

Lecture 6 – Chomsky Normal Form, Pumping lemma for context-free languages

NTIN071 Automata and Grammars

Jakub Bulín (KTIML MFF UK)

Spring 2025

** Adapted from the Czech-lecture slides by Marta Vomlelová with gratitude.
The translation, some modifications, and all errors are mine.*

Recap of Lecture 5

- Grammars: general, context-sensitive, context-free, right-linear (regular) – Chomsky hierarchy
- The language of a grammar, derivation
- Right-linear grammars correspond to FA (and so do left/linear)
- Linear grammars are stronger
- Context-free grammars: parse tree and its yield
- (un)ambiguous grammars, inherently ambiguous languages

2.6 Chomsky Normal Form

Chomsky normal form

The **Chomsky normal form (ChNF)** of a context-free grammar:

- all rules of the form $A \rightarrow BC$ or $A \rightarrow a$ ($A, B, C \in V$, $a \in T$)
- no **useless** symbols

Theorem

For every context-free language L such that $L \setminus \{\epsilon\} \neq \emptyset$ there exists a grammar in ChNF that generates $L \setminus \{\epsilon\}$.

Applications:

- Test membership in L : the **CYK algorithm** (Sakai 1962)
- Prove the **Pumping lemma for context-free languages**

Converting to ChNF

Take any context-free grammar for L and simplify (**in this order!**):

1. eliminate **ϵ -productions** $A \rightarrow \epsilon$ [here we lose $\epsilon \in L$]
2. eliminate **unit productions** $A \rightarrow B$
3. eliminate **useless** symbols
 - 3a. **unreachable** [from the start symbol]
 - 3b. **nongenerating** [a word over terminals]

Now we have a **reduced** grammar. To get to ChNF, we further:

4. **separate** terminals from bodies
5. **break up** longer bodies

Step 1: Eliminate ϵ -productions

A variable $A \in V$ is **nullable** if $A \Rightarrow^* \epsilon$. An algorithm to find them:

basis: for every ϵ -production $A \rightarrow \epsilon$ mark A as nullable

induct: if $B \rightarrow C_1 \dots C_k \in \mathcal{P}$ where all C_i are nullable, B is nullable

To eliminate ϵ -productions: 1. find nullable variables, 2. remove ϵ -productions, 3. process every production $A \rightarrow X_1 \dots X_k \in \mathcal{P}$:

- let $J \subseteq \{1, \dots, k\}$ be the positions of all nullable variables
- for every $J' \subseteq J$ create a copy of the production where X_j for $j \in J'$ are deleted, except if $J = \{1, \dots, k\}$ require $J' \neq \emptyset$

Example: $\mathcal{P} = \{S \rightarrow AB, A \rightarrow aAB \mid \epsilon, B \rightarrow ABBA \mid \epsilon\}$

$S \rightarrow AB \mid A \mid B \quad A \rightarrow aAB \mid aA \mid aB \mid a$

$B \rightarrow ABBA \mid ABA \mid ABB \mid BBA \mid AA \mid AB \mid BA \mid BB \mid A \mid B$

Step 2: Eliminate unit productions

Idea: for a unit production $A \rightarrow B$ copy rules for B with head A , but unit productions can be composed, we need transitive closure:

Unit pairs $\mathcal{U} \subseteq V \times V$ are defined as follows:

- $(A, B) \in \mathcal{U}$ for every unit production $A \rightarrow B \in \mathcal{P}$
- if $(A, B) \in \mathcal{U}$ and $(B, C) \in \mathcal{U}$, then $(A, C) \in \mathcal{U}$

To eliminate unit productions:

1. find all unit pairs \mathcal{U}
2. remove all unit productions
3. for every unit pair $(A, B) \in \mathcal{U}$ and production $B \rightarrow \beta \in \mathcal{P}$ add the production $A \rightarrow \beta$ to \mathcal{P}

Step 2: Eliminate unit productions – an example

$$E \rightarrow T \mid E + T$$

$$F \rightarrow I \mid (E)$$

$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

$$T \rightarrow F \mid T * F$$

unit pairs:

$$(E, E), (E, F), (E, I), (E, T),$$

$$(F, F), (F, I),$$

$$(I, I),$$

$$(T, F), (T, I), (T, T)$$

the result:

$$E \rightarrow E + T \mid T * F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

$$F \rightarrow (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

$$T \rightarrow T * F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

Step 3: Eliminate useless symbols

- $X \in V \cup T$ is a **useful** symbol (in G) if there exists a derivation of the form $S \Rightarrow^* \alpha X \beta \Rightarrow^* w$ for some $w \in T^*$
- X is **useless** if it is not useful
- X is **generating** if $X \Rightarrow^* w$ for some $w \in T^*$
- X is **reachable** if $S \Rightarrow^* \alpha X \beta$ for some $\alpha, \beta \in (V \cup T)^*$

Observe:

- useful \Leftrightarrow generating and reachable
- useless \Leftrightarrow nongenerating or unreachable (we eliminate both)
- all terminals are generating

