
A Tight Regret Analysis of Non-Parametric Repeated Contextual Brokerage

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Abstract

We study a contextual version of the repeated brokerage problem. In each interaction, two traders with private valuations for an item seek to buy or sell based on the learner’s—a broker—proposed price, which is informed by some contextual information. The broker’s goal is to maximize the traders’ net utility—also known as the gain from trade—by minimizing regret compared to an oracle with perfect knowledge of traders’ valuation distributions. We assume that traders’ valuations are zero-mean perturbations of the unknown item’s current market value—which can change arbitrarily from one interaction to the next—and that similar contexts will correspond to similar market prices. We analyze two feedback settings: full-feedback, where after each interaction the traders’ valuations are revealed to the broker, and limited-feedback, where only transaction attempts are revealed. For both feedback types, we propose algorithms achieving tight regret bounds. We further strengthen our performance guarantees by providing a tight $1/2$ -approximation result showing that the oracle that knows the traders’ valuation distributions achieves at least $1/2$ of the gain from trade of the omniscient oracle that knows in advance the actual realized traders’ valuations.

1 INTRODUCTION

We investigate repeated brokerage with contextual information, where a broker (the learner) is tasked with facilitating commerce between prospective traders. This classic setting models commerce in Over-The-Counter (OTC) markets of stock, energy, and rare minerals, to name a few, which are responsible for a massive amount of the overall world’s business volume (Lucas Jr, 1989; Weill, 2020; www.bis.org, 2022).

In this problem, during each interaction t , two traders who own identical copies of an item for which they hold private valuations V_t and W_t reach out to the broker. The traders’ goal is to make a profit by trying to sell a copy of their item if the proposed price is higher than their valuation or buy a new copy if the opposite is true. The broker observes some contextual information \mathbf{x}_t modeling the item in question and the market conditions and uses it along with his past knowledge to propose a trading price P_t to the traders. If the price is below one of the two traders’ valuations and above the other, the trader with the highest valuation buys the item from the other trader at price P_t . The broker strives to maximize the so-called *gain from trade*, i.e., the sum of the net utilities gained by the traders. Consistently with the existing literature (Bachoc et al., 2024b), we assume that traders’ valuations are independent zero-mean perturbations of market values μ_t , which can change arbitrarily over time. But unlike previous works that assume a parametric (linear) relationship between contexts and market values, we only suppose that similar contexts will correspond to similar market prices. The goal of the broker is to minimize the *regret*, defined as the loss in efficiency between the total gain from trade achieved by their strategy and the one of an idealized oracle that chooses the optimal price at each interaction given an exact knowledge of *the traders’ valuation distribution*. We also consider an even more powerful oracle that has perfect knowledge of the *realizations* of the traders’ valuation, and discuss how the techniques we develop apply to this setting.

We study two variants of this problem: the *full-feedback* setting, where the valuations of the traders are revealed to the broker after each interaction, and the *limited-feedback* setting, where nothing other than the fact that the traders attempted to buy or sell is revealed to the learner after each interaction.

1.1 Formal setting

We study the following online learning problem.

Online Protocol 1 Contextual Brokerage

- 1: Two traders arrive with private valuations V_t, W_t
 - 2: The broker observes a context \mathbf{x}_t
 - 3: The broker proposes a trading price P_t
 - 4: A trade occurs iff $\min\{V_t, W_t\} \leq P_t \leq \max\{V_t, W_t\}$
 - 5: The broker observes some feedback
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Consistently with the existing literature, we set the reward associated with each interaction as the *gain from trade*: the sum of the net utilities of the traders. Formally, for any $p, v, w \in [0, 1]$, letting $v \vee w := \max\{v, w\}$ and $v \wedge w := \min\{v, w\}$, the utility of a price p when the valuations of the traders are v and w is

$$g(p, v, w) := \underbrace{(v \vee w - p)}_{\text{buyer's net gain}} + \underbrace{p - v \wedge w}_{\text{seller's net gain}} \mathbb{I}\{v \wedge w \leq p \leq v \vee w\} \\ = (v \vee w - v \wedge w) \mathbb{I}\{v \wedge w \leq p \leq v \vee w\}.$$

The aim of the learner is to post prices P_t (depending the history up to time $t-1$, the current context \mathbf{x}_t and, possibly, some internal randomization) that minimize the *regret* against the best sequence of deterministic prices¹, irrespectively of the underlying instance determining contexts and traders' valuations. Formally: for any time horizon $T \in \mathbb{N}$, we define the *regret* as

$$R_T := \sup_{(\mathbf{x}_t, V_t, W_t)_{t \in \mathbb{N}} \in \mathcal{J}} \left(\sup_{p_1, \dots, p_T \in [0, 1]} \mathbb{E} \left[\sum_{t=1}^T g(p_t, V_t, W_t) \right] - \mathbb{E} \left[\sum_{t=1}^T g(P_t, V_t, W_t) \right] \right),$$

where the expectations are taken with respect to the randomness in $(V_t, W_t)_{t \in \mathbb{N}}$ and, possibly, the internal randomization used to choose the trading prices $(P_t)_{t \in \mathbb{N}}$, and the first supremum is over the instance set \mathcal{J} which consists of all sequences $(\mathbf{x}_t, V_t, W_t)_{t \in \mathbb{N}}$ of contexts and traders' valuations such that:

¹Economically, this benchmark models the best choice of an oracle that knows the distributions but not the realizations of the valuations. We will prove later that this is not too different from comparing against the best random prices $p_1, \dots, p_T \in [0, 1]$ (that do have access to the realizations of the valuations in hindsight).

1. For all $t \in \mathbb{N}$, the context \mathbf{x}_t belongs to $[0, 1]^d$.
2. There exists a sequence of *market values* μ_1, μ_2, \dots in $[0, 1]$ such that, for all $t, t' \in \mathbb{N}$, market values $\mu_t, \mu_{t'}$, and contexts $\mathbf{x}_t, \mathbf{x}_{t'}$, it holds that $|\mu_t - \mu_{t'}| \leq L \|\mathbf{x}_t - \mathbf{x}_{t'}\|_\infty$. To lighten the notation, we assume $L = 1$ without loss of generality.
3. The traders' valuations $V_1, W_1, V_2, W_2, \dots$ form an independent sequence of random variables and, for all $t \in \mathbb{N}$, the traders' valuations V_t and W_t are $[0, 1]$ -valued random variables admitting densities upper bounded by some constant $M > 0$ with a common expectation equal to the current market value $\mu_t \in [0, 1]$.

Finally, we consider the two most studied types of feedback in the bilateral trade literature. Specifically, at each round t , only after having posted the price P_t , the learner receives either:

- *Full feedback*, i.e., the valuations V_t and W_t of the two current traders are disclosed.
- *Limited feedback*, i.e., only the indicator functions $\mathbb{I}\{P_t \leq V_t\}$ and $\mathbb{I}\{P_t \leq W_t\}$ are disclosed.

Discussion on modeling assumptions. In Item 1, we assume that the context space is $[0, 1]^d$ merely for the sake of convenience, and without loss of generality. Our theory can be extended straightforwardly to any bounded context space at the cost of a more cumbersome notation. In Item 2, we assume that similar contexts relate to similar market prices. This natural modeling assumption quantifies the intuitive expectation that, e.g., if the broker knows that today's traders are trying to trade 99%-pure gold, their valuations, on average, will be close to yesterday's valuations for 98%-pure gold, and likely far from last weeks's valuation for 70%-pure iron. In Item 3, we allow for fluctuations of the perceived market price from the perspective of the traders. Note that we do not require the sequence of valuations to be i.i.d.. Also note that the assumption that valuations admit a bounded density cannot be lifted, as it has been shown that even in a simplified, special case of our setting (Bolić et al., 2024, Theorem 9), learning becomes impossible with limited feedback when this assumption is removed.

Discussion on feedback models. The information gathered in the full feedback model reflects *direct revelation mechanisms*, where traders disclose their valuations V_t and W_t prior to each round, but the price determined by the mechanism at time t is guaranteed to be based solely on the previous valuations $V_1, W_1, \dots, V_{t-1}, W_{t-1}$. Conversely, the limited feedback model reflects *posted price mechanisms*. In this

model, traders only indicate their willingness to buy or sell at the posted price, and their valuations V_t and W_t remain undisclosed.

1.2 Our contributions

Under the assumptions described in Section 1.1, and with the goal of designing *simple* and *interpretable* optimal algorithms, we make the following contributions.

1. For the full-feedback setting, we design the BiAve algorithm (Algorithm 2) and show an upper bound on its regret after T interactions of order $T^{\frac{d}{d+2}}$, where d is the dimension of the context space (Theorem 1).
2. We prove the optimality of our result in the full-feedback setting, showing that no other algorithms can achieve a regret of smaller order than BiAve (Theorem 2).
3. For the limited-feedback setting, we design the ExBis algorithm (Algorithm 3) and show an upper bound on its regret after T interactions of order $T^{\frac{d+2}{d+4}}$ (Theorem 3).
4. We prove the optimality of our result in the limited-feedback setting, showing that no other algorithms can achieve a regret of smaller order than ExBis (Theorem 4).
5. Finally, we discuss an even stronger benchmark, known as *first-best*, where the oracle is omniscient and chooses the optimal price at each interaction with exact knowledge of what *the realizations of traders' valuations* will be, therefore, never missing a trading opportunity. We prove that the classic benchmark oracle that chooses prices with exact knowledge of the traders' distributions earns a gain from trade that is at least $1/2$ of that of this omniscient oracle (Theorem 5). This yields, in particular, that the performance of our learning algorithms is at most $1/2$ away from that of an omniscient oracle that knows everything about all traders. Furthermore, we prove that this $1/2$ -approximation factor is unimprovable (Theorem 6). To the best of our knowledge, this is the first work on online learning in the brokerage setting where this kind of approximation result is achieved.

1.3 Related work

The literature on bilateral trade is extremely rich and has experienced a steady growth since the fundamental work of Myerson and Satterthwaite (1983). Classically, bilateral trade has been explored in the one-shot

setting, mainly from a game-theoretic and approximation perspective (Colini-Baldeschi et al., 2016, 2017; Blumrosen and Mizrahi, 2016; Brustle et al., 2017; Colini-Baldeschi et al., 2020; Babaioff et al., 2020; Dütting et al., 2021; Deng et al., 2022; Kang et al., 2022; Archbold et al., 2023). For a fairly complete overview on this literature, see, e.g., Cesa-Bianchi et al. (2023a). On the other hand, a recent stream of literature explored bilateral trade in a repeated setting through the lens of online learning. Being the most relevant for our work, we focus on this literature.

In Cesa-Bianchi et al. (2021); Azar et al. (2022); Cesa-Bianchi et al. (2023a,b); Bernasconi et al. (2024); Cesa-Bianchi et al. (2024), the authors examined the non-contextual repeated bilateral trade problem with pre-defined seller and buyer roles: at each interaction, a new seller/buyer pair arrives, the broker proposes a trading price, and the current item is traded if and only if the proposed price is higher than the private valuation of the seller and lower than the private valuation of the buyer. When this happens, the buyer pays the trading price to the seller, the seller gives the item to the buyer, and the broker is rewarded with the gain from trade, i.e., the sum of the seller's and buyer's utility. In Cesa-Bianchi et al. (2021, 2023a), the authors investigated and obtained sharp regret bounds when sellers' and buyers' valuations, represented by two random sequences of numbers $(S_t)_{t \in \mathbb{N}}, (B_t)_{t \in \mathbb{N}}$, form two i.i.d. sequences, while showing that the adversarial case is unlearnable in general. Azar et al. (2022) managed to obtain learnability in the adversarial case by relaxing the notion of regret to the one of 2-regret. When the platform can post two different prices to sellers and buyers, but still not being allowed to subsidize trades, Cesa-Bianchi et al. (2023b, 2024) achieved learnability using the usual notion of regret when the adversary belongs to the class of *smoothed* adversaries. Bernasconi et al. (2024) managed to achieve learnability in the adversarial case by allowing the platform to subsidize trade, as long as the subsidizing comes from revenue obtained from previous sellers and buyers interactions. On a different direction, Bachoc et al. (2024a) investigated how to achieve *fairness* in the repeated bilateral trade problem between seller's and buyer's earnings by rewarding the broker with the minimum between the seller's utility and the buyer's utility, instead of the gain from trade.

