Lecture 6 – Chomsky Normal Form, Pumping lemma for context-free languages

NTIN071 Automata and Grammars

Jakub Bulín (KTIML MFF UK)

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Recap of Lecture 5

- Grammars: general, context-sensitive, context-free, right-linear (regular) – Chomsky hierarchy
- The language of a grammar, derivation
- Right-linear grammars correspond to FA (and so do left/linear)
- Linear grammars are stronger
- Context-free grammars: parse tree and its yield
- (un)ambiguous grammars, inherently ambiguous languages

2.6 Chomsky Normal Form

Chomsky normal form

The Chomsky normal form (ChNF) of a context-free grammar:

- all rules of the form $A \to BC$ or $A \to a$ ($A, B, C \in V$, $a \in T$)
- no useless symbols

Theorem

For every context-free language L such that $L \setminus \{\epsilon\} \neq \emptyset$ there exists a grammar in ChNF that generates $L \setminus \{\epsilon\}$.

Applications:

- Test membership in L: the CYK algorithm (Sakai 1962)
- Prove the Pumping lemma for context-free languages

Converting to ChNF

Take any context-free grammar for L and simplify (in this order!):

1. eliminate ϵ-productions $A \to \epsilon$ [here we lose $\epsilon \in L$]
2. eliminate unit productions $A \to B$
3. eliminate useless symbols
   - 3a. nongenerating [a word over terminals]
   - 3b. unreachable [from the start symbol]

Now we have a reduced grammar. To get to ChNF, we further:

4. separate terminals from bodies
5. break up longer bodies

Step 1: Eliminate ϵ-productions

A variable $A \in V$ is nullable if $A \Rightarrow^* \epsilon$. An algorithm to find them:

basis: for every ϵ-production $A \to \epsilon$ mark A as nullable

induct: if $B \to C_1 \dots C_k \in \mathcal{P}$ where all C_i are nullable, B is nullable

To eliminate ϵ-productions: 1. find nullable variables, 2. remove ϵ-productions, 3. process every production $A \to X_1 \dots X_k \in \mathcal{P}$:

- let $J \subseteq \{1, \dots, k\}$ be the positions of all nullable variables
- for every $J' \subseteq J$ create a copy of the production where X_j for $j \in J'$ are deleted, except if $J = \{1, \dots, k\}$ require $J' \neq \emptyset$

Example: $\mathcal{P} = \{S \to AB, A \to aAB \mid \epsilon, B \to ABBA \mid \epsilon\}$

$S \to AB \mid A \mid B$ $A \to aAB \mid aA \mid aB \mid a$
$B \to ABBA \mid ABA \mid ABB \mid BBA \mid AA \mid AB \mid BA \mid BB \mid A \mid B$

Step 2: Eliminate unit productions

Idea: for a unit production $A \to B$ copy rules for B with head A, but unit productions can be composed, we need transitive closure:

Unit pairs $\mathcal{U} \subseteq V \times V$ are defined as follows:

- $(A, B) \in \mathcal{U}$ for every unit production $A \to B \in \mathcal{P}$
- if $(A, B) \in \mathcal{U}$ and $(B, C) \in \mathcal{U}$, then $(A, C) \in \mathcal{U}$

To eliminate unit productions:

1. find all unit pairs $\mathcal{U}$
2. remove all unit productions
3. for every unit pair $(A, B) \in \mathcal{U}$ and production $B \to \beta \in \mathcal{P}$ add the production $A \to \beta$ to $\mathcal{P}$

Step 2: Eliminate unit productions – an example

$E \to T \mid E + T$

$F \to I \mid (E)$

$I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

$T \to F \mid T * F$

unit pairs:

$(E, E), (E, F), (E, I), (E, T),$

$(F, F), (F, I),$

$(I, I),$

$(T, F), (T, I), (T, T)$

the result:

$E \to E + T \mid T * F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

$I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

$F \to (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

$T \to T * F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

Step 3: Eliminate useless symbols

- $X \in V \cup T$ is a useful symbol (in G) if there exists a derivation of the form $S \Rightarrow^* \alpha X \beta \Rightarrow^* w$ for some $w \in T^*$
- X is useless if it is not useful
- X is generating if $X \Rightarrow^* w$ for some $w \in T^*$
- X is reachable if $S \Rightarrow^* \alpha X \beta$ for some $\alpha, \beta \in (V \cup T)^*$

Observe:

- useful $\Leftrightarrow$ generating and reachable
- useless $\Leftrightarrow$ nongenerating or unreachable (we eliminate both)
- all terminals are generating

Step 3: Eliminate useless symbols – the algorithm

1. Find all generating symbols:

   basis: mark all terminals $a \in T$ as generating

   induct: for every production $A \to \beta$ where every symbol in the body β is generating, mark the head A as generating (incl. $A \to \epsilon$)

