## Problem 1

In [ ]:

```
import numpy as np
from numpy import log2

def Trapezoid_data(x_data, y_data, panels) :
    n = len(x_data) - 1

    if panels > n :
        print(f"Panel must be less then [{n}] : [{panels}]")
        return

    Area = 0
    for i in range(len(x_data) - 1) :
        width = x_data[i + 1] - x_data[i]
        height = y_data[i] + y_data[i + 1]

        area = width * height / 2
        Area += area

    return Area

def Trapezoid_func(func, a, b, panels) :
    x_data = np.linspace(a, b, panels + 1)
    y_data = func(x_data)

    return Trapezoid_data(x_data=x_data, y_data=y_data, panels=panels)

def Recursive_Trapezoid_func(func, a, b, k, I_old=None) :

    H = b - a
    if k == 1 :
        return (func(a) + func(b)) * H / 2

    n = 2**(k - 2)
    h = (b - a) / n
    x = a + h / 2

    if I_old == None :
        I_old = Trapezoid_func(func=func, a=a, b=b, panels=n)

    sum = 0
    for i in range(n) :
        sum += func(x)
        x += h

    return (I_old + h * sum) / 2

def Recursive_Trapezoid_data(x_data, y_data, I_old=None) :
    H = x_data[-1] - x_data[0]
    n = len(x_data) - 1

    if n != 2**int(log2(n)) :
        print("x_data must have lenght of [2**i + 1] : [{}]".format(len(x_data)))
        return
    elif n == 1 :
        return (y_data[0] + y_data[-1]) * H / 2

    if I_old == None :
        I_old = Trapezoid_data(x_data=x_data, y_data=y_data, panels=2**int(log2(n) - 1))

    sum = 0
```

```python
    for i in range(1, n, 2) :
        sum += H / n * y_data[i]

    return I_old / 2 + sum
```

In [ ]:
```python
def Romberg_2D_func(func, a, b, n) :
    R_matrix = np.zeros(shape=(n,n), dtype=float)
    R_matrix[0,0] = Recursive_Trapezoid_func(func, a, b, k=1)
    for i in range(1, n) :
        R_matrix[i, 0] = Recursive_Trapezoid_func(
            func, a, b, k=i+1, I_old=R_matrix[i - 1, 0]
        )

    for j in range(1, n) :
        for i in range(j, n) :
            R_matrix[i,j] = (
                4**j * R_matrix[i, j - 1] - R_matrix[i - 1, j - 1]
            ) / (4**j - 1)

    return R_matrix
```

In [ ]:
```python
from numpy import log, tan, pi

def f(x) :
    return log(1 + tan(x))

R = Romberg_2D_func(f, 0, pi / 4, n=4)

print(R, end="\n\n")
print("Most accurate : [{:.10f}]".format(R[-1,-1]))
```

```
[[0.27219826 0.         0.         0.        ]
 [0.27219826 0.27219826 0.         0.        ]
 [0.27219826 0.27219826 0.27219826 0.        ]
 [0.27219826 0.27219826 0.27219826 0.27219826]]

Most accurate : [0.2721982613]
```

## Problem 2

In [ ]:
```python
def Simpson_13(x_data, y_data) :

    n = len(x_data)

    if n == 0 :
        h = (x_data[2] - x_data[0]) / 2
        return (y_data[0] + 4 * y_data[1] + y_data[2]) * h / 3

    sum = 0
    for i in range(0, n - 2, 2) :
        h = (x_data[i + 2] - x_data[i]) / 2
        sum += (y_data[i] + 4 * y_data[i + 1] + y_data[i + 2]) * h / 3

    return sum

def Simpson_38(x_data, y_data) :
    h = np.average([x_data[i + 1] - x_data[i] for i in range(len(x_data) - 1)])

    return (y_data[0] + 3 * y_data[1] + 3 * y_data[2] + y_data[3]) * 3 * h / 8

def Simpson(x_data, y_data) :
    n = len(x_data) - 1
```

```python
    if n % 2 != 0 :
        return Simpson_38(x_data[:4], y_data[:4]) + Simpson_13(x_data[3:], y_data[3:])
    else :
        return Simpson_13(x_data, y_data)
```

