NAIL062 P&P Logic: Worksheet 10 – Resolution in predicate logic

Teaching goals: After completing, the student

- understands the concept of unification and can perform the Unification Algorithm
- knows the necessary concepts from the resolution method in predicate logic (resolution rule, resolvent, resolution proof/refutation, resolution tree), can formally define them, give examples, and explain the differences compared to propositional logic
- can apply the resolution method to solve a given problem (word problems, etc.), performing all necessary steps (conversion to PNF, Skolemization, conversion to CNF)
- can construct a resolution refutation of a given (possibly infinite) CNF formula (if it exists), can draw the resolution tree including the unifications used
- can extract an unsatisfiable conjunction of ground instances of axioms from a res. tree
- knows the notion of LI-resolution, can find an LI-refutation of a given theory (if exists)
- has become familiar with selected concepts from model theory

In-class problems

Problem 1. *Every barber shaves all those who do not shave themselves. No barber shaves anyone who shaves themselves.* Formalize and prove by resolution that: *There are no barbers.*

Solution. *First, we choose a suitable language. In the text we identify a property of objects "x is a barber" and a relation between two objects "x shaves y". We use the language $L = \langle B, S\rangle$ without equality, where B is a unary relational symbol, $B(x)$ means "x is a barber", S is a binary relational symbol, $S(x, y)$ means "x shaves y".*

In this language we formalize the statements from the problem:

- *Every barber shaves all those who do not shave themselves:*
$$\varphi_1 \;=\; (\forall x)(B(x) \to (\forall y)(\neg S(y, y) \to S(x, y)))$$
- *No barber shaves anyone who shaves themselves:*
$$\varphi_2 \;=\; \neg(\exists x)(B(x) \land (\exists y)(S(x, y) \land S(y, y)))$$
- *There are no barbers:*
$$\psi \;=\; \neg(\exists x)B(x)$$

Our goal is to show that in the theory $T = \{\varphi_1, \varphi_2\}$ the sentence ψ holds. We prove this by contradiction, starting with the theory $T \cup \{\neg\psi\} = \{\varphi_1, \varphi_2, \neg\psi\}$. Using Skolemization we obtain an equisatisfiable CNF formula S, then find its resolution refutation, showing that S and hence $T \cup \{\neg\psi\}$ is unsatisfiable.

Convert to PNF, Skolemize, remove universal quantifiers, and convert to CNF:

- $\varphi_1 \;\rightsquigarrow\; B(x) \to (\neg S(y, y) \to S(x, y)) \;\sim\; \neg B(x) \lor S(y, y) \lor S(x, y)$
- $\varphi_1 \;\rightsquigarrow\; \neg(B(x) \land S(x, y) \land S(y, y)) \;\sim\; \neg B(x) \lor \neg S(x, y) \lor \neg S(y, y)$
- $\neg\psi \rightsquigarrow\; B(c)$ *(where c is a new constant symbol)*

In set notation we have:

$$S = \{\{\neg B(x), S(y, y), S(x, y)\}, \{\neg B(x), \neg S(x, y), \neg S(y, y)\}, \{B(c)\}\}$$

Resolution refutation:

$$\{B(c)\}, \{\neg B(x), S(y, y), S(x, y)\}, \{S(y, y), S(c, y)\}, \{\neg B(x), \neg S(x, y'), \neg S(y', y')\},$$
$$\{\neg S(c, y'), \neg S(y', y')\}, \square$$

The first two clauses are from S, the third is their resolvent using the unification $\{x/c\}$. The fourth clause is a variant of a clause from S, variable y renamed to y' to satisfy the technical condition of disjoint variable sets in resolved clauses. The fifth clause is the resolvent of the first and fourth clauses using unification $\{x/c\}$. The last, empty clause $\square$ is the resolvent of clauses 3 and 5 with unification $\{y/c, y'/c\}$.

