Think Like A Physicist

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2024-01-01

Table of contents

Foreword

First Principles Thinking is a teachable and incredibly powerful algorithm for efficiently solving real world problems and building the future.

It is the best algorithm I know to make a brain useful.

These notes are compiled from the International Physics Olympiad Australian training program held in Canberra c. 2005. I have found, over the intervening 18 years, that the skills and knowledge taught during this intensive two week program have formed the basis of my ability to apply physics to everyday problems.

I have assembled these notes with the intention to create the most singularly terse summary of undergraduate physics ever written, accessible to motivated professionals who know how to Google. Necessarily, they do not cover the lab or test-taking aspects of the Olympiad training program.

I hope you find thinking like a physicist to be as rewarding as I do.

— Casey Handmer, 25 November, 2023.

Part I

Foundations

1 How to Think Like a Physicist

1.1 SI Units

Table 1.1: SI Base Units

Base Quantity	Base Unit Name	Symbol
Length	meter	m
Time	second	s
Mass	kilogram	kg
Electric Current	ampere	A
Thermodynamic Temperature	kelvin	K
Amount of Substance	mole	mol
Luminous Intensity	candela	cd

The five fundamental units are length (meter, m), time (second, s), mass (kilogram, kg), charge (coulomb, C), and temperature (kelvin, K). Ampere (A), mole (mol), and candela (cd) are derived from these base units.

All other units are derived from combinations of these fundamental units.

Table 1.2: Derived Units

Concept	Unit	Derivation
Force	Newton (N)	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$
Energy	Joule (J)	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
Power	Watt (W)	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$
Current	Ampere (A)	$\mathrm{C} \cdot \mathrm{s}^{-1}$

1.2 Dimensional Analysis and Sanity Checking

Vital tool for both checking (always!!) and fudging. It is part of a consistency check — units on both sides of the equation must be the same. A successful dimension check is followed by a :).

Similarly, equations with vectors must have the same vector shape on each side.

Math and formulas are a tool, not the final authority. Never formula fit. Understand what is going on and check that the answers are sensible.

1.3 Solving Problems with the 7 Ds and the little s

Using this basic algorithm to approach problems from first principles is the single biggest update you are likely to make to your thinking, ever.[1]

1. **Diagrams** — Big, 2/3 page, as many as you need. Load the problem into your GPU.
2. **Directions** — Mark it (negative/positive).
3. **Definitions and Given Data** — Put it on the page (all of them).
4. **Diagnosis** — How? What type of problem is this? E.g. conservation principles, force laws, angular momentum.
5. **Derivation** — Write down the fundamental equations, the diagnosis expressed as symbols. You will need as many equations as variables. Add to the diagram if necessary. Check dimensions.
6. **Determination** (or D'algebra) — Math manipulations to get the answer. Box it!
7. **Dimensions** — **Check dimensions and limiting cases/sanity check**. $LHS = RHS$ then :).
8. **substitutions** — Only if necessary, do rough calculation by hand and check units, include an error term.

Example Problem 1

A body, mass m, slides without friction from rest at height H to lower height h, colliding with a horizontal spring of strength k. By how much does it compress?

> ⚠ TODO: Diagram
>
> Diagram showing: (1) Initial state — mass m at height H on a frictionless ramp. (2) Final state — mass at height h compressing a horizontal spring by distance x. Include direction arrows for gravity and coordinate axes.

Definitions & Data

[1] A more thorough exploration of this algorithm can be found at Casey Handmer's blog post In Space, No One Can Reason By Analogy.

Variable	Description
m	Mass of object
H	Initial height
h	Final height
k	Spring constant
g	Acceleration due to gravity ($9.81\,\mathrm{m/s^2}$)
x	Compression of spring
U_{gpe}	Gravitational potential energy
U_E	Elastic potential energy
E	Energy

Diagnosis

Conservation of Energy (E is conserved).

$$U_{gpe} \to U_E$$

$$mgH = mgh + \frac{1}{2}kx^2$$

Derivation

$$F = -kx$$

$$\begin{aligned} U &= -\int F\,\mathrm{d}x \\ &= -\int_{x_i}^{x_f} kx\,\mathrm{d}x \\ &= \frac{1}{2}kx_f^2 \end{aligned}$$

Determination

$$mg\left(H-h\right) = \frac{1}{2}kx^2$$

$$\frac{2mg\left(H-h\right)}{k} = x^2$$

$$\boxed{x = \sqrt{\frac{2mg(H-h)}{k}}}$$

Dimensions

$$\begin{aligned} L &= \sqrt{\frac{MLT^{-2}\,(L)}{MT^{-2}}} \\ &= \sqrt{L^2} = L \quad \checkmark \end{aligned}$$

Limiting Cases

- $H \to h$: $H - h = 0 \implies x = 0$ — makes sense!
- $m \to 0 \implies x \to 0$ — makes sense!
- $k \to \infty \implies x \to 0$ — makes sense!
- $k \to 0 \implies x \to \infty$ — makes sense!

Substitution

Not needed.

Example Problem 2

A man (mass m) is pulling a piano (mass M) up a hill (angle θ to horizontal), using a pulley. The coefficient of kinetic friction is μ_k.

> ⚠ TODO: Diagram
>
> Diagram showing: (1) Inclined plane at angle θ. (2) Piano (mass M) on the incline connected by string over pulley to man (mass m) hanging vertically. Include direction arrows and coordinate axes.

> ⚠ TODO: Free Body Diagrams
>
> (1) Man: weight mg down, tension T up. (2) Piano: weight Mg decomposed into components along/perpendicular to incline, normal force N, friction $\mu_k N$ opposing motion, tension T up the incline.

Definitions & Data

- m: Mass of man (kg)
- M: Mass of piano (kg)
- θ: Angle of incline from horizontal
- μ_k: Coefficient of kinetic friction
- g: Acceleration due to gravity
- T: Tension in string
- N: Normal force
- a: Acceleration of the system

Diagnosis

Forces: free body diagram. Conservation of string.

- Man: $F = ma$
- Piano: component of gravity along incline, normal force, friction $f_k = \mu_k N$, tension T.

Derivation

For the man:

$$ma = mg - T$$
$$T = m(g - a)$$

For the piano:

$$N = Mg\cos\theta$$
$$\sum F = Mg\sin\theta + \mu_k N - mg$$

Determination

$$\boxed{a = g\left(\frac{M\sin\theta + \mu_k M\cos\theta - m}{M + m}\right)}$$

Dimensions

$$LHS = LT^{-2}, \quad RHS = LT^{-2}\left(\frac{M + M - M}{M}\right) \quad \checkmark$$

Limiting Cases

- If $M \gg m$: acceleration to the right (piano dominates). Makes sense.
- If $m \gg M$: acceleration negative (man pulls piano up). Makes sense.
- $\theta = 0$: acceleration depends on friction vs man's weight. Makes sense.
- $\theta = 90°$: reduces to Atwood machine. Makes sense.

This working includes an error caught during the limiting cases check, and left in to show how this works.

2 Math Toolkit

2.1 Differentiation — Chain Rule

a)

$$\frac{d}{dr}\left(6r^3(\cos 2r + 5r)^7\right) = 6r^2(\cos 2r + 5r)^6(3\cos 2r + 2r(25 - 7\sin 2r))$$

b)

$$x = (5te^{2t+1} + 2)^8$$

$$v = \frac{dx}{dt} = 40e^{2t+1}(1+2t)(5te^{2t+1} + 2)^7$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 40e^{2t+1}(5te^{2t+1} + 2)^6\left(8(1+t) + (7 + 32t(1+t))5e^{2t+1}\right)$$

2.2 Partial Derivatives

Simply take derivatives with respect to each variable, but consider extreme points if finding maxima or minima.

$$f(x,y) = 5xy + ye^x$$

$$\frac{\partial}{\partial x}f(x,y) = 5y + ye^x$$

$$\frac{\partial}{\partial y}f(x,y) = 5x + e^x$$

2.3 Integration

Substitution

$$\int_{1/2}^{1} \sqrt{1-x^2}\,dx \quad \text{Let } x = \cos u, \quad dx = -\sin u\,du$$

$$\Rightarrow I = \int_{\pi/2}^{\pi/6} \sqrt{1-\cos^2 u} \cdot (-\sin u)\,du = \int_{\pi/6}^{\pi/2} \sin^2 u\,du$$

Using $\cos 2u = 1 - 2\sin^2 u$:

$$I = \frac{1}{2}\int_{\pi/6}^{\pi/2} (1-\cos 2u)\,du = \frac{1}{2}\left[-\frac{\sin 2u}{2} - u\right]_{\pi/6}^{\pi/2} = \frac{\sqrt{3}}{8} - \frac{\pi}{6}$$

Integration by Parts

Basically the inverse of the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int \tan^{-1} x\,dx$$

Let $u = \tan^{-1} x$, $dv = dx$.

$$x = \tan u, \quad \frac{dx}{du} = 1 + \tan^2 u = 1 + x^2$$

$$du = \frac{1}{1+x^2}\,dx, \quad v = x$$

$$\Rightarrow \int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

2.4 Kinematic Equations

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Set a constant and $v(t=0) = v_0 = u$:

$$dv = a\,dt, \quad \int_u^v dv = \int_0^t a\,dt, \quad \boxed{v = u + at}$$

$$dx = (u + at)\,dt, \quad \int_{x_0}^x dx = \int_0^t (u + at)\,dt, \quad \boxed{x = x_0 + ut + \frac{1}{2}at^2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$$

$$a\,dx = v\,dv, \quad \int_0^x a\,dx = \int_u^v v\,dv, \quad \boxed{v^2 = u^2 + 2ax}$$

2.5 Differential Equations

There are infinite varieties of DEs, but nearly all physically relevant ones can be solved quickly by inspection. Here is a very incomplete summary of linear equations.

First Order: Rearrange and integrate.

Second Order: Simple Harmonic Motion!

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Guess the answer of the form:

$$x = A\cos\omega t + B\sin\omega t$$

or

$$x = Ae^{i\omega t + \phi} = Ae^{i\omega t} + Be^{-i\omega t}$$

For example, springs. Nearly everything in nature is a spring of one form or another.

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

So by inspection, $\omega^2 = k/m$.

Another example: Solve $v = ktx$.

$$\frac{dx}{dt} = ktx, \quad \frac{dx}{x} = kt\,dt, \quad \int_{x_0}^{x} \frac{dx}{x} = k\int_0^t t\,dt$$

$$2\ln(x/x_0) = kt^2, \quad \boxed{x = x_0 e^{kt^2/2}}$$

2.6 Coordinate Systems

Plane Polar Coordinates (2D)

For a point P in the plane, the polar coordinates (r, θ) are related to the Cartesian coordinates (x, y) by:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

or inversely:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Cylindrical Polar Coordinates

For axial symmetry:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Spherical Polar Coordinates

For spherical symmetry, the coordinates are defined as:

$$0 \le r, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi$$

where r is the radius, θ is the polar angle, and ϕ is the azimuthal angle.

2.7 Multivariate Integration

Univariate integration is one dimensional:

$$\int f'(x)\,dx = f(x) + C$$

Areas and volumes require multivariate integration.

$$\text{Area} = \iint_A dA = \iint_A dx\,dy = \int_c^d \left(\int_a^b dx \right) dy = (b-a)(d-c)$$

Polar Coordinates

For an infinitesimal area element dA in polar coordinates:

$$dA = r\,dr\,d\theta$$

Volumes

The volume of a sphere:

$$\text{Volume} = \iiint_V dV = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta\,dr\,d\theta\,d\phi = \int_0^R r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi \cdot \frac{R^3}{3} \cdot [-\cos\theta]_0^\pi = \frac{4}{3}\pi R^3$$

Flux Integrals

One can also integrate an arbitrary scalar or vector function over some area or volume.

Vector field integrations are performed over areas to find fluxes, and are only concerned with the normal (perpendicular) component of the field:

$$I = \iint \vec{f} \cdot \hat{n}\,dA$$

Part II

Classical Physics

3 Mechanics

3.1 Energy, Conservation, and Forces

3.1.1 Energy

Energy is the ability to do work — that which is conserved. It comes in two broad types:

- **Kinetic energy** — energy of motion: $K = \frac{1}{2}mv^2$
- **Potential energy** — stored energy associated with position or configuration

Energy has units of $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} = \mathrm{J}$ (joules). It is a scalar.