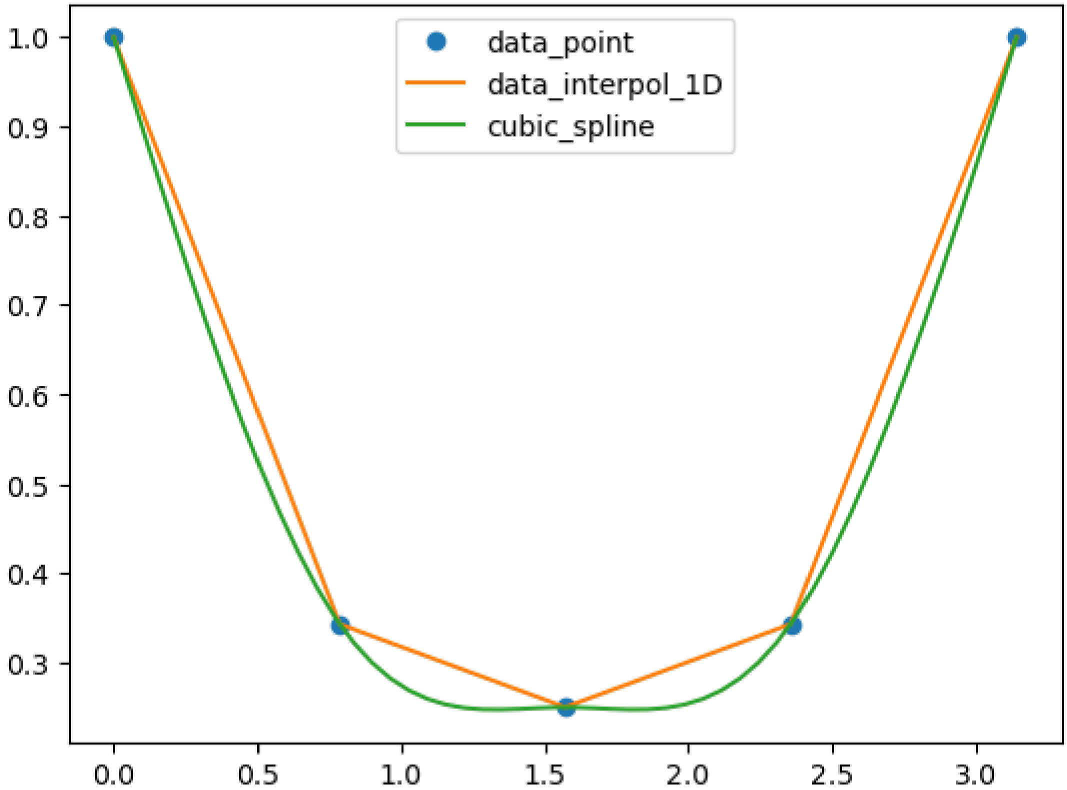
In [ ]:
```python
from numpy import log2

def Romberg_2D_data(x_data, y_data) :
    if len(x_data) - 1 != 2**int(log2(len(x_data) - 1)) :
        print("x_data must have lenght of [2**i + 1] : [{}]".format(len(x_data)))
        return

    n = int(log2(len(x_data) - 1)) + 1

    R_matrix = np.zeros(shape=(n, n), dtype=float)

    H = x_data[-1] - x_data[0]

    for i in range(n) :
        sum = 0
        for j in range(0, len(x_data), 2**(n - i - 1)) :
            sum += 2 * y_data[j]
        sum -= y_data[0] + y_data[-1]

        R_matrix[i,0] = sum * H / (2**(i + 1))

    for j in range(1, n) :
        for i in range(j, n) :
            R_matrix[i,j] = (
                4**j * R_matrix[i, j - 1] - R_matrix[i - 1, j - 1]
            ) / (4**j - 1)

    return R_matrix
```

