Lecture 12 – Universal and Diagonal language, Undecidability, Post Correspondence Problem

NTIN071 Automata and Grammars

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* Adapted from the Czech-lecture slides by Marta Vomlelová with gratitude.
The translation, some modifications, and all errors are mine.

Recap of Lecture 11

- Recursively enumerable languages are exactly those generated by (Type 0) grammars
  - TM to G: simulate moves on a reversed non-terminal copy of ω, generate sufficient space, cleanup if accepting state
  - G to TM: generate all strings, check if any of them represents a valid derivation of ω (sentential forms separated by $\#$)
- Context-sensitive languages:
  - context-sensitive grammars are equivalent to monotone grammars
  - Linear Bounded Automaton (LBA): nondeterministic TM with tape limited to the length of input
  - constructions: monotone grammar to LBA, LBA to monotone grammar
- Intro to computability: an overview
- decision problem $\leftrightsquigarrow$ the language of all 'YES' instances
- machine-readable encoding of TMs

The Diagonal language

Let $\text{decode}(w)$ be the TM M such that $\text{code}(M) = w$. (Recall: if w is not a valid code, then $\text{decode}(w)$ is a fixed one-state TM with no instructions.) Then:

$$L_D = \{w \mid w \notin L(\text{decode}(w))\}$$

Theorem

L_D is not recursively enumerable.

Proof idea: there cannot exist a TM recognizing L_D: running it on its own code would lead to Barber's paradox

"The program accepts all programs that don't accept themselves. Does the program accept itself?"

Proof that $L_D = \{w \mid w \notin L(\text{decode}(w))\}$ is not RE

Proof: Assume for contradiction that $L_D = L(M)$ for some M. Let $w = \text{code}(M)$. Then $L_D = \{w \mid w \notin L(M)\}$. Is $w \in L(M)$?

$$w \in L(M) \iff w \in L_D \iff w \notin L(M) \qquad \square$$

Why 'diagonal'? A variant of Cantor's diagonal argument. Order all TMs by $M_i = \text{decode}(w_i)$. Does M_i accept w_i?


Diagonal

A TM for L_D would be one of the rows but differs from each row in the diagonal element. (Same as the proof that $\mathbb{R}$ is uncountable.)

The Universal Turing Machine

The Universal Turing Machine U can simulate any TM (given by its code) on any input. More precisely, U accepts exactly inputs of the form $\langle \mathrm{code}(M), w \rangle$ where $w \in L(M)$.

The construction: four tapes

1. input tape (w and the encoded transitions of M)

2. simulated tape of M, symbols encoded as 0^i, separated by 1s

3. state of M, again represented by 0^i

4. scratch tape
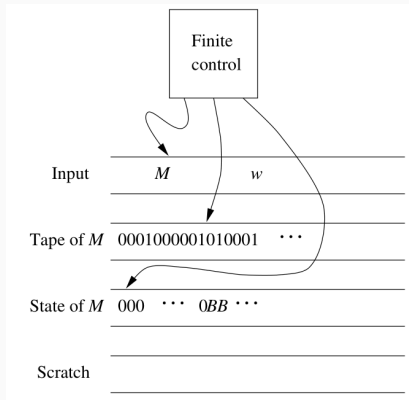
The operation of U

Initialize:

- Check if the input code is valid, if not, halt without accepting

- Initialize Tape 2 with w in its encoded form: 10 for 0 in w, 100 for 1
  Blanks are left blank and replaced with 1000 only 'on demand'
  Move 2nd head to the first simulated cell.

- Place 0 (the start state of M) on Tape 3.

Simulate moves of M:

- Search Tape 1 for the appropriate transition $0^i10^j10^k10^\ell10^m$, where
  0^i on Tape 3, 0^j on Tape 2.

- Change the content of Tape 3 to 0^k.

- Replace 0^j on Tape 2 by 0^ℓ. Scratch tape to manage spacing.

- Move head on Tape 2 to the next 1 left or right, depending on m.

Termination: If M has no transition matching simulated state & tape symbol, halt without accepting. If M enters accepting state, U accepts.

