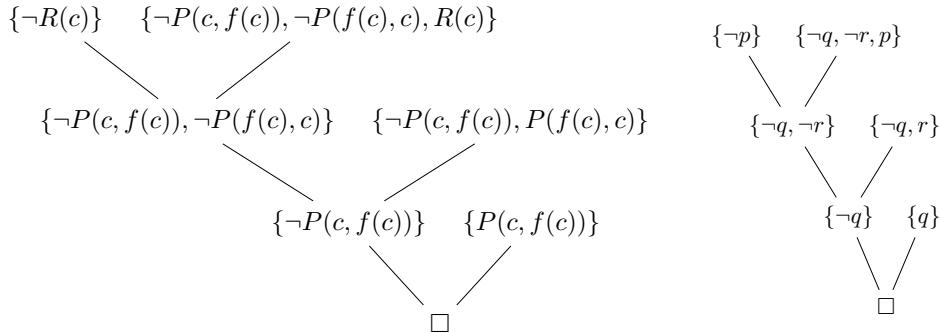
- (d) To obtain the conjunction of ground instances of axioms we can use the constructed resolution refutation. For each leaf of the resolution tree that is labelled by a clause  $C$  (up to renaming it is a clause from  $S$ ), we apply to  $C$ , in order, all the unifications that occur on the path from that leaf to the root:

- $\neg R(x) \cdot \{x'/x\} \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg R(c)$
- $\neg P(x', y) \vee \neg P(y, x') \vee R(x') \cdot \{x'/x\} \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg P(c, f(c)) \vee \neg P(f(c), c) \vee R(c)$
- $\neg P(c, y') \vee P(y', c) \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg P(c, f(c)) \vee P(f(c), c)$
- $P(x, f(x)) \cdot \{x/c, y/f(c)\} = P(c, f(c))$

If variables remain in some clauses, we substitute any constant term for them. Thus we obtain the unsatisfiable conjunction of ground instances of clauses from  $S$ :

$$\neg R(c) \wedge (\neg P(c, f(c)) \vee \neg P(f(c), c) \vee R(c)) \wedge (\neg P(c, f(c)) \vee P(f(c), c)) \wedge P(c, f(c))$$

Its resolution refutation “at the propositional level” has the same structure as the resolution refutation of  $S$ :



If we wanted to have ground instances of axioms of the original theory  $T$ , we would have to check which axiom was used to obtain each clause and apply the same unifications to it:

$$\neg R(c) \wedge (P(c, f(c)) \rightarrow P(f(c), c)) \wedge (P(c, f(c)) \wedge P(f(c), c) \rightarrow R(c)) \wedge P(c, f(c))$$

#### EXTRA PRACTICE

**Problem 4.** Find a resolution refutation:

$$S = \{\{P(a, x, f(y)), P(a, z, f(h(b))), \neg Q(y, z)\}, \{\neg Q(h(b), w), H(w, a)\}, \{\neg H(v, a)\}, \{\neg P(a, w, f(h(b))), H(x, a)\}, \{P(a, u, f(h(u))), H(u, a), Q(h(b), b)\}\}$$

**Problem 5.** Let  $L = \langle <, a, b, c \rangle$  be without equality, where  $a, b, c$  are constant symbols ('apples/bananas/cherries') and  $x < y$  expresses "fruit  $y$  is better than fruit  $x$ ". We know:

- (i) The relation “being better” is a strict partial order (irreflexive, asymmetric, transitive).
- (ii) Pears are better than apples.

Prove by resolution: (iii) If cherries are better than bananas, then apples aren't better than cherries.

- (a) Express statements (i), (ii), (iii) as open formulas in the language  $L$ .
- (b) Using these formulas, find a CNF formula  $S$  that is unsatisfiable exactly when (i) and (ii) imply (iii). Write  $S$  in set representation.
- (c) Prove by resolution that  $S$  is unsatisfiable. Illustrate the refutation with a resolution tree, indicate the unification used at each step. Hint: 4 resolution steps are enough.

