

# Lecture 9 – Closure properties of context-free languages, Dyck languages

NTIN071 Automata and Grammars

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*\* Adapted from the Czech-lecture slides by Marta Vomlelová with gratitude.  
The translation, some modifications, and all errors are mine.*

## Recap of Lecture 8

- Pushdown automata accept exactly context-free languages (constructions: CFG to PDA and PDA to CFG)
- A deterministic pushdown automaton (DPDA)
- DPDA recognize a proper subclass of context-free languages, accepts by empty stack iff prefix-free and accepts by final state (Deterministic PDA + acceptance by empty stack does not even cover regular languages!)
- Deterministic PDA have unambiguous grammars
- The landscape of languages
- Converting between representations of context-free languages
- Undecidable problems about context-free languages (preview)

## **2.12 Closure properties of context-free languages**

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# Closed under union, concatenation, iteration, reverse

## Theorem

*If  $L, L' \subseteq \Sigma^*$  are context-free, then so are  $L \cup L'$ ,  $L.L'$ ,  $L^*$ ,  $L^+$ ,  $L^R$ .*

**Proof:** Let  $G, G'$  be CFG generating  $L, L'$  such that  $V \cap V' = \emptyset$ .  
Take a new start symbol  $S_{new} \notin V \cup V'$ .

- **union**  $L \cup L'$ : add the rule  $S_{new} \rightarrow S_1 \mid S_2$
- **concatenation**  $L.L'$ : add  $S_{new} \rightarrow S_1 S_2$
- **iteration**  $L^*$ : add  $S_{new} \rightarrow SS_{new} \mid \epsilon$
- **positive iteration**  $L^+$ : add  $S_{new} \rightarrow SS_{new} \mid S$
- **reverse**  $L^R$ : reverse the bodies of all production rules  
(i.e.,  $A \rightarrow \beta$  becomes  $A \rightarrow \beta^R$ )

□

# Substitution and homomorphism

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## Recall the definitions

A (string) **substitution** is a mapping  $\sigma: \Sigma^* \rightarrow \mathcal{P}(Y^*)$  where

- $\Sigma$  and  $Y$  are finite alphabets,  $Y = \bigcup_{x \in \Sigma} Y_x$
- for each  $x \in \Sigma$ ,  $\sigma(x)$  is a language over  $Y_x$
- $\sigma(\epsilon) = \{\epsilon\}$  and  $\sigma(u.v) = \sigma(u).\sigma(v)$

For a language  $L \subseteq \Sigma^*$ ,  $\sigma(L) = \bigcup_{w \in L} \sigma(w) \subseteq Y^*$ .

A (string) **homomorphism** is defined similarly but each letter is mapped to a single word,  $h: \Sigma^* \rightarrow Y^*$  where  $h(x) \in Y_x^*$  for  $x \in \Sigma$ ,  $h(\epsilon) = \epsilon$  and  $h(u.v) = h(u).h(v)$ . Then  $h(L) = \{h(w) \mid w \in L\}$ .

The **inverse homomorphism** applied to a language  $L' \subseteq Y^*$ :

$$h^{-1}(L') = \{w \in \Sigma^* \mid h(w) \in L'\}$$

## Example: substitution

### Example

Consider  $G = (\{E\}, \{a, +, (, )\}, \{E \rightarrow E + E \mid (E) \mid a\}, E)$ . Let us have the following substitution:

- $\sigma(a) = L(G_a)$ , where

$$G_a = (\{I\}, \{a, b, 0, 1\}, \{I \rightarrow I0 \mid I1 \mid Ia \mid Ib \mid a \mid b\}, I)$$

- $\sigma(+) = \{-, *, :, \div, \text{ mod } \}$
- $\sigma(( ) = \{ \{ \}$
- $\sigma( ) = \{ \}$

Take  $(a + a) + a \in L(G)$ . Note that  $(a + a) + a \notin \sigma(L(G))$ , because  $+ \notin \sigma(+)$ . But e.g.  $(a001 - bba) * b1 \in \sigma((a + a) + a) \subseteq \sigma(L(G))$

What if we modify the definition:  $\sigma(( ) = \{ (, [ , \sigma( ) = \{ ), ] \}$ ?