In [ ]:
```python
x_data = np.array([0, pi / 4, pi / 2, 3 * pi / 4, pi], dtype=float)
y_data = np.array([1, 0.3431, 0.25, 0.3431, 1], dtype=float)

trapz = Trapezoid_data(x_data, y_data, panels=4)
simp = Simpson(x_data, y_data)
romberg = Romberg_2D_data(x_data, y_data)[-1,-1]
print(
    "By Trapezoidal rule (4 panels): ".ljust(35),
    "[{:.8f}]".format(trapz).ljust(10)
)
print(
    "By Simpson's rule : ".ljust(35),
    "[{:.8f}]".format(simp).ljust(10)
)

print(
    "By Romberg integration : ".ljust(35),
    "[{:.8f}]".format(romberg).ljust(10),
    end="\n\n"
)

print("We CAN NOT evalute it's accuracy since we have few data.")
```

```
By Trapezoidal rule (4 panels):     [1.52068792]
By Simpson's rule :                 [1.37308543]
By Romberg integration :            [1.35990470]

We CAN NOT evalute it's accuracy since we have few data.
```

In [ ]:
```
print("For example, let's consider the cubic spline with data", end="\n\n")

import matplotlib.pyplot as plt
from scipy.interpolate import CubicSpline

CS = CubicSpline(x_data, y_data, bc_type='natural')
plt.plot(x_data, y_data, 'o', label="data_point")
plt.plot(x_data, y_data, label="data_interpol_1D")
plt.plot(
    np.linspace(min(x_data), max(x_data)),
    CS(np.linspace(min(x_data), max(x_data))), label="cubic_spline"
)
plt.legend()
plt.show()

print("You can see there's huge gap between cubic splines and 1 order interpolation.")
print("Since we don't know the true function of data,")
print("it's far more dangerous to say which method has highest accuracy.")
```

```
For example, let's consider the cubic spline with data
```

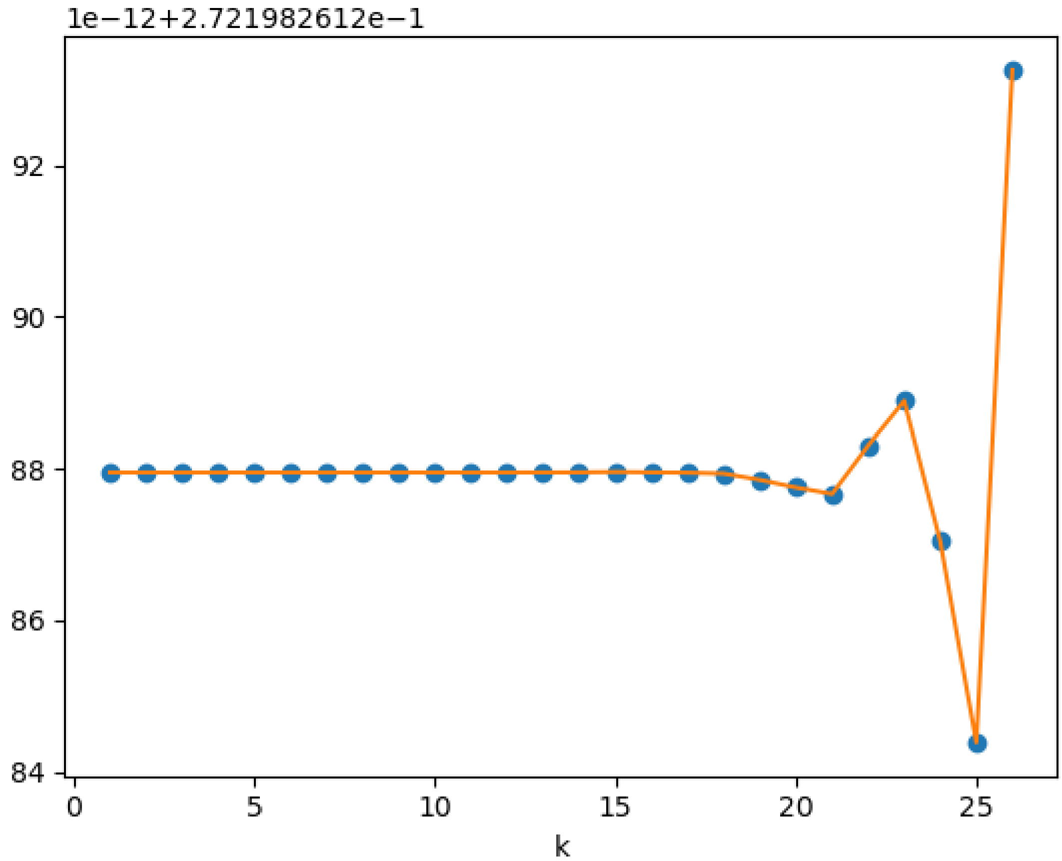



```
You can see there's huge gap between cubic splines and 1 order interpolation.
Since we don't know the true function of data,
it's far more dangerous to say which method has highest accuracy.
```

