NTIN071 A&G: Tutorial 14 – TODO Intro to complexity

Problem 1. Show that the class P is closed under union, intersection, and complement.

Problem 2. Show that the class NP is closed under union and intersection.

Problem 3. Show that the class P is closed under Kleene star. That is, if $A \in P$, then

$$A^* = \{w_1 \dots w_k \mid k \geq 0, w_i \in A \text{ for all } i \leq k\} \in \mathrm{P}.$$

(Hint: Design a dynamic algorithm filling a table T where $T[i, j] = 1$ if and only if $w_i \dots w_j \in A^*$.)

Problem 4. Show that the class NP is closed under Kleene star.

Problem 5. Show that the problems CLIQUE, INDEPENDENT-SET, and VERTEX-COVER defined below are polynomially inter-reducible.

> CLIQUE
>
> In: A graph $G = (V, E)$ and an integer $k \geq 0$.
> Q: Does G contain (as a subgraph) the complete graph (clique) on at least k vertices?

> INDEPENDENT-SET
>
> In: A graph $G = (V, E)$ and an integer $k \geq 0$.
> Q: Does G contain an independent set of size at least k, i.e., $S \subseteq V$, $|S| \geq k$ with no edge connecting a pair of vertices from S?

> VERTEX-COVER
>
> In: A graph $G = (V, E)$ and an integer $k \geq 0$.
> Q: Does G have a vertex cover of size at most k, i.e., $S \subseteq V$, $|S| \leq k$ containing at least one vertex from every edge?

Problem 6. Show that the problem VERTEX-COVER is polynomially reducible to the problem DOMINATING-SET defined below.

DOMINATING-SET	
In:	A graph $G = (V, E)$ and an integer $k \geq 0$.
Q:	Does G contain a set of vertices $S \subseteq V$ of size at most k such that every $v \in V \setminus S$ has a neighbor in S?

Problem 7. It is well known that the problem HAMILTONIAN-CYCLE defined below is an NP-complete problem. Use this fact to show that the problems ORIENTED-HAMILTONIAN-CYCLE, (s,t)-HAMILTONIAN-PATH, and HAMILTONIAN-PATH defined further below are NP-complete as well.

HAMILTONIAN-CYCLE	
In:	An (unoriented) graph $G = (V, E)$.
Q:	Does G contain a Hamiltonian cycle, i.e., a cycle containing every vertex?

ORIENTED-HAMILTONIAN-CYCLE	
In:	An oriented graph $G = (V, E)$.
Q:	Does G contain an oriented Hamiltonian cycle, i.e., an oriented cycle containing every vertex?

(s,t)-HAMILTONIAN-PATH	
In:	An (unoriented) graph $G = (V, E)$ and a pair of vertices $s, t \in V$.
Q:	Does G contain a Hamiltonian path from s to t, i.e., a path that starts in s, ends in t, and visits every vertex exactly once?

HAMILTONIAN-PATH	
In:	An (unoriented) graph $G = (V, E)$.
Q:	Does G contain a Hamiltonian path, i.e., a path that visits every vertex exactly once?

Problem 8. Show that the problem HAMILTONIAN-CYCLE is polynomially reducible to the problem SAT defined below.

SAT
IN: A propositional formula φ in conjunctive normal form (CNF).
Q: Is φ satisfiable?

Problem 9. Show that the problem HAMILTONIAN-CYCLE is polynomially reducible to the problem TRAVELING-SALESPERSON defined below.

TRAVELING-SALESPERSON
IN: A list of cities $C = \{c_1, \ldots, c_n\}$, distances $d(c_i, c_j) \in \mathbb{N}$ between each pair of cities, and $D \in \mathbb{N}$.
Q: Is there a route of length at most D that visits every city exactly once and returns to the origin city?

Problem 10. Show that the problems INTEGER-PROGRAMMING and BINARY-INTEGER-PROGRAMMING defined below are NP-hard. You can use a reduction from one of the following problems: SAT, CLIQUE, INDEPENDENT-SET, or VERTEX-COVER.

Moreover, show that the problem BINARY-INTEGER-PROGRAMMING is in NP (and thus it is NP-complete). The problem INTEGER-PROGRAMMING is in NP as well but it is not that easy to prove. Why?

INTEGER-PROGRAMMING
IN: A $k \times n$ integer matrix A and an integer vector b of length k.
Q: Is there an integer vector x of length n such that $Ax \geq b$?

BINARY-INTEGER-PROGRAMMING
IN: A $k \times n$ integer matrix A and an integer vector b of length k.
Q: Is there a vector $x \in \{0, 1\}^n$ such that $Ax \geq b$?

Problem 11. Show that the problem GRAPH-COLORING defined below is NP-complete.

GRAPH-COLORING
IN: A graph $G = (V, E)$ and $k \in \mathbb{N}$.
Q: Can we color vertices of G with at most k colors so that there are no monochromatic edges?

Show that the problem COVER-ORIENTED-CYCLES defined below is NP-complete. (Hint: Use a reduction from VERTEX-COVER. Do not forget to show that COVER-ORIENTED-CYCLES is in NP.)

COVER-ORIENTED-CYCLES

In: An oriented graph $G = (V, E)$ and an integer $k \geq 0$.
Q: Is there a set of vertices $S \subseteq V$, $|S| \leq k$, containing at least one vertex from every oriented cycle in G?

Consider the PARTITION-PROBLEM defined below.

(a) Show that the PARTITION-PROBLEM is polynomially reducible to the KNAPSACK problem defined further below.
(b) Show that the PARTITION-PROBLEM is polynomially reducible to the SCHEDULING problem defined even further below.

PARTITION-PROBLEM

In: A (finite) set A and a value $s(a) \in \mathbb{N}$ associated with each $a \in A$.
Q: Is it possible to partition A into two parts of the same value? More precisely, is there $A' \subseteq A$ such that $\sum_{a \in A'} s(a) = \sum_{a \in A \setminus A'} s(a)$?

KNAPSACK

In: A (finite) set A, where each $a \in A$ has an associated size $s(a) \in \mathbb{N}$ and value $v(a) \in \mathbb{N}$, and two positive integers: capacity C and value V.
Q: Is there a subset $A' \subseteq A$ such that $\sum_{a \in A'} s(a) \leq C$ and $\sum_{a \in A'} v(a) \geq V$?

SCHEDULING

In: A number of processors $m \in \mathbb{N}$, a (finite) set of tasks T where each task $x \in T$ has an associated duration $d(x) \in \mathbb{N}$, and a positive integer D.
Q: Is it possible to partition the set T into m parts $T_1, \ldots, T_m$ (i.e., pairwise disjoint sets covering T) such that $\sum_{x \in T_i} d(x) \leq D$ for every $1 \leq i \leq m$?