There are many forms of potential energy, each associated with a fundamental force:

Table 3.1: Forms of Potential Energy

Force	Potential Energy
Gravitational	$U = mgh$ (near surface), $U = -\frac{Gm_1m_2}{r}$ (general)
Elastic (spring)	$U = \frac{1}{2}kx^2$
Electrostatic	$U = \frac{kq_1q_2}{r}$
Weak nuclear	(particle physics)
Strong nuclear	(particle physics)

3.1.2 Potential Energy Diagrams

A potential energy diagram plots $U(x)$ vs position x. Key features:

- **Equilibrium points** occur where $F = -\frac{dU}{dx} = 0$
- **Stable equilibrium**: local minimum (restoring force)
- **Unstable equilibrium**: local maximum
- A **potential well** traps particles with insufficient energy to escape
- The harmonic oscillator potential $U = \frac{1}{2}kx^2$ is the simplest potential well; square wells also exist (quantum mechanics)

3.1.3 Gravitational Potential Energy

By convention, $U = 0$ at $r \to \infty$. Since gravity is attractive, bringing masses closer *decreases* their potential energy, so:

$$\boxed{U = -\frac{Gm_1m_2}{r}}$$

The force is related to the potential by:

$$F = -\frac{dU}{dr} = -\frac{Gm_1m_2}{r^2}\,\hat{r}$$

The negative sign means the force points toward decreasing r (attractive).

Example: On the surface of the Earth.

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 8 \times 10^{1}}{(6.4 \times 10^{6})^2} \approx 588\ \mathrm{N}$$

Change in gravitational PE between two radii r_1 and r_2:

$$\Delta U = -\frac{Gm_1m_2}{r_2} + \frac{Gm_1m_2}{r_1} = Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

3.1.4 Force, Energy, and Work

Force is the derivative of potential energy:

$$F = -\frac{dU}{dx}$$

Conversely, potential energy is the negative integral of force:

$$dU = -F \cdot dx \qquad \Rightarrow \qquad U = -\int F \cdot dx \quad \text{(always)}$$

Work is defined as the integral of force along a displacement:

$$W = \int \vec{F} \cdot d\vec{x}$$

How much work does the Moon do sitting on the road due to the Earth? Zero — no displacement, no work.

3.1.5 Conservation of Energy

Energy is always conserved:

$$\Delta E = 0 \qquad E_i = E_f \quad \text{for an isolated system}$$

$$\Delta E_{\text{system}} + \Delta E_{\text{environment}} = 0$$

Conservation principles (like conservation of energy) are extremely powerful problem-solving tools.

Example: Mass on a spring. Consider a mass that can have gravitational PE, elastic PE, kinetic energy, bulk KE, and thermal energy:

> ⚠ TODO: Diagram
>
> Vertical mass-spring system showing the different energy types: gravitational PE at the top, elastic PE in the spring, and kinetic energy of the mass.

3.1.6 Hooke's Law and Spring Energy

The restoring force of a spring:

$$F_{\text{spring}} = -F_{\text{app}} = -k\Delta x$$

$$F = -kx$$

The work done compressing/stretching a spring (equivalently, the elastic PE stored):

$$W = \int_0^x F\,dx$$

$$\boxed{W = \frac{1}{2}kx^2}$$

3.1.7 Mechanical Energy and Conservative Forces

A **conservative force** is one where only the endpoints matter — the work done is path-independent. For conservative forces, we can define a potential energy, and mechanical energy $E = U + K$ is conserved.

Gravitational and elastic forces are conservative. Electrostatic forces are also conservative.

Non-conservative (dissipative) forces convert mechanical energy into thermal and microscopic energy. Friction and air resistance are examples.

$$U = Mgh \qquad KE = \frac{1}{2}mv^2$$

When choosing a reference point for potential energy, you must be consistent — pick one and stick with it.

3.2 Newton's Laws and Forces

3.2.1 Potential vs. Potential Energy

An important distinction:

- **Potential energy** is an *extensive* property — depends on the specific object ($U_{\text{grav}} = -\frac{Gm_1m_2}{r}$)
- **Potential** is an *intensive* property — depends only on the source ($V_{\text{grav}} = -\frac{Gm}{r}$)

Intensive units = extensive units / mass (or charge).

3.2.2 Conservation Laws

Many quantities are conserved in isolated systems:

- Energy
- Momentum
- Angular momentum
- Mass
- Charge
- (and others: strangeness, baryon number, etc.)

3.2.3 Newton's Laws

1. Every body has constant velocity unless acted upon by a non-zero net force (inertia).
2. $\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$
3. For every action, there is an equal and opposite reaction.

Force has units of $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2} = \mathrm{N}$.

Weight $\neq$ mass. Weight is mg.

3.2.4 Free Body Diagrams

Rules for drawing free body diagrams (FBDs):

- Draw a box or lump (depending on state)
- One body, simple, all forces on the same object on the diagram
- Use vector component forces — only forces acting on the object
- Only include *forces*, not velocities or accelerations
- Everything you experience (forces) except for gravity is electromagnetic in nature

Example: Car going around a corner at constant speed. What is the net force acting on the car, and what is its origin? The force is **centripetal**, directed toward the center of the turn, provided by friction between the tires and road.

3.2.5 Normal Force

The normal (contact) force acts perpendicular to a surface. It is the surface's response to prevent objects from passing through.

- Usually equal to the weight component perpendicular to the surface
- It has a *maximum* value — the surface can be overwhelmed (e.g. breaking through)

Example: Elevator.

$$\sum F = ma$$

If the elevator accelerates upward at acceleration a:

$$N - mg = ma \qquad \Rightarrow \qquad N = m(g + a)$$

3.2.6 Friction

Static friction acts parallel to the surface to prevent relative motion:

- Cold welding of microscopic surface bumps
- $F_s \leq \mu_s N$ (it adjusts up to a maximum)

Kinetic friction involves continuous formation and breakage of bonds:

- $F_k = \mu_k N$ (constant for a given pair of surfaces)
- This is an *approximation* (only a model)

Example: Box on back of a ute (truck).

Maximum acceleration before sliding:

$$F_{\text{max}} = \mu_s N = \mu_s mg$$

$$a_{\text{max}} = \mu_s g \quad \text{(independent of mass)}$$

3.2.7 Air Resistance

$$F_{\text{air}} = \frac{1}{2}\rho v^2 C_D A$$

At **terminal velocity**, the drag force equals the weight: $F_{\text{air}} = mg$.

3.2.8 Tension

In an ideal rope:

- Force at one end is instantly transmitted without loss — every part feels the same tension
- $\theta_{\text{ideal rope}} = 0$ (massless, frictionless)
- Ideal ropes have no mass and are inextensible ($T = \text{const}$ throughout)
- Ideal pulleys act to change direction but nothing else
- $N \leq F_{\text{pull}}$: varies depending on applied force
- Two components of contact force (normal and friction) — minimum energy

3.2.9 Momentum

$$\vec{p} = m\vec{v} \qquad \text{units: } \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1} = \mathrm{N} \cdot \mathrm{s}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

No force $\Rightarrow$ no change in momentum.

Example: Two carts, one mass m, the other mass $2m$, pushed by equal force for 3 seconds on an air track.

Equal momenta: $p_1 = p_2$

$$KE \propto \frac{p^2}{2m}$$

Since p is the same, the lighter cart has more KE. It travels distance $2d$ while the heavier travels d.

3.2.10 Newton's Third Law

Forces act on *different* objects:

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

Gravity and normal force on the same object are **not** a Newton's third law pair — they have different origins.

$$\frac{d\vec{p}}{dt} = 0 \quad \text{if } F_{\mathrm{AB}} = -F_{\mathrm{BA}} \quad \Rightarrow \quad \frac{dp_A}{dt} = -\frac{dp_B}{dt}$$

Therefore, **momentum is always conserved** in an isolated system.

3.3 Collisions

3.3.1 Elastic Collisions

In elastic collisions, both momentum and kinetic energy are conserved (gravity, conservative forces, Newton's cradle, gravitational slingshot).

Isolated system, two bodies:

$$\sum_i m_i v_i = C$$

$$m_A v_{Ai} + m_B v_{Bi} = m_A v_{Af} + m_B v_{Bf}$$

$$\frac{1}{2} m_A v_{Ai}^2 + \frac{1}{2} m_B v_{Bi}^2 = \frac{1}{2} m_A v_{Af}^2 + \frac{1}{2} m_B v_{Bf}^2$$

Solving these two equations simultaneously gives the final velocities:

$$\boxed{v_{Af} = \frac{m_A - m_B}{m_A + m_B} v_{Ai} + \frac{2m_B}{m_A + m_B} v_{Bi}}$$

$$\boxed{v_{Bf} = \frac{2m_A}{m_A + m_B} v_{Ai} + \frac{m_B - m_A}{m_A + m_B} v_{Bi}}$$

Limiting cases:

- If the masses are equal ($m_A = m_B$): they swap velocities
- If the masses are vastly different ($M \gg m$): $v_{Bf} \approx v_{Bi}$, the heavy object is barely affected
- If B is initially at rest ($v_{Bi} = 0$) and $M \gg m$: $v_{Af} \approx -v_{Ai}$ (bounces back), $v_{Bf} \approx 0$

3.3.2 Inelastic Collisions

In inelastic collisions, kinetic energy is lost (to deformation, heat, sound, etc.), but **momentum is still conserved**.

Example: Rain falling into a moving sports car (coasting).

As rain accumulates, the speed decreases. If you drill a hole to let the water out, the car does *not* speed back up — unless the water is pumped out stationary with respect to the road.

For a perfectly inelastic collision (objects stick together):

$$m_A v_{Ai} + m_B v_{Bi} = (m_A + m_B) v_f$$

$$\boxed{v_f = \frac{m_A v_{Ai} + m_B v_{Bi}}{m_A + m_B}}$$

3.4 Rotational Mechanics

3.4.1 Rotational Kinematics

A **rigid body** has every point moving in a circular path, with the centers of those circles on a common axis. It can be approximated as a single point for pure rotation.

Variables:

Table 3.2: Linear vs. Rotational Quantities

Linear	Rotational	Relation
x (displacement)	θ (angle, rad)	$s = r\theta$
v (velocity)	ω (angular velocity, rad/s)	$v = r\omega$
a (acceleration)	α (angular acceleration, rad/s^2)	$a = r\alpha$

Positive direction is arbitrary but must be defined (right-hand rule).

Equations of motion (identical in form to linear kinematics):

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Angular velocity direction given by the **right-hand rule**: curl fingers in the direction of rotation, thumb points along the axis.

3.4.2 Torque

Torque is the rotational analogue of force. It depends on where and in what direction the force is applied:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\tau| = rF \sin\theta$$

where θ is the angle between $\vec{r}$ and $\vec{F}$.

3.4.3 Moment of Inertia

The moment of inertia I is the rotational analogue of mass — it measures resistance to angular acceleration.

- Single particle: $I = mr^2$
- Collection of particles: $I = \sum_i m_i r_i^2$
- Solid body: $I = \iiint r^2 \, dm$

3.4.3.1 Parallel Axis Theorem

$$I = I_{cm} + Mh^2$$

where h is the distance between the center-of-mass axis and the parallel axis.

3.4.3.2 Perpendicular Axis Theorem

For a thin, flat object (2D):

$$I_z = I_x + I_y$$

3.4.4 Moment of Inertia of a Sphere

For a uniform solid sphere of mass M and radius R, using cylindrical polar coordinates:

$$I = \iiint r_\perp^2 \, dm = \rho \iiint r_\perp^2 \, dV$$

Carrying out the integration:

$$I = 2\pi\rho \int_0^R \left(\frac{(R^2 - z^2)^2}{4} \right) dz = \frac{8\pi\rho}{15} R^5$$

Since $M = \rho V = \rho \frac{4}{3}\pi R^3$:

$$\boxed{I = \frac{2}{5} M R^2}$$

3.4.5 Newton's Second Law for Rotation

$$\tau = I\alpha$$

This is the rotational analogue of $F = ma$.

Example: Rolling Without Slipping

A ball of mass M and radius R rolls without slipping down an incline of angle θ.

> ⚠ TODO: Diagram
>
> Ball rolling down an incline. Show forces: weight Mg at center (downward), normal force N perpendicular to surface, friction f at contact point (up the incline). Coordinate axes: x along incline (positive downhill), y perpendicular to incline.

Rolling constraint: $v = R\omega$, so $a = R\alpha$.

