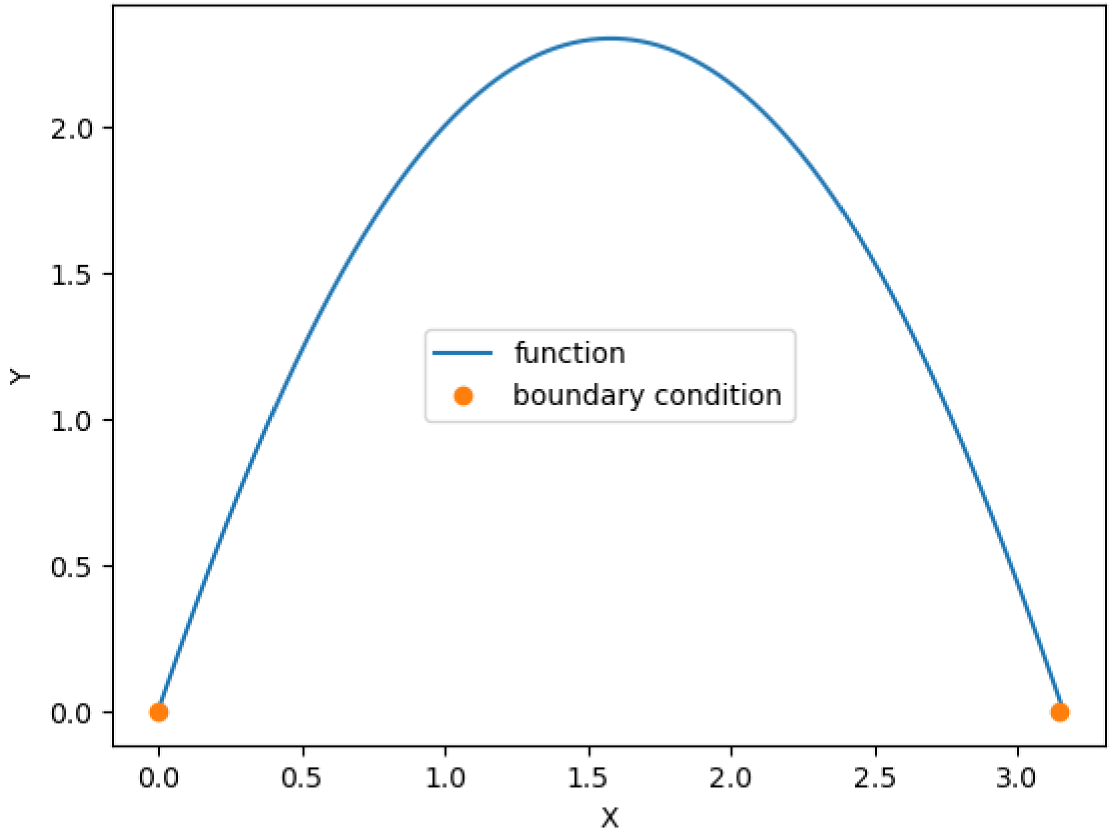
Problem 1 :

