PHY64 Experiment 1: The Cavendish Experiment

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I. INTRODUCTION

In 1797, British scientist Henry Cavendish became the first person to successfully measure the gravitational interaction between two bodies in a laboratory. The apparatus used by Cavendish consisted of a torsion balance with two small lead spheres attached to either end, and two larger lead spheres positioned on alternate sides of the horizontal arm of the balance. By reflecting a light beam off of a mirror attached to the pendulum and observing the change of position of the reflected beam on a distant surface, Cavendish was able to observe a small difference in the equilibrium position of the pendulum with the larger masses in place. Using this method, Cavendish was successfully able to calculate Newton's gravitational constant G within 1% of the currently accepted value.

In this experiment, we replicate Cavendish's experiment using the TEL-Atomic torsion pendulum that mimics the experimental setup used in 1797. The TEL-Atomic apparatus allows us to observe the oscillations of the pendulum by way of an oscillating voltage readout on a computer. Using the same technique as Cavendish did, we then calibrate the oscillations in voltage to the pendulum's angle to determine the force between the pairs of masses. Thus, using Newton's Law of Gravitation, we may calculate the value of the gravitational constant G.

II. THEORY

Consider a rod with two spherical masses of radius r at each end. The rod has linear mass density σ and length $L = 2(d + r)$, where d is the distance from the center of the rod to each sphere. The two masses have mass m. Two large tungsten masses, each with mass M and radius R, are placed near the smaller masses. The rod is allowed to rotate about its center, which we will use as the origin, with the rod and all masses existing in the x-y plane. Each small spherical mass has moment of inertia $I_{\text{sphere}} = 2mr^2/5 + md^2$, and the beam has moment of inertia $I_{\text{beam}} = L\sigma(L^2 + w^2)/12$. Thus, the total moment of inertia is

$$I = \frac{2}{5}m(2r^2 + 5d^2) + \frac{L\sigma}{12}(L^2 + w^2). \tag{1}$$

We can then determine the equilibrium point by solving the following equation:

$$\sum_i \boldsymbol{\tau}_i = 0, \tag{2}$$

where we sum over all of the torques acting on the system.

Consider the system with only one of the tungsten spheres. Due to symmetry, the other sphere will result in the same torque acting on the system, and so we need only multiply by two at the end. Let $\boldsymbol{r} = d\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$ be the position of the tungsten sphere. The two smaller spheres are at positions $\boldsymbol{r}_1 = d\cos\theta\,\hat{\boldsymbol{x}} + d\sin\theta\,\hat{\boldsymbol{y}}$ and $\boldsymbol{r}_2 = -d\cos\theta\,\hat{\boldsymbol{x}} - d\sin\theta\,\hat{\boldsymbol{y}}$. Thus, the separations between the tungsten spheres and the smaller spheres are

$$\boldsymbol{R}_1 = \boldsymbol{r} - \boldsymbol{r}_1 = d(1 - \cos\theta)\hat{\boldsymbol{x}} + (b - d\sin\theta)\hat{\boldsymbol{y}}$$

and

$$\boldsymbol{R}_2 = \boldsymbol{r} - \boldsymbol{r}_2 = d(1 + \cos\theta)\hat{\boldsymbol{x}} + (b + d\sin\theta)\hat{\boldsymbol{y}}.$$

The forces acting on each small sphere are

$$\boldsymbol{F}_k = \frac{GMm}{R_k^2}\,\hat{\boldsymbol{R}}_k, \tag{3}$$

where $k = 1$ or 2 depending on the sphere we look at. The corresponding torques are therefore

$$\boldsymbol{\tau}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k. \tag{4}$$

We now look to determine the torque acting on the rod itself. A small mass element of the rod has mass $dm = \sigma\,dl$, where dl is an infinitesimal length. Choose a point l on the rod, with $-d \le l \le d$. Then the position of that point is $\boldsymbol{p} = l\cos\theta\,\hat{\boldsymbol{x}} + l\sin\theta\,\hat{\boldsymbol{y}}$. The force acting on that particular point on the rod is

$$d\boldsymbol{F} = \frac{GM\sigma}{(\boldsymbol{r} - \boldsymbol{p})^2}\frac{\boldsymbol{r} - \boldsymbol{p}}{|\boldsymbol{r} - \boldsymbol{p}|}\,dl.$$