Translational (along the incline):

$$Mg \sin\theta - f = Ma$$

Rotational (about center of mass):

$$fR = I\alpha = I\frac{a}{R}$$

$$f = \frac{Ia}{R^2}$$

Substituting:

$$Mg\sin\theta - \frac{Ia}{R^2} = Ma$$

$$a = \frac{Mg\sin\theta}{M + I/R^2} = \frac{g\sin\theta}{1 + I/(MR^2)}$$

For a solid sphere with $I = \frac{2}{5}MR^2$:

$$\boxed{a = \frac{5}{7}g\sin\theta}$$

Objects with smaller $I/(MR^2)$ roll faster. A sliding frictionless block ($I = 0$) has $a = g\sin\theta$, the maximum.

3.4.6 Rotational Kinetic Energy

For pure rotation:

$$K = \frac{1}{2}I\omega^2$$

For combined rotation and translation:

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_{cm}^2$$

Rotational work:

$$W = \int \tau\, d\theta$$

Comparison of rolling objects down a ramp. The total kinetic energy is $K = \frac{1}{2}mv^2(1+k)$ where $k = I/(mR^2)$:

Table 3.3: Moments of Inertia for Rolling Objects

Object	I	$k = I/(mR^2)$
Solid sphere	$\frac{2}{5}MR^2$	2/5
Solid cylinder	$\frac{1}{2}MR^2$	1/2
Hoop	MR^2	1

The lower k, the higher the translational speed — the **sphere is fastest**.

3.4.7 Angular Momentum

Angular momentum is the rotational analogue of linear momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

For a rigid body with axial symmetry:

$$L = I\omega$$

Angular momentum and torque:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

This can be broken into components: $\sum \tau_x = \frac{dL_x}{dt}$, etc.

Steiner's theorem (parallel axis for angular momentum):

$$L = L_{cm} + MR_{cm} \times v_{cm}$$

3.4.8 Conservation of Angular Momentum

If $\tau_{\text{ext}} = 0$, then $\frac{dL}{dt} = 0$, so $L = \text{const}$.

$$I_i\omega_i = I_f\omega_f$$

Example: Girl on a merry-go-round. A girl of mass m walks from the rim (radius R) toward the center of a merry-go-round (moment of inertia I_0, initial angular velocity ω_0).

At the rim: $L = (I_0 + mR^2)\omega_0$

At radius r: $L = (I_0 + mr^2)\omega$

$$\omega = \frac{(I_0 + mR^2)\omega_0}{I_0 + mr^2}$$

As $r \to 0$: ω increases (she spins faster as she moves inward).

Angular momentum does **not** imply circular motion — objects moving in a straight line have angular momentum relative to any external point.

3.4.9 Precession

⚠ TODO: Diagram

Spinning top with gravity pulling down at center of mass. Show the torque $\tau = rmg\sin\theta$ causing the angular momentum vector $\vec{L}$ to precess around the vertical axis. Include: the precession circle traced by the tip of $\vec{L}$, the angle θ from vertical, and the precession angular velocity ω_p.

Precession is caused by a small lateral torque that changes the *direction* of the angular momentum vector without affecting its magnitude.

For a spinning top tilted at angle θ from vertical:

- Torque: $\tau = rmg\sin\theta$
- The torque causes $\vec{L}$ to rotate horizontally: $dL = L\sin\theta\,d\phi$

$$\omega_p = \frac{d\phi}{dt}$$

Since $\tau\,dt = dL = L\sin\theta\,d\phi$:

$$\omega_p\,dt = \frac{dL}{L\sin\theta} = \frac{\tau\,dt}{L\sin\theta}$$

$$\boxed{\omega_p = \frac{rmg}{L} = \frac{rmg}{I\omega}}$$

The precession rate is inversely proportional to L (the speed of the top) — a slower top precesses in wider, faster circles.

3.5 Gravity and Orbits

3.5.1 Newton's Law of Gravitation

Gravity is an attractive force between any two masses:

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\,\hat{r}$$

Shell theorem: For multiple masses, use vector superposition (or potentials). Spherically symmetric objects behave exactly like point masses (when you are outside them).

3.5.2 Gravitational Field

The gravitational field is the force per unit mass — an *intensive*, vector field:

$$\vec{g} = -\frac{GM}{r^2}\,\hat{r}$$

Field lines:

- Show direction of force at a point in space
- Emanate from masses
- Never cross around sharp edges
- Density of field lines gives relative field strength

Flux is the measure of field lines passing through a surface:

$$\Phi_{\text{grav}} = \oint \vec{g}\cdot d\vec{A}$$

3.5.3 Gauss's Law for Gravity

The net flux through a closed surface depends only on the enclosed mass:

$$\boxed{\oint \vec{g} \cdot d\vec{A} = -4\pi G M_{\text{enclosed}}}$$

Use this (with a good choice of Gaussian surface) when there is symmetry.

3.5.4 Gravitational Field Inside and Outside a Sphere

Outside the Earth (or any spherically symmetric body):

$$g = \frac{GM}{r^2}$$

Identical to a point mass at the center.

Inside a hollow shell:

$$g = 0$$

Apply Gauss's law: the enclosed mass is zero, so the field inside a uniform shell vanishes. This is a remarkable result.

Inside a uniform solid sphere (at radius $r < R$):

The enclosed mass is $M_{\text{enc}} = M\frac{r^3}{R^3}$ (assuming uniform density), so:

$$g = \frac{GM_{\text{enc}}}{r^2} = \frac{GM}{R^3}\, r$$

The field increases linearly with r inside the sphere.

3.5.5 Gravitational Potential Energy (General)

Potential energy is extensive (depends on both masses). Define $U = 0$ at $r \to \infty$:

$$U = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = -\int_{\infty}^{r} \frac{GMm}{r^2}\, dr$$

$$\boxed{U = -\frac{GMm}{r}}$$

3.5.6 Gravitational Potential

Potential is the potential energy per unit mass — an *intensive* scalar:

$$V = \frac{U}{m} = -\frac{GM}{r}$$

Table 3.4: Extensive vs. Intensive Quantities

Property	Extensive	Intensive
Energy	E (J)	—
Gravitational	U (J)	V (J/kg)
Electrostatic	U (J)	V (V)

3.5.7 Equipotentials

- Lines or surfaces of equal potential
- Perpendicular to field lines (e.g., perpendicular to gravity)
- In a conservative field, journeys that start and end on the same equipotential require no net energy

> ⚠ TODO: Diagram
>
> Equipotential lines around a point mass, showing concentric circles with field lines (arrows) pointing radially inward, perpendicular to the equipotentials. Include a hilly landscape analogy showing contour lines.

3.5.8 Escape Velocity

The velocity needed to escape a gravitational field entirely (reach $r \to \infty$ with $v = 0$). By conservation of energy:

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{r} = 0$$

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

3.5.9 Kepler's Laws

1. All planets move in **elliptical orbits** with the Sun at one focus.
2. A line joining a planet to the Sun sweeps out **equal areas in equal times** (conservation of angular momentum).
3. The square of the orbital period is proportional to the cube of the semi-major axis:

$$T^2 \propto a^3$$

Proof of Kepler's Third Law (for circular orbits):

For a circular orbit, gravitational force provides centripetal acceleration:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = mr\omega^2$$

$$\omega^2 = \frac{GM}{r^3} \qquad \Rightarrow \qquad \frac{4\pi^2}{T^2} = \frac{GM}{r^3}$$

$$\boxed{\frac{T^2}{r^3} = \frac{4\pi^2}{GM}}$$

3.6 Fluids

3.6.1 Pressure

Fluids are things that flow. We describe them using **pressure** and **volume** instead of mass and force.

$$P = \frac{F}{A}$$

Units: $\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2} = \text{Pa}$ (pascals). Also: $1\ \text{atm} = 760\ \text{mmHg} = 1.4\ \text{PSI} = 1.01 \times 10^5\ \text{Pa}$.

Pressure is a **scalar** — it has no direction (it acts equally in all directions at a point in a fluid).

Gauge pressure vs. **absolute pressure:**

- Gauge pressure is measured relative to atmospheric pressure
- Absolute = gauge + atmospheric

- Example: blood pressure of 120 mmHg is *gauge*; tire pressure of 12 kPa is *gauge*; barometric pressure of 102 kPa is *absolute*

3.6.2 Hydrostatic Pressure

The pressure due to a static fluid column, from the weight of fluid above:

$$F = W = mg = \rho V g = \rho A h g$$

$$P = \frac{F}{A} = \rho g h$$

$$\boxed{P = \rho g h + P_0}$$

where P_0 is the pressure at the top of the fluid (e.g. atmospheric pressure).

Key insight: Pressure depends only on *depth*, not on the shape of the container.

Example: Designing a dam. For a wall of width w and depth h, the pressure at depth d is $P = \rho g d$. The total force requires integrating over the wall area.

3.6.3 Archimedes' Principle

A submerged object displaces its own volume of fluid. This allows volume estimation of irregular objects (and fat content of people).

Buoyancy: A body fully or partially submerged is buoyed up by a force equal to the *weight of the fluid it displaces*. Air also exerts a buoyancy force.

$$F_{\text{buoy}} = \rho_{\text{fluid}} V_{\text{displaced}}\, g$$

For a floating object: $F_{\text{buoy}} = W$, so:

$$\rho_{\text{fluid}} V_{\text{displaced}} g = mg$$

Conditions:

- $\rho_{\text{object}} > \rho_{\text{fluid}}$: sinks ($W > F_b$)
- $\rho_{\text{object}} < \rho_{\text{fluid}}$: floats ($F_b > W$, partially submerged)
- $\rho_{\text{object}} = \rho_{\text{fluid}}$: neutral buoyancy (total submersion)

For a partially submerged object: $F_b = W$, so $\rho_{\text{fluid}} V_{\text{displaced}} g = m_{\text{object}} g$, or equivalently $V_{\text{displaced}}/V_{\text{object}} = \rho_{\text{object}}/\rho_{\text{fluid}}$.

Example: Ice cubes. Two identical glasses of water, one with an ice cube, one without. Do they have the same water level? Yes — the ice displaces exactly its own weight in water. When it melts, $W_{\text{ice}} = W_{\text{displaced water}}$, so the level stays the same.

Buoyancy Example: Helium Airship

What volume of He is needed to lift a 1000 kg airship?

> ⚠ TODO: Diagram
>
> Airship (blimp shape) with volume V_H of helium, payload mass M. Show forces: buoyancy $F_b = \rho_a V_H g$ upward, weight $Mg + \rho_H V_H g$ downward.

At equilibrium ($\sum F = 0$):

$$\rho_a V_H g - Mg - \rho_H V_H g = 0$$

$$V_H(\rho_a - \rho_H) = M$$

$$V_H = \frac{M}{\rho_a - \rho_H}$$

3.6.4 Real vs. Ideal Fluids

Ideal Fluid	Real Fluid
Non-viscous (no internal friction)	Viscous
Incompressible (density is constant)	May be compressible
Laminar, non-rotational flow	Can become turbulent

Viscous fluids slow down — important in fuel, oil, blood flow. Viscosity usually decreases with increasing temperature (engine oil gets runnier when warm).

3.6.5 Continuity Equation

For an incompressible fluid, what goes in must come out. The **volume flow rate** is constant:

$$Q = Av = \text{constant} \qquad \text{units: } \mathrm{m^3 s^{-1}}$$

$$\boxed{A_1 v_1 = A_2 v_2}$$

If the pipe narrows, the fluid speeds up.

3.6.6 Bernoulli's Equation

Once a flow is established, two conservation principles apply:

- **Continuity**: conservation of mass (volume flow rate is constant)
- **Bernoulli's equation**: conservation of energy (energy density is constant)

$$\boxed{P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$$

This is conservation of energy density along a streamline:

- P: pressure energy (electrostatic analogy)
- $\frac{1}{2}\rho v^2$: kinetic energy density
- $\rho g h$: gravitational potential energy density

Note: The pressure in Bernoulli's equation is the pressure the walls of the pipe exert on the fluid. This is *not* conservation of momentum (a common misconception).

Example: Water flowing through a pipe. At a constriction, v increases (continuity), so P must decrease (Bernoulli). This is a pressure drop: $\Delta P = -\frac{1}{2}\rho v^2$.

