Lecture 9 – Closure properties of context-free languages, Dyck languages

NTIN071 Automata and Grammars

Jakub Bulín (KTIML MFF UK)

Spring 2025

** Adapted from the Czech-lecture slides by Marta Vomlelová with gratitude. The translation, some modifications, and all errors are mine.*

Recap of Lecture 8

- Pushdown automata accept exactly context-free languages (constructions: CFG to PDA and PDA to CFG)
- A deterministic pushdown automaton (DPDA)
- DPDA recognize a proper subclass of context-free languages, accepts by empty stack iff prefix-free and accepts by final state (Deterministic PDA + acceptance by empty stack does not even cover regular languages!)
- Deterministic PDA have unambiguous grammars
- The landscape of languages
- Converting between representations of context-free languages
- Undecidable problems about context-free languages (preview)

2.12 Closure properties of context-free languages

Closed under union, concatenation, iteration, reverse

Theorem

If $L, L' \subseteq \Sigma^*$ are context-free, then so are $L \cup L'$, $L.L'$, L^*, L^+, L^R.

Proof: Let G, G' be CFG generating L, L' such that $V \cap V' = \emptyset$. Take a new start symbol $S_{new} \notin V \cup V'$.

- union $L \cup L'$: add the rule $S_{new} \to S_1 \mid S_2$
- concatenation $L.L'$: add $S_{new} \to S_1 S_2$
- iteration L^*: add $S_{new} \to S S_{new} \mid \epsilon$
- positive iteration L^+: add $S_{new} \to S S_{new} \mid S$
- reverse L^R: reverse the bodies of all production rules (i.e., $A \to \beta$ becomes $A \to \beta^R$) $\square$

Substitution and homomorphism

Recall the definitions

A (string) substitution is a mapping $\sigma\colon \Sigma^* \to \mathcal{P}(Y^*)$ where

- Σ and Y are finite alphabets, $Y = \bigcup_{x\in\Sigma} Y_x$
- for each $x \in \Sigma$, $\sigma(x)$ is a language over Y_x
- $\sigma(\epsilon) = \{\epsilon\}$ and $\sigma(u.v) = \sigma(u).\sigma(v)$

For a language $L \subseteq \Sigma^*$, $\sigma(L) = \bigcup_{w\in L} \sigma(w) \subseteq Y^*$.

A (string) homomorphism is defined similarly but each letter is mapped to a single word, $h\colon \Sigma^* \to Y^*$ where $h(x) \in Y_x^*$ for $x \in \Sigma$, $h(\epsilon) = \epsilon$ and $h(u.v) = h(u).h(v)$. Then $h(L) = \{h(w) \mid w \in L\}$.

The inverse homomorphism applied to a language $L' \subseteq Y^*$:

$$h^{-1}(L') = \{w \in \Sigma^* \mid h(w) \in L'\}$$

Example: substitution

Example

Consider $G = (\{E\}, \{a, +, (,)\}, \{E \to E + E \mid (E) \mid a\}, E)$. Let us have the following substitution:

- $\sigma(a) = L(G_a)$, where

$$G_a = (\{I\}, \{a, b, 0, 1\}, \{I \to I0 \mid I1 \mid Ia \mid Ib \mid a \mid b\}, I)$$

- $\sigma(+) = \{-, *, :, \div, \mod\}$
- $\sigma(() = \{(\}$
- $\sigma()) = \{)\}$

Take $(a + a) + a \in L(G)$. Note that $(a + a) + a \notin \sigma(L(G))$, because $+ \notin \sigma(+)$. But e.g. $(a001 - bba) * b1 \in \sigma((a + a) + a) \subseteq \sigma(L(G))$

What if we modify the definition: $\sigma(() = \{(, [\}$, $\sigma()) = \{),]\}$?