$$y'' + \sin y + 1 = 0, \;\; y(0) = 0, \; y(\pi) = 0$$

In [ ]:

```
# shooting method

from numpy import pi, sin
import matplotlib.pyplot as plt

def bisect(y, x_start, x_end, func, max_iter=200, tol=1.0e-9) :

    y_start = func(x_start)
    y_end = func(x_end)
    x_root = None

    for _ in range(max_iter) :
        x_mid = (x_start + x_end) / 2
        y_mid = func(x_mid)

        if abs(y - y_mid) <= tol :
            x_root = x_mid
            break

        if y_mid * y_start >= 0 :
            x_start = x_mid
            y_start = y_mid

        else :
            x_end = x_mid
            y_end = y_mid

    if x_root == None :
        raise ValueError("Failed to converge.")

    return x_root

def shooting(x_start, x_end, h, y_dif_init, bc, func_calc, func_dif, tol=1.0e-9) :
    y0 = bc[0]
    y_end = bc[1]
    def wrapper(y1_input) :
        return func_calc(x_start, x_end, y0, y1_input, h, func_dif)

    y_dif = bisect(y_end, y_dif_init[0], y_dif_init[1], func=wrapper, tol=tol)

    return y_dif

def calc_yb(x_start, x_end, y0, y1, h, func) :
    while x_start <= x_end + h :
        y2 = func(x_start, y0)
        y1 += y2 * h
        y0 += y1 * h
        x_start += h
    return y0

def range_func(x_start, x_end, y0, y1, h, func) :
    x_range = []
    y_range = []
    while x_start <= x_end + h :
        x_range.append(x_start)
        y_range.append(y0)
        y2 = func(x_start, y0)
```

```
        y1 += y2 * h
        y0 += y1 * h
        x_start += h

    return x_range, y_range

def dif(x, y) :
    return -1 * (sin(y) + 1)

h = 0.01
y_dif = shooting(0, pi, h, [2.5, 3], [0, 0], func_calc=calc_yb, func_dif=dif, tol=1.0e-12)

x_range, y_range = range_func(0, pi, 0, y_dif, h, dif)

plt.plot(x_range, y_range, label="function")
plt.plot([0, pi], [0, 0], 'o', label="boundary condition")
plt.xlabel("X")
plt.ylabel("Y")
plt.legend()
plt.show()
```



Problem 2 :

$$\frac{d^4v}{dx^4} - \frac{N}{EI}\frac{d^2v}{dx^2} = \frac{w_0}{EI}$$

with boundary condition

$$v_{(x=0)} = v_{(x=L)} = 0, \quad \frac{d^2v}{dx^2}_{(x=0)} = \frac{d^2v}{dx^2}_{(x=L)} = 0$$

problem say eqaution can be shwon as,

$$\frac{d^4y}{d\xi^4} - \beta\frac{d^2y}{d\xi^2} = 1$$

with boundary condition

$$y_{(\xi=0)} = y_{(\xi=1)} = 0, \quad \left.\frac{d^2y}{d\xi^2}\right|_{(\xi=0)} = \left.\frac{d^2y}{d\xi^2}\right|_{(\xi=1)} = 0$$

as

$$\xi = \frac{x}{L}, \quad y = \frac{EI}{w_0L^4}v, \quad \beta = \frac{NL^2}{EI}$$

Let us use finite-difference method. We can rewrite equations.

$$\frac{d^4y_i}{d\xi_i^4} = \frac{y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}}{h^4}$$

$$\frac{d^2y_i}{d\xi_i^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Therefore, equation can be shown as

$$\frac{y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}}{h^4} - \beta\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = 1$$

$$y_{i-2} - (4 + h^2\beta)y_{i-1} + (6 + 2h^2\beta)y_i - (4 + h^2\beta)y_{i+1} + y_{i+2} = h^4$$

Applying boundary condition $y_{i=0} = y_{i=m} = 0, \; y^{(2)}_{i=0} = y^{(2)}_{i=m} = 0$

$$y_0^{(2)} = \frac{y_{-1} - 2y_0 + y_1}{h^2} = 0 \; \rightarrow \; y_{-1} = y_1$$

$$y_m^{(2)} = \frac{y_{m-1} - 2y_m + y_{m+1}}{h^2} = 0 \; \rightarrow \; y_{m+1} = y_{m-1}$$

$$i = 1 \; \rightarrow \; y_{-1} - (4 + h^2\beta)y_0 + (6 + 2h^2\beta)y_1 - (4 + h^2\beta)y_2 + y_3 = h^4,$$

$$i = 1 \; \rightarrow \; -(4 + h^2\beta)y_0 + (7 + 2h^2\beta)y_1 - (4 + h^2\beta)y_2 + y_3 = h^4$$

$$i = m-1 \; \rightarrow \; y_{m-3} - (4 + h^2\beta)y_{m-2} + (6 + 2h^2\beta)y_{m-1} - (4 + h^2\beta)y_m + y_{m+1} = h^4$$

$$i = m-1 \; \rightarrow \; y_{m-3} - (4 + h^2\beta)y_{m-2} + (7 + 2h^2\beta)y_{m-1} - (4 + h^2\beta)y_m = h^4$$

