Think Like A Physicist

Casey Handmer Jonathan Whitmore

2024-01-01

Table of contents

Foreword

First Principles Thinking is a teachable and incredibly powerful algorithm for efficiently solving real world problems and building the future.

It is the best algorithm I know to make a brain useful.

These notes are compiled from the International Physics Olympiad Australian training program held in Canberra c. 2005. I have found, over the intervening 18 years, that the skills and knowledge taught during this intensive two week program have formed the basis of my ability to apply physics to everyday problems.

I have assembled these notes with the intention to create the most singularly terse summary of undergraduate physics ever written, accessible to motivated professionals who know how to Google. Necessarily, they do not cover the lab or test-taking aspects of the Olympiad training program.

I hope you find thinking like a physicist to be as rewarding as I do.

— Casey Handmer, 25 November, 2023.

Part I

Foundations

1 How to Think Like a Physicist

1.1 SI Units

Table 1.1: SI Base Units

Base Quantity	Base Unit Name	Symbol
Length	meter	m
Time	second	s
Mass	kilogram	kg
Electric Current	ampere	A
Thermodynamic Temperature	kelvin	K
Amount of Substance	mole	mol
Luminous Intensity	candela	cd

The five fundamental units are length (meter, m), time (second, s), mass (kilogram, kg), charge (coulomb, C), and temperature (kelvin, K). Ampere (A), mole (mol), and candela (cd) are derived from these base units.

All other units are derived from combinations of these fundamental units.

Table 1.2: Derived Units

Concept	Unit	Derivation
Force	Newton (N)	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$
Energy	Joule (J)	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
Power	Watt (W)	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$
Current	Ampere (A)	$\mathrm{C} \cdot \mathrm{s}^{-1}$

1.2 Dimensional Analysis and Sanity Checking

Vital tool for both checking (always!!) and fudging. It is part of a consistency check — units on both sides of the equation must be the same. A successful dimension check is followed by a :).

Similarly, equations with vectors must have the same vector shape on each side.

Math and formulas are a tool, not the final authority. Never formula fit. Understand what is going on and check that the answers are sensible.

1.3 Solving Problems with the 7 Ds and the little s

Using this basic algorithm to approach problems from first principles is the single biggest update you are likely to make to your thinking, ever.[1]

1. **Diagrams** — Big, 2/3 page, as many as you need. Load the problem into your GPU.
2. **Directions** — Mark it (negative/positive).
3. **Definitions and Given Data** — Put it on the page (all of them).
4. **Diagnosis** — How? What type of problem is this? E.g. conservation principles, force laws, angular momentum.
5. **Derivation** — Write down the fundamental equations, the diagnosis expressed as symbols. You will need as many equations as variables. Add to the diagram if necessary. Check dimensions.
6. **Determination** (or D'algebra) — Math manipulations to get the answer. Box it!
7. **Dimensions** — **Check dimensions and limiting cases/sanity check**. $LHS = RHS$ then :).
8. **substitutions** — Only if necessary, do rough calculation by hand and check units, include an error term.

Example Problem 1

A body, mass m, slides without friction from rest at height H to lower height h, colliding with a horizontal spring of strength k. By how much does it compress?

> ⚠ TODO: Diagram
>
> Diagram showing: (1) Initial state — mass m at height H on a frictionless ramp. (2) Final state — mass at height h compressing a horizontal spring by distance x. Include direction arrows for gravity and coordinate axes.

Definitions & Data

[1] A more thorough exploration of this algorithm can be found at Casey Handmer's blog post In Space, No One Can Reason By Analogy.

Variable	Description
m	Mass of object
H	Initial height
h	Final height
k	Spring constant
g	Acceleration due to gravity ($9.81\,\mathrm{m/s}^2$)
x	Compression of spring
U_{gpe}	Gravitational potential energy
U_E	Elastic potential energy
E	Energy

Diagnosis

Conservation of Energy (E is conserved).

$$U_{gpe} \to U_E$$

$$mgH = mgh + \frac{1}{2}kx^2$$

Derivation

$$F = -kx$$

$$\begin{aligned} U &= -\int F\,\mathrm{d}x \\ &= -\int_{x_i}^{x_f} kx\,\mathrm{d}x \\ &= \frac{1}{2}kx_f^2 \end{aligned}$$

Determination

$$mg\left(H-h\right) = \frac{1}{2}kx^2$$

$$\frac{2mg\left(H-h\right)}{k} = x^2$$

$$\boxed{x = \sqrt{\frac{2mg(H-h)}{k}}}$$

Dimensions

$$\begin{aligned} L &= \sqrt{\frac{MLT^{-2}\,(L)}{MT^{-2}}} \\ &= \sqrt{L^2} = L \quad \checkmark \end{aligned}$$

Limiting Cases

- $H \to h$: $H - h = 0 \implies x = 0$ — makes sense!
- $m \to 0 \implies x \to 0$ — makes sense!
- $k \to \infty \implies x \to 0$ — makes sense!
- $k \to 0 \implies x \to \infty$ — makes sense!

