Continued Fractions Generated By Primes and Better Approximations For The Metallic Ratios

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Contents

Abstract

In this paper I will search for exact values of continued fractions built from different families for primes and find ways to connect them. The notation in this paper will stay mostly consistent(Gauss's Kettenbruch), though there may be use of the linear form of simple($b_n = 1$) continued fractions. Decimal expansions of these fractions have been found but no exact values have ever been documented.This paper will assume some unproved conjectures as being true, mainly the twin primes conjecture.

1 Introduction

The notation in this paper for a continued fraction may vary, but for most all of the paper it will be denoted by Gauss's Kettenbruch notation:

$$a_0 + \mathop{\mathrm{K}}_{n=1}^{N} \frac{a_n}{b_n} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_1}{a_2 + \ddots + \frac{a_N}{b_N}}}$$

2 Continued Fractions Of The First Kind Built From Primes

Let us denote a function $p(n)$, let it give us the nth prime number, we can define this function as the Willan's prime function.

$$p(n) = 1 + \sum_{i=1}^{2^n} \left\lfloor \left(\frac{n}{\sum_{j=1}^{i} \left\lfloor \cos^2 \frac{(j-1)!+1}{j} \pi \right\rfloor} \right)^{1/n} \right\rfloor$$

Using $p(n)$ we can now create a continued fraction F_1 that is built from the primes with our a_0 term being 2, our b_n terms being 1. Thus we arrive at this fraction:

$$F_1 = 2 + \operatorname*{K}_{n=2}^{\infty} \frac{1}{p_n} = 2 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{\ddots}}}$$

Let us now use this definition of F_1 to create a function $F_1(n)$ for $n \geq 2$ which by definition truncates this continued fraction by order n, i.e. the upper bound of the K-Sum(Kettenbruch Sum) is n. Now our problem becomes finding this limit: $\lim_{n\to\infty} F_1(n)$, we can see by the Śleszyński–Pringsheim theorem that this limit does converge to a finite value. If you evaluate $F_1(n)$ for large n you can get sufficient decimal approximations of F_1 but the aim of this paper is to find exact values for fractions spawning from the same family.

2.1 Search For An Exact Value Of F_1

We can truncate the value of F_1 to an extent to get a decimal approximation of 2.3088, but when truncating this fraction across a set of consecutive values there is no viable pattern that could be used to find the exact value based off the only the truncated values: $F_1(n)$. Though we can conclude that $2.2 < F_1 < 2.6$, this can be seen even with the one truncation of F_1 shown above. Although these bounds still leave an error of 0.1 which we shall aim to reduce thought this section. With this form of F_1 it is customary to find a recursive formula expressing some x as a truncated continued fractions with elements $f_n(x)$ where f_n denotes a recursive sequence based of an initial function $f(x)$, and an initial condition $f(c) = d$, then the recursive formula is simplified to be some quadratic of the initial function $f(x)$ and an initial condition $f(c) = d$. But with this continued fraction F_1 there is not an easily aperient function $\varphi(\chi)$, this is due to the non constant distance between primes, although a recursive formula may still be possible using the Willan's prime formula $p(n)$ or the prime counting function $\pi(n)$, but it would not be in a form expressible by a quadratic equation, ergo it mostly likely would not have a closed form recursive equation expressed in elementary functions.

Lemma 2.1. *Let Λ be an infinite fraction and let $n \in \mathbb{N}$, this implies there must be a function $\Lambda(n)$ which truncates Λ. This implies that there must exist a m such that $\Lambda(m-1) < \Lambda < \Lambda(m+1)$*

Note: This lemma may be proven later in the paper, but for now I will just assume it is true.

Lemma 2.1 guarantees that you can find an m such that the error in you approximation of Λ is as small as required, this may also help you in finding the exact value of Λ given you know the exact values of $\Lambda(m-1), \Lambda(m+1)$ for a sufficiently large value of m.

Corollary 2.1.1. *As m increases the error in Lemma 2.1. decreases.*

If you take the limit of m, the error in Lemma 2.1. goes to zero, this can be seen by the definition of the squeeze theorem, this follows from our definition of $\Lambda(n)$ being the truncated Λ. The definition of F_1 right now is very unpleasing because of the form of $p(n)$ and the K-Sum. We can simply the $p(n)$ to be:

$$p(n) = 1 + \sum_{i=1}^{2^n} \left\lfloor \left(\frac{n}{\sum_{j=1}^{i} \left\lfloor \cos^2 \frac{(j-1)!+1}{j}\pi \right\rfloor} \right)^{1/n} \right\rfloor = 1 + \sum_{j=1}^{2^n} \left\lfloor \left(\frac{n}{j}\right)^{1/n} \right\rfloor$$

Using this new formula for $p(n)$ we can start to calculate some values of $F_1(n)$:

Table 1: Values of $F_1(n)$ for $n = 2$ to 10

n	$F_1(n)$
2	$\frac{7}{3}$
3	$\frac{30}{13}$
4	$\frac{157}{68}$
5	$\frac{972}{421}$
6	$\frac{6961}{3015}$
7	$\frac{56760}{24541}$
8	$\frac{516901}{223884}$
9	$\frac{4325220}{1873186}$
10	$\frac{57999271}{25121075}$

Playing around with our formula of $F_1(n)$ and using the notion of a continued fraction we arrive at the recursive formula:

$$F_1(n) = 2 + \frac{1}{3+\frac{1}{\frac{1}{F_1(n-1)-2}+\frac{1}{1+\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor}}}, \text{ with } F_1(2) = \frac{7}{3}$$