Bolić et al. (2024) introduced the non-contextual repeated bilateral trade setting where the traders have unspecified seller's and buyer's role (brokerage) and obtained sharp learning rates in the i.i.d. setting when the reward function is the gain from trade. Cesari and Colomboni (2025) focused on the same non-contextual setting, but with the different objective of maximiz-

ing the total number of trades. By investigating a contextual bilateral trade with unspecified seller and buyer roles with the gain from trade as reward function, Bachoc et al. (2024b) is the closest to our setting. However, they assume a linear (and hence, parametric) relationship between contexts and marker values, while we relax this assumption by imposing only that close enough contexts give rise to close enough market values.

Our work also shares some similarities with contextual bandits (Slivkins, 2011). Our largest differentiation compared to contextual bandits is provided by our results in the limited feedback setting (Section 3, which we recommend reaching to better understand the following discussion). There, the concepts of exploration and exploitation are different from the contextual bandit setting. With bandits, exploitation consists in choosing an arm which is currently expected to be the best. Nevertheless, the feedback of exploitation rounds is still a realization from an arm and can be used to learn the arm mean. In contrast, in our setting, exploitation consists in playing our estimate of the mean μ_t for context \mathbf{x}_t , and this has no value at all for learning the mean. Hence, our feedback for exploitation rounds are not used later on in Algorithm 3 (ExBis), since the mean estimates on Lines 15 and 16 only use feedback from exploration rounds. In contrast, in (Slivkins, 2011), Algorithm 1, Line 12, each feedback can be used to update the various estimators for the algorithm. Regarding exploration, our exploration rounds consist in playing uniformly distributed prices (which is a feature specific to our limited feedback in the brokerage problem, and absent from bandit algorithms). Only the feedback from these rounds can be used, but we bound their instantaneous regret by the maximal possible value 1. In contrast, with bandits algorithms, exploration would consist in playing an arm which is not expected to be optimal, but can still be close to optimal with a regret bound much better than 1. On exploration/exploitation, another difference is that in our setting each round is tagged as one or the other (Lines 14 and 17 in Algorithm 3), while in (Slivkins, 2011) and more broadly with bandits, each round is a mix of both, for instance with an upper confidence bound rule (see Slivkins 2011, Section 4.2). Finally, the multiple differences above have an impact on the regret rates that are achieved. In (Slivkins, 2011), the worst rate is $T^{(1+d_c)/(2+d_c)}$ (see (7), there) for the dimension d_c of the space where the Lipschitz mean function is defined. In our setting, the rate is $T^{(d+2)/(d+4)}$.

1.4 Techniques and challenges

In the full-feedback case, Lemma 1 in Appendix E can be used to reduce our problem to a full-feedback online adversarial contextual regression problem where, at each time step t , the learner is presented with a new context \mathbf{x}_t , asked to make a prediction y_t , suffers a corresponding loss $\ell_t(y_t)$, then observes ℓ_t . Existing techniques (Hazan and Megiddo, 2007; Cesa-Bianchi et al., 2017) study this problem with the goal of competing against the best Lipschitz policy that maps contexts to predictions. However, a black-box application of these techniques requires noiseless feedback, i.e., the learner needs to reconstruct exactly the loss function at the end of each round (which in our setting would be the *expected* gain from trade function). In contrast, in our setting, after having access to the traders' valuations V_t and W_t , we observe only noisy realizations of the market price μ_t , which in turn translates into the fact that we observe only *noisy* realizations of the associated loss functions in the aforementioned reduction. To circumvent this problem, we take a different route and devise *ad hoc* techniques to estimate the value of our reward function at specific points. Specifically, we partition the context space in dyadic cells, and use the feedback we receive from previous rounds to predict the value of the reward function for contexts that belong to the same cell. Importantly, when sufficiently many points land in the same region, the dyadic partition is adaptively refined to increase the precision of the estimates by relying only on the information retrieved from the closest contexts. A suitable choice of the criterion to further split the dyadic cells gives the optimal rate. For the lower bound, it is important to note that we do not have direct control on the reward functions, but we need to devise suitable instances of traders' distributions in order to produce hard instances for our problem. Once this is done, a lattice of sufficiently-spaced contexts is built and the contexts on this lattice are repeatedly presented to the learner. Finally, we determine a suitable horizon-dependent tuning of the number of points in this lattice and of the number of times that each of these contexts should be repeatedly presented so that the learner cannot infer any non-trivial information about the market value associated to the next context in the lattice.

In the limited feedback case there is a further layer of complexity: the feedback we receive depends on the posted price and is not even enough to directly reconstruct *bandit* feedback, i.e., the realized gain associated with the action we performed. For this reason, we cannot directly rely on existing techniques to solve bandit online adversarial contextual regression problems, but we need to devise novel techniques tai-

lored to our problem. Specifically, we first show that a Monte Carlo sampling procedure can be performed to reconstruct an estimate of the market value associated to a certain context. On the other hand, it is important to note that this exploration procedure is costly (it requires posting prices with low gain from trade). By relying on an adaptive dyadic partition of the context space, we show how to properly balance exploration rounds (where we use this Monte Carlo procedure to estimate the reward function) and exploitation rounds (where we use this information to increase our total reward) to obtain optimal regret bounds. The lower bound construction relies on the same lattice construction we used in the full-feedback case, but with the further layer of complexity of devising instances where, given the limited feedback, potentially optimal actions do not reveal *any* meaningful information about their actual optimality, forcing the learner to explore in costly regions in order to obtain that piece of information, which loosely resembles the exploration/exploitation dilemma of the so-called revealing action problem (Cesa-Bianchi and Lugosi, 2006). This further layer of complexity requires a different tuning (with respect to the full feedback case) of the points in the lattice and the number of rounds in a row that contexts are presented.

Finally, the $1/2$ -approximation result of the *first-best* cannot be deduced from the corresponding existing literature on bilateral trade: the closest result to ours in this literature provides approximations of the *first-best* when the traders have definite seller and buyer roles, and, crucially, it is required that they share the same distribution (Kang and Vondrák, 2019). In contrast, in our case, each trader is allowed to sell and buy, and while they share the same expected value, they do not share the same distribution. For this reason, devise an entirely new proof to deduce our approximation result of the *first-best*.

2 FULL FEEDBACK

In this section, we analyze how efficiently the broker can learn by leveraging full feedback.

2.1 BiAve and regret upper bound

In this section, we introduce and analyze our BiAve algorithm for the full-feedback setting. We begin with some notation. For a subset $\mathcal{S} \subseteq \mathbb{R}$, we let $\mathcal{S}^- = \inf \mathcal{S}$, $\mathcal{S}^+ = \sup \mathcal{S}$, and $\text{length}(\mathcal{S}) = \mathcal{S}^+ - \mathcal{S}^-$. For a subset $\mathcal{S} \subseteq \mathbb{R}^d$ of the form $\mathcal{S} := \mathcal{I}_1 \times \dots \times \mathcal{I}_d$, where $\mathcal{I}_1, \dots, \mathcal{I}_d$ are left-closed and right-open intervals of same length, we define $\text{length}(\mathcal{S})$ as this common length, and we define $\text{Bisect}(\mathcal{S})$ as the set containing the 2^d hypercubes

specified below

$$\text{Bisect}(\mathcal{S}) := \{\mathcal{I}_{1,a_1} \times \dots \times \mathcal{I}_{d,a_d} \mid a_1, \dots, a_d \in \{-, +\}\},$$

where, for $j \in [d]$, $\mathcal{I}_{j,-} := [\mathcal{I}_j^-, (\mathcal{I}_j^- + \mathcal{I}_j^+)/2)$ and $\mathcal{I}_{j,+} := [(\mathcal{I}_j^- + \mathcal{I}_j^+)/2, \mathcal{I}_j^+)$. We consider dyadic hypercubes—which for brevity, we call *cells* in what follows—of the form $\prod_{j=1}^d [k_j 2^{-i}, (k_j + 1) 2^{-i})$, for $k_1, \dots, k_d \in \{0, \dots, 2^i - 1\}$ and $i \in \{0, 1, 2, \dots\}$, and we say that this i is the *level* of the cell. Given a family of cells \mathcal{F} , we say that $\mathcal{C} \in \mathcal{F}$ is *terminal* if no other cell in the family is properly contained in \mathcal{C} . If \mathcal{C} is a cell of level $i \geq 1$, its *parent* is the only cell of level $i - 1$ that contains it, which we denote by \mathcal{C}' . By convention, we say that the cell $[0, 1)^d$ is the parent of itself, so if $\mathcal{C} = [0, 1)^d$ then $\mathcal{C}' = \mathcal{C}$. The pseudocode of our BiAve algorithm is present in Algorithm 2. At a

Algorithm 2 BiAve (Bisect and average)

initialization: Set $\mathcal{F} := \{[0, 1)^d\}$

- 1: The environment reveals a context $\mathbf{x}_1 \in [0, 1)^d$
 - 2: Post price $P_1 := \frac{1}{2}$
 - 3: Observe V_1 and W_1
 - 4: **for** $t = 2, 3, \dots$ **do**
 - 5: The environment reveals a context $\mathbf{x}_t \in [0, 1)^d$
 - 6: Let \mathcal{C}_t be the terminal cell in \mathcal{F} such that $\mathbf{x}_t \in \mathcal{C}_t$
 - 7: Let i_t be the level of \mathcal{C}_t
 - 8: Let n_t be the number of $s \in [t - 1]$ such that $\mathbf{x}_s \in \mathcal{C}_t$
 - 9: **if** $\mathcal{C}_t = [0, 1)^d$ **then**
 - 10: Let $n'_t := n_t$
 - 11: **else**
 - 12: Let q_t be the time at which the parent cell \mathcal{C}'_t of \mathcal{C}_t was bisected
 - 13: Let n'_t be the number of $s \in [q_t]$ s.t. $\mathbf{x}_s \in \mathcal{C}'_t$
 - 14: **if** $n_t \geq n'_t$ **then**
 - 15: Post price $P_t := \frac{1}{2n_t} \sum_{s=1}^{t-1} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}_t\} (V_s + W_s)$
 - 16: **else if** $n_t < n'_t$ **then**
 - 17: Post price $P_t := \frac{1}{2n'_t} \sum_{s=1}^{q_t} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}'_t\} (V_s + W_s)$
 - 18: **if** $\sqrt{n_t} \geq 2^{i_t}$ **then**
 - 19: Add the family of cells $\text{Bisect}(\mathcal{C}_t)$ to \mathcal{F}
 - 20: Observe V_t and W_t
-

high level, the algorithm discretizes the context space $[0, 1)^d$ into cells of *locally-adaptive granularity*. The price P_t proposed at any time step t is computed as one of two possible empirical averages, depending on whether or not the number of past contexts that fell into the terminal cell \mathcal{C}_t containing the current context \mathbf{x}_t is larger than the number of contexts that fell into the parent cell \mathcal{C}'_t of \mathcal{C}_t before \mathcal{C}'_t was bisected (Lines 14 and 16). If sufficiently many contexts fell into \mathcal{C}_t , then P_t is chosen as the empirical average of all valuations observed in past rounds s where contexts \mathbf{x}_s fell into

\mathcal{C}_t (Line 15). Otherwise, P_t is chosen as the empirical average of the valuations observed in past rounds s where contexts \mathbf{x}_s fell into the parent cell \mathcal{C}'_t , up to the time where \mathcal{C}'_t was bisected (Line 17). Finally, as soon as “too many” contexts have fallen into the same terminal cell (Line 18), the algorithm bisects it to increase the granularity of the estimation in that context region (Line 19). We now provide theoretical guarantees for the performance of BiAve.

