Lecture 10 – Turing Machines

NTIN071 Automata and Grammars

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Recap of Lecture 9

- Closure properties of context-free languages (including substitution, homomorphism, inverse homomorphism)
- Also closure properties of deterministic CFLs
- Dyck languages, a characterization of context-free languages

Chapter 3: Turing Machines

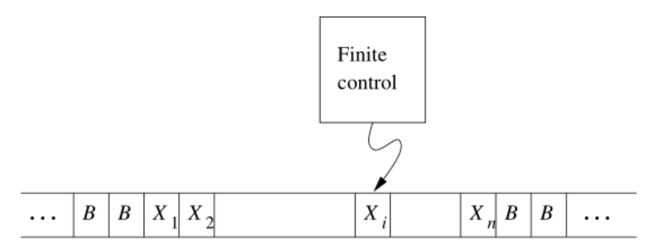
3.1 Turing machine

History and motivation

1931–1936 Gödel, Church, Turing, Kleene: formalize 'algorithms'

Turing machine: a general model of any computer

- a two-way infinite tape (sequential memory)
- a head to read/write, moves in both directions
- a control unit (finite state)



Other formalizations: RAM, λ-calculus, partially recursive functions

Computability theory: what problems can['t] computers solve?

The definition

A Turing Machine (TM) is $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ where:

- Q is a finite, nonempty set of states
- Σ is a finite, nonempty input alphabet
- Γ is a finite, nonempty tape alphabet, $\Gamma \supseteq \Sigma$, $Q \cap \Gamma = \emptyset$
- $\delta\colon (Q \setminus F) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the (partial) transition function, i.e., one instruction is $\delta(q, X) = (p, Y, D)$ where:
    - $q \in Q \setminus F$ is the current state [no transitions out of final states]
    - $X \in \Gamma$ is the tape symbol in the current cell
    - $p \in Q$ is the next state to switch to
    - $Y \in \Gamma$ is the tape symbol to rewrite X with in the current cell
    - $D \in \{L, R\}$ is the direction in which the head then moves
- $q_0 \in Q$ is the start state
- $B \in \Gamma \setminus \Sigma$ is the blank symbol, initially written in all but finitely many cells that hold the input symbols
- $F \subseteq Q$ are the final or accepting states

Describing computation: configurations

Recall computation graph: vertices=configurations, arcs=moves $\vdash$

A configuration of a TM is a finite string

$$X_1 X_2 \dots X_{i-1} q X_i X_{i+1} \dots X_n$$

- $q \in Q$ is the current state
- $X_1 \dots X_n \in \Gamma^*$ describe the contents of the relevant portion of the tape, that is, between
  - the first (leftmost) non-blank symbol or head position, and
  - the last (rightmost) non-blank symbol or head position
- the tape head is scanning the i-th symbol $X_i \in \Gamma$

Describing computation: moves

For moves of a TM M, use same notation as for PDA: $\vdash_M, \vdash_M^*, \vdash^*$

- For $\delta(q, X_i) = (p, Y, L)$:

$$X_1 X_2 \dots X_{i-1} q X_i X_{i+1} \dots X_n \vdash_M X_1 X_2 \dots X_{i-2} p X_{i-1} \mathbf{Y} X_{i+1} \dots X_n$$

- For $\delta(q, X_i) = (p, Y, R)$:

$$X_1 X_2 \dots X_{i-1} q X_i X_{i+1} \dots X_n \vdash_M X_1 X_2 \dots X_{i-1} \mathbf{Y} p X_{i+1} \dots X_n$$

And $\vdash_M^*$ is a reflexive, transitive closure of $\vdash_M$ (oriented path in the computation graph).

initial configuration: $q_0 w$ for the input word $w \in \Sigma^*$
accepting configurations: those where $q \in F$, any tape contents (i.e., in our definition, the TM doesn't need to 'clean' the tape)

The language, an example

The language recognized by a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ is:

$$L(M) = \{w \in \Sigma^* \mid q_0 w \vdash_M^* \alpha p \beta, p \in F, \alpha, \beta \in \Gamma^*\}$$