## Example: homomorphism

### Example

$G = (\{E\}, \{a, +, (, )\},$   
 $\{E \rightarrow E + E \mid (E) \mid a\}, E)$

- $h(a) = \epsilon$
- $h(+) = \epsilon$
- $h(( ) = \text{left}$
- $h( ) = \text{right}$

- $h((a + a) + a) = \text{leftright},$
- $h^{-1}(\text{leftright}) \ni (a + +)a.$

### Example

$G = (\{E\}, \{a, +, (, )\},$   
 $\{E \rightarrow a + E \mid (E) \mid a\}, E)$

- $h_2(a) = a$
- $h_2(+) = +$
- $h_2(( ) = \epsilon$
- $h_2( ) = \epsilon$

Are the following regular?

- $L(G)$
- $h_2(L(G))$
- $h_2^{-1}(h(L(G)))$

Is  $h_2^{-1}(h_2(L(G))) = L(G)$ ?



# Closure under substitution and homomorphism

## Theorem

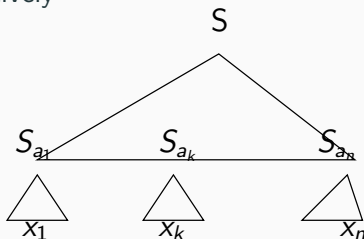
Let  $L \subseteq \Sigma^*$  be a context-free language.

- (i) If  $\sigma$  is a substitution on  $\Sigma$  such that  $\sigma(a)$  is context-free for all  $a \in \Sigma$ , then  $\sigma(L)$  is context-free.
- (ii) If  $h$  is a homomorphism on  $\Sigma$ , then  $h(L)$  is context-free.

**Idea:** replace terminals  $a \in T$  in a parse tree with corresponding parse trees for  $\sigma(a)$  or  $h(a)$ , respectively

**Proof:** (ii) follows immediately from (i), define  $\sigma(a) = \{h(a)\}$

(i) construct a context-free grammar for  $\sigma(L)$  [next slide]



## Proof: construct the grammar for $\sigma(L)$

Let us have the following context-free grammars, assume all their variable sets are (pairwise) disjoint:

- $G = (V, T, \mathcal{P}, S)$  generating  $L$
- $G_a = (V_a, T_a, \mathcal{P}_a, S_a)$  generating  $\sigma(a)$ , for  $a \in T$

Construct a grammar  $G' = (V', T', \mathcal{P}', S')$  where

- $V' = V \cup (\bigcup_{a \in T} V_a)$
- $T' = \bigcup_{a \in \Sigma} T_a$
- $\mathcal{P}' = (\bigcup_{a \in \Sigma} \mathcal{P}_a) \cup \mathcal{P}''$  where  $\mathcal{P}''$  is obtained from  $\mathcal{P}$  by replacing every terminal  $a$  by the variable  $S_a$
- $S' = S$

Clearly,  $G'$  generates the language  $\sigma(L)$ .

□

## Example

Consider the language

$$L = \{a^i b^j \mid 0 \leq i \leq j\}$$

and the following substitution:

- $\sigma(a) = L_a = \{c^i d^i \mid i \geq 0\}$
- $\sigma(b) = L_b = \{c^i \mid i \geq 0\}$

The grammars  $G$ ,  $G_a$ , and  $G_b$ :

- $S \rightarrow aSb \mid Sb \mid \epsilon$
- $S_a \rightarrow cS_a d \mid \epsilon$
- $S_b \rightarrow cS_b \mid \epsilon$

Then the grammar  $G'$  for  $\sigma(L)$  consists of the following rules:

$$S \rightarrow S_a S S_b \mid S S_b \mid \epsilon, \quad S_a \rightarrow c S_a d \mid \epsilon, \quad S_b \rightarrow c S_b \mid \epsilon$$

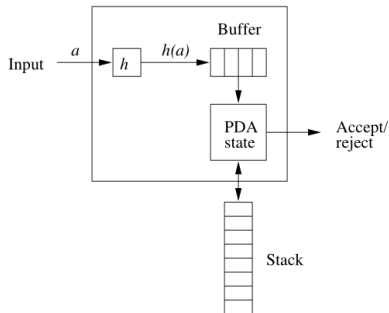
# Closure under inverse homomorphism

## Theorem

Let  $L \subseteq Y^*$  be a context-free language and  $h: \Sigma^* \rightarrow Y^*$  a homomorphism. Then  $h^{-1}(L)$  is also context-free. Moreover, if  $L$  is deterministic, then so is  $h^{-1}(L)$ .