Typically, we represent the refutation as a resolution tree, indicating the unifications used:

$$\begin{tikzcd}[column sep=small]
\{B(c)\} \arrow[dr, dash] & & \{\neg B(x), S(y,y), S(x,y)\} \arrow[dl, dash] & \{B(c)\} \arrow[dr, dash] & & \{\neg B(x), \neg S(x,y'), \neg S(y',y')\} \arrow[dl, dash] \\
 & \{x/c\}\ \ \{S(y,y), S(c,y)\} \arrow[drr, dash] & & & \{\neg S(c,y'), \neg S(y',y')\}\ \ \{x/c\} \arrow[dl, dash] & \\
 & & & \{y/c, y'/c\}\ \ \square & &
\end{tikzcd}$$

Problem 2. The following statements describe a genetic experiment:

(i) Every sheep was either born from another sheep or cloned (but not both).
(ii) No cloned sheep gave birth.

We want to show by resolution that: *(iii) If a sheep gave birth, it was itself born.* Specifically:

(a) Express as sentences φ_1, φ_2, φ_3 in $L = \langle P, K\rangle$ without equality (P is binary, K unary rel. symbol, $P(x, y)$ means 'sheep x gave birth to sheep y', $K(x)$ 'sheep x was cloned').
(b) Using Skolemization of these sentences or their negations, construct a set of clauses S (possibly in an extended language) that is unsatisfiable exactly when $\{\varphi_1, \varphi_2\} \models \varphi_3$.
(c) Find a resolution refutation of S, draw the resolution tree with unifications used.
(d) Does S have an LI-refutation?

Solution. *Note that all objects are sheep, so no predicate for 'being a sheep' is needed. The procedure is similar to the previous example:*

(a) There are several ways to formulate the formulas; following the text closely, we get:

$$\begin{aligned}
\varphi_1 &= (\forall x)(((\exists y)P(y, x) \lor K(x)) \land \neg((\exists z)P(z, x) \land K(x))) \\
\varphi_2 &= \neg(\exists x)(K(x) \land (\exists y)P(x, y)) \\
\varphi_3 &= (\forall x)((\exists y)P(x, y) \to (\exists z)P(z, x))
\end{aligned}$$

(b) Start from the theory $\{\varphi_1, \varphi_2, \neg\varphi_3\}$ (proof by contradiction). Convert to PNF, Skolemize, remove universal quantifiers, convert to CNF, and represent as sets:

- $\varphi_1 \sim (\forall x)(\exists y)(\forall z)((P(y, x) \lor K(x)) \land \neg(P(z, x) \land K(x))) \rightsquigarrow (P(f(x), x) \lor K(x)) \land \neg(P(z, x) \land K(x)) \sim \{\{P(f(x), x), K(x)\}, \{\neg P(z, x), \neg K(x)\}\}$
- $\varphi_2 \sim (\forall x)(\forall y)\neg(K(x) \land P(x, y)) \sim \{\{\neg K(x), \neg P(x, y)\}\}$
- $\neg\varphi_3 \sim (\exists x)(\exists y)(\forall z)\neg(P(x, y) \to P(z, x)) \rightsquigarrow \neg(P(c, d) \to P(z, c)) \sim \{\{P(c, d)\}, \{\neg P(z, c)\}\}$

$$S = \{\{P(f(x), x), K(x)\}, \{\neg P(z, x), \neg K(x)\}, \{\neg K(x), \neg P(x, y)\}, \{P(c, d)\}, \{\neg P(z, c)\}\}$$

(c) Resolution tree for $S \vdash_R \square$:

$$\{P(f(x),x),K(x)\} \qquad \{\neg P(z,c)\}$$

$$\{x/c, z/f(c)\}\ \{K(c)\} \qquad \{\neg K(x), \neg P(x,y)\}$$

$$\{x/c, z/f(c)\}\ \{\neg P(c,y)\} \qquad \{P(c,d)\}$$

$$\{y/d\}\ \square$$

(d) *Yes, in (c) we constructed an LI-refutation. Even if we had not, existence of an LI-refutation follows from the completeness theorem of LI-resolution for Horn formulas; our CNF S is Horn.*

Problem 3. Let $T = \{\neg(\exists x)R(x),\ (\exists x)(\forall y)(P(x,y) \to P(y,x)),\ (\forall x)((\exists y)(P(x,y) \wedge P(y,x)) \to R(x)),\ (\forall x)(\exists y)P(x,y)\}$ be a theory in the language $L = \langle P, R\rangle$ without equality.