A language is recursively enumerable if it is recognized by some TM

Example

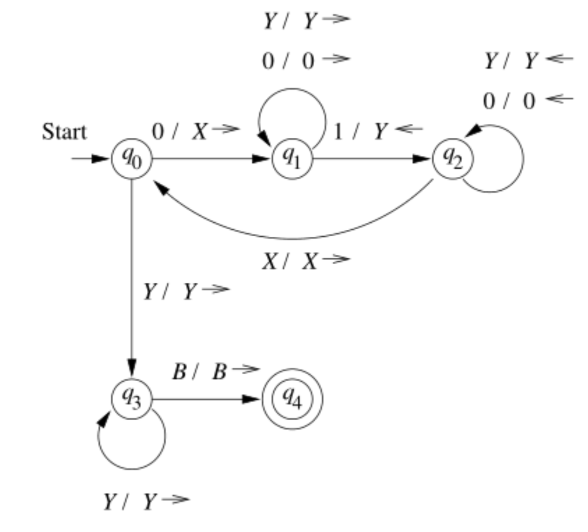
The following TM accepts the language $L = \{0^n 1^n \mid n \geq 1\}$:

$$M = (\{q_0, q_1, q_2, q_3, q_4\}, \{0, 1\}, \{0, 1, X, Y, B\}, \delta, q_0, B, \{q_4\})$$

δ	0	1	X	Y	B
q_0	(q_1, X, R)	–	–	(q_3, Y, R)	–
q_1	$(q_1, 0, R)$	(q_2, Y, L)	–	(q_1, Y, R)	–
q_2	$(q_2, 0, L)$	–	(q_0, X, R)	(q_2, Y, L)	–
q_3	–	–	–	(q_3, Y, R)	(q_4, B, R)
q_4	–	–	–	–	–

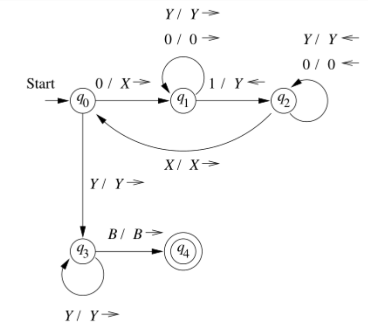
Transition diagram

nodes are states, arcs $q \rightarrow p$ are labeled by X/YD for all $\delta(q, X) = (p, Y, D)$ (use $D \in \{\leftarrow, \rightarrow\}$ instead of $\{L, R\}$)



The program explained

Recognizes $L = \{0^n 1^n \mid n > 0\}$.



On tape always $X^*0^*Y^*1^*$.

Repeatedly rewrite a 0 to X,
and the corresponding 1 to Y:

q_0: rewrite 0 to X, switch to q_1
q_1: search forward for the first 1, rewrite to Y, switch to q_2
q_2: search backward for the last X, go forward, switch to q_0

If q_0 sees 0, continue as above, if it sees Y, switch to q_3

q_3: moves to the end to check that there are no remaining 1s

- if q_3 finds B, switch to q_4, accept (accepting state)
- if q_3 finds 1, fail (no instruction, not accepting state)

Computation examples: $w = 0011$ and $w = 0010$

$$
\begin{aligned}
& q_0 0011 \vdash \\
& X q_1 011 \vdash \\
& X 0 q_1 11 \vdash \\
& X q_2 0 Y 1 \vdash \\
& q_2 X 0 Y 1 \vdash \\
& X q_0 0 Y 1 \vdash \\
& X X q_1 Y 1 \vdash \\
& X X Y q_1 1 \vdash \\
& X X q_2 Y Y \vdash \\
& X q_2 X Y Y \vdash \\
& X X q_0 Y Y \vdash \\
& X X Y q_3 Y \vdash \\
& X X Y Y q_3 B \vdash \\
& X X Y Y B q_4 B \quad \text{... accepted}
\end{aligned}
$$

$$
\begin{aligned}
& q_0 0010 \vdash \\
& X q_1 010 \vdash \\
& X 0 q_1 10 \vdash \\
& X q_2 0 Y 0 \vdash \\
& q_2 X 0 Y 0 \vdash \\
& X q_0 0 Y 0 \vdash \\
& X X q_1 Y 0 \vdash \\
& X X Y q_1 0 \vdash \\
& X X Y 0 q_1 B \quad \text{... fail (no instruction)}
\end{aligned}
$$