Recursive languages are closed under complement

Lemma

If L is recursive, then $\overline{L}$ is recursive as well.
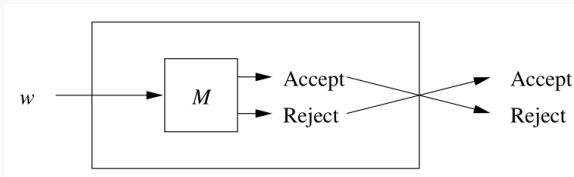
Proof: Given M deciding L, construct M' deciding L'. Since M always halts, if it does not accept, the reason is missing transition.

M' has a single, new accepting state q_{ACCEPT}. For every non-accepting state of M and every tape symbol X such that $\delta(q, X)$ is undefined, redefine $\delta'(q, X) = (q_{\text{ACCEPT}}, X, L)$.

Clearly, $L(M') = \overline{L}$. Since M is guaranteed to halt, so is M'. $\quad\square$

Theorem

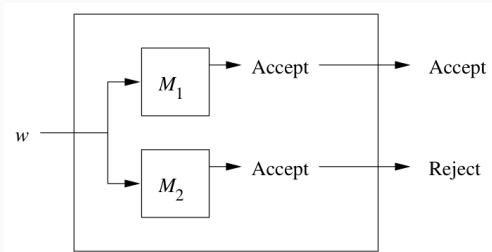*L is recursive iff both L and $\overline{L}$ are recursively enumerable.*

Proof: $\Rightarrow$ Follows from the lemma.

$\Leftarrow$ Let $L = L(M_1)$ and $\overline{L} = L(M_2)$. For an input w, simulate both M_1 and M_2 (two tapes, states with two components). L and $\overline{L}$ are complementary, one of M_1 or M_2 will halt and accept.

- If M_1 accepts, accept.
- If M_2 accepts, reject. □

The Universal language and its undecidability

$L_U = \{\langle \mathrm{code}(M), w \rangle \mid w \in L(M)\}$ RE but not R

$\overline{L_D} = \{w \mid w \in L(\mathrm{decode}(w))\}$ RE but not R

$L_D = \{w \mid w \notin L(\mathrm{decode}(w))\}$ not RE

Theorem

The Universal language is recursively enumerable but not recursive. Same is true for complement of the Diagonal language.

Proof:

- L_U is RE: it is recognized by the Universal TM U
- $\overline{L_D}$ is RE: rewrite input w to $\langle w, w \rangle = w111w$, then run on U
- $\overline{L_D}$ is not recursive: if it were, by Post's theorem L_D would be RE (actually R), but we know it is not
- L_U is not recursive: if it were, $\overline{L_D}$ would be recursive (rewrite w to $\langle w, w \rangle$, run on the hypothetical M deciding L_U) $\square$

Reductions between decision problems

Definition

A reduction R is an algorithm mapping all instances of P_1 to instances of P_2 that always halts, and for every instance w of P_1 outputs an instance $R(w)$ of P_2 such that:

- w is a YES instance of P_1 iff $R(W)$ is a YES instance of P_2
- w is a NO instance of P_1 iff $R(W)$ is a NO instance of P_2

(Technically, $R = f_M$ for some TM M that always halts.)

Example The mapping $w \rightsquigarrow \langle w, w \rangle = w111w$ (from the previous proof) can clearly be done algorithmically. It is a reduction from $\overline{L_D}$ to L_U (and also from L_D to $\overline{L_U}$).

Only easy reduce to easy, hard only reduce to hard

Theorem

If there is a reduction from P_1 to P_2, then:

(i) If P_1 is not decidable then neither is P_2.

(ii) If P_2 is decidable, then so is P_1.

(iii) If P_1 is not partially decidable then neither is P_2.

(iv) If P_2 is partially decidable, then so is P_1.

(i&ii) Let P_1 be undecidable. If P_2 were decidable, we could combine the reduction from P_1 to P_2 with the algorithm deciding P_2 to construct an algorithm that decides P_1.

(iii&iv) Assume P_1 is not partially decidable, but P_2 is. Similarly as above, we could combine the reduction and the algorithm for P_2 to get an algorithm partially deciding P_1–a contradiction.$\square$

"Does the given program halt for the given input?'

An instance of the Halting Problem Halt: $\langle \mathrm{code}(M), w \rangle \in \{0,1\}^*$.
The answer is YES iff M halts on input w; otherwise it is NO.

Theorem

The Halting Problem is undecidable.

(Note that it is partially decidable: we can simulate using U.)