Then if we move around some of the terms to make turn it from a continued fraction to linear equation, giving us the functional equation:

$$-7(F_1(n-1)-2)+3F_1(n)(F_1(n-1)-2)=$$
$$7p(n)-3F_1(n)p(n)-p(n)(F_1(n-1)-2)(F_1(n)-2)$$
$$\text{Where } p(n) \text{ is the Willan's prime function and } F_1(2)=\tfrac{7}{3}.$$

This is sadly not in a polynomial form meaning we cannot find what $F_1(n)$ is, although we may be able to solve for $F_1(n)$ via a function equation. Our main problems are the $F_1(n-1)$ and $p(n)$ terms, we can eliminate one of these problems by setting $p(n)$ as a constant function because we are solving for $F_1(n)$, thus the $p(n)$ just contribute to the final form of $F_1(n)$. The $F_1(n-1)$ terms however are not straight forward, this is because of unknown form of $F_1(n)$ this means we can't be for certain what $F_1(n-1)$ is, but we can solve for $F_1(n-1)$ in terms of $F_1(n)$ and subsitute it into the equation getting:

$$F_1(n-1)=\tfrac{6F_1(n)-p(n)F_1(n)+3p(n)-14}{3F_1(n)+p(n)F_1(n)-2p(n)-7}$$

Thus giving us:

$$3p(n)F_1(n)^2+3F_1(n)^2-26p(n)F_1(n)+28p(n)=0$$

Isolating $F_1(n)$ gives us: $F_1(n)=\frac{13p(n)\pm\sqrt{p(n)(85p(n)-84)}}{3(p(n)+1)}$ Now that we have a closed form for $F_1(n)$ we can solve for F_1 by taking the limit of $F_1(n)$ as n increases.

$$F_1=\lim_{n\to\infty}\frac{13p(n)\pm\sqrt{p(n)(85p(n)-84)}}{3(p(n)+1)}$$

This inherently does not make sense because it is asking us to find the ∞th prime number which does not make sense, but we can use the Willian's prime formula to find the limit, this still does not help to much because that still approaches ∞ but we can see that the limit is of the form $\frac{\infty}{\infty}$ so we can use L'Hôpital's rule and the limit becomes:

$$F_1=\lim_{n\to\infty}\frac{p'(n)\left[13\pm\frac{(170p(n)-84)(p(n)+1)-2(p(n)+1)\sqrt{p(n)(85p(n)-84)}}{2(p(n)+1)\sqrt{p(n)(85p(n)-84)}}\right]}{3(p(n)+1)}$$

But this still does not resolve the indeterminate form, the next viable way to solve this limit is to expand $p(n)$ and try to use a substitution that makes the limit resolvable. Expanding the $p(n)$ terms in our original limit gives the function:

$$\frac{13p(n)\pm\sqrt{p(n)(85p(n)-84)}}{3(p(n)+1)}=$$
$$\frac{14+\displaystyle\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor\pm\sqrt{1+\displaystyle\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor\left(1+85\displaystyle\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor\right)}}{3\left(\displaystyle\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor+2\right)}$$

If we make the substitution $\frac{1}{n} = x$ then the summation terms become $\frac{1}{x^x}$ because the upper bounds of the summations go to one, and what n tends to changes to x tends to 0. So we get the new limit:

$$F_1 = \lim_{x\to 0} \frac{\frac{14x^x+1}{x^x} \pm \sqrt{1+\frac{1}{x^x}(1+\frac{85}{x^x})}}{\frac{3}{x^x}+6}$$

This is a solvable form because $\lim_{x\to 0} x^x = 1$. So our F_1 becomes:

$$\boxed{F_1 = \lim_{x\to 0} \frac{\frac{14x^x+1}{x^x} \pm \sqrt{1+\frac{1}{x^x}(1+\frac{85}{x^x})}}{\frac{3}{x^x}+6} = \frac{15\pm\sqrt{87}}{9}}$$

If we use the positive answer $\frac{15+\sqrt{87}}{9}$ we get approximately 2.59 which follows somewhat from the table above, where each value was constantly increasing from about 2.3. Thus we arrive at $F_1 = 2 + \frac{1}{3+\frac{1}{5+\frac{1}{\ddots}}} = \frac{15\pm\sqrt{87}}{9}$

2.1.1 The Significance Of F_1

Given the values of F_1, how does this motivate other problems in mathematics. Well this value of F_1 can be used to solve for other types of continued fractions such as the fraction built from the same primes but with alternating coefficients, namely it provides upper and lower bounds of that value:

$$\left|\frac{15-\sqrt{87}}{9}\right| < 2 + \left|\operatorname*{K}_{n=2}^{\infty} \frac{1}{(-1)^{n+1}p(n)}\right| < \left|\frac{15+\sqrt{87}}{9}\right|$$

Lemma 2.2. *Given a non-alternating continuous fraction Λ converges to two values λ_1, λ_2, then one of the values that the alternating continuous fraction Λ' converges to Λ'_1, then $|\lambda_1| < |\Lambda'_1| < |\lambda_2|$*

Corollary 2.2.1. *Given a continued fraction function Λ and it truncated function $\Lambda(n)$ and it's alternating counterpart Λ' and $\Lambda'(n)$, then $\Lambda'(n) \leqq \Lambda(n)$ for finite n.*

2.2 The Alternating Counterpart Of F_1 With Period 1

Now that we know that the function $F_1(n)$ is absolutely convergent on the interval $n \in [2, \infty)$ thus we can concretely say that $F'_1(n)$ also converges on the same interval where $F'_1(n)$ is it's alternating counterpart. We can also conclude that $\frac{15-\sqrt{87}}{9} < F_1 < \frac{15+\sqrt{87}}{9}$ and that for any finite n, $F'_1(n) \leqq F_1(n)$. Now we shall follow the same approach as with F_1, the only difference is now our a_n terms are:

$$1 + \sum_{j=1}^{2^n} \left\lfloor \left(\frac{(-1)^{n+1}n}{j}\right)^{1/n} \right\rfloor$$