Recognizing regular and context-free languages

Regular languages:

- simulate a DFA, move always right, never write on the tape
- if we see B, we are at the end of input: if the DFA is in accepting state, switch to a new accepting state q_F
- (note: in a TM, the accepting state q_F cannot have outgoing transitions; in a DFA it is allowed)

Example

$L = \{a^{2n} \mid n \geq 0\}$ recognized by the following TM:
$M = (\{q_0, q_1, q_F\}, \{a\}, \{a, B\}, \delta, q_0, B, \{q_F\})$ with transitions

- $\delta(q_0, B) = (q_F, B, R)$
- $\delta(q_0, a) = (q_1, a, R)$
- $\delta(q_1, a) = (q_0, a, R)$

Context-free languages: simulate a PDA, simulate an auxiliary tape to hold the stack contents (how?? later)

Turing machines with output

Turing Machines can give output, i.e., compute a (partial) function

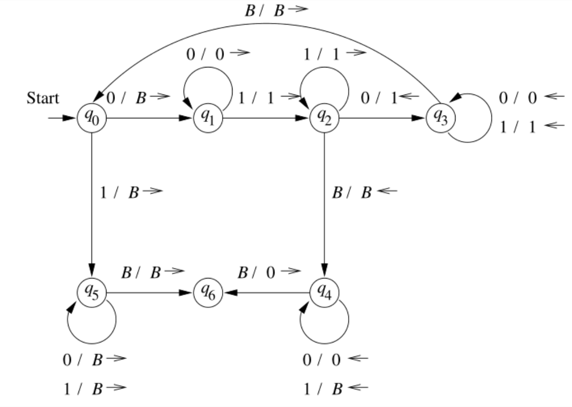
$$f_M : \Sigma^* \to \Gamma^*$$

where $f_M(w)$ is defined as follows:

- if M halts, then $f_M(w)$ equals the contents of the tape at the end of computation (everything between the first and last non-blank symbol, or $f_M(w) = \epsilon$ if the tape is all blanks)
- if M does not halt, then $f_M(w)$ is undefined

Note: the set of accepting states F is ignored, often omitted

Example: computing **monus** $m \dot{-} n = max(m - n, 0)$



m, n encoded in unary
at the start: $0^m 1 0^n$
at the end: $0^{m \dot{-} n}$
find leftmost 0, delete
search right for a 1
if found, continue

find a 0, rewrite by 1
return left
if no 0 found, either left or right:
right: replace all 1s by B
left ($m < n$): replace all 1s and 0s
by B (leave the tape blank)

Halting, recursively enumerable and recursive languages

Definition

A TM halts if it enters a state q, scanning a tape symbol X, and there is no transition in this situation, i.e., $\delta(q, X)$ is undefined.

A TM halts whenever it gets to an accepting state (no outgoing transitions allowed). In general, we cannot require that a TM always halts, even if it does not accept.

(Until a TM halts, we do not know whether it will accept or not.)

Definition

A language L is:

- recursively enumerable if it is recognized by some TM
- recursive if there exists at TM M that recognizes L and *halts on every input* $w \in \Sigma^*$; in that case, we also say M decides L

We will see that...

$$\text{context-sensitive} \subsetneq \text{recursive} \subsetneq \text{recursively enumerable} \subsetneq \text{all languages}$$

- Every context-sensitive language is recursive.
- Not all recursive languages are context sensitive.
- Every recursive language is recursively enumerable.
- Not all recursively enumerable languages are recursive.
- A language is recursively enumerable, iff it is generated by a Type 0 grammar in the Chomsky hierarchy.
- Not all languages are recursively enumerable.