Substitution

Not needed.

Example Problem 2

A man (mass m) is pulling a piano (mass M) up a hill (angle θ to horizontal), using a pulley. The coefficient of kinetic friction is μ_k.

> ⚠ TODO: Diagram
>
> Diagram showing: (1) Inclined plane at angle θ. (2) Piano (mass M) on the incline connected by string over pulley to man (mass m) hanging vertically. Include direction arrows and coordinate axes.

> ⚠ TODO: Free Body Diagrams
>
> (1) Man: weight mg down, tension T up. (2) Piano: weight Mg decomposed into components along/perpendicular to incline, normal force N, friction $\mu_k N$ opposing motion, tension T up the incline.

Definitions & Data

- m: Mass of man (kg)
- M: Mass of piano (kg)
- θ: Angle of incline from horizontal
- μ_k: Coefficient of kinetic friction
- g: Acceleration due to gravity
- T: Tension in string
- N: Normal force
- a: Acceleration of the system

Diagnosis

Forces: free body diagram. Conservation of string.

- Man: $F = ma$
- Piano: component of gravity along incline, normal force, friction $f_k = \mu_k N$, tension T.

Derivation

For the man:

$$ma = mg - T$$

$$T = m(g - a)$$

For the piano:

$$N = Mg\cos\theta$$

$$\sum F = Mg\sin\theta + \mu_k N - mg$$

Determination

$$\boxed{a = g\left(\frac{M\sin\theta + \mu_k M\cos\theta - m}{M + m}\right)}$$

Dimensions

$$LHS = LT^{-2}, \quad RHS = LT^{-2}\left(\frac{M + M - M}{M}\right) \quad \checkmark$$

Limiting Cases

- If $M \gg m$: acceleration to the right (piano dominates). Makes sense.
- If $m \gg M$: acceleration negative (man pulls piano up). Makes sense.
- $\theta = 0$: acceleration depends on friction vs man's weight. Makes sense.
- $\theta = 90°$: reduces to Atwood machine. Makes sense.

This working includes an error caught during the limiting cases check, and left in to show how this works.

2 Math Toolkit

2.1 Differentiation — Chain Rule

a)

$$\frac{d}{dr}\left(6r^3(\cos 2r + 5r)^7\right) = 6r^2(\cos 2r + 5r)^6(3\cos 2r + 2r(25 - 7\sin 2r))$$

b)

$$x = (5te^{2t+1} + 2)^8$$

$$v = \frac{dx}{dt} = 40e^{2t+1}(1 + 2t)(5te^{2t+1} + 2)^7$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 40e^{2t+1}(5te^{2t+1} + 2)^6\left(8(1 + t) + (7 + 32t(1 + t))5e^{2t+1}\right)$$

2.2 Partial Derivatives

Simply take derivatives with respect to each variable, but consider extreme points if finding maxima or minima.

$$f(x, y) = 5xy + ye^x$$

$$\frac{\partial}{\partial x}f(x, y) = 5y + ye^x$$

$$\frac{\partial}{\partial y}f(x, y) = 5x + e^x$$

2.3 Integration

Substitution

$$\int_{1/2}^{1} \sqrt{1-x^2}\,dx \quad \text{Let } x = \cos u, \quad dx = -\sin u\,du$$

$$\Rightarrow I = \int_{\pi/2}^{\pi/6} \sqrt{1-\cos^2 u}\cdot(-\sin u)\,du = \int_{\pi/6}^{\pi/2} \sin^2 u\,du$$

Using $\cos 2u = 1 - 2\sin^2 u$:

$$I = \frac{1}{2}\int_{\pi/6}^{\pi/2} (1-\cos 2u)\,du = \frac{1}{2}\left[-\frac{\sin 2u}{2} - u\right]_{\pi/6}^{\pi/2} = \frac{\sqrt{3}}{8} - \frac{\pi}{6}$$

Integration by Parts

Basically the inverse of the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int \tan^{-1} x\,dx$$

Let $u = \tan^{-1} x$, $dv = dx$.

$$x = \tan u, \quad \frac{dx}{du} = 1 + \tan^2 u = 1 + x^2$$

$$du = \frac{1}{1+x^2}\,dx, \quad v = x$$

$$\Rightarrow \int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

2.4 Kinematic Equations

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Set a constant and $v(t=0) = v_0 = u$:

$$dv = a\,dt, \quad \int_u^v dv = \int_0^t a\,dt, \quad \boxed{v = u + at}$$

$$dx = (u + at)\,dt, \quad \int_{x_0}^x dx = \int_0^t (u + at)\,dt, \quad \boxed{x = x_0 + ut + \frac{1}{2}at^2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$$

$$a\,dx = v\,dv, \quad \int_0^x a\,dx = \int_u^v v\,dv, \quad \boxed{v^2 = u^2 + 2ax}$$

2.5 Differential Equations

There are infinite varieties of DEs, but nearly all physically relevant ones can be solved quickly by inspection. Here is a very incomplete summary of linear equations.