So the eqaution can be shown as

$$\begin{cases} y_0 = 0, \quad (i = 0) \\ -(4 + h^2\beta)y_0 + (7 + 2h^2\beta)y_1 - (4 + h^2\beta)y_2 + y_3 = h^4, \quad (i = 1) \end{cases}$$

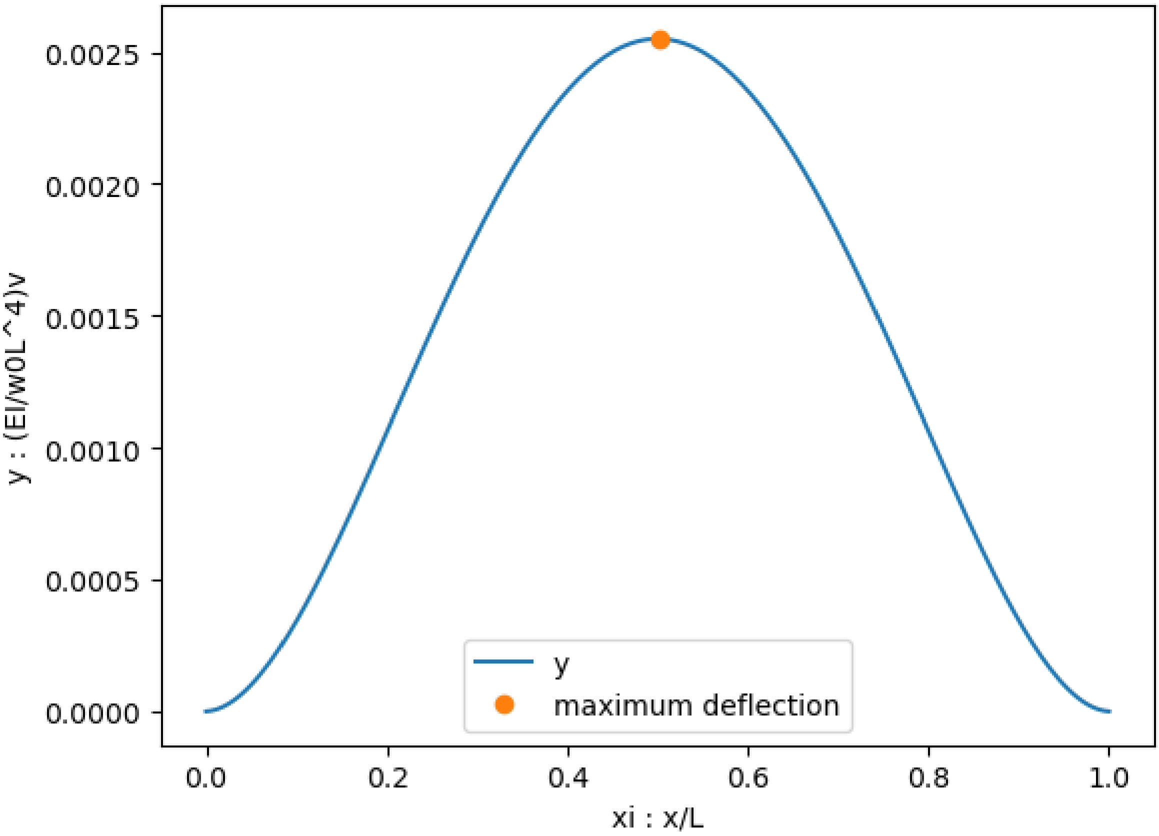
$$y_{i-2} - (4 + h^2\beta)y_{i-1} + (6 + 2h^2\beta)y_i - (4 + h^2\beta)y_{i+1} + y_{i+2} = h^4 \quad (i = 2, \dots m-2)$$

$$\begin{cases} y_{m-3} - (4 + h^2\beta)y_{m-2} + (7 + 2h^2\beta)y_{m-1} - (4 + h^2\beta)y_m = h^4, \quad (i = m-1) \\ y_m = 0, \quad (i = m) \end{cases}$$