Example: homomorphism

Example

$G = (\{E\}, \{a, +, (,)\},$
$\{E \to E + E \mid (E) \mid a\}, E)$

- $h(a) = \epsilon$
- $h(+) = \epsilon$
- $h(() = \mathit{left}$
- $h()) = \mathit{right}$

- $h((a + a) + a) = \mathit{leftright}$,
- $h^{-1}(\mathit{leftright}) \ni (a + +)a$.

Example

$G = (\{E\}, \{a, +, (,)\},$
$\{E \to E + E \mid (E) \mid a\}, E)$

- $h_2(a) = a$
- $h_2(+) = +$
- $h_2(() = \epsilon$
- $h_2()) = \epsilon$

Are the following regular?

- $L(G)$
- $h_2(L(G))$
- $h_2^{-1}(h_2(L(G)))$

Is $h_2^{-1}(h_2(L(G))) = L(G)$?

Closure under substitution and homomorphism

Theorem

Let $L \subseteq \Sigma^*$ be a context-free language.

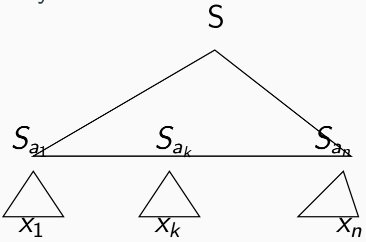
(i) *If σ is a substitution on Σ such that $\sigma(a)$ is context-free for all $a \in \Sigma$, then $\sigma(L)$ is context-free.*

(ii) *If h is a homomorphism on Σ, then $h(L)$ is context-free.*

Idea: replace terminals $a \in T$ in a parse tree with corresponding parse trees for $\sigma(a)$ or $h(a)$, respectively

Proof: (ii) follows immediately from (i), define $\sigma(a) = \{h(a)\}$

(i) construct a context-free grammar for $\sigma(L)$ [next slide]



Proof: construct the grammar for $\sigma(L)$

Let us have the following context-free grammars, assume all their variable sets are (pairwise) disjoint:

- $G = (V, T, \mathcal{P}, S)$ generating L
- $G_a = (V_a, T_a, \mathcal{P}_a, S_a)$ generating $\sigma(a)$, for $a \in T$

Construct a grammar $G' = (V', T', \mathcal{P}', S')$ where

- $V' = V \cup (\bigcup_{a \in T} V_a)$
- $T' = \bigcup_{a \in \Sigma} T_a$
- $\mathcal{P}' = (\bigcup_{a \in \Sigma} P_a) \cup \mathcal{P}''$ where $\mathcal{P}''$ is obtained from $\mathcal{P}$ by replacing every terminal a by the variable S_a
- $S' = S$

Clearly, G' generates the language $\sigma(L)$. $\square$

Example

Consider the language

$$L = \{a^i b^j \mid 0 \leq i \leq j\}$$

and the following substitution:

- $\sigma(a) = L_a = \{c^i d^i \mid i \geq 0\}$
- $\sigma(b) = L_b = \{c^i \mid i \geq 0\}$

The grammars G, G_a, and G_b:

- $S \to aSb \mid Sb \mid \epsilon$
- $S_a \to cS_ad \mid \epsilon$
- $S_b \to cS_b \mid \epsilon$

Then the grammar G' for $\sigma(L)$ consists of the following rules:

$$S \to S_aSS_b \mid SS_b \mid \epsilon, \qquad S_a \to cS_ad \mid \epsilon, \qquad S_b \to cS_b \mid \epsilon$$

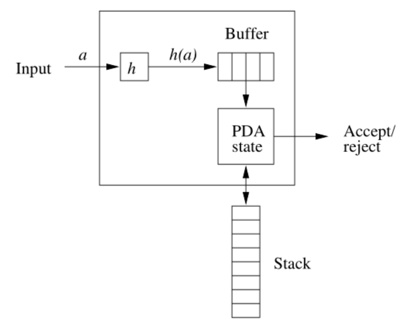
Closure under inverse homomorphism

Theorem

Let $L \subseteq Y^*$ be a context-free language and $h\colon \Sigma^* \to Y^*$ a homomorphism. Then $h^{-1}(L)$ is also context-free. Moreover, if L is deterministic, then so is $h^{-1}(L)$.