3.6.7 Bernoulli Applications

Venturi effect: Where fluid speeds up, pressure drops. This explains:

- Lift on an aerofoil (faster flow over top → lower pressure → net upward force)
- Roofs blowing off in storms (fast wind over roof, still air underneath)
- Aeroplane lift: $F_{\text{lift}} \approx \frac{1}{2}\rho v^2 A$

> ⚠ TODO: Diagram
>
> Aerofoil cross-section showing streamlines compressed above (faster, lower pressure) and wider below (slower, higher pressure). Include net upward force arrow.

Open windows on the lee side to equalize pressure (how aeroplanes fly, how roofs stay on).

Bernoulli's equation can only be applied along a continuous streamline.

3.6.8 Poiseuille's Law

For viscous flow through a pipe of radius r and length ΔL, the volume flow rate is:

$$\boxed{Q = \frac{\pi r^4}{8\eta}\frac{\Delta P}{\Delta L}}$$

where η is the dynamic viscosity. Note the extremely strong r^4 dependence — doubling the radius increases flow by a factor of 16.

Flow vs. flux: Flow (Q) is volume per time; flux is flow per unit area.

4 Thermodynamics

4.1 Temperature and Heat

4.1.1 Temperature

Temperature measures how hot or cold something is — more precisely, it determines the direction of heat flow between two systems in thermal contact.

A **thermometer** works by exploiting a physical property that varies with temperature (e.g., the volume of a gas at constant pressure). The **constant-volume gas thermometer** defines temperature via the pressure of a gas at fixed volume:

$$T_{\text{gas}} = T_{\text{ref}} \frac{P}{P_{\text{ref}}} \quad \text{(proportional to pressure)}$$

At the **triple point of water** ($T_{\text{trip}} = 273.16$ K), we fix the scale. As gas pressure is reduced to zero, all gas thermometers converge — this defines the **ideal gas temperature scale**.

4.1.2 Zeroth Law of Thermodynamics

Why do different thermometers agree? The **zeroth law**:

> If systems A and B are each in thermal equilibrium with system C, then A and B are in thermal equilibrium with each other.

This guarantees that temperature is well-defined and standardized. The Celsius scale is related to Kelvin by:

$$T_C = T_K - 273.15 \text{ K}$$

4.1.3 Heat

Heat (Q) is the transfer of thermal energy, measured in joules. Heat flows between two bodies that are not in thermal equilibrium — always from hot to cold.

4.1.4 Heat Transfer

There are three ways heat can flow:

Conduction — heat flows through a material without bulk motion. The rate of heat flow:

$$H = \frac{Q}{t} = \frac{kA\,\Delta T}{\Delta x}$$

where k is the thermal conductivity, A the cross-sectional area, ΔT the temperature difference, and Δx the thickness.

Radiation — heat transfer via electromagnetic waves. The Stefan–Boltzmann law gives the radiated intensity:

$$I = \varepsilon\sigma T^4$$

where $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$ is the Stefan–Boltzmann constant and ε is the emissivity ($0 \leq \varepsilon \leq 1$; $\varepsilon = 1$ for a blackbody). Radiated power is proportional to T^4.

Convection — heat transfer by bulk fluid motion. Heated fluid becomes less dense and rises in bubbles (turbulence, convection cells). This drives radiators, atmospheric circulation, and many other phenomena.

4.1.5 Thermal Expansion

Most materials expand when heated.

Linear expansion: for a rod of length L,

$$\Delta L = \alpha L \Delta T$$

where α is the coefficient of linear expansion. Note that holes get bigger too.

Volume expansion: for a solid with dimensions $a \times b \times c$,

$$V = abc, \qquad V + \Delta V = (a + \alpha a\Delta T)(b + \alpha b\Delta T)(c + \alpha c\Delta T)$$

Expanding and keeping only first-order terms in ΔT:

$$\Delta V \approx 3\alpha V\Delta T = \beta V\Delta T$$

where $\beta \approx 3\alpha$ is the coefficient of volume expansion.

For **liquids** (no linear expansion applies), $\Delta V = \beta V \Delta T$ directly. Note that β is generally temperature-dependent, and for water, $\beta > 0$ usually but $\beta < 0$ near 4°C (water's anomalous expansion).

4.1.6 Phases and Latent Heat

Matter exists in three common phases: **solid**, **liquid**, and **gas**. Phase transitions occur at specific temperatures and pressures.

Latent heat (ℓ) is the heat per kilogram required for a phase change:

- ℓ_f — latent heat of fusion (solid $\leftrightarrow$ liquid)
- ℓ_v — latent heat of vaporization (liquid $\leftrightarrow$ gas)

4.1.7 Heat Capacity

Heat capacity C is the heat required to raise the temperature of a system by one degree:

$$C = \frac{\delta Q}{\delta T} \quad [\text{J/K}] \quad (\text{extensive})$$

Specific heat capacity c is heat capacity per unit mass (or per mole):

$$\delta Q = mc\,\delta T, \qquad c = \frac{C}{m} \quad [\text{J/(kg} \cdot \text{K)}] \quad (\text{intensive})$$

4.1.8 First Law of Thermodynamics

For a process between two equilibrium states A and B, the quantity $Q+W$ is the same regardless of the path taken:

$$\Delta U = Q + W$$

where Q is heat added to the system and W is work done on the system. Internal energy U is a **state function** — it depends only on the current state, not on how the system got there.

4.2 Ideal Gas and Kinetic Theory

4.2.1 The Ideal Gas Law

An **ideal gas** satisfies:

$$\boxed{PV = NkT = nRT}$$

where:

Table 4.1: Ideal Gas Parameters

Symbol	Meaning
P	Pressure
V	Volume
N	Number of molecules
n	Number of moles
$k = 1.38 \times 10^{-23}$ J/K	Boltzmann constant
$R = 8.314$ J/(K $\cdot$ mol)	Gas constant
$N_A = 6.022 \times 10^{23}$ mol^{-1}	Avogadro's number

Note $R = N_A k$.

Assumptions of the ideal gas model:

1. Particles have negligible volume and exert no long-range forces
2. Total number of particles is large
3. Newtonian mechanics applies
4. Interactions occur through collisions of negligible duration
5. Equipartition theorem holds

4.2.2 Equipartition Theorem

On average, each degree of freedom contributes $\frac{1}{2}kT$ of energy per molecule:

$$\varepsilon = \frac{f}{2}kT$$

where f is the number of degrees of freedom.

Table 4.2: Degrees of Freedom

Molecule	f	Example
Monatomic	3	He (3 translational)
Diatomic	5	N_2 (3 translational + 2 rotational)
Polyatomic	6	H_2O (3 translational + 3 rotational)

The total internal energy of an ideal gas is:

$$\boxed{U = \frac{f}{2}NkT = \frac{f}{2}nRT}$$

4.2.3 Kinetic Theory of Gases

Consider N molecules in a box of side L. A molecule with velocity v_x bouncing off a wall transfers momentum $\Delta p = 2mv_x$ each round trip (time $\Delta t = 2L/v_x$).

The force from one molecule:

$$F = \frac{\Delta p}{\Delta t} = \frac{2mv_x}{2L/v_x} = \frac{mv_x^2}{L}$$

Summing over all N molecules and using $P = F/A = F/L^2$:

$$P = \frac{Nm\overline{v_x^2}}{L^3} = \frac{Nm\overline{v_x^2}}{V}$$

Since the motion is isotropic, $\overline{v_x^2} = \frac{1}{3}\overline{v^2}$, so:

$$\boxed{P = \frac{1}{3}\frac{Nm\overline{v^2}}{V} = \frac{1}{3}\rho v_{\text{rms}}^2}$$

where $v_{\text{rms}} = \sqrt{\overline{v^2}}$ is the root-mean-square speed.

4.2.4 Dalton's Law

For a mixture of gases, the total pressure is the sum of the partial pressures:

$$P_{\text{total}} = \sum_i P_i$$

4.2.5 RMS Speed

From $P = NkT/V$ and $P = \frac{1}{3}Nm\overline{v^2}/V$:

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT \qquad \Longrightarrow \qquad v_{\mathrm{rms}} = \sqrt{\frac{3kT}{m}}$$

The total translational kinetic energy is:

$$K_{\mathrm{trans}} = \frac{3}{2}NkT = \frac{3}{2}PV$$

4.3 Thermodynamic Processes

4.3.1 PV Diagrams and Work

The first law of thermodynamics:

$$dU = \delta Q + \delta W$$

where U is a state function (path-independent). On a PV diagram, the work done *on* the gas is:

$$\boxed{W = -\int_{V_1}^{V_2} P\,dV = -(\text{area under curve})}$$

The sign convention: $W > 0$ when work is done *on* the gas (compression), $W < 0$ when the gas expands.

TODO: Diagram

PV diagram showing work as area under the curve for an expansion process, with labeled axes P and V.

4.3.2 Special Processes

Table 4.3: Thermodynamic Processes

Process	Condition	Work	Heat
Isochoric	$dV = 0$	$W = 0$	$Q = \Delta U$
Isobaric	$dP = 0$	$W = -P\Delta V$	$Q = \Delta U + P\Delta V$
Isothermal	$dT = 0,\ PV = \text{const}$	$W = -NkT \ln \frac{V_2}{V_1}$	$Q = -W$
Adiabatic	$\delta Q = 0$	$W = \Delta U$	$Q = 0$

4.3.3 Isothermal Process

For an ideal gas at constant temperature ($PV = NkT = \text{const}$):

$$W_{\text{isotherm}} = -\int_{V_1}^{V_2} \frac{NkT}{V}\, dV = -NkT \ln \frac{V_2}{V_1}$$

Since $dT = 0$, the internal energy doesn't change ($\Delta U = 0$ for an ideal gas), so:

$$Q = -W = NkT \ln \frac{V_2}{V_1}$$

4.3.4 Specific Heat Capacities

At constant volume ($dV = 0$, so $W = 0$):

$$dU = \delta Q = nC_v\, dT$$

Since $U = \frac{f}{2} nRT$, we get $dU = \frac{f}{2} nR\, dT$, giving:

$$\boxed{C_v = \frac{f}{2} R}$$

At constant pressure ($dP = 0$):

$$dU = \delta Q + \delta W = nC_p\, dT - PdV$$

Using $PV = nRT$, at constant pressure $PdV = nR\, dT$. Then:

$$\frac{f}{2} nR\,dT = nC_p\,dT - nR\,dT$$

$$\boxed{C_p = \left(\frac{f}{2} + 1\right) R}$$

The **heat capacity ratio**:

$$\boxed{\gamma = \frac{C_p}{C_v} = \frac{f+2}{f}}$$

Table 4.4: Heat Capacity Ratios

Gas type	f	γ
Monatomic	3	5/3
Diatomic	5	7/5
Polyatomic	6	4/3

4.3.5 Adiabatic Process

For an adiabatic process ($\delta Q = 0$), starting from the first law and using $dU = \frac{f}{2} nR\,dT$:

$$dU = \delta W = -PdV$$

$$\frac{f}{2} nR\,dT = -PdV$$

Using $d(PV) = PdV + VdP = nR\,dT$, we can write:

$$\frac{f}{2}(PdV + VdP) = -PdV$$

$$\frac{f}{2} VdP = -\left(\frac{f}{2} + 1\right) PdV$$

$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

Integrating:

$$\ln P + \gamma \ln V = \text{const}$$

$$\boxed{PV^\gamma = \text{constant}}$$

On a PV diagram, adiabats are steeper than isotherms (since $\gamma > 1$).

> ⚠ TODO: Diagram
>
> PV diagram comparing an isotherm and an adiabat through the same point, showing the adiabat is steeper.

Definitions:

- **Equilibrium**: no macroscopic changes; temperature is well-defined
- **Quasistatic**: system is always in equilibrium; infinitely slow changes

4.4 Heat Engines and the Carnot Cycle

4.4.1 Heat Engines

A **heat engine** absorbs heat $|Q_H|$ from a hot reservoir, does work $|W|$, and rejects heat $|Q_C|$ to a cold reservoir. By conservation of energy:

$$|Q_H| = |W| + |Q_C|$$

> ⚠ TODO: Diagram
>
> Schematic of a heat engine: hot reservoir (T_H) at top, cold reservoir (T_C) at bottom, engine in middle with arrows showing Q_H in, W out, Q_C out.

The **efficiency** of a heat engine:

$$e = \frac{|W|}{|Q_H|} = \frac{|Q_H| - |Q_C|}{|Q_H|} = 1 - \frac{|Q_C|}{|Q_H|}$$

4.4.2 Refrigerators

A **refrigerator** is a heat engine run in reverse: it uses work $|W|$ to transfer heat from cold to hot.