In [ ]:
```
import numpy as np

def gen_matrix(mesh_num, h, beta, bc) :

    A = (6 + 2 * h**2 * beta) * np.eye(N=mesh_num, k=0)
    A -= (4 + h**2 * beta) * np.eye(N=mesh_num, k=1) \
        + (4 + h**2 * beta) * np.eye(N=mesh_num, k=-1)
    A += np.eye(N=mesh_num, k=2) + np.eye(N=mesh_num, k=-2)

    B = np.zeros(shape=(mesh_num), dtype=float) + h**4
    A[0,:] = 0
    A[-1,:] = 0
    A[0,0], A[-1,-1] = 1, 1
    B[0], B[-1] = bc[0], bc[1]

    return A, B

def finite_dif_fourth(x_start, x_end, mesh_num, beta, bc) :

    x_range = np.linspace(x_start, x_end, mesh_num)
    h = x_range[1] - x_range[0]

    A, B = gen_matrix(mesh_num, h, beta, bc)

    return x_range, np.linalg.solve(A, B)
```

2 - (a)

In [ ]:
```
import matplotlib.pyplot as plt

mesh_num = 200
beta = 1.6529
bc = [0, 0]
x_range, y_range = finite_dif_fourth(0, 1, mesh_num, beta, bc)

max_index = np.where(y_range == y_range.max())

plt.plot(x_range, y_range, label="y")
plt.plot(x_range[max_index], y_range[max_index], 'o', label="maximum deflection")
plt.xlabel("xi : x/L")
plt.ylabel("y : (EI/w0L^4)v")
plt.legend()
plt.show()

print("Since other variables like EI, N, L were not given,")
print("we can only plot numerically via XI and Y dimension.")
print("Maximum deflection of y : [{:.5f}]".format(*y_range[max_index]))
print("At xi : [{:.5f}]".format(*x_range[max_index]))
```

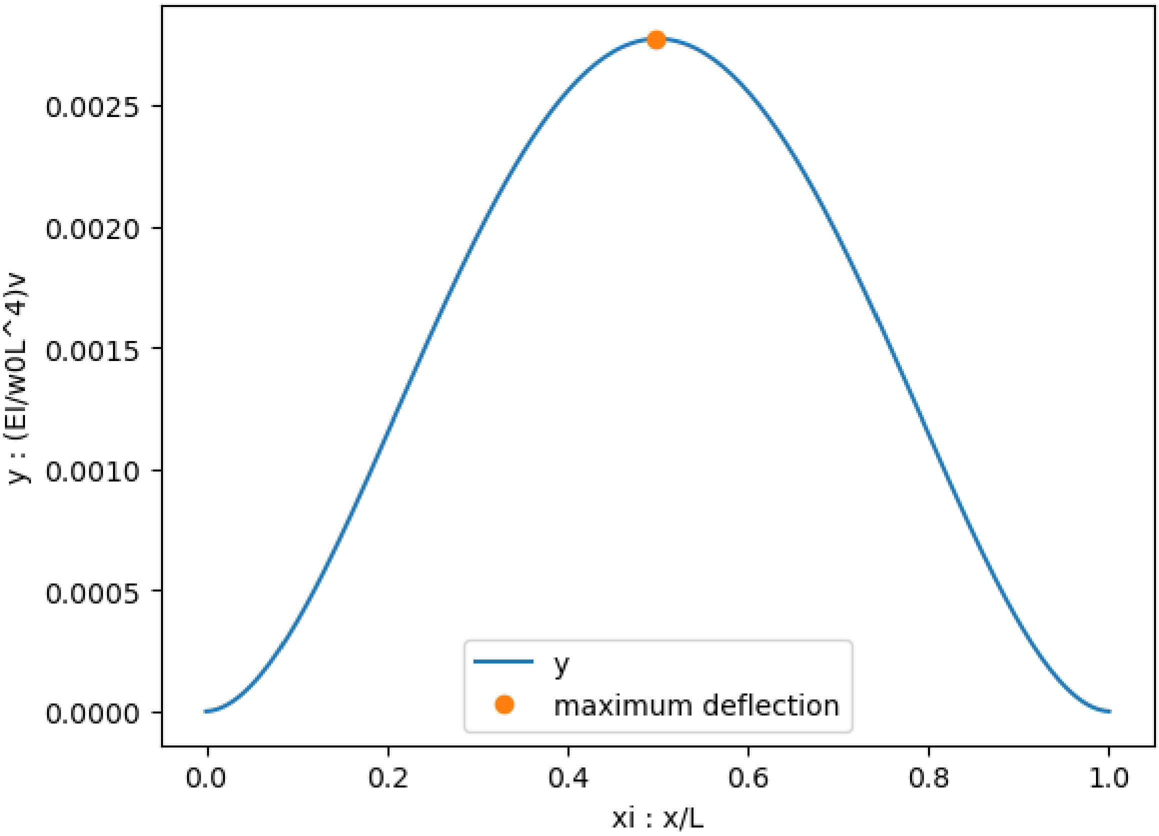



```
Since other variables like EI, N, L were not given,
we can only plot numerically via XI and Y dimension.
Maximum deflection of y : [0.00255]
At xi : [0.50251]
```