Theorem 1. *In the full-feedback setting, if we run the BiAve algorithm for T time steps, its regret satisfies*

$$R_T = O\left(T^{\frac{d}{d+2}}\right).$$

For a full proof of this result, see Appendix A.

Proof sketch. We begin by claiming that the optimal price to propose at any time $t \in \mathbb{N}$ is $P_t := \mu_t$, where μ_t is the market price at time t (see Lemma 1 in Appendix E), and that posting any other price p would result in a instantaneous regret of order $O((\mu_t - p)^2)$, i.e., quadratic in the distance from the market price μ_t (again, see Lemma 1 in Appendix E). We also note that the updates of \mathcal{F} during a run of the algorithm are deterministic, since they only depend on the adversarial sequence of contexts $\mathbf{x}_1, \mathbf{x}_2, \dots$. Using these facts, we can prove that the two different rules the algorithm uses to determine its proposed prices P_t (on Lines 15 and 17) are sufficiently accurate approximations of μ_t .

Given that both rules are empirical averages of past traders’ valuations coming from rounds in which contexts fell in a common cell, this boils down to quantifying the bias and variance. First, although the empirical averages are *biased* estimates of market prices μ_t , this bias can be controlled. In the case of Line 15, all the observations V_s, W_s are associated to contexts \mathbf{x}_s in a cell of diameter $O(2^{-i_t})$, so that by the Lipschitz property of μ_t , the squared bias has order $O(2^{-2i_t})$. By independence, the variance has order $O(\frac{1}{n_t})$. Our bisection condition at Line 18 then ensures that the variance also has order $O(2^{-2i_t})$. We have a similar conclusion for Line 17.

The last main step of the proof is to consider any potential cell \mathcal{C} and to count the number $n_{\mathcal{C}}$ of time steps $t \in [T]$ where this cell is equal to \mathcal{C}_t on Line 6. Then we can show $n_{\mathcal{C}} \leq 2^{2i_{\mathcal{C}}}$ with $i_{\mathcal{C}}$ being the level of \mathcal{C} (from Line 18). By partitioning the cumulated regret according to all these possible cells \mathcal{C} , that can be indexed as $\{\mathcal{C}_{i,j}\}_{i=0,j=1}^{\infty,2^{id}}$, we obtain an upper bound of the order

$$\sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} 2^{-2i},$$

with the constraints that

$$n_{\mathcal{C}_{i,j}} \leq 2^{2i} \quad \text{and} \quad \sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} \leq T.$$

Some final technicalities yield the theorem. \square

2.2 Regret lower bound: BiAve is optimal

In this section, we prove the optimality of BiAve in the full-feedback setting, by showing that any other algorithm will pay a regret of order at least $T^{\frac{d}{d+2}}$.

Theorem 2. *In the full-feedback setting, for any time horizon T , any algorithm suffers regret*

$$R_T = \Omega\left(T^{\frac{d}{d+2}}\right).$$

For a full proof of this result, see Appendix B.

Proof sketch. The key idea of the lower bound is to pack the context space $[0, 1]^d$ with a lattice of k equispaced points. The environment then selects contexts as follows. For any given time horizon T , it begins by revealing one of the contexts \mathbf{x}' for T/k consecutive rounds, then moves on to a different context \mathbf{x}'' and reveals this second context for T/k rounds, and so on until all k contexts in the lattice have been revealed for T/k rounds each. This way, for each of the k points in the lattice, a learner will observe T/k noisy realizations of its corresponding market value. Leveraging this construction, we show that no learner is able to confidently distinguish market values corresponding to consecutive points in the lattice (and, *a fortiori* this cannot be done for contexts that are further away) if their corresponding market values are too close. The idea is then to select a sequence of market values compatible with the contexts (i.e., that are close enough if contexts are close enough) that are still far enough to guarantee a sufficiently high regret. We do this by setting a threshold level and, for each context in the lattice, randomly raising or lowering the corresponding market value by a small constant. The result then follows by tuning the number of elements in the lattice and by proving that these random perturbations of a threshold market value can be done in a way that respects our modeling assumptions. \square

3 LIMITED FEEDBACK

In this section, we analyze how efficiently the broker can learn by leveraging limited feedback.

3.1 ExBis and regret upper bound

In this section, we introduce and analyze our ExBis algorithm for the limited-feedback setting. In addi-

tion to the bisection notation presented in Section 2.1, we introduce here the i.i.d. sequence of $[0, 1]$ -valued uniform random variables U_1, U_2, \dots that are used by ExBis as random seeds, and that are independent of the sequence of traders' valuations $V_1, W_1, V_2, W_2, \dots$. The pseudocode of our ExBis algorithm is presented in Algorithm 3. The algorithm leverages the adap-

Algorithm 3 ExBis (Exploit, Explore, and Bisect)

initialization: Set $\mathcal{F} := \{[0, 1]^d\}$

- 1: The environment reveals a context $\mathbf{x}_1 \in [0, 1]^d$
- 2: Post $P_1 := U_1$ and add $\text{Bisect}([0, 1]^d)$ to \mathcal{F}
- 3: Observe $\tilde{V}_1 := \mathbb{I}\{P_1 \leq V_1\}$ and $\tilde{W}_1 := \mathbb{I}\{P_1 \leq W_1\}$
- 4: **for** $t = 2, 3, \dots$ **do**
- 5: The environment reveals a context $\mathbf{x}_t \in [0, 1]^d$
- 6: Let \mathcal{C}_t be the terminal cell in \mathcal{F} such that $\mathbf{x}_t \in \mathcal{C}_t$
- 7: Let i_t be the level of \mathcal{C}_t
- 8: Let m_t be the number of $s \in [t-1]$ such that $m_s < 2^{4i_s}$ (exploiting) **and** $\mathcal{C}_s = \mathcal{C}_t$
- 9: Let \mathcal{E}_t be the set of $s \in [t-1]$ such that $m_s \geq 2^{4i_s}$ (exploring) **and** $\mathbf{x}_s \in \mathcal{C}_t$
- 10: Let n_t be the cardinality of \mathcal{E}_t
- 11: Let q_t be the time at which the parent cell \mathcal{C}'_t of \mathcal{C}_t was bisected
- 12: Let \mathcal{E}'_t be the set of $s \in [q_t]$ such that $m_s \geq 2^{4i_s}$ (exploring) **and** $\mathbf{x}_s \in \mathcal{C}'_t$
- 13: Let n'_t be the cardinality of \mathcal{E}'_t
- 14: **if** $m_t < 2^{4i_t}$ **then** \triangleright (exploiting)
- 15: **if** $n_t \geq n'_t$ **then** Post $P_t := \frac{1}{2n_t} \sum_{s \in \mathcal{E}_t} (\tilde{V}_s + \tilde{W}_s)$
- 16: **if** $n_t < n'_t$ **then** Post $P_t := \frac{1}{2n'_t} \sum_{s \in \mathcal{E}'_t} (\tilde{V}_s + \tilde{W}_s)$
- 17: **else if** $m_t \geq 2^{4i_t}$ **then** \triangleright (exploring)
- 18: Post $P_t := U_t$
- 19: **if** $n_t \geq 2^{2i_t} - 1$ **then** Add $\text{Bisect}(\mathcal{C}_t)$ to \mathcal{F}
- 20: Observe $\tilde{V}_t := \mathbb{I}\{P_t \leq V_t\}$ and $\tilde{W}_t := \mathbb{I}\{P_t \leq W_t\}$

tive granularity insights of BiAve, with the additional challenges arising from the limited feedback. Readers familiar with multiarmed bandits might have noticed that the limited feedback is less informative than the already limited bandit feedback, not even being sufficient to compute the reward function at the posted price. To get around this roadblock, ExBis reserves exploration rounds (Line 17) where uniform prices are posted to allow gathering estimates of the market value at the cost of a high instantaneous regret. In the exploitation rounds (Line 14), instead, the algorithm posts an empirical average of the estimates of the market value gathered in past rounds where contexts were close enough to the current one. Finally, the algorithm locally increases the granularity of the estimates by bisecting areas where sufficiently many contexts have fallen. We now provide theoretical guarantees for the performance of ExBis.

Theorem 3. *In the limited-feedback setting, if we run*

the ExBis algorithm for T time steps, its regret satisfies

$$R_T = O\left(T^{\frac{d+2}{d+4}}\right).$$

For a full proof of this result, see Appendix C.

Proof sketch. Similarly to the proof of Theorem 1, we begin by observing that the optimal price to propose at any time t is the market price μ_t , that posting any other price p would result in an instantaneous regret of order $O((\mu_t - p)^2)$, and that the updates of \mathcal{F} , as well as all the quantities appearing in the algorithm (with the only exceptions of \tilde{V} , \tilde{W} , and P_t) are deterministic, since they only depend on the adversarial sequence of contexts $\mathbf{x}_1, \mathbf{x}_2, \dots$.

Similarly as when proving Theorem 1, we consider any potential cell $\mathcal{C}_{i,j}$ and the cumulated regret over all time steps $t \in [T]$ where this cell is equal to \mathcal{C}_t on Line 6 of Algorithm 3. An important difference compared to Theorem 1 is that now the regret comes both from exploration and exploitation rounds. For the sequence of time steps $t \in [T]$ where $\mathcal{C}_{i,j} = \mathcal{C}_t$ on Line 6, Algorithm 3 starts with many exploitation rounds (Line 14), and only when a sufficient number of them is achieved, do exploration rounds occur (Line 17). As a consequence, we can show that it is sufficient to bound the regret stemming from exploitation rounds only.

As for the proof of Theorem 1, the instantaneous regret of one exploitation round is $O(2^{-2i})$. The number of these exploitation rounds for $\mathcal{C}_{i,j}$, $n_{\mathcal{C}_{i,j}}$, is bounded by $O(2^{4i})$ from Line 17 (in the proof of Theorem 1, the bound was $O(2^{2i})$ which explains the final difference of order of bounds between the two theorems). In the end, we obtain an upper bound of the order

$$\sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} 2^{-2i},$$

with the constraints that

$$n_{\mathcal{C}_{i,j}} \leq 2^{4i} \quad \text{and} \quad \sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} \leq T.$$

As for the proof of Theorem 1, some final technicalities yield the theorem. \square

3.2 Regret lower bound: ExBis is optimal

In this section, we prove the optimality of ExBis in the full-feedback setting, by showing that any other algorithm will pay a regret of order at least $T^{\frac{d+2}{d+4}}$.

