Lecture 2 – Myhill–Nerode theorem, Equivalent and Minimal Representations

NTIN071 Automata and Grammars

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Recap of Lecture 1

- Deterministic Finite Automaton (DFA): $A = (Q, \Sigma, \delta, q_0, F)$
- Extended transition function δ^*
- The language recognized by the DFA A is the language

$$L(A) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}$$

- Languages recognized by some DFA are called regular
- Finite automata encode only finite information, but can recognize infinite languages
- Product automaton, intersection of reg. languages is regular
- Pumping lemma for regular languages (prove nonregularity)
- PL not a characterization, some nonregular can be pumped
- A regular language is infinite iff it contains a word of length $n \leq |w| \leq 2n$ where $n = \#$states of a recognizing automaton

1.4 Myhill–Nerode Theorem

Characterize regular languages

How to recognize if a given language is regular? So far, we can construct a DFA recognizing the language, or use the Pumping lemma for contradiction.

We would like to have a characterization (which the Pumping Lemma is not!).

Luckily, as we'll see, every regular language comes with an implicit automaton 'hiding' in the set of all words over its alphabet.

Congruences on words

Let Σ be a finite alphabet and $\sim$ an equivalence relation on Σ^* (reflexive, symmetric, transitive). Then:

- $\sim$ is a right congruence iff $(\forall u, v, w \in \Sigma^*) u \sim v \Rightarrow uw \sim vw$.
- $\sim$ has finite index iff the partition $\Sigma^* / \sim$ has a finite number of classes.
- the class containing a word u is denoted $[u]_\sim$ or simply $[u]$

The theorem

Theorem (Mihyl–Nerode theorem)

Let Σ be a finite alphabet and $L \subset \Sigma^*$ a language over Σ. The following statements are equivalent:

(i) *L is regular,*

(ii) *there exists a right congruence $\sim$ on Σ^* with finite index such that L is a union of some classes of the partition $\Sigma^* / \sim$.*

Proof idea: Group together words that end in the same state when we start reading from q_0.

The proof

(i)$\Rightarrow$(ii) from an automaton to a right congruence of finite index

- we define $u \sim v \equiv \delta^*(q_0, u) = \delta^*(q_0, v)$
- it is indeed a right congruence (from the definition of δ^*)
- it has a finite index (Q is finite)
- $L = \{w \mid \delta^*(q_0, w) \in F\} = \bigcup_{q \in F}\{w \mid \delta^*(q_0, w) = q\}$ $= \bigcup_{q \in F}[w \mid \delta^*(q_0, w) = q]_\sim$.

(ii)$\Rightarrow$(i) from a right congruence of finite index to an automaton

- the alphabet is Σ, states Q are the congruence classes $\Sigma^*/\sim$
- the initial state $q_0 = [\epsilon]$, final states $F = \{c_1, \dots, c_n\}$ where $L = \bigcup_{i=1,\dots,n} c_i$
- trans. function $\delta([u], x) = [ux]$ (well-defined right congruence)
- to show that $L(A) = L$, using $\delta^*([\epsilon], w) = [w]$: $w \in L \Leftrightarrow w \in \bigcup_{i=1,\dots,n} c_i \Leftrightarrow w \in c_1 \vee \dots w \in c_n \Leftrightarrow$ $[w] = c_1 \vee \dots [w] = c_n \Leftrightarrow [w] \in F \Leftrightarrow w \in L(A)$ $\square$

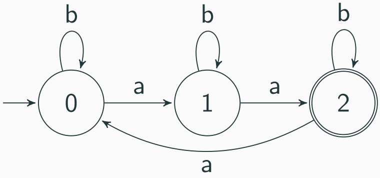
Application: proof of regularity

Example

Construct an automaton that recognizes the language
$L = \{w \in \{a, b\}^* \mid |w|_a = 3k + 2 \text{ for some } k \geq 0\}$.

$$u \sim v \text{ iff } |u|_a \equiv |v|_a \pmod 3$$

- equivalence classes are 0,1,2
- L corresponds to the class 2
- a – transitions to the next class
- b – stay in the same class.



Application: proof of nonregularity

Example

Show that $L = \{u \in \{a, b, c\}^* \mid u = a^+ b^i c^i \text{ or } u = b^i c^j\}$ is not regular. (Note that the first letter can be pumped.)

Suppose for contradiction that L is regular. Let $\sim_L$ be a right congruence of finite index where L is a union of some $\sim_L$-classes.

Consider the set of words $S = \{a, ab, abb, \ldots\} = \{ab^n \mid n \in \mathbb{N}\}$.

