PHY64 Experiment 1: The Cavendish Experiment

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I. INTRODUCTION

In 1797, British scientist Henry Cavendish became the first person to successfully measure the gravitational interaction between two bodies in a laboratory. The apparatus used by Cavendish consisted of a torsion balance with two small lead spheres attached to either end, and two larger lead spheres positioned on alternate sides of the horizontal arm of the balance. By reflecting a light beam off of a mirror attached to the pendulum and observing the change of position of the reflected beam on a distant surface, Cavendish was able to observe a small difference in the equilibrium position of the pendulum with the larger masses in place. Using this method, Cavendish was successfully able to calculate Newton's gravitational constant G within 1% of the currently accepted value.

In this experiment, we replicate Cavendish's experiment using the TEL-Atomic torsion pendulum that mimics the experimental setup used in 1797. The TEL-Atomic apparatus allows us to observe the oscillations of the pendulum by way of an oscillating voltage readout on a computer. Using the same technique as Cavendish did, we then calibrate the oscillations in voltage to the pendulum's angle to determine the force between the pairs of masses. Thus, using Newton's Law of Gravitation, we may calculate the value of the gravitational constant G.

II. THEORY

We have a long rod of length $2d$ and width w, with two spherical masses of radius r at each end. The rod has linear mass density σ, and the two masses have mass m. Two large tungsten masses, each with mass M and radius R, are placed near the smaller masses. The rod is allowed to rotate about its center, which we will use as the origin, with the rod and all masses existing in the x-y plane. Each small spherical mass has moment of inertia $I_{\text{sphere}} = 2mr^2/5 + md^2$, and the beam has moment of inertia $I_{\text{beam}} = 2d\sigma(4d^2 + w^2)/12$. Thus, the total moment of inertia is

$$I = \frac{2}{5}m(2r^2 + 5d^2) + \frac{1}{6}d\sigma(w^2 + 4d^2). \tag{1}$$

We can then determine the equilibrium point by solving the following equation:

$$\sum_i \boldsymbol{\tau}_i = I\frac{d^2\boldsymbol{\theta}}{dt^2}, \tag{2}$$

where we sum over all of the torques acting on the system.

Consider the system with only one of the tungsten spheres. Due to symmetry, the other sphere will result in the same torque acting on the system, and so we need only multiply by two at the end. Let $\boldsymbol{r} = d\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$ be the position of the tungsten sphere. The two smaller spheres are at positions $\boldsymbol{r}_1 = d\cos\theta\,\hat{\boldsymbol{x}} + d\sin\theta\,\hat{\boldsymbol{y}}$ and $\boldsymbol{r}_2 = -d\cos\theta\,\hat{\boldsymbol{x}} - d\sin\theta\,\hat{\boldsymbol{y}}$. Thus, the separations between the tungsten spheres and the smaller spheres are

$$\boldsymbol{R}_1 = \boldsymbol{r} - \boldsymbol{r}_1 = d(1-\cos\theta)\hat{\boldsymbol{x}} + (b - d\sin\theta)\hat{\boldsymbol{y}}$$

and

$$\boldsymbol{R}_2 = \boldsymbol{r} - \boldsymbol{r}_2 = d(1+\cos\theta)\hat{\boldsymbol{x}} + (b + d\sin\theta)\hat{\boldsymbol{y}}.$$

The forces acting on each small sphere are

$$\boldsymbol{F}_k = \frac{GMm}{R_k^2}\hat{\boldsymbol{R}}_k, \tag{3}$$

where $k = 1$ or 2 depending on the sphere we look at. The corresponding torques are therefore

$$\boldsymbol{\tau}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k. \tag{4}$$

We now look to determine the torque acting on the rod itself. A small mass element of the rod has mass $dm = \sigma\,dl$, where dl is an infinitesimal length. Choose a point l on the rod, with $-d \le l \le d$. Then the position of that point is $\boldsymbol{p} = l\cos\theta\,\hat{\boldsymbol{x}} + l\sin\theta\,\hat{\boldsymbol{y}}$. The force acting on that particular point on the rod is

$$d\boldsymbol{F} = \frac{GM\sigma}{(\boldsymbol{r}-\boldsymbol{p})^2}\frac{\boldsymbol{r}-\boldsymbol{p}}{|\boldsymbol{r}-\boldsymbol{p}|}\,dl.$$