**Idea:** simulate a PDA  $M$  for  $L$

- read a letter  $a \Rightarrow$  place  $h(a)$  into an inner buffer
- simulate  $M$ , but with input taken from the buffer
- if buffer empties, read the next letter from real input
- accept iff empty buffer and  $M$  is in accepting state



**NB:** buffer is **finite**  $\Rightarrow$  can be encoded in the states:

**state** = (state, buffer contents) 10

## Proof: the construction

Let  $M = (Q, Y, \Gamma, \delta, q_0, Z_0, F)$  (by final state). We define a PDA

$$M' = (Q', \Sigma, \Gamma, \delta', [q_0, \epsilon], Z_0, F \times \{\epsilon\})$$

where the set of states is the following ( $u$  is the buffer)

$$Q' = \{[q, u] \mid q \in Q, u \in Y^*, (\exists a \in \Sigma)(\exists v \in Y^*) h(a) = vu\}$$

and the transition function is defined as follows:

- [re]fill buffer:

$$\delta'([q, \epsilon], a, Z) = \{([q, h(a)], Z)\}$$

- read from buffer:

$$\begin{aligned} \delta'([q, u], \epsilon, Z) = & \{([p, u], \gamma) \mid (p, \gamma) \in \delta(q, \epsilon, Z)\} \\ & \cup \{([p, v], \gamma) \mid (p, \gamma) \in \delta(q, b, Z), u = bv\} \end{aligned}$$

For a DPDA  $M$ , the resulting  $M'$  is also deterministic. □

## Closure properties: it's complicated

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# CFLs not closed under intersection

## Example

$$L = \{0^n 1^n 2^n \mid n \geq 1\} = \{0^n 1^n 2^i \mid n, i \geq 1\} \cap \{0^i 1^n 2^n \mid n, i \geq 1\}$$

$L$  is not context-free, even though both operands of the intersection are context-free:

$L_1 = \{0^n 1^n 2^i \mid n, i \geq 1\}$  generated by  $G = (\{S, A, B\}, \{0, 1\}, \mathcal{P}, S)$  with production rules

$$\mathcal{P} = \{S \rightarrow AB, A \rightarrow 0A1 \mid \epsilon, B \rightarrow 2B \mid \epsilon\}$$

$L_2 = \{0^n 1^n 2^i \mid n, i \geq 1\}$  generated similarly using production rules

$$\mathcal{P} = \{S \rightarrow AB, A \rightarrow 0A \mid \epsilon, B \rightarrow 1B2 \mid \epsilon\}$$

## Simulating two PDAs in parallel

Regular languages are closed under intersection, because we can simulate two DFAs in parallel. Why not PDAs?

- the FA units can be merged (same as for DFAs)
- reading input can be merged (one automaton can wait)
- but two stacks cannot be simulated on one stack!

In fact, 'PDAs with two stacks' are equivalent to **Turing machines**, can recognize any **recursively enumerable** language  $L \in \mathcal{L}_0$ .

⋮

But what if one of the PDAs does not really use its stack?



# Intersection of a context-free and a regular language

## Theorem

*Let  $L$  be a context-free language and  $R$  a regular language. Then  $L \cap R$  is context free. Moreover, if  $L$  is deterministic, so is  $L \cap R$ .*

**Proof:** Let  $L = L(P)$  for a PDA  $P = (Q_1, \Sigma, \Gamma, \delta_1, q_1, Z_0, F_1)$  and  $R = L(A)$  for a DFA  $A = (Q_2, \Sigma, \delta_2, q_2, F_2)$ . Construct a PDA

$$M = (Q_1 \times Q_2, \Sigma, \Gamma, \delta, (q_1, q_2), Z_0, F_1 \times F_2)$$

where we have a transition  $\delta((q_1, q_2), a, X) \ni ((r_1, r_2), \gamma)$  iff either

- (i)  $a \neq \epsilon$  and  $(r_1, \gamma) \in \delta_1(q_1, a, X)$  and  $r_2 = \delta_2(q_2, a)$ , or
- (ii)  $a = \epsilon$  and  $(r_1, \gamma) \in \delta_1(q_1, \epsilon, X)$  and  $r_2 = q_2$

In (i) both automata read input, in (ii)  $P$  works on its stack while  $A$  waits. Clearly,  $L(M) = L(P) \cap L(A)$  ( $P$  and  $R$  run in parallel).  $\square$

## An application: proving non-context-freeness

### Example

$L = \{0^i 1^j 2^k 3^\ell \mid i = 0 \text{ or } j = k = \ell\}$  is not context-free.