First Order: Rearrange and integrate.

Second Order: Simple Harmonic Motion!

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Guess the answer of the form:

$$x = A\cos\omega t + B\sin\omega t$$

or

$$x = Ae^{i\omega t + \phi} = Ae^{i\omega t} + Be^{-i\omega t}$$

For example, springs. Nearly everything in nature is a spring of one form or another.

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

So by inspection, $\omega^2 = k/m$.

Another example: Solve $v = ktx$.

$$\frac{dx}{dt} = ktx, \quad \frac{dx}{x} = kt\,dt, \quad \int_{x_0}^{x} \frac{dx}{x} = k\int_0^t t\,dt$$

$$2\ln(x/x_0) = kt^2, \quad \boxed{x = x_0 e^{kt^2/2}}$$

2.6 Coordinate Systems

Plane Polar Coordinates (2D)

For a point P in the plane, the polar coordinates (r, θ) are related to the Cartesian coordinates (x, y) by:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

or inversely:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Cylindrical Polar Coordinates

For axial symmetry:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Spherical Polar Coordinates

For spherical symmetry, the coordinates are defined as:

$$0 \le r, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi$$

where r is the radius, θ is the polar angle, and ϕ is the azimuthal angle.

2.7 Multivariate Integration

Univariate integration is one dimensional:

$$\int f'(x)\,dx = f(x) + C$$

Areas and volumes require multivariate integration.

$$\text{Area} = \iint_A dA = \iint_A dx\,dy = \int_c^d \left(\int_a^b dx \right) dy = (b-a)(d-c)$$

Polar Coordinates

For an infinitesimal area element dA in polar coordinates:

$$dA = r\,dr\,d\theta$$

Volumes

The volume of a sphere:

$$\text{Volume} = \iiint_V dV = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta\,dr\,d\theta\,d\phi = \int_0^R r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi \cdot \frac{R^3}{3} \cdot [-\cos\theta]_0^\pi = \frac{4}{3}\pi R^3$$

Flux Integrals

One can also integrate an arbitrary scalar or vector function over some area or volume.

Vector field integrations are performed over areas to find fluxes, and are only concerned with the normal (perpendicular) component of the field:

$$I = \iint \vec{f} \cdot \hat{n}\,dA$$

Part II

Classical Physics

3 Mechanics

3.1 Energy, Conservation, and Forces

3.1.1 Energy

Energy is the ability to do work — that which is conserved. It comes in two broad types:

- **Kinetic energy** — energy of motion: $K = \frac{1}{2}mv^2$
- **Potential energy** — stored energy associated with position or configuration

Energy has units of $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} = \mathrm{J}$ (joules). It is a scalar.

There are many forms of potential energy, each associated with a fundamental force:

Table 3.1: Forms of Potential Energy

Force	Potential Energy
Gravitational	$U = mgh$ (near surface), $U = -\frac{Gm_1m_2}{r}$ (general)
Elastic (spring)	$U = \frac{1}{2}kx^2$
Electrostatic	$U = \frac{kq_1q_2}{r}$
Weak nuclear	(particle physics)
Strong nuclear	(particle physics)

3.1.2 Potential Energy Diagrams

A potential energy diagram plots $U(x)$ vs position x. Key features:

- **Equilibrium points** occur where $F = -\frac{dU}{dx} = 0$
- **Stable equilibrium**: local minimum (restoring force)
- **Unstable equilibrium**: local maximum
- A **potential well** traps particles with insufficient energy to escape
- The harmonic oscillator potential $U = \frac{1}{2}kx^2$ is the simplest potential well; square wells also exist (quantum mechanics)

3.1.3 Gravitational Potential Energy

By convention, $U = 0$ at $r \to \infty$. Since gravity is attractive, bringing masses closer *decreases* their potential energy, so:

$$\boxed{U = -\frac{Gm_1 m_2}{r}}$$

The force is related to the potential by:

$$F = -\frac{dU}{dr} = -\frac{Gm_1 m_2}{r^2}\,\hat{r}$$

The negative sign means the force points toward decreasing r (attractive).

Example: On the surface of the Earth.

$$F = \frac{Gm_1 m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 8 \times 10^1}{(6.4 \times 10^6)^2} \approx 588 \text{ N}$$

Change in gravitational PE between two radii r_1 and r_2:

$$\Delta U = -\frac{Gm_1 m_2}{r_2} + \frac{Gm_1 m_2}{r_1} = Gm_1 m_2 \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

3.1.4 Force, Energy, and Work

Force is the derivative of potential energy:

$$F = -\frac{dU}{dx}$$

Conversely, potential energy is the negative integral of force:

$$dU = -F \cdot dx \qquad \Rightarrow \qquad U = -\int F \cdot dx \quad \text{(always)}$$

Work is defined as the integral of force along a displacement:

$$W = \int \vec{F} \cdot d\vec{x}$$

How much work does the Moon do sitting on the road due to the Earth? Zero — no displacement, no work.