Proof: Reduce the undecidable problem $\overline{L_D}$ to Halt. Given an instance w of $\overline{L_D}$, let $M = \mathrm{decode}(w)$. Modify M to get M' such that if M halts without accepting, M' goes to an infinite loop.

Set $R(w) = \langle \mathrm{code}(M'), w \rangle$. Clearly, it can be done algorithmically.

- If $w \in \overline{L_D}$, i.e. $w \in L(M)$, then M' accepts (thus halts) on w.
- If $w \notin \overline{L_D}$, i.e. $w \notin L(M)$, then either M doesn't halt or halts without accepting. In either case M' doesn't halt on w.   □

Accepts no inputs?

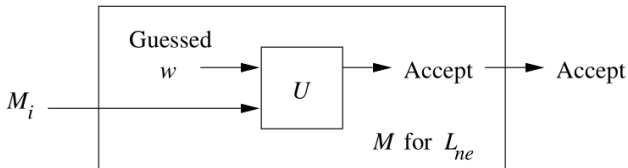- $L_e = \{\mathrm{code(M)} \mid L(M) = \emptyset\}$
- $L_{ne} = \{\mathrm{code(M)} \mid L(M) \neq \emptyset\} = \overline{L_e}$

Theorem

(i) L_e is not recursively enumerable.

(ii) L_{ne} is recursively enumerable but not recursive.

Proof: As $L_{ne} = \overline{L_e}$, (i) follows from (ii) by Post's theorem.

L_{ne} is RE: nondeterministically guess $w \in L(M)$, verify using U

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Proof cont'd

L_{ne} is not recursive: reduction from undecidable $\overline{L_D}$

Given $w = \mathrm{code(M)}$, $R(w)$ is a TM M' that ignores its input, rewrites the input tape with w, and simulates M on w.
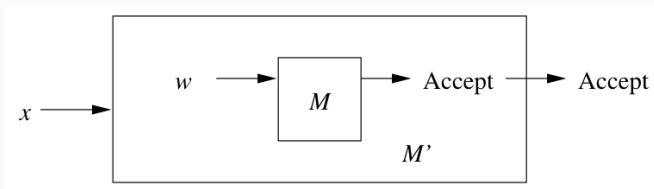
- If $w \in \overline{L_D}$, i.e. $w \in L(M)$, then $R(w)$ always accepts.
- If $w \notin \overline{L_D}$, i.e. $w \notin L(M)$, then $R(w)$ never accepts. $\qquad \square$

Rice's theorem

Which properties of programs are decidable?

None of them!

(Except for trivial properties true/false for all programs.)

We have all the tools, but not the time to prove Rice's theorem.

"Theoretically, static analysis of programs cannot be done automatically?"

Undecidable problems about context-free languages

Post correspondence problem

Several problems about context-free grammars are undecidable, e.g. is $L(G_1) \cap L(G_2) = \emptyset$? We show that by reduction from a suitable undecidable problem:

Post correspondence problem (PCP)

- **Instance:** two same-length lists of words over a finite alphabet Σ: $A = w_1, w_2, \ldots, w_k$, $B = x_1, x_2, \ldots, x_k$
- **Question:** is there a finite sequence of positive integers $i_1, \ldots, i_m$ ($m \geq 1$) such that $w_{i_1} w_{i_2} \ldots w_{i_m} = x_{i_1} x_{i_2} \ldots x_{i_m}$?
- call $i_1, \ldots, i_m$ a solution, (w_i, x_i) a corresponding pair

Theorem

The Post Correspondence Problem is undecidable.

Proof idea: Reduction from L_U by simulating computation of any TM on any input as a PCP instance. [detailes later, if there's time]

Solvable instance:

- solution: $2, 1, 1, 3$ (forms 101111110)
- another solution: 2, 1, 1, 3, 2, 1, 1, 3

partial solution: $i_1, \ldots, i_r$ such that one of $w_{i_1} \ldots w_{i_r}$, $x_{i_1} \ldots x_{i_r}$ is a prefix of the other
observe: prefixes of solns are partial solns

	List A	List B
i	w_i	x_i
1	1	111
2	10111	10
3	10	0

Unsolvable instance:

- $i_1 = 1$, $A : 10$, $B : 101$ (1st letter is 1)
- if $i_2 = 1$, $A : 1010$, $B : 101101$
- if $i_2 = 2$, $A : 10011$, $B : 101111$
- so $i_2 = 3$, $A : 10101$, $B : 101011$
- same situation as after $i_1 = 1$, no way to get same length

	List A	List B
i	w_i	x_i
1	10	101
2	011	11
3	101	011

Context-free languages and (un)decidability

Let G, G' be context-free grammars, R a regular expression, Σ a finite alphabet, w a word (all given on the input).

