Sample tutorial test: Predicate logic

Time limit: 45 minutes. Total points: 100.

1. We know that:

   (i) Aristotle is Greek, Caesar is Roman, and Dido is Carthaginian.

   (ii) No Greek is a Roman.

   (iii) No Carthaginian is Greek.

   (iv) Only Carthaginians were born in Carthage.

   We want to prove using resolution that:

   (v) There exists someone who was not born in Carthage and is not a Roman.

   Specifically:

   (a) Express the statements as sentences $\varphi_1, \ldots, \varphi_5$ in the language $L = \langle G, R, C, B, a, c, d\rangle$ without equality, where G, R, C, B are unary relation symbols, $G(x), R(x), C(x)$ and $B(x)$ mean (in order) "x is Greek / Roman / Carthaginian" and "x was born in Carthage", and a, c, d are constant symbols denoting Aristotle, Caesar, and Dido. *(15 points)*

   (b) Using Skolemization, find an open theory T (possibly in a larger language) which is unsatisfiable if and only if $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \models \varphi_5$. Convert T into CNF and write it in set notation. *(10 points)*

   (c) Prove by resolution that T is unsatisfiable. Represent the resolution refutation with a resolution tree. For each step, indicate the unification used. *(20 points)*

2. Let $T = \{(\exists x)(P(x) \rightarrow Q(x)),\ (\exists x)(\neg R(x) \rightarrow \neg Q(x))\}$ be a theory in the language $L = \langle P, Q, R\rangle$ without equality, where P, Q, R are unary relation symbols, and let φ be the sentence $(\exists x)(P(x) \rightarrow R(x))$.

   (a) Construct a finished tableau from the theory T with $F\varphi$ at the root. *(25 points)*

   (b) Is φ valid in T? Is it inconsistent with T? Is it independent in T? Justify all answers. *(10 points)*

   (c) Does the theory T have a complete conservative extension? Provide an example or justify why not. *(10 points)*

3. Let $\mathcal{A} = \langle \mathbb{Z}, \mathrm{abs}^A\rangle$ be a structure in the language $L = \langle \mathrm{abs}\rangle$ with equality, where abs is a unary function symbol and abs^A is the absolute value function on $\mathbb{Z}$. Give an example of a nontrivial (i.e., other than $\emptyset$ and $\mathbb{Z}$) set definable in $\mathcal{A}$ without parameters. Provide the defining formula. *(10 points)*