Theorem 4. *In the limited-feedback setting, for any time horizon T , any algorithm suffers regret*

$$R_T = \Omega\left(T^{\frac{d+2}{d+4}}\right).$$

For a full proof of this result, see Appendix D

Proof sketch. For this lower bound, we leverage the same lattice construction we used in the full-feedback case. The main difference is that now the feedback depends on the algorithm in a way that allows us to build two different sequences of traders' valuations distributions (i.e., instances) at each lattice point with the following high-level properties: A price p_1 is optimal in the first instance and suboptimal on the second; A price p_2 is optimal in the second instance and suboptimal in the first; Neither p_1 nor p_2 reveal any meaningful feedback; there exists a third price p_0 , which is highly suboptimal in both instances but, every time it is chosen, it reveals some information about the underlying instance being the first or the second one. Tuning everything properly and showing that one such construction can be obtained without violating our modeling assumptions gives the result. \square

4 $\frac{1}{2}$ -APPROXIMATION OF FIRST-BEST

In this section, we show that our theory yields a $\frac{1}{2}$ -approximation of the performance of an omniscient oracle with perfect information about the *realizations* of the traders' valuations. This powerful oracle is known in game theory and economics as *first-best*. We also prove that our $\frac{1}{2}$ -approximation of the first-best is tight, i.e., that no approximation rate better than $\frac{1}{2}$ can be obtained in general.

Theorem 5. *Suppose that V and W are two bounded non-negative independent random variables admitting bounded densities, with cumulative distributions F and G , respectively. Assume that $\mathbb{E}[V] = \mathbb{E}[W] =: \mu$. Then*

$$\begin{aligned} \max_{p \in [0,1]} \mathbb{E}[g(p, V, W)] &= \mathbb{E}[g(\mu, V, W)] \\ &\geq \frac{1}{2} \cdot \mathbb{E}[|W - V|] = \frac{1}{2} \cdot \mathbb{E}\left[\max_{p \in [0,1]} g(p, V, W)\right]. \end{aligned}$$

Proof. Integrating by part twice, we get

$$\begin{aligned} &\int_0^{+\infty} F(\lambda) (1 - G(\lambda)) d\lambda \\ &= \lim_{u \rightarrow \infty} \left[\int_0^\lambda F(v) dv (1 - G(\lambda)) \right]_{\lambda=0}^u \\ &\quad + \int_0^{+\infty} \int_0^\lambda F(v) dv dG(\lambda) \\ &= \int_0^{+\infty} \int_0^\lambda F(v) dv dG(\lambda) = \mathbb{E}\left[\int_0^W F(v) dv\right] \\ &= \mathbb{E}\left[\left[-(W - v)F(v)\right]_{v=0}^W + \int_0^W (W - v) dF(v)\right] \end{aligned}$$

$$\begin{aligned} &= \mathbb{E}\left[\int_0^W (W - v) dF(v)\right] \\ &= \mathbb{E}\left[\int_0^{+\infty} (W - v) \mathbb{I}\{v \leq W\} dF(v)\right] \\ &= \mathbb{E}\left[\mathbb{E}[(W - V) \mathbb{I}\{V \leq W\} \mid W]\right] \\ &= \mathbb{E}[(W - V) \mathbb{I}\{V \leq W\}]. \end{aligned}$$

Analogously, switching the role of V and W , we can prove that

$$\int_0^{+\infty} G(\lambda) (1 - F(\lambda)) d\lambda = \mathbb{E}[(V - W) \mathbb{I}\{W \leq V\}].$$

It follows that

$$\begin{aligned} \mathbb{E}[|W - V|] &= \mathbb{E}[(W - V) \mathbb{I}\{V \leq W\}] \\ &\quad + \mathbb{E}[(V - W) \mathbb{I}\{W \leq V\}] \\ &= \int_0^{+\infty} F(\lambda) (1 - G(\lambda)) d\lambda + \int_0^{+\infty} G(\lambda) (1 - F(\lambda)) d\lambda. \end{aligned} \tag{1}$$

Now, given that $(F + G)(0) = 0$, that $\lim_{u \rightarrow +\infty} (F + G)(u) = 2$ and that $F + G$ is continuous, there exists (and we fix) $p \in (0, +\infty)$ such that $(F + G)(p) = 1$. Then

$$\begin{aligned} \mathbb{E}[g(\mu, V, W)] &\geq \mathbb{E}[g(p, V, W)] \\ &= \int_0^p (F + G)(\lambda) d\lambda + (\mu - p)(F + G)(p) \\ &= \int_0^p (F + G)(\lambda) d\lambda + \mu - p \\ &= \int_0^p (F + G)(\lambda) d\lambda + \frac{1}{2} \int_0^{+\infty} (1 - F(\lambda) + 1 - G(\lambda)) d\lambda - p \\ &= \int_0^p (F + G)(\lambda) d\lambda + \frac{1}{2} \int_0^p (1 - F(\lambda) + 1 - G(\lambda)) d\lambda \\ &\quad + \frac{1}{2} \int_p^{+\infty} (1 - F(\lambda) + 1 - G(\lambda)) d\lambda - p \\ &= \frac{1}{2} \int_0^p F(\lambda) d\lambda + \frac{1}{2} \int_0^p G(\lambda) d\lambda \\ &\quad + \frac{1}{2} \int_p^{+\infty} (1 - F(\lambda)) d\lambda + \frac{1}{2} \int_p^{+\infty} (1 - G(\lambda)) d\lambda \\ &\geq \frac{1}{2} \int_0^p F(\lambda) (1 - G(\lambda)) d\lambda + \frac{1}{2} \int_0^p G(\lambda) (1 - F(\lambda)) d\lambda \\ &\quad + \frac{1}{2} \int_p^{+\infty} G(\lambda) (1 - F(\lambda)) d\lambda + \frac{1}{2} \int_p^{+\infty} F(\lambda) (1 - G(\lambda)) d\lambda \\ &= \frac{1}{2} \int_0^{+\infty} F(\lambda) (1 - G(\lambda)) d\lambda + \frac{1}{2} \int_0^{+\infty} G(\lambda) (1 - F(\lambda)) d\lambda \\ &= \frac{1}{2} \cdot \mathbb{E}[|W - V|], \end{aligned}$$

where the first inequality and the first equality follow from Lemma 1 in Appendix E, the second equality by the definition of p , the second inequality from the fact that $0 \leq F \leq 1$ and $0 \leq G \leq 1$, and the last equality from Equation (1). \square

The following lemma shows that the previous $\frac{1}{2}$ -approximation guarantee is unimprovable in general.

Theorem 6. *For each $\varepsilon > 0$, there exist two independent $[0, 1]$ -valued random variables V and W admitting bounded densities and with common expectation μ such that*

$$\mathbb{E}[g(\mu, V, W)] \leq \left(\frac{1}{2} + \varepsilon\right) \cdot \mathbb{E}[|W - V|].$$

For a full proof of this result, see Appendix E.

Proof sketch. The proof leverages the fact that V and W are free to have different distributions as long as they share the same expected value. The idea is to determine two different distributions (say, the first for V and the second for W) whose shared expectation is $1/2$ and, while the first is symmetric around $1/2$ and highly concentrated near 0 and 1 , the second one is still symmetric around $1/2$ but highly concentrated around $1/2$. From Lemma 1 (in Appendix E) we know that the best fixed price is $1/2$. By posting $1/2$, the broker manages to let trades happen when V has a value close to 0 and W has a value slightly greater than $1/2$, or when V has a value close to 1 and W has a value slightly smaller than $1/2$, leading to an expected reward that is slightly above $1/4$. On the other hand, the first-best manages to let trades happen in all the previous cases, but also when V is close to 0 and W is slightly smaller than $1/2$ and when V is close to 1 and W is slightly bigger than $1/2$, for an extra expected reward that is slightly below $1/4$. \square

5 CONCLUSIONS AND FUTURE WORK

In this paper, we investigated a *Lipschitz* contextual brokerage problem, extending the classical brokerage problem to a non-parametric contextual setting. We designed two algorithms, BiAve and ExBis, to minimize regret in full and limited feedback settings, respectively. Our results provide tight regret bounds, specifically $O(T^{d/(d+2)})$ for the full-feedback setting and $O(T^{(d+2)/(d+4)})$ for the limited-feedback setting, demonstrating the optimality of these approaches. Furthermore, we established a $\frac{1}{2}$ -approximation factor between the performance of our algorithms and an omniscient oracle, proving that this approximation is unimprovable.

Our findings offer significant theoretical contributions to the study of brokerage problems in online learning and commerce applications involving contextual information. By relaxing parametric assumptions and focusing on non-parametric methods, we offer broad applicability to real-world over-the-counter (OTC) mar-

kets, where trade conditions and valuations are often influenced by contextual factors.

A natural extension of this work would involve relaxing the assumptions about market value fluctuations to accommodate more general stochastic processes. This could allow the model to be applied in more volatile or uncertain market environments. Finally, while this work fully fleshed out the full and limited feedback settings, future work could explore other feedback models, such as partial trader disclosures, which could yield additional insights into regret minimization approaches in economics.

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Checklist

1. For all models and algorithms presented, check if you include:
 - (a) A clear description of the mathematical setting, assumptions, algorithm, and/or model. [Yes—see Sections 1.1, 2.1 and 3.1]
 - (b) An analysis of the properties and complexity (time, space, sample size) of any algorithm. [No]
 - (c) (Optional) Anonymized source code, with specification of all dependencies, including external libraries. [No]
2. For any theoretical claim, check if you include:
 - (a) Statements of the full set of assumptions of all theoretical results. [Yes—see Section 1.1 and Theorems 1–6]
 - (b) Complete proofs of all theoretical results. [Yes—we included at least sketches of all proofs in the main, and all missing technical details in the appendix]
 - (c) Clear explanations of any assumptions. [Yes—see Section 1.1]
3. For all figures and tables that present empirical results, check if you include:
 - (a) The code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL). [Not Applicable]
 - (b) All the training details (e.g., data splits, hyperparameters, how they were chosen). [Not Applicable]
 - (c) A clear definition of the specific measure or statistics and error bars (e.g., with respect to the random seed after running experiments multiple times). [Not Applicable]
 - (d) A description of the computing infrastructure used. (e.g., type of GPUs, internal cluster, or cloud provider). [Not Applicable]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets, check if you include:
 - (a) Citations of the creator If your work uses existing assets. [Not Applicable]
 - (b) The license information of the assets, if applicable. [Not Applicable]
 - (c) New assets either in the supplemental material or as a URL, if applicable. [Not Applicable]
 - (d) Information about consent from data providers/curators. [Not Applicable]
 - (e) Discussion of sensible content if applicable, e.g., personally identifiable information or offensive content. [Not Applicable]
5. If you used crowdsourcing or conducted research with human subjects, check if you include:
 - (a) The full text of instructions given to participants and screenshots. [Not Applicable]
 - (b) Descriptions of potential participant risks, with links to Institutional Review Board (IRB) approvals if applicable. [Not Applicable]
 - (c) The estimated hourly wage paid to participants and the total amount spent on participant compensation. [Not Applicable]

A PROOF OF THEOREM 1

Note first that the evolution of \mathcal{F} during a run of BiAve is deterministic, since the decision to refine it or not (Line 18) depends only on the adversarial sequence of contexts $\mathbf{x}_1, \mathbf{x}_2, \dots$. For the same reason, for any time step $t \in \mathbb{N}$, $\mathcal{C}_t, \mathcal{C}'_t, i_t, n_t, q_t$ (when defined), and n'_t are deterministic. Then, for any time step t where the property $n_t \geq n'_t$ on Line 14 holds (which is, again, a deterministic event), Lemma 1 implies that the instantaneous regret of BiAve satisfies

$$\begin{aligned} & \sup_{p_t \in [0,1]} \mathbb{E}[g(p_t, V_t, W_t)] - \mathbb{E}[g(P_t, V_t, W_t)] \\ & \leq M \mathbb{E} \left[\left(\mu_t - \frac{1}{2n_t} \sum_{s=1}^{t-1} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}_t\} (V_s + W_s) \right)^2 \right] \\ & = M \left(\mathbb{E} \left[\mu_t - \frac{1}{n_t} \sum_{s=1}^{t-1} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}_t\} \mu_s \right] \right)^2 \\ & \quad + M \text{Var} \left[\frac{1}{2n_t} \sum_{s=1}^{t-1} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}_t\} (V_s + W_s) \right] \\ & \leq M 2^{-2i_t} + \frac{M}{2n_t}, \end{aligned}$$

where in the last inequality, we used the fact that the sum is over rounds $s \in [t]$ such that $\mathbf{x}_s \in \mathcal{C}_t$ and that cell \mathcal{C}_t has level i_t , which implies, by Item 2 in our model, that $|\mu_s - \mu_t| \leq 2^{-i_t}$. Moreover, in the same time steps t where the property $n_t \geq n'_t$ on Line 14 holds, since $\sqrt{n'_t} \geq 2^{i_t-1}$ by Line 18, we obtain that $\sqrt{n_t} \geq 2^{i_t-1}$, which plugged into the right-hand side of the previous chain of inequalities yields that the instantaneous regret of BiAve is upper bounded by $3M \cdot 2^{-2i_t}$.

