NAIL062 P&P Logika: Worksheet 3 – Algebra of propositions, SAT

Teaching goals: After completing, the student

- understands the relationship between propositions/theories up to [T-]equivalence and sets of models (the so-called algebra of propositions), can apply in concrete examples
- can encode a given problem as an instance of SAT
- has gained practical experience with using a SAT solver
- understands the algorithm for solving 2-SAT using the implication graph (including finding all models), and can apply it to an example
- understands the algorithm for solving Horn-SAT using unit propagation, and can apply it to an example
- understands the DPLL algorithm and can apply it to an example

In-class problems

Problem 1. Let $|\mathbb{P}| = n$ and let $\varphi \in \mathrm{VF}_{\mathbb{P}}$ be a proposition such that $|M(\varphi)| = k$. Determine (up to equivalence):

(a) the number of propositions ψ such that $\varphi \models \psi$ or $\psi \models \varphi$,
(b) the number of theories over $\mathbb{P}$ in which φ holds,
(c) the number of complete theories over $\mathbb{P}$ in which φ holds,
(d) the number of theories T over $\mathbb{P}$ such that $T \cup \{\varphi\}$ is consistent.

Now, consider a contradictory theory $\{\varphi, \psi\}$ where $|M(\psi)| = p$. Compute (up to equivalence):

(e) the number of propositions χ such that $\varphi \vee \psi \models \chi$,
(f) the number of theories in which $\varphi \vee \psi$ holds.

Solution. *(a) The condition can be expressed in terms of model sets:* $\mathrm{M}(\varphi) \subseteq \mathrm{M}(\psi)$ *or* $\mathrm{M}(\psi) \subseteq \mathrm{M}(\varphi)$. *There are* 2^n *total models, and* $|\mathrm{M}(\varphi)| = k$. *We want to count possible sets* $\mathrm{M}(\psi)$. *The condition* $\mathrm{M}(\varphi) \subseteq \mathrm{M}(\psi)$ *holds for* 2^{2^n-k} *sets (i.e. the number of supersets of the given* k*-element set inside a* 2^n*-element universe), and the condition* $\mathrm{M}(\psi) \subseteq \mathrm{M}(\varphi)$ *holds for* 2^k *sets. We must subtract one to avoid double-counting the case* $\mathrm{M}(\psi) = \mathrm{M}(\varphi)$. *Altogether there are* $2^{2^n-k} + 2^k - 1$ *possible model sets, hence that many propositions* ψ *up to equivalence.*

(b) $T \models \varphi$ *iff* $\mathrm{M}(T) \subseteq \mathrm{M}(\varphi)$*; the number of such model sets* $\mathrm{M}(T)$ *is* 2^k.

(c) Additionally we require $|\mathrm{M}(T)| = 1$*; the number of* 1*-element subsets of the* k*-element set is* k.

(d) In terms of models the condition says $\mathrm{M}(T \cup \{\varphi\}) \neq \emptyset$. *Since* $\mathrm{M}(T \cup \{\varphi\}) = \mathrm{M}(T) \cap \mathrm{M}(\varphi)$, *we count how many possible sets* $\mathrm{M}(T)$ *have a nonempty intersection with the* k*-element set* $\mathrm{M}(\varphi)$. *One way to express this is* $(2^k - 1) \cdot 2^{2^n-k}$, *where* $2^k - 1$ *counts the nonempty possible intersections* $\mathrm{M}(T) \cap \mathrm{M}(\varphi)$ *and* 2^{2^n-k} *counts arbitrary choices for membership of models outside* $\mathrm{M}(\varphi)$.

(e) Since $\{\varphi, \psi\}$ *is contradictory we have* $\emptyset = \mathrm{M}(\varphi, \psi) = \mathrm{M}(\varphi) \cap \mathrm{M}(\psi)$. *We count sets* $\mathrm{M}(\chi)$ *with* $\mathrm{M}(\varphi \vee \psi) \subseteq \mathrm{M}(\chi)$. *By the Lindenbaum–Tarski algebra* $\mathrm{M}(\varphi \vee \psi) = \mathrm{M}(\varphi) \cup \mathrm{M}(\psi)$. *Disjointness gives* $|\mathrm{M}(\varphi) \cup \mathrm{M}(\psi)| = k + p$, *so the number of choices for* $\mathrm{M}(\chi)$ *is* $2^{2^n-(k+p)}$.