We can see just by the structure of a_n that F_1' will not be a real number, this still coincides with Lemma 2.2 because $|z| \in \mathbb{R}$ for a complex z. Thus we get to the recursive formula for $F_1'(n)$:

$$F_1'(n) = 2 + \frac{1}{3 - \frac{1}{\frac{1}{F_1'(n-1)-2} + \frac{1}{1+\sum_{j=1}^{2^n}\left\lfloor\left(\frac{(-1)^{n+1}n}{j}\right)^{1/n}\right\rfloor}}}, \text{ with } F_1'(2) = \frac{7}{3}$$

The solving for $F_1'(n-1)$ gives us:

$$F_1'(n-1) = \frac{5p_1(n)F_1'(n) - 11p_1(n) - 6F_1'(n) + 14}{p_1(n)F_1'(n) - 2p_1(n) - 3F_1'(n) + 7}$$

Where $p_1(n)$ is our alternating prime function. Then substituting what in for $F_1'(n-1)$ and simplifying gives us:

$$F_1'(n) = \frac{5p_1(n)F_1'(n) - 11p_1(n) - 6F_1'(n) + 14}{p_1(n)F_1'(n) - 2p_1(n) - 3F_1'(n) + 7}$$

Then we just have to solve for $F_1'(n)$ in terms of $p_1(n)$:

$$F_1'(n) = \frac{7p_1(n) - 13 \pm \sqrt{(5p_1(n)+1)(p_1(n)+1)}}{2(p_1(n)-3)}$$

Now that we have isolated $F_1'(n)$ now we just have to find the limit as n tends to infinity, but when doing this we run into the same type of problem that was seen in the derivation of F_1, we have to find the ∞th alternating prime, so to combat this we again will expand the $p_1(n)$ terms and look for a possible substitution that could eliminate this problem. So we get:

$$\frac{7p_1(n) - 13 \pm \sqrt{(5p_1(n)+1)(p_1(n)+1)}}{2(p_1(n)-3)} =$$

$$\frac{\sum_{j=1}^{2^n}\left\lfloor\left(\frac{(-1)^{n+1}n}{j}\right)^{1/n}\right\rfloor - 5 \pm \sqrt{\left(5\sum_{j=1}^{2^n}\left\lfloor\left(\frac{(-1)^{n+1}n}{j}\right)^{1/n}\right\rfloor + 6\right)\left(2 + \sum_{j=1}^{2^n}\left\lfloor\left(\frac{(-1)^{n+1}n}{j}\right)^{1/n}\right\rfloor\right)}}{2\left(\sum_{j=1}^{2^n}\left\lfloor\left(\frac{(-1)^{n+1}n}{j}\right)^{1/n}\right\rfloor - 2\right)}$$

Let us again use the substitution $1/n = x$ then our summation terms become:

$$\sum_{j=1}^{\infty}\left\lfloor\left(\frac{(-1)^{\left(\frac{x+1}{x}\right)}}{jx}\right)^{x}\right\rfloor$$

The limits now tend towards 0. so we arrive at the limit:

$$\lim_{x\to 0}\frac{\sum_{j=1}^{\infty}\left\lfloor\left(\frac{(-1)^{\left(\frac{x+1}{x}\right)}}{jx}\right)^{x}\right\rfloor - 5 \pm \sqrt{\left(5\sum_{j=1}^{\infty}\left\lfloor\left(\frac{(-1)^{\left(\frac{x+1}{x}\right)}}{jx}\right)^{x}\right\rfloor + 6\right)\left(2 + \sum_{j=1}^{\infty}\left\lfloor\left(\frac{(-1)^{\left(\frac{x+1}{x}\right)}}{jx}\right)^{x}\right\rfloor\right)}}{2\left(\sum_{j=1}^{\infty}\left\lfloor\left(\frac{(-1)^{\left(\frac{x+1}{x}\right)}}{jx}\right)^{x}\right\rfloor - 2\right)}$$

Note: The infinity in the upper bounds of the summation terms is really $1/x$ but as x tends to zero $1/x$ tends to infinity. In this form we have to take the limit of $\frac{x+1}{x}$ as x tends to 0, which gives us one, but all the summation terms are raised to the power of x so all the summation terms become $\frac{1}{x}$, thus we have the sum limit:

$$\lim_{x\to 0} \frac{\frac{1}{x} - 5 \pm \sqrt{\left(\frac{5}{x}+6\right)\left(2+\frac{1}{x}\right)}}{\frac{2}{x} - 4}$$

Simplifying gives us:

$$\boxed{F_1' = \lim_{x\to 0} \frac{1 - 5x \pm \sqrt{12x^2 + 16x + 5}}{2 - 4x} = \frac{1 \pm \sqrt{5}}{2}}$$

This answer does follow from Lemma 2.2 for one answer namely the golden ratio ϕ.

2.3 The Alternating Counterpart Of F_1 With Period 2

With our new result F_1' we can set new bounds for a new fraction F_1'' where the double prime means that we have $a_0 - Q$ where Q is the fractional part, we call this period 2, and F_1' is of period 1.