2 - (b)

In [ ]:

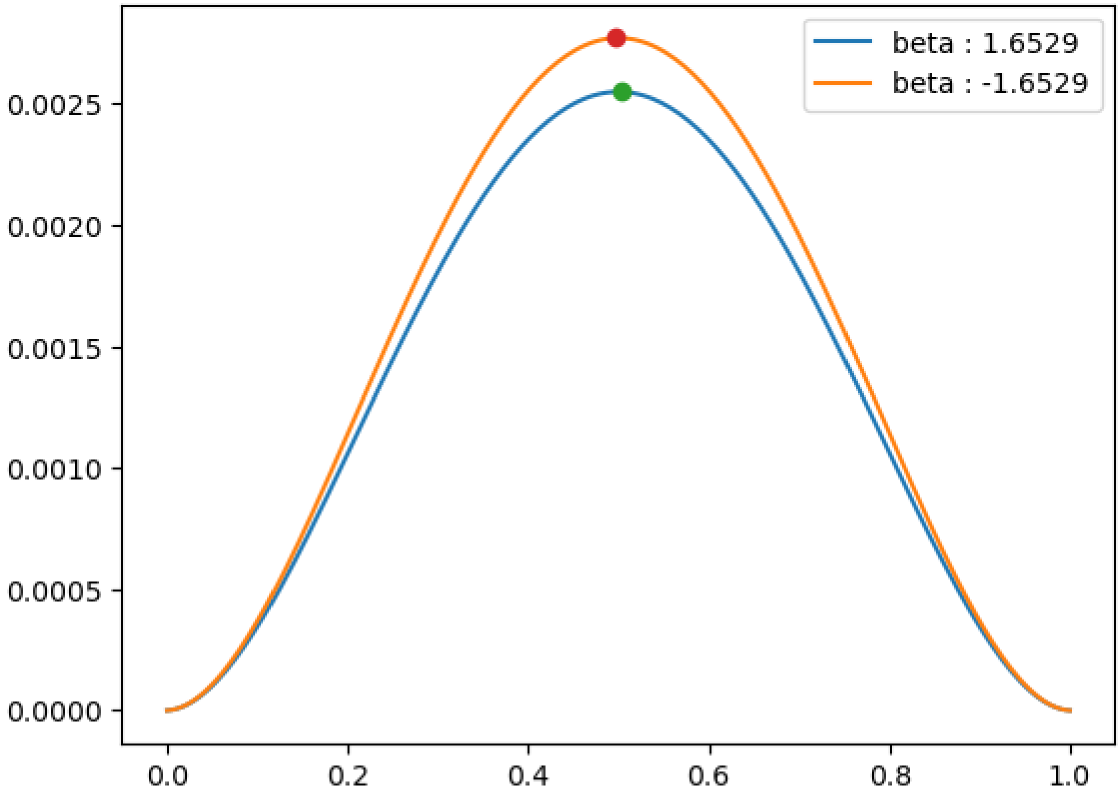
```python
import matplotlib.pyplot as plt

mesh_num = 200
beta = -1.6529
bc = [0, 0]
x_range, y_range = finite_dif_fourth(0, 1, mesh_num, beta, bc)

max_index = np.where(y_range == y_range.max())

plt.plot(x_range, y_range, label="y")
plt.plot(x_range[max_index], y_range[max_index], 'o', label="maximum deflection")
plt.xlabel("xi : x/L")
plt.ylabel("y : (EI/w0L^4)v")
plt.legend()
plt.show()

print("Since other variables like EI, N, L were not given,")
print("we can only plot numerically via XI and Y dimension.")
print("Maximum deflection of y : [{:.5f}]".format(*y_range[max_index]))
print("At xi : [{:.5f}]".format(*x_range[max_index]))
```




```
Since other variables like EI, N, L were not given,
we can only plot numerically via XI and Y dimension.
Maximum deflection of y : [0.00277]
At xi : [0.49749]
```