For any two $i \neq j$ there is a string (c^i) distinguishing the words (in/out of the language): $ab^i c^i \in L$ but $ab^j c^i \notin L$

No two elements of S can be in the same class of $\sim_L$ (L would split the class). Since S is infinite, this contradicts finite index of $\sim_L$. $\square$

1.5 Equivalent and Minimal Representations

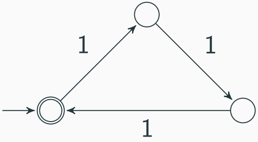
Equivalent automata

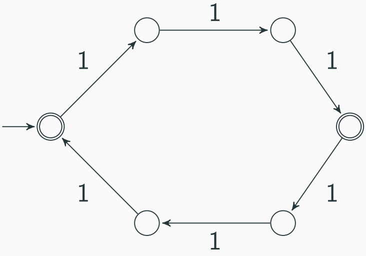
Definition

Finite automata A, B are equivalent iff they recognize the same language, that is $L(A) = L(B)$.

Example

$L = \{w \in \{1\}^* \mid |w| = 3k \text{ for some } k \geq 0\}$





Automata homomorphism

Definition

Let A_1, A_2 be DFAs. A surjective mapping $h : Q_1 \to Q_2$ is an (automata) homomorphism, if it satisfies:

- $h(\delta_1(q, x)) = \delta_2(h(q), x)$
- $h(q_{0_1}) = q_{0_2}$
- $q \in F_1 \Leftrightarrow h(q) \in F_2$

A bijective homomorphism is called an isomorphism.

(Isomorphic automata only differ by the 'names' of the states.)

Theorem (Automata Equivalence Theorem)

Let A_1, A_2 be DFAs. If there exists a homomorphism from A_1 to A_2, then A_1 and A_2 are equivalent.

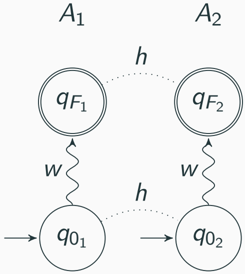
The proof

For any $w \in \Sigma^*$, $q \in Q_1$, we can prove by finite induction that

$$h(\delta_1^*(q, w)) = \delta_2^*(h(q), w)$$

Then the following holds:

$$\begin{aligned}
w \in L(A_1) \quad &\Leftrightarrow \quad \delta_1^*(q_{0_1}, w) \in F_1 \\
&\Leftrightarrow \quad h(\delta_1^*(q_{0_1}, w)) \in F_2 \\
&\Leftrightarrow \quad \delta_2^*(h(q_{0_1}), w) \in F_2 \\
&\Leftrightarrow \quad \delta_2^*(q_{0_2}, w) \in F_2 \\
&\Leftrightarrow \quad w \in L(A_2)
\end{aligned}$$



$\square$

Reducing automata

The smallest DFA recognizing a given language?

Start with any DFA.

Two steps:

- remove unreachable states
- merge equivalent (indistinguishable) states

The reduced DFA is unique (up to automata isomorphism).

(Un)reachable states

Definition (Reachable states)

Let's have a DFA $A = (Q, \Sigma, \delta, q_0, F)$ and $q \in Q$. The state q is reachable iff there exists $w \in \Sigma^*$ such that $\delta^*(q_0, w) = q$.

Algorithm (Reachable States – BFS on the state diagram)

- *set* $M_0 = \{q_0\}$
- *repeat* $M_{i+1} = M_i \cup \{q \in Q \mid (\exists p \in M_i, \exists x \in \Sigma)\ \delta(p, x) = q\}$
- *until* $M_{i+1} = M_i$
- *return* M_i

Proof of correctness and completeness.

Corectness: $M_0 \subseteq M_1 \subseteq \ldots \subseteq Q$ and consist of reachable states.
Completeness: Let q be reachable. Let $w = x_1 \ldots x_n$ be shortest such that $\delta^*(q_0, x_1 \ldots x_n) = q$. As $\delta^*(q_0, x_1 \ldots x_i) \in M_i \setminus M_{i-1}$ we get $\delta^*(q_0, x_1 \ldots x_n) = q \in M_n$. $\square$

(In)distinguishable states

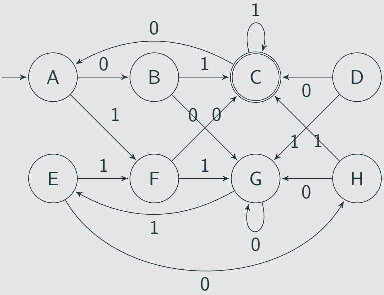
Definition (State equivalence)

States $p, q \in Q$ of a DFA A are equivalent (indistinguishable), if for all words w: $\delta^*(p, w) \in F \iff \delta^*(q, w) \in F$

Observation

State equivalence is indeed reflexive, symmetric and transitive.