Lemma 2.3. *Given an alternating continuous fraction of period one Λ' that has two solutions λ_1 and λ_2, and it's alternating counterpart Λ'':*

$$|\lambda_1| < |\Lambda''| < |\lambda_2|$$

Corollary 2.3.1. *Given a continuous fraction Λ, and it's alternating counterparts Λ', Λ'', then*

$$|\Lambda''| < |\Lambda'| < |\Lambda|$$

With Corollary 2.3.1 and Lemma 2.3 if you know the values of Λ and Λ' you can generate the best bounds possible for Λ''. Thus we get the equation:

$$\text{Error Bound : F}_1'' \qquad \tfrac{1+\sqrt{5}}{2} < F_1'' < \tfrac{15-\sqrt{87}}{9}$$

Notice that we our using both F_1 and F_1' this is to create the smallest possible error. Solving for a recursive formula of F_1'' will be of the same form as the problems above.

$$F_1''(n) = 2 - \frac{1}{3+\frac{1}{\frac{1}{F_1''(n-1)-2} - \frac{1}{1+\sum_{j=1}^{2n}\left\lfloor\left(\frac{(-1)^n n}{j}\right)^{1/n}\right\rfloor}}}, \text{ with } F_1''(2) = \tfrac{5}{3}$$

Then solving for $F_1''(n-1)$ gives us:

$$F_1''(n-1) = \frac{7F_1''(n) - 11 + F_1''(n)p_2(n) - p_2(n)}{2F_1''(n) - 3 - F_1''(n)p_2(n) + 2p_2(n)}$$

Where $p_2(n)$ is the prime function of period 2. Inputting $F_1''(n)$ and simplifying gives us:

$$F_1''(n) = \frac{p_2(n)^2+p_2(n)+18\pm\sqrt{p_2(n)^4+2p_2(n)^3+2p_2(n)^2+9}}{9}$$

We again must expand $p_2(n)$ and try to find a simplification for the limit. Looking at the $p_2(n)$ terms, if we again substitute $n = 1/x$ as it is of the same form, the summation terms again just become $1/x$. Thus we get the limit:

$$\lim_{x\to 0} \frac{\frac{1}{x^2} + \frac{1}{x} + 18 \pm \sqrt{\frac{1}{x^4} + \frac{2}{x^3} + \frac{2}{x^2} + 9}}{9} =$$
$$\lim_{x\to 0} \frac{18x^2 + x + 1 \pm \sqrt{9x^4 + 2x^2 + 2x + 1}}{9x^2}$$

In the form of this limit you are going to get a division by zero so we are going to have to use L'Hôpital's rule twice thus we get the limit:

$$\boxed{F_1'' = \lim_{x\to 0} \frac{36 \pm 162x^6 + 108x^4 + 36x^3 + 288x^2 + 12x + 5}{18\sqrt{(9x^4 + 6x^2 + 2x + 1)^3}} = \frac{36 \pm 5}{18}}$$

If we are looking at the positive value of F_1'' it follows with our error bound.

2.4 The Shifted Counterparts Of F_1 , F_1' , and F_1''

We can not only find the alternating counterparts of a continued fraction, we can also find the value of a shifted fraction, i.e. if you were to shift a fraction by m $a_0 \to a_m$ and subsequently $a_n \to a_{n+m}$ Thus we get the formula for a shifted continued fraction of type one:

$$\Omega_\tau \Lambda_{\tau,P_k,m,\Gamma} = a_m + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{P_k \Omega_\tau b_{n+m}}{P_k \Omega_\tau a_{n+m}} \text{ for } 0 \leqq k \leqq 2,\ 1 \leqq \tau \leqq 3 \text{ and, } \gamma_1, \gamma_0 \in \mathbb{F}$$

Where P_k denotes what period(period k) the fraction is alternating at where a non alternating fraction is period 0, Ω_τ denotes the warp which will be touched on later., τ is the type of the fraction which will be expanded on later, and Γ denotes the gap over a field $\mathbb{F}$ which will also be expanded on later. Note we are using a change of notation in this formula because of the confusing notation using ticks, so when in this notation P_0 is 0 ticks, P_1 is one tick, and P_2 is two ticks. Thus we obtain the formula for $F_{1,m}, F_{1.m}'$ and $F_{1,m}''$ or in our $\Omega_\tau \Lambda_{\tau,P_k,m,\Gamma}$ notation, $F_{1,0,m,[0,0]}, F_{1,1,m,[0,0]}$ and $F_{1,2,m,[0,0]}$:

$$F_{1,m} = p(1+m) + \underset{n=2+m}{\overset{\infty}{\mathrm{K}}} \frac{1}{p(n)}$$
$$F_{1,m}' = p(1+m) + \underset{n=2+m}{\overset{\infty}{\mathrm{K}}} \frac{1}{p_1(n)}$$
$$F_{1,m}'' = p(1+m) - \underset{n=2+m}{\overset{\infty}{\mathrm{K}}} \frac{1}{p_2(n)}$$

Remark. *The $\Omega_\tau \Lambda_{\tau,P_k,m,\Gamma}$ notation is somewhat an of notation, though it does allow for you to be able to see exactly what is happening with the given fraction. In this paper the $\Omega_\tau \Lambda_{\tau,P_k,m,\Gamma}$ notation will be used but it will be used sparingly.*

Let us try to derive general formulas for $F_{1,m}, F'_{1,m}$ and $F''_{1,m}$. Note we must have $m \in \mathbb{N}$ but for any Λ a shift may not just be a natural number.

$$\boxed{F_{1,m}(n) = p(1+m) + \cfrac{1}{p(2+m) + \cfrac{1}{\frac{1}{F_{1,m}(n-1)-2} + \frac{1}{1+p(k)}}}} \text{ with}$$

$$F_{1,m}(2) = p(1+m) + \tfrac{1}{p(2+m)}$$

$$\boxed{F'_{1,m}(n) = p(1+m) + \cfrac{1}{p(2+m) - \cfrac{1}{\frac{1}{F'_{1,m}(n-1)-2} + \frac{1}{1+p(k)}}}} \text{ with}$$

$$F'_{1,m}(2) = p(1+m) + \tfrac{1}{p(2+m)}$$

$$\boxed{F''_{1,m}(n) = p(1+m) - \cfrac{1}{p(2+m) + \cfrac{1}{\frac{1}{F''_{1,m}(n-1)-2} + \frac{1}{1+p(k)}}}} \text{ with}$$

$$F''_{1,m}(2) = p(1+m) - \tfrac{1}{p(2+m)}$$

Where $k = m + n + 3$

Given these general formulas you can solve for $F_{1,m}(n-1)$ and its counterparts and then isolate the $F_{1,m}(n)$ and it's counterparts in terms of $p(n)$ by the same method used above. This derivation will not be present in this paper because of it similarity of the derivations above.