(a) Find an open equisatisfiable theory T' for T by Skolemization.
(b) Convert T' to an equivalent theory S in CNF. Represent S as sets.
(c) Find a resolution refutation of S. Indicate the unification used at each step.
(d) Find an unsatisfiable conjunction of the ground instances of clauses from S. *Hint: use the unifications from (c).*

Solution. *(a) By Skolemization we get:*

$$T' = \{\neg R(x),\ P(c,y) \to P(y,c),\ P(x,y) \wedge P(y,x) \to R(x),\ P(x,f(x))\}$$

(Note for the third axiom: $(\exists y)$ in the antecedent of the implication changes to $(\forall y)$.)

(b) Easily convert to CNF:

$$S = \{\{\neg R(x)\}, \{\neg P(c,y), P(y,c)\}, \{\neg P(x,y), \neg P(y,x), R(x)\}, \{P(x,f(x))\}\}$$

(c) Resolution tree for $S \vdash_R \square$:

$$\{\neg R(x)\} \qquad \{\neg P(x',y), \neg P(y,x'), R(x')\}$$

$$\{x'/x\}\ \{\neg P(x,y), \neg P(y,x)\} \qquad \{\neg P(c,y'), P(y',c)\}$$

$$\{x/c, y'/y\}\ \{\neg P(c,y)\} \qquad \{P(x,f(x))\}$$

$$\{x/c, y/f(c)\}\ \square$$

(Note the variable renaming to ensure disjoint variable sets in resolved clauses.)

(d) To find a conjunction of ground instances of the original theory, apply the unifications from the resolution tree to each leaf clause C:

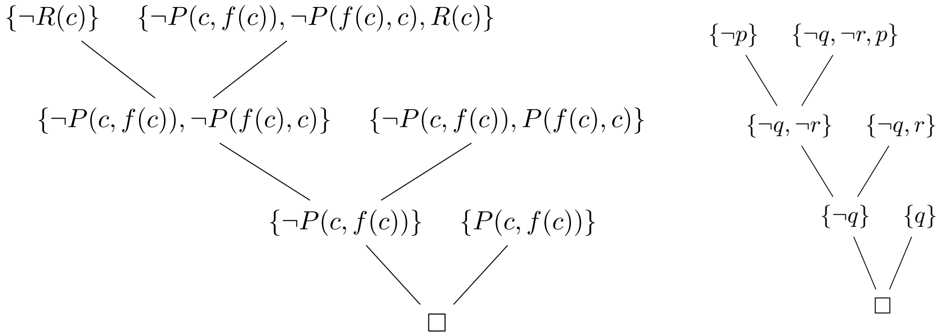
- $\neg R(x) \cdot \{x'/x\} \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg R(c)$

- $\neg P(x', y) \vee \neg P(y, x') \vee R(x') \cdot \{x'/x\} \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg P(c, f(c)) \vee \neg P(f(c), c) \vee R(c)$
- $\neg P(c, y') \vee P(y', c) \cdot \{x/c, y'/y\} \cdot \{x/c, y/f(c)\} = \neg P(c, f(c)) \vee P(f(c), c)$
- $P(x, f(x)) \cdot \{x/c, y/f(c)\} = P(c, f(c))$

If variables remain, substitute arbitrary constants. The resulting unsatisfiable conjunction of ground instances:

$$\neg R(c) \wedge (\neg P(c, f(c)) \vee \neg P(f(c), c) \vee R(c)) \wedge (\neg P(c, f(c)) \vee P(f(c), c)) \wedge P(c, f(c))$$

Its resolution refutation at the propositional level has the same structure as that of S:



To get the ground instances of the original theory T, apply the same unifications to the axioms from which the clauses originated:

$$\neg R(c) \wedge (P(c, f(c)) \rightarrow P(f(c), c)) \wedge (P(c, f(c)) \wedge P(f(c), c) \rightarrow R(c)) \wedge P(c, f(c))$$

Extra practice

Problem 4. Find a resolution refutation:

$$S = \{\{P(a, x, f(y)), P(a, z, f(h(b))), \neg Q(y, z)\},\ \{\neg Q(h(b), w), H(w, a)\},\ \{\neg H(v, a)\},$$
$$\{\neg P(a, w, f(h(b))), H(x, a)\},\ \{P(a, u, f(h(u))), H(u, a), Q(h(b), b)\}\}$$

Problem 5. Let $L = \langle <, j, h, s \rangle$ be without equality, where j, h, q are constant symbols ('apples/pears/plums') and $x < y$ expresses *"fruit y is better than fruit x"*. We know that:

(i) The relation "being better" is a strict partial order (irreflexive, asymmetric, transitive).
(ii) Pears are better than apples.