We now prove that a similar bound holds in the complementary set of rounds $t \in \mathbb{N}$ in which property $n_t < n'_t$ on Line 16 is true; indeed, in any of these rounds t , reasoning as above, the instantaneous regret of BiAve satisfies

$$\begin{aligned} & \sup_{p_t \in [0,1]} \mathbb{E}[g(p_t, V_t, W_t)] - \mathbb{E}[g(P_t, V_t, W_t)] \\ & \leq M \mathbb{E} \left[\left(\mu_t - \frac{1}{2n'_t} \sum_{s=1}^{q_t} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}'_t\} (V_s + W_s) \right)^2 \right] \\ & = M \mathbb{E} \left[\mu_t - \frac{1}{n'_t} \sum_{s=1}^{q_t} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}'_t\} \mu_s \right]^2 \\ & \quad + M \text{Var} \left[\frac{1}{2n'_t} \sum_{s=1}^{q_t} \mathbb{I}\{\mathbf{x}_s \in \mathcal{C}'_t\} (V_s + W_s) \right] \\ & \leq M 2^{-2(i_t-1)} + \frac{M}{2n'_t}. \end{aligned}$$

Because in time steps t in which property $n_t < n'_t$ on Line 16 is true, the parent cell \mathcal{C}'_t was bisected, we have

from Line 18 that $\sqrt{n'_t} \geq 2^{i_t-1}$, which plugged into the right-hand side of the previous chain of inequalities yields that the instantaneous regret of BiAve is upper bounded by $6M \cdot 2^{-2i_t}$.

Therefore, for *all* time steps $t \in \mathbb{N}$, the instantaneous regret of BiAve satisfies

$$\sup_{p_t \in [0,1]} \mathbb{E}[g(p_t, V_t, W_t)] - \mathbb{E}[g(P_t, V_t, W_t)] \leq 6M \cdot 2^{-2i_t}.$$

We now show that this is sufficient to prove that the regret (i.e., the worst-case sum over $t \in [T]$ of all instantaneous regrets) is upper bounded by $T^{d/(d+2)}$, up to constants.

To see it, begin by considering any cell \mathcal{C} that can be obtained by successive bisections of $[0, 1]^d$ (i.e., one of the cells that could be generated by BiAve, introduced at the beginning of Section 2.1), and denote by $n_{\mathcal{C}}$ the number of time steps $t \in [T]$ where this cell is equal to \mathcal{C}_t on Line 6; then, from Line 18, we have $\sqrt{n_{\mathcal{C}}} \leq 2^{i_{\mathcal{C}}}$, with $i_{\mathcal{C}}$ being the level of \mathcal{C} . Therefore, we can bound the regret as follows. For any level $i \in \mathbb{N}$, let $\mathcal{C}_{i,1}, \dots, \mathcal{C}_{i,2^{id}}$ be the 2^{id} cells of the form $[b_1, b_1 + 1/2^i) \times \dots \times [b_d, b_d + 1/2^i)$, for $b_1, \dots, b_d \in \{0, \dots, \frac{2^i-1}{2^i}\}$. Then, putting everything together, we get

$$R_T \leq 6M \sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} 2^{-2i}, \quad (2)$$

with the constraints that (letting $\mathbb{N}_0 := \{0, 1, 2, \dots\}$):

$$n_{\mathcal{C}_{i,j}} \leq 2^{2i}, \quad \forall i \in \mathbb{N}_0, \forall j \in [2^{id}], \quad \text{and} \quad \sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} = T. \quad (3)$$

Let k be the smallest integer such that

$$\sum_{i=1}^k 2^{id} 2^{2i} \geq T.$$

Note that we have

$$\frac{1 - 2^{(k+1)(d+2)}}{1 - 2^{d+2}} \geq T$$

and thus

$$2^{(k+1)(d+2)} \geq 1 + (2^{d+2} - 1)T.$$

Also we have, by definition of k ,

$$\sum_{i=1}^{k-1} 2^{id} 2^{2i} < T.$$

Thus

$$\frac{1 - 2^{k(d+2)}}{1 - 2^{d+2}} < T$$

and consequently

$$2^{k(d+2)} < 1 + (2^{d+2} - 1)T.$$

By the constraints in Equation (3) and the definition of k , Equation (2) implies that

$$\begin{aligned} \frac{1}{6M} R_T &\leq \sum_{i=0}^k 2^{id} 2^{2i} 2^{-2i} \\ &= \frac{1 - 2^{(k+1)d}}{1 - 2^d} \\ &\leq \frac{1}{2^d - 1} 2^{(k+1)d} \\ &= \frac{1}{2^d - 1} 2^{k(d+2)} 2^{d-2k} \\ &\leq \frac{1 + (2^{d+1} - 1)T}{2^d - 1} 2^{d-2k}. \end{aligned}$$

Now, recalling that

$$2^{(k+1)(d+2)} \geq 1 + (2^{d+2} - 1)T,$$

we obtain

$$2^{k(d+2)} \geq \frac{2^{d+2} - 1}{2^{d+2}} T$$

and consequently

$$2^k \geq \left(\frac{2^{d+2} - 1}{2^{d+2}} \right)^{\frac{1}{d+2}} T^{\frac{1}{d+2}}.$$

Hence

$$\begin{aligned} \frac{1}{6M} R_T &\leq \frac{2^d (1 + (2^{d+1} - 1))}{2^d - 1} T 2^{-2k} \\ &\leq \frac{1 + (2^{d+1} - 1)}{\left(\frac{2^{d+2} - 1}{2^{d+2}} \right)^{\frac{2}{d+2}}} \cdot T^{1 - \frac{2}{d+2}} \\ &= \frac{2^{d+1}}{\left(\frac{2^{d+2} - 1}{2^{d+2}} \right)^{\frac{2}{d+2}}} \cdot T^{\frac{d}{d+2}} \\ &= \frac{4 \cdot 2^{d+1}}{(2^{d+2} - 1)^{\frac{2}{d+2}}} \cdot T^{\frac{d}{d+2}} \\ &= \frac{4 \cdot 2^{d+1}}{4 \left(1 - \frac{1}{2^{d+2}} \right)^{\frac{2}{d+2}}} \cdot T^{\frac{d}{d+2}} \\ &\leq \frac{2}{\left(\frac{7}{8} \right)^{2/3}} 2^d \cdot T^{\frac{d}{d+2}}, \end{aligned}$$

which, using the numerical inequality $6 \cdot 2 \cdot \left(\frac{8}{7} \right)^{2/3} \leq 14$, yields

$$R_T \leq 14 \cdot M \cdot 2^d \cdot T^{\frac{d}{d+2}} = O\left(T^{\frac{d}{d+2}}\right)$$

and concludes the proof.

B PROOF OF THEOREM 2

Fix $T \in \mathbb{N}$. Assume without loss of generality that $K := T^{\frac{1}{d+2}}$ is an integer, and note that K^d divides T . Let $n := \frac{T}{K^d} = T^{2/(d+2)} = K^2 \in \mathbb{N}$ and $\varepsilon := n^{-1/2} = T^{-1/(d+2)}$. Let $f_{\pm\varepsilon} := 1 \mp \varepsilon \mathbb{I}_{[\frac{1}{7}, \frac{3}{14}]} \pm \varepsilon \mathbb{I}_{(\frac{3}{14}, \frac{2}{7}]}$. Note that $0 \leq f_{\pm\varepsilon} \leq 2$ and $\int_0^1 f_{\pm\varepsilon}(x) dx = 1$, hence $f_{\pm\varepsilon}$ is a valid density on $[0, 1]$ bounded by $M = 2$. We will denote the corresponding probability measure by $\mathcal{D}_{\pm\varepsilon}$ and define $\mu_{\pm\varepsilon} := \int_{[0,1]} x d\mathcal{D}_{\pm\varepsilon}(x) = \frac{1}{2} \pm \frac{\varepsilon}{196}$. Consider for each $q \in [0, 1]$, an i.i.d. sequence $(B_{q,t})_{t \in \mathbb{N}}$ of Bernoulli random variables of parameter q , an i.i.d. sequence $(\tilde{B}_t)_{t \in \mathbb{N}}$ of Bernoulli random variables of parameter $1/7$, an i.i.d. sequence $(U_t)_{t \in \mathbb{N}}$ of uniform random variables on $[0, 1]$, such that $((B_{q,t})_{t \in \mathbb{N}}, q \in [0, 1], (\tilde{B}_t)_{t \in \mathbb{N}}, (U_t)_{t \in \mathbb{N}})$ is an independent family. Let $\varphi: [0, 1] \rightarrow [0, 1]$ be such that, if U is a uniform random variable on $[0, 1]$, then the distribution of $\varphi(U)$ has density $\frac{7}{6} \cdot \mathbb{I}_{[0,1] \setminus [1/7, 2/7]}$ (which exists by the Skorokhod representation theorem, Williams 1991, Section 17.3). For each $t \in \mathbb{N}$, define

$$\begin{aligned} G_{\pm\varepsilon,t} &:= \left(\frac{2 + U_t}{14} (1 - B_{\frac{1 \pm \varepsilon}{2}, t}) \right. \\ &\quad \left. + \frac{3 + U_t}{14} B_{\frac{1 \pm \varepsilon}{2}, t} \right) \tilde{B}_t + \varphi(U_t) (1 - \tilde{B}_t), \end{aligned} \quad (4)$$

$V_{\pm\varepsilon,t} := G_{\pm\varepsilon,2t-1}$, $W_{\pm\varepsilon,t} := G_{\pm\varepsilon,2t}$, $\xi_{\pm\varepsilon,t} := V_{\pm\varepsilon,t} - \mu_{\pm\varepsilon}$, and $\zeta_{\pm\varepsilon,t} := W_{\pm\varepsilon,t} - \mu_{\pm\varepsilon}$.

In the following, if a_1, \dots, a_{K^d} is a sequence of elements, we will use the notation $a_{1:K^d}$ as a shorthand for (a_1, \dots, a_{K^d}) . For each $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, each $i \in [K^d]$, and each $j \in [n]$, define the random variables $\xi_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \xi_{\varepsilon_i, j+(i-1)n}$ and $\zeta_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \zeta_{\varepsilon_i, j+(i-1)n}$. One can check that the family $(\xi_{j+(i-1)n}^{\varepsilon_{1:K^d}}, \zeta_{j+(i-1)n}^{\varepsilon_{1:K^d}})_{t \in [T], \varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}}$ is an independent family, and for each $i \in [K^d]$ and each $j \in [n]$ the two random variables $\xi_{j+(i-1)n}^{\varepsilon_{1:K^d}}, \zeta_{j+(i-1)n}^{\varepsilon_{1:K^d}}$ are zero mean with common distribution given by a shift by μ_{ε_i} of $\mathcal{D}_{\varepsilon_i}$. For each $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, for each $i \in [K^d]$ and $j \in [n]$, let $V_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \mu_{\varepsilon_i} + \xi_{j+(i-1)n}^{\varepsilon_{1:K^d}}$ and $W_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \mu_{\varepsilon_i} + \zeta_{j+(i-1)n}^{\varepsilon_{1:K^d}}$. Note that these last two random variables are $[0, 1]$ -valued zero-mean perturbations of μ_{ε_i} with shared density given by f_{ε_i} , and hence bounded by 2.