2.4.1 The Late Shifted Counterparts of F_1, F'_1 and F''_1

We can generalize our formulas for $F_{1,m}$ and it's alternating counterparts to $F_{1,M}, F'_{1,M}$ and $F''_{1,M}$ where M is some interval in $\mathbb{N}$.

Lemma 2.4. *If we have the continued fraction $\Lambda_{\tau,P_k,M,\Gamma}$ such that M is an interval and the value $[m_1, m_2], \mu$ where $\Lambda(m_1)$ is an initial condition. Then Λ is shifted by μ starting at m_2*

2.4.2 The Semi-Shifted Counterparts of F_1, F'_1 and F''_1

We can further generalize the lemma above by allowing m_1 not to be an initial condition and changing our notation to $[m_1, m_2]_\beta , \mu$ where β is either 0 denoting in, or 1 denoting out.

Corollary 2.4.1. *If we have the continued fraction $\Lambda_{\tau,P_k,M,\Gamma}$ such that M is an interval and the value defined by $[m_1, m_2]_\beta , \mu$. This gives us a shifted fraction Λ where either only the interval $[m_1, m_2]$ is shifted by μ or everything but the interval $[m_1, m_2]$ is shifted by μ*

2.4.3 The Multi-Shifted Counterparts of F_1, F_1' and F_1''

We can obtain the most general form of a shifted fraction. To do this we change M from $[m_1, m_2]_\beta$, μ to $\{[m_n, m_{n+1}]_0$, μ_n , $\epsilon\}_{n\in\mathbb{N}}$ this gives us a infinite set of intervals and values where the M intervals and values tell how much the values in that interval are shifted by, and the ϵ value determine how much every other value is shifted by.

2.5 The Gapped Counterparts of F_1, F_1' and F_1''

We can also remove values from a fraction in addition to shifting terms. Using our $\Omega_\tau\Lambda_{\tau,P_k,m,\Gamma}$ notation, the gap is Γ which is defined to be an interval $[\gamma_1, \gamma_2]$ where those values are removed from Λ. We can derive the formula for the gapped versions of F_1, F_1' and F_1'' for a given interval Γ.

$$\boxed{F_{1,\Gamma} = \Gamma\langle p(n)\rangle + \mathop{\mathrm{K}}_{k=\Gamma\langle 2\leqq h\rangle}^{\infty} \frac{1}{\Gamma\langle p(k)\rangle}}$$

$$\boxed{F_{1,\Gamma}' = \Gamma\langle p(n)\rangle + \mathop{\mathrm{K}}_{k=\Gamma\langle 2\leqq h\rangle}^{\infty} \frac{1}{\Gamma\langle p_1(k)\rangle}}$$

$$\boxed{F_{1,\Gamma}'' = \Gamma\langle p(n)\rangle - \mathop{\mathrm{K}}_{k=\Gamma\langle 2\leqq h\rangle}^{\infty} \frac{1}{\Gamma\langle p_2(k)\rangle}}$$

Where $[a, b]\langle\hat{x}\rangle$ is the first value of $\hat{x}$ in $[a, b]$ and $[a, b]\langle x \leqq \hat{x}\rangle$ is the first value of $\hat{x}$ that satisfies the inequality and the interval.

Remark. *We can not find a general formula for $F_{1,\Gamma}, F_{1,\Gamma}'$ or $F_{1,\Gamma}''$ for a given gap $[\gamma_1, \gamma_2]$ because of the uncertainty of if a value will stay present or be part of the gap, but we can find a formula using the $[a, b]\langle\hat{x}|x\rangle$ where $|$ is a operator.*

$$\boxed{F_{1,\Gamma}(n) = \Gamma\langle 2 \leqq \chi\rangle + \cfrac{1}{\Gamma\langle 3 \leqq \chi\rangle + \cfrac{1}{\frac{1}{\Gamma\langle F_1(n-1)-2\rangle} + \frac{1}{p_{0,\Gamma}(n)}}}}$$

$$\boxed{F_{1,\Gamma}'(n) = \Gamma\langle 2 \leqq \chi\rangle + \cfrac{1}{\Gamma\langle 3 \leqq \chi\rangle - \cfrac{1}{\frac{1}{\Gamma\langle F_1'(n-1)-2\rangle} + \frac{1}{p_{1,\Gamma}(n)}}}}$$

$$\boxed{F_{1,\Gamma}''(n) = \Gamma\langle 2 \leqq \chi\rangle - \cfrac{1}{\Gamma\langle 3 \leqq \chi\rangle + \cfrac{1}{\frac{1}{\Gamma\langle F_1''(n-1)-2\rangle} + \frac{1}{p_{2,\Gamma}(n)}}}}$$

Where $p_\Gamma(n)$ is $p(n)$ under the gap Γ.

2.5.1 The Multi-Gapped Counterparts of F_1, F_1' and F_1''

We can also set Γ to be a set of intervals insted of just one so that there are multiple different gaps in the fraction. We define $\Gamma = \{\nu_1, \nu_2, \nu_3, \dots\}$ such that every ν is an interval.