(f) $\mathrm{M}(T)$ *must be a subset of the* $(k+p)$*-element set* $\mathrm{M}(\varphi \vee \psi)$, *hence there are* 2^{k+p} *possibilities.*

Problem 2. Build the implication graph of the given 2-CNF formula. Is it satisfiable? If yes, find some solution: (a) the proposition φ below, (b) $\varphi \wedge \neg p_1$, (c) $\varphi \wedge \neg p_1 \wedge (p_1 \vee p_2)$.

$$\varphi = (p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee \neg p_1)$$

Solution. *(a) Construct the implication graph. One finds two strongly connected components: $C = \{p_1, p_2, \neg p_3, p_4, \neg p_5\}$ and $\overline{C} = \{\neg p_1, \neg p_2, p_3, \neg p_4, p_5\}$, and there are no edges between them. After contracting SCCs we obtain a two-vertex DAG $\mathcal{G}^*$ with no edges; it has two topological orders $(C, \overline{C})$ and $(\overline{C}, C)$, which correspond to the models $(0,0,1,0,1)$ and $(1,1,0,1,0)$ respectively.*

(b) The SCCs are the same, but adding $\neg p_1$ forces an edge $C \to \overline{C}$ in $\mathcal{G}^*$, so the only topological order is $(C, \overline{C})$, yielding the model $(0,0,1,0,1)$.

(c) With the extra clause the implication graph becomes strongly connected; its single SCC contains complementary literals, so the formula is unsatisfiable.

Problem 3. Use unit propagation to decide whether the following Horn formula is satisfiable. If yes, find a satisfying assignment.

$$(\neg p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2) \wedge p_1 \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge$$
$$(p_1 \vee \neg p_2 \vee \neg p_4) \wedge (\neg p_2 \vee \neg p_3 \vee \neg p_4) \wedge (p_4 \vee \neg p_5 \vee \neg p_6)$$

Solution. *Perform unit propagation step by step starting from the unit p_1. Propagating units over the clauses yields assignments for $p_1, p_2, p_3, \neg p_4$; the remaining clause becomes $\neg p_5 \vee \neg p_6$. Assigning either $p_5 = 0$ or $p_6 = 0$ makes the whole formula true. For instance the models include $\{(1,1,1,0,0,1), (1,1,1,0,1,0), (1,1,1,0,1,1)\}$.*

Problem 4. Use the DPLL algorithm to decide if the following CNF formula is satisfiable:

$$(\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2) \wedge (p_2 \vee \neg p_3) \wedge (p_1 \vee p_3)$$

Solution. *The formula has no unit clauses and no pure literals, so we must branch, say on p_1:*

- *For $\varphi \wedge p_1$: unit propagation yields $\neg p_2 \wedge p_2 \wedge (p_2 \vee \neg p_3)$. Propagating $\neg p_2$ produces $\square \wedge \neg p_3$, hence a contradiction (empty clause), so this branch is unsatisfiable.*
- *For $\varphi \wedge \neg p_1$: unit propagation yields $\neg p_2 \wedge (p_2 \vee \neg p_3) \wedge p_3$. Propagating $\neg p_2$ gives $\neg p_3 \wedge p_3$, again a contradiction, so this branch is also unsatisfiable.*

Both branches lead to contradiction, therefore the formula is unsatisfiable.

Problem 5. Given a directed graph, we want to determine whether it is acyclic and, if so, find a topological ordering. Encode this problem as SAT.

