PHY64 Experiment 1: The Cavendish Experiment

Matthew S. E. Peterson and Jackson Burzynski
(Dated: February 21, 2016)

I. INTRODUCTION

II. THEORY

We have a long rod of length $2d$ and width w, with two spherical masses of radius r at each end. The rod has linear mass density σ, and the two masses have mass m. Two large tungsten masses, each with mass M and radius R, are placed near the smaller masses. The rod is allowed to rotate about its center, which we will use as the origin, with the rod and all masses existing in the x-y plane. Each small spherical mass has moment of inertia $I_{\text{sphere}} = 2mr^2/5 + md^2$, and the beam has moment of inertia $I_{\text{beam}} = 2d\sigma(4d^2 + w^2)/12$. Thus, the total moment of inertia is

$$I = \frac{2}{5}m(2r^2 + 5d^2) + \frac{1}{6}d\sigma(w^2 + 4d^2). \tag{1}$$

We can then determine the equilibrium point by solving the following equation:

$$\sum_i \boldsymbol{\tau}_i = I\frac{d^2\boldsymbol{\theta}}{dt^2}, \tag{2}$$

where we sum over all of the torques acting on the system.

Consider the system with only one of the tungsten spheres. Due to symmetry, the other sphere will result in the same torque acting on the system, and so we need only multiply by two at the end. Let $\boldsymbol{r} = d\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$ be the position of the tungsten sphere. The two smaller spheres are at positions $\boldsymbol{r}_1 = d\cos\theta\,\hat{\boldsymbol{x}} + d\sin\theta\,\hat{\boldsymbol{y}}$ and $\boldsymbol{r}_2 = -d\cos\theta\,\hat{\boldsymbol{x}} - d\sin\theta\,\hat{\boldsymbol{y}}$. Thus, the separations between the tungsten spheres and the smaller spheres are

$$\boldsymbol{R}_1 = \boldsymbol{r} - \boldsymbol{r}_1 = d(1-\cos\theta)\hat{\boldsymbol{x}} + (b - d\sin\theta)\hat{\boldsymbol{y}}$$

and

$$\boldsymbol{R}_2 = \boldsymbol{r} - \boldsymbol{r}_2 = d(1+\cos\theta)\hat{\boldsymbol{x}} + (b + d\sin\theta)\hat{\boldsymbol{y}}.$$

The forces acting on each small sphere are

$$\boldsymbol{F}_k = \frac{GMm}{R_k^2}\,\hat{\boldsymbol{R}}_k, \tag{3}$$

where $k = 1$ or 2 depending on the sphere we look at. The corresponding torques are therefore

$$\boldsymbol{\tau}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k. \tag{4}$$

We now look to determine the torque acting on the rod itself. A small mass element of the rod has mass $dm = \sigma\,dl$, where dl is an infinitesimal length. Choose a point l on the rod, with $-d \le l \le d$. Then the position of that point is $\boldsymbol{p} = l\cos\theta\,\hat{\boldsymbol{x}} + l\sin\theta\,\hat{\boldsymbol{y}}$. The force acting on that particular point on the rod is

$$d\boldsymbol{F} = \frac{GM\sigma}{(\boldsymbol{r}-\boldsymbol{p})^2}\frac{\boldsymbol{r}-\boldsymbol{p}}{|\boldsymbol{r}-\boldsymbol{p}|}\,dl.$$

We therefore find the torque to be

$$\boldsymbol{\tau}_{\text{rod}} = \int \boldsymbol{p}\times d\boldsymbol{F} = GM\sigma\int_{-d}^{d}\frac{\boldsymbol{p}\times\boldsymbol{r}}{|\boldsymbol{r}-\boldsymbol{p}|^3}\,dl \tag{5}$$

Expanding the terms, this integral is

$$GM\sigma\int_{-d}^{d}\frac{l(b\cos\theta - d\sin\theta)\hat{\boldsymbol{z}}}{[l^2 - 2l(d\cos\theta + b\sin\theta) + b^2 + d^2]^{3/2}}\,dl.$$

Let $\alpha = d\cos\theta + b\sin\theta$ and $\beta^2 = b^2 + d^2$. For convenience, we will ignore the constant factors for now. We can rewrite the above as

$$\int_{-d}^{d}\frac{l\,dl}{[(l-\alpha-\beta)(l-\alpha+\beta)]^{3/2}}.$$

Using the substitution $u = l - \alpha$, this can be integrated to find

$$\int\frac{u+\alpha}{(u^2-\beta^2)^{3/2}}\,du = -\left.\frac{\alpha(l-\alpha)+\beta^2}{\beta^2\sqrt{(l-\alpha)^2-\beta^2}}\right|_{-d}^{d}.$$

III. DATA

IV. ANALYSIS

V. ERROR

VI. CONCLUSION