We therefore find the torque to be

$$\boldsymbol{\tau}_{\text{rod}} = \int \boldsymbol{p}\times d\boldsymbol{F} = GM\sigma\int_{-d}^{d}\frac{\boldsymbol{p}\times\boldsymbol{r}}{|\boldsymbol{r}-\boldsymbol{p}|^3}\,dl \tag{5}$$

Expanding the terms, this integral is

$$GM\sigma\int_{-d}^{d}\frac{l(b\cos\theta - d\sin\theta)\hat{\boldsymbol{z}}}{[l^2 - 2l(d\cos\theta + b\sin\theta) + b^2 + d^2]^{3/2}}\,dl.$$

Let $\alpha = d\cos\theta + b\sin\theta$ and $\beta^2 = b^2 + d^2$. For convenience, we will ignore the constant factors for now. We can rewrite the above as

$$\int_{-d}^{d}\frac{l\,dl}{[(l-\alpha-\beta)(l-\alpha+\beta)]^{3/2}}.$$

Using the substitution $u = l - \alpha$, this can be integrated to find

$$\int\frac{u+\alpha}{(u^2-\beta^2)^{3/2}}\,du = -\left.\frac{\alpha(l-\alpha)+\beta^2}{\beta^2\sqrt{(l-\alpha)^2-\beta^2}}\right|_{-d}^{d}.$$

III. DATA

Using Mathematica we fit our data to a damped cosine wave to determine the displacement from equilibrium δ, the angular frequency ω, and the decay parameter β. The data sets and their fits are shown below.

FIG. 1. Driven Oscillations: $\beta = 0.001166$, $\omega = 0.03221$, $\delta = -0.04917$

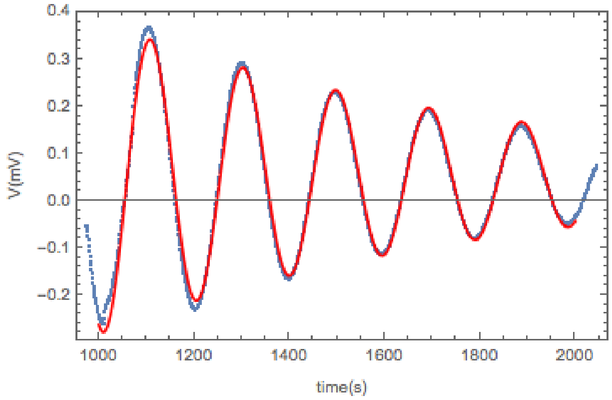


FIG. 2. Counterclockwise Oscillations Trial 1: $\beta = 0.001260$, $\omega = 0.03240$, $\delta = -0.03934$

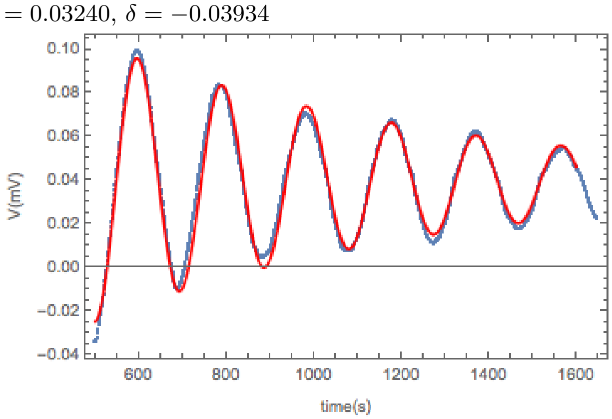


FIG. 3. Counterclockwise Oscillations Trial 2: $\beta = 0.001199$, $\omega = 0.03212$, $\delta = -0.03436$

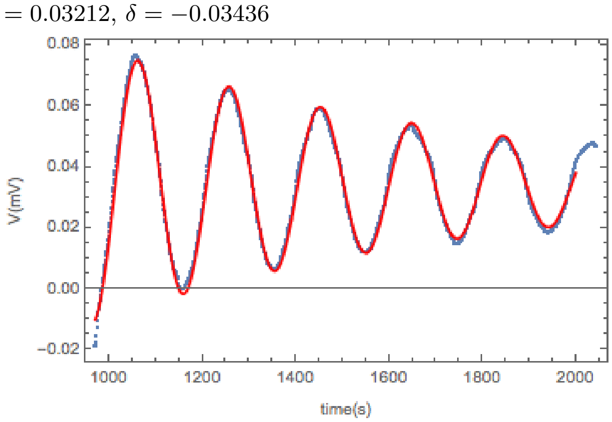


FIG. 4. Clockwise Oscillations Trial 1: $\beta = 0.001130$, $\omega = 0.03222$, $\delta = 0.02973$