Crucially, we assume that the learner knows what will be the sequence of the contexts in advance, and hence we can restrict the proof to deterministic algorithms without any loss of generality. Specifically, define, for all $(i_1, \dots, i_d) \in [K]^d$ and $j \in [n]$, the lattice points

$$\mathbf{x}_{i_1, \dots, i_d, j} := \left(\frac{i_1 - 1}{K}, \dots, \frac{i_d - 1}{K} \right).$$

Then, define the contexts $(\mathbf{x}_t)_{t \in [T]}$ as the contexts corresponding to the lexicographic (increasing) ordering of the indices of $(\mathbf{x}_{i_1, \dots, i_d, j})_{(i_1, \dots, i_d, j) \in [K]^d \times [n]}$, where vectors of indices (i_1, \dots, i_d, j) are thought of as digits composing a numerical string. In words, the first n contexts $\mathbf{x}_1, \dots, \mathbf{x}_n$ are all equal to $\mathbf{x}_{1, \dots, 1, 1} = \dots = \mathbf{x}_{1, \dots, 1, n} = (0, \dots, 0)$, the next n contexts $\mathbf{x}_{n+1}, \dots, \mathbf{x}_{2n}$ are $\mathbf{x}_{1, \dots, 1, 2, 1} = \mathbf{x}_{1, \dots, 1, 2, n} = (0, \dots, 1/K)$, and so on, until the last n contexts $\mathbf{x}_{T-n+1} = \dots = \mathbf{x}_T$ that are all equal to $(\frac{K-1}{K}, \dots, \frac{K-1}{K})$. Now, notice that since $\varepsilon = 1/K$ then for any $\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}$ we have that the valuations $V_1^{\varepsilon_{1:K^d}}, W_1^{\varepsilon_{1:K^d}}, \dots, V_T^{\varepsilon_{1:K^d}}, W_T^{\varepsilon_{1:K^d}}$ are consistent with our model, in the sense that they are zero-mean noisy perturbations of the market values μ_t defined by parameterizing every $t \in [T]$ by $t = j + (i-1)n$, with $(i, j) \in [K^d] \times [n]$, and letting $\mu_t := \mu_{\varepsilon_i}$, and these market values and contexts satisfy Item 2 in our model definition.

We will show that for each algorithm for contextual brokerage with full feedback and each time horizon T , if $R_T^{\varepsilon_{1:K^d}}$ is the regret of the algorithm at time horizon T when the traders' valuations are $V_1^{\varepsilon_{1:K^d}}, W_1^{\varepsilon_{1:K^d}}, \dots, V_T^{\varepsilon_{1:K^d}}, W_T^{\varepsilon_{1:K^d}}$, then $\max_{\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}} R_T^{\varepsilon_{1:K^d}} = \Omega(T^{\frac{d}{d+2}})$ with our choices of ε and K .

To do it, we first denote, for any $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, $p \in [0, 1]$, and $t \in [T]$, $\text{GFT}_t^{\varepsilon_{1:K^d}}(p) := g(p, V_t^{\varepsilon_{1:K^d}}, W_t^{\varepsilon_{1:K^d}})$.

Then, by Lemma 1, we have, for all $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, $i \in [K^d]$, $j \in [n]$, and $p \in [0, 1]$,

$$\begin{aligned} \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)] &= 2 \int_0^p \int_0^\lambda f_{\varepsilon_i}(s) ds d\lambda \\ &\quad + 2(\mu_{\varepsilon_i} - p) \int_0^p f_{\varepsilon_i}(s) ds, \end{aligned}$$

which, together with the fundamental theorem of calculus —(Bass, 2013, Theorem 14.16), noting that $p \mapsto \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)]$ is absolutely continuous with derivative defined a.e. by $p \mapsto 2(\mu_{\varepsilon_i} - p)f_{\varepsilon_i}(p)$ — yields, for any $p \in [2/7, 1]$,

$$\mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(\mu_{\varepsilon_i})] - \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)] = |\mu_{\varepsilon_i} - p|^2. \quad (5)$$

Hence note that for all $\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}$, $i \in [K^d]$, $j \in [n]$, and $p < \frac{1}{2}$, if $\varepsilon_i > 0$, a direct verification shows that

$$\mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(1/2)] \geq \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)] + \Omega(\varepsilon^2). \quad (6)$$

Similarly, for all $\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}$, $i \in [K^d]$, $j \in [n]$,

and $p > \frac{1}{2}$, if $\varepsilon_i < 0$, then

$$\mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(1/2)] \geq \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)] + \Omega(\varepsilon^2). \quad (7)$$

In words, in the $\varepsilon_i = \varepsilon$ (resp., $\varepsilon_i = -\varepsilon$) case, the optimal price for the rounds $1 + (i-1)n, \dots, in$ belongs to the region $(\frac{1}{2}, 1]$ (resp., $[0, \frac{1}{2})$). Then, with a standard information-theoretic argument, since $\Omega(1/\varepsilon^2) = \Omega(n)$ samples are needed to determine the sign of ε_i (with high probability), any algorithm will pay a regret of at least $\Omega(\frac{1}{\varepsilon^2} \cdot \varepsilon^2) = \Omega(1)$ for each point in the lattice, for a total regret of $\Omega(1 \cdot K^d) = \Omega(T^{d/(d+2)})$.

C PROOF OF THEOREM 3

Note first that the decisions to either explore or exploit (Lines 14 and 17), the decisions to whether or not bisect (Line 19) during a run of ExBis, and $\mathcal{C}_t, i_t, m_t, \mathcal{E}_t, n_t, \mathcal{C}'_t, q_t, \mathcal{E}'_t, n'_t$ (for all $t \in [T]$) are deterministic, since they are only determined by the adversarial sequence of contexts $\mathbf{x}_1, \dots, \mathbf{x}_T$.

Hence, for any time step $t \in [T]$ where the property $n_t \geq n'_t$ on Line 15 holds (which, again, is a deterministic event), we have that $n_t \geq n'_t \geq 2^{2(i_t-1)}$, since the parent cell \mathcal{C}'_t of \mathcal{C}_t was bisected on Line 19, with Line 17 holding. Proceeding as in Appendix A, this implies that, for any time step $t \in [T]$ where the property $n_t \geq n'_t$ on Line 15 holds, the instantaneous regret of ExBis satisfies

$$\begin{aligned} \sup_{p_t \in [0, 1]} \mathbb{E}[g(p_t, V_t, W_t)] - \mathbb{E}[g(P_t, V_t, W_t)] \\ \leq M 2^{-2i_t} + \frac{M}{2} 2^{-2(i_t-1)} = 3M \cdot 2^{-2i_t}. \end{aligned}$$

Proceeding once more as in Appendix A, we obtain that for all time steps $t \in [T]$ such that property $n_t < n'_t$ on Line 16 holds, the instantaneous regret of ExBis satisfies

$$\begin{aligned} \sup_{p_t \in [0, 1]} \mathbb{E}[g(p_t, V_t, W_t)] - \mathbb{E}[g(P_t, V_t, W_t)] \\ \leq M 2^{-2(i_t-1)} + \frac{M}{2} 2^{-2(i_t-1)} = 6M \cdot 2^{-2i_t}. \end{aligned}$$

Therefore, the instantaneous regret of ExBis in *all* exploiting rounds $t \in [T]$ such that $m_t < 2^{4i_t}$ on Line 14 satisfies

$$\sup_{p_t \in [0, 1]} \mathbb{E}[g(p_t, V_t, W_t)] - \mathbb{E}[g(P_t, V_t, W_t)] \leq 6M \cdot 2^{-2i_t}. \quad (8)$$

Now, for any cell \mathcal{C} of level $i_{\mathcal{C}}$ obtained by successive bisections of $[0, 1]^d$, let $\mathcal{F}_{\mathcal{C}}$ be the set of all time steps $s \in [T]$ for which $\mathcal{C}_s = \mathcal{C}$ on Line 6, $n_{\mathcal{C}} := |\mathcal{F}_{\mathcal{C}}|$, and note

that the sum of the instantaneous regrets of ExBis over time steps $t \in \mathcal{F}_C$ such that $m_t \geq 2^{4i_t}$ on Line 17 (i.e., total regret due to *exploration* in rounds t where $\mathcal{C}_t = C$) is less than or equal to the sum of the upper bounds in Equation (8) of the instantaneous regrets of ExBis over time steps $t \in \mathcal{F}_C$ such that $m_t < 2^{4i_t}$ on Line 14 (i.e., total regret due to *exploitation* in rounds t where $\mathcal{C}_t = C$), since each exploration time $t \in \mathcal{F}_C$ is preceded by 2^{4i_c} exploitation times (by Lines 8 and 14), yielding a total regret due to exploitation of $6M \cdot 2^{4i_c} \cdot 2^{-2i_c} = 6M \cdot 2^{2i_c}$, whereas the maximal number of exploration times is 2^{2i_c} (by Lines 17 and 19).

Hence, it is sufficient to control the sum over all exploitation rounds of the upper bounds in Equation (8) to obtain an upper bound on the regret of ExBis, up to a factor of 2. With this in mind, and with the same notation $(\mathcal{C}_{i,j})_{i \in \{0,1,\dots\}, j \in [2^{di}]}$ introduced in Appendix A for the family of all cells that can be obtained by successive bisections of $[0,1]^d$, let \tilde{R}_T be the upper bound on the total regret due to exploitation defined as the sum over all exploitation rounds (i.e., over all rounds $t \in [T]$ such that $m_t < 2^{4i_t}$ on Line 14) of the upper bounds in Equation (8). We have

$$\tilde{R}_T \leq 6M \sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} 2^{-2i}, \quad (9)$$

with the constraints that (letting $\mathbb{N}_0 := \{0, 1, 2, \dots\}$):

$$n_{\mathcal{C}_{i,j}} \leq 2^{4i}, \quad \forall i \in \mathbb{N}_0, \forall j \in [2^{id}], \quad \text{and} \quad \sum_{i=0}^{\infty} \sum_{j=1}^{2^{id}} n_{\mathcal{C}_{i,j}} \leq T. \quad (10)$$

Let k be the smallest integer such that

$$\sum_{i=1}^k 2^{id} 2^{4i} \geq T.$$

Then

$$\frac{1 - 2^{(k+1)(d+4)}}{1 - 2^{d+4}} \geq T$$

or, equivalently,

$$2^{(k+1)(d+4)} \geq 1 + (2^{d+4} - 1)T.$$

By definition of k , we also have

$$\sum_{i=1}^{k-1} 2^{id} 2^{4i} < T.$$

Then

$$\frac{1 - 2^{k(d+4)}}{1 - 2^{d+4}} < T$$

or, equivalently,

$$2^{k(d+4)} < 1 + (2^{d+4} - 1)T.$$

By the constraints in Equation (10) and the definition of k , Equation (9) implies that

$$\begin{aligned} \frac{\tilde{R}_T}{6M} &\leq \sum_{i=0}^k 2^{id} 2^{4i} 2^{-2i} \\ &= \frac{1 - 2^{(k+1)(d+2)}}{1 - 2^{d+2}} \\ &\leq \frac{1}{2^{d+2} - 1} 2^{(k+1)(d+2)} \\ &= \frac{1}{2^{d+2} - 1} 2^{k(d+4)} 2^{d+2-2k} \\ &\leq \frac{1 + (2^{d+4} - 1)T}{2^{d+2} - 1} 2^{d+2-2k}. \end{aligned}$$

