1. Lagrange's method and cubic splines exercises

Given the following data

x = [0, 1, 2]
y = [1, 4, 3]

a) Use the Langrange interpolation method to find analytically the second order polynomial that fits the data.
b) Find the analytical expressions of the cubic splines assuming that the curvatures at the endpoints $k_0 = k_2 = 0$ are zero.
c) Find the analytical expressions of the cubic splines assuming the curvature for the left point $k_0 = 0$ to be zero and the middle point $k_1 = -5$.
d) Plot the polynomials obtained analytically in a), b) and c) and by computing the cubic splines using the numerical libraries.

Hint: Use the following formulas for each piecewise interval. The Lagrange's polynomial is given by

$$P_n(x) = \sum_{i=0}^{n} y_i \ell_i(x)$$

where $\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$ with $i = 0, 1, \dots, n$.

The equation of a cubic spline in a given interval between [i, i+1] for $i = 0, 1, \dots, n$ is

$$f_{i,i+1}(x) = \frac{k_i}{6}\left[\frac{(x-x_{i+1})^3}{x_i-x_{i+1}} - (x-x_{i+1})(x_i-x_{i+1})\right] - \frac{k_{i+1}}{6}\left[\frac{(x-x_i)^3}{x_i-x_{i+1}} - (x-x_i)(x_i-x_{i+1})\right] + \frac{y_i(x-x_{i+1})-y_{i+1}(x-x_i)}{x_i-x_{i+1}}$$

and the equations for the curvatures are

$$k_{i-1}\left(x_{i-1}-x_i\right) + 2k_i\left(x_{i-1}-x_{i+1}\right) + k_{i+1}\left(x_i-x_{i+1}\right) = 6\left(\frac{y_{i-1}-y_i}{x_{i-1}-x_i} - \frac{y_i-y_{i+1}}{x_i-x_{i+1}}\right)$$

for $i = 1, 2, \dots, n-1$.

2. Polynomial interpolation and cubic splines.

Program the code to find the n^{th} order polynomial that interpolates the following $n+1$ data points. Then program the code to find the natural cubic splines. Plot both results with 30 interpolating points.

x = [0, 1, 2, 3, 4]
y = [0, 1, 0, 2, 0]

3. Use Newton's method to find the polynomial that fits the following points:

x = [-3 , 2, -1, 3, 1]
y = [0, 5, -4, 12, 0]

Newton's polynomials are defined as

$$P_0(x) = a_n \quad P_k(x) = a_{n-k} + (x - x_{n-k})P_{k-1}(x), \quad k = 1, 2, \cdots, n$$

and the coeffecients are given by:

$$a_0 = y_0 \quad a_1 = \nabla y_1 \quad a_2 = \nabla^2 y_2 \quad \cdots \quad a_n = \nabla y_n$$

$$\nabla y_i = \frac{y_i - y_0}{x_i - x_0} \quad i = 1, 2, \cdots, n$$

$$\nabla^2 y_i = \frac{\nabla y_i - \nabla y_1}{x_i - x_1} \quad i = 2, 3, \cdots, n$$

$$\nabla^3 y_i = \frac{\nabla y_i - \nabla y_2}{x_i - x_2} \quad i = 3, 4, \cdots, n$$

$$\vdots$$

$$\nabla y_n = \frac{\nabla^{n-1} y_n - \nabla^{n-1} y_{n-1}}{x_n - x_{n-1}}$$

x_0	y_0		
x_1	y_1	∇y_1	
x_2	y_2	∇y_2	$\nabla^2 y_2$

a) Find the coefficients of Newton's polynomial
b) Write the code to find the Newton's polynomial function and print the y value when x=0.5.
c) Plot an interpolating polynomial between $x \in [-3, 3]$ with 20 points. Mark with symbols the input data points and make sure that the interpolated function passes through them.