2. Remove all nongenerating symbols and rules containing them
3. Find all reachable symbols

   basis: mark S as reachable

   induct: for every production $A \to \beta$ where the head A is reachable mark every symbol in the body β as reachable

4. Remove all unreachable symbols and rules containing them

- The order is important! Eliminating nongenerating symbols can create new unreachable symbols, but not vice versa.
- **Example:** eliminate nongenerating B, then unreachable A

$$S \to AB \mid a \qquad S \to a \qquad S \to a$$
$$A \to b \qquad A \to b$$

Steps 4 & 5: Separate terminals and break up long bodies

Step 4: Separate terminals from bodies

For every $a \in T$, introduce a new variable V_a and the rule $V_a \to a$.

For every rule $A \to \beta$ with $|\beta| \geq 2$, replace every terminal a by V_a.

Step 5: Break up longer bodies

Replace every rule $A \to B_1 \dots B_k$ with $k \geq 3$ with:

$$\begin{aligned} A &\to B_1 C_1 \\ C_1 &\to B_2 C_2 \\ &\vdots \\ C_{k-2} &\to B_{k-1} B_k \end{aligned}$$

where $C_1, \dots, C_{k-2}$ are new variables (only used for this purpose).
[Alternatively, instead of a chain use a binary tree.]

Conversion to Chomsky Normal Form

ChNF: only useful symbols and rules $A \to BC$ or $A \to a$

Theorem

For every context-free language L such that $L \setminus \{\epsilon\} \neq \emptyset$ there exists a grammar in ChNF that generates $L \setminus \{\epsilon\}$.

Proof.

Take a context-free grammar G for L. Modify it by applying steps 1, 2, 3a, 3b, 4, and 5, in order. Clearly, the result is in ChNF. After step 1 we get G' such that $L(G') = L(G) \setminus \{\epsilon\}$; the remaining steps produce equivalent grammars. Steps 2-5 don't add any ϵ-productions, 3-5 don't add unit productions, 3b-5 don't add nongenerating symbols, 4-5 don't add useless, etc. $\square$

Note: If we only apply 1, 2, 3a, and 3b, we get a reduced grammar: only useful symbols, no ϵ-productions, no unit productions.

Example

$I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

$F \to I \mid (E)$

$T \to F \mid T * F$

$E \to T \mid E + T$

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reduce + separate

$I \to a \mid b \mid IA \mid IB \mid IZ \mid IU$
$F \to LER \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$
$T \to TMF \mid LER \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$
$E \to EPT \mid TMF \mid LER \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$

$A \to a$, $B \to b$, $Z \to 0$, $U \to 1$, $P \to +$, $M \to *$, $L \to ($, $R \to)$

break up longer bodies

$F \to LC_3 \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$
$T \to TC_2 \mid LC_3 \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$
$E \to EC_1 \mid TC_2 \mid LC_3 \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$
$C_1 \to PT$
$C_2 \to MF$
$C_3 \to ER$
$I, A, B, Z, U, P, M, L, R$ same as on the left side

2.7 Pumping lemma for context-free languages

Pumping lemma for context-free languages

Theorem (Pumping Lemma for Context Free Languages)

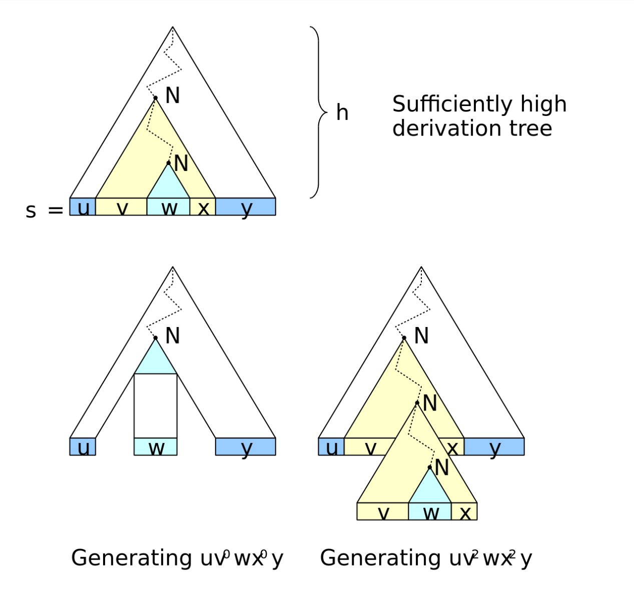
Let $L \subseteq \Sigma^*$ be context-free. Then there exists $n \in \mathbb{N}$ s.t. for any $z \in L, |z| \geq n$ there are $u, v, w, x, y \in \Sigma^*$ s.t. $z = uvwxy$ and:

(i) $|vwx| \leq n$ *(ii)* $|vx| > 0$ *(iii)* $uv^iwx^iy \in L$ *for all* $i \geq 0$

Proof idea: Take a ChNF grammar for L. If $z \in L$ is long enough, a parse tree for z must contain a path from S to a leaf (terminal) of length $|V| + 1$. Some nonterminal $N \in V$ repeats on this path giving two subtrees with root N: a larger one containing a smaller one. Replace the larger with a copy of the smaller ($i = 0$) or the smaller with a copy of the larger ($i = 2$).