- (d) Find a conjunction of ground instances of axioms of  $S$  that is unsatisfiable.  
 (e) Is  $S$  refutable by LI-resolution?

**Problem 6.** Let  $T = \{\varphi\}$  be in  $L = \langle U, c \rangle$  with equality, where  $U$  is unary relational and  $c$  is a constant symbol, and  $\varphi$  expresses “*There are at least 5 elements for which  $U(x)$  holds.*”

- (a) Find two non-equivalent complete simple extensions of  $T$ .  
 (b) Is the theory  $T$  openly axiomatizable? Give justification.

**Problem 7.** Let  $T = \{U(x) \rightarrow U(f(x)), (\exists x)U(x), \neg(f(x) = x), \varphi\}$  be a theory in the language  $L = \langle U, f \rangle$  with equality, where  $U$  is a unary relational symbol,  $f$  is a unary function symbol, and  $\varphi$  expresses that “there are at most 4 elements.”

- (a) Is the theory  $T$  an extension of the theory  $S = \{(\exists x)(\exists y)(\neg x = y \wedge U(x) \wedge U(y)), \varphi\}$  in the language  $L' = \langle U \rangle$ ? Is it a conservative extension? Justify.  
 (b) Is the theory  $T$  openly axiomatizable? Justify.

**Problem 8.** Let  $T = \{(\forall x)(\exists y)S(y) = x, S(x) = S(y) \rightarrow x = y\}$  be a theory in the language  $L = \langle S \rangle$  with equality, where  $S$  is a unary function symbol.

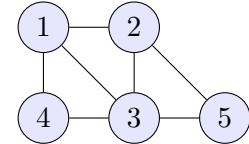
- (a) Find an extension  $T'$  of the theory  $T$  by definition of a new unary function symbol  $P$  such that  $T' \models S(S(x)) = y \leftrightarrow P(P(y)) = x$ .  
 (b) Is the theory  $T'$  openly axiomatizable? Give justification.

**Problem 9.** Let  $T$  be an extension of the theory  $DeLO^-$  (i.e., dense linear orders with a minimal element and without a maximal element) by a new axiom  $c \leq d$  in the language  $L = \langle \leq, c, d \rangle$  with equality, where  $c, d$  are new constant symbols.

- (a) Are  $(\exists x)(x \leq d \wedge x \neq d)$  and  $(\forall x)(x \leq d)$  valid / contradictory / independent in  $T$ ?  
 (b) Write two non-equivalent complete simple extensions of the theory  $T$ .

**Problem 10.** Consider the following graph.

- (a) Find all automorphisms.  
 (b) Which subsets of the set of vertices  $V$  are definable? Give the defining formulas. (*Hint: Use (a).*)  
 (c) Which binary relations on  $V$  are definable?



#### FOR FURTHER THOUGHT

**Problem 11.** Let  $T = \{(\forall x)(\exists y)S(y) = x, S(x) = S(y) \rightarrow x = y\}$  be a theory in the language  $L = \langle S \rangle$  with equality, where  $S$  is a unary function symbol.

- (a) Let  $\mathcal{R} = \langle \mathbb{R}, S \rangle$ , where  $S(r) = r + 1$  for  $r \in \mathbb{R}$ . For which  $r \in \mathbb{R}$  is the set  $\{r\}$  definable in  $\mathcal{R}$  from the parameter 0?  
 (b) Is the theory  $T$  openly axiomatizable? Give justification.  
 (c) Is the extension  $T'$  of  $T$  by the axiom  $S(x) = x$  an  $\omega$ -categorical theory? Is  $T'$  complete?  
 (d) For which  $0 < n \in \mathbb{N}$  does there exist an  $L$ -structure  $\mathcal{B}$  of size  $n$  elementarily equivalent to  $\mathcal{R}$ ? Does there exist a countable structure  $\mathcal{B}$  elementarily equivalent to  $\mathcal{R}$ ?