Lecture 8 – Equivalence of PDA and CFG, Deterministic PDA

NTIN071 Automata and Grammars

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Recap of Lecture 7

- Testing membership in a context-free language: the Cocke-Younger-Kasami algorithm
- Testing emptiness and finiteness of a context-free language
- Pushdown automaton: extend an ϵ-NFA with a stack memory (potentially infinite), pop the top symbol, decide based on (q, a, X), can push a finite string of stack symbols
- Acceptance by final state $L(P)$ and by empty stack $N(P)$, conversion between the two options

2.10 Equivalence of PDA and context-free grammars

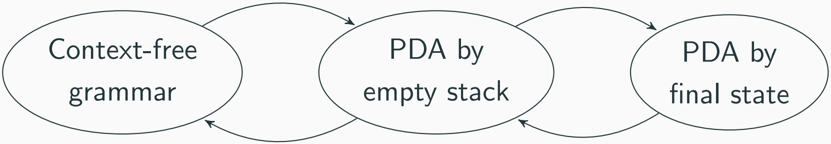
Theorem

The following statements about $L \subset \Sigma^*$ are equivalent:

(i) *There exists a context-free grammar such that $L(G) = L$.*

(ii) *There exists a PDA such that $L(P) = L$.*

(iii) *There exists a PDA such that $N(P) = L$.*



We have already shown $(ii) \Leftrightarrow (iii)$. To prove equivalence with a context-free grammar, we use acceptance by empty stack.

Context-free grammar to pushdown automaton

CFG to PDA

The construction

Given $G = (V, T, \mathcal{P}, S)$, construct $P = (\{q\}, T, V \cup T, \delta, q, S)$:

(1) for each $A \in V$, $\delta(q, \epsilon, A) = \{(q, \beta) \mid A \to \beta \in \mathcal{P}\}$

[apply rule]

(2) for each $a \in T$, $\delta(q, a, a) = \{(q, \epsilon)\}$

[match terminal]

How it works:

- a leftmost derivation is simulated by the PDA
- current sentential form = part of input read + stack contents
- see a variable: apply rule, a terminal: read & pop from stack

An example

Example

$I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1,$
$E \to I \mid E * E \mid E + E \mid (E)$

$\Sigma = \{a, b, 0, 1, (,), +, *\}$, $\Gamma = \Sigma \cup \{I, E\}$, δ is defined as follows:

- $\delta(q, \epsilon, I) = \{(q, a), (q, b), (q, Ia), (q, Ib), (q, I0), (q, I1)\}$
- $\delta(q, \epsilon, E) = \{(q, I), (q, E * E), (q, E + E), (q, (E))\}$
- $\delta(q, s, s) = \{(q, \epsilon)\}$ for all $s \in \Sigma$ (e.g. $\delta(q, +, +) = \{(q, \epsilon)\}$)
- $\delta(q, x)$ is empty otherwise

Leftmost derivation: $E \Rightarrow E * E \Rightarrow I * E \Rightarrow a * E \Rightarrow a * I \Rightarrow a * b$

The sequence of configurations:

$(q, a * b, E) \vdash (q, a * b, E * E) \vdash (q, a * b, I * E) \vdash (q, a * b, a * E)$
$\vdash (q, *b, *E) \vdash (q, b, E) \vdash (q, b, I) \vdash (q, b, b) \vdash (q, \epsilon, \epsilon)$

Start with a leftmost derivation $S = \gamma_1 \Rightarrow_{lm} \ldots \Rightarrow_{lm} \gamma_n = w$.

Prove by induction on i that $(q, w, S) \vdash_P^* (q, v_i, \alpha_i)$, where $\gamma_i = u_i\alpha_i$ is the i-th sentential form and $u_i v_i = w$.

If γ_i contains only terminals, set $\gamma_i = w = u_i, v_i = \epsilon = \alpha_i$. Otherwise, write $\gamma_i = u_i A\alpha_i$, where $u_i \in T^*$ and $A \in V$ is the leftmost variable.

By induction we have $(q, w, S) \vdash_P^* (q, v_i, A\alpha_i)$, $w = u_i v_i$.

For the step $\gamma_i \Rightarrow_{lm} \gamma_{i+1}$ we used some rule $A \to \beta \in P$. The PDA replaces A on the stack with β, moves to configuration $(q, v_i, \beta\alpha_i)$.

We pop all terminals $v \in \Sigma^*$ from the beginning of $\beta\alpha$ (matching them with the input): $v_i = vv_{i+1}$ and $\beta\alpha = v\alpha_{i+1}$

We got to $(q, v_{i+1}, \alpha_{i+1})$, corresponds to the sentential form γ_{i+1}.

Proof that $N(P) = L(G)$ **(ii)** $w \in N(P) \Rightarrow w \in L(G)$

Prove that if $(q, u, X) \vdash_P^* (q, \epsilon, \epsilon)$, then $X \Rightarrow_G^* u$. By induction on the number of moves. **Basis** $n = 1$ move:

- $X = a \in \Sigma$: $\delta(q, a, a) \ni (q, \epsilon)$, $u = a$, 0-step derivation
- $X = A \in \Gamma$: $\delta(q, \epsilon, A) \ni (q, \epsilon)$ coming from $A \to \epsilon \in \mathcal{P}$, $u = \epsilon$

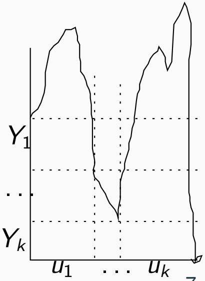
Induction step $n > 1$ moves: if the first move is [match terminal], don't extend the derivation, if it is [apply rule]: A on top of stack was replaced by $\beta = Y_1 Y_2 \ldots Y_k$, for a rule $A \to \beta \in \mathcal{P}$.