Remark. *With a gapped partial fraction function $\Lambda_\Gamma(n)$, it's graph is non-continuous unless each endpoint of each gap are related to each other by a function, come in regular intervals, and gapped for any n. We will touch on this later.*

2.6 The Warped Counterparts of F_1, F_1' and F_1''

We may also change a fraction Λ by performing a warp $\Omega_\tau\Lambda$, this Ω_τ is the warp operator and has three forms depending on τ (which is the same τ that tells the type of the fraction) :

$$\Omega_1 = (1, \omega_2)$$
$$\Omega_2 = (\omega_1, 1)$$
$$\Omega_3 = (\omega_1, \omega_2)$$

We will only focus on Ω_1 for now but the definitions for all omegas are show below.

$$\Omega_1\Lambda = \operatorname*{K}_{n=m}^{M} \frac{b_n}{\omega_2 a_n}$$
$$\Omega_2\Lambda = \operatorname*{K}_{n=m}^{M} \frac{\omega_1 b_n}{a_n}$$
$$\Omega_3\Lambda = \operatorname*{K}_{n=m}^{M} \frac{\omega_1 b_n}{\omega_2 a_n}$$

Thus we can now pull values from a fraction, shift those values around, and scale those values. We can find formulas for $\Omega_\tau F_1, \Omega_\tau F_1'$ and $\Omega_\tau F_1''$ for any $1 \leqq \tau \leqq 3$:

$$\Omega_1 F_1 = 2\omega_2 + \operatorname*{K}_{n=2}^{\infty} \frac{1}{\omega_2 p(n)}$$
$$\Omega_2 F_1 = 2\omega_1 + \operatorname*{K}_{n=2}^{\infty} \frac{\omega_1}{p(n)}$$
$$\Omega_3 F_1 = 2\left(\tfrac{\omega_1+\omega_2}{2}\right) + \operatorname*{K}_{n=2}^{\infty} \frac{\omega_1}{\omega_2 p(n)}$$
$$\Omega_1 F_1' = 2\omega_2 + \operatorname*{K}_{n=2}^{\infty} \frac{1}{\omega_2 p_1(n)}$$
$$\Omega_2 F_1' = 2\omega_1 + \operatorname*{K}_{n=2}^{\infty} \frac{\omega_1}{p_1(n)}$$

$$\Omega_3 F_1' = 2\left(\tfrac{\omega_1+\omega_2}{2}\right) + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{\omega_1}{\omega_2 p(n)}$$
$$\Omega_1 F_1'' = 2\omega_2 - \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{1}{\omega_2 p_2(n)}$$
$$\Omega_2 F_1'' = 2\omega_1 - \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{\omega_1}{p_2(n)}$$
$$\Omega_3 F_1'' = 2\left(\tfrac{\omega_1+\omega_2}{2}\right) - \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{\omega_1}{\omega_2 p_2(n)}$$

Notice that we always multiply the a_0 term by the non-one ω and when they are both non-one we average them.

2.6.1 The Multi-Warped Counterparts of F_1, F_1' and F_1''

We can also extend our warping operator Ω_τ to be a set of intervals with corresponding values that uses the $[a,b]\langle\hat{x}\rangle$ function to check if a value is in one of the intervals and if it is in that interval then it applies the warp. Notice that this also creates a discontinuity in the graph of the partial fractions of degree n generated by $\Lambda(n)$, this will also be touched on later.

3 Continued Fractions Of The Second and Third Kind Built From Primes

We have seen that we can build fraction of the first kind ($b_n = 1$) from the prime numbers and find there value, but we can also do this for fractions of the second($a_n = 1$) and third($a_n, b_n \neq 1$) kinds. We use somewhat of the same method to calculate these numbers, except that any recursive formula we can derive from fractions of the second and third kind will be of a fundamentally different structure but it will still result in a quadratic limit that we need to find. We can also find error bounds for continued fractions of the second and third kind but they require different restrictions, namely we need to specify the growth rate($\mathcal{O}(f(n))$) of a_n and b_n.

Lemma 3.1. *If Λ_2 is a type two fraction and Λ_3 is a type three fraction, and $\Lambda_2 = 1 + \mathrm{K}_{n=0}^{\infty} \frac{b_n}{1}$ and $\Lambda_3 = \alpha_0 + \mathrm{K}_{n=0}^{\infty} \frac{\beta_n}{\alpha_n}$ then the following inequalities must hold for any Λ_2, Λ_3:*

$$\textit{If } \mathcal{O}(\beta n) > \mathcal{O}(\alpha_n) > \mathcal{O}(b_n):$$

$$|\Lambda_2| > |\Lambda_3| \tag{1}$$

$$\textit{If } \mathcal{O}(b_n) < \mathcal{O}(\alpha_n) < \mathcal{O}(\beta_n):$$

$$|\Lambda_3| > |\Lambda_2| \tag{2}$$

Notice the use of absolute value bars, this is because a fraction continued fraction can be complex, so we use the complex modulus.

3.1 The Values Of F_2, F_2' and F_2''

We can calculate the values of F_2, F_2' and F_2'' by doing the same method we used previously, but the structure of the recursive formula generated by $F_2 F_2'$ and F_2'' will not be fundamentally of the same form as above, namely we our now searching for a recursive formula that generates the b_n terms as opposed to the a_n terms. We define F_2, F_2' and F_2'' to be:

$$F_2 = 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{p(n)}{1}$$

$$F_2' = 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{p(n)}{(-1)^n}$$

$$F_2'' = 1 - \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{p(n)}{(-1)^{n+1}}$$

