Short definitions in constraint languages

Jakub Bulín[a], joint work with Michael Kompatscher

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[1] J. Bulín and M. Kompatscher: *Short definitions in constraint languages*, arXiv:2305.01984 (May 2023), accepted to MFCS 2023

The what and the why

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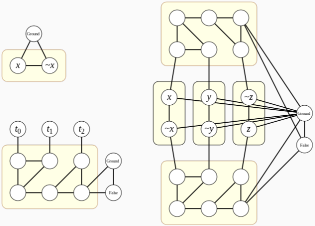

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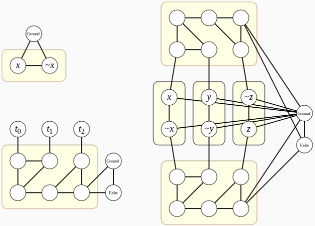

Theorem (Jeavons, Cohen, Gyssens JACM 1997)

If $\Delta \subseteq \langle \Gamma \rangle$*, then* $\mathrm{CSP}(\Delta)$ *reduces to* $\mathrm{CSP}(\Gamma)$*.*

The examples

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Γ has short definitions, if $\exists$ polynomial $p(n)$ such that each $R \in \langle\Gamma\rangle_n$ has a pp-definition $\phi(x_1, \ldots, x_n)$ of length $|\phi| \leq p(n)$.

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$\Rightarrow$ pp-definitions of length $O(n^2)$

The conjecture and the result

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- Γ_{2SAT} is invariant under the 2-edge function called majority: [2]
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(general k-edge is a "combination" of those two types of behavior)

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Conjecture (B., Kompatscher)

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True if the algebra of polymorphisms of Γ generates a residually finite variety.[3] **Corollary** True if $|A| = 3$.

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The proof

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[4]Step IV. is the only place where we need residual finiteness. Otherwise, in “$x + y = u$” the domain for u may grow too fast (in general, “$x + y \neq y + x$”).

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- $\mathbf{A}$ has pp-definitions of length $O(n^k)$ iff $\{\mathbf{A_1}, \ldots \mathbf{A_k}\}$ does, etc. (some technical work needed here)

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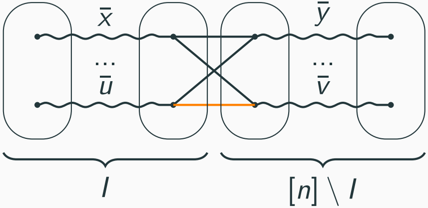

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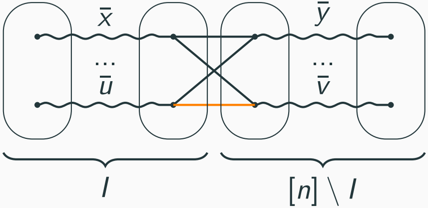

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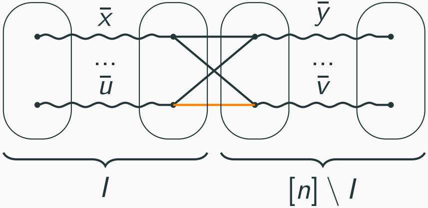

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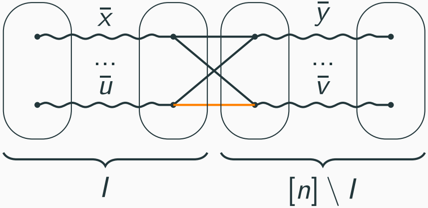

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Easy: C.p.r.'s can be defined from reduced c.p.r.'s in $O(n)$

Key Lemma: If R is a reduced c.p.r., then for any $I \subset [n]$ the algebra $\mathbf{A_I} = {}^{\mathrm{proj}_I\, R}/_{\sim_I}$ is SI. (multisorted Kearnes, Szendrei)

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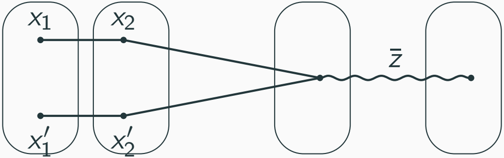

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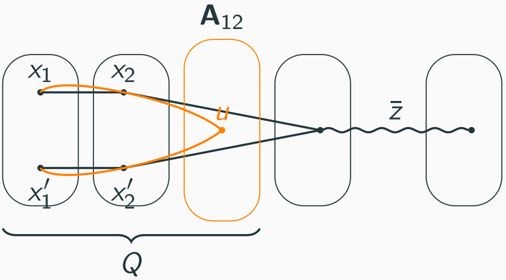

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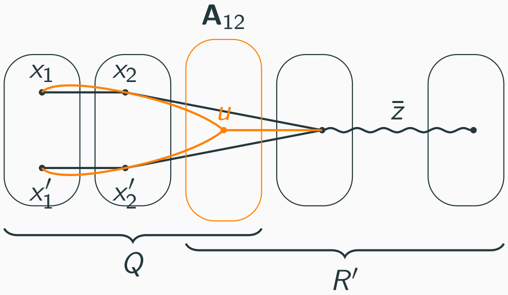

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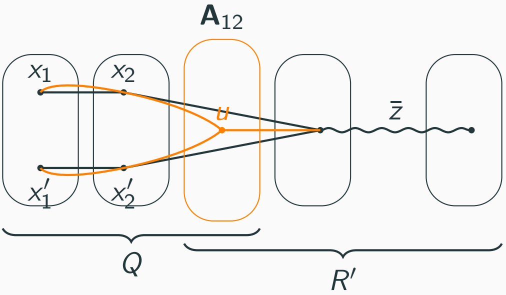

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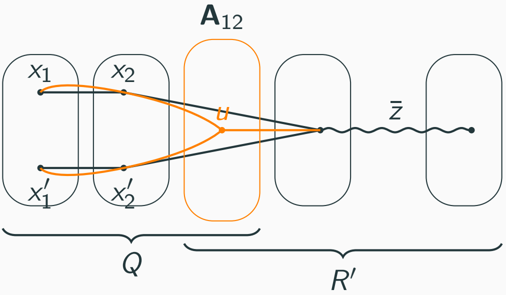

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By Key Lemma, $\mathbf{A_{12}} = \mathrm{proj}_{12} R/\sim_{12}$ is SI, so by residual finiteness it is in $\mathrm{HS}(\mathbf{A}^N)$. Thus $Q \in \Gamma'$; the arity of R' is $n-1$.

The application

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$\mathbf{A}$ is a finite algebra (e.g. the polymoprhism algebra of Γ)

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- True for $A = \{0, 1\}$, otherwise open