3.1.5 Conservation of Energy

Energy is always conserved:

$$\Delta E = 0 \qquad E_i = E_f \quad \text{for an isolated system}$$

$$\Delta E_{\text{system}} + \Delta E_{\text{environment}} = 0$$

Conservation principles (like conservation of energy) are extremely powerful problem-solving tools.

Example: Mass on a spring. Consider a mass that can have gravitational PE, elastic PE, kinetic energy, bulk KE, and thermal energy:

> ⚠ TODO: Diagram
>
> Vertical mass-spring system showing the different energy types: gravitational PE at the top, elastic PE in the spring, and kinetic energy of the mass.

3.1.6 Hooke's Law and Spring Energy

The restoring force of a spring:

$$F_{\text{spring}} = -F_{\text{app}} = -k\Delta x$$

$$F = -kx$$

The work done compressing/stretching a spring (equivalently, the elastic PE stored):

$$W = \int_0^x F\,dx$$

$$\boxed{W = \frac{1}{2}kx^2}$$

3.1.7 Mechanical Energy and Conservative Forces

A **conservative force** is one where only the endpoints matter — the work done is path-independent. For conservative forces, we can define a potential energy, and mechanical energy $E = U + K$ is conserved.

Gravitational and elastic forces are conservative. Electrostatic forces are also conservative.

Non-conservative (dissipative) forces convert mechanical energy into thermal and microscopic energy. Friction and air resistance are examples.

$$U = Mgh \qquad KE = \frac{1}{2}mv^2$$

When choosing a reference point for potential energy, you must be consistent — pick one and stick with it.

3.2 Newton's Laws and Forces

3.2.1 Potential vs. Potential Energy

An important distinction:

- **Potential energy** is an *extensive* property — depends on the specific object ($U_{\text{grav}} = -\frac{Gm_1m_2}{r}$)
- **Potential** is an *intensive* property — depends only on the source ($V_{\text{grav}} = -\frac{Gm}{r}$)

Intensive units = extensive units / mass (or charge).

3.2.2 Conservation Laws

Many quantities are conserved in isolated systems:

- Energy
- Momentum
- Angular momentum
- Mass
- Charge
- (and others: strangeness, baryon number, etc.)

3.2.3 Newton's Laws

1. Every body has constant velocity unless acted upon by a non-zero net force (inertia).
2. $\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$
3. For every action, there is an equal and opposite reaction.

Force has units of $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2} = \mathrm{N}$.

Weight $\neq$ mass. Weight is mg.

3.2.4 Free Body Diagrams

Rules for drawing free body diagrams (FBDs):

- Draw a box or lump (depending on state)
- One body, simple, all forces on the same object on the diagram
- Use vector component forces — only forces acting on the object
- Only include *forces*, not velocities or accelerations
- Everything you experience (forces) except for gravity is electromagnetic in nature

Example: Car going around a corner at constant speed. What is the net force acting on the car, and what is its origin? The force is **centripetal**, directed toward the center of the turn, provided by friction between the tires and road.

3.2.5 Normal Force

The normal (contact) force acts perpendicular to a surface. It is the surface's response to prevent objects from passing through.

- Usually equal to the weight component perpendicular to the surface
- It has a *maximum* value — the surface can be overwhelmed (e.g. breaking through)

Example: Elevator.

$$\sum F = ma$$

If the elevator accelerates upward at acceleration a:

$$N - mg = ma \qquad \Rightarrow \qquad N = m(g + a)$$

3.2.6 Friction

Static friction acts parallel to the surface to prevent relative motion:

- Cold welding of microscopic surface bumps
- $F_s \leq \mu_s N$ (it adjusts up to a maximum)

Kinetic friction involves continuous formation and breakage of bonds:

- $F_k = \mu_k N$ (constant for a given pair of surfaces)
- This is an *approximation* (only a model)

Example: Box on back of a ute (truck).

Maximum acceleration before sliding:

$$F_{\text{max}} = \mu_s N = \mu_s mg$$

$$a_{\text{max}} = \mu_s g \quad \text{(independent of mass)}$$

3.2.7 Air Resistance

$$F_{\text{air}} = \frac{1}{2}\rho v^2 C_D A$$

At **terminal velocity**, the drag force equals the weight: $F_{\text{air}} = mg$.

3.2.8 Tension

In an ideal rope:

- Force at one end is instantly transmitted without loss — every part feels the same tension
- $\theta_{\text{ideal rope}} = 0$ (massless, frictionless)
- Ideal ropes have no mass and are inextensible ($T = \text{const}$ throughout)
- Ideal pulleys act to change direction but nothing else
- $N \leq F_{\text{pull}}$: varies depending on applied force
- Two components of contact force (normal and friction) — minimum energy

3.2.9 Momentum

$$\vec{p} = m\vec{v} \qquad \text{units: kg} \cdot \text{m} \cdot \text{s}^{-1} = \text{N} \cdot \text{s}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

No force $\Rightarrow$ no change in momentum.