By contradiction, assume  $L$  is context-free.

The language  $L_1 = \{01^j 2^k 3^\ell \mid i, j, k \geq 0\}$  is regular (e.g. a regular grammar  $\{S \rightarrow 0B, B \rightarrow 1B \mid C, C \rightarrow 2C \mid D, D \rightarrow 3D \mid \epsilon\}$ ).

But  $L \cap L_1 = \{01^i 2^i 3^i \mid i \geq 0\}$  is not context-free, a contradiction with the previous theorem.  $\square$

In fact,  $L$  is a **context-sensitive** language:

$S \rightarrow \epsilon \mid 0 \mid 0A \mid B_1 \mid C_1 \mid D_1$

$A \rightarrow 0 \mid 0A \mid P, P \rightarrow 1PCD \mid 1CD$

$B_1 \rightarrow 1 \mid 1B_1 \mid C_1$

$C_1 \rightarrow 2 \mid 2C_1 \mid D_1$

$D_1 \rightarrow 3 \mid 3D_1$

$DC \rightarrow CD$  rewrite as

context-sensitive rules

$DC \rightarrow XC, XC \rightarrow XY,$

$XY \rightarrow CY, CY \rightarrow CD$

$1C \rightarrow 12$

$2C \rightarrow 22$

$2D \rightarrow 23$

$3D \rightarrow 33$

# CFLs are not closed under difference nor complement

## Theorem

*The class of the context-free languages is not closed under difference, nor complement.*

**Proof:**  $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ , closure under complement would imply closure under intersection. For difference, use  $\overline{L} = \Sigma^2 - L$ .  $\square$

**NB:** PDA is non-deterministic, switching accepting/non-accepting states does not work.

## Proposition

*If  $L$  is context-free and  $R$  regular, then  $L - R$  is context-free.*

**Proof:**  $L - R = L \cap \overline{R}$ , and  $\overline{R}$  is also regular.  $\square$

# DCFLs are closed under complement

## Theorem

*The complement of a deterministic CFL is also deterministic.*

**Proof:** Idea: accept iff the original DPDA rejects.

But we need to:

- catch failure due to empty stack by a new bottom of the stack
- recognize possible  $\epsilon$ -transition cycle by a counter
- at the end of input, check if we are in an accepting state, or transitioned out of it using  $\epsilon$ -transitions; in that case, reject



# DCFLs are not closed under union nor intersection

**Recall:** intersection of a DCFL and a regular language is a DCFL

## Example

$L = \{a^i b^j c^k \mid i \neq j \text{ or } j \neq k \text{ or } i \neq k\}$ , a union of three DCFLs, is context-free but not a DCFL.

**Proof:**  $\bar{L} \cap L(a^* b^* c^*) = \{a^i b^j c^k \mid i = j = k\}$  would be a DCFL; it's not even context-free (Pumping lemma). □

**Note:** This also implies that DCFLs are not closed under intersection (de Morgan laws:  $L_1 \cup L_2 = \overline{\bar{L}_1 \cap \bar{L}_2}$ ).

# DCFLs are not closed under homomorphism

## Example

Consider languages  $L_1 = \{a^i b^j c^k \mid i \neq j\}$ ,  $L_2 = \{a^i b^j c^k \mid j \neq k\}$ ,  $L_3 = \{a^i b^j c^k \mid i \neq k\}$  which are deterministic context-free.

- The language  $0L_1 \cup 1L_2 \cup 2L_3$  is a DCFL, construct a DPDA.
- The language  $L_1 \cup L_2 \cup L_3$  is not a DCFL, otherwise also  $\overline{L_1 \cup L_2 \cup L_3} \cap L(a^* b^* c^*) = \{a^i b^j c^k \mid i = j = k\}$  would be.