## Problem 3

In [ ]:
```
import numpy as np
import matplotlib.pylab as plt
from numpy import pi, tan, log

def f(x) :
    return log(1 + tan(x))

integ_list = [Recursive_Trapezoid_func(f, 0, pi/4, k=1)]
for i in range(25) :
    integ_list.append(Recursive_Trapezoid_func(f, 0, pi/4, k=i+1, I_old=integ_list[-1]))
```

```python
plt.plot([i + 1 for i in range(len(integ_list))], integ_list, 'o')
plt.plot([i + 1 for i in range(len(integ_list))], integ_list)
plt.xlabel("k")
plt.show()

print("It seems like there's no improvemence adding more panels.")
print("But after k=17, rapid change has been occurred.")
```

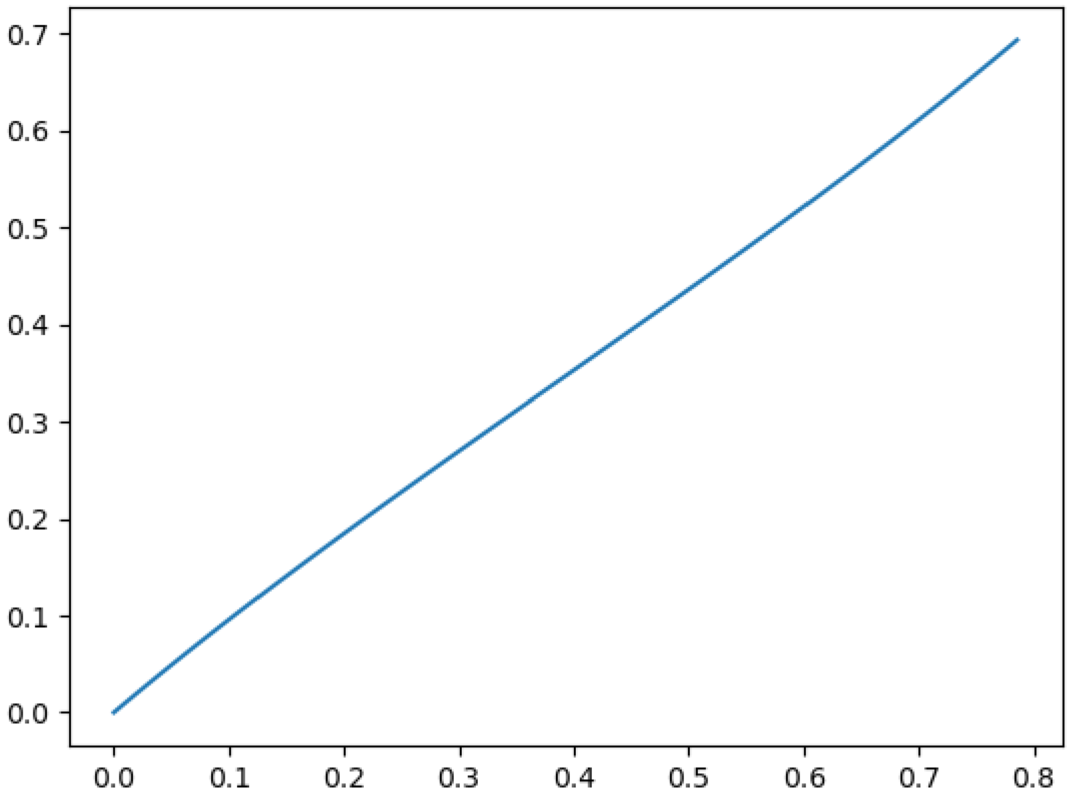



```
It seems like there's no improvemence adding more panels.
But after k=17, rapid change has been occurred.
```