Prove by resolution: *(iii) If plums are better than pears, then apples aren't better than plums.*

(a) Express statements (i), (ii), (iii) as open formulas in the language L.
(b) Using these formulas, find a CNF formula S that is unsatisfiable exactly when (i), (ii) imply (iii). Write S in set representation.
(c) Prove by resolution that S is unsatisfiable. Illustrate the refutation with a resolution tree, indicate the unification used at each step. *Hint: 4 resolution steps are enough.*
(d) Find the conjunction of the basic instances of the axioms of S that is unsatisfiable.
(e) Is S refutable by LI-resolution?

Problem 6. Let $T = \{\varphi\}$ be in $L = \langle U, c \rangle$ with equality, where U is unary relational and c is a constant symbol, and φ expresses *"There are at least 5 elements for which $U(x)$ holds."*

(a) Find two non-equivalent simple complete extensions of T.
(b) Is the theory T openly axiomatizable? Give justification.

Problem 7. Let $T = \{U(x) \to U(f(x)), (\exists x)U(x), \neg(f(x) = x), \varphi\}$ be a theory in the language $L = \langle U, f \rangle$ with equality, where U is a unary relational symbol, f is a unary function symbol, and φ expresses that "there are at most 4 elements."

(a) Is the theory T an extension of the theory $S = \{(\exists x)(\exists y)(\neg x = y \wedge U(x) \wedge U(y)), \varphi\}$ in the language $L' = \langle U \rangle$? Is it a conservative extension? Justify.

(b) Is the theory T openly axiomatizable? Justify.

Problem 8. Let $T = \{(\forall x)(\exists y)S(y) = x,\ S(x) = S(y) \to x = y\}$ be a theory in the language $L = \langle S \rangle$ with equality, where S is a unary function symbol.

(a) Find an extension T' of the theory T by defining a new unary function symbol P such that $T' \models S(S(x)) = y \leftrightarrow P(P(y)) = x$.

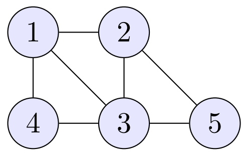
(b) Is the theory T' openly axiomatizable? Give justification.

Problem 9. Let T be an extension of the theory $DeLO^-$ (i.e., dense linear orders with a minimal element and without a maximal element) by a new axiom $c \leq d$ in the language $L = \langle \leq, c, d \rangle$ with equality, where c, d are new constant symbols.

(a) Are the sentences $(\exists x)(x \leq d \wedge x \neq d)$ and $(\forall x)(x \leq d)$ true / false / independent in T?

(b) Write two non-equivalent simple complete extensions of the theory T.

Problem 10. Consider the following graph.



(a) Find all automorphisms.

(b) Which subsets of the set of vertices V are definable? Give the defining formulas. *(Hint: Use (a).)*

(c) Which binary relations on V are definable?

For further thought

Problem 11. Let $T = \{(\forall x)(\exists y)S(y) = x,\ S(x) = S(y) \to x = y\}$ be a theory in the language $L = \langle S \rangle$ with equality, where S is a unary function symbol.

(a) Let $\mathcal{R} = \langle \mathbb{R}, S \rangle$, where $S(r) = r + 1$ for $r \in \mathbb{R}$. For which $r \in \mathbb{R}$ is the set $\{r\}$ definable in $\mathcal{R}$ from the parameter 0?

(b) Is the theory T openly axiomatizable? Give justification.

(c) Is the extension T' of T by the axiom $S(x) = x$ an ω-categorical theory? Is T' complete?

(d) For which $0 < n \in \mathbb{N}$ does there exist an L-structure $\mathcal{B}$ of size n elementarily equivalent to $\mathcal{R}$? Does there exist a countable structure $\mathcal{B}$ elementarily equivalent to $\mathcal{R}$?