3.2 Variants of TMs

Construction tricks

Construction trick: storage in the FA unit

The following TM recognizes the language $L = L(01^* + 10^*)$; it remembers 0, 1, or B in its state:

$$M = (\{q_0, q_1\} \times \{0, 1, B\}, \{0, 1\}, \{0, 1, B\}, \delta, [q_0, B], B, \{[q_1, B]\})$$

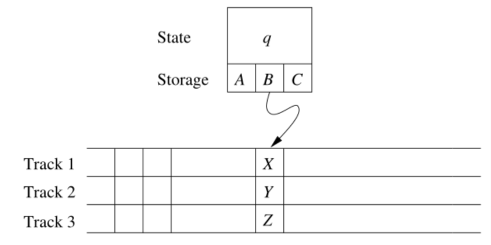
δ	0	1	B
$\rightarrow [q_0, B]$	$([q_1, 0], 0, R)$	$([q_1, 1], 1, R)$	
$[q_1, 0]$		$([q_1, 0], 1, R)$	$([q_1, B], B, R)$
$[q_1, 1]$	$([q_1, 1], 0, R)$		$([q_1, B], B, R)$
$*[q_1, B]$			

In general, we can store a finite number of variables with finitely many possible values (e.g. Boolean, input symbols, etc.): the state is a tuple, entries are values of the variables.

Construction trick: tape with multiple tracks

To split the tape into two tracks, each of which can hold a tape symbol: $\Gamma' = \Gamma \cup \{ \genfrac{}{}{0pt}{}{X}{Y} \mid X, Y \in \Gamma \}$. At the beginning, traverse the input changing a to $\genfrac{}{}{0pt}{}{B}{a}$, then return.

Or say $\Gamma = \{0, 1, B\}$ and we want to put a mark $*$ over certain digits. Then $\Gamma' = \{0, 1, B, \genfrac{}{}{0pt}{}{}{0}, \genfrac{}{}{0pt}{}{*}{0}, \genfrac{}{}{0pt}{}{}{1}, \genfrac{}{}{0pt}{}{*}{1}\}$. (We write $[X, Y]$ for $\genfrac{}{}{0pt}{}{X}{Y}$.)



NB: This is different from (but will be used to simulate!) multiple tapes with heads moving independently.

Example: $L_{wcw} = \{wcw \mid w \in \{0,1\}^+\}$

Put a mark '$*$' over the letter being checked, store it in memory. (We skip the preprocessing, assume a is already $[B, a]$.)

$M = (\{q_0, \ldots, q_9\} \times \{0, 1, B\}, \{[B, 0], [B, 1], [B, c]\}, \{B, *\} \times \{0, 1, B, c\}, \delta, [q_1, B], [B, B], \{[q_9, B]\})$ where δ is $(a, b \in \Sigma)$:

- $\delta([q_1, B], [B, a]) = ([q_2, a], [*, a], R)$ pick up the symbol a
- $\delta([q_2, a], [B, b]) = ([q_2, a], [B, b], R)$ move right, search for c
- $\delta([q_2, a], [B, c]) = ([q_3, a], [B, c], R)$ continue right, state changed
- $\delta([q_3, a], [*, b]) = ([q_3, a], [*, b], R)$ continue right
- $\delta([q_3, a], [B, a]) = ([q_4, B], [*, a], L)$ match symbols, clear memory
- $\delta([q_4, B], [*, a]) = ([q_4, B], [*, a], L)$ go left
- $\delta([q_4, B], [B, c]) = ([q_5, B], [B, c], L)$ c found, continue left
- are all symbols left and right checked? branch adequately

Example continued

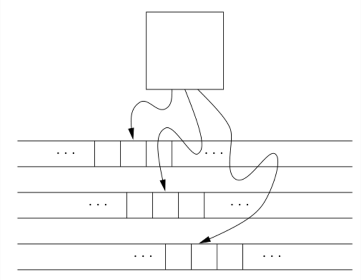
- are all symbols left and right checked? branch adequately
- $\delta([q_5, B], [B, a]) = ([q_6, B], [B, a], L)$ left symbol unchecked
- $\delta([q_6, B], [B, a]) = ([q_6, B], [B, a], L)$ proceed left
- $\delta([q_6, B], [*, a]) = ([q_1, B], [*, a], R)$ start again
- $\delta([q_5, B], [*, a]) = ([q_7, B], [*, a], R)$ symbol left from c checked, go right
- $\delta([q_7, B], [B, c]) = ([q_8, B], [B, c], R)$ proceed right
- $\delta([q_8, B], [*, a]) = ([q_8, B], [*, a], R)$ proceed right
- $\delta([q_8, B], [B, B]) = ([q_8, B], [B, B], R)$ accept