Now, recalling that

$$2^{(k+1)(d+4)} \geq 1 + (2^{d+4} - 1)T,$$

we obtain

$$2^{k(d+4)} \geq \frac{(2^{d+4} - 1)T}{2^{d+4}}$$

and consequently

$$2^k \geq \left(\frac{2^{d+4} - 1}{2^{d+4}} \right)^{\frac{1}{d+4}} T^{\frac{1}{d+4}}.$$

Hence

$$\begin{aligned} \frac{\tilde{R}_T}{6M} &\leq \frac{2^{d+2} (1 + (2^{d+4} - 1)T)}{2^{d+2} - 1} T 2^{-2k} \\ &\leq \frac{2^{d+2} (1 + (2^{d+4} - 1)T)}{2^{d+2} - 1} \cdot T^{1 - \frac{2}{d+4}} \\ &= \frac{2^{d+2} (1 + (2^{d+4} - 1)T)}{2^{d+2} - 1} \cdot T^{\frac{d+2}{d+4}} \\ &= \frac{2^{d+2} 2^{d+4}}{2^{d+2} - 1} \cdot T^{\frac{d+2}{d+4}} \\ &= \frac{2^{d+2}}{(1 - \frac{1}{2^{d+4}})^{\frac{2}{d+4}}} \cdot T^{\frac{d+2}{d+4}} \\ &= 2^4 \cdot 2^d \cdot \frac{2^{d+2}}{2^{d+2} - 1} \cdot T^{\frac{d+2}{d+4}} \\ &\leq 2^4 \cdot 2^d \cdot \frac{2^{1+2}}{2^{1+2} - 1} \cdot T^{\frac{d+2}{d+4}} \\ &= 2^4 \cdot 2^d \cdot \frac{32}{7 \cdot 31^{2/5}} \cdot T^{\frac{d+2}{d+4}} \end{aligned}$$

where, in the last inequality, we used the fact that the function $d \mapsto \frac{2^{d+2}}{2^{d+2} - 1} / (1 - \frac{1}{2^{d+4}})^{\frac{2}{d+4}}$ is decreasing on $d \geq 1$. This immediately implies that

$$R_T \leq 2\tilde{R}_T \leq 223 \cdot M \cdot 2^d \cdot T^{\frac{d+2}{d+4}} = O\left(T^{\frac{d+2}{d+4}}\right).$$

D PROOF OF THEOREM 4

The proof shares a similar construction as the proof of Theorem 2, with a different tuning and additional arguments at the end to account for the limited feedback. Nevertheless, we give the full details here for ease of exposition.

Fix $T \in \mathbb{N}$. Assume without loss of generality that $K := T^{\frac{1}{d+4}}$ is an integer, and note that K^d divides T . Let $n := \frac{T}{K^d} = T^{4/(d+4)} = K^4 \in \mathbb{N}$ and $\varepsilon := n^{-1/4} = T^{-1/(d+4)}$. Let $f_{\pm\varepsilon} := 1 \mp \varepsilon \mathbb{I}[\frac{1}{7}, \frac{3}{14}] \pm \varepsilon \mathbb{I}[\frac{3}{14}, \frac{2}{7}]$. Note that $0 \leq f_{\pm\varepsilon} \leq 2$ and $\int_0^1 f_{\pm\varepsilon}(x) dx = 1$, hence $f_{\pm\varepsilon}$ is a valid density on $[0, 1]$ bounded by $M = 2$. We will denote the corresponding probability measure by $\mathcal{D}_{\pm\varepsilon}$ and define $\mu_{\pm\varepsilon} := \int_{[0,1]} x d\mathcal{D}_{\pm\varepsilon}(x) = \frac{1}{2} \pm \frac{\varepsilon}{196}$. Consider for each $q \in [0, 1]$, an i.i.d. sequence $(B_{q,t})_{t \in \mathbb{N}}$ of Bernoulli random variables of parameter q , an i.i.d. sequence $(\tilde{B}_t)_{t \in \mathbb{N}}$ of Bernoulli random variables of parameter $1/7$, an i.i.d. sequence $(U_t)_{t \in \mathbb{N}}$ of uniform random variables on $[0, 1]$, such that $((B_{q,t})_{t \in \mathbb{N}, q \in [0,1]}, (\tilde{B}_t)_{t \in \mathbb{N}}, (U_t)_{t \in \mathbb{N}})$ is an independent family. Let $\varphi: [0, 1] \rightarrow [0, 1]$ be such that, if U is a uniform random variable on $[0, 1]$, then the distribution of $\varphi(U)$ has density $\frac{7}{6} \cdot \mathbb{I}_{[0,1] \setminus [1/7, 2/7]}$ (which exists by the Skorokhod representation theorem, Williams 1991, Section 17.3). For each $t \in \mathbb{N}$, define

$$G_{\pm\varepsilon,t} := \left(\frac{2+U_t}{14} (1 - B_{\frac{1\pm\varepsilon}{2},t}) + \frac{3+U_t}{14} B_{\frac{1\pm\varepsilon}{2},t} \right) \tilde{B}_t + \varphi(U_t) (1 - \tilde{B}_t), \quad (11)$$

$V_{\pm\varepsilon,t} := G_{\pm\varepsilon,2t-1}$, $W_{\pm\varepsilon,t} := G_{\pm\varepsilon,2t}$, $\xi_{\pm\varepsilon,t} := V_{\pm\varepsilon,t} - \mu_{\pm\varepsilon}$, and $\zeta_{\pm\varepsilon,t} := W_{\pm\varepsilon,t} - \mu_{\pm\varepsilon}$.

In the following, if a_1, \dots, a_{K^d} is a sequence of elements, we will use the notation $a_{1:K^d}$ as a shorthand for (a_1, \dots, a_{K^d}) . For each $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, each $i \in [K^d]$, and each $j \in [n]$, define the random variables $\xi_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \xi_{\varepsilon_i, j+(i-1)n}$ and $\zeta_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \zeta_{\varepsilon_i, j+(i-1)n}$. One can check that the family $(\xi_t^{\varepsilon_{1:K^d}}, \zeta_t^{\varepsilon_{1:K^d}})_{t \in [T], \varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}}$ is an independent family, and for each $i \in [K^d]$ and each $j \in [n]$ the two random variables $\xi_{j+(i-1)n}^{\varepsilon_{1:K^d}}, \zeta_{j+(i-1)n}^{\varepsilon_{1:K^d}}$ are zero mean with common distribution given by a shift by μ_{ε_i} of $\mathcal{D}_{\varepsilon_i}$. For each $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, for each $i \in [K^d]$ and $j \in [n]$, let $V_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \mu_{\varepsilon_i} + \xi_{j+(i-1)n}^{\varepsilon_{1:K^d}}$ and $W_{j+(i-1)n}^{\varepsilon_{1:K^d}} := \mu_{\varepsilon_i} + \zeta_{j+(i-1)n}^{\varepsilon_{1:K^d}}$. Note that these last two random variables are $[0, 1]$ -valued zero-mean perturbations of μ_{ε_i} with shared density given by f_{ε_i} , and hence bounded by 2.

Crucially, we assume that the learner knows what will be the sequence of the contexts in advance, and hence

we can restrict the proof to deterministic algorithms without any loss of generality. Specifically, define, for all $(i_1, \dots, i_d) \in [K]^d$ and $j \in [n]$, the lattice points

$$\mathbf{x}_{i_1, \dots, i_d, j} := \left(\frac{i_1 - 1}{K}, \dots, \frac{i_d - 1}{K} \right).$$

Then, define the contexts $(\mathbf{x}_t)_{t \in [T]}$ as the contexts corresponding to the lexicographic (increasing) ordering of the indices of $(\mathbf{x}_{i_1, \dots, i_d, j})_{(i_1, \dots, i_d, j) \in [K]^d \times [n]}$, where vectors of indices (i_1, \dots, i_d, j) are thought of as digits composing a numerical string. In words, the first n contexts $\mathbf{x}_1, \dots, \mathbf{x}_n$ are all equal to $\mathbf{x}_{1, \dots, 1, 1} = \dots = \mathbf{x}_{1, \dots, 1, n} = (0, \dots, 0)$, the next n contexts $\mathbf{x}_{n+1}, \dots, \mathbf{x}_{2n}$ are $\mathbf{x}_{1, \dots, 1, 2, 1} = \mathbf{x}_{1, \dots, 1, 2, n} = (0, \dots, 1/K)$, and so on, until the last n contexts $\mathbf{x}_{T-n+1} = \dots = \mathbf{x}_T$ that are all equal to $(\frac{K-1}{K}, \dots, \frac{K-1}{K})$.

Now, notice that since $\varepsilon = 1/K$ then for any $\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}$ we have that the valuations $V_{1:K^d}^{\varepsilon_{1:K^d}}, W_{1:K^d}^{\varepsilon_{1:K^d}}, \dots, V_{T:K^d}^{\varepsilon_{1:K^d}}, W_{T:K^d}^{\varepsilon_{1:K^d}}$ are consistent with our model, in the sense that they are zero-mean noisy perturbations of the market values μ_t defined by parameterizing every $t \in [T]$ by $t = j + (i-1)n$, with $(i, j) \in [K^d] \times [n]$, and letting $\mu_t := \mu_{\varepsilon_i}$, and these market values and contexts satisfy Item 2 in our model definition.

We will show that for each algorithm for contextual brokerage with limited feedback and each time horizon T , if $R_T^{\varepsilon_{1:K^d}}$ is the regret of the algorithm at time horizon T when the traders' valuations are $V_{1:K^d}^{\varepsilon_{1:K^d}}, W_{1:K^d}^{\varepsilon_{1:K^d}}, \dots, V_{T:K^d}^{\varepsilon_{1:K^d}}, W_{T:K^d}^{\varepsilon_{1:K^d}}$, then $\max_{\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}} R_T^{\varepsilon_{1:K^d}} = \Omega(T^{\frac{d+2}{d+4}})$ with our choices of ε and K .

To do it, we first denote, for any $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, $p \in [0, 1]$, and $t \in [T]$, $\text{GFT}_t^{\varepsilon_{1:K^d}}(p) := g(p, V_t^{\varepsilon_{1:K^d}}, W_t^{\varepsilon_{1:K^d}})$.