In [ ]:

```python
print("The reason of rapid change is due to the function it self")

x_range = np.linspace(0, pi / 4, 100)
y = f(x_range)

plt.plot(x_range, y)
plt.show()

print("When you look closer to the function,")
print("it looks perfect line but there's small curvature.")
print("Since function is almost linear, adding few panels will affect almost nothing.")
print("However if we add enough panels to recognize function's curvature,")
print("it will affect immediately.")
print("That's why the rapid chnage has been occurred.")
```

```
The reason of rapid change is due to the function it self
```

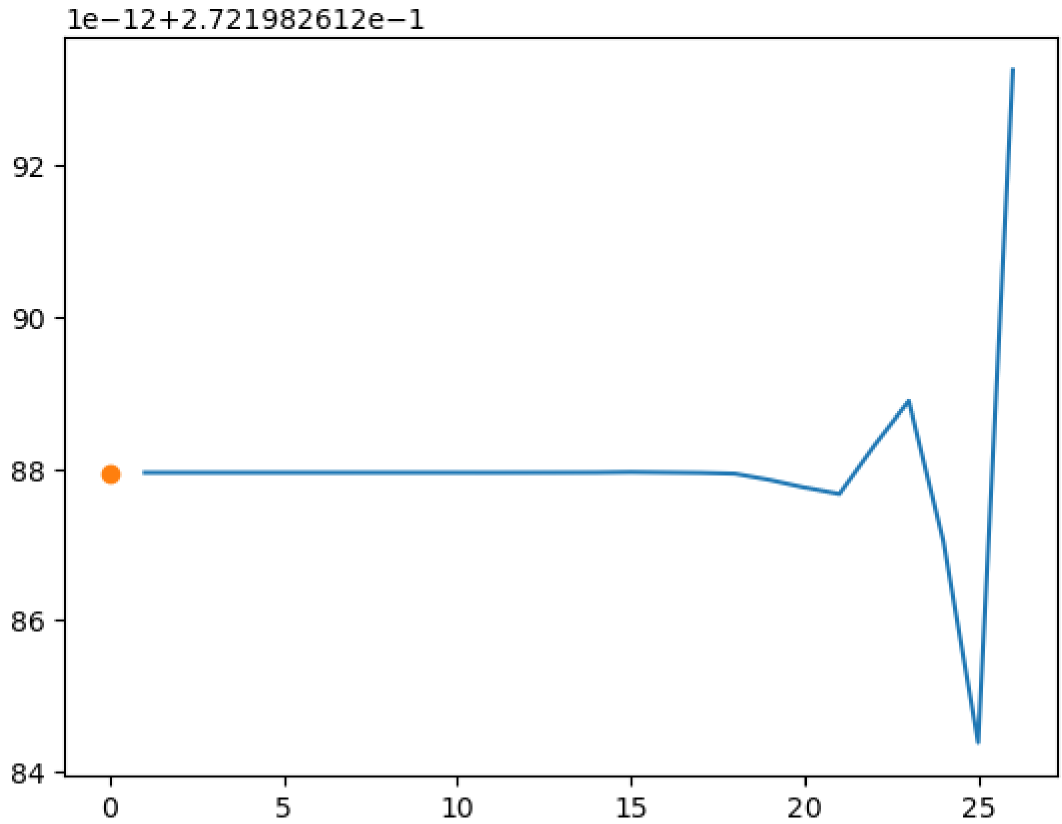


```
When you look closer to the function,
it looks perfect line but there's small curvature.
Since function is almost linear, adding few panels will affect almost nothing.
However if we add enough panels to recognize function's curvature,
it will affect immediately.
That's why the rapid chnage has been occurred.
```