In [ ]:

```
# comparing both plot

mesh_num = 200
beta = [1.6529, -1.6529]
bc = [0, 0]

x1_range, y1_range = finite_dif_fourth(0, 1, mesh_num, beta[0], bc)
x2_range, y2_range = finite_dif_fourth(0, 1, mesh_num, beta[1], bc)

max_index1 = np.where(y1_range == y1_range.max())
max_index2 = np.where(y2_range == y2_range.max())

plt.plot(x1_range, y1_range, label="beta : {}".format(beta[0]))
plt.plot(x2_range, y2_range, label="beta : {}".format(beta[1]))
plt.plot(x1_range[max_index1], y1_range[max_index1], 'o')
plt.plot(x2_range[max_index2], y2_range[max_index2], 'o')
plt.legend()
plt.show()
```



problem 3 :

$$-\frac{d^2\psi}{d\xi^2} = (\xi^2 - E_n')\psi$$

where

$$\xi = \sqrt{\frac{mw}{\hbar}}x, \quad E_n' = \frac{2E_n}{\hbar w} = 2n+1, \quad n = 0, 1, 2\ldots$$

with boundary condition : $\psi \to 0$ when $x \to \pm\infty$

$$\frac{d^2 y_i}{d\xi_i^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$-\frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{h^2} = (\xi^2 - 2n - 1)\psi_i$$

$$\psi_{i-1} - (2 - h^2(\xi^2 - 2n - 1))\psi_i + \psi_{i+1} = 0$$

In [ ]:
```
a = -10
b = 10
N = 100
x = np.linspace(a,b,N)
h = x[1]-x[0]
T = np.zeros((N-2)**2).reshape(N-2,N-2)
for i in range(N-2):
    for j in range(N-2):
```

```
        if i==j:
            T[i,j]= -2
        elif np.abs(i-j)==1:
            T[i,j]=1
        else:
            T[i,j]=0

V = np.zeros((N-2)**2).reshape(N-2,N-2)
for i in range(N-2):
    for j in range(N-2):
        if i==j:
            V[i,j]= x[i+1]**2
        else:
            V[i,j]=0
H = -T/(2*h**2) + V
val,vec=np.linalg.eig(H)
z = np.argsort(val)
z = z[0:5]
energies=(val[z]/val[z][0])

for i in range(len(z)):
    y = []
    y = np.append(y,vec[:,z[i]])
    y = np.append(y,0)
    y = np.insert(y,0,0)
    plt.plot(x,y,lw=3, label="n = {}".format(i))
    plt.xlabel('x', size=14)
    plt.ylabel('$\psi$(x)',size=14)
plt.legend()
plt.title(
    'normalized wavefunctions for a harmonic oscillator using finite difference method',
    size=14
)
plt.show()
```