Then, by Lemma 1, we have, for all $\varepsilon_1, \dots, \varepsilon_{K^d} \in \{-\varepsilon, \varepsilon\}$, $i \in [K^d]$, $j \in [n]$, and $p \in [0, 1]$,

$$\begin{aligned} \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)] &= 2 \int_0^p \int_0^\lambda f_{\varepsilon_i}(s) ds d\lambda \\ &\quad + 2(\mu_{\varepsilon_i} - p) \int_0^p f_{\varepsilon_i}(s) ds, \end{aligned}$$

which, together with the fundamental theorem of calculus —(Bass, 2013, Theorem 14.16), noting that $p \mapsto \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)]$ is absolutely continuous with derivative defined a.e. by $p \mapsto 2(\mu_{\varepsilon_i} - p)f_{\varepsilon_i}(p)$ — yields, for any $p \in [2/7, 1]$,

$$\mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(\mu_{\varepsilon_i})] - \mathbb{E}[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p)] = |\mu_{\varepsilon_i} - p|^2. \quad (12)$$

Hence note that for all $\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}$, $i \in [K^d]$, $j \in [n]$, and $p < \frac{1}{2}$, if $\varepsilon_i > 0$, a direct verification shows

that

$$\mathbb{E} \left[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(1/2) \right] \geq \mathbb{E} \left[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p) \right] + \Omega(\varepsilon^2). \quad (13)$$

Similarly, for all $\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}$, $i \in [K^d]$, $j \in [n]$, and $p > \frac{1}{2}$, if $\varepsilon_i < 0$, then

$$\mathbb{E} \left[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(1/2) \right] \geq \mathbb{E} \left[\text{GFT}_{j+(i-1)n}^{\varepsilon_{1:K^d}}(p) \right] + \Omega(\varepsilon^2). \quad (14)$$

Furthermore, a direct verification shows that, for each $\varepsilon_{1:K^d} \in \{-\varepsilon, \varepsilon\}^{K^d}$ and $t \in [T]$,

$$\max_{p \in [0,1]} \mathbb{E} \left[\text{GFT}_t^{\varepsilon_{1:K^d}}(p) \right] - \max_{p \in [\frac{1}{7}, \frac{2}{7}]} \mathbb{E} \left[\text{GFT}_t^{\varepsilon_{1:K^d}}(p) \right] \geq \frac{1}{50} = \Omega(1). \quad (15)$$

In words, in the $\varepsilon_i = \varepsilon$ (resp., $\varepsilon_i = -\varepsilon$) case, the optimal price for the rounds $1 + (i-1)n, \dots, in$ belongs to the region $(\frac{1}{2}, 1]$ (resp., $[0, \frac{1}{2})$). By posting prices in the wrong region $[0, \frac{1}{2}]$ (resp., $[\frac{1}{2}, 1)$) in the $\varepsilon_i = \varepsilon$ (resp., $\varepsilon_i = -\varepsilon$) case, the learner incurs a $\Omega(\varepsilon^2) = \Omega(1/K^2)$ instantaneous regret by (13) (resp., (14)). Then, in order to attempt suffering less than $\Omega(1/K^2 \cdot n) = \Omega(1/\varepsilon^2)$ cumulative regret in the rounds $1 + (i-1)n, \dots, in$, the algorithm would have to detect the sign of ε_i and play accordingly. We will show now that even this strategy will not improve the regret of the algorithm (by more than a constant) because of the cost of determining the sign of ε_i with the available feedback. Since for any $i \in [K^d]$ and $j \in [n]$, the feedback received from the two traders at time $j + (i-1)n$ by posting a price p is $\mathbb{I}\{p \leq V_{j+(i-1)n}^{\varepsilon_{1:K^d}}\}$ and $\mathbb{I}\{p \leq W_{j+(i-1)n}^{\varepsilon_{1:K^d}}\}$, the only way to obtain information about (the sign of) ε_i is to post in the costly ($\Omega(1)$ -instantaneous regret by Equation (15)) sub-optimal region $[\frac{1}{7}, \frac{2}{7}]$. However, posting prices in the region $[\frac{1}{7}, \frac{2}{7}]$ at time $j + (i-1)n$ can't give more information about (the sign of) ε_i than the information carried by $V_{j+(i-1)n}^{\varepsilon_{1:K^d}}$ and $W_{j+(i-1)n}^{\varepsilon_{1:K^d}}$, which, in turn, can't give more information about (the sign of) ε_i than the information carried by the two Bernoullis $B_{\frac{1+\varepsilon_i}{2}, 2(j+(i-1)n)-1}$ and $B_{\frac{1+\varepsilon_i}{2}, 2(j+(i-1)n)}$. Also, notice that only during rounds $1 + (i-1)n, \dots, in$ it is possible to extract information about the sign of ε_i . With a standard information-theoretic argument, in order to distinguish the sign of ε_i having access to i.i.d. Bernoulli random variables of parameter $\frac{1+\varepsilon_i}{2}$ requires $\Omega(1/\varepsilon^2)$ samples. Hence we are forced to post at least $\Omega(1/\varepsilon^2)$ prices in the costly region $[\frac{1}{7}, \frac{2}{7}]$ during the rounds $1 + (i-1)n, \dots, in$ suffering a regret of $\Omega(1/\varepsilon^2) \cdot \Omega(1) = \Omega(1/\varepsilon^2)$. Putting everything together, no matter what the strategy, each algorithm will pay at least $\Omega(1/\varepsilon^2)$ regret in each epoch $1 + (i-1)n, \dots, in$ for every $i \in [K^d]$, resulting in an overall regret of $K^d \cdot \Omega(1/\varepsilon^2) = \Omega(T^{\frac{d+2}{d+4}})$.

E MISSING DETAILS FROM SECTION 4

In the proof of our approximation theorem, we leverage the following result from (Bachoc et al., 2024b, Lemma 1).

Lemma 1. *Suppose that V and W are two $[0, 1]$ -valued independent random variables with possibly different densities but both bounded by the same constant $M \geq 1$, and such that $\mathbb{E}[V] = \mathbb{E}[W] =: \mu$. Denote by F (resp., G) the cumulative distribution function of V (resp., W). Then, for each $p \in [0, 1]$, it holds that*

$$\mathbb{E}[g(p, V, W)] = \int_0^p (F + G)(\lambda) d\lambda + (\mu - p)(F + G)(p)$$

and

$$0 \leq \mathbb{E}[g(\mu, V, W) - g(p, V, W)] \leq M |\mu - p|^2.$$

We now prove the optimality of our $\frac{1}{2}$ -approximation result.

Proof of Theorem 6. Fix $\delta \in (0, 1/6)$ and the probability density functions:

$$f_\delta: [0, 1] \rightarrow \mathbb{R},$$

$$x \mapsto f_\delta(x) := \frac{1}{2\delta} \mathbb{I}\{0 \leq x \leq \delta\} + \frac{1}{2\delta} \mathbb{I}\{1 - \delta \leq x \leq 1\}$$

and

$$g_\delta: [0, 1] \rightarrow \mathbb{R},$$

$$x \mapsto g_\delta(x) := \frac{1}{2\delta} \mathbb{I}\left\{\frac{1}{2} - \delta \leq x \leq \frac{1}{2} + \delta\right\}.$$

Let V_δ and W_δ be two independent random variables with probability density functions f_δ and g_δ , respectively. Then:

$$\begin{aligned} & \mathbb{E}[|V_\delta - W_\delta|] \\ &= \int_{[0,1]^2} |v - w| f_\delta(v) g_\delta(w) dv dw \\ &= \frac{1}{4\delta^2} \int_{[0,\delta] \cup [1-\delta,1]} \left(\int_{[\frac{1}{2}-\delta, \frac{1}{2}+\delta]} |v - w| dw \right) dv \\ &= \frac{1}{4\delta^2} \int_0^\delta \left(\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} (w - v) dw \right) dv \\ &\quad + \frac{1}{4\delta^2} \int_{1-\delta}^1 \left(\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} (v - w) dw \right) dv \\ &= \frac{1}{4\delta^2} \int_0^\delta \left[\frac{(w - v)^2}{2} \right]_{w=\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} dv \\ &\quad + \frac{1}{4\delta^2} \int_{1-\delta}^1 \left[-\frac{(v - w)^2}{2} \right]_{w=\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} dv \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\delta^2} \int_0^\delta \left(\frac{(\frac{1}{2} + \delta - v)^2}{2} - \frac{(\frac{1}{2} - \delta - v)^2}{2} \right) dv \\
&\quad + \frac{1}{4\delta^2} \int_{1-\delta}^1 \left(-\frac{(v - (\frac{1}{2} + \delta))^2}{2} + \frac{(v - (\frac{1}{2} - \delta))^2}{2} \right) dv \\
&= \frac{1}{4\delta^2} \int_0^\delta (1 - 2v)\delta dv + \frac{1}{4\delta^2} \int_{1-\delta}^1 (2v - 1)\delta dv \\
&= \frac{1}{4\delta} \int_0^\delta (1 - 2v) dv + \frac{1}{4\delta} \int_{1-\delta}^1 (2v - 1) dv \\
&= \frac{1 - \delta}{2}
\end{aligned}$$

that $\delta \in (0, 1/6)$, we get

$$\begin{aligned}
\mathbb{E}[g(\mu, V_\delta, W_\delta)] &= \frac{1}{4} = \left(\frac{1}{2} + \varepsilon\right) \frac{1 - \delta}{2} \\
&= \left(\frac{1}{2} + \varepsilon\right) \mathbb{E}[|V_\delta - W_\delta|]. \quad \square
\end{aligned}$$

Instead, the reward accrued when posting the expectation $\mu := \mathbb{E}[V_\delta] = \mathbb{E}[W_\delta] = \frac{1}{2}$ is:

$$\begin{aligned}
&\mathbb{E}[g(\mu, V_\delta, W_\delta)] \\
&= \int_{[0,1]^2} (v \vee w - v \wedge w) \mathbb{I}\{v \wedge w \leq \mu \leq v \vee w\} f_\delta(v) g_\delta(w) dv dw \\
&= \frac{1}{4\delta^2} \int_{[\frac{1}{2}-\delta, \frac{1}{2}+\delta]} \left(\int_{[0,\delta] \cup [1-\delta,1]} (v \vee w - v \wedge w) \mathbb{I}\{v \wedge w \leq \mu \leq v \vee w\} dv \right) dw \\
&= \frac{1}{4\delta^2} \int_{[\frac{1}{2}-\delta, \frac{1}{2}+\delta]} \left(\int_{[0,\delta]} (w - v) \mathbb{I}\{v \leq \mu \leq w\} dv \right. \\
&\quad \left. + \int_{[1-\delta,1]} (v - w) \mathbb{I}\{w \leq \mu \leq v\} dv \right) dw \\
&= \frac{1}{4\delta^2} \int_{[\frac{1}{2}-\delta, \frac{1}{2}+\delta]} \left(\int_{[0,\delta]} (w - v) \mathbb{I}\{\mu \leq w\} dv \right. \\
&\quad \left. + \int_{[1-\delta,1]} (v - w) \mathbb{I}\{w \leq \mu\} dv \right) dw \\
&= \frac{1}{4\delta^2} \int_{[\frac{1}{2}, \frac{1}{2}+\delta]} \left(\int_{[0,\delta]} (w - v) dv \right) dw \\
&\quad + \frac{1}{4\delta^2} \int_{[\frac{1}{2}-\delta, \frac{1}{2}]} \left(\int_{[1-\delta,1]} (v - w) dv \right) dw \\
&= \frac{1}{4\delta^2} \int_{[\frac{1}{2}, \frac{1}{2}+\delta]} \left(\frac{w^2}{2} - \frac{(w - \delta)^2}{2} \right) dw \\
&\quad + \frac{1}{4\delta^2} \int_{[\frac{1}{2}-\delta, \frac{1}{2}]} \left(\frac{(1 - w)^2}{2} - \frac{(1 - \delta - w)^2}{2} \right) dw \\
&= \frac{1}{4}.
\end{aligned}$$

We will now show that, for all $\varepsilon \in (0, 1/10)$, there exist two independent $[0, 1]$ -valued random variables V and W with expectation $1/2$ and admitting bounded densities such that $\mathbb{E}[g(\mu, V, W)] = (\frac{1}{2} + \varepsilon) \cdot \mathbb{E}[|W - V|]$ (which immediately implies that, for all $\varepsilon > 0$, there exist two independent $[0, 1]$ -valued random variables V and W with bounded densities and common expectation such that $\mathbb{E}[g(\mu, V, W)] \leq (\frac{1}{2} + \varepsilon) \cdot \mathbb{E}[|W - V|]$).

Indeed, for all $\varepsilon \in (0, 1/10)$, setting $\delta := \frac{2\varepsilon}{1+2\varepsilon}$ and noting