Idea: simulate a PDA M for L

- read a letter $a \Rightarrow$ place $h(a)$ into an inner buffer
- simulate M, but with input taken from the buffer
- if buffer empties, read the next letter from real input
- accept iff empty buffer and M is in accepting state



NB: buffer is finite $\Rightarrow$ can be encoded in the states:
state = (state, buffer contents)

Proof: the construction

Let $M = (Q, Y, \Gamma, \delta, q_0, Z_0, F)$ (by final state). We define a PDA

$$M' = (Q', \Sigma, \Gamma, \delta', [q_0, \epsilon], Z_0, F \times \{\epsilon\})$$

where the set of states is the following (u is the buffer)

$$Q' = \{[q, u] \mid q \in Q, u \in Y^*, (\exists a \in \Sigma)(\exists v \in Y^*)\ h(a) = vu\}$$

and the transition function is defined as follows:

- [re]fill buffer:

$$\delta'([q, \epsilon], a, Z) = \{([q, h(a)], Z)\}$$

- read from buffer:

$$\begin{aligned}\delta'([q, u], \epsilon, Z) = &\{([p, u], \gamma) \mid (p, \gamma) \in \delta(q, \epsilon, Z)\} \\ &\cup \{([p, v], \gamma) \mid (p, \gamma) \in \delta(q, b, Z), u = bv\}\end{aligned}$$

For a DPDA M, the resulting M' is also deterministic. $\square$

Closure properties: it’s complicated

CFLs not closed under intersection

Example

$L = \{0^n1^n2^n \mid n \geq 1\} = \{0^n1^n2^i \mid n, i \geq 1\} \cap \{0^i1^n2^n \mid n, i \geq 1\}$

L is not context-free, even though both operands of the intersection are context-free:

$L_1 = \{0^n1^n2^i \mid n, i \geq 1\}$ generated by $G = (\{S, A, B\}, \{0, 1\}, \mathcal{P}, S)$ with production rules

$$\mathcal{P} = \{S \to AB, A \to 0A1 \mid \epsilon, B \to 2B \mid \epsilon\}$$

$L_2 = \{0^n1^n2^i \mid n, i \geq 1\}$ generated similarly using production rules

$$\mathcal{P} = \{S \to AB, A \to 0A \mid \epsilon, B \to 1B2 \mid \epsilon\}$$

Simulating two PDAs in parallel

Regular languages are closed under intersection, because we can simulate two DFAs in parallel. Why not PDAs?

- the FA units can be merged (same as for DFAs)
- reading input can be merged (one automaton can wait)
- but two stacks cannot be simulated on one stack!

In fact, 'PDAs with two stacks' are equivalent to Turing machines, can recognize any recursively enumerable language $L \in \mathcal{L}_0$.

$\vdots$

But what if one of the PDAs does not really use its stack?

Intersection of a context-free and a regular language

Theorem

Let L be a context-free language and R a regular language. Then $L \cap R$ is context free. Moreover, if L is deterministic, so is $L \cap R$.

Proof: Let $L = L(P)$ for a PDA $P = (Q_1, \Sigma, \Gamma, \delta_1, q_1, Z_0, F_1)$ and $R = L(A)$ for a DFA $A = (Q_2, \Sigma, \delta_2, q_2, F_2)$. Construct a PDA

$$M = (Q_1 \times Q_2, \Sigma, \Gamma, \delta, (q_1, q_2), Z_0, F_1 \times F_2)$$

where we have a transition $\delta((p_1, p_2), a, X) \ni ((r_1, r_2), \gamma)$ iff either

(i) $a \neq \epsilon$ and $(r_1, \gamma) \in \delta_1(p_1, a, X)$ and $r_2 = \delta_2(p_2, a)$, or
(ii) $a = \epsilon$ and $(r_1, \gamma) \in \delta_1(p_1, \epsilon, X)$ and $r_2 = p_2$

In (i) both automata read input, in (ii) P works on its stack while A waits. Clearly, $L(M) = L(P) \cap L(A)$ (P and R run in parallel). $\square$

An application: proving non-context-freeness

Example

$L = \{0^i 1^j 2^k 3^\ell \mid i = 0 \text{ or } j = k = \ell\}$ is not context-free.