Example: Two carts, one mass m, the other mass $2m$, pushed by equal force for 3 seconds on an air track.

Equal momenta: $p_1 = p_2$

$$KE \propto \frac{p^2}{2m}$$

Since p is the same, the lighter cart has more KE. It travels distance $2d$ while the heavier travels d.

3.2.10 Newton's Third Law

Forces act on *different* objects:

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

Gravity and normal force on the same object are **not** a Newton's third law pair — they have different origins.

$$\frac{d\vec{p}}{dt} = 0 \quad \text{if } F_{\text{AB}} = -F_{\text{BA}} \quad \Rightarrow \quad \frac{dp_A}{dt} = -\frac{dp_B}{dt}$$

Therefore, **momentum is always conserved** in an isolated system.

3.3 Collisions

3.3.1 Elastic Collisions

In elastic collisions, both momentum and kinetic energy are conserved (gravity, conservative forces, Newton's cradle, gravitational slingshot).

Isolated system, two bodies:

$$\sum_i m_i v_i = C$$

$$m_A v_{Ai} + m_B v_{Bi} = m_A v_{Af} + m_B v_{Bf}$$

$$\frac{1}{2} m_A v_{Ai}^2 + \frac{1}{2} m_B v_{Bi}^2 = \frac{1}{2} m_A v_{Af}^2 + \frac{1}{2} m_B v_{Bf}^2$$

Solving these two equations simultaneously gives the final velocities:

$$\boxed{v_{Af} = \frac{m_A - m_B}{m_A + m_B}\, v_{Ai} + \frac{2m_B}{m_A + m_B}\, v_{Bi}}$$

$$\boxed{v_{Bf} = \frac{2m_A}{m_A + m_B}\, v_{Ai} + \frac{m_B - m_A}{m_A + m_B}\, v_{Bi}}$$

Limiting cases:

- If the masses are equal ($m_A = m_B$): they swap velocities
- If the masses are vastly different ($M \gg m$): $v_{Bf} \approx v_{Bi}$, the heavy object is barely affected
- If B is initially at rest ($v_{Bi} = 0$) and $M \gg m$: $v_{Af} \approx -v_{Ai}$ (bounces back), $v_{Bf} \approx 0$

3.3.2 Inelastic Collisions

In inelastic collisions, kinetic energy is lost (to deformation, heat, sound, etc.), but **momentum is still conserved**.

Example: Rain falling into a moving sports car (coasting).

As rain accumulates, the speed decreases. If you drill a hole to let the water out, the car does *not* speed back up — unless the water is pumped out stationary with respect to the road.

For a perfectly inelastic collision (objects stick together):

$$m_A v_{Ai} + m_B v_{Bi} = (m_A + m_B) v_f$$

$$\boxed{v_f = \frac{m_A v_{Ai} + m_B v_{Bi}}{m_A + m_B}}$$

3.4 Rotational Mechanics

3.4.1 Rotational Kinematics

A **rigid body** has every point moving in a circular path, with the centers of those circles on a common axis. It can be approximated as a single point for pure rotation.

Variables:

Table 3.2: Linear vs. Rotational Quantities

Linear	Rotational	Relation
x (displacement)	θ (angle, rad)	$s = r\theta$
v (velocity)	ω (angular velocity, rad/s)	$v = r\omega$
a (acceleration)	α (angular acceleration, rad/s^2)	$a = r\alpha$

Positive direction is arbitrary but must be defined (right-hand rule).

Equations of motion (identical in form to linear kinematics):

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Angular velocity direction given by the **right-hand rule**: curl fingers in the direction of rotation, thumb points along the axis.

3.4.2 Torque

Torque is the rotational analogue of force. It depends on where and in what direction the force is applied:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\tau| = rF \sin\theta$$

where θ is the angle between $\vec{r}$ and $\vec{F}$.

3.4.3 Moment of Inertia

The moment of inertia I is the rotational analogue of mass — it measures resistance to angular acceleration.

- Single particle: $I = mr^2$
- Collection of particles: $I = \sum_i m_i r_i^2$
- Solid body: $I = \iiint r^2 \, dm$

3.4.3.1 Parallel Axis Theorem

$$I = I_{cm} + Mh^2$$

where h is the distance between the center-of-mass axis and the parallel axis.

3.4.3.2 Perpendicular Axis Theorem

For a thin, flat object (2D):

$$I_z = I_x + I_y$$

3.4.4 Moment of Inertia of a Sphere

For a uniform solid sphere of mass M and radius R, using cylindrical polar coordinates:

$$I = \iiint r_\perp^2 \, dm = \rho \iiint r_\perp^2 \, dV$$

Carrying out the integration:

$$I = 2\pi\rho \int_0^R \left(\frac{(R^2 - z^2)^2}{4} \right) dz = \frac{8\pi\rho}{15} R^5$$

Since $M = \rho V = \rho \frac{4}{3} \pi R^3$:

$$\boxed{I = \frac{2}{5} M R^2}$$

3.4.5 Newton's Second Law for Rotation

$$\tau = I\alpha$$

This is the rotational analogue of $F = ma$.