The **coefficient of performance**:

$$K = \frac{|Q_C|}{|W|} = \frac{|Q_C|}{|Q_H| - |Q_C|}$$

4.4.3 Second Law of Thermodynamics

Kelvin–Planck form: No process can convert heat completely into work (no perfect heat engine).

Clausius form: No process can transfer heat from cold to hot with no other effect (no perfect refrigerator).

These two statements are equivalent.

4.4.4 The Carnot Cycle

Carnot constructed a conceptual engine from two isotherms and two adiabats:

> ⚠ TODO: Diagram
>
> PV diagram of Carnot cycle: four stages labeled A→B (isothermal expansion at T_H), B→C (adiabatic expansion), C→D (isothermal compression at T_C), D→A (adiabatic compression). Shaded area = net work.

1. **A → B**: Isothermal expansion at T_H (absorb Q_H)
2. **B → C**: Adiabatic expansion ($T_H \to T_C$)
3. **C → D**: Isothermal compression at T_C (reject Q_C)
4. **D → A**: Adiabatic compression ($T_C \to T_H$)

For the isothermal stages:

$$Q_{AB} = NkT_H \ln \frac{V_B}{V_A} > 0, \qquad Q_{CD} = NkT_C \ln \frac{V_D}{V_C} < 0$$

For the adiabatic stages, $PV^\gamma = \text{const}$ with $PV = NkT$ gives $TV^{\gamma-1} = \text{const}$:

$$T_H V_B^{\gamma-1} = T_C V_C^{\gamma-1}, \qquad T_H V_A^{\gamma-1} = T_C V_D^{\gamma-1}$$

Dividing: $\frac{V_B}{V_A} = \frac{V_C}{V_D}$, so the logarithms in Q_{AB} and Q_{CD} have the same magnitude. The efficiency:

$$e = 1 - \frac{|Q_C|}{|Q_H|} = 1 - \frac{T_C \ln(V_C/V_D)}{T_H \ln(V_B/V_A)} = 1 - \frac{T_C}{T_H}$$

$$\boxed{e_{\text{Carnot}} = 1 - \frac{T_C}{T_H}}$$

4.4.5 Carnot Refrigerator

Running the Carnot cycle in reverse gives a refrigerator with coefficient of performance:

$$\boxed{K_{\text{Carnot}} = \frac{T_C}{T_H - T_C}}$$

4.4.6 Carnot's Theorem

> The efficiency of any heat engine operating between two temperatures can never exceed the efficiency of a Carnot engine operating between the same two temperatures.

Proof by contradiction: Suppose an engine has efficiency $e' > e_{\text{Carnot}}$. Use it to drive a Carnot refrigerator. The combined system would transfer heat from cold to hot with no work input — violating the Clausius form of the second law.

The same argument shows that the Carnot refrigerator has the highest possible coefficient of performance.

4.4.7 Key Property of the Carnot Cycle

For any Carnot cycle:

$$\frac{|Q_H|}{T_H} = \frac{|Q_C|}{T_C}, \qquad \text{or equivalently} \qquad \frac{Q_H}{T_H} + \frac{Q_C}{T_C} = 0$$

The Carnot cycle is **reversible**. Any reversible process can be approximated as a sum of infinitesimal Carnot cycles, which means:

$$\oint \frac{\delta Q}{T} = 0 \quad \text{(for any reversible cycle)}$$

4.5 Entropy

4.5.1 Entropy as a State Function

Since $\oint \frac{\delta Q}{T} = 0$ for reversible processes, $\frac{\delta Q}{T}$ defines an exact differential for a state function — **entropy**:

$$dS = \frac{\delta Q_{\text{rev}}}{T}$$

The entropy change between two states A and B is:

$$\Delta S = \int_A^B \frac{\delta Q_{\text{rev}}}{T}$$

and is independent of the path taken (as long as we use a reversible path to calculate it).

4.5.2 Second Law in Terms of Entropy

For a reversible process, $\Delta S_{\text{system}} + \Delta S_{\text{surroundings}} = 0$.

For an irreversible process, consider heat flow: if $T_{\text{sys}} < T_{\text{surr}}$ (heat flows in), then $dQ_{\text{sys}} > 0$ and $dQ_{\text{surr}} < 0$, with $|dQ_{\text{surr}}/T_{\text{surr}}| < |dQ_{\text{sys}}/T_{\text{sys}}|$ since $T_{\text{surr}} > T_{\text{sys}}$. Therefore:

$$\boxed{\Delta S_{\text{universe}} \geq 0}$$

with equality only for reversible processes. This is the **second law of thermodynamics**: in any process between equilibrium states, the total entropy of the universe cannot decrease.

4.5.3 Entropy and Adiabatic Processes

For an adiabatic process, $\delta Q = 0$, so $dS = \delta Q/T = 0$. A reversible adiabatic process is therefore **isentropic** ($\Delta S = 0$).

4.5.4 The Carnot Cycle on a T–S Diagram

On a temperature-entropy (T–S) diagram, the Carnot cycle is a rectangle:

> ⚠ TODO: Diagram
>
> T–S diagram of Carnot cycle: rectangle with isotherms (horizontal lines at T_H and T_C) and adiabats (vertical lines). Shaded area = net work. $Q = \int T dS$ along isotherms.

The heat exchanged is:

$$Q = \int T dS = \text{area under curve on T–S diagram}$$

The net work equals the enclosed area of the cycle on the T–S diagram (just as it equals the enclosed area on the PV diagram).

4.6 Phase Diagrams

4.6.1 Ideal vs Real Gases

The ideal gas law $PV = NkT$ is an excellent approximation in many regimes but fails at **extremes** — very high pressure or very low temperature — where intermolecular forces and finite molecular volume matter. Under these conditions, gases can **liquefy**.

4.6.2 Phase Diagram

A **phase diagram** plots pressure vs temperature, showing the regions where each phase (solid, liquid, gas) is stable.

> ⚠ TODO: Diagram
>
> Phase diagram showing solid, liquid, and gas regions. Label the triple point, critical point, lines of coexistence (sublimation, melting, boiling), and the special features for water (negative slope of melting curve).

Key features:

- **Triple point**: the unique (P, T) where all three phases coexist in equilibrium
- **Critical point**: beyond this, no distinction between liquid and gas (supercritical fluid)
- **Lines of coexistence**: boundaries between phases (sublimation, melting, boiling lines)
- For **water**, the melting curve has a negative slope (ice is less dense than water), which is unusual

4.6.3 Phase Transitions on a T–S Diagram

During a constant-pressure phase change:

> ⚠ TODO: Diagram
>
> T–S diagram showing constant-pressure heating: temperature rises, then plateaus during phase change (latent heat absorbed at constant T increases entropy), then rises again.

- Temperature increases until the phase boundary is reached
- During the phase change, temperature remains constant while entropy increases (latent heat is absorbed: $\Delta S = Q/T = m\ell/T$)
- After the phase change completes, temperature rises again

The **van der Waals model** accounts for intermolecular attraction and finite molecular volume, providing a qualitative description of the gas-to-liquid transition.

5 Waves & Oscillations

5.1 Simple Harmonic Motion

5.1.1 What Is an Oscillation?

An oscillation is caused by a **restoring force** — a force that acts to return a system to some equilibrium position. At equilibrium the net force is zero; displace the system and it gets pushed back.

> ⚠ TODO: Diagram
>
> Potential energy well (U vs x) with a ball at the minimum. Show the restoring force arrows pointing inward on both sides of the equilibrium point.

Key variables:

Table 5.1: Oscillation Variables

Symbol	Meaning
T	Period — time for one complete oscillation
f	Frequency — cycles per second (Hz)
ω	Angular frequency (rad/s)
x_0	Amplitude — maximum displacement from equilibrium
ϕ	Phase constant
λ	Wavelength

The period and frequency are related by $T = 1/f$, and the angular frequency is $\omega = 2\pi f$.

5.1.2 The Simple Harmonic Oscillator

The simplest case: a restoring force proportional to displacement (Hooke's law):

$$F = -kx$$

Newton's second law gives:

$$m\ddot{x} = -kx \quad \implies \quad \ddot{x} + \frac{k}{m}x = 0$$

The general solution is:

$$\boxed{x(t) = x_0 \cos(\omega t + \phi), \qquad \omega = \sqrt{\frac{k}{m}}}$$

where x_0 and ϕ are set by initial conditions.

5.1.3 Velocity and Acceleration in SHM

Differentiating:

$$v(t) = \dot{x} = -x_0 \omega \sin(\omega t + \phi)$$

$$a(t) = \ddot{x} = -x_0 \omega^2 \cos(\omega t + \phi)$$

> ⚠ TODO: Diagram
>
> Three aligned graphs of x, v, and a vs t for SHM. Show x as a cosine, v as a negative sine (leading x by $\pi/2$), and a as a negative cosine (opposite phase to x). Label amplitudes x_0, $x_0\omega$, $x_0\omega^2$.

Note the relationships: v leads x by $\pi/2$, and a is exactly out of phase with x. Finding the minima and maxima of these functions is a standard exercise.

5.1.4 Potential Energy in SHM

From $F = -kx$, the potential energy is:

$$U = -\int F\,dx = \int_0^x kx'\,dx'$$

$$\boxed{U = \frac{1}{2}kx^2} = \frac{1}{2}kx_0^2 \cos^2(\omega t + \phi)$$

5.1.5 Energy Conservation in SHM

The total mechanical energy is conserved:

$$E_T = U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

Substituting the SHM solutions:

$$E_T = \frac{1}{2}kx_0^2 \cos^2(\omega t + \phi) + \frac{1}{2}m\omega^2 x_0^2 \sin^2(\omega t + \phi)$$

Since $m\omega^2 = k$, this simplifies to:

$$\boxed{E_T = \frac{1}{2}kx_0^2}$$

The total energy is constant and proportional to the square of the amplitude. Energy sloshes back and forth between kinetic and potential, but their sum never changes.

5.2 The Simple Pendulum

> ⚠ TODO: Diagram
>
> Simple pendulum: mass m hanging from a string of length L at angle θ from vertical. Show the weight mg decomposed into components along and perpendicular to the string. Label the arc length $x = L\theta$.

For a pendulum of length L at angle θ from vertical, the restoring force along the arc is:

$$F = -mg\sin\theta$$

For **small angles** ($\theta \lesssim 15°$), $\sin\theta \approx \theta$, so:

$$F \approx -mg\theta = -\frac{mg}{L}x$$

where $x = L\theta$ is the arc-length displacement. This is Hooke's law with effective spring constant $k_{\text{eff}} = mg/L$, giving:

$$\omega^2 = \frac{k_{\text{eff}}}{m} = \frac{g}{L}$$

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

Note that the period is independent of mass and (for small angles) independent of amplitude. This is why pendulums make good clocks.

5.3 Traveling Waves

5.3.1 What Is a Wave?

Waves transport **energy and information** without transporting matter. A mechanical wave travels through an elastic medium — particles in the medium oscillate about their equilibrium positions but do not travel with the wave.

Types:

- **Transverse** — particle displacement perpendicular to wave propagation (e.g., waves on a string, electromagnetic waves)
- **Longitudinal** — particle displacement parallel to wave propagation (e.g., sound waves, compression in a spring)

Waves can also be classified by:

- **Dimension** — 1D (string), 2D (water surface, concentric circles), 3D (sound, spherical wavefronts)
- **Periodicity** — periodic or pulse
- **Shape of wavefront** — plane, cylindrical, spherical

5.3.2 The Traveling Wave Equation

A sinusoidal wave traveling in the $+x$ direction:

$$\boxed{y(x,t) = y_0 \sin(kx - \omega t)}$$

where:

- y_0 = amplitude

- $k = \frac{2\pi}{\lambda}$ = wave number
- $\omega = 2\pi f$ = angular frequency

The **phase velocity** (speed of a given phase of the wave):

$$\boxed{v = \frac{\omega}{k} = \lambda f}$$

A snapshot at fixed t shows the spatial pattern; a fixed point at constant x oscillates in time.

5.3.3 Waves on a Stretched String

For a string under tension F_T with linear mass density μ (mass per unit length):

$$\boxed{v = \sqrt{\frac{F_T}{\mu}}}$$

This is one of the most important wave speed formulas. Greater tension means faster waves; greater mass density means slower waves.

