NTIN071 A&G: Tutorial 12 – Intro to complexity theory

Teaching goals: The student is able to

- give formal definitions of $\mathtt{TIME}(f(n))$ and $\mathtt{SPACE}(f(n))$
- define the complexity classes $\mathtt{P}$, $\mathtt{NP}$ (both verifier and NTM-based), co-$\mathtt{NP}$
- define polynomial-time reductions, $\mathtt{NP}$-hardness, $\mathtt{NP}$-completeness
- design polynomial-time reductions between problems
- decide whether complexity classes are closed under various operations

In-class problems

Problem 1. Show that the problems CLIQUE, INDEPENDENT-SET, and VERTEX-COVER defined below are polynomial-time inter-reducible.

CLIQUE	
In:	A graph $G = (V, E)$ and an integer $k \geq 0$.
Q:	Does G contain (as a subgraph) the complete graph (clique) on at least k vertices?

INDEPENDENT-SET	
In:	A graph $G = (V, E)$ and an integer $k \geq 0$.
Q:	Does G contain an independent set of size at least k, i.e., $S \subseteq V$, $\lvert S\rvert \geq k$ with no edge connecting a pair of vertices from S?

VERTEX-COVER	
In:	A graph $G = (V, E)$ and an integer $k \geq 0$.
Q:	Does G have a vertex cover of size at most k, i.e., $S \subseteq V$, $\lvert S\rvert \leq k$ containing at least one vertex from every edge?

Problem 2. Use the well-known fact that HAMILTONIAN-CYCLE is $\mathtt{NP}$-complete to show that ORIENTED-HAMILTONIAN-CYCLE, (s, t)-HAMILTONIAN-PATH, and HAMILTONIAN-PATH are $\mathtt{NP}$-complete as well.

HAMILTONIAN-CYCLE	
In:	An (unoriented) graph $G = (V, E)$.
Q:	Does G contain a Hamiltonian cycle, i.e., a cycle containing every vertex?

ORIENTED-HAMILTONIAN-CYCLE	
In:	An oriented graph $G = (V, E)$.
Q:	Does G contain an oriented Hamiltonian cycle, i.e., an oriented cycle containing every vertex?

(s,t)-HAMILTONIAN-PATH	
In:	An (unoriented) graph $G = (V, E)$ and a pair of vertices $s, t \in V$.
Q:	Does G contain a Hamiltonian path from s to t, i.e., a path that starts in s, ends in t, and visits every vertex exactly once?

HAMILTONIAN-PATH	
In:	An (unoriented) graph $G = (V, E)$.
Q:	Does G contain a Hamiltonian path, i.e., a path that visits every vertex exactly once?

Problem 3. Show that the class P is closed under union, intersection, and complement.

Problem 4. Show that the class NP is closed under union and intersection.

Extra Practice and Thinking

Problem 5. Show that VERTEX-COVER is polynomial-time reducible to DOMINATING-SET.

DOMINATING-SET	
In:	A graph $G = (V, E)$ and an integer $k \geq 0$.
Q:	Does G contain a set of vertices $S \subseteq V$ of size at most k such that every $v \in V \setminus S$ has a neighbor in S?

Problem 6. Show that HAMILTONIAN-CYCLE is polynomial-time reducible to TRAVELING-SALESPERSON.

TRAVELING-SALESPERSON	
In:	A list of cities $C = \{c_1, \ldots, c_n\}$, distances $d(c_i, c_j) \in \mathbb{N}$ between each pair of cities, and $D \in \mathbb{N}$.
Q:	Is there a route of length at most D that visits every city exactly once and returns to the origin city?

Problem 7. Show that HAMILTONIAN-CYCLE is polynomial-time reducible to SAT.

Problem 8. Show that GRAPH-COLORING is NP-complete.

GRAPH-COLORING	
In:	A graph $G = (V, E)$ and $k \in \mathbb{N}$.
Q:	Can we color vertices of G with at most k colors so that there are no monochromatic edges?

Problem 9. Show that the class P is closed under iteration. That is, if $L \in$ P, then L^* is also in P. (Hint: Design a table-filling algorithm where $T[i, j] = 1$ iff $a_i \ldots a_j \in L^*$.)