Example: Rolling Without Slipping

A ball of mass M and radius R rolls without slipping down an incline of angle θ.

⚠ TODO: Diagram

Ball rolling down an incline. Show forces: weight Mg at center (downward), normal force N perpendicular to surface, friction f at contact point (up the incline). Coordinate axes: x along incline (positive downhill), y perpendicular to incline.

Rolling constraint: $v = R\omega$, so $a = R\alpha$.

Translational (along the incline):

$$Mg \sin\theta - f = Ma$$

Rotational (about center of mass):

$$fR = I\alpha = I\frac{a}{R}$$

$$f = \frac{Ia}{R^2}$$

Substituting:

$$Mg\sin\theta - \frac{Ia}{R^2} = Ma$$

$$a = \frac{Mg\sin\theta}{M + I/R^2} = \frac{g\sin\theta}{1 + I/(MR^2)}$$

For a solid sphere with $I = \frac{2}{5}MR^2$:

$$\boxed{a = \frac{5}{7}g\sin\theta}$$

Objects with smaller $I/(MR^2)$ roll faster. A sliding frictionless block ($I = 0$) has $a = g\sin\theta$, the maximum.

3.4.6 Rotational Kinetic Energy

For pure rotation:

$$K = \frac{1}{2}I\omega^2$$

For combined rotation and translation:

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_{cm}^2$$

Rotational work:

$$W = \int \tau\, d\theta$$

Comparison of rolling objects down a ramp. The total kinetic energy is $K = \frac{1}{2}mv^2(1+k)$ where $k = I/(mR^2)$:

Table 3.3: Moments of Inertia for Rolling Objects

Object	I	$k = I/(mR^2)$
Solid sphere	$\frac{2}{5}MR^2$	2/5
Solid cylinder	$\frac{1}{2}MR^2$	1/2
Hoop	MR^2	1

The lower k, the higher the translational speed — the **sphere is fastest**.

3.4.7 Angular Momentum

Angular momentum is the rotational analogue of linear momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

For a rigid body with axial symmetry:

$$L = I\omega$$

Angular momentum and torque:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

This can be broken into components: $\sum \tau_x = \frac{dL_x}{dt}$, etc.

Steiner's theorem (parallel axis for angular momentum):

$$L = L_{cm} + MR_{cm} \times v_{cm}$$

3.4.8 Conservation of Angular Momentum

If $\tau_{\text{ext}} = 0$, then $\frac{dL}{dt} = 0$, so $L = \text{const}$.

$$I_i\omega_i = I_f\omega_f$$

Example: Girl on a merry-go-round. A girl of mass m walks from the rim (radius R) toward the center of a merry-go-round (moment of inertia I_0, initial angular velocity ω_0).

At the rim: $L = (I_0 + mR^2)\omega_0$

At radius r: $L = (I_0 + mr^2)\omega$

$$\omega = \frac{(I_0 + mR^2)\omega_0}{I_0 + mr^2}$$

As $r \to 0$: ω increases (she spins faster as she moves inward).

Angular momentum does **not** imply circular motion — objects moving in a straight line have angular momentum relative to any external point.

3.4.9 Precession

⚠ TODO: Diagram

Spinning top with gravity pulling down at center of mass. Show the torque $\tau = rmg\sin\theta$ causing the angular momentum vector $\vec{L}$ to precess around the vertical axis. Include: the precession circle traced by the tip of $\vec{L}$, the angle θ from vertical, and the precession angular velocity ω_p.

Precession is caused by a small lateral torque that changes the *direction* of the angular momentum vector without affecting its magnitude.

For a spinning top tilted at angle θ from vertical:

- Torque: $\tau = rmg\sin\theta$
- The torque causes $\vec{L}$ to rotate horizontally: $dL = L\sin\theta\, d\phi$

$$\omega_p = \frac{d\phi}{dt}$$

Since $\tau\, dt = dL = L\sin\theta\, d\phi$:

$$\omega_p \, dt = \frac{dL}{L \sin\theta} = \frac{\tau \, dt}{L \sin\theta}$$

$$\boxed{\omega_p = \frac{rmg}{L} = \frac{rmg}{I\omega}}$$

The precession rate is inversely proportional to L (the speed of the top) — a slower top precesses in wider, faster circles.

3.5 Gravity and Orbits

3.5.1 Newton's Law of Gravitation

Gravity is an attractive force between any two masses:

$$\vec{F} = -\frac{Gm_1 m_2}{r^2}\,\hat{r}$$

Shell theorem: For multiple masses, use vector superposition (or potentials). Spherically symmetric objects behave exactly like point masses (when you are outside them).