What is long enough? If $|z| > 2^{k-1}$, then the depth of the tree is $k + 1$. (All inner nodes not immediately above a leaf are binary!)

The proof in a picture



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The proof

If $L = \emptyset$ and $L = \{\epsilon\}$ trivial, take $n = 1$. Otherwise take a ChNF grammar for L. Set $n = 2^{|V|-1} + 1$. Let $z \in L$ with $|z| \geq n$.

A parse tree for z contains a path from S to a terminal t of length at least $|V| + 1$. At least two of the last $|V| + 1$ nonterminals on this path must be the same. Let A^1, A^2 be such a pair that is closest to t. Let T^1, T^2 be the subtrees rooted at A^1, A^2.

The path from A^1 to t is the longest one in T^1 and has length at most $(k + 1)$. Thus $|vwx| \leq n$.

There are two paths from A^1 (ChNF!): one leads to T^2, the other to the rest, it must generate at least one letter (no ϵ-productions). Thus $|vx| > 0$.

The proof cont'd

The word $z = uvwxy$ is derived as follows:

- $A^2 \Rightarrow^* w$
- $A^1 \Rightarrow^* vA^2x \Rightarrow^* vwx$
- $S \Rightarrow^* uA^1y \Rightarrow^* uvA^2xy \Rightarrow^* uvwxy$

For $i = 0$: replace T^1 by T^2

$$S \Rightarrow^* uA^2y \Rightarrow^* uwy$$

For $i = 2$: replace T^2 by a copy of T^1

$$S \Rightarrow^* uA^1y \Rightarrow^* uvA^1xy \Rightarrow^* uvvA^2xxy \Rightarrow^* uvvwxxy$$

For $i \geq 3$ repeat the above. $\square$

Application: proving a laguage is not context-free

Example

The language $L = \{0^n 1^n 2^n \mid n \geq 0\}$ is not context-free.

Suppose for contradiction that it is. Let n be constant from the Pumping lemma. Choose $z = 0^n 1^n 2^n \in L$. Clearly $|z| \geq n$.

The Pumping lemma gives us a split $z = uvwxy$ satisfying (i)–(iii). Since $|vwx| \leq n$, the pumped part vx contains at most two of the symbols $0, 1, 2$. Pumping will violate equal number of symbols. $\square$

Example

The language $L = \{0^i 1^j 2^k \mid 0 \leq i \leq j \leq k\}$ is not context-free.

Similar as above, also $z = 0^n 1^n 2^n$, at most two symbols pumped:

- if 0 or 1 are pumped, but 2 is not: pump up $(i = 2)$
- if 1 or 2 are pumped, but 0 is not: pump down $(i = 0)$ $\square$

More examples

Example

$L = \{0^j 1^k 2^j 3^k \mid j, k \geq 0\}$ is not context-free.

Similar as before, choose $z = 0^n 1^n 2^n 3^n$, vx must contain some symbol. But from $|vwx| \leq n$ we know that it can contain neither both 0 and 2, nor both 1 and 3. In any case, the equal number of symbols 0 and 2 or 1 and 3 is violated. $\square$

Example

$L = \{ww \mid w \in \{0,1\}^*\}$ is not context-free.

Choose $z = 0^n 1^n 0^n 1^n$, then $|z| \geq n$. The pumped part can cover neither both blocks of 0s nor both blocks of 1s. Four cases to consider: vx contains a symbol from the 1st block of 0s, 1st block of 1s, 2nd 0s, 2nd 1s. In all cases we get a violation. $\square$

It is not a characterization

The Pumping lemma is again only an implication, not equivalence:

Example

$L = \{a^i b^j c^k d^\ell | i = 0 \text{ or } j = k = \ell\}$ can be pumped. But it is not context-free.

$i = 0 : b^j c^k d^l$ can be pumped in any letter
$i > 0 : a^i b^n c^n d^n$ can be pumped in a^*

What to do in such cases?

- Ogden's lemma: generalize Pumping lemma, mark some of the letters, some marked symbol is pumped
- use closure properties of context-free languages

Summary of Lecture 6

- Reducing a grammar: removing ϵ-productions, unit productions, useless symbols
- Chomsky Normal Form of a context-free grammar
- Pumping lemma for context-free languages, application: proving non-context-freeness