We therefore find the torque to be

$$\boldsymbol{\tau}_{\text{rod}} = \int \boldsymbol{p} \times d\boldsymbol{F} = GM\sigma\int_{-d}^{d}\frac{\boldsymbol{p} \times \boldsymbol{r}}{|\boldsymbol{r} - \boldsymbol{p}|^3}\,dl \tag{5}$$

Expanding the terms, this integral is

$$GM\sigma\int_{-L/2}^{L/2}\frac{l(\hat{\boldsymbol{p}} \times \boldsymbol{r})}{[l^2 - 2l(d\cos\theta + b\sin\theta) + b^2 + d^2]^{3/2}}\,dl.$$

Let $\alpha = d\cos\theta + b\sin\theta$ and $r = |\boldsymbol{r}|$. For convenience, we will ignore the constant factors for now. We can rewrite the above as

$$\int_{-d}^{d}\frac{l\,dl}{[(l - \alpha - r)(l - \alpha + r)]^{3/2}}.$$

Using the substitution $u = l - \alpha$, this can be integrated to find

$$\int \frac{u + \alpha}{(u^2 - r^2)^{3/2}}\,du = -\left.\frac{\alpha(l - \alpha) + r^2}{r^2\sqrt{(l - \alpha)^2 - r^2}}\right|_{-L/2}^{L/2}.$$

Thus, we have that

$$\boldsymbol{\tau}_{\text{Rod}} = -GM\sigma(\hat{\boldsymbol{p}} \times \boldsymbol{r}) \left. \frac{\alpha(l-\alpha)+r^2}{r^2\sqrt{(l-\alpha)^2-r^2}} \right|_{l=-L/2}^{l=L/2}. \tag{6}$$

We finally consider the torque caused by the twisting of a thin tungsten wire holding up the rod. Using a torsional pendulum as the model, we expect the torque to have the form

$$\boldsymbol{\tau}_{\text{twist}} = -\lambda(\theta-\theta_0)\,\hat{\boldsymbol{z}}, \tag{7}$$

where θ_0 is the equilibrium angle of the rod when the two large tungsten spheres are not present. The value of λ must be determined experimentally.

The total torque is thus

$$\sum \boldsymbol{\tau}_i = 2(\boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 + \boldsymbol{\tau}_{\text{rod}}) + \boldsymbol{\tau}_{\text{twist}} = 0.$$

Solving for G yields

$$G = \frac{1}{2}\lambda(\theta-\theta_0) \times \left[Mm\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right) - \frac{M\sigma}{r^2}|\hat{\boldsymbol{p}} \times \boldsymbol{r}| \left(\frac{\alpha(2\alpha+L)-2r^2}{\sqrt{(2\alpha+L)^2-4r^2}} + \frac{\alpha(L-2\alpha)+2r^2}{\sqrt{(L-2\alpha)^2-4r^2}} \right) \right]^{-1}. \tag{8}$$

After λ, θ and θ_0 are determined experimentally, this equation will be used to calculate the value of G.

III. RESULTS

Using Mathematica, we fit our data to a damped cosine wave to determine the displacement from equilibrium δ, the angular frequency ω, and the decay parameter β. That is, the fit equation was of the form

$$V(t) = Ae^{-\beta t}\cos(\omega t + \phi) - \delta. \tag{9}$$

The data sets and their fits are shown below.

Using the voltage data from the driven oscillations, we calibrate the device by comparing the maximal and minimal values of V with the corresponding values of θ obtained by observing the change in the position of the reflected laser beam. The relation between these two variables is linear, given by $V = K\theta + C$, where K and C are fitting parameters.

To determine the torsional constant λ, we use the fact that the period of oscillation is dependent on λ by the equation

$$T = 2\pi\sqrt{\frac{I}{\lambda}}.$$

Rearranging gives

$$\lambda = I\left(\frac{2\pi}{T}\right)^2 = I\omega_0^2,$$

where ω_0 is the natural frequency of oscillation. From the fitting parameter, we know the damping coefficient β and the angular frequency ω, where