3.5.2 Gravitational Field

The gravitational field is the force per unit mass — an *intensive*, vector field:

$$\vec{g} = -\frac{GM}{r^2}\,\hat{r}$$

Field lines:

- Show direction of force at a point in space
- Emanate from masses
- Never cross around sharp edges
- Density of field lines gives relative field strength

Flux is the measure of field lines passing through a surface:

$$\Phi_{\text{grav}} = \oint \vec{g} \cdot d\vec{A}$$

3.5.3 Gauss's Law for Gravity

The net flux through a closed surface depends only on the enclosed mass:

$$\boxed{\oint \vec{g} \cdot d\vec{A} = -4\pi G M_{\text{enclosed}}}$$

Use this (with a good choice of Gaussian surface) when there is symmetry.

3.5.4 Gravitational Field Inside and Outside a Sphere

Outside the Earth (or any spherically symmetric body):

$$g = \frac{GM}{r^2}$$

Identical to a point mass at the center.

Inside a hollow shell:

$$g = 0$$

Apply Gauss's law: the enclosed mass is zero, so the field inside a uniform shell vanishes. This is a remarkable result.

Inside a uniform solid sphere (at radius $r < R$):

The enclosed mass is $M_{\text{enc}} = M\frac{r^3}{R^3}$ (assuming uniform density), so:

$$g = \frac{GM_{\text{enc}}}{r^2} = \frac{GM}{R^3}\,r$$

The field increases linearly with r inside the sphere.

3.5.5 Gravitational Potential Energy (General)

Potential energy is extensive (depends on both masses). Define $U = 0$ at $r \to \infty$:

$$U = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = -\int_{\infty}^{r} \frac{GMm}{r^2}\,dr$$

$$\boxed{U = -\frac{GMm}{r}}$$

3.5.6 Gravitational Potential

Potential is the potential energy per unit mass — an *intensive* scalar:

$$V = \frac{U}{m} = -\frac{GM}{r}$$

Table 3.4: Extensive vs. Intensive Quantities

Property	Extensive	Intensive
Energy	E (J)	—
Gravitational	U (J)	V (J/kg)
Electrostatic	U (J)	V (V)

3.5.7 Equipotentials

- Lines or surfaces of equal potential
- Perpendicular to field lines (e.g., perpendicular to gravity)
- In a conservative field, journeys that start and end on the same equipotential require no net energy

⚠ TODO: Diagram

Equipotential lines around a point mass, showing concentric circles with field lines (arrows) pointing radially inward, perpendicular to the equipotentials. Include a hilly landscape analogy showing contour lines.

3.5.8 Escape Velocity

The velocity needed to escape a gravitational field entirely (reach $r \to \infty$ with $v = 0$). By conservation of energy:

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{r} = 0$$

$$\boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

3.5.9 Kepler's Laws

1. All planets move in **elliptical orbits** with the Sun at one focus.
2. A line joining a planet to the Sun sweeps out **equal areas in equal times** (conservation of angular momentum).
3. The square of the orbital period is proportional to the cube of the semi-major axis:

$$T^2 \propto a^3$$

Proof of Kepler's Third Law (for circular orbits):

For a circular orbit, gravitational force provides centripetal acceleration:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = mr\omega^2$$

$$\omega^2 = \frac{GM}{r^3} \qquad \Rightarrow \qquad \frac{4\pi^2}{T^2} = \frac{GM}{r^3}$$

$$\boxed{\frac{T^2}{r^3} = \frac{4\pi^2}{GM}}$$

3.6 Fluids

3.6.1 Pressure

Fluids are things that flow. We describe them using **pressure** and **volume** instead of mass and force.

$$P = \frac{F}{A}$$

Units: $\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2} = \text{Pa}$ (pascals). Also: $1 \text{ atm} = 760 \text{ mmHg} = 1.4 \text{ PSI} = 1.01 \times 10^5 \text{ Pa}$.

Pressure is a **scalar** — it has no direction (it acts equally in all directions at a point in a fluid).

Gauge pressure vs. **absolute pressure:**

- Gauge pressure is measured relative to atmospheric pressure
- Absolute = gauge + atmospheric

- Example: blood pressure of 120 mmHg is *gauge*; tire pressure of 12 kPa is *gauge*; barometric pressure of 102 kPa is *absolute*

3.6.2 Hydrostatic Pressure

The pressure due to a static fluid column, from the weight of fluid above:

$$F = W = mg = \rho V g = \rho A h g$$

$$P = \frac{F}{A} = \rho g h$$

$$\boxed{P = \rho g h + P_0}$$

where P_0 is the pressure at the top of the fluid (e.g. atmospheric pressure).

Key insight: Pressure depends only on *depth*, not on the shape of the container.

Example: Designing a dam. For a wall of width w and depth h, the pressure at depth d is $P = \rho g d$. The total force requires integrating over the wall area.

3.6.3 Archimedes' Principle

A submerged object displaces its own volume of fluid. This allows volume estimation of irregular objects (and fat content of people).

