Lecture 5 – Formal grammars, regular and context-free grammars

NTIN071 Automata and Grammars

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Spring 2025

** Adapted from the Czech-lecture slides by Marta Vomlelová with gratitude. The translation, some modifications, and all errors are mine.*

Recap of Lecture 4

- regular expressions
- Kleene's theorem (two variants)
- constructions: RE to ϵ-NFA, DFA to RE
- state elimination algorithm
- string substitution, homomorphism, inverse homomorphism
- decision properties

Chapter 2: Grammars

2.1 Formal grammars

Palindromes are not regular

palindrome: $w = w^R$, e.g. `racecar`, `step_on_no_pets`

Example

The language $L_{\text{pal}} = \{w \in \{0,1\}^* \mid w = w^R\}$ is not regular.

(A standard Pumping lemma argument using $w = 0^n 1 0^n$.)

How to represent L_{pal}? We can use a (context-free) grammar, $G = (\{S\}, \{0,1\}, \mathcal{P}, S)$ with the following set of production rules:

$$\begin{aligned}\mathcal{P} = \{S &\to \epsilon,\\ S &\to 0,\\ S &\to 1,\\ S &\to 0S0,\\ S &\to 1S1\}\end{aligned}$$

For brevity, we also write $\mathcal{P} = \{S \to \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}$.

Formal grammar: the definition

A formal (generative) grammar: $G = (V, T, \mathcal{P}, S)$ where

- V is a finite nonempty set of nonterminals (variables)
- T is a finite set of terminal symbols (terminals), $V \cap T = \emptyset$
- $S \in V$ is the start symbol
- $\mathcal{P}$ is a finite set of production rules of the form $\alpha \to \beta$ where
  - $\alpha \in (V \cup T)^+ \setminus T^+$, the head (must contain some variable!)
  - $\beta \in (V \cup T)^*$, the body

A grammar is context-free (a CFG) if the head is a single variable, i.e., the rules are of the form $A \to \beta$ for $A \in V$ and $\beta \in (V \cup T)^*$.

The production rules thus represent a recursive definition of the language, starting from the start symbol (see the example).

Derivation, the language of a grammar

Let $G = (V, T, \mathcal{P}, S)$ be a grammar.

- γ one-step derives δ (write $\gamma \Rightarrow_G \delta$ or just $\gamma \Rightarrow \delta$) if $\gamma = \eta\alpha\nu$ and $\delta = \eta\beta\nu$ for some $\alpha \to \beta \in \mathcal{P}$ and $\eta, \nu \in (V \cup T)^*$
- γ derives δ (write $\gamma \Rightarrow_G^* \delta$ or just $\gamma \Rightarrow^* \delta$) if there are $\beta_1, \ldots, \beta_n \in (V \cup T)^*$ s.t. $\gamma = \beta_1 \Rightarrow \beta_2 \Rightarrow \cdots \Rightarrow \beta_n = \delta$ (Note that always $\gamma \Rightarrow^* \gamma$.)
- the sequence $\beta_1, \ldots, \beta_n$ is a derivation of δ from γ, it is minimal if $\beta_i \neq \beta_j$ for $i \neq j$
- a sentential form is any $\delta \in (V \cup T)^*$ such that $S \Rightarrow_G^* \delta$

The language generated by G, $L(G)$ consists of words over the terminals derivable from the start symbol:

$$L(G) = \{\omega \in T^* \mid S \Rightarrow_G^* \omega\}$$

(Similarly, for any $A \in V$ define $L(A) = \{\omega \in T^* \mid A \Rightarrow_G^* \omega\}$.)