Step 3: Eliminate useless symbols – the algorithm

1. Find all generating symbols:

basis: mark all terminals $a \in T$ as generating

induct: for every production $A \rightarrow \beta$ where every symbol in the body β is generating, mark the head A as generating (incl. $A \rightarrow \epsilon$)

2. Remove all **nongenerating** symbols and rules containing them

3. Find all reachable symbols

basis: mark S as reachable

induct: for every production $A \rightarrow \beta$ where the head A is reachable mark every symbol in the body β as reachable

4. Remove all **unreachable** symbols and rules containing them

- The order is important! Eliminating nongenerating symbols can create new unreachable symbols, but not vice versa.
- **Example:** eliminate nongenerating B , then unreachable A

$$S \rightarrow AB \mid a$$

$$A \rightarrow b$$

$$S \rightarrow a$$

$$A \rightarrow b$$

$$S \rightarrow a$$

Steps 4 & 5: Separate terminals and break up long bodies

Step 4: Separate terminals from bodies

For every $a \in T$, introduce a new variable V_a and the rule $V_a \rightarrow a$.

For every rule $A \rightarrow \beta$ with $|\beta| \geq 2$, replace every terminal a by V_a .

Step 5: Break up longer bodies

Replace every rule $A \rightarrow B_1 \dots B_k$ with $k \geq 3$ with:

$$A \rightarrow B_1 C_1$$

$$C_1 \rightarrow B_2 C_2$$

$$\vdots$$

$$C_{k-2} \rightarrow B_{k-1} B_k$$

where C_1, \dots, C_{k-2} are new variables (only used for this purpose).

[Alternatively, instead of a chain use a binary tree.]

Conversion to Chomsky Normal Form

ChNF: only useful symbols and rules $A \rightarrow BC$ or $A \rightarrow a$

Theorem

For every context-free language L such that $L \setminus \{\epsilon\} \neq \emptyset$ there exists a grammar in ChNF that generates $L \setminus \{\epsilon\}$.

Proof.

Take a context-free grammar G for L . Modify it by applying steps 1, 2, 3a, 3b, 4, and 5, in order. Clearly, the result is in ChNF. After step 1 we get G' such that $L(G') = L(G) \setminus \{\epsilon\}$; the remaining steps produce equivalent grammars. Steps 2-5 don't add any ϵ -productions, 3-5 don't add unit productions, 3b-5 don't add nongenerating symbols, 4-5 don't add useless, etc. \square

Note: If we only apply 1, 2, 3a, and 3b, we get a **reduced** grammar: only useful symbols, no ϵ -productions, no unit productions.

Example

$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$
$$F \rightarrow I \mid (E)$$
$$T \rightarrow F \mid T * F$$
$$E \rightarrow T \mid E + T$$

reduce + separate

$$I \rightarrow a \mid b \mid IA \mid IB \mid IZ \mid IU$$
$$F \rightarrow LER \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$$
$$T \rightarrow TMF \mid LER \mid a \mid b \mid IA \mid IB \mid$$
$$IZ \mid IU$$
$$E \rightarrow EPT \mid TMF \mid LER \mid a \mid b \mid$$
$$IA \mid IB \mid IZ \mid IU$$
$$A \rightarrow a, B \rightarrow b, Z \rightarrow 0, U \rightarrow 1,$$
$$P \rightarrow +, M \rightarrow *, L \rightarrow (, R \rightarrow)$$

break up longer bodies

$$F \rightarrow LC_3 \mid a \mid b \mid IA \mid IB \mid IZ \mid IU$$
$$T \rightarrow TC_2 \mid LC_3 \mid a \mid b \mid IA \mid IB \mid$$
$$IZ \mid IU$$
$$E \rightarrow EC_1 \mid TC_2 \mid LC_3 \mid a \mid b \mid IA \mid$$
$$IB \mid IZ \mid IU$$
$$C_1 \rightarrow PT$$
$$C_2 \rightarrow MF$$
$$C_3 \rightarrow ER$$

$I, A, B, Z, U, P, M, L, R$ same as on
the left side

2.7 Pumping lemma for context-free languages

Pumping lemma for context-free languages

Theorem (Pumping Lemma for Context Free Languages)

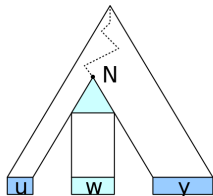
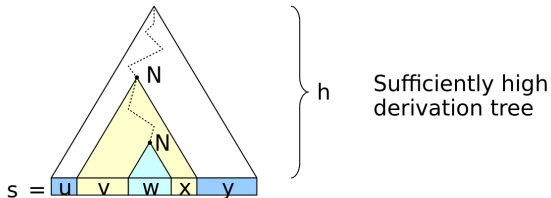
Let $L \subseteq \Sigma^$ be context-free. Then there exists $n \in \mathbb{N}$ s.t. for any $z \in L$, $|z| \geq n$ there are $u, v, w, x, y \in \Sigma^*$ s.t. $z = uvwxy$ and:*

(i) $|vwx| \leq n$ (ii) $|vx| > 0$ (iii) $uv^iwx^iy \in L$ for all $i \geq 0$

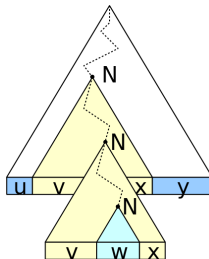
Proof idea: Take a ChNF grammar for L . If $z \in L$ is long enough, a parse tree for z must contain a path from S to a leaf (terminal) of length $|V| + 1$. Some nonterminal $N \in V$ repeats on this path giving two subtrees with root N : a larger one containing a smaller one. Replace the larger with a copy of the smaller ($i = 0$) or the smaller with a copy of the larger ($i = 2$).