But  $h(0L_1 \cup 1L_2 \cup 2L_3) = L_1 \cup L_2 \cup L_3$  for the homomorphism:

- $h(0) = \epsilon$ ,  $h(1) = \epsilon$ ,  $h(2) = \epsilon$ ,
- $h(s) = s$  for all other symbols.

**But recall:** DCFLs are closed under inverse homomorphism.

## Closure properties: summary

language	regular (RL)	context-free	deterministic CFL
union	YES	YES	NO
intersection	YES	NO	NO
$\cap$ with RL	YES	YES	YES
complement	YES	NO	YES
homomorphism	YES	YES	NO
inverse hom.	YES	YES	YES

## 2.13 Dyck languages

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## Definition

The **Dyck language**  $D_n$  is defined over  $\Sigma_n = \{a_1, a_1^|, \dots, a_n, a_n^|\}$  using the context-free grammar with the following rules:

$$S \rightarrow \epsilon \mid SS \mid a_1Sa_1^| \mid \dots \mid a_nSa_n^|$$

- the Dyck language  $D_n$  captures correctly parenthesized expressions with  $n$  types of parentheses
- we use it to describe computations of an arbitrary PDA
- to show that any context-free language can be expressed as:

$$L = h(D_n \cap R)$$

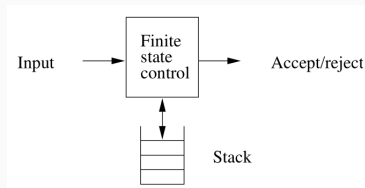
- the regular language  $R$  describes all computation steps
- the Dyck language selects only valid computations
- the homomorphism  $h$  cleans auxiliary symbols

# A characterization of context-free languages

If we only push to the stack, **never pop**: it suffices to remember the top symbol; thus we only need **finite memory**, i.e. a FA

If we also need to pop:

- finite memory is not enough
- but memory access is not arbitrary, a stack is LIFO (last in–first out)



Expand the computation with a stack to a linear structure:

$X$  symbol pushed

$X^{-1}$  symbol popped

Pushing and popping form a pair that behaves like parentheses:

$$\underbrace{ZZ^{-1}} \underbrace{B \underbrace{AA^{-1} CC^{-1}} B^{-1}}$$

# The theorem

## Theorem

*For any context-free language  $L$  there exist a regular language  $R$ , a Dyck language  $D$  and a homomorphism  $h$  s.t.  $L = h(D \cap R)$ .*

**Proof:** Start with a PDA recognizing  $L$ , accepting by empty stack. First, convert to instructions of the form:

$$\delta(q, a, Z) \in (p, w), |w| \leq 2$$

(Instructions pushing more symbols can be split using new states).

Let  $R^|$  consist of all expressions of the form

$$q^{-1}aa^{-1}Z^{-1}BAp$$

for instruction  $\delta(q, a, Z) \ni (p, AB)$ , and similarly for instructions  $\delta(q, a, Z) \in (p, A)$  and  $\delta(q, a, Z) \in (p, \epsilon)$ . (For  $a = \epsilon$  omit  $aa^{-1}$ .)

Now define the regular language as  $R = Z_0q_0(R^|)^*Q^{-1}$ .

## Proof cont'd

The Dyck language  $D$  is over the alphabet

$$\Sigma \cup \Sigma^{-1} \cup Q \cup Q^{-1} \cup \Gamma \cup \Gamma^{-1}$$

The language  $D \cap R = D \cap Z_0 q_0 (R^I)^* Q^{-1}$  describes valid computations, e.g.

$$\underbrace{Z_0 \overbrace{q_0 q_0^{-1}} aa^{-1} Z_0^{-1}} B \underbrace{A \overbrace{pp^{-1}} bb^{-1} A^{-1} \overbrace{qq^{-1}} cc^{-1} B^{-1} \overbrace{rr^{-1}}}$$

The homomorphism  $h$  selects the input word being read:

$$h(a) = a \quad \text{for input symbols } a \in \Sigma$$

$$h(y) = \epsilon \quad \text{for all other symbols}$$



## Summary of Lecture 9

- Closure properties of context-free languages (including substitution, homomorphism, inverse homomorphism)
- Also closure properties of deterministic CFLs
- Dyck languages, a characterization of context-free languages