2.2 Chomsky hierarchy

Chomsky hierarchy (of grammars)

Restricting the form of production rules:

- Type 0: **general grammars**
  - $\alpha_1 A \alpha_2 \to \beta$ where $A \in V$, $\alpha_1, \alpha_2, \beta \in (V \cup T)^*$
  - recursively enumerable languages $\mathcal{L}_0$
- Type 1: **context-sensitive grammars**
  - $\alpha_1 A \alpha_2 \to \alpha_1 \gamma \alpha_2$, $A \in V$, $\gamma \in (V \cup T)^+$, $\alpha_1, \alpha_2 \in (V \cup T)^*$
  - note: the variable must be rewritten to at least one symbol
  - sometimes we allow $S \to \epsilon$, then S cannot appear in any body
  - context-sensitive languages $\mathcal{L}_1$
- Type 2: **context-free grammars**
  - $A \to \beta$ where $A \in V$, $\beta \in (V \cup T)^*$
  - context-free languages $\mathcal{L}_2$
- Type 3: **right-linear grammars** (aka regular grammars)
  - $A \to \omega B$ or $A \to \omega$ where $A, B \in V$, $\omega \in T^*$
  - regular languages $\mathcal{L}_3$

The classes are ordered by (strict) inclusion

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \mathcal{L}_3$$

- context-sensitive languages are recursively enumerable: the head of a CSG contains a variable
- context-free languages are context-sensitive: the context α_1, α_2 is empty; we can eliminate ϵ-productions $A \to \epsilon$
- regular languages are context-free: body can have any form
- strict inclusion: we will give examples later

2.3 Regular grammars

Right-linear grammars

A grammar G is right-linear (regular, type 3), if its production rules are of the form $A \to \omega B$ or $A \to \omega$ where $A, B \in V$, $\omega \in T^*$.

(At most one variable in the body, it can only be at the end.)

Example

$G = (\{S, A, B\}, \{0, 1\}, \mathcal{P}, S)$ where

$$\mathcal{P} = \{S \to 0S \mid 1A \mid \epsilon,\ A \to 0A \mid 1B,\ B \to 0B \mid 1S\}$$

A derivation of $01101 \in L(G)$:

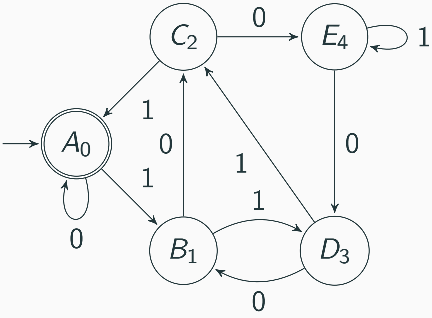
$$S \Rightarrow 0S \Rightarrow 01A \Rightarrow 011B \Rightarrow 0110B \Rightarrow 01101S \Rightarrow 01101$$

Corresponds to FA: nonterminals are states, generate means read.

Theorem

A language is regular, iff it is generated by a right-linear grammar.

Example: binary numbers divisible by 5



$$
\begin{aligned}
A &\rightarrow 1B \mid 0A \mid \epsilon \\
B &\rightarrow 0C \mid 1D \\
C &\rightarrow 0E \mid 1A \\
D &\rightarrow 0B \mid 1C \\
E &\rightarrow 0D \mid 1E
\end{aligned}
$$

Derivation examples

$$
\begin{aligned}
&A \Rightarrow 0A \Rightarrow 0 && (n = 0) \\
&A \Rightarrow 1B \Rightarrow 10C \Rightarrow 101A \Rightarrow 101 && (n = 5) \\
&A \Rightarrow 1B \Rightarrow 10C \Rightarrow 101A \Rightarrow 1010A \Rightarrow 1010 && (n = 10) \\
&A \Rightarrow 1B \Rightarrow 11D \Rightarrow 111C \Rightarrow 1111A \Rightarrow 1111 && (n = 15)
\end{aligned}
$$

Finite automaton to right-linear grammar

Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$ define a right-linear grammar $G = (Q, \Sigma, \mathcal{P}, q_0)$, i.e. nonterminals are states, with productions:

- $p \to aq$ for all transitions $\delta(p, a) = q$
- $p \to \epsilon$ for every final state $p \in F$