By contradiction, assume L is context-free.

The language $L_1 = \{01^j 2^k 3^\ell \mid i, j, k \geq 0\}$ is regular (e.g. a regular grammar $\{S \to 0B, B \to 1B \mid C, C \to 2C \mid D, D \to 3D \mid \epsilon\}$.

But $L \cap L_1 = \{01^i 2^i 3^i \mid i \geq 0\}$ is not context-free, a contradiction with the previous theorem. $\square$

In fact, L is a context-sensitive language:

$S \to \epsilon \mid 0 \mid 0A \mid B_1 \mid C_1 \mid D_1$
$A \to 0 \mid 0A \mid P,\ P \to 1PCD \mid 1CD$
$B_1 \to 1 \mid 1B_1 \mid C_1$
$C_1 \to 2 \mid 2C_1 \mid D_1$
$D_1 \to 3 \mid 3D_1$

$DC \to CD$ rewrite as context-sensitive rules
$DC \to XC,\ XC \to XY,$
$XY \to CY,\ CY \to CD$

$1C \to 12$
$2C \to 22$
$2D \to 23$
$3D \to 33$

CFLs are not closed under difference nor complement

Theorem

The class of the context-free languages is not closed under difference, nor complement.

Proof: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$, closure under complement would imply closure under intersection. For difference, use $\overline{L} = \Sigma^* - L$. $\square$

NB: PDA is non-deterministic, switching accepting/non-accepting states does not work.

Proposition

If L is context-free and R regular, then $L - R$ is context-free.

Proof: $L - R = L \cap \overline{R}$, and $\overline{R}$ is also regular. $\square$

DCFLs are closed under complement

Theorem

The complement of a deterministic CFL is also deterministic.

Proof: Idea: accept iff the original DPDA rejects.

But we need to:

- catch failure due to empty stack by a new bottom of the stack
- recognize possible ϵ-transition cycle by a counter
- at the end of input, check if we are in an accepting state, or transitioned out of it using ϵ-transitions; in that case, reject

□

DCFLs are not closed under union nor intersection

Recall: intersection of a DCFL and a regular language is a DCFL

Example

$L = \{a^i b^j c^k | i \neq j \text{ or } j \neq k \text{ or } i \neq k\}$, a union of three DCFLs, is context-free but not a DCFL.

Proof: $\overline{L} \cap L(a^* b^* c^*) = \{a^i b^j c^k \mid i = j = k\}$ would be a DCFL; it's not even context-free (Pumping lemma). □

Note: This also implies that DCFLs are not closed under intersection (de Morgan laws: $L_1 \cup L_2 = \overline{\overline{L_1} \cap \overline{L_2}}$).

DCFLs are not closed under homomorphism

Example

Consider languages $L_1 = \{a^i b^j c^k \mid i \neq j\}$, $L_2 = \{a^i b^j c^k \mid j \neq k\}$, $L_3 = \{a^i b^j c^k \mid i \neq k\}$ which are deterministic context-free.

- The language $0L_1 \cup 1L_2 \cup 2L_3$ is a DCFL, construct a DPDA.
- The language $L_1 \cup L_2 \cup L_3$ is not a DCFL, otherwise also $\overline{L_1 \cup L_2 \cup L_3} \cap L(a^* b^* c^*) = \{a^i b^j c^k \mid i = j = k\}$ would be.

But $h(0L_1 \cup 1L_2 \cup 2L_3) = L_1 \cup L_2 \cup L_3$ for the homomorphism:

- $h(0) = \epsilon$, $h(1) = \epsilon$, $h(2) = \epsilon$,
- $h(s) = s$ for all other symbols.