5.3.4 Energy and Power in Waves

The kinetic energy density of a transverse wave on a string:

$$\frac{dK}{dx} = \frac{1}{2}\mu\omega^2 y_0^2 \cos^2(kx - \omega t)$$

The time-averaged power transmitted by the wave:

$$\boxed{P_{\text{avg}} = \frac{1}{2}\mu\omega^2 y_0^2 v}$$

5.3.5 Intensity

For waves spreading in 3D, the **intensity** I is the power per unit area:

$$I = \frac{P_{\text{avg}}}{A}$$

For a point source radiating uniformly in all directions (spherical wavefront):

$$I = \frac{P_{\text{avg}}}{4\pi r^2}$$

Intensity falls off as $1/r^2$ — the inverse square law. Since intensity is proportional to amplitude squared, the amplitude of a spherical wave falls off as $1/r$.

5.4 Superposition and Interference

5.4.1 The Superposition Principle

When two or more waves overlap, the resultant displacement at any point is the **sum of the individual displacements**:

$$y_{\text{total}} = y_1 + y_2 + \cdots$$

This holds for all linear wave equations (i.e., when the restoring force is proportional to displacement).

5.4.2 Interference

When waves of the same frequency overlap:

- **Constructive interference** — waves in phase, amplitudes add
- **Destructive interference** — waves out of phase, amplitudes cancel

5.5 Standing Waves

5.5.1 Formation

Two waves of equal amplitude and frequency traveling in **opposite directions** superimpose to form a standing wave:

$$y_1 = y_0 \sin(kx - \omega t), \qquad y_2 = y_0 \sin(kx + \omega t)$$

Using the product-to-sum identity:

$$\boxed{y(x,t) = 2y_0 \sin(kx) \cos(\omega t)}$$

This is the product of a spatial part $\sin(kx)$ (the "shape") and a temporal part $\cos(\omega t)$ (the "breathing"). The wave does not travel — it oscillates in place.

- **Nodes** — points of zero amplitude at all times ($\sin(kx) = 0$, i.e., $kx = n\pi$)
- **Antinodes** — points of maximum displacement ($\sin(kx) = \pm 1$, i.e., $kx = (n + \frac{1}{2})\pi$)

5.5.2 Reflection at Boundaries

- **Fixed boundary** — a transverse wave undergoes a phase change of π upon reflection (inverted)
- **Free boundary** — no phase change upon reflection

⚠ TODO: Diagram

Pulse reflection at fixed vs free boundaries. Show incoming and reflected pulses, with inversion at fixed end and no inversion at free end.

5.5.3 Resonance

A medium of finite length L supports standing waves only at specific frequencies — the **resonant frequencies** (harmonics).

Two fixed ends (or two free ends) — e.g., a stretched string:

$$L = n\frac{\lambda}{2}, \qquad n = 1, 2, 3, \ldots$$

$$\boxed{f_n = \frac{nv}{2L}}$$

The $n = 1$ mode is the **fundamental**; higher n are overtones.

One closed end, one open end — e.g., an organ pipe:

$$L = (2n-1)\frac{\lambda}{4}, \qquad n = 1, 2, 3, \ldots$$

$$\boxed{f_n = \frac{(2n-1)v}{4L}}$$

Only odd harmonics are present.

> ⚠ TODO: Diagram
>
> Standing wave patterns for the first three harmonics of a string fixed at both ends, and for a pipe closed at one end and open at the other. Label nodes and antinodes.

5.6 Beats

When two waves of **nearly equal frequency** are superimposed, they alternately go in and out of phase, producing a pulsating pattern called **beats**.

For two waves $y_1 = A\cos(\omega_1 t)$ and $y_2 = A\cos(\omega_2 t)$, the sum is:

$$y = y_1 + y_2 = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)$$

using the product-to-sum identity $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$.

> ⚠ TODO: Diagram
>
> Beat pattern: two waves of slightly different frequency (top), and their superposition showing the amplitude envelope (bottom). Label the beat frequency.

The result is a wave at the **average frequency** $\bar{f} = (f_1 + f_2)/2$, with an amplitude that oscillates at the **beat frequency**:

$$\boxed{f_{\text{beat}} = |f_1 - f_2|}$$

You hear one beat per cycle of the envelope. Since the envelope goes through two maxima per cycle of $\cos(\frac{\omega_1-\omega_2}{2}t)$, the beat frequency is the difference (not half the difference) of the two frequencies.

5.7 The Doppler Effect

The **Doppler effect** is the apparent shift in frequency due to relative motion between a source and an observer. Let V be the speed of sound in the medium.

5.7.1 Moving Source, Stationary Observer

A source moving at speed V_s toward the observer compresses the wavelength in front:

$$\lambda' = \lambda - \frac{V_s}{f} = \lambda\left(1 - \frac{V_s}{V}\right)$$

The observed frequency increases:

$$f' = \frac{V}{\lambda'} = f\frac{V}{V - V_s}$$

5.7.2 Moving Observer, Stationary Source

An observer moving at speed V_o toward the source encounters crests more frequently:

$$f' = \frac{V + V_o}{\lambda} = f\frac{V + V_o}{V}$$

5.7.3 General Doppler Formula

For both source and observer moving (with velocities taken as positive when toward each other):

$$\boxed{f' = f\frac{V - V_o}{V - V_s}}$$

where V_o is positive when the observer moves **away** from the source, and V_s is positive when the source moves **toward** the observer. (Sign convention: define the positive direction as from source to observer. Then $V_s > 0$ means the source chases the wave, reducing wavelength; $V_o > 0$ means the observer flees the wave, reducing observed frequency.)

Limiting cases:

- Source approaching ($V_s > 0$, $V_o = 0$): $f' > f$ — pitch increases
- Source receding ($V_s < 0$, $V_o = 0$): $f' < f$ — pitch decreases
- Observer approaching ($V_o < 0$, $V_s = 0$): $f' > f$
- $V_s \to V$: $f' \to \infty$ — sonic boom (Mach 1)

Part III

Electromagnetism & Optics

6 Electrostatics

6.1 Electric Charge

Charge is a fundamental, conserved property of particles. It comes in two types — positive and negative — and is measured in **coulombs** (C). The fundamental unit of charge is the electron charge:

$$e = 1.6 \times 10^{-19} \text{ C}$$

Charges exert forces on each other at a distance: like charges repel, unlike charges attract. These interactions can be dealt with using vectors.

6.2 Coulomb's Law

The force between two point charges q_1 and q_2 separated by distance r:

$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}}$$

where ε_0 is the **permittivity of free space** ($\varepsilon_0 \approx 8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$) and $\hat{r}$ points from one charge to the other. The force is positive (repulsive) for like charges and negative (attractive) for unlike charges.

Note the structural similarity to Newton's law of gravitation — both are inverse-square laws. Coulomb's law deals with charge; gravity deals with mass.

6.3 The Electric Field

Coulomb's law describes a force between two charges. But we often want an **intensive** property — one that doesn't depend on the test charge. Divide by the test charge to get the **electric field**:

$$\boxed{\vec{E} = \frac{\vec{F}}{q} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}}$$

The electric field is the force per unit charge at a point in space. It exists whether or not a test charge is present.

6.3.1 Superposition

The electric field obeys the **principle of superposition**: the total field from multiple charges is the vector sum of the individual fields:

$$\vec{E}_{\text{total}} = \sum_i \vec{E}_i = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{r}_i$$

6.3.2 Field Lines

Electric field lines are a visualization tool:

- Start at positive charges (sources), end at negative charges (sinks)
- Never cross
- Always hit conducting surfaces at 90°
- Closer lines indicate stronger field

⚠ TODO: Diagram

Electric field lines for: (1) a single positive charge (radially outward), (2) a dipole (field lines from + to −), (3) two like charges (lines repelling). Show density of lines proportional to field strength.

6.3.3 Conductors

Conductors allow charges to flow freely. In **electrostatic equilibrium**:

- The electric field inside a conductor is **zero** ($\vec{E} = 0$)
- All excess charge resides on the **surface**
- Charge clusters more densely at sharp points (higher curvature $\rightarrow$ stronger field)
- The field just outside the surface is perpendicular to it

6.3.4 Continuous Charge Distributions

For many charges, replace the sum with an integral. Define charge densities:

Table 6.1: Charge Density Notation

Symbol	Meaning	Units
λ	Linear charge density	C/m
σ	Surface charge density	C/m^2
ρ	Volume charge density	C/m^3

The field from a continuous distribution:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r^2} \hat{r}$$

6.4 Gauss's Law

6.4.1 Electric Flux

The **electric flux** through a surface measures how much field "flows through" it:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

where $d\vec{A}$ is the outward-pointing area element of a closed surface. When $\vec{E}$ is constant and parallel to $d\vec{A}$, this simplifies to $\Phi_E = EA$.

6.4.2 The Law

Gauss's law relates the flux through any closed surface to the enclosed charge:

$$\boxed{\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}}$$

This is always true, but it is most *useful* when symmetry makes $\vec{E}$ constant over the Gaussian surface and parallel (or perpendicular) to $d\vec{A}$.

Compare with the gravitational analogue: $\oint \vec{g} \cdot d\vec{A} = -4\pi G M_{\text{enc}}$.

6.4.3 Application: Point Charge

Choose a spherical Gaussian surface of radius r centered on charge q. By symmetry, $\vec{E}$ is radial and constant on the sphere:

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$$

This recovers Coulomb's law — Gauss's law contains it.

6.4.4 Application: Infinite Line Charge

For an infinite line with linear charge density λ, choose a cylindrical Gaussian surface of radius r and length L. The flux through the end caps is zero (field is parallel to caps); the curved surface contributes:

$$E(2\pi r L) = \frac{\lambda L}{\varepsilon_0}$$

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}}$$

The field falls off as $1/r$, not $1/r^2$ — the line charge is effectively one-dimensional.

> ⚠ TODO: Diagram
>
> Gaussian surfaces for: (1) spherical surface around a point charge, (2) cylindrical surface around an infinite line charge. Show $\vec{E}$ and $d\vec{A}$ vectors on the surfaces, with end-cap contributions labeled as zero for the cylinder.

6.4.5 Application: Charged Conducting Sphere

Inside ($r < R$): A Gaussian surface inside the conductor encloses no charge (all charge is on the surface), so $E = 0$.

Outside ($r > R$): The field is identical to that of a point charge Q at the center:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

6.4.6 Faraday Cage

A hollow conducting shell shields its interior from external electric fields. If a charge q is placed inside the cavity, it induces $-q$ on the inner surface and $+q$ on the outer surface, but external fields have no effect on the interior. This is the principle behind the **Faraday cage**.

6.5 Electric Potential

6.5.1 From Force to Potential Energy

The work done by the electric force on charge q moving through displacement $d\vec{s}$:

$$dW = \vec{F} \cdot d\vec{s} = q\vec{E} \cdot d\vec{s}$$

The potential energy change is $dU = -dW$, so:

$$\Delta U = -q \int_a^b \vec{E} \cdot d\vec{s}$$

6.5.2 Electric Potential (Voltage)

Divide by charge to get an **intensive** quantity — the **electric potential** V:

$$\boxed{V = -\int \vec{E} \cdot d\vec{s}}$$

The potential is a scalar (just add contributions!), which makes it much easier to work with than the vector field $\vec{E}$. The relationship between potential energy and potential is simply $U = qV$.

The potential is path-independent — it depends only on the endpoints. This is because the electrostatic force is **conservative**.

6.5.3 Potential of a Point Charge

For a point charge Q at the origin:

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}}$$

For multiple point charges, superposition gives:

$$V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i}$$

Since V is a scalar, there are no vector components to worry about — just add the numbers.

6.5.4 From Potential Back to Field

The electric field is the negative gradient of the potential:

$$\boxed{\vec{E} = -\nabla V}$$

In Cartesian coordinates:

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

In one dimension, $E = -dV/dx$.

6.5.5 Equipotentials

Surfaces of constant potential are called **equipotentials**. Key properties:

- $\vec{E}$ is always perpendicular to equipotential surfaces
- No work is done moving a charge along an equipotential
- A nonzero potential does *not* imply a nonzero field (e.g., inside a charged conductor V is constant but $E = 0$)

6.6 Capacitors

A **capacitor** stores charge (and energy) by maintaining a potential difference between two conductors.