We cannot rewrite F_2 as $\frac{p(n)}{1} = p(n)$ because of the nature of a continued fraction, also notice that we only apply the period shift on the a_n terms so we do not need to involve our $p_1(n)$ or $p_2(n)$ functions. This allows makes us be able to substitute the same for every limit and have minimal complications for each resulting limit. We can also guess that $F_2 > F_1$ because of the growth rate of the prime functions which when in the numerator(b_n) of the partial fractions surpass the value of the denominator(a_n), but they grow at a slow enough rate as to not become smaller overall when the partial fractions are evaluated. We can say the $F_2 > F_1$ but we can't say for what value of F_1 or that the inequality holds, thus we must calculate F_2, but then when we calculate F_2 we can use Lemma 2.2 to create a good error bound for F_2'. We can assume based on our other continued fractions that we will have to find a quadratic limit, but we can not say for certain what form it will be in, but we can guess that it will be of the final form:

$$F_2(n) = P(F_2(n), p(n)) \tag{3}$$

Where $P(x, y)$ is of the form $ax^n y^m + bx^{n-1} y^{m-1} + \cdots + cxy + d$

We can then assume that the limit will be of the form:

$$\frac{Q(p(n)) + \sqrt{M(p(n))}}{L(p(n))} \tag{4}$$

Where $Q(x)$ and $M(x)$ are of the form $ax^n + bx^{n-1} + ... + cx + d$ and $L(x) = ax + b$

Solving for the recursive formula gives us:

$$F_2(n) = 1 + \tfrac{p(n)}{F_2(n-1)}$$
With $F_2(1) = 3$

Solving for $F_2(n-1)$,

$$F_2(n-1) = \tfrac{p(n)}{F_2(n)-1}$$

Then substituting results in,

$$F_2(n) = 1 + \tfrac{p(n)F_2(n)-p(n)}{p(n)}$$

This formula does follow from (3) but we don't need to take the limit because this simplifies to 1, thus $\boxed{F_2 = 1}$. We can then say $\mathop{\Large\mathrm{K}}_{n=1}^{\infty} \frac{p(n)}{1} = 0$ which means that that the growth rate $\mathcal{O}(Q(F_2)) \lll k$ where k is a constant, and $Q(F_2)$ is the partial fraction part minus a_0. Thus we may assume that $F_2' \leqq F_2$. We define,

$$F_2' = 1 + \mathop{\Large\mathrm{K}}_{n=1}^{\infty} \frac{p(n)}{(-1)^n}$$

We can also guess that it will be of somewhat of the same form, although we can almost guarantee that $\mathfrak{Re}(F_2') < 1$ which would follow from Lemma 2.2. Note were using the $\mathfrak{Re}(z)$ because we cannot be certain that F_2' just by looking at the structure, but we can concretely say that $\mathfrak{Re}(F_2')$ is less than $\mathfrak{Re}(F_2)$. We can solve for the recursive structure for $F_2'(n)$:

$$F_2'(n) = \tfrac{(-1)^n p(n-1) + p(n)p(n-2)}{p(n)(-1)^n - 1}$$

We notice that this formula is not recursive, i.e. it gives us a closed form for $F_2'(n)$, although this is of the form (4) which we expected, the only difference is the square root but we can just square $p(n)p(n-2)$ and then square root it giving us the expected form. Thus we end up at the limit:

$$F_2' = \lim_{n\to\infty} \frac{(-1)^n p(n-1) + p(n)p(n-2)}{p(n)(-1)^n - 1}$$

We need to again make a substitution that this time eliminates the fact that the limit of $(-1)^n$ is undetermined as well as the $p(n)$ problem. Explaining the problem gives us:

$$\frac{(-1)^n\left(1+\sum_{j=1}^{2^{n-1}}\left\lfloor\left(\frac{n-1}{j}\right)^{\frac{1}{n-1}}\right\rfloor\right)+\left(1+\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor\right)\left(1+\sum_{j=1}^{2^{n-2}}\left\lfloor\left(\frac{n-2}{j}\right)^{\frac{1}{n-2}}\right\rfloor\right)}{\left(1+\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor\right)(-1)^n-1}$$

We can make the first simplification $p(n) = n, p(n-1) = n-1, p(n-2) = n-2$ because as $n \to \infty$ that statement becomes true.

$$\frac{(-1)^n\left(1+\sum_{j=1}^{2^{n-1}}\left\lfloor\left(\frac{n-1}{j}\right)^{\frac{1}{n-1}}\right\rfloor\right)+\left(1+\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor\right)\left(1+\sum_{j=1}^{2^{n-2}}\left\lfloor\left(\frac{n-2}{j}\right)^{\frac{1}{n-2}}\right\rfloor\right)}{\left(1+\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor\right)(-1)^n-1} = \boxed{\frac{(-1)^n(n-1)+n(n-2)}{(-1)^n n-1}}$$

Now we need to find a substitution for $(-1)^n$, we can substitute $n = \frac{1}{x+1}$ thus we get the limit:

$$\boxed{F_2' = \lim_{x\to 0}\frac{(-1)^{\frac{1-x}{x}}(\frac{1}{x+1}-1)+\frac{\frac{1}{x+1}-2}{x+1}}{(-1)^{\frac{1-x}{x}}\frac{1}{x+1}-1} = \frac{1}{2}}$$

This answer does follow from (2) and our assumptions about F_2'. We can now make another assumption about F_2'', namely that $F_2'' \leqq \frac{1}{2}$ we can guess that $0 < F_2'' < \frac{1}{2}$ although we cannot guarantee that this is true. We can find a formula for $F_2''(n)$,

$$F_2''(n) = 1 - \frac{p(0)}{-1+\frac{p(1)(p(n)-1)}{p(n)-1+(-1)^n 1\, p(2)}}$$

This formula does require us to find $p(0)$ which would not normally be possible, but the Willan's prime formula does allow for this, although it must be thought of as a limit because of the $1/n$ term giving us,

$$p(0) = 1 + \lim_{n\to 0}\sum_{j=1}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor = 2$$