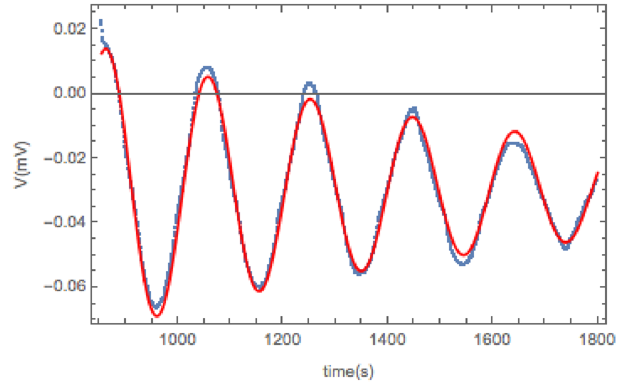
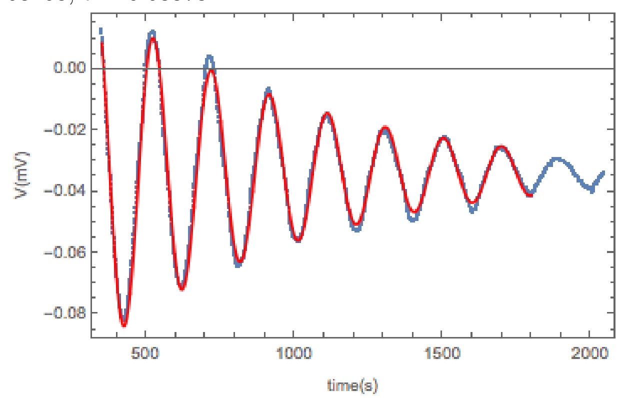
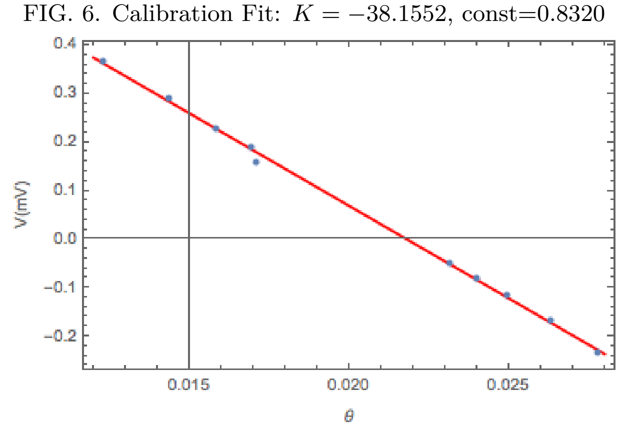


FIG. 5. Clockwise Oscillations Trial 2: $\beta = 0.001385$, $\omega = 0.03208$, $\delta = 0.03373$



Using the voltage data from the driven oscillations, we calibrate the device by comparing the maximal and minimal values of V with the corresponding values of θ obtained by observing the change in the position of the reflected laser beam. The relation between these two variables is linear, given by $V = K\theta +$ const.

FIG. 6. Calibration Fit: $K = -38.1552$, const=0.8320



IV. ANALYSIS

To determine the gravitational force between the lead and tungsten spheres, we first must calculate the torsional constant λ given by

$$\lambda = I\left(\frac{2\pi}{T}\right)^2.$$

where T is the (undamped) period of oscillation. The damped frequency w_1 is given by

$$w_1 = \sqrt{w_0^2 - \beta^2}$$

where w_0 is the natural frequency and β is the damping constant. Taking the mean of the fitting parameters obtained by our 5 data sets, we observe that $\omega_1 = 0.03221$ and $\beta = 0.001223$. Thus,

$$w_0 = \sqrt{w_1^2 + \beta^2} = 0.03223$$

and

$$T = \frac{2\pi}{w_0} = 194.944.$$

Using this value of the period and the moment of intertia caluclated in section II, we observe that

$$\lambda = 1.45722 \times 10^{-7}\ \mathrm{J}$$

By Equation (??), we have that

$$\tau_{\text{ext}} - \lambda(\theta - \theta_0) = 0$$

V. ERROR

VI. CONCLUSION