To show that $L(A) = L(G)$:

- For the empty word:

  $\epsilon \in L(A)$ iff $q_0 \in F$ iff $q_0 \to \epsilon \in \mathcal{P}$ iff $\epsilon \in L(G)$

- For a word $w = a_1 \dots a_n$: $a_1 \dots a_n \in L(A)$ iff there are $q_0, \dots, q_n \in Q$ s.t. $\delta(q_i, a_{i+1}) = q_{i+1}$ for $i < n$ and $q_n \in F$. This means that $q_0 \Rightarrow a_1 q_1 \Rightarrow \dots \Rightarrow a_1 \dots a_n q_n \Rightarrow a_1 \dots a_n$ is derivation of $a_1 \dots a_n$, which shows that $a_1 \dots a_n \in L(G)$. $\square$

(Note: Same construction works for NFA or ϵ-NFA.)

Right linear grammar to finite automaton

Given a right-linear grammar we construct a ϵ-NFA.

Encoding productions based on their form:

- $A \to aB$ are encoded directly as transitions
- $A \to \epsilon$ (ϵ-productions) define accepting states
- $A \to B$ (unit productions) correspond to ϵ-transitions

Productions with more terminals, $A \to a_1 \dots a_n B$ or $A \to a_1 \dots a_n$:

- introduce new variables $Y_1, Y_2, \dots, Y_{n+1}$
- replace with $A \to a_1 Y_2$, $Y_2 \to a_2 Y_3$, $\dots$, $Y_{n-1} \to a_{n-1} Y_n$, and finally either $Y_n \to a_n B$ or $Y_n \to a_n Y_{n+1}$, $Y_{n+1} \to \epsilon$

Similarly, $A \to a$ can be rewritten to $A \to aY$, $Y \to \epsilon$.

(Think of a state diagram but edges labelled with words, subdivide them. For edges pointing nowhere, add a new final state.)

Standardization of a right-linear grammar

Sometimes we want to get rid of unit productions too, this can be done by taking transitive closure (same as removing ϵ-transitions).

We call grammars G and G' equivalent if $L(G) = L(G')$.

Lemma

For any right-linear grammar G there exist an equivalent G' which only has productions of the form $A \to aB$ or $A \to \epsilon$.

Formalizing the previous slide, define $G' = (V', T, \mathcal{P}', S)$ where V' contains the original variables V plus all new variables used for encoding. Productions $\mathcal{P}'$ are as described.

To remove unit productions ($A \to B$), take the transitive closure $U(A) = \{B \in V \mid A \Rightarrow^* B\}$. For every production $B \to \gamma \in \mathcal{P}$ with $B \in U(A)$ add the production $A \to \gamma$ to $\mathcal{P}'$. $\square$

Formalizing the construction of an automaton

Given a right-linear grammar, first standardize it: $G = (V, T, \mathcal{P}, S)$ with productions only of the form $A \to aB$ or $A \to \epsilon$.

Define an NFA $A = (Q, \Sigma, \delta, S_0, F)$, where $Q = V$, $\Sigma = T$, $S_0 = \{S\}$, $F = \{A \mid A \to \epsilon \in \mathcal{P}\}$, and the transitions are:

$$\delta(A, a) = \{B \mid A \to aB \in \mathcal{P}\} \text{ for } A \in V, a \in T$$

To show that $L(G) = L(A)$: For the empty word, $\epsilon \in L(G)$ iff $S \to \epsilon \in \mathcal{P}$ iff $S \in F$ iff $\epsilon \in L(A)$. Otherwise, $w = a_1 \dots a_n \in L(G)$ iff there is a derivation $S \Rightarrow a_1 X_1 \Rightarrow \cdots \Rightarrow a_1 \dots a_n X_n \Rightarrow a_1 \dots a_n$.