In [ ]:

```python
from scipy.integrate import quad

print("But since we only need the integration with tolerance, we can ignore this change.")
print("Comparing with scipy.integrate.quad, ", end="\n\n")
print(
    "Using scipy : ".rjust(25),
    "[{}]".format(quad(f, 0, pi / 4)[0]).ljust(25)
)
print(
    "Using trapezoidal : ".rjust(25),
    "[{}]".format(integ_list[-1]).ljust(25),
    end="\n\n"
)

print("You shall notice the error is negligible.")

plt.plot([i + 1 for i in range(len(integ_list))], integ_list)
plt.plot([0], quad(f, 0, pi/4)[0], 'o')
plt.show()
```

```
But since we only need the integration with tolerance, we can ignore this change.
Comparing with scipy.integrate.quad,

           Using scipy :  [0.27219826128795027]
     Using trapezoidal :  [0.27219826129325536]

You shall notice the error is negligible.
```




## Problem 4

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n} A_i f(x_i)$$

$$x_i = \frac{b+a}{2} + \frac{b-a}{2}\xi_i$$

| $\pm\xi_i$ | $A_i$ | $\pm\xi_i$ | $A_i$ |
|---|---|---|---|
| $n=1$ | | $n=4$ | |
| 0.577 350 | 1.000 000 | 0.000 000 | 0.568 889 |
| $n=2$ | | 0.538 469 | 0.478 629 |
| 0.000 000 | 0.888 889 | 0.906 180 | 0.236 927 |
| 0.774 597 | 0.555 556 | $n=5$ | |
| $n=3$ | | 0.238 619 | 0.467 914 |
| 0.339 981 | 0.652 145 | 0.661 209 | 0.360 762 |
| 0.861 136 | 0.347 855 | 0.932 470 | 0.171 324 |

**Table 6.3.** Nodes and weights for Gauss–Legendre quadrature.

$$f(x) = \frac{\ln(x)}{x^2 - 2x + 2}$$

$$\int_a^b \frac{\ln(x)}{x^2 - 2x + 2}dx$$

In [ ]:

```
import math
import numpy as np
def gaussNodes(m,tol=10e-9):

    def legendre(t,m):
        p0 = 1.0; p1 = t
        for k in range(1,m):
            p = ((2.0*k + 1.0)*t*p1 - k*p0)/(1.0 + k )
            p0 = p1; p1 = p
        dp = m*(p0 - t*p1)/(1.0 - t**2)
        return p,dp
    A = np.zeros(m)
    x = np.zeros(m)
    nRoots = int((m + 1)/2)
    for i in range(nRoots):
        t = math.cos(math.pi*(i + 0.75)/(m + 0.5))
        for j in range(30):
            p,dp = legendre(t,m)
            dt = -p/dp; t = t + dt
            if abs(dt) < tol:
                x[i] = t; x[m-i-1] = -t
                A[i] = 2.0/(1.0 - t**2)/(dp**2)
                A[m-i-1] = A[i]
                break
    return x,A

def gaussQuad(f,a,b,m):
    c1 = (b + a)/2.0
    c2 = (b - a)/2.0
    x,A = gaussNodes(m)
    sum = 0.0
    for i in range(len(x)):
        sum = sum + A[i]*f(c1 + c2*x[i])
    return c2*sum
```