Example



C, G distinguishable, $\delta^*(C, \epsilon) \in F$, $\delta^*(G, \epsilon) \notin F$

A,G too: $\delta^*(A, 01) = C$ accepting, $\delta^*(G, 01) = E$ not accepting.

A,E equivalent (for ϵ, $1*$ obvious, 0 goes to non-accepting, 01 & 00 meet in the same state)

Recognizing state equivalence

Distinguish accepting from nonaccepting. Then go backwards.

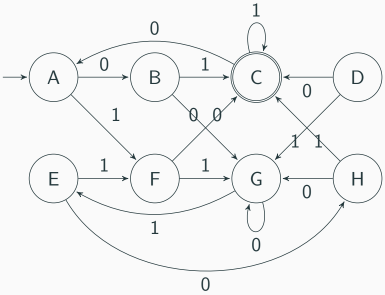
Algorithm (Finding distinguishable states in a DFA)

- *Basis: If $p \in F$ is accepting and $q \notin F$ is not, the pair $\{p, q\}$ is distinguishable.*
- *Induction: Let $p, q \in Q$ and $a \in \Sigma$. If $r = \delta(p, a)$ and $s = \delta(q, a)$ are distinguishable, then so are p and q. (Repeat until no newly distinguished pair.)*

Example 1/4

1. Accepting vs. non–accepting

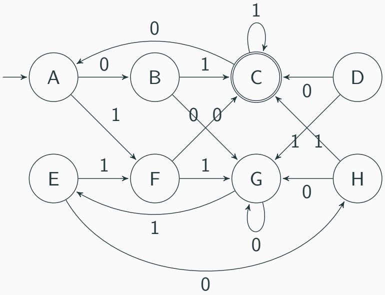
	A	B	C	D	E	F	G
B							
C	×	×					
D			×				
E			×				
F			×				
G			×				
H			×				



Example 2/4

2. $\delta(q,1) \in \mathcal{F}$ for $q \in \{B, C, H\}$

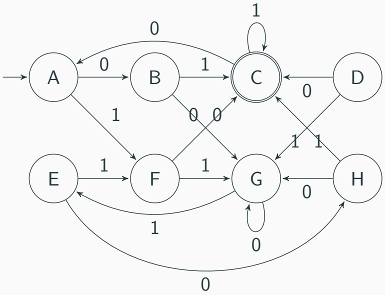
B	×						
C	×	×					
D		×	×				
E		×	×				
F		×	×				
G		×	×				
H	×		×	×	×	×	×
	A	B	C	D	E	F	G



Example 3/4

3. $\delta(q, 0) \in \mathcal{F}$ for $q \in \{D, F\}$

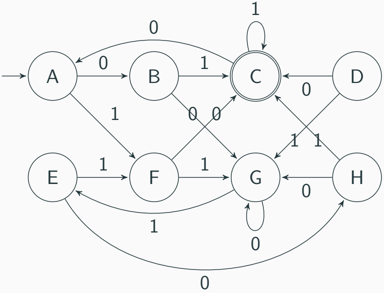
	A	B	C	D	E	F	G
B	×						
C	×	×					
D	×	×	×				
E		×	×	×			
F	×	×	×		×		
G		×	×	×		×	
H	×		×	×	×	×	×



Example 4/4

4. B and G are distinguishable, $\delta(A, 0) = B$, $\delta(G, 0) = G$, therefore A, G are distinguishable. Similarly, $\delta(*, 0)$ for E, G goes to distinguishable states H, G.

B	×						
C	×	×					
D	×	×	×				
E		×	×	×			
F	×	×	×		×		
G	×	×	×	×	×	×	
H	×		×	×	×	×	×
	A	B	C	D	E	F	G



Equivalent pairs:

(A, E), (B, H), (D, F)

Correctness

Theorem

A pair of states is not distinguished by the algorithm, if and only if the states are equivalent.

Proof.

$\Leftarrow$ Clearly, only distinguishable pairs are distinguished.

$\Rightarrow$ Induction on the length of a shortest distinguishing word. If p, q are distinguished by $w = \epsilon$, then the algorithm distinguishes them. Now let $w = a_1 \dots a_k$. By induction, $r = \delta(p, a_1)$ and $s = \delta(q, a_1)$ are distinguished by the algorithm. But then the algorithm distinguishes p, q in the next round (following a_1-transitions backwards). $\square$

Complexity

The time complexity is polynomial in the number of states n.

- In one iteration, we consider all pairs, that is $O(n^2)$.
- In each iteration we add a cross, that means no more than $O(n^2)$ iterations.
- Together, $O(n^4)$.

The algorithm may be sped up to $O(n^2)$ by memorizing states that depend on the pair $\{r, s\}$ and following the list backwards.

Exercise

- Describe the $O(n^2)$ algorithm hinted above.
- The algorithm can also compute, for each distinguishable pair, the shortest word distinguishing that pair.