6.6.1 Parallel-Plate Capacitor

Consider two parallel plates of area A separated by distance d, with surface charge density $\pm\sigma$. By Gauss's law (using a pillbox surface), the field between the plates is uniform:

$$E = \frac{\sigma}{\varepsilon_0}$$

The potential difference is:

$$V = Ed = \frac{\sigma d}{\varepsilon_0} = \frac{qd}{\varepsilon_0 A}$$

The **capacitance** — charge stored per unit voltage:

$$\boxed{C = \frac{q}{V} = \frac{\varepsilon_0 A}{d}}$$

Larger area and smaller separation give larger capacitance.

6.6.2 Energy Stored in a Capacitor

Building up charge on a capacitor requires work against the growing potential. Adding charge dq when the voltage is $V = q/C$:

$$dU = V dq = \frac{q}{C} dq$$

Integrating from 0 to Q:

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2}$$

6.7 Dielectrics

To increase capacitance without changing geometry, insert a **dielectric** — an insulating material that becomes polarized in an electric field.

6.7.1 Polarization

When an external field is applied to a dielectric, the molecular dipoles align with the field. This is called **polarization**. The internal dipoles cancel in the bulk, leaving a net surface charge that opposes the applied field.

The **polarization** $\vec{P}$ is the dipole moment per unit volume. The bound surface charge density is:

$$\sigma_{\text{bound}} = \vec{P} \cdot \hat{n}$$

6.7.2 Linear Dielectrics

For linear dielectrics, $\vec{P} = \chi_e \varepsilon_0 \vec{E}$, where χ_e is the **electric susceptibility**. The total field inside the dielectric is reduced:

$$E_{\text{total}} = E_0 - \frac{\sigma_{\text{bound}}}{\varepsilon_0} = \frac{E_0}{1 + \chi_e} = \frac{E_0}{\kappa}$$

where $\kappa = 1 + \chi_e$ is the **dielectric constant**. The permittivity of the material is:

$$\varepsilon = \kappa\varepsilon_0$$

With a dielectric, the parallel-plate capacitance becomes:

$$\boxed{C = \frac{\kappa\varepsilon_0 A}{d} = \kappa C_0}$$

The dielectric multiplies the capacitance by κ.

6.8 Electric Dipoles

An **electric dipole** consists of two equal and opposite charges $+q$ and $-q$ separated by distance d. The **dipole moment** is:

$$\vec{p} = q\vec{d}$$

where $\vec{d}$ points from the negative to the positive charge.

6.8.1 Torque on a Dipole

In a uniform external field $\vec{E}$, the net force on a dipole is zero, but there is a torque:

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

The torque acts to align the dipole with the field.

6.8.2 Potential Energy of a Dipole

The potential energy of a dipole in a uniform field:

$$\boxed{U = -\vec{p} \cdot \vec{E} = -pE\cos\theta}$$

Minimum energy (stable equilibrium) when $\vec{p} \parallel \vec{E}$ ($\theta = 0$); maximum energy when antiparallel ($\theta = \pi$).

6.8.3 Dipole Potential and Field

Far from the dipole ($r \gg d$), the potential is:

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}}$$

Note the $1/r^2$ dependence — faster falloff than a point charge ($1/r$).

The electric field of a dipole (in spherical coordinates):

$$\boxed{\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

This $1/r^3$ dependence (one power faster than a monopole's $1/r^2$) is characteristic of dipole fields. Along the axis ($\theta = 0$), $E = 2p/(4\pi\varepsilon_0 r^3)$; in the equatorial plane ($\theta = \pi/2$), $E = p/(4\pi\varepsilon_0 r^3)$.

7 Circuits & Magnetism

> **Status**
>
> Content to be transcribed from IPhO notes.

8 Optics

8.1 Geometrical Optics

8.1.1 The Ray Model of Light

When light encounters obstacles or apertures much larger than its wavelength, we can ignore wave effects (interference, diffraction) and treat light as traveling in straight lines called **rays**. This is the **ray model** — the foundation of geometrical optics.

At a boundary between two media, a ray can be:

- **Reflected** — bounced back into the same medium
- **Refracted** (transmitted) — bent as it passes into the new medium

8.1.2 Reflection

The **law of reflection**: the angle of incidence equals the angle of reflection, both measured from the normal to the surface:

$$\boxed{\theta_i = \theta_r}$$

> ⚠ TODO: Diagram
>
> Ray hitting a surface with incident and reflected rays. Show the normal line, θ_i and θ_r on either side of it.

Phase change upon reflection depends on the relative wave speeds:

- If $v_1 < v_2$ (going from slower to faster medium): reflected wave undergoes a phase change of π (inverted)
- If $v_1 > v_2$ (going from faster to slower medium): no phase change

8.1.3 Refraction and Snell's Law

When light crosses a boundary between two media, its speed changes. The **refractive index** n of a medium is the ratio of the speed of light in vacuum to the speed in the medium:

$$n = \frac{v_{\text{vacuum}}}{v}$$

Since frequency is set by the source and does not change across a boundary, only the wavelength and speed change.

Using Fermat's principle (light takes the path of minimum time) or geometry, we derive **Snell's law**:

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

where θ_1 and θ_2 are the angles from the normal in media 1 and 2, respectively.

> ⚠ TODO: Diagram
>
> Refraction at a boundary: incident ray in medium n_1, refracted ray bending toward/away from normal in medium n_2. Show angles θ_1 and θ_2.

8.1.4 Total Internal Reflection

When light travels from a denser medium ($n_1 > n_2$) to a less dense one, there is a **critical angle** θ_c beyond which all light is reflected — no transmission occurs:

$$\sin\theta_c = \frac{n_2}{n_1} \quad (\text{only for } n_1 > n_2)$$

$$\boxed{\theta_c = \arcsin\left(\frac{n_2}{n_1}\right)}$$

At $\theta_1 = \theta_c$, the refracted ray travels along the surface ($\theta_2 = 90°$). For $\theta_1 > \theta_c$, total internal reflection occurs. This is the principle behind **optical fibers** — light bounces along the core by repeated total internal reflection.

8.1.5 Fermat's Principle

> A light ray traveling from one point to another follows a path such that the time required is a minimum (or maximum) compared to other paths.

Mathematically: $dt/dx = 0$ along the ray path. Fermat's principle can be used to derive both the law of reflection and Snell's law.

8.2 Image Formation

Geometrical optics deals with the formation of images using reflection (mirrors) and refraction (lenses). We can understand optical instruments using **ray tracing**.

8.2.1 Objects and Images

- **Objects** — things that emit or reflect light (the source)
- **Images** — what you see (light on the back of your retina)
  - **Real image** — light actually converges there; can be projected on a screen
  - **Virtual image** — light appears to diverge from there, but doesn't actually pass through; cannot be projected
- **Magnification** — ratio of image size to object size; can be > 1 or < 1

8.2.2 Plane Mirrors

A plane mirror produces a virtual image that is:

- The same size as the object ($|M| = 1$)
- The same distance behind the mirror as the object is in front
- Laterally inverted (left–right reversal)

The minimum mirror length needed to see your full body is **half your height** — the mirror only needs to span from halfway between your eyes and the top of your head to halfway between your eyes and your feet.

Multiple mirrors:

- Two mirrors at right angles produce 3 images
- Two parallel mirrors produce infinitely many images (fading with distance)

8.3 Curved Mirrors

8.3.1 Concave Mirrors

A concave (converging) mirror has a focal length related to its radius of curvature:

$$f = \frac{R}{2}$$

⚠ TODO: Diagram

Concave mirror showing center of curvature C, focal point F, and principal axis. Show how parallel rays converge at F.

Ray diagrams — three special rays from the top of the object:

1. **Parallel ray** — reflects through the focal point
2. **Focal ray** — passes through F, reflects back parallel to the axis
3. **Radial ray** — travels through the center of curvature C, reflects back on itself

Any two of these rays locate the image.

Three cases for a concave mirror:

1. Object beyond C — image is real, inverted, reduced
2. Object between C and F — image is real, inverted, magnified
3. Object inside F — image is virtual, upright, magnified

8.3.2 The Mirror Equation

$$\boxed{\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}}$$

where s is the object distance, s' is the image distance, and f is the focal length.

Sign conventions:

- For a concave mirror, f is positive
- For a convex mirror, f is negative
- If $s' > 0$, the image is on the same side as the object (real)
- If $s' < 0$, the image is on the opposite side (virtual)

8.3.3 Magnification

$$\boxed{M = -\frac{s'}{s}}$$

- $M > 0$: image is upright
- $M < 0$: image is inverted
- $|M| > 1$: image is magnified
- $|M| < 1$: image is reduced

8.3.4 Convex Mirrors

A convex (diverging) mirror always produces a virtual, upright, reduced image. The focal point is behind the mirror ($f < 0$).

> ⚠ TODO: Diagram
>
> Convex mirror ray diagram showing virtual image formation. The reflected rays diverge, and the virtual image appears behind the mirror.

8.4 Thin Lenses

8.4.1 Refraction at Curved Surfaces

When light refracts through a curved surface, the sign conventions are:

- s is positive on the incident (incoming) side
- s' is positive on the transmission side
- R is positive for a convex surface, negative for concave

8.4.2 The Lensmaker's Equation

For a thin lens with refractive index n in air, with radii of curvature R_1 and R_2:

$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

The **diopter** $D = 1/f$ (in m^{-1}) measures the optical power of a lens.

- **Converging lens** (double convex): $f > 0$ — brings parallel rays to a focus

- **Diverging lens** (double concave): $f < 0$ — spreads parallel rays apart

8.4.3 The Thin Lens Equation

The same equation as for mirrors:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

with magnification $M = -s'/s$.

8.4.4 Ray Tracing for Lenses

Three special rays from the top of the object:

1. **Parallel ray** — refracts through the focal point on the far side
2. **Focal ray** — passes through F on the near side, refracts parallel to the axis
3. **Central ray** — passes straight through the center of the lens (undeviated, because the surfaces are locally parallel)

⚠ TODO: Diagram

Converging lens ray diagram showing the three special rays forming a real, inverted image. Also show the case where the object is inside F, producing a virtual, upright, magnified image.

8.4.5 The Eye and Vision

The eye is a lens system that focuses images on the retina. The ciliary muscles adjust the lens curvature to focus at different distances (**accommodation**).

- **Short-sighted** (myopia) — eye too long or lens too strong; corrected with a diverging lens
- **Long-sighted** (hyperopia) — eye too short or lens too weak; corrected with a converging lens

People naturally become long-sighted with age as the lens stiffens.

8.4.6 Optical Instruments

Combination of lenses: line up images in cascade — the image of the first lens becomes the object of the second.

Refracting telescope: two converging lenses separated by $L = f_1 + f_2$ (matching focal points). The angular magnification is:

$$m = \frac{f_1}{f_2}$$

where f_1 is the objective focal length and f_2 is the eyepiece focal length. For high magnification, use a long-focal-length objective and a short-focal-length eyepiece.

Reflecting telescope: uses a concave mirror instead of an objective lens. Advantages:

- No **chromatic aberration** (different wavelengths refract at different angles through a lens, but all reflect identically from a mirror)
- Mirrors can be made larger and are structurally stronger

8.5 Physical Optics: Interference

When the size of obstacles or apertures is comparable to the wavelength of light, we must account for the wave nature of light. This is **physical optics**.

8.5.1 Light as an Electromagnetic Wave

Light is a transverse wave of coupled electric and magnetic fields. In vacuum:

$$v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

The **E** and **B** fields are perpendicular to each other and to the direction of propagation. As with all waves, the traveling wave solution is:

$$E(x, t) = E_0 \sin(kx - \omega t)$$

The **intensity** (time-averaged power per unit area) is proportional to the square of the electric field amplitude:

$$I = \frac{1}{2} \varepsilon_0 c E_0^2$$

Note the factor of 1/2 from time-averaging. What we actually see (and what detectors measure) is the intensity, not the field.

8.5.2 Huygens' Construction

Every point on a wavefront acts as a source of secondary spherical wavelets. The envelope of these wavelets at a later time gives the new wavefront.

This principle explains:

- **Diffraction** — bending of waves around obstacles and through apertures
- **Refraction** — change in direction when wavelet speeds change at a boundary

8.5.3 Coherence

For interference to be observable, the light sources must be **coherent** — they must maintain a constant phase relationship $\delta(t)$ over time. Lasers are coherent; incandescent bulbs are not.