In [ ]:

```
from numpy import log, pi

def f(x) :
    return log(x) / (x**2 - 2 * x + 2)

node_2 = gaussQuad(f, 1, pi, 2)
node_4 = gaussQuad(f, 1, pi, 4)

print("2 node : [{}]".format(node_2))
print("4 node : [{}]".format(node_4))
```

```
2 node : [0.6067250072484867]
4 node : [0.5847680362120717]
```

## Problem 5

In [ ]:

```
from numpy import pi, sqrt, exp

def f(x) :
    return exp(-1 * x**2)

k = 1
value = sqrt(pi)
pred = Recursive_Trapezoid_func(f, -10, 10, k=k)
tol = 1.0e-10
while abs(value - pred) >= tol :
    k += 1
```

```
    pred = Recursive_Trapezoid_func(f, -10, 10, k=k, I_old=pred)

    if k >= 20 :
        print("failed to converge")
        break

print("k : [{}]".format(k))
print("sqrt(pi) : ".ljust(20), "[{}]".format(value).ljust(20))
print("Trapezoid : ".ljust(20), "[{}]".format(pred).ljust(20))
print("Error : ".ljust(20), "[{}]".format(value - pred).ljust(20))
```

```
k : [6]
sqrt(pi) :           [1.7724538509055159]
Trapezoid :          [1.772453850943242]
Error :              [-3.7726044510577594e-11]
```

## Problem 6

In [ ]:

```
x_data = np.arange(0, 0.81, 0.16)
y_data = np.array([0.2, 1.2969, 1.7434, 3.1860, 3.1819, 0.232])

print("x_data : ", x_data)

simp = Simpson(x_data=x_data, y_data=y_data)

print("Simpson : [{}]".format(simp))
```

```
x_data :  [0.   0.16 0.32 0.48 0.64 0.8 ]
Simpson : [1.611512666666667]
```

## Problem 7

In [ ]:

```
import numpy as np

def romberg(f,a,b,tol=1.0e-6):
    def richardson(r,k):
        for j in range(k-1,0,-1):
            const = 4.0**(k-j)
            r[j] = (const*r[j+1] - r[j])/(const - 1.0)
        return r

    r = np.zeros(21)
    r[1] = Recursive_Trapezoid_func(f, a, b, 1)
    r_old = r[1]
    for k in range(2,21):
        r[k] = Recursive_Trapezoid_func(f, a, b, k=k, I_old=r[k-1])
        r = richardson(r,k)
        if abs(r[1]-r_old) < tol*max(abs(r[1]),1.0):
            return r[1],2**(k-1)
        r_old = r[1]
    print("Romberg quadrature did not converge")
```

In [ ]:

```
# using romberg integration

G = 6.673e-11    # J * m / kg^2
M = 5.972e+24    # kg
R = 6371e+3      # km --> m
m = 200          # kg

def U(x) :
    return G * M * m / x**2      # J / m
```

```python
H = [h * 1.0e+3 for h in [550, 1150, 340]]      # km --> m
Num = [1600, 2800, 7500]
Energy = 0

for height, num in zip(H, Num) :
    integ = romberg(U, R, R + height, tol=1.0e-9)[0]
    Energy += integ * num

print("Required total work :", "[{:.5e}] kJ".format(Energy * 1.0e-3))
```

```
Required total work : [1.17002e+10] kJ
```

In [ ]:

```python

```