Recall that the following are decidable:

- $L(G) \ni w$? [the CYK algorithm]
- $L(G) = \emptyset$? [check if S is a generating symbol]

Theorem

The following are undecidable:

(a) $L(G) = \Sigma^*$?

(b) $L(G) = L(R)$?

(c) $L(G) \supseteq L(R)$?

(d) $L(G) = L(G')$?

(e) $L(G) \supseteq L(G')$?

(f) $L(G) \cap L(G') \neq \emptyset$?

(g) *Is G ambiguous?*

We prove undecidability by reduction from the PCP problem.

Given a PCP instance $(A = w_1, \ldots, w_k, B = x_1, \ldots, x_k)$, fix a set of new terminal symbols $S = \{s_1, \ldots, s_k\}$ to represent indices.

The list language L_A for A consists of all words of the form $w_{i_1} \ldots w_{i_m} s_{i_m} \ldots s_{i_1}$ where $m \geq 1$ and $i_j \in \{1, \ldots, k\}$.

- L_A is generated by $G_A = (\{A\}, \Sigma \cup S, \mathcal{P}_A, A)$ with rules

$$\mathcal{P}_A = \{A \to w_1 A a_1 \mid \cdots \mid w_k A a_k \mid w_1 a_1 \mid \cdots \mid w_k a_k\}$$

- L_A is recognized by a deterministic PDA (first store letters from Σ on the stack, then for each s_j pop w_j^R from the stack)
- same is true for analogously defined L_B
- the PCP instance (A, B) has a solution iff $L_A \cap L_B \neq \emptyset$,

$$z = w_{i_1} \ldots w_{i_m} s_{i_m} \ldots s_{i_1}$$
$$= x_{i_1} \ldots x_{i_m} s_{i_m} \ldots s_{i_1} \in L(A) \cap L(B)$$

Proof of (g): Undecidability of ambiguiuty of G

Recall: ambiguous iff $\exists z \in L(G)$ with two different parse trees

Reduction from PCP: given $(A = w_1, \ldots, w_k, B = x_1, \ldots, x_k)$
construct $G = (\{S, A, B\}, \Sigma \cup \{S\}, \mathcal{P}, S)$ with rules

$$\mathcal{P} = \{S \to A \mid B\} \cup \mathcal{P}_A \cup \mathcal{P}_B$$

G is ambiguous iff the PCP instance (A, B) has a solution: Every $z \in L_A$ has a unique parse tree from A (given by the s_{i_j}'s), same for B.

Different parse trees for $z \in L(G)$ mean:

- $z = w_{i_1} \ldots w_{i_m} s_{i_m} \ldots s_{i_1} \in L_A$, and
- $z = x_{i_1} \ldots x_{i_m} s_{i_m} \ldots s_{i_1} \in L_B$, thus
- $z \in L_A \cap L_B$

Which happens iff (A, B) has as solution. $\qquad\square$ 20
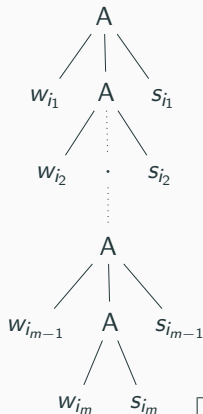
Proof of (f): Undecidability of $L(G) \cap L(G') \neq \emptyset$

Reduction from PCP: the given PCP instance (A, B) has a solution iff $L(G_A) \cap L(G_B) \neq \emptyset$ ☐

Proof of (a): Undecidability of $L(G) = \Sigma^*$

Recall that L_A, L_B are deterministic. Since DCFLs are closed under complement, $L = \overline{L_A} \cup \overline{L_B}$ is context-free. Let $L = L(G)$. Then:

$$(A, B) \text{ has a solution } \Leftrightarrow L_A \cap L_B \neq \emptyset \Leftrightarrow L(G) \neq (\Sigma \cup S)^* \quad ☐$$

Note: We proved undecidability of "$L(G) \neq \Sigma^*$?". Undecidability of "$L(G) = \Sigma^*$?" follows easily: they are almost complementary, except for 'invalid inputs':

- Check if input is a valid encoding of G and Σ, if not, reject.
- If it is valid, run algorithm for "$\neq$" and reverse its answer.