Application: testing equality of regular languages

- regular languages L, M are given by some representations
- from those construct DFA A_L, A_M recognizing L, M
- we can assume $Q_L \cap Q_L = \emptyset$ (otherwise rename the states)
- run the following algorithm:

Algorithm (Testing equivalence of automata)

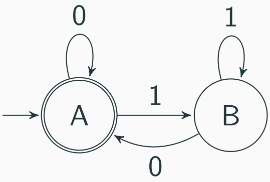
- *Construct a DFA* $B = (Q_L \cup Q_M, \Sigma, \delta_L \cup \delta_M, q_L, F_L \cup F_M)$ *as a union of states and transitions; select one (any) initial state.*
- *Test if* ${q_0}_L$ *and* ${q_0}_M$ *are equivalent. (The automata are equivalent iff their initial states are equivalent in* B*.)*

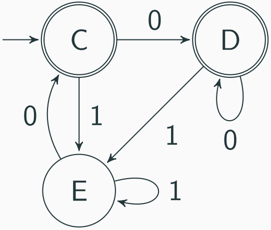
NB: Renaming states gives an isomorphic (hence equivalent) automaton. So we can always assume disjoint states. Alternatively, we can use disjoint union: $Q_B = Q_L \dot{\sqcup} Q_M = Q_L \times \{0\} \cup Q_M \times \{1\}$.

Example

Example

Two different DFAs recognizing $L = \{\epsilon\} \cup \{w0 \mid w \in \{0,1\}^*\}$.





B	×			
C		×		
D		×		
E	×		×	×
	A	B	C	D

Reduced automaton

Definition (Reduced DFA)

A DFA A is reduced iff all states are reachable, no two distinct states are equivalent, and there is no 'fail' state from which no accepting state would be reachable.

It is a reduct of a DFA B iff it is reduced and equivalent to B.

Theorem (DFA minimization)

(i) *Any two equivalent reduced automata are isomorphic.*

(ii) *Any DFA accepting at least one word has a reduct, unique up to automata isomorphism.*

Proof.

(i) Any $q \in Q_1$ is reachable. Find a word w s.t. $q = \delta_1^*(q_{0_1}, w)$. Define $h(q) = \delta_2^*(q_{0_2}, w)$. Check that h is an isomorphism.
(ii) The construction described on the next slide works. $\square$

Algorithm: constructing the reduct

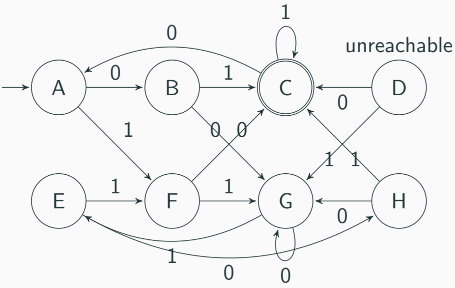
input: a DFA B

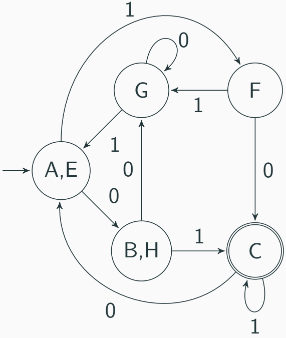
output: a DFA A which is the reduct of B

1. eliminate from B all unreachable states
2. find the indistinguishability partition on the remaining states
3. construct the reduct B:

- Q_B are the equivalence classes
- q_{0_B} is the class containing the initial state of A
- final states F_B are the classes containing some state from F_A
- the transition function: for any $a \in \Sigma$ and $S \in Q_B$ choose arbitrary $q \in S$ and define $\delta_B(S, a) = [\delta_A(q, a)]$, i.e., the class containing $\delta_A(q, a) \in Q_A$; note that this class is the same for any choice of $q \in S$ since they are all equivalent
- if there's a 'fail' state from which no final states can be reached [and if we allow partial transition functions], remove it

Example





B	×					
C	×	×				
E		×	×			
F	×	×	×	×		
G	×	×	×	×	×	
H	×		×	×	×	×
	A	B	C	E	F	G

Equivalence classes:

$$\{A, E\}, \{B, H\}, \{C\}, \{F\}, \{G\}$$

Summary of Lecture 2

- Mihyll–Nerode theorem (DFAs $\leftrightarrow$ right congruences of Σ^* of finite index where L is a union of classes)
- Equivalent automata (recognize the same language), automata homomorphism (implies automata equivalence).
- Finding reachable states: BFS on the state diagram
- Finding equivalent (indistinguishable) states: a table-filling algorithm
- Testing equivalence of DFAs, equality of regular languages
- Reduced (minimum–state) DFA, an algorithm to reduce a given DFA (using the equivalent states algorithm)