Equivalently, in A there are states $X_0, X_1, \dots, X_n \in Q$ such that $X_0 = S \in S_0, X_n \in F$ and $X_{i+1} \in \delta(X_i, a_i)$. But this means that $a_1 \dots a_n \in L(A)$. $\square$

(Note: Easier to leave unit productions in, construct an ϵ-NFA: $\delta(A, \epsilon) = \{B \mid A \to B \in \mathcal{P}\}$.)

Linear grammars an linear languages

A context-free grammar is

- left-linear if productions are of the form $A \to B\omega$ or $A \to \omega$,
- linear if productions are of the form $A \to \omega B\omega'$ or $A \to \omega$

where $A, B \in V$ and $\omega, \omega' \in T^*$. A language is linear if it is generated by some linear grammar.

- Left-linear grammars generate regular languages. (L is regular iff L^R is, reversing bodies gives a right-linear grammar.)
- But not every linear language is regular! Example: $L = \{0^n1^n \mid n \geq 1\}$, linear rules $S \to 0S1 \mid 01$
- Observe: productions can be split to left-linear and right-linear
- Not every context-free language is linear, for example $L = \{w \in \{0,1\}^* \mid |w|_0 = |w|_1\}$. Context-free grammar later, to prove non-linearity use a version of PL for linear languages.

2.4 Context-free grammars

Example: simple expressions

Recall that in a CFG, the head always consists of a single variable.

Consider $G = (\{E, I\}, \{+, *, (,), a, b, 0, 1\}, \mathcal{P}, E)$ where

$$\begin{aligned} \mathcal{P} = \{ & E \to I, \\ & E \to E + E, \\ & E \to E + E, \\ & E \to (E), \\ & \\ & I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1 \} \end{aligned}$$

- Rules 1-4 describe expressions E.
- Rules 5-10 describe identifiers I, correspond to the regular expression $(\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b} + \mathbf{0} + \mathbf{1})^*$.

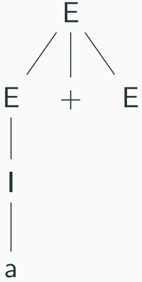
Parse trees

The tree is the data structure of choice to represent the source program in a compiler, facilitating translation into executable code.

Derivations from CFG naturally correspond to trees. Apply a production $\rightsquigarrow$ append symbols from body as children of head.

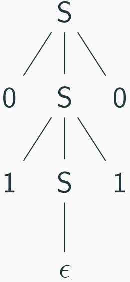
(a) Expresssions:

$E \Rightarrow^* a + E$



(b) Palindromes:

$S \Rightarrow^* 0110$



The definition

A parse tree for a CFG is a labeled ordered tree such that:

- inner nodes are labeled by variables from V
- the root is labeled by the start symbol S
- leaves have labels from $V \cup T \cup \{\epsilon\}$
- if a leaf is labeled ϵ, it must be the only child of its parent
- if an inner node is labeled A and its children $X_1, \ldots, X_k$ (ordered left-to-right), then $A \to X_1, \ldots, X_k \in P$

The yield of a parse tree is the string $\gamma \in (V \cup T)^*$ obtained by reading the labels on all leaves left-to-right.

Note: yields containing only terminals $\leftrightsquigarrow$ words from the language

Leftmost and rightmost derivations

Leftmost derivation $\Rightarrow_{lm}$, $\Rightarrow^*_{lm}$: in each step rewrite the leftmost (first) variable; rightmost $\Rightarrow_{rm}$, $\Rightarrow^*_{rm}$: rewrite the rightmost (last)