Proof of (b), (c): Undecidability of $L(G) = L(R)$, $L(G) \supseteq L(R)$

Reduction from $L(G) = \Sigma^*$, create a regular expression R such that $L(R) = \Sigma^*$

Proof of (d), (e): Undecidability of $L(G) = L(G')$, $L(G) \supseteq L(G')$

Reduction from $L(G) = \Sigma^*$, create a grammar G' that generates all words in Σ^* $\qquad\qquad\qquad\qquad\qquad\square$

Note: "$L(G) \subseteq L(R)$?" is decidable:

$$L(G) \subseteq L(R) \;\Leftrightarrow\; L(G) \cap \overline{L(R)} = \emptyset$$

and $L(G) \cap \overline{L(R)}$ is context-free (intersection with regular).

Undecidability of the Post Correspondence Problem

Modified Post Correspondence Problem (MPCP)

A solution $(i_1, i_2, \ldots, i_m)$ to a PCP instance
$(A = w_1, \ldots, w_k, B = x_1, \ldots, x_k)$ is initial if $i_1 = 1$, i.e.,

$$\mathbf{w_1} w_{i_1} w_{i_2} \ldots w_{i_m} = \mathbf{x_1} x_{i_1} x_{i_2} \ldots x_{i_m}$$

Modified PCP: Does a given PCP instance have an initial solution?

Example: This PCP instance has no initial solution:

	A	B
i	w_i	x_i
1	1	111
2	10111	10
3	10	0

Why? A:1, B:111 $\leadsto$ A:11, B:111111 $\leadsto$ A:111, B:111111111

The lengths won't match. Other choices lead to mismatch.

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Reducing MPCP to PCP

From (A, B) construct (C, D) such that (A, B) has an initial solution iff (C, D) has a solution.

The construction: Add new symbols $*, \$ \notin \Sigma$. Define y_i from w_i by adding $*$ after each letter, and z_i from x_i adding $*$ before each letter ($1 \leq i \leq k$). Define $y_0 = *y_1, z_0 = z_1, y_{k+1} = \$, z_{k+1} = *\$$.

Example:

i	List A	List B	List C	List D
	w_i	x_i	y_i	z_i
0			*1*	*1*1*1
1	1	111	1*	*1*1*1
2	10111	10	1*0*1*1*1*	*1*0
3	10	0	1*0*	*0
4			\$	*\$

Clearly, $(i_1, i_2, \ldots, i_m)$ is an initial solution to (A, B) iff $(0, i_1, i_2, \ldots, i_m, k + 1)$ is a solution to (C, D). $\square$ 24

Undecidability of MPCP

Reduction from L_U: Given a TM M and input w, construct (A, B). Assume M never writes B and never moves left of inital position.

List A	List B	
$\#$	$\#q_0 w\#$	
X	X	for all tape symbols $X \in \Gamma$
$\#$	$\#$	
qX	Yp	for $\delta(q, X) = (p, Y, R)$
ZqX	pZY	for $\delta(q, X) = (p, Y, L)$, $Z \in \Gamma$ tape symbol
$q\#$	$Yp\#$	for $\delta(q, B) = (p, Y, R)$
$Zq\#$	$pZY\#$	for $\delta(q, B) = (p, Y, L)$, $Z \in \Gamma$ tape symbol
XqY	q	$q \in F$ accepting state
Xq	q	$q \in F$
qY	q	$q \in F$
$q\#\#$	$q\#$	$q \in F$

Example: $M, w \rightsquigarrow (A, B)$

$M = (\{q_1, q_2, q_3\}, \{0, 1\}, \{0, 1, B\}, \delta, q_1, B, \{q_3\})$, input word $w = 01$

	0	1	B
$\to q_1$	$(q_2, 1, R)$	$(q_2, 0, L)$	$(q_2, 1, L)$
q_2	$(q_3, 0, L)$	$(q_1, 0, R)$	$(q_2, 0, R)$
$*q_3$	–	–	–