Multi-tape Turing Machines

A multi-tape TM

Initial configuration:

- input on first tape, others blank
- first head scans the first input letter
- FA unit in the initial state



One step:

- FA unit switches to the new state
- on each tape rewrite independently
- each head moves independently

Transition function: $\delta\colon (Q \setminus F) \times \Gamma^n \to Q \times \Gamma^n \times \{L, R\}^n$

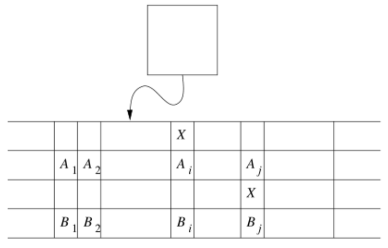
Theorem

Any language recognized by a multi-tape TM is also recognized by some (single-tape) Turing machine.

Proof: simulate using multiple tracks, mark head positions

Split the single tape into $2k$ tracks

- odd: mark i^{th} head position
- even tracks: i^{th} tape contents



To simulate one step, visit all heads and store in the FA unit:

- the simulated state
- the number of head marks to the left of us
- for every i, the symbol under i^{th} head

Then we know enough to simulate one step. We visit all heads again to rewrite and move them. □

Simulating n steps of a k-tape TM takes $O(n^2)$ moves. (One step takes $4n + 2k$, heads no further than $2n$; read, write, move marks).

Nondeterministic Turing Machines

Nondeterministic Turing Machine

Definition

A nondeterministic Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ where $Q, \Sigma, \Gamma, q_0, B, F$ are the same as for a determ. TM, and

$$\delta : (Q - F) \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

A word $w \in \Sigma^*$ is accepted by the nondeterminisic TM M iff there exists an accepting sequence of moves $q_0 w \vdash^* \alpha p \beta$, $p \in F$.

Theorem

For every nondeterministic Turing machine M_N there exist a deterministic TM M_D such that $L(M_N) = L(M_D)$.

Proof idea: Subset construction not possible, the tape is infinite. Instead, BFS of the computation graph. In k steps, at most m^k configs for $m = \max_{q \in Q, x \in \Gamma} |\delta(q, x)|$. Store on a tape & process.

Proof

Breadth-first search of all possible computations of M_N (configurations reachable from the initial configuration).

Use a two-tape TM:

- **first tape:** maintain a queue of open vertices (configurations) separated by a special symbol
- **second tape:** scratch tape for processing one configuration

To process one configuration:

- copy it to the scratch tape, dequeue (erase)
- read the state and scanned symbol
- if accepting state, accept and halt
- for each possible transition of δ_N: perform the move and enqueue the resulting config (write at the end of tape 1)

[The transition function δ_N is encoded in δ_D.]

One-way infinite tape

Theorem

Every recursively enumerable language is also recognized by a TM whose head never moves left of its initial position.

X_0	X_1	X_2	$\dots$
$*$	X_{-1}	X_{-2}	$\dots$

Proof idea: Two-track tape, lower track for negative indices. Remember in state if positive or negative. If negative, L means moving to the right and vice versa. Special cases around 0. $\square$

Exercise: Consider a variant of Turing Machine with an infinite 2D board instead of a tape, moves $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$. Show how to simulate by the standard TM.

Summary of Lecture 10

- Turing machine: two-way infinite tape, read, write, move head
- Accept iff in a final state; configurations
- TMs with output, computing a function
- Recursively enumerable vs. recursive languages (always halt).
- Construction tricks:
    - storage in state
    - multiple tracks (on a single tape)
- Variants of TMs:
    - multi-tape (independent heads),
    - nondeterministic (accept iff some choices lead to final state)