Example: Same productions for each variable but different order

- leftmost: $E \Rightarrow_{lm} E * E \Rightarrow_{lm} I * E \Rightarrow_{lm} a * E \Rightarrow_{lm} a * (E) \Rightarrow_{lm} a * (E + E) \Rightarrow_{lm} a * (I + E) \Rightarrow_{lm} a * (a + E) \Rightarrow_{lm} \Rightarrow_{lm} a * (a + I) \Rightarrow_{lm} a * (a + I0) \Rightarrow_{lm} a * (a + I00) \Rightarrow_{lm} a * (a + b00)$
- rightmost: $E \Rightarrow_{rm} E * E \Rightarrow_{rm} E * (E) \Rightarrow_{rm} E * (E + E) \Rightarrow_{rm} E * (E + I) \Rightarrow_{rm} E * (E + I0) \Rightarrow_{rm} E * (E + I00) \Rightarrow_{rm} E * (E + b00) \Rightarrow_{rm} \Rightarrow_{rm} E * (I + b00) \Rightarrow_{rm} E * (a + b00) \Rightarrow_{rm} I * (a + b00) \Rightarrow_{rm} a * (a + b00)$

Derivations and parse trees

Theorem

Given a context-free grammar $G = (V, T, P, S)$ and $\gamma \in (V \cup T)^*$, the following are equivalent:

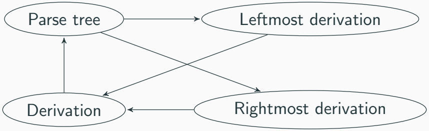
(i) $A \Rightarrow^* \gamma$

(ii) $A \Rightarrow^*_{lm} \gamma$

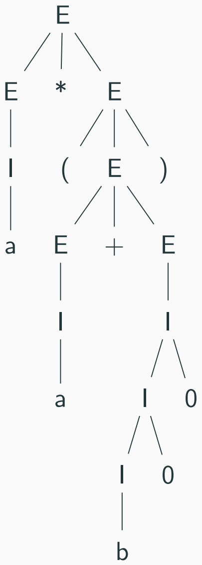
(iii) $A \Rightarrow^*_{rm} \gamma$

(iv) *There is a parse tree with root A and yield γ.*

Proof (ii)⇒(i) and (iii)⇒(i) are trivial. We will show (i)⇒(iv) and (iv)⇒(ii) [analogously (iv)⇒(iii)].



Parse tree to leftmost derivation: an example



- Root: $E \Rightarrow_{lm} E * E$
- Leftmost child of the root: $E \Rightarrow_{lm} I \Rightarrow_{lm} a$
- Rightmost child of the root:
  $E \Rightarrow_{lm} (E) \Rightarrow_{lm} (E{+}E) \Rightarrow_{lm} (I{+}E) \Rightarrow_{lm} (a{+}E)$
  $\Rightarrow_{lm} (a + I) \Rightarrow_{lm} (a + I0) \Rightarrow_{lm} (a + I00) \Rightarrow_{lm}$
  $(a + b00)$
- Leftmost integrated to root:
  $E \Rightarrow_{lm} E * E \Rightarrow_{lm} I * E \Rightarrow_{lm} a * E$
- Full derivation:
  $E \Rightarrow_{lm} E * E \Rightarrow_{lm} I * E \Rightarrow_{lm} a * E \Rightarrow_{lm}$
  $\Rightarrow_{lm} a{*}(E) \Rightarrow_{lm} a{*}(E + E) \Rightarrow_{lm} a{*}(I + E) \Rightarrow_{lm}$
  $\Rightarrow_{lm} a{*}(a{+}E) \Rightarrow_{lm} a{*}(a{+}I) \Rightarrow_{lm} a{*}(a{+}I0) \Rightarrow_{lm}$
  $\Rightarrow_{lm} a * (a + I00) \Rightarrow_{lm} a * (a + b00)$

Parse tree to leftmost derivation: the proof

Observe: If $\beta_1 \Rightarrow \beta_2 \Rightarrow \cdots \Rightarrow \beta_n$ is a derivation, then for any $\alpha, \alpha' \in (V \cup T)^*$, $\alpha\beta_1\alpha' \Rightarrow \alpha\beta_2\alpha' \Rightarrow \cdots \Rightarrow \alpha\beta_n\alpha'$ is a derivation.

Suppose we have a parse tree with root A and yield γ. Induction on the depth of the tree.