List A	List B
#	$\#q_101\#$
0	0
1	1
#	#

List A	List B	source
q_10	$1q_2$	$\delta(q_1, 0) = (q_2, 1, R)$
$0q_11$	q_200	$\delta(q_1, 1) = (q_2, 0, L)$
$1q_11$	q_210	$\delta(q_1, 1) = (q_2, 0, L)$
$0q_1\#$	$q_201\#$	$\delta(q_1, B) = (q_2, 1, L)$
$1q_1\#$	$q_211\#$	$\delta(q_1, B) = (q_2, 1, L)$
$0q_20$	q_300	$\delta(q_2, 0) = (q_3, 0, L)$
$1q_20$	q_310	$\delta(q_2, 0) = (q_3, 0, L)$
q_21	$0q_1$	$\delta(q_2, 1) = (q_1, 0, R)$
$q_2\#$	$0q_2\#$	$\delta(q_2, B) = (q_2, 0, R)$

List A	List B
$0q_30$	q_3
$0q_31$	q_3
$1q_30$	q_3
$1q_31$	q_3
$0q_3$	q_3
$1q_3$	q_3
q_30	q_3
q_31	q_3
$q_3\#\#$	#

$\vdots$

MPCP simulating the TM

List A	List B	source
$q_1 0$	$1 q_2$	$\delta(q_1, 0) = (q_2, 1, R)$
$0 q_1 1$	$q_2 0 0$	$\delta(q_1, 1) = (q_2, 0, L)$
$1 q_1 1$	$q_2 1 0$	$\delta(q_1, 1) = (q_2, 0, L)$
$0 q_1 \#$	$q_2 0 1 \#$	$\delta(q_1, B) = (q_2, 1, L)$
$1 q_1 \#$	$q_2 1 1 \#$	$\delta(q_1, B) = (q_2, 1, L)$
$0 q_2 0$	$q_3 0 0$	$\delta(q_2, 0) = (q_3, 0, L)$
$1 q_2 0$	$q_3 1 0$	$\delta(q_2, 0) = (q_3, 0, L)$
$q_2 1$	$0 q_1$	$\delta(q_2, 1) = (q_1, 0, R)$
$q_2 \#$	$0 q_2 \#$	$\delta(q_2, B) = (q_2, 0, R)$

List A	List B
$\#$	$\# q_1 0 1 \#$
0	0
1	1
$\#$	$\#$
$0 q_3 0$	q_3
$0 q_3 1$	q_3
$1 q_3 0$	q_3
$1 q_3 1$	q_3
$0 q_3$	q_3
$1 q_3$	q_3
$q_3 0$	q_3
$q_3 1$	q_3
$q_3 \# \#$	$\#$

- Accepting path of M:

  $q_1 0 1 \vdash 1 q_2 1 \vdash 1 0 q_1 \vdash 1 q_2 0 1 \vdash q_3 1 0 1$

$A: \quad \# q_1 0 1 \# 1 q_2 1 \# 1 0 q_1 \# 1 q_2 0 1 \# q_3 1 0 1 \# q_3 0 1 \# q_3 1 \# q_3 \# \#$

$B: \quad \# q_1 0 1 \# 1 q_2 1 \# 1 0 q_1 \# 1 q_2 0 1 \# q_3 1 0 1 \# q_3 0 1 \# q_3 1 \# q_3 \# \#$

Undecidability of PCP

Theorem

The Post Correspondence Problem is undecidable.

Proof: We have already shown that MPCP reduces to PCP. To show that L_U reduces to MPCP, we must verify that:

M accepts w $\Leftrightarrow$ the constructed (A, B) has an initial solution

$\Rightarrow$ If $w \in L(M)$, start with the initial pair and simulate the computation of M on input w.

$\Leftarrow$ Given an initial solution to (A, B), there is a corresponding computation of M on w:

- MPCP must start with the initial pair
- while $q \notin F$, rules for cleaning cannot be used
- the partial solution is of the form A:x,B:xy, i.e., B is longer
- thus we must end in an accepting state $\qquad \square$

Summary of Lecture 12

- the Diagonal language L_D is not recursively enumerable
- the Universal language L_U, the Universal TM: simulate any M on any w
- recursive languages are closed under complement
- Post's theorem: L recursive iff both $L, \overline{L}$ are RE
- L_U, $\overline{L_D}$ are recursively enumerable but not recursive
- reductions between decision problems
- the Halting problem is undecidable
- (Rice's thm: nontriv. properties of programs are undecidable)
- Undecidable problems about context-free grammars
- Source of undecidability: Post Correspondence Problem