Solution. *Sketch of a solution. Use the language $\mathbb{P} = \{p_{uv} \mid u, v \in V\}$ where p_{uv} means "vertex u appears strictly before v in the ordering". Enforce that this is a strict (irreflexive, antisymmetric, transitive) order by axioms:*

- *$\neg p_{vv}$ for all $v \in V$,*
- *$p_{uv} \to \neg p_{vu}$ for all $u, v \in V$,*
- *$(p_{uv} \wedge p_{vw}) \to p_{uw}$ for all $u, v, w \in V$.*

Ensure every graph edge goes forward in the order:

- *p_{uv} for each edge $(u, v) \in E$.*

Convert the above axioms to CNF. In set notation one convenient CNF representation is

$$S = \{\{\neg p_{vv}\}, \{\neg p_{uv}, \neg p_{vu}\}, \{\neg p_{uv}, \neg p_{vw}, \neg p_{uw}\} \mid u, v, w \in V\} \cup \{\{p_{uv}\} \mid (u, v) \in E\}.$$

Extra practice

Problem 6. Consider the following propositions φ and ψ over $\mathbb{P} = \{p, q, r, s\}$:

$$\varphi = (\neg p \vee q) \rightarrow (p \wedge r)$$
$$\psi = s \rightarrow q$$

(a) Determine the number (up to equivalence) of propositions χ over $\mathbb{P}$ such that $\varphi \wedge \psi \models \chi$.
(b) Determine the number (up to equivalence) of complete theories T over $\mathbb{P}$ such that $T \models \varphi \wedge \psi$.
(c) Find an axiomatization for each (up to equivalence) complete theory T over $\mathbb{P}$ such that $T \models \varphi \wedge \psi$.

Problem 7. Using unit propagation, find all models of:

$$(\neg a \vee \neg b \vee c \vee \neg d) \wedge (\neg b \vee c) \wedge d \wedge (\neg a \vee \neg c \vee e) \wedge$$
$$(\neg c \vee \neg d) \wedge (\neg a \vee \neg d \vee \neg e) \wedge (a \vee \neg b \vee \neg e)$$

Problem 8. Solve using the implication graph as in Example **??**, and also using the DPLL algorithm as in Example **??**:

(a) $(p_1 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (\neg p_3 \vee p_1) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_4 \vee p_5) \wedge (\neg p_5 \vee p_1)$
(b) $(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1 \wedge \neg p_7$

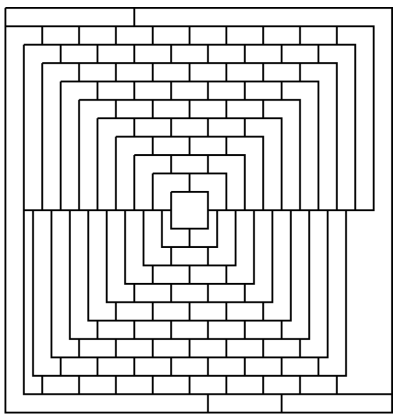
Problem 9. Can the numbers 1 to n be colored with two colors so that there is no monochromatic solution of the equation $a + b = c$ for any $1 \leq a < b < c \leq n$? Construct a propositional CNF formula φ_n that is satisfiable iff such a coloring exists. Try $n = 8$ first.

Try at home: Write a script that generates φ_n in DIMACS CNF format. Use a SAT solver to find the smallest n for which such a coloring does not exist (i.e., every 2-coloring contains a monochromatic triple $a < b < c$ with $a + b = c$).

Problem 10. The four-color theorem states that the following maps can be colored with four colors so that no two adjacent regions share the same color. Find such a coloring using a SAT solver.

(a) Map of regions (krajů) of the Czech Republic

(b) A harder instance



For further thought

Problem 11. For a given proposition φ in CNF, find a 3-CNF formula φ' such that φ' is satisfiable if and only if φ is satisfiable. Describe an efficient algorithm for constructing φ' given φ (i.e., a *reduction* from the SAT problem to the 3-SAT problem).

Problem 12. Encode the problem of sorting a given n-tuple of integers into SAT.

Problem 13. Encode into SAT the well-known riddle about a farmer who needs to transport a wolf, a goat, and a cabbage across a river.