Buoyancy: A body fully or partially submerged is buoyed up by a force equal to the *weight of the fluid it displaces*. Air also exerts a buoyancy force.

$$F_{\text{buoy}} = \rho_{\text{fluid}} V_{\text{displaced}}\, g$$

For a floating object: $F_{\text{buoy}} = W$, so:

$$\rho_{\text{fluid}} V_{\text{displaced}} g = mg$$

Conditions:

- $\rho_{\text{object}} > \rho_{\text{fluid}}$: sinks ($W > F_b$)
- $\rho_{\text{object}} < \rho_{\text{fluid}}$: floats ($F_b > W$, partially submerged)
- $\rho_{\text{object}} = \rho_{\text{fluid}}$: neutral buoyancy (total submersion)

For a partially submerged object: $F_b = W$, so $\rho_{\text{fluid}} V_{\text{displaced}} g = m_{\text{object}} g$, or equivalently $V_{\text{displaced}}/V_{\text{object}} = \rho_{\text{object}}/\rho_{\text{fluid}}$.

Example: Ice cubes. Two identical glasses of water, one with an ice cube, one without. Do they have the same water level? Yes — the ice displaces exactly its own weight in water. When it melts, $W_{\text{ice}} = W_{\text{displaced water}}$, so the level stays the same.

Buoyancy Example: Helium Airship

What volume of He is needed to lift a 1000 kg airship?

> ⚠ TODO: Diagram
>
> Airship (blimp shape) with volume V_H of helium, payload mass M. Show forces: buoyancy $F_b = \rho_a V_H g$ upward, weight $Mg + \rho_H V_H g$ downward.

At equilibrium ($\sum F = 0$):

$$\rho_a V_H g - Mg - \rho_H V_H g = 0$$

$$V_H(\rho_a - \rho_H) = M$$

$$V_H = \frac{M}{\rho_a - \rho_H}$$

3.6.4 Real vs. Ideal Fluids

Ideal Fluid	Real Fluid
Non-viscous (no internal friction)	Viscous
Incompressible (density is constant)	May be compressible
Laminar, non-rotational flow	Can become turbulent

Viscous fluids slow down — important in fuel, oil, blood flow. Viscosity usually decreases with increasing temperature (engine oil gets runnier when warm).

3.6.5 Continuity Equation

For an incompressible fluid, what goes in must come out. The **volume flow rate** is constant:

$$Q = Av = \text{constant} \qquad \text{units: m}^3\text{s}^{-1}$$

$$\boxed{A_1 v_1 = A_2 v_2}$$

If the pipe narrows, the fluid speeds up.

3.6.6 Bernoulli's Equation

Once a flow is established, two conservation principles apply:

- **Continuity**: conservation of mass (volume flow rate is constant)
- **Bernoulli's equation**: conservation of energy (energy density is constant)

$$\boxed{P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$$

This is conservation of energy density along a streamline:

- P: pressure energy (electrostatic analogy)
- $\frac{1}{2}\rho v^2$: kinetic energy density
- $\rho g h$: gravitational potential energy density

Note: The pressure in Bernoulli's equation is the pressure the walls of the pipe exert on the fluid. This is *not* conservation of momentum (a common misconception).

Example: Water flowing through a pipe. At a constriction, v increases (continuity), so P must decrease (Bernoulli). This is a pressure drop: $\Delta P = -\frac{1}{2}\rho v^2$.

3.6.7 Bernoulli Applications

Venturi effect: Where fluid speeds up, pressure drops. This explains:

- Lift on an aerofoil (faster flow over top $\rightarrow$ lower pressure $\rightarrow$ net upward force)
- Roofs blowing off in storms (fast wind over roof, still air underneath)
- Aeroplane lift: $F_{\text{lift}} \approx \frac{1}{2}\rho v^2 A$

⚠ TODO: Diagram

Aerofoil cross-section showing streamlines compressed above (faster, lower pressure) and wider below (slower, higher pressure). Include net upward force arrow.

Open windows on the lee side to equalize pressure (how aeroplanes fly, how roofs stay on).

Bernoulli's equation can only be applied along a continuous streamline.

3.6.8 Poiseuille's Law

For viscous flow through a pipe of radius r and length ΔL, the volume flow rate is:

$$\boxed{Q = \frac{\pi r^4}{8\eta}\frac{\Delta P}{\Delta L}}$$

where η is the dynamic viscosity. Note the extremely strong r^4 dependence — doubling the radius increases flow by a factor of 16.

Flow vs. flux: Flow (Q) is volume per time; flux is flow per unit area.

4 Thermodynamics

Status

Content to be transcribed from IPhO notes.

5 Waves & Oscillations

Status

Content to be transcribed from IPhO notes.

Part III

Electromagnetism & Optics

6 Electrostatics

> **Status**
>
> Content to be transcribed from IPhO notes.

7 Circuits & Magnetism

Status

Content to be transcribed from IPhO notes.

8 Optics

Status

Content to be transcribed from IPhO notes.

Part IV

Modern Physics

9 Modern Physics & Relativity

Status

Content to be transcribed from IPhO notes.

References