$$\omega^2 = \omega_0^2 - \beta^2.$$

Thus, we find the natural frequency from the fit parameters to be

$$\omega_0^2 = \omega^2 + \beta^2.$$

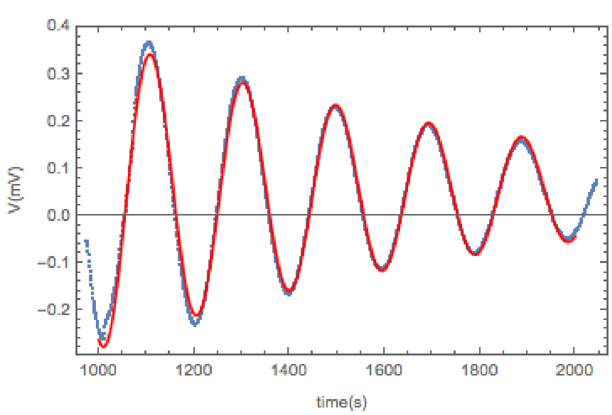

FIG. 1. Driven Oscillations: $\beta = 0.001166$, $\omega = 0.03221$, $\delta = -0.04917$

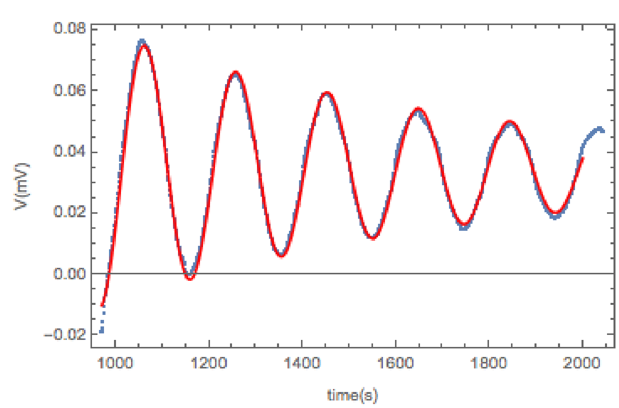

FIG. 2. Counterclockwise Oscillations Trial 2: $\beta = 0.001199$, $\omega = 0.03212$, $\delta = -0.03436$

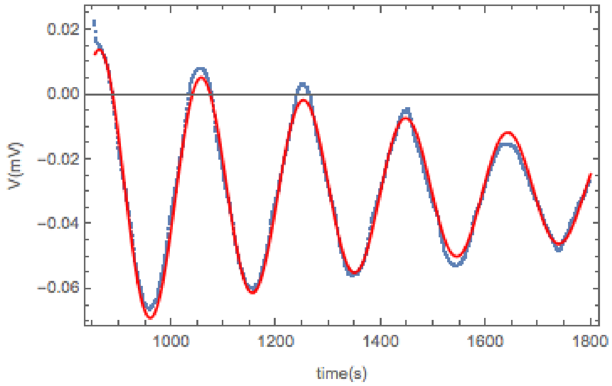

FIG. 3. Clockwise Oscillations Trial 1: $\beta = 0.001130$, $\omega = 0.03222$, $\delta = 0.02973$

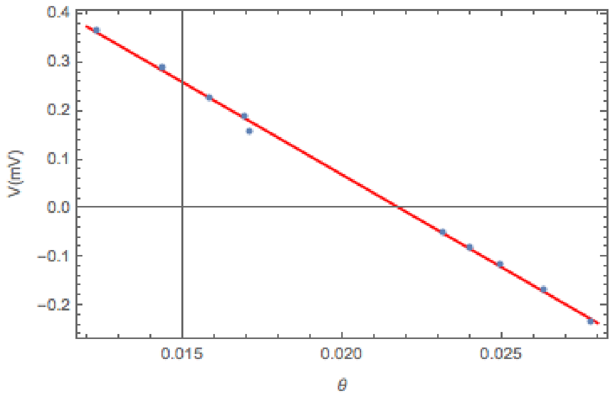

FIG. 4. Calibration Fit: $K = -38.1552\,\mathrm{mV/rad}$, $C = 0.8320\,\mathrm{mV}$

We experimentally determined that $\omega_0 = 32.23 \times 10^{-3}\,\mathrm{rad\,s^{-1}}$ and $\beta = 1.223 \times 10^{-3}\,\mathrm{s^{-1}}$. Using (1) we find

$$I = 1.403 \times 10^{-4}\,\mathrm{kg\,m^2}$$

and therefore we have that

$$\lambda = 1.457 \times 10^{-7}\,\mathrm{N\,m/rad}.$$

From (9), we can determine the voltage at equilibrium to simply be the fitting parameter δ.

IV. ERROR

V. CONCLUSION