8.5.4 Superposition and Interference

When two coherent waves overlap:

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

Any waveform can be built up from a set of simple sinusoidal waves (**Fourier analysis**). For two waves of the same amplitude and frequency but with a phase difference ϕ:

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

- $\phi = 0$: **constructive interference** — amplitude doubles
- $\phi = \pi$: **destructive interference** — amplitude is zero
- Intermediate ϕ: something in between

8.5.5 Phasors

A **phasor** is a rotating vector of length E_0 that rotates at frequency ω. The projection onto one axis gives the instantaneous field:

$$E(t) = E_0 \sin(\omega t)$$

To add two waves, **add their phasors as vectors**. The resultant phasor gives the amplitude and phase of the combined wave. This is far easier than trigonometric manipulation, especially for many waves.

8.5.6 The Double Slit

> ⚠ TODO: Diagram
>
> Double slit setup: monochromatic light incident on two slits separated by d. Show path difference $d \sin\theta$ to a point P on a distant screen at angle θ.

Two narrow slits separated by distance d, illuminated by coherent monochromatic light. The **path difference** to a distant point at angle θ is:

$$\Delta = d \sin\theta$$

- **Constructive interference** (bright fringes): $d \sin\theta = m\lambda$, where $m = 0, \pm 1, \pm 2, \ldots$
- **Destructive interference** (dark fringes): $d \sin\theta = (m + \frac{1}{2})\lambda$

Using phasors, the phase difference between the two waves is:

$$\phi = \frac{2\pi d \sin\theta}{\lambda}$$

Adding two phasors of equal amplitude E_0 at angle ϕ using the cosine rule gives a resultant amplitude $E_R = 2E_0 \cos(\phi/2)$. Since intensity goes as amplitude squared:

$$\boxed{I = 4I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)}$$

where I_0 is the intensity from a single slit.

8.5.7 Diffraction Gratings

A **diffraction grating** is a large number N of closely spaced slits (typically hundreds or thousands per mm). The maxima occur at the same positions as the double slit:

$$d \sin\theta = m\lambda, \qquad m = 0, \pm 1, \pm 2, \ldots$$

But with N slits, the peaks become much **sharper**. The angular half-width of each peak is:

$$\boxed{\delta\theta = \frac{\lambda}{N d \cos\theta}}$$

More slits means sharper peaks and better wavelength resolution.

⚠ TODO: Diagram

Intensity patterns for $N = 2$, 3, and many slits. Show how peaks narrow and secondary maxima appear as N increases. Include phasor polygons for each case.

Phasor picture: For N slits, the phasors form a polygon. At the first minimum, the phasors close into a regular polygon — the total phase accumulated is 2π, giving $\phi_{\text{min}} = 2\pi/N$ per slit.

8.5.8 Spectrometers and Dispersion

A diffraction grating acts as a **spectrometer** — it separates light by wavelength. The angular separation of different wavelengths is characterized by the **dispersion**:

$$\boxed{D = \frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta}}$$

where m is the diffraction order. Note that dispersion does not depend on N — more slits make the peaks sharper, but don't move them.

The **resolving power** of a grating is its ability to distinguish two closely spaced wavelengths:

$$\boxed{R = \frac{\lambda}{\Delta\lambda} = Nm}$$

where N is the number of slits and m is the order. To resolve finer wavelength differences, use more slits or work at higher order.

8.5.9 Thin Film Interference

When light reflects from a thin film (e.g., a soap bubble or oil slick), interference occurs between rays reflected from the top and bottom surfaces.

> ⚠ TODO: Diagram
>
> Thin film of thickness t and refractive index n_2 between media n_1 and n_3. Show the two reflected rays and the path difference.

Key considerations:

- A ray reflecting from a boundary going to a higher n gets a phase change of π (equivalent to adding $\lambda/2$ to the path)
- A ray reflecting from a boundary going to a lower n has no phase change
- The path difference through the film is $2t$ (multiplied by n_2 for the optical path)

For a film of refractive index n and thickness t:

- If only one surface gives a phase change: constructive interference when $2nt = m\lambda$
- If both or neither surfaces give a phase change: constructive interference when $2nt = (m + \frac{1}{2})\lambda$

Anti-reflection coatings: Choose n and t so that the reflected rays interfere destructively. For a quarter-wave coating ($t = \lambda/(4n)$), reflected light is suppressed.

8.5.10 The Michelson Interferometer

A **beam splitter** divides a beam into two paths. Each beam reflects from a mirror and returns. The recombined beams interfere, and the pattern depends on the path difference $d_2 - d_1$.

> ⚠ TODO: Diagram
>
> Michelson interferometer: source, beam splitter at 45°, two mirrors at right angles, and detector. Show the two beam paths.

Moving one mirror by $\lambda/2$ shifts the pattern by one fringe. This allows extremely precise distance measurements — down to fractions of a wavelength.

8.6 Diffraction

Diffraction occurs when waves encounter obstacles or apertures comparable in size to their wavelength. While interference deals with a finite number of discrete sources, diffraction treats a continuous distribution.

8.6.1 Huygens' Principle and X-Ray Diffraction

Huygens' principle: every point on a wavefront is a source of secondary wavelets. For a large object, the shadow behind it is sharp. For a small object (comparable to λ), waves spread out — diffraction is significant.

X-ray diffraction uses short-wavelength radiation ($\lambda \sim 0.1$ nm) to probe crystal structure. **Bragg's law** gives the condition for constructive interference from crystal planes separated by spacing d:

$$\boxed{m\lambda = 2d\sin\theta}$$

where θ is measured from the crystal planes (not the normal). This can determine crystal structure and atomic spacing.

8.6.2 Single Slit Diffraction

⚠ TODO: Diagram

Single slit of width D with Fraunhofer diffraction. Show incoming plane wave, the slit, and the diffraction pattern on a distant screen. Label the central maximum and first minima.

For a single slit of width D, **Fraunhofer diffraction** (far-field, $L \gg D$) gives minima at:

$$\boxed{D\sin\theta = m\lambda, \qquad m = \pm 1, \pm 2, \ldots}$$

Note: $m = 0$ is the central maximum, not a minimum.

Phase difference between rays from opposite edges of the slit:

$$\Delta\phi = \frac{2\pi D\sin\theta}{\lambda}$$

8.6.3 Single Slit Intensity

Using the phasor approach: divide the slit into infinitely many point sources. At $\theta = 0$, all phasors line up (maximum intensity). At the first minimum, the phasors curl into a complete circle (net amplitude zero). At the second minimum, they form two complete circles.

For an arbitrary angle, the phasor integration gives the intensity:

$$\boxed{I = I_m \left[\frac{\sin(\alpha)}{\alpha}\right]^2, \qquad \alpha = \frac{\pi D \sin\theta}{\lambda}}$$

where I_m is the intensity at the central maximum. The function $\mathrm{sinc}^2(\alpha) = [\sin(\alpha)/\alpha]^2$ has:

- Central maximum at $\alpha = 0$ (using L'Hôpital: $\sin\alpha/\alpha \to 1$)
- Zeros at $\alpha = m\pi$, i.e., $D\sin\theta = m\lambda$
- Secondary maxima that are much weaker than the central peak

8.6.4 Combined Double Slit and Diffraction

In reality, each slit has a finite width D, so the double-slit pattern is **modulated** by the single-slit diffraction envelope:

$$I = I_m \left[\frac{\sin\alpha}{\alpha}\right]^2 \cos^2\left(\frac{\phi}{2}\right)$$

where $\alpha = \pi D \sin\theta/\lambda$ (single-slit envelope) and $\phi = 2\pi d \sin\theta/\lambda$ (double-slit interference).

8.6.5 Circular Aperture: The Airy Disk

Telescopes and microscopes have circular apertures, so diffraction produces a circular pattern called the **Airy disk** instead of the linear pattern from a slit.

The first minimum of the Airy disk occurs at:

$$\boxed{\sin\theta = 1.22\frac{\lambda}{d}}$$

where d is the diameter of the aperture. The factor 1.22 (compared to 1.00 for a slit) comes from the geometry of the circular aperture (Bessel functions).

8.6.6 The Rayleigh Criterion

Two point sources are just **resolved** when the central maximum of one falls on the first minimum of the other. This gives the minimum angular separation:

$$\theta_R \approx 1.22\frac{\lambda}{d}$$

For **telescopes**: increase d (larger aperture) to resolve finer details. For **microscopes**: use a smaller λ (e.g., electron microscopes, where the de Broglie wavelength is much smaller than visible light).

8.7 Polarization

8.7.1 What Is Polarization?

Light is a transverse wave — the electric field oscillates perpendicular to the direction of propagation. **Polarization** describes the direction of this oscillation.

Most light sources (sun, incandescent bulbs) are **unpolarized** — the electric field direction varies randomly. Waves from a single antenna or laser, however, are **polarized** (the field oscillates in a preferred direction).

> ⚠ TODO: Diagram
>
> Unpolarized light (random field directions) vs linearly polarized light (field oscillates in one plane). Show the propagation direction and field vectors.

Methods of producing polarized light:

1. **Absorption** (Polaroid sheets)
2. **Reflection** (Brewster's angle)
3. **Scattering** (sunlight)
4. **Birefringence** (double refraction in crystals)

8.7.2 Polarization by Absorption

A **Polaroid sheet** contains long-chain molecules that absorb the component of the electric field parallel to the chains, transmitting only the component perpendicular to them (the **transmission axis**).

For **unpolarized** light incident on a single polarizer, the transmitted intensity is:

$$I = \frac{I_0}{2}$$

because on average, half the intensity is in each polarization direction.

8.7.3 Malus's Law

For **already polarized** light passing through a second polarizer (analyzer) at angle θ to the polarization direction:

$$\boxed{I = I_0 \cos^2 \theta}$$

This is **Malus's law**. When the polarizers are parallel ($\theta = 0$), all light passes. When crossed ($\theta = 90°$), no light passes.

Three polarizers: Surprisingly, inserting a third polarizer between two crossed polarizers at an intermediate angle can transmit some light. For example, with the middle polarizer at 45°:

$$I = I_0 \cdot \frac{1}{2} \cdot \cos^2 45° \cdot \cos^2 45° = \frac{I_0}{8}$$

8.7.4 Polarization by Reflection: Brewster's Angle

When unpolarized light reflects from a surface, the reflected light is partially polarized. At a special angle called **Brewster's angle**, the reflected light is **completely polarized** (perpendicular to the plane of incidence):

$$\boxed{\theta_B = \arctan\left(\frac{n_2}{n_1}\right)}$$

At Brewster's angle, the reflected and refracted rays are perpendicular ($\theta_1 + \theta_2 = 90°$).

Practical application: polarized sunglasses block horizontally polarized glare from reflective surfaces (roads, water) because reflected light is preferentially polarized in the horizontal plane.

8.7.5 Polarization by Scattering

When light scatters off air molecules, the scattered light is partially polarized. The electric field of the incoming wave causes the molecule to oscillate, and the re-radiated light is polarized in the plane perpendicular to the scattering direction.

This explains why the sky appears polarized — looking at 90° from the sun, the scattered light is most strongly polarized.

Scattering also explains why the **sky is blue**: the scattering intensity goes as $1/\lambda^4$ (Rayleigh scattering), so shorter wavelengths (blue) scatter much more than longer wavelengths (red).

8.7.6 Birefringence

In some crystalline materials (e.g., calcite), the refractive index depends on the polarization direction of the light. This is **birefringence** (double refraction).

- **Ordinary ray** (n_o) — refractive index is the same in all directions; obeys Snell's law normally
- **Extraordinary ray** (n_e) — refractive index varies with direction relative to the **optic axis**; does not obey Snell's law in the usual sense

The two rays travel at different speeds through the crystal, acquiring a relative phase difference.

8.7.7 Circular Polarization and Quarter-Wave Plates

When two perpendicular linearly polarized components are out of phase, the result is **elliptically polarized light**. The tip of the electric field vector traces an ellipse as the wave propagates.

Special cases:

- $\phi = 0$ or π: linearly polarized
- $\phi = \pm\pi/2$ with equal amplitudes: **circularly polarized** — the field vector rotates in a circle
- Left or right circular polarization depends on which component leads

A **quarter-wave plate** is a birefringent slab whose thickness is chosen so that the ordinary and extraordinary rays acquire a relative phase shift of $\pi/2$ (a quarter wavelength). Sending linearly polarized light at 45° to the optic axis through a quarter-wave plate produces circularly polarized light, and vice versa.

Part IV

Modern Physics

9 Modern Physics & Relativity

Status

Content to be transcribed from IPhO notes.

References