What is long enough? If $|z| > 2^{k-1}$, then the depth of the tree is $k + 1$. (All inner nodes not immediately above a leaf are binary!)

The proof in a picture



Generating $uv^0 wx^0 y$



Generating $uv^2 wx^2 y$

The proof

If $L = \emptyset$ and $L = \{\epsilon\}$ trivial, take $n = 1$. Otherwise take a ChNF grammar for L . Set $n = 2^{|V|-1} + 1$. Let $z \in L$ with $|z| \geq n$.

A parse tree for z contains a path from S to a terminal t of length at least $|V| + 1$. At least two of the last $|V| + 1$ nonterminals on this path must be the same. Let A^1, A^2 be such a pair that is closest to t . Let T^1, T^2 be the subtrees rooted at A^1, A^2 .

The path from A^1 to t is the longest one in T^1 and has length at most $(k + 1)$. Thus $|vwx| \leq n$.

There are two paths from A^1 (ChNF!): one leads to T^2 , the other to the rest, it must generate at least one letter (no ϵ -productions). Thus $|vx| > 0$.

The proof cont'd

The word $z = uvwxy$ is derived as follows:

- $A^2 \Rightarrow^* w$
- $A^1 \Rightarrow^* vA^2x \Rightarrow^* vwx$
- $S \Rightarrow^* uA^1y \Rightarrow^* uvA^2xy \Rightarrow^* uvwxy$

For $i = 0$: replace T^1 by T^2

$$S \Rightarrow^* uA^2y \Rightarrow^* uwy$$

For $i = 2$: replace T^2 by a copy of T^1

$$S \Rightarrow^* uA^1y \Rightarrow^* uvA^1xy \Rightarrow^* uvvA^2xxy \Rightarrow^* uvvwxxxy$$

For $i \geq 3$ repeat the above.



Application: proving a language is not context-free

Example

The language $L = \{0^n 1^n 2^n \mid n \geq 0\}$ is not context-free.

Suppose for contradiction that it is. Let n be constant from the Pumping lemma. Choose $z = 0^n 1^n 2^n \in L$. Clearly $|z| \geq n$.

The Pumping lemma gives us a split $z = uvwxy$ satisfying (i)–(iii). Since $|vwx| \leq n$, the pumped part vx contains at most two of the symbols 0, 1, 2. Pumping will violate equal number of symbols. \square

Example

The language $L = \{0^i 1^j 2^k \mid 0 \leq i \leq j \leq k\}$ is not context-free.

Similar as above, also $z = 0^n 1^n 2^n$, at most two symbols pumped:

- if 0 or 1 are pumped, but 2 is not: pump up ($i = 2$)
- if 1 or 2 are pumped, but 0 is not: pump down ($i = 0$)

More examples

Example

$L = \{0^j 1^k 2^j 3^k \mid j, k \geq 0\}$ is not context-free.

Similar as before, choose $z = 0^n 1^n 2^n 3^n$, vx must contain some symbol. But from $|vwx| \leq n$ we know that it can contain neither both 0 and 2, nor both 1 and 3. In any case, the equal number of symbols 0 and 2 or 1 and 3 is violated. \square

Example

$L = \{ww \mid w \in \{0, 1\}^*\}$ is not context-free.

Choose $z = 0^n 1^n 0^n 1^n$, then $|z| \geq n$. The pumped part can cover neither both blocks of 0s nor both blocks of 1s. Four cases to consider: vx contains a symbol from the 1st block of 0s, 1st block of 1s, 2nd 0s, 2nd 1s. In all cases we get a violation. \square

It is not a characterization

The Pumping lemma is again only an implication, not equivalence:

Example

$L = \{a^i b^j c^k d^\ell \mid i = 0 \text{ or } j = k = \ell\}$ can be pumped. But it is not context-free.

$i = 0 : b^j c^k d^\ell$ can be pumped in any letter

$i > 0 : a^i b^n c^n d^n$ can be pumped in a^*

What to do in such cases?

- **Ogden's lemma**: generalize Pumping lemma, mark some of the letters, some marked symbol is pumped
- use closure properties of context-free languages

Summary of Lecture 6

- Reducing a grammar: removing ϵ -productions, unit productions, useless symbols
- Chomsky Normal Form of a context-free grammar
- Pumping lemma for context-free languages, application: proving non-context-freeness