Split $u = u_1 \ldots u_k$ s.t. while popping Y_i we read u_i, i.e. $(q, u_i u_{i+1} \ldots u_k, Y_i) \vdash^* (q, u_{i+1} \ldots u_k, \epsilon)$

Thus also $(q, u_i, Y_i) \vdash^* (q, \epsilon, \epsilon)$, by induction assumption we get $Y_i \Rightarrow^* u_i$. Together:

$$A \Rightarrow Y_1 Y_2 \ldots Y_k \Rightarrow^* u_1 Y_2 \ldots Y_k \Rightarrow^* \ldots \Rightarrow^* u_1 u_2 \ldots u_k$$

$\square$

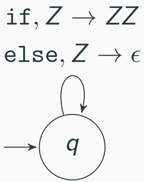


Pushdown automaton to context-free grammar

An example

Given $P = (\{q\}, \{\texttt{if}, \texttt{else}\}, \{Z\}, \delta, q, Z)$

$$\delta(q, \texttt{if}, Z) = \{(q, ZZ)\}$$
$$\delta(q, \texttt{else}, Z) = \{(q, \epsilon)\}$$



Construct $G = (V, \{\texttt{if}, \texttt{else}\}, \mathcal{P}, S)$

- variables: $V = \{S, [qZq]\}$
- production rules:
  - $S \to [qZq]$
  - $[qZq] \to \texttt{else}$
  - $[qZq] \to \texttt{if}[qZq][qZq]$

In this example, S and $[qZq]$ generate the same words, so we can simplify: $G = (\{S\}, \{\texttt{if}, \texttt{else}\}, \{S \to \texttt{if}SS \mid \texttt{else}\}, S)$

PDA to CFG: the construction

- key event: pop a symbol X, while changing from state q to r
- variables: $[qXr]$ for $q, r \in Q$ and $X \in \Gamma$, plus a new variable S

$$L([qXr]) = \{w \in \Sigma^* \mid (q, w, X) \vdash_P^* (r, \lambda, \lambda)\}$$

- S to choose (guess) in which state the stack is emptied

The construction

Given $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$, construct $G = (V, T, \mathcal{P}, S)$ where $V = \{S\} \cup \{[pXq] \mid p, q \in Q, X \in \Gamma\}$ and the productions are:

(i) for every state $p \in Q$ add $S \to [q_0 X p]$

(ii) for every transition $(p, Y_1 Y_2 \ldots Y_k) \in \delta(q, a, X)$ (incl. $a = \epsilon$) and all k-tuples of states $p_1, \ldots, p_{k-1}, p_k \in Q$ add

$$[qXp_k] \to a[pY_1p_1][p_1Y_2p_2]\ldots[p_{k-1}Y_kp_k]$$

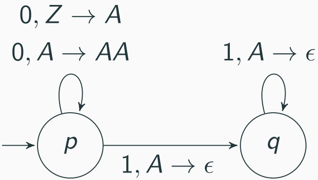
In particular, for $(p, \epsilon) \in \delta(q, a, X)$ (i.e., $k = 0$) add $[qXp] \to a$.

Another example: $\{0^n1^n \mid n > 0\}$

δ	Productions	
	$S \to [pZp] \mid [pZq]$	(1)
$\delta(p, 0, Z) \ni (p, A)$	$[pZp] \to 0[pAp]$	(2)
	$[pZq] \to 0[pAq]$	(3)
$\delta(p, 0, A) \ni (p, AA)$	$[pAp] \to 0[pAp][pAp]$	(4)
	$[pAp] \to 0[pAq][qAp]$	(5)
	$[pAq] \to 0[pAp][pAq]$	(6)
	$[pAq] \to 0[pAq][qAq]$	(7)
$\delta(p, 1, A) \ni (q, \epsilon)$	$[pAq] \to 1$	(8)
$\delta(q, 1, A) \ni (q, \epsilon)$	$[qAq] \to 1$	(9)

Derivation of 0011:

$$S \Rightarrow^{(1)} [pZq] \Rightarrow^{(3)} 0[pAq]$$
$$\Rightarrow^{(7)} 00[pAq][qAq]$$
$$\Rightarrow^{(8)} 001[qAq] \Rightarrow^{(9)} 0011$$

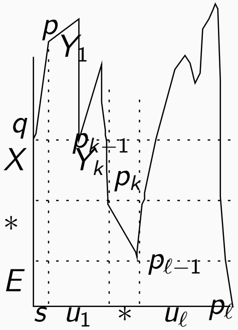


Sketch of proof that $L(G) = N(P)$

It suffices to show that:

$$[qXp] \Rightarrow^* w \quad \text{iff} \quad (q, w, X) \vdash^* (p, \epsilon, \epsilon)$$

In both directions, the proof is done by induction (number of moves/steps).



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2.11 Deterministic pushdown automata

Definition (Deterministic PDA)

A pushdown automaton $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is deterministic (a DPDA) iff both of the following hold:

- The set of possible transitions $\delta(q, a, X)$ is at most one-element for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$.
- If $\delta(q, a, X) \neq \emptyset$ for some $a \in \Sigma$, then $\delta(q, \epsilon, X) = \emptyset$.

TODO

Summary of Lecture 8

- Pushdown automata accept exactly context-free languages (constructions: CFG to PDA and PDA to CFG)