Note: we have $p(0), p(1) = 2$ this is a direct result of the Willan's prime formula, when evaluated at integers in succession gives us the sequence: $2, 2, 3, 5, 7, 11, 13, 17, 23, \ldots$ Plugging that into our formula for $F_2''(n)$ and simplifying gives us:

$$F_2''(n) = \frac{3(-1)^{n+1}-p(n)+1}{p(n)-3(-1)^{n+1}}$$

We again need to find a simplification that eliminates the $(-1)^{n+1}$ and $p(n)$ problems. Expanding gives us:

$$\frac{2+3(-1)^{n+1}-\sum_{j=0}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor}{1+\sum_{j=0}^{2^n}\left\lfloor\left(\frac{n}{j}\right)^{1/n}\right\rfloor-3(-1)^{n+1}}$$

If we again substitute $x = 1/n$ we get:

$$\boxed{F_2'' = -\lim_{x\to 0} \frac{1+\sum_{j=0}^{\infty}\left\lfloor \left(\frac{1}{xj}\right)^{x}\right\rfloor}{4+\sum_{j=0}^{\infty}\left\lfloor \left(\frac{1}{xj}\right)^{x}\right\rfloor} = 1 - \cfrac{2}{1+\cfrac{3}{1-\cfrac{5}{\ddots}}} = -1}$$

This answer follows from our guess, $F_2'' < \frac{1}{2}$. From what we have seen with our type two fractions, we can make the conjecture:

Conjecture 1. If Λ an continued fraction of type two, and Λ' and Λ'' are it's alternating counterparts then:

$$\Lambda' = \frac{\Lambda + 3\Lambda''}{4} \tag{5}$$

Note: For the rest of this paper we will be assuming this conjecture to be valid.
Are value of F_2'' does follow from (5).

3.2 The Values Of F_3, F_3' and F_3''

We now arrive at the K-Sums that form F_3, F_3' and F_3'':

$$F_3 = 2 + \cfrac{3}{5+\cfrac{7}{11+\cfrac{13}{\ddots}}} = 2 + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{p(n)}{p(n+1)}$$

$$F_3' = 2 + \cfrac{3}{5-\cfrac{7}{11+\cfrac{13}{\ddots}}} = 2 + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{p(n)}{(-1)^{n+1}p(n+1)}$$

$$F_3'' = 2 - \cfrac{3}{5+\cfrac{7}{11-\cfrac{13}{\ddots}}} = 2 - \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{p(n)}{(-1)^{n}p(n+1)}$$

We can start to solve for as recursive formula for F_3:

$$F_3(k) = 2 + \frac{1}{\frac{1}{F_3(k-1)-2} + \frac{p(k-1)}{p(k-2)}\left(\frac{1}{p(k)} - \frac{1}{p(k-1)}\right)}$$

Solving for $F_3(k-1)$ an substituting:

$$F_3(k-1) = 2 + \frac{1}{\frac{1}{F_3(k)-2} - \frac{p(k-1)}{p(k-2)}\left(\frac{1}{p(k)} - \frac{1}{p(k-1)}\right)}$$

$$F_3(k) = 2 + \frac{1}{\frac{1}{\left(2+\frac{1}{\frac{1}{F_3(k)-2} - \frac{p(k-1)}{p(k-2)}\left(\frac{1}{p(k)} - \frac{1}{p(k-1)}\right)}\right)-2} + \frac{p(k-1)}{p(k-2)}\left(\frac{1}{p(k)} - \frac{1}{p(k-1)}\right)} =$$

$$2 + \frac{p(k-2)p(k)(F_3(k)-2)}{p(k-2)p(k)-(p(k-1)-p(k))(F_3(k)-2)+(p(k-1)-p(k))(F_3(k)-2)}$$

Lastly solving for $F_3(k)$:

$$F_3(k) = \tfrac{p(k)^2}{p(k)+1} \Longrightarrow F_3 = \lim_{k\to\infty} \frac{p(k)^2}{p(k)+1}$$

We again make the substitution, $1/n = x$:

$$p(n) = 1 + \sum_{j=1}^{2^n} \left\lfloor \left(\frac{n}{j}\right)^{1/n} \right\rfloor = 1 + \sum_{j=1}^{\infty} \left\lfloor \left(\frac{1}{xj}\right)^{1/x} \right\rfloor$$

Note: I have the second sum going to infinity because as $n \to \infty$ $2^n \to \infty$. Finally we get:

$$\boxed{F_3 = \lim_{x\to 0} \frac{\left(1 + \sum_{j=1}^{\infty} \left\lfloor \left(\frac{1}{xj}\right)^{1/x} \right\rfloor\right)^2}{2 + \sum_{j=1}^{\infty} \left\lfloor \left(\frac{1}{xj}\right)^{1/x} \right\rfloor} = \infty}$$

This result does hold with the Śleszyński–Pringsheim theorem, thus we get F_3 to be divergent.

4 The Significance of F_1'

4.1 Approximations of ϕ using F_1'

4.1.1 The New Irrationality of ϕ

4.1.2 New Approximations For The Metallic Ratios

5 Continued Fractions Truncated to Form Functions

5.1 Properties of These Functions

5.2 Discontinuities In The Graphs of $\Lambda(n)$

5.3 Calculus On These Functions

5.3.1 Approximation Methods For Integrals and Derivatives of These Functions

5.4 Uses For These Functions

6 Other Fractions Generated By Special Primes

6.1 Fractions Built From The Twin Primes

6.1.1 Fraction Built From n-tuples of Primes

6.2 Fractions Built From The Gaussian Primes

6.3 Fractions Built From The Mersenne Primes

6.4 Fractions Built From The Pythagorean Primes

7 Tables Of Values