Base: depth 1, root A with children that read γ

$A \to \gamma$ is a production, thus $A \Rightarrow_{lm} \gamma$ is a one-step derivation

Induction step: depth $n > 1$, root A with children $X_1, X_2, \ldots, X_k$

- If X_i is a terminal, define $\gamma_i = X_i$
- If X_i is a variable, then by induction $X_i \Rightarrow^*_{lm} \gamma_i$.

To construct the leftmost derivation, show by induction on $i = 1, \ldots, k$ that $A \Rightarrow^*_{lm} \gamma_1\gamma_2 \ldots \gamma_i X_{i+1} X_{i+2} \ldots X_k$

Finally, when $i = k$, the result is a leftmost derivation of γ from A.

The induction within the induction

Assuming that $A \Rightarrow^*_{lm} \gamma_1\gamma_2 \dots \gamma_{i-1}X_iX_{i+1}X_{i+2}\dots X_k$, show

$$A \Rightarrow^*_{lm} \gamma_1\gamma_2 \dots \gamma_i X_{i+1}X_{i+2}\dots X_k$$

- If X_i is a terminal, do nothing, just increment i.
- If X_i is a variable, rewrite the derivation

$$X_i \Rightarrow_{lm} \alpha_1 \Rightarrow_{lm} \alpha_2 \cdots \Rightarrow_{lm} \gamma_i$$

to the following:

$$\begin{aligned} \gamma_1\gamma_2 \dots \gamma_{i-1}X_iX_{i+1}X_{i+2}\dots X_k &\Rightarrow_{lm} \\ \gamma_1\gamma_2 \dots \gamma_{i-1}\alpha_1X_{i+1}X_{i+2}\dots X_k &\Rightarrow_{lm} \\ &\vdots \\ \Rightarrow_{lm} \gamma_1\gamma_2 \dots \gamma_{i-1}\gamma_iX_{i+1}X_{i+2}\dots X_k & \end{aligned}$$

(To construct a rightmost derivation, go from $i = k$ down to 1.)

Derivation to parse tree

Suppose we have a derivation $A = \beta_1 \Rightarrow \beta_2 \Rightarrow \cdots \Rightarrow \beta_n = \gamma$.

We construct a parse tree with root A and yield γ by induction on the number of steps n in the derivation.

Base ($n = 1$)**:** A is a single-vertex parse tree

Induction step ($n > 1$) **:** We have $A \Rightarrow^* \beta_{n-1} \Rightarrow \beta_n$.

Suppose $\beta_{n-1} = \alpha C \alpha'$ and $\beta_n = \alpha \delta \alpha'$ for a production $C \to \delta$.

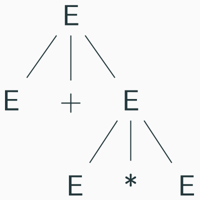
By induction, we have a parse tree with root A and yield $\alpha C \alpha'$. Find the leaf corresponding to C and append to it new leaves labelled by the symbols from δ.

This shows that (i)$\Rightarrow$(iv), and thus the theorem is proved. $\square$

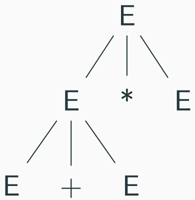
2.5 Ambiguity in grammars

Ambiguity: an example

$E \Rightarrow E + E \Rightarrow E + E * E$



$E \Rightarrow E * E \Rightarrow E + E * E$



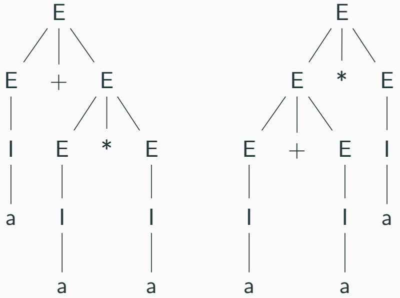
- Two different parse trees for the same expression.
- Important difference: $1 + (2 * 3) = 7$ but $(1 + 2) * 3 = 9$
- Imagine source file interpretable as two different programs.
- This grammar can be modified to remove ambiguity.
- Different derivations with the same parse tree are not an issue (e.g. left-most and right-most).