But recall: DCFLs are closed under inverse homomorphism.

Closure properties: summary

language	regular (RL)	context–free	deterministic CFL
union	YES	YES	NO
intersection	YES	NO	NO
$\cap$ with RL	YES	YES	YES
complement	YES	NO	YES
homomorphism	YES	YES	NO
inverse hom.	YES	YES	YES

2.13 Dyck languages

Dyck languages

Definition

The Dyck language D_n is defined over $\Sigma_n = \{a_1, a_1^|, \ldots, a_n, a_n^|\}$ using the context-free grammar with the following rules:

$$S \to \epsilon \mid SS \mid a_1 S a_1^| \mid \ldots \mid a_n S a_n^|$$

- the Dyck language D_n captures correctly parenthesized expressions with n types of parentheses
- we use it to describe computations of an arbitrary PDA
- to show that any context-free language can be expressed as:

$$L = h(D_n \cap R)$$

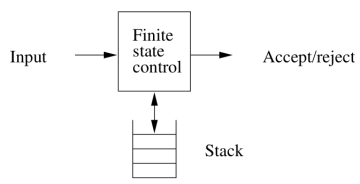
  - the regular language R describes all computation steps
  - the Dyck language selects only valid computations
  - the homomorphism h cleans auxiliary symbols

A characterization of context-free languages

If we only push to the stack, never pop: it suffices to remember the top symbol; thus we only need finite memory, i.e. a FA

If we also need to pop:

- finite memory is not enough
- but memory access is not arbitrary, a stack is LIFO (last in–first out)



Expand the computation with a stack to a linear structure:

X symbol pushed
X^{-1} symbol popped

Pushing and popping form a pair that behaves like parentheses:

$$\underbrace{ZZ^{-1}}\underbrace{B\,\underbrace{AA^{-1}}\,\underbrace{CC^{-1}}\,B^{-1}}$$

The theorem

Theorem

For any context-free language L there exist a regular language R, a Dyck language D and a homomorphism h s.t. $L = h(D \cap R)$.

Proof: Start with a PDA recognizing L, accepting by empty stack. First, convert to instructions of the form:

$$\delta(q, a, Z) \in (p, w), |w| \leq 2$$

(Instructions pushing more symbols can be split using new states).

Let $R^|$ consist of all expressions of the form

$$q^{-1}aa^{-1}Z^{-1}BAp$$

for instruction $\delta(q, a, Z) \ni (p, AB)$, and similarly for instructions $\delta(q, a, Z) \in (p, A)$ and $\delta(q, a, Z) \in (p, \epsilon)$. (For $a = \epsilon$ omit aa^{-1}.)

Now define the regular language as $R = Z_0 q_0 (R^|)^* Q^{-1}$.

Proof cont'd

The Dyck language D is over the alphabet

$$\Sigma \cup \Sigma^{-1} \cup Q \cup Q^{-1} \cup \Gamma \cup \Gamma^{-1}$$

The language $D \cap R = D \cap Z_0 q_0 (R^{|})^* Q^{-1}$ describes valid computations, e.g.

$$\underbrace{Z_0 \overbrace{q_0 q_0^{-1}} aa^{-1} Z_0^{-1}} \underbrace{B \underbrace{A \overbrace{pp^{-1}} bb^{-1} A^{-1}} \overbrace{qq^{-1}} cc^{-1} B^{-1}} \overbrace{rr^{-1}}$$

The homomorphism h selects the input word being read:

$$\begin{aligned} h(a) &= a && \text{for input symbols } a \in \Sigma \\ h(y) &= \epsilon && \text{for all other symbols} \end{aligned}$$

$\square$

Summary of Lecture 9

- Closure properties of context-free languages (including substitution, homomorphism, inverse homomorphism)
- Also closure properties of deterministic CFLs
- Dyck languages, a characterization of context-free languages