"The chef made her duck."

Amiguous context-free grammars

A context-free grammar G is ambiguous if for some $\omega \in L(G)$ there exist two different parse trees with root S and yield ω. Otherwise the grammar is unambiguous.

Example: our grammar for simple expressions is ambiguous, $\omega = a + a * a \in L(G)$ is yielded by both of the following parse trees:

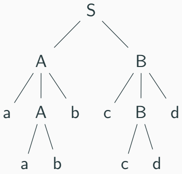


Inherent ambiguity of context-free languages

A context-free language L is unambiguous if there exists an unambiguous grammar generating it, and L is inherently ambiguous otherwise (i.e., if every CFG for L is ambiguous).

Example: $L = \{a^n b^n c^k d^k \mid n, k \geq 1\} \cup \{a^n b^k c^k d^n \mid n, k \geq 1\}$ is inherently ambiguous. Here is an ambiguous grammar with two parse trees for $\omega = aabbccdd$.

$S \to AB \mid C$
$A \to aAb \mid ab$
$B \to cBd \mid cd$
$C \to aCd \mid aDd$
$D \to bDc \mid bc$



Why inherently? Idea: any grammar will generate at least some $a^n b^n c^n d^n$ in the two different ways.

Removing ambiguity

- There is no algorithm deciding if a given CFG is ambiguous.
- There exist inherently ambiguous context-free languages (see the example above).
- There are certain techniques for removing ambiguity.

Different causes for ambiguity:

- The precedence of operators is not respected.
- A sequence of identical operators associates from left or right.
- E.g. for rules $S \rightarrow$ `if then` S `else` $S \mid$ `if then` $S \mid \epsilon$, the word `if then if then else` has two meanings: `if then (if then else)` or `if then (if then) else`

  Possible solutions:

  - syntax error (Algol 60)
  - `else` belongs to the closer `if` (rules ordered by preference)
  - parentheses, `begin`—`end`, or indentation (Python)

Enforcing precedence

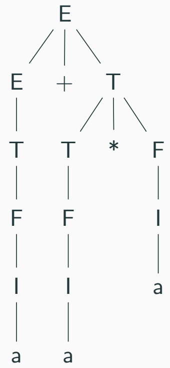
Introduce a new variable for each level of ‘binding strength’: ‘

- a factor is an expression that cannot be broken by any operator: identifiers, parenthesized expressions
- a term is an expression not broken by $+$
- an expression can be broken by either $*$ or $+$.

An unambiguous grammar:

$I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
$F \to I \mid (E)$
$T \to F \mid T * F$
$E \to T \mid E + T$

right: the only parse tree for $a + a * a$



Unambiguity and compilers

Compiling an expression (a stack for intermediate results + two registers):

(1)	$E \to E + T$	... `pop r1; pop r2; add r1,r2; push r2`
(2)	$E \to T$	
(3)	$T \to T * F$	... `pop r1; pop r2; mul r1,r2; push r2`
(4)	$T \to F$	
(5)	$F \to (E)$	
(6)	$F \to a$	... `push a`

- 'a+a*a' is obtained by applying rules 1,2,4,6,3,4,6,6
- reverse the sequence and choose only code-generating rules: 6,6,3,6,1
- now replace the rules with the corresponding code:
  `push a; push a; pop r1; pop r2; mul r1,r2; push r2; push a; pop r1; pop r2; add r1,r2; push r2`

Summary of Lecture 5

- Grammars: general, context-sensitive, context-free, right-linear (regular) – Chomsky hierarchy
- The language of a grammar, derivation
- Right-linear grammars correspond to FA (and so do left/linear)
- Linear grammars are stronger
- Context-free grammars: parse tree and its yield
- (un)ambiguous grammars, inherently ambiguous languages